QCD and Renormalisation Group Methods

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by

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Chapter 1

Introduction

In the following lecture, **renormalisation group methods** in the context of quantum field theory will be discussed. Most concepts and applications will be investigated on the basis of QCD, the theory of the strong interactions. The following material is mainly taken from the following textbooks and review articles [1–9]. Further references will be mentioned when they are required for a specific topic. Along the lecture, many **problems** will be presented which should be worked out explicitly in order to deepen the understanding of the subject, or practice computational skills. Often, the problems ask to perform a computation in a special case, in order to get the main idea, but then in the solution, more general cases are discussed, since the additional structure might shed further light on the interpretation of the results.

Historically, the method of the renormalisation group was first introduced by Stückelberg and Petermann [10] as well as Gell-Mann and Low [11], as a means to deal with the failure of perturbation theory in quantum electrodynamics at very high energies. Consider for example the photon vacuum polarisation function $\Pi_{\mu\nu}(q^2)$ (to be discussed further below). In perturbation theory, an n-loop contribution to this amplitude contains a factor α^n and up to n factors of $\ln(q^2/m_e^2)$ so that perturbation theory will break down when $\alpha|\ln(q^2/m_e^2)|$ is large, even though the fine structure constant α itself is small. The root of the problem lies in the very different scales considered: the electron mass m_e on the one hand and the momentum transfer $q^2 \gg m_e^2$ on the other. The renormalisation group methods will represent a way how to attack problems where largely different scales are involved.

The first chapter of the lecture will set up the stage for the discussion of the renormalisation group equation. After introducing the basics of QCD, the most common renor-

malisation procedure, namely dimensional regularisation will be discussed and applied to the one-loop renormalisation of the quark- and gluon propagators. Then, in the second chapter, the renormalisation group equation will be derived, and the dependence of the gauge coupling and quark masses on the renormalisation scale, the so-called "running", will be investigated in detail. Finally, in the third chapter, applications of the renormalisation group method will be presented. The consequences of the renormalisation group invariance for a physical quantity, the e^+e^--cross section to hadrons, will be worked out. Renormalisation group improvements of perturbation theory are discussed, and, as further applications, finite energy sum rules and contour-improved perturbation theory. Renormalisation group methods can also be applied in the framework of the operator product expansion, the renormalisation of composite operators and in particular the weak effective Hamiltonian. These topics will be the subject of the last sections.

1.1 QCD basics

The fundamental degrees of freedom in Quantum Chromodynamics (QCD) are the matter fields, the quarks, as well as the fields that mediate the strong interactions, the gluons. The classical Lagrange density of QCD is very similar to that of QED,

$$\mathcal{L}_{\rm QCD}(x) \;=\; -\frac{1}{4}\, G^a_{\mu\nu}(x)\, G^{\mu\nu\, a}(x) + \sum_A \Big[\, \frac{i}{2}\, \bar{q}^A(x)\gamma^\mu \overleftrightarrow{D}_\mu\, q^A(x) - m_A\, \bar{q}^A(x)\, q^A(x) \,\Big]\,. \quad (1.1)$$

Let us introduce the notation in some detail. $G^a_{\mu\nu}(x)$ is the field strength tensor corresponding to the gluon field $B^a_\mu(x)$, defined by

$$G^a_{\mu\nu}(x) \;\equiv\; \partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x) + g f^{abc} B^b_\mu(x) B^c_\nu(x)\,. \quad (1.2)$$

The Greek letters denote Lorentz indices, running from 0 to 3. The Latin letters $a, b, c = 1, \ldots, 8$ will denote colour indices in the adjoint representation of the colour group SU(3). Next, the f^{abc} are the structure constants of the colour group, which satisfy the relation

$$[\, T^a, T^b \,] \;=\; i\, f^{abc}\, T^c\,, \quad (1.3)$$

with T^a being the generators of SU(3). A particular representation – the fundamental representation – can be given in terms of the hermitian, traceless, 3×3 Gell-Mann matrices λ^a:

$$T^a \;=\; \frac{\lambda^a}{2} \qquad \text{with} \qquad {\rm Tr}\,[\lambda^a \lambda^b] \;=\; 2\,\delta^{ab}\,. \quad (1.4)$$

Finally, g is the coupling constant of QCD which quantifies the interaction strength between quarks and gluons and also the gluon self-coupling which arises due to the

non-abelian nature of the gauge group.

In the term containing the quarks fields $q^A_{\alpha i}(x)$, A denotes the flavour index running over the known quark flavours up, down, strange, charm, bottom and top. As far as the spinor (α) and colour (i) indices are concerned, we have used a matrix notation in order to suppress further indices. The main relation we shall need for the Dirac γ-matrices γ_μ (and which defines what is known as the Clifford-algebra) is given by

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu = 2\, g_{\mu\nu} \mathbb{1}\,. \tag{1.5}$$

Here $g_{\mu\nu} \equiv \mathrm{diag}\,(1,-1,-1,-1)$ represents the Minkowski-metric of flat space time. (Sometimes this is denoted by $\eta_{\mu\nu}$ with $g_{\mu\nu}$ being the one of curved space-time, but we shall not need the distinction below.) The covariant derivative D_μ (matrix in colour space) takes the form

$$D_\mu = \partial_\mu \mathbb{1} - i\, g\, \frac{\lambda^a}{2} B^a_\mu\,, \tag{1.6}$$

with the bidirectional arrow implying that the derivative acts to the right and, with a minus sign, also to left. Finally, m_A represents the mass term for a flavour of type A.

The classical QCD Lagrangian is constructed such as to maintain local SU(3)-colour gauge transformation. (This symmetry can easily be extended to an arbitrary number of colours N_c with the corresponding gauge group being SU(N_c).) Employing the Euler-Lagrange equations for the quark and gluon fields, one arrives at the following classical equations of motion:

$$\Big[\, i\,\gamma^\mu D_\mu - m_A \,\Big] q^A(x) = 0\,, \tag{1.7}$$

$$\Big[\, D^\mu, G^a_{\mu\nu}(x) \Big] = -\frac{1}{2}\, g \sum_A \bar{q}^A(x) \gamma_\nu \lambda^a q^A(x)\,. \tag{1.8}$$

The first equation corresponds to the Dirac equation in an external gluon background, and the second describes the coupling of the gauge field to a matter source. Thus the QCD Lagrangian describes free Dirac spinors and the non-abelian gauge field, together with their interaction as well as the self-interaction of the gluon field, due to the non-abelian term in eq. (1.2).

Now we can proceed with quantising the QCD Lagrangian. However, due to the gauge invariance, this is not straightforward. Like in QED, there are no problems with the spinor fields (quarks). If $\psi(x)$ represents a generic spinor field, the corresponding quantum field can be written in terms of creation and annihilation operators as:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})} \sum_\lambda \Big[\, u(\vec{p},\lambda)\, a(\vec{p},\lambda)\, \mathrm{e}^{-ipx} + v(\vec{p},\lambda)\, b^+(\vec{p},\lambda)\, \mathrm{e}^{ipx} \,\Big]\,, \tag{1.9}$$

where the integration ranges over the positive sheet of the mass hyperboloid $\Omega_+(m) = \{p \,|\, p^2 = m^2,\, p^0 > 0\}$ with m being the mass of the quark. The four-spinors $u(\vec{p},\lambda)$ and $v(\vec{p},\lambda)$ are solutions to the free Dirac equation in momentum space,

$$[\not{p} - m\,]u(\vec{p},\lambda) \;=\; 0 \qquad \text{and} \qquad [\not{p} + m\,]v(\vec{p},\lambda) \;=\; 0\,, \tag{1.10}$$

with λ representing the helicity state of the spinors. Employing the commutation relations for the creation and annihilation operators together with the above equations, one can show that the propagator (Green function) for the free spinor field is given by:

$$\begin{aligned} i\,S^{(0)}(x-y) \;&\equiv\; \langle 0|T\{\psi(x)\,\bar{\psi}(y)\}|0\rangle \;\equiv\; \overbrace{\psi(x)\,\bar{\psi}}(y) \\ &=\; i\int \frac{d^4p}{(2\pi)^4}\,\frac{(\not{p}+m)}{(p^2-m^2+i0)}\,\mathrm{e}^{-ip(x-y)} \;\equiv\; i\int \frac{d^4p}{(2\pi)^4}\,S^{(0)}(p)\,\mathrm{e}^{-ip(x-y)}\,, \end{aligned} \tag{1.11}$$

where the T-operator implies time-ordering of the operators in curly brackets and the underline denotes the so-called "contraction" of two field operators, being another notation for the vacuum-expectation value of the time-ordered product. The $i0$-prescription to shift the pole of the propagator is required in order to maintain causality.

The quantisation of the gauge field is much more cumbersome. Applying the canonical quantisation procedure in this case, since the conjugate momenta corresponding to $B^a_0(x)$ are zero – $\partial^0 B^a_0(x)$ does not appear in the Lagrangian ($G^a_{\mu\nu}(x)$ is antisymmetric!) – manifest covariance of the theory is lost. This problem is related to the fact that the vector field $B^a_\mu(x)$ describes four degrees of freedom, whereas the massless physical gluon field should only have two. Similar to QED, the problem can be solved by adding a gauge-fixing term $\mathcal{L}_{\mathrm{gf}}$ to the Lagrangian, with the explicit form:

$$\mathcal{L}_{\mathrm{gf}} \;=\; -\,\frac{1}{2a}\,[\partial^\mu B^a_\mu(x)][\partial^\nu B^a_\nu(x)]\,. \tag{1.12}$$

Proceeding with the canonical quantisation, like in QED, the Fock-space of states has indefinite metric. As was first proved by Gupta and Bleuler, however, in QED this does not matter since the probability of observing a longitudinal photon is exactly cancelled by the one of observing a time-like photon, such that the theory is unitary in the physical subspace of transverse photons.

In QCD, due to the self-coupling of the gluons, the Gupta-Bleuler mechanism is no longer valid. In order to restore unitarity in the physical subspace of the gluons, one is forced to introduce a supplementary set of massless non-physical fields, the so-called Faddeev-Popov ghosts $c^a(x)$, which obey Fermi-Dirac statistic and only couple to the gluons. The

corresponding piece $\mathcal{L}_{\rm ghost}$ which needs to be added to the Lagrangian reads:

$$\mathcal{L}_{\rm ghost} \;=\; -\,[\partial^\mu \bar{c}^a(x)]\partial_\mu c^a(x) + g f^{abc}[\partial^\mu \bar{c}^a(x)]c^b(x)B^c_\mu(x)\,. \tag{1.13}$$

Following now the different steps of the quantisation procedure, the free propagators for the quark, gluon and ghost fields are found to be:

$$i\,S^{(0)AB}_{ij}(x-y) \;\equiv\; \langle 0|T\{q^A_i(x)\,\bar{q}^B_j(y)\}|0\rangle \;\equiv\; \overbrace{q^A_i(x)\,\bar{q}^B_j}(y) \;=\; \delta_{ij}\delta^{AB} i\,S^{(0)}(x-y)\,, \tag{1.14}$$

$$\begin{aligned} i\,D^{(0)ab}_{\mu\nu}(x-y) \;&\equiv\; \langle 0|T\{B^a_\mu(x)B^b_\nu(y)\}|0\rangle \;\equiv\; \overbrace{B^a_\mu(x)B^b_\nu}(y) \;\equiv\; i\,\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\,D^{(0)}_{\mu\nu}(k)\,{\rm e}^{-ik(x-y)} \\ &=\; i\,\delta^{ab}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2+i0)}\,\Big[-g_{\mu\nu}+(1-a)\frac{k_\mu k_\nu}{(k^2+i0)}\Big]\,{\rm e}^{-ik(x-y)}\,, \end{aligned} \tag{1.15}$$

$$\begin{aligned} i\,\tilde{D}^{(0)ab}(x-y) \;&\equiv\; \langle 0|T\{c^a(x)c^b(y)\}|0\rangle \;\equiv\; \overbrace{c^a(x)\bar{c}^b}(y) \;\equiv\; i\,\delta^{ab}\int\frac{d^4q}{(2\pi)^4}\,\tilde{D}^{(0)}(q)\,{\rm e}^{-iq(x-y)} \\ &=\; i\,\delta^{ab}\int\frac{d^4q}{(2\pi)^4}\,\frac{-1}{(q^2+i0)}\,{\rm e}^{-iq(x-y)}\,, \end{aligned} \tag{1.16}$$

where the free fermion propagator was already given in eq. (1.11) above. Until now, we have implicitly assumed, that all products of field operators at equal space-time points are normal-ordered, which means that all creation operators are commuted such that they stand left of the annihilation operators. This way such products are renormalised, and contractions of field operators at equal points vanish by definition. We shall for the moment accept this procedure, but comment further on it below when the renormalisation of composite operators will be discussed.

In a conventional approach, the next step would be to state the Feynman rules for the interaction vertices as well as the symmetry factors and signs to be applied. However, we shall derive these later from the interaction part of the Lagrangian, so that there is no need to present them here. Calculating loop corrections to processes with the QCD Lagrangian discussed above will lead to divergences. In order to obtain finite physical quantities from the Lagrangian, the expressions arising from it have to be **regularised** (the divergences have to be made manifest) and they have to be **renormalised** (subtracted, which means redefined into the parameters which appear in the Lagrangian). This is what we shall discuss in the next two sections.

1.2 Dimensional regularisation

More than 30 years after its invention [12–14], for several reasons, dimensional regularisation has become the method of choice for regularising multi-loop Feynman diagrams. (In contrast to the lattice, where the lattice spacing provides a natural cutoff.) One of the reasons is that dimensional regularisation preserves gauge invariance, which is an important ingredient in proving the renormalisability of the theory. Also from the calculational point of view, it is the method which is implemented most easily. Therefore, in what follows, we shall employ dimensional regularisation to address the problem of regulating and renormalising the divergencies which arise when calculating loop-amplitudes in perturbation theory.

The main idea behind dimensional regularisation is simple. The whole theory formally has to be formulated for an arbitrary space-time dimension D, where for the moment we assume that the dimension is an integer, but we will soon see that the expressions which arise can even be continued to non-integer dimensions. The divergencies which arise then appear as poles of the resulting expressions if D goes to the physical dimension 4. Thus when expanding the result in $(D-4) \equiv 2\varepsilon$, the poles have been made manifest. There are various places where modifications due to the change $4 \to D$ have to be applied.

Problem 1.1: Knowing that the action in D space-time dimensions, $\int \mathcal{L}(x)\, d^D x$, should be dimensionless, calculate the dimensions of the quark, gluon and ghost fields, as well as of the gauge coupling g, in units of mass or momentum. (Throughout, we employ natural units with $c \equiv 1$ and $\hbar \equiv 1$.)

Problem 1.2: With the result of problem 1.1, calculate the mass dimension of the free momentum space quark and gluon propagators $S^{(0)}(p)$ and $D^{(0)}_{\mu\nu}(k)$ respectively.

Generally, dimensional analysis is a powerful tool, and we shall often pursue that later.

