MA1 Integrály

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1 Integrály

$$\int f(x)\,\mathrm{d}x \;\dots\; \text{neurčitý integrál}$$
$$\int f(x)\,\mathrm{d}x = F(x) \;\dots\; \text{primitivní funkce k funkci } f(x)$$
$$F'(x) = f(x) \;\dots\; \text{derivace primitivní funkce, } F(x) \text{, je funkce } f(x)$$

Pravidlo sumy

$$\int \Big(f(x) \pm g(x)\Big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

Pravidlo konstanty

$$\int k f(x)\,\mathrm{d}x = k \int f(x)\,\mathrm{d}x$$

Metoda Per Partes

$$(f \cdot g)' = f'g + fg'$$
$$\int (f \cdot g)' = \int f'g + \int fg'$$
$$f \cdot g = \int f'g + \int fg'$$

$$\textbf{I.}\;\int f'g = fg - \int fg'$$
$$\textbf{II.}\;\int fg' = fg - \int f'g$$

Subtituční metoda

Pokud pro funkce f a g platí, že

1. funkce f má primitivní funkci F na intervalu (a, b),
2. funkce g je na intervalu (α, β) diferenciovatelná
3. a $g((\alpha, \beta)) \subset (a, b)$,

pak funkce $f(g(x)) \cdot g'(x)$ má primitivní funkci na intervalu (α, β) a platí

$$\int f(g(x)) \cdot g'(x) = F(g(x))\,.$$

$$\int f(g(x))\, g'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t\,,\ \text{kde}$$
$$t = g(x)\,;\ \mathrm{d}t = g'(x)\,\mathrm{d}x.$$

2 Příklady

Vypočítej integrál

$$\int \pi x^{2024}\,\mathrm{d}x$$

$$\begin{aligned} &= \int \pi x^{2024}\,\mathrm{d}x && (1)\\ &= \pi \int x^{2024}\,\mathrm{d}x && (2)\\ &= \pi \frac{x^{2025}}{2025} && (3)\\ &= \frac{\pi}{2025} x^{2025} + C && (4) \end{aligned}$$

Vypočítej integrál

$$\int (3x-2)e^{x}\,\mathrm{d}x$$

$$f(x) = 3x - 2 \rightarrow f'(x) = 3$$
$$g'(x) = e^{x} \rightarrow g(x) = e^{x}$$

Vybrání funkce $f(x)$ a $g(x)$.

$$\begin{aligned} &= (3x-2)e^{x} - \int 3e^{x}\,\mathrm{d}x && (5)\\ &= (3x-2)e^{x} - 3e^{x} && (6)\\ &= e^{x}\left((3x-2)-3\right) && (7)\\ &= e^{x}\left(3x-5\right) + C && (8) \end{aligned}$$

Vypočítej integrál

$$\int (2x+3)\cos 3x\,\mathrm{d}x$$

$$f(x) = 2x + 3 \rightarrow f'(x) = 2$$
$$g'(x) = \cos 3x \rightarrow g(x) = -\frac{1}{3}\sin 3x$$

Vybrání funkce $f(x)$ a $g(x)$

$$\begin{aligned} &= -\frac{1}{3}\sin 3x(2x+3) + \frac{2}{3}\int \sin 3x\,\mathrm{d}x && (9)\\ &= -\frac{1}{3}\sin 3x(2x+3) + \frac{2}{3}\left(-\frac{1}{3}\cos 3x\right) && (10)\\ &= -\frac{1}{3}\sin 3x(2x+3) - \frac{2}{9}\cos 3x + C && (11) \end{aligned}$$

Vypočítej integrál

$$\int (x^2 - x)e^{4x}\,\mathrm{d}x$$

$$f(x) = x^2 - x \to f'(x) = 2x - 1$$
$$g'(x) = e^{4x} \to g(x) = \frac{1}{4}e^{4x}$$

1. Vybrání funkce $f(x)$ a $g(x)$

$$= \frac{1}{4}(x^2 - x)e^{4x} - \frac{1}{4}\int (2x-1)e^{4x}\,\mathrm{d}x \tag{12}$$