As a next step, we have to consider the **Dirac-algebra**, which also has to be carried out in an arbitrary dimension D. Formally, the defining equation takes the same form as in 4 dimensions,

$$\{\gamma_\mu, \gamma_\nu\} \;\equiv\; \gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu \;=\; 2\, g_{\mu\nu} \mathbb{1}\,, \tag{1.17}$$

but now the space-time indices μ, ν should be considered as running from $0, 1, 2, \ldots, D-1$. Furthermore, we have the trivial relations:

$$g_{\mu\nu} g^{\mu\nu} \;=\; D \qquad \text{and} \qquad \gamma_\mu \gamma^\mu \;=\; D\, \mathbb{1}\,, \tag{1.18}$$

the second of which immediately follow from (1.17), together with the first relation. The anti-commutation rules for the γ-matrices can be employed to explicitly carry out contractions of Lorentz indices which appear twice. As an **example**, calculate:

$$\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma^\mu = 2\gamma_\lambda\gamma_\rho\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda\gamma_\rho\gamma^\mu = 2\gamma_\lambda\gamma_\rho\gamma_\nu - 2\gamma_\nu\gamma_\rho\gamma_\lambda + \gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\rho\gamma^\mu \tag{1.19}$$

$$= 2\gamma_\lambda\gamma_\rho\gamma_\nu - 2\gamma_\nu\gamma_\rho\gamma_\lambda + (2-D)\gamma_\nu\gamma_\lambda\gamma_\rho = -2\gamma_\rho\gamma_\lambda\gamma_\nu + (4-D)\gamma_\nu\gamma_\lambda\gamma_\rho\,.$$

Problem 1.3: In a similar fashion, explicitly perform the contractions in the Dirac-matrix expressions $\gamma_\mu\gamma_\nu\gamma^\mu$ and $\gamma_\mu\gamma_\nu\gamma_\lambda\gamma^\mu$.

In practice, it is very tedious to carry out the contractions for long strings of γ-matrices. However, computer algebra programs exist to perform this task, even for the D-dimensional γ-algebra. (For example `Tracer.m` written in Mathematica which has been developed by the author.) (Maybe **Tracer-demonstration** if beamer is available.)

In what follows, we shall also need to calculate traces over strings of γ-matrices. These can be calculated by employing the cyclicity of the trace and the anti-commutation rule (1.17). Like in 4 space-time dimensions, also for an arbitrary dimension D, the trace over an odd number of γ-matrices vanishes. Thus we only need to consider traces over an even number of γ-matrices. Let us give two explicit examples. Firstly,

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu] = \mathrm{Tr}[\gamma_\nu\gamma_\mu] = \frac{1}{2}\,\mathrm{Tr}[\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu] = g_{\mu\nu}\mathrm{Tr}[\mathbb{1}] = 4g_{\mu\nu}\,. \tag{1.20}$$

Here, we have somewhat arbitrarily set $\mathrm{Tr}[\mathbb{1}] \equiv 4$. In principle, one could also take other choices, for example $\mathrm{Tr}[\mathbb{1}] \equiv 2^{D/2}$, which is the dimension of the irreducible representation of (1.17) for even D, and agrees with the first choice at $D = 4$, as it should. However, both choices only differ by terms of order ε, and when multiplied with a possible divergence in ε, the second choice yields additional constant contributions. As we shall later see in more detail, these constants can be absorbed in the renormalisation, and define what will be called the renormalisation scheme. In all our future calculations, the first choice, which has become more standard, will be adopted.

Now, as a second example, let us calculate the trace over a string of four γ-matrices:

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] = 8g_{\mu\nu}g_{\lambda\omega} - \mathrm{Tr}[\gamma_\nu\gamma_\mu\gamma_\lambda\gamma_\omega] = 8g_{\mu\nu}g_{\lambda\omega} - 8g_{\mu\lambda}g_{\nu\omega} + \mathrm{Tr}[\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\omega]$$

$$= 8g_{\mu\nu}g_{\lambda\omega} - 8g_{\mu\lambda}g_{\nu\omega} + 8g_{\mu\omega}g_{\nu\lambda} - \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega]\,.$$

Thus it follows that:

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] = 4(g_{\mu\nu}g_{\lambda\omega} - g_{\mu\lambda}g_{\nu\omega} + g_{\mu\omega}g_{\nu\lambda})\,. \tag{1.21}$$

As should be clear from these examples, the evaluation of the trace does not introduce additional dependencies on the dimension D because only the relation (1.17) has been used. This statement must obviously also hold true for an arbitrary string of γ-matrices.

An intricate problem for dimensional regularisation which should be mentioned, although it will not be needed in what follows, is the definition of the chirality (or handedness) operator γ_5. (Then $(\mathbb{1} \mp \gamma_5)/2$ are the operators which project out states of definite chirality.) One possible definition of γ_5 in 4 dimensions is given by

$$\gamma_5 \equiv \frac{i}{4!}\, \varepsilon_{\mu\nu\lambda\omega} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\omega \,, \tag{1.22}$$

where $\varepsilon_{\mu\nu\lambda\omega}$ is the totally antisymmetric Levi-Cività tensor with $\varepsilon^{0123} = 1$. From this definition, one can show that γ_5 anti-commutes with all other γ-matrices and satisfies:

$$\{\gamma_5, \gamma_\mu\} \;=\; 0\,, \qquad (\gamma_5)^2 \;=\; \mathbb{1} \qquad \text{and} \qquad \mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] \;=\; 4i\varepsilon_{\mu\nu\lambda\omega}\,. \tag{1.23}$$

Now, one could try to proceed with defining γ_5 anti-commuting with all other γ-matrices also for an arbitrary dimension D. However, together with the cyclicity of the trace, this requirement leads to inconsistencies.

Let us begin with the calculation of $\mathrm{Tr}[\gamma_5]$. This can be done by considering $\mathrm{Tr}[\gamma_5\gamma_\mu\gamma^\mu]$, and employing cyclicity of the trace and the anticommutation relation:

$$\mathrm{Tr}[\gamma_5\gamma_\mu\gamma^\mu] \;=\; D\,\mathrm{Tr}[\gamma_5] \;=\; -\,\mathrm{Tr}[\gamma_\mu\gamma_5\gamma^\mu] \;=\; -\,D\,\mathrm{Tr}[\gamma_5]\,. \tag{1.24}$$

From this result, we conclude that $D\,\mathrm{Tr}[\gamma_5] = 0$, and hence $\mathrm{Tr}[\gamma_5] = 0$ except eventually at $D = 0$. Since we demand that we are able to analytically continue our expressions in the dimension D, we would conclude that $\mathrm{Tr}[\gamma_5] = 0$ should hold for all D. In an analogous fashion, one can derive the relations:

$$(D-2)\,\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu] \;=\; 0 \qquad \text{and} \qquad (D-4)\,\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] \;=\; 0\,. \tag{1.25}$$

From the second of these relations, we would conclude that $\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\omega] \;=\; 0$ for all D, because of the analytic continuation property. However, this is in conflict with the result for this trace in $D = 4$ (1.23). Therefore, we either have to give up the property that our expressions can be continued analytically from an arbitrary dimension D to the 4 physical space-time dimensions, or we have to abandon the property that γ_5 anticommutes with all other γ-matrices.

There are several ways out of this problem, one of which was already proposed in the original papers by 't Hooft and Veltman. Namely to define γ_5 such that it anti-commutes

with the 4-dimensional subset of γ-matrices and that it commutes with the remaining $(D-4)$-dimensional ones. (HV-scheme.) This leads to somewhat cumbersome computations, and in addition care must be taken, since the rules lead to spurious anomalies and the breaking of chiral invariance as well as corresponding Ward-identities which have to be restored by hand. But for practical computations, the rules can be implemented into the algebraic programs so that no principle problem remains. In cases where no real anomaly contribution is present, and always even numbers of γ_5's appear in traces, one can also just stick to the conventional use of an anti-commuting γ_5. This is sometimes called "naive" dimensional regularisation or NDR-scheme. A nice discussion of these issues, together with further references, can for example be found in the recent review by Jegerlehner [15].

The final, and crucial, ingredient which is missing in order to give meaning to the approach, is the continuation of the **Feynman integrals** which arise in the calculations to an arbitrary dimension. Although these integrals will only appear later below, let us already discuss a particular integral at this point. Let us consider the following integral:

$$\tilde{I}(1,1;q^2,m^2) \;\equiv\; \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{[p^2-m^2+i0][(q-p)^2-m^2+i0]}\,. \tag{1.26}$$

As we have seen above, the propagators are not modified, when going from $4 \to D$. However, the integration measure, which stems from the Fourier-transform, should be continued to dimension D. However, this would alter the canonical dimension of momentum-space Green functions in whose calculation the integral arises. Now, we insist that the mass-dimension of Green functions is not altered when going from 4 to D dimensions, which forces us to introduce an arbitrary scale parameter μ. The procedure might appear somewhat ad hoc. However, as we shall see below, the integrals always appear multiplied with factors of the gauge coupling g. Also the gauge coupling should be defined dimensionless (like in 4 dimensions) which gives rise to a corresponding factor in μ such that the two modifications would compensate each other. The corresponding expression for the integral in D dimensions then reads:

$$\tilde{I}(1,1;q^2,m^2) \;\equiv\; \mu^{2\varepsilon}\int \frac{d^Dp}{(2\pi)^D}\,\frac{1}{[p^2-m^2+i0][(q-p)^2-m^2+i0]}\,. \tag{1.27}$$

Two further comments are in order: the continuation of the factor $(2\pi)^4$ is not really necessary, but has become convention. Different choices would again lead to additional constant terms which could be redefined in the renormalisation to be discussed below. The second comment is of more general nature. The regularisation of the divergent momentum integrals always necessitates the introduction of an additional scale. In the traditional

Pauli-Villars regularisation which historically was employed to renormalise QED, this happened to be the cutoff Λ, in dimensional regularisation, we are forced to introduce the scale μ, and as already mentioned above, on the lattice we have as a natural cutoff the lattice spacing which provides the additional scale. Nonetheless, the need of an additional scale in the problem is common to all approaches. Sometimes the appearance of such a new scale in the problem is called "dimensional transmutation" [16].

Now, let us continue with the calculation of the integral. A convenient way of proceeding to calculate such type of integrals is due to Feynman and consists in rewriting the integrand with the help of the following formula:

$$\frac{1}{a^\alpha b^\beta} \;=\; \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int\limits_0^1 dx\, \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}}\,, \tag{1.28}$$

where $\Gamma(z)$ denotes Eulers Γ-function.

Problem 1.4: Explicitly verify the relation (1.28) for the case $\alpha = 1$ and $\beta = 2$.

Problem 1.5: Show that with the Feynman parametrisation of eq. (1.28), the integral $\tilde{I}(1,1;q^2,m^2)$ can be brought into the form:

$$\tilde{I}(1,1;q^2,m^2) \;=\; \mu^{2\varepsilon} \int\limits_0^1 dx \int \frac{d^Dk}{(2\pi)^D}\, \frac{1}{[k^2-a^2]^2}\,, \tag{1.29}$$

with $a^2 \equiv m^2 - x(1-x)q^2$.

As far as the new type of integral is concerned, we shall directly consider a generalisation which does not pose additional difficulties, and can be solved along the same lines. Let

$$I(\alpha,\beta;a^2) \;\equiv\; \mu^{2\varepsilon} \int \frac{d^Dk}{(2\pi)^D}\, \frac{k^{2\alpha}}{[k^2-a^2+i0]^\beta}\,. \tag{1.30}$$

The first step of solving this integral consists in performing the so-called Wick rotation, which amounts to rotating the time-axis k^0 into an imaginary time direction ik^D. This is permissible as long as no poles are crossed in the imaginary time-plane when performing the rotation. For our integral this is indeed the case. Then we find:

$$I(\alpha,\beta;a^2) \;=\; \mu^{2\varepsilon} \int \frac{dk^0 dk^1 \ldots dk^{D-1}}{(2\pi)^D}\, \frac{\left[(k^0)^2 - \sum\limits_{i=1}^{D} (k^i)^2\right]^\alpha}{\left[(k^0)^2 - \sum\limits_{i=1}^{D} (k^i)^2 - a^2 + i0\right]^\beta}$$

$$= i\,(-1)^{\alpha-\beta}\mu^{2\varepsilon}\int\frac{dk^1\ldots dk^{D-1}dk^D}{(2\pi)^D}\,\frac{\left[(k^D)^2+\sum_{i=1}^{D}(k^i)^2\right]^{\alpha}}{\left[(k^D)^2+\sum_{i=1}^{D}(k^i)^2+a^2\right]^{\beta}}$$

$$= i\,(-1)^{\alpha-\beta}\mu^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\,\frac{k^{2\alpha}}{[k^2+a^2]^{\beta}}\,, \tag{1.31}$$

where the last integral has to be performed in an D-dimensional Euclidean space. Since it only depends on k^2, this integration is best performed in D-dimensional polar coordinates. This yields

$$I(\alpha,\beta;a^2) \;=\; i\,(-1)^{\alpha-\beta}\frac{\mu^{2\varepsilon}}{(2\pi)^D}\int_0^{\infty}dk\int d\Omega\,\frac{k^{2\alpha+D-1}}{[k^2+a^2]^{\beta}}\,. \tag{1.32}$$

The angular integration results in the area of a D-dimensional unit-sphere which is given by $S_D = 2\pi^{(D/2)}/\Gamma(D/2)$. (Try to derive this for yourself.) Finally, we are left with the remaining radial integration. The idea behind solving this integral is to rewrite it in terms of an integral which is Eulers beta-function $B(u,v)$ defined by:

$$B(u,v) \;\equiv\; \int_0^1 z^{u-1}(1-z)^{v-1}dz \;=\; \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}\,. \tag{1.33}$$

This transformation can be achieved with the substitution $k^2 = a^2(1-z)/z$ where z runs from 0 to 1. One then finds:

$$\begin{aligned} I(\alpha,\beta;a^2) \;&=\; i\,(-1)^{\alpha-\beta}\mu^{2\varepsilon}\,\frac{[a^2]^{D/2+\alpha-\beta}}{(4\pi)^{D/2}}\,\frac{\Gamma(\beta-\alpha-D/2)\Gamma(\alpha+D/2)}{\Gamma(\beta)\Gamma(D/2)} \\ &=\; \frac{i}{(4\pi)^2}\,[-a^2]^{\alpha-\beta+2}\left(\frac{4\pi\mu^2}{a^2}\right)^{\varepsilon}\,\frac{\Gamma(\beta-\alpha-2+\varepsilon)\Gamma(\alpha+2-\varepsilon)}{\Gamma(\beta)\Gamma(2-\varepsilon)}\,. \end{aligned} \tag{1.34}$$

Problem 1.6: Derive eq. (1.34) from eq. (1.32).

The important point about our result for $I(\alpha,\beta;a^2)$ is that the dimension D only appears in exponents and in the Γ-functions which allows to analytically continue our result also to non-integer D. Furthermore, if the integral is ill-defined, we can take eq. (1.34) as the definition for it. Divergencies of the integral will then appear as poles in the Γ-functions when $(4-D)/2 = \varepsilon \to 0$.

The result for $I(0,2;a^2)$ can now be employed in order to continue with the calculation

of $\tilde{I}(1,1;q^2,m^2)$. The resulting expression is:

$$\begin{aligned}\tilde{I}(1,1;q^2,m^2) &= \frac{i}{(4\pi)^2}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon}\Gamma(\varepsilon)\int_0^1 dx\ \left[\tfrac{m^2}{q^2}-x(1-x)\right]^{-\varepsilon} \\ &= \frac{i}{(4\pi)^2}\left\{\frac{1}{\varepsilon}-\gamma_{\rm E}+\ln 4\pi-\ln\frac{q^2}{\mu^2}-\int_0^1 dx\ \ln\left[\tfrac{m^2}{q^2}-x(1-x)\right]\right\}. \quad (1.35)\end{aligned}$$

In order to arrive at the second line, we have used the expansion for the Γ-function, $\Gamma(\varepsilon)=1/\varepsilon-\gamma_{\rm E}+\mathcal{O}(\varepsilon)$, and have taken the limit of ε going to zero. Here $\gamma_{\rm E}=0.577215665$ is the Euler constant. Since the divergencies in ε will always appear in the combination with $\gamma_{\rm E}$ and $\ln 4\pi$, we shall introduce the abbreviation

$$\frac{1}{\hat{\varepsilon}} \equiv \frac{1}{\varepsilon}-\gamma_{\rm E}+\ln 4\pi\,. \quad (1.36)$$

The reason for the $\gamma_{\rm E}$ term can be seen from the useful formula for the expansion of the Γ-function

$$\Gamma(1+z) = \exp\Big[-\gamma_{\rm E}\,z+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta_n}{n}\,z^n\Big], \quad (1.37)$$

where ζ_n is the Riemann ζ-function. The remaining integral can be done analytically leading to the final result

$$\tilde{I}(1,1;q^2,m^2) = \frac{i}{(4\pi)^2}\left\{\frac{1}{\hat{\varepsilon}}-\ln\frac{m^2}{\mu^2}+2-\sqrt{1-\tfrac{4m^2}{q^2}}\,\ln\frac{\sqrt{1-\frac{4m^2}{q^2}}+1}{\sqrt{1-\frac{4m^2}{q^2}}-1}\right\}. \quad (1.38)$$

The important function $\tilde{I}(1,1;q^2,m^2)$ is also known as the *scalar one-loop integral with equal masses*, and we shall see below, that it arises in the context of the hadronic e^+e^- cross section as well as the quark-loop contribution to the one-loop correction for the gluon propagator. Already note that in eq. (1.38) we have a logarithm of m^2/μ^2. Thus if for some reason we would like to chose the arbitrary scale μ^2 very small, or very large, compared to m^2, we would have the problem of a large logarithm, already mentioned in the beginning. Now we have discussed the question, how to regularise the divergencies in the framework of dimensional regularisation. To obtain finite physical quantities, as the next step, we have to renormalise the parameters of our theory.

1.3 Renormalisation at one-loop

In this section, the one-loop renormalisation of the quark and gluon propagators will be carried out explicitly. From this calculation we will be able to extract the renormalisation

constants Z_m for the quark mass (from now on we shall only consider a single quark flavour q with mass m) and Z_g for the gauge coupling. The renormalisation constants will be needed in the next chapter in the context of the renormalisation group equation, and will determine the "running" of quark mass and gauge coupling at one-loop.

Let us begin to calculate the perturbative correction to the quark propagator in momentum space:

$$S_{ij}(p) \;=\; -\,i\int d^Dx\,\mathrm{e}^{ipx}\langle 0|T\{q_i(x)\bar{q}_j(0)\mathrm{e}^{i\int d^Dz\,\mathcal{L}_{\rm I}(z)}\}|0\rangle\,, \tag{1.39}$$

where i,j denote colour indices and we have suppressed the spinor indices and $\mathcal{L}_{\rm I}(z)$ is the interaction part of the Lagrangian. The only piece of the interaction Lagrangian which contributes at this order is the quark-gluon interaction, and we need this term twice, since the vacuum-expectation value of a single gluon field vanishes. We thus find

$$S^{(1)}_{ij}(p) \;=\; i\,g^2\!\int\!d^Dx\!\int\!d^Dz_1\!\int\!d^Dz_2\,\mathrm{e}^{ipx}[S^{(0)}(x-z_1)\gamma^\mu\tfrac{\lambda^a}{2}S^{(0)}(z_1-z_2)\gamma^\nu\tfrac{\lambda^a}{2}S^{(0)}(z_2)]_{ij}D^{(0)}_{\mu\nu}(z_1-z_2). \tag{1.40}$$

The next step consists in replacing the coordinate-space propagators through their momentum space representation. Then the coordinate-space integrations can be carried out, yielding δ-distributions, which allows to perform three of the momentum-space integrals. This leads to

$$S^{(1)}_{ij}(p) \;=\; i\,g^2\,C_F\,\delta_{ij}\int\frac{d^Dk}{(2\pi)^D}\,[S^{(0)}(p)\gamma^\mu S^{(0)}(p-k)\gamma^\nu S^{(0)}(p)]D^{(0)}_{\mu\nu}(k)\,, \tag{1.41}$$

where we have used the fact that

$$\left[\frac{\lambda^a}{2}\,\frac{\lambda^a}{2}\right]_{ij} \;=\; \frac{(N_c^2-1)}{2N_c}\,\delta_{ij} \;\equiv\; C_F\,\delta_{ij} \;\stackrel{N_c=3}{=}\; \frac{4}{3}\,\delta_{ij}\,, \tag{1.42}$$

with C_F being the Casimir operator of the gauge group $SU(N_c)$ in the fundamental representation. In the diagrammatic language of Feynman diagrams, eq. (1.41) would correspond to: Now, we rewrite the quark propagator in the following form:

$$S_{ij}(p) \;=\; \delta_{ij}\,S^{(0)}(p) + \delta_{ij}\,S^{(0)}(p)\,\Sigma(p)\,S^{(0)}(p) + \ldots \tag{1.43}$$

The new function $\Sigma(p)$ is called the quark selfenergy, and at order g^2 takes the form

$$\Sigma^{(1)}(p) \;=\; i\,g^2\,C_F\int\frac{d^Dk}{(2\pi)^D}\,[\gamma^\mu S^{(0)}(p-k)\gamma^\nu]D^{(0)}_{\mu\nu}(k)\,. \tag{1.44}$$

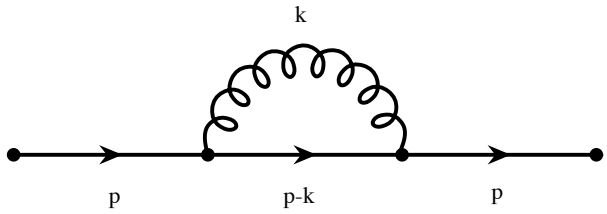

Figure 1.1: 1-loop correction to the quark propagator.

Resumming the selfenergy contributions to the quark propagator, also called Dyson resummation, one obtains:

$$S_{ij}(p) \;=\; \frac{\delta_{ij}}{[\not{p} - m - \Sigma(p) + i0]}\,. \tag{1.45}$$

Problem 1.7: Derive eq. (1.44) from (1.39) yourself.

To proceed with the calculation of the selfenergie $\Sigma(p)$ we insert the free quark and gluon propagators into eq. (1.44):

$$\Sigma^{(1)}(p) \;=\; -\,i\,g^2\mu^{-2\varepsilon}\,C_F\,\mu^{2\varepsilon}\!\int\!\frac{d^Dk}{(2\pi)^D}\,\frac{[\gamma^\mu(\not{p}-\not{k}+m)\gamma^\nu]}{k^2\,[(p-k)^2-m^2]}\,\Big[g_{\mu\nu}-(1-a)\frac{k_\mu k_\nu}{k^2}\Big]\,. \tag{1.46}$$

Problem 1.8: Calculate $\Sigma^{(1)}(p)$ in the Feynman gauge ($a = 1$) and to linear order in the quark mass m. (This is sufficient to determine the renormalisation constant Z_m to one-loop, as we shall see below.)

The derivation of the complete result with full mass and gauge parameter dependence is presented in detail in the book by Pascual and Tarrach [5]. Splitting $\Sigma(p)$ in the following form,

$$\Sigma(p) \;=\; \not{p}\,\Sigma_p(p) + m\,\Sigma_m(p)\,, \tag{1.47}$$

the final result is given by:

$$\Sigma_p^{(1)}(p) \;=\; \frac{C_F}{4}\,\frac{\alpha_s}{\pi}\,a\left\{-\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}-1-\frac{m^2}{p^2}+\Big(1-\frac{m^4}{p^4}\Big)\,\ln\Big(1-\frac{p^2}{m^2}\Big)\right\}, \tag{1.48}$$

$$\Sigma_m^{(1)}(p) \;=\; \frac{C_F}{4}\,\frac{\alpha_s}{\pi}\left\{(3+a)\Big(\frac{1}{\hat{\varepsilon}}-\ln\frac{m^2}{\mu^2}\Big)+4+2a-(3+a)\Big(1-\frac{m^2}{p^2}\Big)\,\ln\Big(1-\frac{p^2}{m^2}\Big)\right\},$$

where we have defined the dimensionless coupling $\alpha_s \equiv g^2\mu^{-2\varepsilon}/(4\pi)$ which is analogous to the fine-structure constant in QED, and conventually known as the QCD coupling

(constant). Expanding for small m^2 and setting $a = 1$, the solution to problem 1.8 is recovered.

QCD is a renormalisable theory which means that all Green functions constructed from the fundamental fields appearing in the Lagrangian can be rendered finite through a redefinition of those fields and parameters appearing in the Lagrangian. Of course, this implies that the redefinitions must be infinite, in order to cancel the divergencies emerging in the loop-corrections to the Green functions. In these lectures, only the mass and gauge coupling renormalisation will be investigated in detail, since they are relevant for the renormalisation group running, to be discussed below. For the general renormalisation program, the reader is referred to the standard textbooks already mentioned at the beginning [2–7].

The only quantities in the Lagrangian which play a role for the one-loop renormalisation of the quark propagator are the quark field and the quark mass. Their renormalisation is defined as follows:

$$q_i(x) \;=\; Z_{2\mathrm{F}}^{1/2}\, q_i^{\mathrm{R}}(x) \qquad \text{and} \qquad m \;=\; Z_m\, m^{\mathrm{R}}\,. \tag{1.49}$$

The quantities on the lhs are the so-called *bare quantities* which lead to divergent Green functions, whereas the quantities on the rhs, with the superscript R are the so-called *renormalised quantities* which result in finite Green functions. The Z-factors are the (infinite) renormalisation constants. Conventionally, the fermion field is renormalised with a factor $Z_{2\mathrm{F}}^{1/2}$ so that the fermion-bilinear, which appears in the Lagrangian, is renormalised with a factor $Z_{2\mathrm{F}}$.

The renormalisation constants can also be expanded in perturbation theory, because they are to be constructed such that they cancel the divergencies order by order:

$$Z_{2\mathrm{F}} \;=\; 1 + Z_{2\mathrm{F}}^{(1)}\,\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \qquad \text{and} \qquad Z_m \;=\; 1 + Z_m^{(1)}\,\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\,. \tag{1.50}$$

Considering the inverse of the quark propagator, the renormalisation constants generate additional terms from the kinetic as well as mass term of the QCD Lagrangian:

$$S^{-1}(p) \;=\; \not{p} + \not{p}\, Z_{2\mathrm{F}}^{(1)}\,\frac{\alpha_s}{\pi} - m^{\mathrm{R}} - m^{\mathrm{R}}\left(Z_m^{(1)} + Z_{2\mathrm{F}}^{(1)}\right)\frac{\alpha_s}{\pi} - \not{p}\,\Sigma_p^{(1)}(p) - m^{\mathrm{R}}\,\Sigma_m^{(1)}(p)\,. \tag{1.51}$$

Demanding that this expression should be finite, together with the results of eq. (1.48), leads to:

$$Z_{2\mathrm{F}}^{(1)} \;=\; -\,a\,\frac{C_F}{4}\,\frac{1}{\hat{\varepsilon}} \qquad \text{and} \qquad Z_m^{(1)} \;=\; -\,\frac{3}{4}\,C_F\,\frac{1}{\hat{\varepsilon}}\,. \tag{1.52}$$

This is the first central result of this section. In writing eq. (1.52), we have taken the choice that the divergent contribution is just proportional to $1/\hat{\varepsilon}$. This choice corresponds to the so-called modified-minimal-subtraction or $\overline{\text{MS}}$-scheme [17], the renormalisation scheme which nowadays is most popular. In general, minimal subtraction schemes are characterised by subtracting the $1/\varepsilon$-poles plus some arbitrary constant, but the subtraction constants stay independent of masses and momenta. Therefore, they are also called *mass-independent* renormalisation schemes. Historically, the scheme first proposed by 't Hooft and Veltman [13] was the minimal-subtraction or MS-scheme, where only the $1/\varepsilon$-poles (without any additional constant) are subtracted.

The mass-independence of minimal schemes was the reason that just taking the linear term for $\Sigma(p)$ in problem 1.8 is sufficient to obtain the renormalisation constants $Z_{2\text{F}}$ and Z_m. Furthermore, from eq. (1.52), we see that $Z_m^{(1)}$ is independent of the gauge parameter a. Therefore, working in the Feynman gauge is sufficient for obtaining Z_m (but not for $Z_{2\text{F}}$). Actually, the independence of Z_m on the gauge parameters holds true to all orders in perturbation theory [18].

For the renormalisation group analysis of the gauge coupling g or α_s, which we shall analyse in the next chapter, one also requires the renormalisation constants Z_g and $Z_{2\text{G}}$ which are defined by:

$$g \;=\; Z_g\, g^{\text{R}} \qquad \text{and} \qquad B_\mu^a(x) \;=\; Z_{2\text{G}}^{1/2} B_\mu^{a\,\text{R}}(x)\,. \tag{1.53}$$

However, for the standard procedure, in the one-loop renormalisation of the gluon propagator, only the renormalisation constant of the gluon field $Z_{2\text{G}}$ is required. To obtain $Z_{2\text{G}}$ one has to consider the renormalisation of vertices in which g appears. Nevertheless, an approach exists which allows to also obtain $Z_{2\text{G}}$ from the renormalisation of the gluon propagator, namely the background-field method (BFM) [19].

The idea of BFM is to split the gluon field into two parts, $B_\mu^a = \bar{B}_\mu^a + A_\mu^a$, where A_μ^a is the so-called classical background field, which is not integrated in the functional integral, and which only appears in external lines. The field $\bar{B}_\mu^a$, on the other hand, is the quantum part of the gluon field, which is integrated in the functional integral, and which appears in the loops. One can now choose a gauge fixing for $\bar{B}_\mu^a$ (which depends on A_μ^a) such that gluon Green functions with A_μ^a as the external field are gauge invariant. In the conventional approach, for example, higher order corrections to the gluon propagator depend on the gauge. As a consequence of this gauge invariance, one can also show that then Z_g and $Z_{2\text{G}}$

are related by

$$Z_g \;=\; Z_{2\mathrm{G}}^{-1/2} \qquad \text{or} \qquad Z_\alpha \;=\; Z_g^2 \;=\; Z_{2\mathrm{G}}^{-1}\,. \tag{1.54}$$

Calculation of the one-loop correction to the gluon propagator in the BF-gauge.

From the calculation of the one-loop renormalisation of the gluon propagator [5], one then finds

$$Z_\alpha \;=\; 1 - \frac{\alpha_s}{\pi}\,\frac{1}{12}\,(11N_c - 2N_f)\frac{1}{\hat{\varepsilon}} + \mathcal{O}(\alpha_s^2)\,. \tag{1.55}$$

In minimal schemes, like Z_m, also Z_α is independent of the gauge parameter to all orders in perturbation theory.