$$f(x) = 2x - 1 \to f'(x) = 2$$
$$g'(x) = e^{4x} \to g(x) = \frac{1}{4}e^{4x}$$

2. Vybrání funkce $f(x)$ a $g(x)$

$$\begin{aligned}
&= \frac{1}{4}(x^2 - x)e^{4x} - \frac{1}{4}\left(\frac{1}{4}(2x-1)e^{4x} - \frac{1}{2}\int e^{4x}\,\mathrm{d}x\right) && (13)\\
&= \frac{1}{4}(x^2 - x)e^{4x} - \frac{1}{4}\left(\frac{1}{4}(2x-1)e^{4x} - \frac{1}{8}e^{4x}\right) && (14)\\
&= \frac{1}{4}(x^2 - x)e^{4x} - \frac{1}{16}(2x-1)e^{4x} + \frac{1}{32}e^{4x} && (15)\\
&= \frac{1}{32}e^{4x}\left(8(x^2 - x) - 2(2x-1) + 1\right) && (16)\\
&= \frac{1}{32}e^{4x}\left(8x^2 - 8x - 4x - 2 + 1\right) && (17)\\
&= \frac{1}{32}e^{4x}\left(8x^2 - 12x - 1\right) + C && (18)
\end{aligned}$$

Vypočítej integrál

$$\int \ln x\,\mathrm{d}x$$

$$f(x) = \ln x \to f'(x) = \frac{1}{x}$$
$$g'(x) = 1 \to g(x) = x$$

Vybrání funkce $f(x)$ a $g(x)$

$$\begin{aligned}
&= x\ln x - \int \frac{1}{x}x\,\mathrm{d}x && (19)\\
&= x\ln x - \int 1\,\mathrm{d}x && (20)\\
&= x\ln x - x && (21)\\
&= x(\ln x - 1) + C && (22)
\end{aligned}$$

Vypočítej integrál

$$\int \ln^2 x \,\mathrm{d}x$$

$$f(x) = \ln^2 x \,\rightarrow\, f'(x) = 2\ln x \frac{1}{x}$$

$$g'(x) = 1 \,\rightarrow\, g(x) = x$$

Vybrání funkce $f(x)$ a $g(x)$

$$x\ln^2 x - \int 2\ln x \cdot \frac{1}{x} x \,\mathrm{d}x \tag{23}$$

$$x\ln^2 x - 2\int \ln x \cdot 1 \,\mathrm{d}x \tag{24}$$

$$x\ln^2 x - 2\int \ln x \,\mathrm{d}x \tag{25}$$

$$x\ln^2 x - 2\left(x\left(\ln x - 1\right)\right) \tag{26}$$

$$x\ln^2 x - 2x\ln x + 2x \tag{27}$$

$$x\left(\ln^2 x - 2\ln x + 2\right) + C \tag{28}$$

Vypočítej integrál

$$\int x \arctan x \,\mathrm{d}x$$

$$f(x) = \arctan x \,\rightarrow\, f'(x) = \frac{1}{1+x^2}$$

$$g'(x) = x \,\rightarrow\, g(x) = \frac{1}{2}x^2$$

Vybrání funkce $f(x)$ a $g(x)$

$$= \frac{1}{2}x^2 \arctan x - \int \frac{1}{1+x^2}\frac{1}{2}x^2 \,\mathrm{d}x \tag{29}$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{1}{1+x^2}x^2 \,\mathrm{d}x \tag{30}$$

$$\frac{1}{1+x^2}x^2 = \frac{x^2}{1+x^2} = \frac{x^2+1-1}{x^2+1} =$$

$$= 1 - \frac{1}{x^2+1}$$

Úprava výrazu

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\left(\int 1 - \frac{1}{x^2+1}\,\mathrm{d}x\right) \tag{31}$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\left(\int 1\,\mathrm{d}x - \int \frac{1}{x^2+1}\,\mathrm{d}x\right) \tag{32}$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}(x - \arctan x) \tag{33}$$