Chapter 2

Renormalisation Group

2.1 Renormalisation group equation

Renormalisation invariance is the statement that physical observables must be independent of the renormalisation scheme which one chooses for their theoretical calculation. For historical reasons [10,11], the renormalisation invariance is usually denominated invariance under the renormalisation group, although as we shall see, the group structure is only defined within classes of renormalisation schemes.

Let us assume that we have calculated a bare Green function $\Gamma(\ldots)$ and have renormalised it in a particular renormalisation scheme R. Then the relation between bare and renormalised Green function is given by

$$\Gamma_{\mathrm{R}}(\ldots) \;=\; Z(\mathrm{R})\,\Gamma(\ldots)\,, \tag{2.1}$$

where $Z(\mathrm{R})$ denotes the appropriate product of renormalisation constants, defined in the R-scheme. Let us now consider a different renormalisation scheme R$'$. Then

$$\Gamma_{\mathrm{R}'}(\ldots) \;=\; Z(\mathrm{R}')\,\Gamma(\ldots)\,, \tag{2.2}$$

and combining eqs. (2.1) and (2.2), we obtain:

$$\Gamma_{\mathrm{R}'}(\ldots) \;=\; Z(\mathrm{R}',\mathrm{R})\,\Gamma_{\mathrm{R}}(\ldots) \qquad \text{with} \qquad Z(\mathrm{R}',\mathrm{R}) \;=\; \frac{Z(\mathrm{R}')}{Z(\mathrm{R})}\,. \tag{2.3}$$

Let us consider the set of all possible $Z(\mathrm{R}',\mathrm{R})$ with R and R$'$ arbitrary. Among the elements of this set there exists a composition law

$$Z(\mathrm{R}'',\mathrm{R}) \;=\; Z(\mathrm{R}'',\mathrm{R}')\,Z(\mathrm{R}',\mathrm{R})\,. \tag{2.4}$$

To each element $Z(\mathrm{R}', \mathrm{R})$, we can associate an inverse

$$Z^{-1}(\mathrm{R}', \mathrm{R}) \;=\; Z(\mathrm{R}, \mathrm{R}')\,, \tag{2.5}$$

and we can define a unit element

$$Z(\mathrm{R}, \mathrm{R}) \;\equiv\; 1\,. \tag{2.6}$$

Notice that the composition law is not defined for two arbitrary elements of the group; the product $Z(R_i, R_j)\, Z(R_k, R_l)$ is not, in general, an element of the group, unless $R_j = R_k$. A set of relations obeying the above relations defines a *groupoid structure* [20].

Let us next consider a physical quantity $R(q, a_s, m)$ where q generically stands for external momenta and $a_s \equiv \alpha_s/\pi$ as well as m denote the renormalised QCD coupling and quark mass. (In what follows, we shall only consider a single quark flavour.) Being a physical quantity, $R(q, a_s, m)$ cannot depend on the arbitrary renormalisation scale μ. This requirement leads to the relation

$$\mu \frac{d}{d\mu} R(q, a_s, m) \;=\; \left\{ \mu \frac{\partial}{\partial \mu} + \mu \frac{da_s}{d\mu} \frac{\partial}{\partial a_s} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} \right\} R(q, a_s, m) \;=\; 0\,. \tag{2.7}$$

Now, we can define the following *renormalisation group functions*:

$$\beta(a_s) \;\equiv\; -\,\mu \frac{da_s}{d\mu} \;=\; \beta_1\, a_s^2 + \beta_2\, a_s^3 + \ldots \qquad \beta\text{-function}, \tag{2.8}$$

$$\gamma(a_s) \;\equiv\; -\,\frac{\mu}{m} \frac{dm}{d\mu} \;=\; \gamma_1\, a_s + \gamma_2\, a_s^2 + \ldots \qquad \text{mass anomalous dimension}. \tag{2.9}$$

Today, the coefficients of the β-function and the mass anomalous dimension are known up to the 4-loop coefficients β_4 and γ_4. Employing these functions, the relation (2.7) can be written as:

$$\left\{ \mu \frac{\partial}{\partial \mu} - \beta(a_s) \frac{\partial}{\partial a_s} - \gamma(a_s)\, m \frac{\partial}{\partial m} \right\} R(q, a_s, m) \;=\; 0\,. \tag{2.10}$$

This equation is called the *renormalisation group equation* (RGE) for $R(q, a_s, m)$. Before we analyse the RGE further, the one-loop coefficients β_1 and γ_1 of the β-function and the mass anomalous dimension will be calculated.

2.2 Running gauge coupling

If we know the coefficients of the β-function and the mass anomalous dimension, we can determine the renormalisation scale dependence of the QCD coupling and quark mass. In

this section, the running of the gauge coupling will be considered and the next section discusses the running of the quark mass.

Making use of the relation between the bare and the renormalised coupling, $\alpha_s = Z_\alpha^{-1}\alpha_s^0$, we find

$$\beta(a_s) \;=\; -\,\mu\,\frac{da_s}{d\mu} \;=\; -\,\mu\,\frac{d(Z_\alpha^{-1}a_s^0)}{d\mu} = \frac{a_s^0}{Z_\alpha^2}\frac{dZ_\alpha}{d\mu} - \frac{\mu}{Z_\alpha}\frac{da_s^0}{d\mu} \;=\; a_s\,\frac{\mu}{Z_\alpha}\frac{dZ_\alpha}{d\mu} + 2\varepsilon a_s\,. \qquad (2.11)$$

The μ-dependence of Z_α only results from a_s. This leads to

$$\beta(a_s) \;=\; -\,a_s\,\beta(a_s)\,\frac{1}{Z_\alpha}\frac{dZ_\alpha}{da_s} + 2\varepsilon a_s\,. \qquad (2.12)$$

Inserting the perturbative expansions for $\beta(a_s)$ and the renormalisation constant Z_α of eq. (1.55),

$$Z_\alpha \;\equiv\; 1 + a_s\,Z_\alpha^{(1)}\,\frac{1}{\hat{\varepsilon}} + \mathcal{O}(a_s^2) \;=\; 1 - a_s\,\frac{1}{12}\,(11N_c - 2N_f)\frac{1}{\hat{\varepsilon}} + \mathcal{O}(a_s^2)\,, \qquad (2.13)$$

into eq. (2.12), the coefficients of the β-function can be extracted by comparing equal powers of a_s on both sides of the equation. This yields:

$$\begin{aligned} \beta_1\,a_s^2 + \ldots \;&=\; -\,a_s\,(2\varepsilon a_s + \beta_1\,a_s^2 + \ldots)\,\frac{Z_\alpha^{(1)}/\varepsilon}{\Big(1 + a_s\,Z_\alpha^{(1)}/\varepsilon + \ldots\Big)} \\ \Rightarrow \qquad \beta_1 \;&=\; -\,2\,Z_\alpha^{(1)} \;=\; \frac{1}{6}\,(11N_c - 2N_f)\,. \end{aligned} \qquad (2.14)$$

The renormalisation group equation for a_s of eq. (2.8) can be integrated immediately by separation of variables:

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta(a_s)} \;=\; -\int_{\mu_1}^{\mu_2}\frac{d\mu}{\mu} \;=\; \ln\frac{\mu_1}{\mu_2}\,. \qquad (2.15)$$

At one-loop this equation can be solved analytically:

$$\begin{aligned} &\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta_1 a_s^2} \;=\; \frac{1}{\beta_1}\left[\,\frac{1}{a_s(\mu_1)} - \frac{1}{a_s(\mu_2)}\,\right] \;=\; \ln\frac{\mu_1}{\mu_2} \\ &\Rightarrow \quad \frac{1}{a_s(\mu_2)} \;=\; \frac{1}{a_s(\mu_1)} - \beta_1\ln\frac{\mu_1}{\mu_2} \quad \Rightarrow \quad a_s(\mu_2) \;=\; \frac{a_s(\mu_1)}{[1 - a_s(\mu_1)\beta_1\ln\frac{\mu_1}{\mu_2}]}\,. \end{aligned} \qquad (2.16)$$

Since the first coefficient of the β-function β_1 is positive, for $\mu_2 > \mu_1$, $a_s(\mu_2)$ decreases logarithmically, and it vanishes for $\mu_2 \to \infty$. This is the celebrated *asymptotic freedom* property of QCD.

The running of a_s can either be characterised by some starting value $a_s(\mu_1)$, or the point Λ where the coupling diverges, which is also called the Landau pole. This is the point where the denominator of eq. (2.16) vanishes, and it is characterised by the relation:

$$1 - a_s(\mu_1)\beta_1 \ln \frac{\mu_1}{\Lambda} \;=\; 0 \qquad \Rightarrow \qquad \Lambda \;=\; \mu_1 \, \mathrm{e}^{-\frac{1}{\beta_1 a_s(\mu_1)}} \,. \tag{2.17}$$

It is a simple exercise to show that Λ does not depend on the scale μ_1, i.e. $d\Lambda/d\mu_1 = 0$. Thus, Λ is a so-called renormalisation group invariant (RGI) quantity. Nevertheless, since Λ increases exponentially with $a_s(\mu)$, and therefore the uncertainty is inflated, in my opinion the use of a starting value $a_s(\mu_1)$ for the renormalisation group running of a_s is preferred.

One feature of minimal subtraction schemes which should be discussed further, namely their mass independence, is simultaneously an advantage and a drawback. In minimal schemes, for the running of α_s, quarks are treated as massless particles until one approaches the threshold for a particular quark. Below the threshold, the corresponding quark is treated as infinitely heavy, and "integrated out" in the framework of *effective theories*. Thus, the quark mass only appears in so-called matching factors on scales around the mass of a particular quark, m_q to $2\,m_q$ and the coupling is discontinuous at the threshold. In non-minimal schemes, like for example momentum subtraction [21], the quark mass appears in the coefficients of the β-function, and a smooth running across quark thresholds is achieved.

Nowadays, it has become convention to compare the running strong coupling at the scale $\mu = M_\mathrm{Z} = 91.188\,\mathrm{GeV}$, the mass of the Z-boson. The present world average reads [22]

$$\alpha_s(M_\mathrm{Z}) \;=\; 0.1187 \pm 0.0020 \,. \tag{2.18}$$

At higher loop orders, the RGE for α_s has to be integrated numerically. Doing this at 4-loop order – and including appropriate matching factors when quark thresholds are crossed – at a typical low-energy scale of the order of $m_p \approx 1\,\mathrm{GeV}$ one finds:

$$\alpha_s(1\,\mathrm{GeV}) \;=\; 0.50^{+0.05}_{-0.04} \,. \tag{2.19}$$

The size of the QCD coupling at 1 GeV, $\alpha_s(1\,\mathrm{GeV}) \approx 0.5$, indicates, that below this scale, α_s is rather large, and the perturbative expansion becomes questionable.

2.3 Running quark mass

As already presented in eq. (2.9), the scale dependence of the quark mass, characterised by the mass anomalous dimension $\gamma(a_s)$, is given by

$$\gamma(a_s) \;=\; -\,\frac{\mu}{m}\,\frac{dm}{d\mu} \;=\; \gamma_1\,a_s + \gamma_2\,a_s^2 + \gamma_3\,a_s^3 + \gamma_4\,a_s^4 + \ldots \tag{2.20}$$

Nowadays, the perturbative coefficients of the mass anomalous dimension are known up to the 4-loop coefficient γ_4.

Problem 2.1: Recalling that the renormalisation constant for the quark mass Z_m is given by $Z_m = 1 - \frac{3}{4}\,C_F a_s \frac{1}{\hat{\varepsilon}} + \mathcal{O}(a_s^2)$, calculate the one-loop coefficient γ_1 in analogy to the one-loop coefficient β_1.

The renormalisation group equation for the quark mass can be integrated directly by separation of variables:

$$\int\limits_{m(\mu_1)}^{m(\mu_2)} \frac{dm}{m} \;=\; \ln\frac{m(\mu_2)}{m(\mu_1)} \;=\; -\int\limits_{\mu_1}^{\mu_2} \frac{d\mu}{\mu}\,\gamma(a_s) \;=\; \int\limits_{a_s(\mu_1)}^{a_s(\mu_2)} da_s\,\frac{\gamma(a_s)}{\beta(a_s)}\,. \tag{2.21}$$

Thus, we obtain:

$$m(\mu_2) \;=\; m(\mu_1)\exp\!\left\{\int\limits_{a_s(\mu_1)}^{a_s(\mu_2)} da_s\,\frac{\gamma(a_s)}{\beta(a_s)}\right\}. \tag{2.22}$$

For minimal subtraction schemes, both the β-function as well as the mass anomalous dimension are flavour independent. Therefore, in ratios of quark masses for different flavours, $m_{q_1}(\mu)/m_{q_2}(\mu)$, the exponential factor cancels. Due to the resulting scale and scheme independence of quark mass ratios in minimal schemes, they can be considered as physical quantities.

Problem 2.2: Explicitly calculate the renormalisation group running for the quark mass to one-loop order.

Chapter 3

Applications

3.1 e^+e^--scattering to hadrons

One of the most studied measurable quantities in QCD is the total e^+e^- cross section scattering into hadrons, and the related quantity $R_q(s)$, which is defined by

$$R_q(s) \;\equiv\; \frac{\sigma(e^+e^- \to \gamma^* \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,, \tag{3.1}$$

where s is the invariant mass squared of the final hadron system. In the following, we shall investigate this physical quantity in some detail, and explore the constraints that the RGE places on its perturbative expansion.

Theoretically, $R_q(s)$ is related to the imaginary part of the vector two-point (or correlation) function $\Pi(s)$ as follows [3]:

$$R_q(s) \;=\; 12\pi\,\mathrm{Im}\,\Pi(s+i0)\,, \tag{3.2}$$

with $s = q^2$ and

$$\Pi_{\mu\nu}(q^2) \;=\; i\mu^{2\varepsilon}\!\int d^Dx\,\mathrm{e}^{iqx}\,\langle 0|T\{j_\mu(x)j_\nu(0)\}|0\rangle \;=\; (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi(q^2)\,. \tag{3.3}$$

Here, $j_\mu(x) =:\bar{q}(x)\gamma_\mu q(x):$ is the vector current with q-quark flavour quantum numbers, physically representing a $q\bar{q}$ quarkonium state that can be produced in e^+e^- scattering or continuum states with vector quantum numbers. The second identity holds since the vector current is conserved, i.e. $\partial^\mu j_\mu(x) = 0$, and thus $q^\mu\,\Pi_{\mu\nu}(q^2)$ has to vanish, which implies the Lorentz structure on the right. Equation (3.3) has already been written in D dimensions and an appropriate scale factor $\mu^{2\varepsilon}$ has been introduced, such that $\Pi_{\mu\nu}(q^2)$ has dimension 2 also in the general case.

As a first step, we shall calculate the vector two-point function $\Pi(q^2)$ at lowest order in perturbation theory. This is most conveniently done by contracting eq. (3.3) with $g^{\mu\nu}$ which leads to:

$$\begin{aligned}\Pi(q^2) &= \frac{-i\mu^{2\varepsilon}}{(D-1)q^2}\int d^Dx\, \mathrm{e}^{iqx}\,\langle 0|T\{j_\mu(x)j^\mu(0)\}|0\rangle \\ &= \frac{-iN_c\,\mu^{2\varepsilon}}{(D-1)q^2}\int d^Dx \mathrm{e}^{iqx}\,\mathrm{Tr}[S(x)\gamma_\mu S(-x)\gamma^\mu] \\ &= \frac{-iN_c\,\mu^{2\varepsilon}}{(D-1)q^2}\int \frac{d^Dp}{(2\pi)^D}\,\frac{\mathrm{Tr}[(\not{p}+m)\gamma_\mu(\not{p}-\not{q}+m)\gamma^\mu]}{(p^2-m^2)[(p-q)^2-m^2]}\,.\end{aligned} \tag{3.4}$$

The general, massive case can in principle be treated employing the scalar one-loop integral with equal masses of eq. (1.38). For simplicity, here, we shall only proceed with the massless case $m = 0$. This case should be a reasonable approximation for the light quarks up, down and strange, whose masses are smaller than Λ_{QCD}. One then finds:

$$\begin{aligned}\Pi(q^2) &= 4iN_c\,\frac{(D-2)}{(D-1)}\,\frac{\mu^{2\varepsilon}}{q^2}\int\frac{d^Dp}{(2\pi)^D}\,\frac{p\cdot(p-q)}{p^2(q-p)^2} \\ &= \frac{N_c}{4\pi^2}\left(\frac{4\pi\mu^2}{-q^2}\right)^{\varepsilon}\frac{(1-\varepsilon)}{(3-2\varepsilon)}\,\frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\,\Gamma(\varepsilon) \\ &= \frac{N_c}{12\pi^2}\left\{\frac{1}{\hat{\varepsilon}}-\ln\frac{(-q^2-i0)}{\mu^2}+\frac{5}{3}+\mathcal{O}(\varepsilon)\right\}.\end{aligned} \tag{3.5}$$

Employing this result, $R_q(s)$ at lowest order is given by

$$R_q(s) \;=\; 12\pi\,\mathrm{Im}\,\Pi(s+i0) \;=\; N_c\,. \tag{3.6}$$

The results of eqs. (3.5) and (3.6) exemplify that $\mathrm{Im}\,\Pi(s)$, and the so-called spectral function $\rho(s) \equiv \mathrm{Im}\,\Pi(s)/\pi$, are physical quantities, meaning that they do not depend on the details of the renormalisation. This is not the case for $\Pi(q^2)$ which should be obvious from its logarithmic μ-dependence. This important point will be further discussed below.

Summing over the contributing number of quark flavours at a given energy, one obtains the total cross section of e^+e^- scattering into hadrons or the corresponding quantity $R_{e^+e^-}$. In this case, the relevant current is the electromagnetic current, which also carries a charge factor Q_q for each contributing quark. One then arrives at the well-known leading order result

$$R_{e^+e^-} \;=\; N_c\sum_q Q_q^2\,. \tag{3.7}$$

For the heavier quarks charm, bottom and top, quark mass corrections should certainly be taken into account.

Outline of the calculation of the order α_s correction to $\Pi(s)$?

More generally, in the massless case, the vector correlation function $\Pi(s)$ can be written as follows:

$$\Pi(s) \;=\; -\frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{nk}\, L^k \qquad \text{with} \qquad L \;\equiv\; \ln\frac{-s}{\mu^2}\,, \tag{3.8}$$

and $a_\mu \equiv a \equiv \alpha_s(\mu^2)/\pi$. (The global minus sign is introduced for later convenience.) From eq. (3.5), we can then read of the lowest order coefficients c_{01} and c_{00}:

$$c_{01} \;=\; 1 \qquad \text{and} \qquad c_{00} \;=\; -\frac{5}{3}\,. \tag{3.9}$$

$\Pi(s)$ itself is not a physical quantity. However, this is the case for the Adler function $D(s)$ (as we shall dicuss in more detail below) and the spectral function $\rho(s)$ where:

$$D(s) \;\equiv\; -\,s\,\frac{d}{ds}\,\Pi(s) \qquad \text{and} \qquad \rho(s) \;\equiv\; \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s+i0)\,. \tag{3.10}$$

The general expansion for the Adler function is given by:

$$D(s) \;=\; \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk}\, L^{k-1}\,. \tag{3.11}$$

It should be clear that the coefficients c_{n0} do not contribute to both the spectral function (because they are real) and to the Adler function (because of the factor k). Thus the coefficients c_{n0} can be considered as "unphysical" in that they do not appear in measurable quantities.

As physical quantities, both $D(s)$ as well as $\rho(s)$ have to satisfy a homogeneous RGE:

$$-\,\mu\,\frac{d}{d\mu}\left\{\begin{array}{c} D(s) \\ \rho(s) \end{array}\right\} \;=\; \left[\,2\,\frac{\partial}{\partial L} + \beta(a)\,\frac{\partial}{\partial a}\,\right]\left\{\begin{array}{c} D(s) \\ \rho(s) \end{array}\right\} \;=\; 0\,, \tag{3.12}$$

where $\beta(a)$ is the QCD β-function, defined as:

$$-\,\mu\,\frac{da}{d\mu} \;\equiv\; \beta(a) \;=\; \beta_1\,a^2 + \beta_2\,a^3 + \beta_3\,a^4 + \beta_4\,a^5 + \ldots \tag{3.13}$$

Numerically, for $N_c = 3$ the first four coefficients are given by [23, 24]:

$$\begin{aligned} \beta_1 \;&=\; \tfrac{11}{2} - \tfrac{1}{3}\,N_f\,, \qquad \beta_2 \;=\; \tfrac{51}{4} - \tfrac{19}{12}\,N_f\,, \qquad \beta_3 \;=\; \tfrac{2857}{64} - \tfrac{5033}{576}\,N_f + \tfrac{325}{1728}\,N_f^2\,, \\ \beta_4 \;&=\; \tfrac{149753}{768} + \tfrac{891}{32}\,\zeta_3 - \Big(\tfrac{1078361}{20736} + \tfrac{1627}{864}\,\zeta_3\Big)N_f + \Big(\tfrac{50065}{20736} + \tfrac{809}{1296}\,\zeta_3\Big)N_f^2 + \tfrac{1093}{93312}\,N_f^3\,. \end{aligned} \tag{3.14}$$

Problem 3.1: Taking into account the RGE for $D(s)$ up to order a^2, and considering the coefficients c_{n1} to be independent, explicitly calculate the constraints on the coefficients c_{12}, c_{22} and c_{23}.

The corresponding relations for the coefficients up to order a^4 can be calculated in an analogous fashion, and are given as follows:

$$\begin{aligned} c_{34} &= 0\,, \quad c_{33} = \frac{\beta_1^2}{12}\,c_{11}\,, \quad c_{32} = -\frac{1}{4}\,(\beta_2\,c_{11} + 2\beta_1\,c_{21})\,, \\ c_{45} &= 0\,, \quad c_{44} = -\frac{\beta_1^3}{32}\,c_{11}\,, \quad c_{43} = \frac{\beta_1}{24}\,(5\beta_2\,c_{11} + 6\beta_1\,c_{21})\,, \\ c_{42} &= -\frac{1}{4}\,(\beta_3\,c_{11} + 2\beta_2\,c_{21} + 3\beta_1\,c_{31})\,. \end{aligned} \tag{3.15}$$

Employing these constraints, up to order a_μ^3, $D(s)$ and $\rho(s)$ take the form:

$$\begin{aligned} D(s) &= \frac{N_c}{12\pi^2}\Big\{\, c_{01} + c_{11}\,a_\mu + \Big(c_{21} - \tfrac{1}{2}\beta_1 c_{11} L\Big)a_\mu^2 \\ &\quad + \Big(c_{31} - \beta_1 c_{21} L + \tfrac{1}{4}(\beta_1^2 L^2 - 2\beta_2 L)c_{11}\Big)a_\mu^3 + \mathcal{O}(a_\mu^4)\,\Big\}\,, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \rho(s) &= \frac{N_c}{12\pi^2}\Big\{\, c_{01} + c_{11}\,a_\mu + \Big(c_{21} - \tfrac{1}{2}\beta_1 c_{11} \bar{L}\Big)a_\mu^2 \\ &\quad + \Big(c_{31} - \beta_1 c_{21} \bar{L} + \tfrac{1}{12}(\beta_1^2(3\bar{L}^2 - \pi^2) - 6\beta_2 \bar{L})c_{11}\Big)a_\mu^3 + \mathcal{O}(a_\mu^4)\,\Big\}\,. \end{aligned} \tag{3.17}$$

In the second case, $\bar{L} \equiv \ln(|s|/\mu^2)$. Since both $D(s)$ and $\rho(s)$ satisfy a homogeneous RGE, the logarithms can be resummed with the choice $\mu^2 = -s \equiv Q^2$, leading to:

$$D(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} c_{n1}\,a_Q^n\,, \tag{3.18}$$

$$\begin{aligned} \rho(s) &= \frac{N_c}{12\pi^2}\Big\{\, c_{01} + c_{11}\,a_Q + c_{21}\,a_Q^2 + \Big(c_{31} - \tfrac{\pi^2}{12}\beta_1^2 c_{11}\Big)a_Q^3 \\ &\quad + \Big(c_{41} - \tfrac{\pi^2}{4}\beta_1^2 c_{21} - \tfrac{5\pi^2}{24}\beta_2\beta_1 c_{11}\Big)a_Q^4 + \mathcal{O}(a_Q^5)\,\Big\}\,, \end{aligned} \tag{3.19}$$

where $a_Q \equiv \alpha_s(Q^2)/\pi$. As should be clear from these equations, the only physically relevant coefficients are the c_{n1}. The coefficient $c_{01} = 1$ has been calculated explicitly above. The subsequent higher order corrections are known analytically up to order α_s^3 and at $N_c = 3$

take the following values:

$$c_{11} \;=\; 1\,, \quad c_{21} \;=\; \tfrac{365}{24} - 11\zeta_3 - (\tfrac{11}{12} - \tfrac{2}{3}\zeta_3)N_f\,,$$

$$c_{31} \;=\; \tfrac{87029}{288} - \tfrac{1103}{4}\zeta_3 + \tfrac{275}{6}\zeta_5 + (-\tfrac{7847}{216} + \tfrac{262}{9}\zeta_3 - \tfrac{25}{9}\zeta_5)N_f + (\tfrac{151}{162} - \tfrac{19}{27}\zeta_3)N_f^2\,. \quad (3.20)$$

In writing the four-loop coefficient c_{31}, we have neglected the small "light-by-light" contribution which when summing over several quark flavours is proportional to $(\sum_f Q_f)^2$, in contrast to the structure of the main contribution (3.7). For the next five-loop coefficient c_{41}, as yet only rough estimates from Padé approximation and other approaches to estimate higher orders exist.

Let us now return to the vector correlator $\Pi(s)$ itself. Due to its explicit μ-dependence, $\Pi(s)$ only satisfies an inhomogeneous RGE:

$$\left[\,2\,\frac{\partial}{\partial L} + \beta(a)\,\frac{\partial}{\partial a}\,\right]\Pi(s) \;=\; \frac{N_c}{12\pi^2}\Big\{\,2\sum_{n=0}^{\infty} c_{n1}\,a^n + \beta_1 c_{10}\,a^2 + (\beta_2 c_{10} + 2\beta_1 c_{20})a^3$$
$$+\,(\beta_3 c_{10} + 2\beta_2 c_{20} + 3\beta_1 c_{30})a^4 + \mathcal{O}(a^5)\,\Big\}\,. \quad (3.21)$$

The solution to the inhomogeneous RGE takes the following form:

$$\Pi(Q^2) \;=\; \frac{N_c}{12\pi^2}\Big\{\,c_{00} + c_{10}\,a_\mu + c_{20}\,a_\mu^2 + c_{30}\,a_\mu^3 + c_{40}\,a_\mu^4 + \ldots$$
$$+\,d_1\Big(\frac{1}{a_Q} - \frac{1}{a_\mu}\Big) + d_2\ln\frac{a_Q}{a_\mu} + d_3(a_Q - a_\mu) + \ldots\,\Big\}\,. \quad (3.22)$$

One possibility to verify the structure of this solution is to reexpand a_Q in terms of a_μ. A comparison with eq. (3.8) then allows to deduce the coefficients d_n. The first three d-coefficients are found to be:

$$d_1 \;=\; \frac{2}{\beta_1}\,c_{01}\,, \qquad d_2 \;=\; \frac{2}{\beta_1^2}\Big(\beta_2 c_{01} - \beta_1 c_{11}\Big)\,,$$

$$d_3 \;=\; \frac{2}{\beta_1^3}\Big((\beta_3\beta_1 - \beta_2^2)c_{01} + \beta_2\beta_1 c_{11} - \beta_1^2 c_{21}\Big)\,. \quad (3.23)$$

As anticipated also the renormalisation group improved expression for $\Pi(Q^2)$ with resummed logarithms displays the explicit renormalisation scale dependence.

3.2 QCD sum rules

The correlation function $\Pi(s)$ can only theoretically be calculated sufficiently far away from the positive real axis, where the perturbative expansion is applicable. On the positive

real axis, $\Pi(s)$ as well as the physical functions $D(s)$ and $\rho(s)$ can have poles and cuts due to physical intermediate states which appear in a respective channel. On the remaining complex s-plane, all functions are analytic.

Now the question arises, how the theoretical function $\Pi_{\text{th}}(s)$ and the experimentally measured function $\rho_{\text{exp}}(s)$ – e.g. through the measurement of $R_{e^+e^-}$ on the positive real s-axis – can be connected. The key idea is to employ the analytic structure of $\Pi(s)$. With the help of Cauchys theorem, one can then show that

$$\Pi(s) \;=\; \int_0^\infty \frac{\rho(s')}{(s'-s-i0)}\, ds' + P(s)\,, \tag{3.24}$$

where $P(s)$ is a polynomial in s which carries the required scale dependence since the integral over $\rho(s)$ on the rhs is a physical quantity, whereas $\Pi(s)$ on the lhs is unphysical. An equation with the structure of (3.24) is often called a dispersion relation analogous to similar relation which arise for example in electrodynamics.

The basic assumption to connect the theoretical description based on quarks and gluons as the fundamental degrees of freedom and the observed hadronic world lies in the existence of the so-called *quark-hadron duality*. This means that a certain physical quantity should be either describable in the quark-gluon picture or in the hadronic picture, and both descriptions should be equivalent. Under this crucial assumption, eq. (3.24) can be turned into:

$$\Pi_{\text{th}}(s) \;=\; \int_0^\infty \frac{\rho_{\text{exp}}(s')}{(s'-s-i0)}\, ds' + P(s)\,, \tag{3.25}$$

the starting point for analyses in the framework of QCD sum rules [25]. One problem for practical applications of eq. (3.25) is the fact that for the calculation of the dispersion integral on the rhs, the spectral function $\rho_{\text{exp}}(s)$ is required for all energies up to infinity. However, experimental information usually is only available up to some threshold energy s_{th}. One way out is to apply transformations to the relation (3.25) such that the energy region above s_{th} gets suppressed. One such transformation is the so-called Borel (or inverse Laplace) transformation which leads to an exponential suppression of higher energies.