$$= \frac{1}{2}\left(x^2 \arctan x - (x - \arctan x)\right) \tag{34}$$

$$= \frac{1}{2}\left(x^2 \arctan x - x + \arctan x\right) + C \tag{35}$$

Vypočítej integrál

$$\int \frac{7x^2}{\sqrt{1+x^3}}\,\mathrm{d}x$$

$$t = 1 + x^3$$
$$\mathrm{d}t = 3x^2\,\mathrm{d}x$$

Substituce

$$= \int \frac{7x^2}{\sqrt{t}}\frac{1}{3x^2}\,\mathrm{d}t \tag{36}$$

$$= \int \frac{7}{\sqrt{t}}\frac{1}{3}\,\mathrm{d}t \tag{37}$$

$$= \frac{1}{3}\int \frac{7}{\sqrt{t}}\,\mathrm{d}t \tag{38}$$

$$= \frac{7}{3}\int \frac{1}{\sqrt{t}}\,\mathrm{d}t \tag{39}$$

$$= \frac{7}{3}\int \frac{1}{t^{\frac{1}{2}}}\,\mathrm{d}t \tag{40}$$

$$= \frac{7}{3}\int t^{-\frac{1}{2}}\,\mathrm{d}t \tag{41}$$

$$= \frac{7}{3}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \tag{42}$$

$$= \frac{7}{3}2t^{\frac{1}{2}} \tag{43}$$

$$= \frac{14}{3}t^{\frac{1}{2}} + C \tag{44}$$

Vypočítej integrál

$$\int \frac{x^9}{(1+x^5)^3}\,\mathrm{d}x$$

$$t = x^5 + 1$$
$$\mathrm{d}t = 5x^4\,\mathrm{d}x$$

Substituce

$$= \int \frac{x^9}{t^3} \frac{1}{5x^4} \, \mathrm{d}t \tag{45}$$

$$= \frac{1}{5} \int \frac{x^9}{t^3} \frac{1}{x^4} \, \mathrm{d}t \tag{46}$$

$$= \frac{1}{5} \int \frac{x^5}{t^3} \, \mathrm{d}t \tag{47}$$

$$= \frac{1}{5} \int \frac{t-1}{t^3} \, \mathrm{d}t \tag{48}$$

$$= \frac{1}{5} \left(\int t^{-2} \, \mathrm{d}t - \int t^{-3} \, \mathrm{d}t \right) \tag{49}$$

$$= \frac{1}{5} \left(-t^{-1} - \frac{t^{-2}}{-2} \right) \tag{50}$$

$$= \frac{1}{5} \left(-t^{-1} + \frac{1}{2} t^{-2} \right) \tag{51}$$

$$= -\frac{1}{5} t^{-1} + \frac{1}{10} t^{-2} + C \tag{52}$$

Vypočítej integrál

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2} \cdot \arccos^3 x}$$

$$\int \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\arccos^3 x} \, \mathrm{d}x \tag{53}$$

$$\tag{54}$$

$$t = \arccos x$$

$$\mathrm{d}t = -\frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x$$

Substituce

$$-\int \frac{1}{t^3} \, \mathrm{d}t = -\int t^{-3} \, \mathrm{d}t = \frac{t^{-2}}{2} + C \tag{55}$$

Vypočítej integrál

$$\int \frac{1}{x^2} \sin \frac{1}{x} \, \mathrm{d}x$$

$$\int x^{-2} \sin x^{-1} \, \mathrm{d}x \tag{56}$$

$$\tag{57}$$

$$t = x^{-1}$$

$$dt = -x^{-2} \, \mathrm{d}x$$

Substituce

$$= \int \frac{x^{-2}}{1} \sin(t) \left(-\frac{1}{x^{-2}}\right) \mathrm{d}t \tag{58}$$

$$= -\int \sin(t) \,\mathrm{d}t \tag{59}$$

$$= -\int \sin t \,\mathrm{d}t \tag{60}$$

$$= -(-\cos t) \tag{61}$$

$$= \cos t \tag{62}$$

$$= \cos \frac{1}{x} + C \tag{63}$$

Vypočítej integrál

$$\int \frac{1}{(1+x)\sqrt{x}} \,\mathrm{d}x$$

$$t = \sqrt{x} = x^{\frac{1}{2}}$$

$$\mathrm{d}t = \frac{1}{2} x^{-\frac{1}{2}} \,\mathrm{d}x = \frac{1}{2\sqrt{x}} \,\mathrm{d}x$$

Substituce

$$= \int \frac{1}{(1+t^2)t} \cdot \frac{1}{\frac{1}{2\sqrt{x}}} \,\mathrm{d}t \tag{64}$$

$$= \int \frac{1}{(1+t^2)t} \cdot \frac{2t}{1} \,\mathrm{d}t \tag{65}$$

$$= 2\int \frac{1}{1+t^2} \,\mathrm{d}t \tag{66}$$

$$= 2 \arctan t \tag{67}$$

$$= 2 \arctan \frac{1}{x} + C \tag{68}$$

Vypočítej integrál

$$\int \tan x \,\mathrm{d}x$$

$$\int \frac{\sin x}{\cos x} \,\mathrm{d}x \tag{69}$$

$$t = \cos x$$

$$\mathrm{d}t = -\sin x \,\mathrm{d}x$$

Substituce

$$= \int \frac{\sin x}{t} \cdot \frac{1}{-\sin x} \,\mathrm{d}t \tag{70}$$

$$= \int \frac{1}{t} \cdot \frac{1}{-1} \,\mathrm{d}t \tag{71}$$

$$= - \int \frac{1}{t} \,\mathrm{d}t \tag{72}$$

$$= -\ln |t| \tag{73}$$

$$= -\ln |\cos x| + C \tag{74}$$