A second approach to exploit the analytic structure of the vector correlator $\Pi(s)$ are the so-called *finite energy sum rules* (FESR). Since $\Pi(s)$ only has a cut for the positive real s-axis, the integral of $w(s)\,\Pi(s)$ over the closed contour as displayed in figure 3.1, where $w(s)$ is a weight function which does not introduce additional poles or cuts, should vanish. Splitting the integration contour into the straight-line segments along the real s-axis and

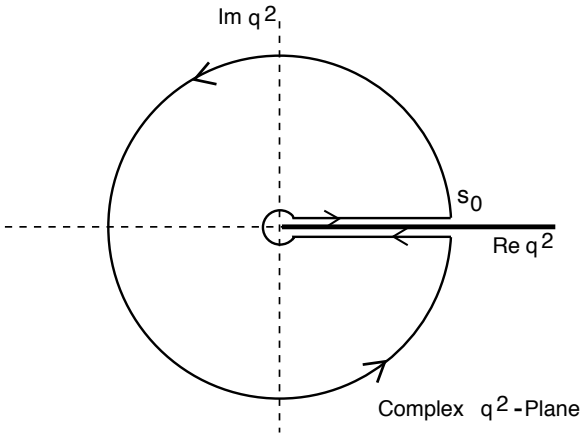

Figure 3.1: Complex integration contour for the finite energy sum rule.

the circle with radius s_0, one derives:

$$R(s_0) \;\equiv\; \int_0^{s_0} w(s)\,\rho(s)\,ds \;=\; \frac{-1}{2\pi i}\oint_{|s|=s_0} w(s)\,\Pi(s)\,ds\,. \tag{3.26}$$

Since the lhs is a physical quantity, so must be the rhs, i.e., the unphysical contributions cancel in the contour integral. The advantage of FESR over the dispersion relation (3.24) is – as the name says – that the spectral function $\rho(s)$ is only required up to a finite energy s_0. The disadvantage being that, depending on the weight function $w(s)$, it may be necessary to evaluate the correlator $\Pi(s)$ close to the real axis, where perturbation theory is no longer applicable.

A spectral integral of the type (3.26) appears in the analysis of the total decay rate of τ-leptons into hadrons:

$$R_\tau(s_0) \;\equiv\; \int_0^{s_0} w_\tau(s)\,\rho(s)\,ds \;\equiv\; \frac{\Gamma(\tau\to \text{hadrons}\,\nu_\tau)}{\Gamma(\tau\to\mu\,\bar\nu_\mu\nu_\tau)}\,. \tag{3.27}$$

For the case of the hadronic τ-decay rate, the relevant weight function $w_\tau(s)$ turns out to be:

$$w_\tau(s) \;=\; \frac{12\pi^2}{s_0}\Big(1-\frac{s}{s_0}\Big)^2\Big(1+2\,\frac{s}{s_0}\Big) \quad\text{with}\quad s_0 \;=\; M_\tau^2\,. \tag{3.28}$$

The fortuitous fact about R_τ is that the weight function introduces a double zero at $s = s_0$, such that there is no need to evaluate $\Pi(s)$ close to the real axis. The dominant contribution to R_τ stems from the vector current correlator (3.8), and thus, for the purpose of demonstrating an application of the renormalisation group equation, it is sufficient to consider this contribution.

Inserting $\Pi(s)$ of eq. (3.8) as well as $w_\tau(s)$ into eq. (3.26), $R_\tau^\Pi(s_0)$ is found to be:

$$\begin{aligned} R_\tau^\Pi(s_0) &= -\frac{i}{\pi}\,N_c \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{nk} \oint\limits_{|x|=1} dx\,(1-x)^2\,(1+2x)\ln^k\!\Big(\frac{-s_0 x}{\mu^2}\Big) \\ &= -\frac{i}{\pi}\,N_c \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{nk} \sum_{l=0}^{k} \binom{k}{l} \ln^{k-l}\frac{s_0}{\mu^2} \oint\limits_{|x|=1} dx\,(1-x)^2\,(1+2x)\ln^l(-x)\,, \\ &\equiv 2N_c \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{nk} \sum_{l=0}^{k} \binom{k}{l} I_l \ln^{k-l}\frac{s_0}{\mu^2}\,, \end{aligned} \tag{3.29}$$

where we have already multiplied $R_\tau^\Pi(s_0)$ by a factor of two since, up to small corrections, the contribution coming from the axialvector current is equal to that of the vector current. The contour integrals I_l are defined by:

$$\begin{aligned} I_l &= \frac{1}{2\pi i} \oint\limits_{|x|=1} dx\,(1-x)^2\,(1+2x)\ln^l(-x) \\ &= \frac{i^l}{2\pi} \int\limits_{-\pi}^{\pi} \alpha^l \Big[\,2\,\mathrm{e}^{4i\alpha} + 3\,\mathrm{e}^{3i\alpha} - \mathrm{e}^{i\alpha}\,\Big] d\alpha \;=\; \frac{1}{2\pi}\,\Big[\,2\,I_{l,4}^{\rm cs} + 3\,I_{l,3}^{\rm cs} - I_{l,1}^{\rm cs}\,\Big]\,. \end{aligned} \tag{3.30}$$

Here, the integrals $I_{l,m}^{\rm cs}$ are given by:

$$I_{l,m}^{\rm cs} \;\equiv\; i^l \int\limits_{-\pi}^{\pi} \alpha^l\,\mathrm{e}^{im\alpha}\,d\alpha \;=\; (-1)^{l+m}\frac{2\,l!}{m^{l+2}} \sum_{k=1}^{[(l+1)/2]} (-1)^k \frac{m^{2k}\pi^{2k-1}}{(2k-1)!}\,, \tag{3.31}$$

where $[n]$ denotes the integer part of n. Particular values of the integrals I_l are:

$$I_0 = 0\,, \quad I_1 = \frac{1}{2}\,, \quad I_2 = -\frac{19}{12}\,, \quad I_3 = \frac{265}{48} - \frac{\pi^2}{2}\,, \quad I_4 = -\frac{3355}{144} + \frac{19}{6}\,\pi^2\,,$$

$$I_5 = \frac{205205}{1728} - \frac{1325}{72}\,\pi^2 + \frac{\pi^4}{2}\,, \quad I_6 = -\frac{2479295}{3456} + \frac{16775}{144}\,\pi^2 - \frac{19}{4}\,\pi^4\,. \tag{3.32}$$

The fact that $I_0 = 0$ again ensures that the unphysical coefficients c_{n0} do not contribute to a physical quantity like R_τ.

Making use of partial integration, the contour integral in (3.26) can also be expressed in terms of the Adler function D(s). In the case of the τ decay rate, this yields:

$$R_\tau^D(s_0) \;=\; -6\pi i \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\,D(s_0 x)$$

$$= -\frac{i}{2\pi} N_c \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} \oint\limits_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^{k-1}\Big(\frac{-s_0 x}{\mu^2}\Big)$$

$$= -\frac{i}{2\pi} N_c \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} \sum_{l=0}^{k-1} \binom{k-1}{l} \ln^{k-l-1}\frac{s_0}{\mu^2} \oint\limits_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^l(-x)\,,$$

$$\equiv N_c \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{nk} \sum_{l=0}^{k-1} \binom{k-1}{l} J_l \ln^{k-l-1}\frac{s_0}{\mu^2}\,. \tag{3.33}$$

This time, the contour integrals J_l are defined by:

$$\begin{aligned} J_l &= \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^l(-x) \\ &= \frac{i^l}{2\pi} \int\limits_{-\pi}^{\pi} \alpha^l \Big[\, 1 + 2\,\mathrm{e}^{i\alpha} - 2\,\mathrm{e}^{3i\alpha} - \mathrm{e}^{4i\alpha}\,\Big] d\alpha \\ &= \frac{1}{2\pi} \Big[\, I_{l,0}^{\rm cs} + 2\, I_{l,1}^{\rm cs} - 2\, I_{l,3}^{\rm cs} - I_{l,4}^{\rm cs}\,\Big]\,. \end{aligned} \tag{3.34}$$

The contour integrals I_l and J_l satisfy the relation:

$$J_l = \frac{2}{(l+1)}\, I_{l+1}\,. \tag{3.35}$$

With the help of this relation, it is then easy to see that:

$$R_\tau^V(s_0) \equiv R_\tau^\Pi(s_0) = R_\tau^D(s_0)\,. \tag{3.36}$$

Since $R_\tau^V(s_0)$ is a physical quantity, it also satisfies a homogeneous RGE, and the logarithms can be resummed with the choice $\mu^2 = s_0$. Up to order $a_{s_0}^4$, this leads to:

$$\begin{aligned} R_\tau^V(s_0) &= N_c \Bigg\{ c_{01} + c_{11}\, a_{s_0} + \Big(c_{21} + \tfrac{19}{24}\beta_1 c_{11}\Big) a_{s_0}^2 \\ &\quad + \Big(c_{31} + \tfrac{19}{12}\beta_1 c_{21} + \Big(\tfrac{19}{24}\beta_2 + \Big(\tfrac{265}{288} - \tfrac{\pi^2}{12}\Big)\beta_1^2\Big) c_{11}\Big) a_{s_0}^3 \\ &\quad + \Big(c_{41} + \tfrac{19}{8}\beta_1 c_{31} + \Big(\tfrac{19}{12}\beta_2 + \Big(\tfrac{265}{96} - \tfrac{\pi^2}{4}\Big)\beta_1^2\Big) c_{21} \\ &\qquad + \Big(\tfrac{19}{24}\beta_3 + \Big(\tfrac{1325}{576} - \tfrac{5\pi^2}{24}\Big)\beta_2\beta_1 + \Big(\tfrac{3355}{2304} - \tfrac{19}{96}\pi^2\Big)\beta_1^3\Big) c_{11}\Big) a_{s_0}^4 \Bigg\} \end{aligned} \tag{3.37}$$

$$\begin{aligned} &= N_c \Bigg\{ 1 + a_{s_0} + \Big(\tfrac{769}{48} - 9\,\zeta_3\Big) a_{s_0}^2 + \Big(\tfrac{363247}{1152} - \tfrac{27}{16}\,\pi^2 - \tfrac{2071}{8}\,\zeta_3 + \tfrac{75}{2}\,\zeta_5\Big) a_{s_0}^3 \\ &\quad + \Big(\tfrac{19907171}{6144} + c_{41} - \tfrac{22683}{256}\,\pi^2 - \Big(\tfrac{345405}{128} - \tfrac{729}{16}\,\pi^2\Big)\zeta_3 + \tfrac{12825}{32}\,\zeta_5\Big) a_{s_0}^4 \Bigg\}\,. \end{aligned} \tag{3.38}$$

For the second step, we have employed the coefficients of the β-function (3.14) as well as the c_{n1}-coefficients of eq. (3.20). Before we shall analyse the expression numerically, let us introduce a second approach of resumming large logarithms in the perturbative expression of R_τ.

3.3 Contour improved perturbation theory

Let us again start with the τ decay rate expressed in terms of the Adler function $D(s)$:

$$R_\tau^D(s_0) \;=\; -\,6\pi i \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\,D(s_0 x)\,. \tag{3.39}$$

As discussed above, the Adler function is a physical quantity, and thus satisfies a homogeneous RGE. Therefore, the logarithms in eq. (3.11) can be resummed with the choice $\mu^2 = -s_0 x$, which yields:

$$R_\tau^D(s_0) \;=\; -\,\frac{i}{2\pi}\,N_c \sum_{n=0}^{\infty} c_{n1} \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\,a^n(-s_0 x) \;=\; N_c \sum_{n=0}^{\infty} c_{n1}\,J_n^a(s_0)\,, \tag{3.40}$$

where we have again rewritten the equation in terms of contour integrals $J_n^a(s_0)$, defined as:

$$J_n^a(s_0) \;=\; \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{dx}{x}\,(1-x)^3\,(1+x)\,a^n(-s_0 x)\,. \tag{3.41}$$

To perform the contour integration, we have to analytically continue the strong coupling α_s to the complex s-plane, but this is only a technical problem since the dependence of α_s on s is logarithmic and it has the same cut structure as $\Pi(s)$. Furthermore, the contour integrals can only be calculated numerically, but this also is no problem in principle.

Let us now come to a numerical comparison of the two approaches to resum the logarithms with the help of the renormalisation group. To do this, we require a value for α_s and we have employed $\alpha_s(M_Z) = 0.119$ [22], already discussed above, which corresponds to $\alpha_s(M_\tau) = 0.323$. The perturbative expansions of the *fixed order* result of eq. (3.38) and the *contour improved* expression of eq. (3.40) are then given by:

$$\begin{aligned} & \qquad\quad a^0 \qquad a^1 \qquad\quad a^2 \qquad\quad a^3 \qquad\quad a^4 \\ R_\tau^{\mathrm{FO}} \;&=\; 3\Big[\,1 + 0.103 + 0.055 + 0.029 + 0.012\,\Big] \;=\; 3.597\,, \end{aligned} \tag{3.42}$$

$$R_\tau^{\mathrm{CI}} \;=\; 3\Big[\,1 + 0.140 + 0.027 + 0.011 + 0.004\,\Big] \;=\; 3.544\,, \tag{3.43}$$

where for the fourth-order contribution a current estimate $c_{41} \approx 25$ for the five-loop coefficient has been used.

As is obvious, the contour improved perturbative expansion converges better than the fixed order expansion and higher order contributions are shifted to a lower perturbative order. This is due to the fact that for fixed order, all $\alpha_s(s)$ on the contour are expressed in terms of the fixed $\alpha_s(s_0)$. However, there are known large logarithms due to the running of α_s which can be resummed by resorting to the contour improved approach. Nevertheless, the difference $R_\tau^{\rm FO} - R_\tau^{\rm CI} = 0.053$ is much larger than the last known term in the fixed order expansion although a breakdown of this expansion is not really apparent. This problem might be due to yet unknown higher orders, but it has not been resolved yet and is still under current investigation.

The numerical results of eqs. (3.42) and (3.43) can be compared with the experimental finding $R_\tau^{\rm exp} = 3.642 \pm 0.012$. There are missing contributions to our theoretical expression for R_τ from higher dimensional operators, but they have been calculated and, as is also obvious from our numerics, their contribution is rather small. Nevertheless, if one aims on extracting the strong coupling α_s from the analysis of hadronic τ decays, those corrections should be included. Qualitatively, since our result for $R_\tau^{\rm CI}$ at fixed α_s is smaller than the one for $R_\tau^{\rm FO}$, the contour improved expression would point to a larger value for α_s. In the next section, we shall discuss the renormalisation of composite operators, and will investigate the bi-quark operator $\bar{q}(x)q(x)$ in more detail which is the leading higher dimensional operator contribution in hadronic τ decays.

3.4 Renormalisation of composite operators

Besides with Green functions, we often have to deal with composite operators. One frequent place where composite operators play a role is Wilsons operator product expansion (OPE) [26, 27]. For two operators $O_A(x)$ and $O_B(0)$, generically, the operator product expansion takes the form:

$$O_A(x)\, O_B(0) \overset{x\to 0}{\sim} \sum_i C_i(x)\, O_i(0)\,. \tag{3.44}$$

The x-dependence of the operator product is carried by the complex-number valued so-called Wilson coefficient functions $C_i(x)$. The operator structure resides in the set of local operators $O_i(0)$, usually containing an infinite set of operators which have to have the same quantum numbers as $O_A(x)\, O_B(0)$ on the lhs. The expansion is only in the weak sense, i.e. with respect to matrix elements, and it is applicable if we have a problem which involves short-distances, or conversely, a large momentum scale.

Let us discuss two concrete examples. The first one is the time-ordered product of two vector currents which already appeared in the vector current correlation function of eq. (3.3):

$$T\{j_\mu(x)\, j_\nu(0)\} \;\sim\; C_1(x)\cdot\mathbb{1} + :\bar{q}(x)\gamma_\mu S(x)\gamma_\nu q(0): \;. \tag{3.45}$$

The leading operator in this case is the unit operator and the Wilson coefficient C_1 just corresponds to the usual perturbative part which was already considered above. Performing the small-x expansion in the second term on the rhs yields $C_{\bar{q}q}(x):\bar{q}q(0):$, where for reasons to become clear soon, we have only written the Lorentz-scalar piece of the expansion. Previously, we had taken the expectation value of the current product with respect to the perturbative vacuum $|0\rangle$. Due to the normal ordering of the operators, this expectation value vanishes. The real QCD vacuum $|\Omega\rangle$, however, has a non-trivial structure, and thus the vacuum expectation value $\langle\Omega|:\bar{q}q(0):|\Omega\rangle$ is unequal from zero. (And since the vacuum has Lorentz-scalar quantum numbers, the operators which yield non-vanishing expectation values also must have scalar quantum numbers.) QCD vacuum expectation values of scalar operators are usually called "vacuum condensates" or just condensates, and $\langle\Omega|:\bar{q}q(0):|\Omega\rangle$ is the so-called quark condensate, often abbreviated as $\langle\bar{q}q\rangle$. Before we enter a more detailed discussion of the operator $\bar{q}q$, let us present a second example.

The second example arises in the context of non-leptonic weak transitions and will be discussed in more detail in the next section. In this case, the exchange of a weak W- or Z-boson also leads to a current product of the type $j_\mu^{V-A}(x)\, j^{\mu,V-A}(0)$ where the $(V-A)$ (vector minus axialvector) structure arises due to the weak interaction. The large scale in this problem is provided by the large mass of the weak bosons, as compared to the low-energy scale of QCD. Then performing the expansion for small x, in the case of the strangeness-changing $\Delta S = 1$ transition, the leading operator is of dimension six and has the form: $(\bar{s}\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)d)(0)$. Hadronic matrix elements of operators of this type together with the corresponding Wilson coefficient functions the constitute a weak decay amplitude.

Unlike the Green functions that we have already discussed above, composite operators do not automatically get renormalised through a renormalisation of the Lagrangian parameters (fields, coupling and masses). Usually, they require an additional operator renormalisation. Let us demonstrate that for the specific example of the bi-quark operator $O_{\bar{q}q}(z) \equiv :\bar{q}(z)q(z):$. A convenient way to calculate the renormalisation constant for a composite operator is by calculating Green functions with insertions of the operator in question. The most simple Green function for our example would be:

$$\Gamma^{(0)\,ij}_{\alpha\beta}(q) \;=\; -\int dz \int d^Dx\, \mathrm{e}^{iqx}\langle 0|T\{q^i_\alpha(x)\, O_{\bar{q}q}(z)\, q^j_\beta(0)\}|0\rangle \;=\; \delta^{ij}\,[\,S(q)S(q)\,]_{\alpha\beta}\,. \tag{3.46}$$

The renormalisation constant be be extracted by calculating higher order corrections to the Green function $\Gamma^{ij}_{\alpha\beta}(q)$ and comparing with the lowest-order structure of eq. (3.46).

Problem 3.2: Explicitly calculate the first order α_s correction to the Green function $\Gamma^{ij}_{\alpha\beta}(q)$ of eq. (3.46), and extract the renormalisation constant for the operator $O_{\bar{q}q}$ at that order.

From problem 3.2, we have obtained the renormalisation constant for the bi-quark operator,

$$Z_{\bar{q}q} \;=\; 1 + \frac{3}{4}\, C_F\, a_s\, \frac{1}{\hat{\varepsilon}} + \mathcal{O}(a_s^2)\,, \tag{3.47}$$

which is defined via the relation $:\bar{q}q:=\; Z_{\bar{q}q}\,(:\bar{q}^{\rm R}q^{\rm R}:)^{\rm R}$. As one might notice, at this order $Z_{\bar{q}q}$ turns out to be just the inverse of the mass renormalisation constant Z_m. One can proof that this relation actually holds to all order in perturbation theory. Thus we have:

$$Z_{\bar{q}q} \;=\; Z_m^{-1}\,, \tag{3.48}$$

and it follows that the product $m:\bar{q}q:$ is a quantity that does not depend on the renormalisation [28]. (A so-called *renormalisation group invariant* quantity.) Indeed, as such it can be considered a physical quantity and for the light quarks it can be related to meson masses and meson decay constants. However, one should emphasis that the renormalisation group invariance of $m:\bar{q}q:$ crucially depends on the operator being normal ordered. However, in the framework of the OPE, there are reasons to work with non-normal ordered operators, in order to obtain a clean separation of short-distance physics, encoded in the Wilson coefficients, and long-distance physics, represented through operator matrix elements [29, 30]. But this goes beyond the scope of the tutorial and the interested reader should consult the original literature.

3.5 The weak effective Hamiltonian

All weak transitions of hadrons are influenced by strong interactions, even though the character of QCD effects in weak decays depends strongly on the process under consideration. Generally, weak decays can be divided into three distinct classes: leptonic, semi-leptonic and non-leptonic decays. For simplicity, let us begin with a discussion of purely leptonic decays.

To be specific, consider the decay $K^+ \to \mu^+\nu_\mu$ for which the decay amplitude is given

by

$$A(K^+ \to \mu^+\nu_\mu) \;=\; \frac{G_F}{\sqrt{2}}\, V_{us}^*\, (\bar\nu_\mu \mu^-)_{V-A} \langle 0|(\bar s u)_{V-A}|K^+\rangle\,. \tag{3.49}$$

Here, $(\bar\nu_\mu\mu^-)_{V-A}$ is the leptonic current, whereas $(\bar s u)_{V-A}$ is the hadronic current with the quantum numbers of the K^+-meson, $(V-A)$ indicating the vector minus axial-vector structure of weak currents. Because the leptons do not interact strongly, inclusion of QCD effects only influences the hadronic matrix element, and the *factorised* form of the amplitude, lepton current times matrix element of quark current, remains rigorously valid.

The Kaon mass is relatively small, of the order of the QCD scale $\Lambda_{\overline{\rm MS}}$. Therefore, in this case a calculation of the hadronic matrix element only involves physics at long distances, and a non-perturbative method has to be applied. Conventionally, the current matrix element is parameterised in terms of a single parameter, the Kaon decay constant F_K which contains all non-perturbative physics:

$$\langle 0|(\bar s u)_{V-A}|K^+(p)\rangle \;=\; \langle 0|(\bar s\,\gamma_\mu(1-\gamma_5)u)|K^+(p)\rangle \;=\; -\,iF_K\,p_\mu\,. \tag{3.50}$$

Reasonable estimates for the Kaon decay constant have for example been obtained from the lattice or QCD sum rules. On the other hand, F_K can be determined experimentally from the measured decay rate. The knowledge of analogous matrix elements fixes the decay constants of other mesons.

Semi-leptonic decays are slightly more complicated. Consider the decay $K^+ \to \pi^0 e^+\nu_e$, for which the decay amplitude takes the form

$$A(K^+ \to \pi^0 e^+\nu_e) \;=\; \frac{G_F}{\sqrt{2}}\, V_{us}^*\, (\bar\nu_e e^-)_{V-A} \langle \pi^0|(\bar s u)_{V-A}|K^+\rangle\,. \tag{3.51}$$

As should be clear from the structure of eq. (3.51), also here the factorised form of the amplitude remains valid, and the strong interaction only influences the hadronic matrix element — this time called "form-factor" because the momentum transfer to the lepton pair can vary. Since the matrix element involves two meson states, its evaluation is more difficult. Thus, in general the calculation of form-factors from lattice QCD or sum rules is plagued by larger uncertainties than in the case of decay constants. For this reason, several models for form factors have been invoked. Moreover, in the case of K-decays, chiral perturbation theory turns out to be useful, and the matrix elements involving heavy mesons can be efficiently studied in the heavy quark effective theory.

The most complicated case to analyse are non-leptonic decays, because the factorisation of the matrix elements of dimension-six four-quark operators into the product of current matrix elements no longer holds. Gluons now can connect the quarks from two different

currents. It has proven useful to introduce the concept of the effective Hamiltonian. In the rest of this section, we shall discuss general aspects of the low-energy effective Hamiltonians used in the analyses of non-leptonic weak decays. Specific expressions for flavour-changing $\Delta F = 1$ as well as $\Delta F = 2$ ($F = S$, C, B) transitions have been calculated in the past, most of them at the next-to-leading order in renormalisation group improved perturbation theory [31].

The underlying framework for calculating the low-energy effective Hamiltonian is the operator product expansion [26] together with the renormalisation group equation [10, 11]. Due to the fact that $W^{\pm}$ and Z^0 are very massive, the basic weak interactions take place at very short distances $\mathcal{O}(1/M_{W,Z})$ and the operator product expansion amounts to an expansion in powers of $1/M^2_{W,Z}$ [27]. The effective Hamiltonians to be considered have the following universal structure:

$$\mathcal{H}_{eff} \;=\; \sum_k C_k(\mu) Q_k(\mu)\,, \tag{3.52}$$

where Q_k is a set of local operators contributing to a specific transition in question, and the C_k are the corresponding Wilson coefficient functions. We have already indicated that generally both the coefficient functions C_k, as well as the renormalised operators Q_k, depend on the renormalisation scale μ and the renormalisation scheme, whereas the effective Hamiltonian, being related to physical quantities, is free from these dependencies.

Evaluating the effective Hamiltonian between initial and final state vectors $|i\rangle$ and $|f\rangle$, eq. (3.52) takes the form

$$\langle f|\mathcal{H}_{eff}|i\rangle \;=\; \sum_k C_k(\mu)\langle f|Q_k(\mu)|i\rangle\,. \tag{3.53}$$

The operator product expansion makes it possible to separate the effective Hamiltonian into short-distance contributions, the Wilson coefficient functions C_k, due to asymptotic freedom being calculable in perturbation theory, and a long distance part, being comprised in the hadronic matrix elements $\langle f|Q_k|i\rangle$. Because of confinement, the long-distance contribution is genuinely non-perturbative and for its calculation once more we have to resort to methods which allow the inclusion of low-energy phenomena. Amongst these are lattice gauge theory, QCD sum rules, chiral perturbation theory or the $1/N$ expansion.

Until today, for various reasons all non-perturbative methods only allow to calculate the matrix elements at scales roughly $\mathcal{O}(1\,\mathrm{GeV})$. Since the weak interactions take place at scales $\mathcal{O}(M_W)$, large logarithms appear in the Wilson coefficient functions which have to be summed with the renormalisation group technique. The independence of $\langle f|\mathcal{H}_{eff}|i\rangle$ on

the renormalisation scale μ can be exploited to obtain the following renormalisation group equation:

$$\Big(\mu\frac{d}{d\mu}\,C_k(\mu)\Big)\langle Q_k(\mu)\rangle + C_k(\mu)\Big(\mu\frac{d}{d\mu}\,\langle Q_k(\mu)\rangle\Big) \;=\; 0\,. \tag{3.54}$$

On the other hand, we have the relation between the bare and the renormalised matrix elements:

$$\langle Q_i^B\rangle \;=\; Z_{ij}(\mu)\langle Q_j(\mu)\rangle\,. \tag{3.55}$$

In general, under renormalisation different operators can mix, leading to the renormalisation matrix Z_{ij}. Furthermore, the bare matrix element does not depend on the scale μ, and we get a second renormalisation group equation:

$$\Big(\mu\frac{d}{d\mu}\,Z_{ij}(\mu)\Big)\langle Q_j(\mu)\rangle + Z_{ij}(\mu)\Big(\mu\frac{d}{d\mu}\,\langle Q_j(\mu)\rangle\Big) \;=\; 0\,. \tag{3.56}$$

Defining the anomalous dimension matrix $\hat\gamma$ by

$$\hat\gamma \;\equiv\; \hat Z^{-1}(\mu)\,\mu\frac{d}{d\mu}\,\hat Z(\mu)\,, \tag{3.57}$$

eqs. (3.54) and (3.56) can be combined to give a renormalisation group equation for the Wilson coefficient function:

$$\mu\frac{d}{d\mu}\,\vec C(\mu) \;=\; \hat\gamma^T(\mu)\,\vec C(\mu)\,, \tag{3.58}$$

where the Wilson coefficients have been collected into the vector $\vec C(\mu)$.

Since we only consider mass-independent renormalisation schemes, as already mentioned above, we have $\hat\gamma(\mu) = \hat\gamma(a(\mu))$. Combining eq. (3.58) with the evolution equation for the coupling constant, eq. (2.8), yields:

$$\frac{d}{da}\,\vec C(a) \;=\; \frac{\hat\gamma^T(a)}{\beta(a)}\,\vec C(a)\,. \tag{3.59}$$

It is convenient to define an evolution matrix $\hat U(a_1, a_2)$ by

$$\vec C(a_1) \;=\; \hat U(a_1,a_2)\,\vec C(a_2) \qquad \text{and} \qquad \hat U(a,a) \;=\; \hat 1\,, \tag{3.60}$$

which obeys the same differential equation:

$$\frac{d}{da}\,\hat U(a,a_2) \;=\; \frac{\hat\gamma^T(a)}{\beta(a)}\,\hat U(a,a_2)\,. \tag{3.61}$$

This equation has the following formal solution ($\mu_1 < \mu_2$):

$$\hat{U}(\mu_1, \mu_2) \;=\; T_a \left[\exp\left\{ \int\limits_{a(\mu_2)}^{a(\mu_1)} \frac{\hat{\gamma}^T(x)}{\beta(x)} dx \right\} \right]. \tag{3.62}$$

The operator T_a denotes an ordering in the coupling constant, such that the coupling increases from right to left. This is necessary because the anomalous dimension in the exponential is a matrix. The solution for the renormalisation group improved Wilson coefficient functions finally reads:

$$\vec{C}(\mu_1) \;=\; T_a \left[\exp\left\{ \int\limits_{a(\mu_2)}^{a(\mu_1)} \frac{\hat{\gamma}^T(x)}{\beta(x)} dx \right\} \right] \vec{C}(\mu_2)\,. \tag{3.63}$$

This allows to rewrite the effective Hamiltonian in the form

$$\langle f|\mathcal{H}_{eff}|i\rangle \;=\; \langle f|\vec{Q}^T(\mu_1)|i\rangle\, \hat{U}(\mu_1, \mu_2)\, \vec{C}(\mu_2)\,. \tag{3.64}$$

Let us recapitulate what has been achieved so far: the effective low-energy Hamiltonian has been explicitly separated into three pieces; the Wilson coefficient functions $\vec{C}(\mu_2)$, the evolution matrix $\hat{U}(\mu_1, \mu_2)$ and the matrix elements of the operators contributing to a certain process, $\langle f|\vec{Q}(\mu_1)|i\rangle$. Splitting the scales of coefficient functions and matrix elements has the advantage that the Wilson coefficients can now be calculated in pure perturbation theory at some high energy scale $\mu_2 \approx \mathcal{O}(M_W)$ or $\mathcal{O}(m_t)$. Then the coefficient functions can be evolved with the help of the evolution matrix to some low-energy scale $\mu_1 \approx \mathcal{O}(1\,\text{GeV})$, $\mathcal{O}(m_c)$ or $\mathcal{O}(m_b)$, at which the process takes place, and all long-distance physics, being contained in the matrix elements evaluated at this scale, can be calculated by the non-perturbative methods mentioned above. The explicit calculations show that the perturbative expansion breaks down for scales $\mu_1 \lesssim 1\,\text{GeV}$. Therefore, the coefficient functions should not be evolved below this scale.

To summarise, the calculation of the effective Hamiltonian for non-leptonic weak decays consists of three steps:

i) Calculation of the Wilson coefficient functions $\vec{C}(\mu)$ at some high energy scale $\mathcal{O}(M_W)$ or $\mathcal{O}(m_t)$ in perturbative QCD,

ii) calculation of the anomalous dimension matrix $\hat{\gamma}$ corresponding to the operators in question, which is required to construct the evolution matrix $\hat{U}(\mu_1, \mu_2)$, and

iii) calculation of the matrix elements $\langle f|\vec{Q}(\mu)|i\rangle$ at a low-energy scale $\mathcal{O}(1\,\mathrm{GeV})$, $\mathcal{O}(m_c)$ or $\mathcal{O}(m_b)$ by means of some non-perturbative method.

All three components which constitute the effective Hamiltonian depend, obviously, on the renormalisation scale, but also on the renormalisation scheme. Again, like the scheme dependence of the QCD scale Λ_{QCD}, here, this scheme dependence is first seen at the next-to-leading order. Furthermore, if one performs perturbative calculations in the Standard Model, one has to deal with the Dirac matrix γ_5. In dimensional regularisation [13], so far the only procedure with which actual higher-loop calculations have been carried out, the definition of γ_5 is problematic, and various choices of differing theoretical soundness exist. Different treatments of γ_5 can also be considered as different schemes. In the following, we shall mainly give results for the calculations using a naively anti commuting γ_5 (NDR scheme). Practically, however, due to algebraic inconsistencies of this definition of γ_5, it has to be applied with great care — alas, not so naively. In addition, we shall also comment on a mathematically consistent definition of γ_5, originally due to 't Hooft and Veltman (HV scheme) [13]. Calculating in two different schemes has the merit that it provides a very strong check on the correctness of physical results, which certainly have to be independent of this choice. For further details, the reader is referred to the original publications.

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Appendix A

Solutions to the exercises

Solution 1.1 If the operator dim[...] represents the mass dimension of the quantity under investigation, one finds ($D = 4 - 2\varepsilon$):

$$\dim[q_i^A(x)] \;=\; \frac{1}{2}\,(D-1) \;=\; \frac{3}{2} - \varepsilon\,, \qquad \dim[B_\mu^a(x)] \;=\; \frac{D}{2} - 1 \;=\; 1 - \varepsilon\,,$$

$$\dim[c^a(x)] \;=\; \frac{D}{2} - 1 \;=\; 1 - \varepsilon\,, \qquad \dim[g] \;=\; 2 - \frac{D}{2} \;=\; \varepsilon\,.$$

Solution 1.2 $\dim[S^{(0)}(p)] \;=\; -1\,, \qquad \dim[D_{\mu\nu}^{(0)}(k)] \;=\; -2\,.$

Thus the momentum space propagators retain the same mass dimension as in 4 space-time dimensions. This is not the case for the coordinate space propagators $S^{(0)}(x)$ and $D_{\mu\nu}^{(0)}(x)$!

Solution 1.3 $\gamma_\mu\gamma_\nu\gamma^\mu \;=\; (2-D)\gamma_\nu\,,$

$$\gamma_\mu\gamma_\nu\gamma_\lambda\gamma^\mu \;=\; 2\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda\gamma^\mu \;=\; 4g_{\nu\lambda}\mathbb{1} + (D-4)\gamma_\nu\gamma_\lambda\,.$$

Solution 1.4

$$\frac{1}{ab^2} \;=\; 2\int\limits_0^1 \frac{(1-x)\,dx}{[ax+b(1-x)]^3} \;=\; 2\int\limits_0^1 \frac{z\,dz}{[a+(b-a)z]^3}$$

$$= \frac{2}{(b-a)}\int\limits_0^1 \frac{[a+(b-a)z-a]}{[a+(b-a)z]^3}\,dz \;=\; \frac{2}{(b-a)}\left[\int\limits_0^1 \frac{dz}{[a+(b-a)z]^2} - \int\limits_0^1 \frac{a\,dz}{[a+(b-a)z]^3}\right]$$

$$= \frac{2}{(b-a)^2}\left[\frac{-1}{[a+(b-a)z]}\bigg|_0^1 + \frac{a}{2[a+(b-a)z]^2}\bigg|_0^1\right]$$

$$= \frac{2}{(b-a)^2}\left[\frac{1}{a}-\frac{1}{b}+\frac{a}{2b^2}-\frac{1}{2a}\right] = \frac{1}{ab^2} \quad \surd$$

Making use of the hypergeometric function ${}_2F_1(a,b;c;z)$, the relation can also be shown in the general case:

$$\begin{aligned}\frac{1}{a^\alpha b^\beta} &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^1 dx\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{1}{b^{\alpha+\beta}}\int\limits_0^1 dx\,x^{\alpha-1}(1-x)^{\beta-1}[1-(1-\tfrac{a}{b})x]^{-\alpha-\beta} \\ &= \frac{1}{b^{\alpha+\beta}}\,{}_2F_1(\alpha+\beta,\alpha;\alpha+\beta;1-\tfrac{a}{b}) = \frac{1}{b^{\alpha+\beta}}\left(\frac{b}{a}\right)^\alpha = \frac{1}{a^\alpha b^\beta} \qquad \surd\end{aligned}$$

Solution 1.8 For the calculation of the quark selfenergie, we shall need two types of massless Feynman integrals, which we first want to present in the general case for future purposes:

$$\begin{aligned}\hat{I}(\alpha,\beta;p) &\equiv \mu^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\,\frac{1}{[k^2+i0]^\alpha[(p-k)^2+i0]^\beta} \\ &= \frac{i}{4\pi^2}\left(\frac{4\pi\mu^2}{-p^2}\right)^\varepsilon\frac{1}{p^{2(\alpha+\beta-2)}}\,\frac{\Gamma(2-\alpha-\varepsilon)\Gamma(2-\beta-\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(4-\alpha-\beta-2\varepsilon)}\,\Gamma(\alpha+\beta-2+\varepsilon)\end{aligned}$$

$$\hat{I}_\mu(\alpha,\beta;p) \equiv \mu^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\,\frac{k_\mu}{[k^2+i0]^\alpha[(p-k)^2+i0]^\beta} = p_\mu\,\frac{(2-\alpha-\varepsilon)}{(4-\alpha-\beta-2\varepsilon)}\,\hat{I}(\alpha,\beta;p)$$

The first expression can be derived by introducing Feynman parameters and the making use of the formula (1.34) for $I(\alpha,\beta,a^2)$. The remaining x-integration can be performed analogously to the derivation of (1.34).

To derive the second expression, one can use a trick which is quite useful in general. Since the result can only depend on the momentum p, and must transform like a Lorentz vector, we can already say that it must have the structure $A(p^2)\,p_\mu$. Now, we can multiply both sides with $2p^\mu$, yielding:

$$\mu^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\,\frac{2p\cdot k}{[k^2+i0]^\alpha[(p-k)^2+i0]^\beta} = 2p^2A(p^2)\,.$$

The numerator under the integral can be rewritten as: $2p\cdot k = p^2+k^2-(p-k)^2$, which results in three integrals of the type $\hat{I}(\alpha,\beta;p)$. After some massaging of the Γ-functions, one arrives at the given integral formula for $\hat{I}_\mu(\alpha,\beta;p)$.

The two special cases which are needed for the calculation of $\Sigma^{(1)}(p)$ are:

$$\hat{I}(1,1;p) \;=\; \frac{i}{4\pi^2}\left(\frac{4\pi\mu^2}{-p^2}\right)^{\varepsilon}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\,\Gamma(\varepsilon) \qquad \text{and} \qquad \hat{I}_\mu(1,1;p) \;=\; \frac{p_\mu}{2}\,\hat{I}(1,1;p)\,.$$

This can be inserted into our expression for $\Sigma^{(1)}(p)$:

$$\begin{aligned}\Sigma^{(1)}(p) \;&=\; -\,i\,g^2\mu^{-2\varepsilon}\,C_F\,\mu^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\,\frac{[\gamma_\mu(\not{p}-\not{k}+m)\gamma^\mu]}{k^2\,[(p-k)^2-m^2]}\\ &=\; i\,g^2\mu^{-2\varepsilon}\,C_F\Big[(1-\varepsilon)\not{p}-(4-2\varepsilon)m\Big]\hat{I}(1,1;p)\\ &=\; \frac{g^2\mu^{-2\varepsilon}}{(4\pi)^2}\,C_F\Big[(-1+\varepsilon)\not{p}+(4-2\varepsilon)m\Big]\left\{\,\frac{1}{\hat{\varepsilon}}-\ln\frac{-p^2}{\mu^2}+2+\mathcal{O}(\varepsilon)\,\right\}\end{aligned}$$

Splitting up $\Sigma(p)$ according to eq. (1.47), one finds:

$$\Sigma_p^{(1)}(p) \;=\; \frac{C_F}{4}\,\frac{\alpha_s}{\pi}\left\{-\frac{1}{\hat{\varepsilon}}+\ln\frac{-p^2}{\mu^2}-1\right\}, \quad \Sigma_m^{(1)}(p) \;=\; \frac{C_F}{4}\,\frac{\alpha_s}{\pi}\left\{\frac{4}{\hat{\varepsilon}}-4\ln\frac{-p^2}{\mu^2}+6\right\},$$

where we have defined the dimensionless coupling $\alpha_s \equiv g^2\mu^{-2\varepsilon}/(4\pi)$.

Solution 2.1 Like for the coupling a_s, we start with the relation between the bare and the renormalised quark mass: $m = Z_m^{-1}m^0$ where m^0 corresponds to the bare quark mass. Inserting this in the definition of the mass anomalous dimension yields:

$$\begin{aligned}\gamma(a_s) \;&=\; -\,\frac{\mu}{m}\,\frac{dm}{d\mu} \;=\; -\,\frac{\mu}{(Z_m^{-1}m^0)}\,\frac{d(Z_m^{-1}m^0)}{d\mu} \;=\; \frac{\mu}{Z_m}\,\frac{Z_m}{d\mu} \;=\; \frac{\mu}{Z_m}\,\frac{da_s}{d\mu}\,\frac{Z_m}{da_s}\\ &=\; -\,\frac{\beta(a_s)}{Z_m}\,\frac{Z_m}{da_s} \;=\; -\,2a_s\varepsilon\left(-\frac{3}{4}\right)C_F\,\frac{1}{\hat{\varepsilon}}+\mathcal{O}(a_s^2) \;=\; \frac{3}{2}\,C_F\,a_s+\mathcal{O}(a_s^2)\,.\end{aligned}$$

Thus we otain: $\gamma_1 \;=\; \frac{3}{2}\,C_F \overset{N_c=3}{=} 2\,.$

Solution 2.2 At one-loop order, the exponential factor for the running of the quark mass takes the form:

$$\exp\left\{\int\limits_{a_s(\mu_1)}^{a_s(\mu_2)} da_s\,\frac{\gamma(a_s)}{\beta(a_s)}\right\} \;=\; \exp\left\{\frac{\gamma_1}{\beta_1}\int\limits_{a_s(\mu_1)}^{a_s(\mu_2)}\frac{da_s}{a_s}\right\} \;=\; \exp\left\{\frac{\gamma_1}{\beta_1}\ln\frac{a_s(\mu_2)}{a_s(\mu_1)}\right\} \;=\; \left(\frac{a_s(\mu_2)}{a_s(\mu_1)}\right)^{\gamma_1/\beta_1}.$$

The running of the quark mass is the given by:

$$m(\mu_2) \;=\; m(\mu_1)\left(\frac{a_s(\mu_2)}{a_s(\mu_1)}\right)^{\gamma_1/\beta_1}\Big[\,1+\mathcal{O}(a_s)\,\Big]\,.$$

Solution 3.1 Up to order a^2, $D(s)$ of eq. (3.11) is given by:

$$D(s) \;=\; \frac{N_c}{12\pi^2}\Big[\, c_{01} + a_\mu(c_{11}+2c_{12}L) + a_\mu^2(c_{21}+2c_{22}L+3c_{23}L^2)\,\Big]\,.$$

Inserting this expression into the RGE (3.12), up to the desired order we find:

$$4a_\mu c_{12} + 2a_\mu^2(2c_{22}+6c_{23}L) + \beta_1 a_\mu^2(c_{11}+2c_{12}L) + \mathcal{O}(a_\mu^3) \;=\; 0\,.$$

At order a_μ, only the c_{12}-term is present. Therefore, we obtain the condition $c_{12} = 0$. Then, also at $\mathcal{O}(a_\mu^2 L)$ the c_{23}-term is the only remaining one and thus has to vanish: $c_{23} = 0$. Finally, the pure order a_μ^2 term gives the non-trivial relation $c_{22} = \beta_1 c_{11}/4$.

Solution 3.1 The renormalisation of the external legs is taken care of by the renormalisation of the quark fields in the operator through the renormalisation constant Z_{2F}. Thus we only have to calculate the vertex correction where the gluon line goes from one quark to the other. Suppressing spinor as well as colour indices, the one-loop correction $\Gamma^{(1)}$ is given by:

$$\begin{aligned}
\Gamma^{(1)}(q) \;&=\; g^2\!\int d^Dz\, d^Dz_1\, d^Dz_2\, d^Dx\, \mathrm{e}^{\mathrm{i}qx}\, \langle 0|T\{q(x)(\bar q\gamma^\mu t^a B^a_\mu q)(z_1)\!:\!\bar q q(z)\!:\!(\bar q\gamma^\nu t^b B^b_\nu q)(z_2) q(0)\}|0\rangle \\
&=\; ig^2 C_F\!\int d^Dz\, d^Dz_1\, d^Dz_2\, d^Dx\, \mathrm{e}^{\mathrm{i}qx}\, [S(x-z_1)\gamma^\mu S(z_1-z)S(z-z_2)\gamma^\nu S(z_2)] D_{\mu\nu}(z_1-z_2) \\
&=\; ig^2 C_F\!\int \frac{d^Dk}{(2\pi)^D}\, [S(q)\gamma^\mu S(q-k)S(q-k)\gamma^\nu S(q)] D_{\mu\nu}(k)
\end{aligned}$$

Analogous to the self-energy for the quark propagator, also in this case we can define a self-energy for the Green function $\Gamma(q)$ according to:

$$\Gamma(q) \;\equiv\; S(q)[\,\mathbb{1} + \Sigma_{\bar q q}(q)\,]\,S(q)\,.$$

At first order α_s, we then deduce:

$$\begin{aligned}
\Sigma^{(1)}_{\bar q q}(q) \;&=\; ig^2 C_F\,\mu^{2\varepsilon}\!\int \frac{d^Dk}{(2\pi)^D}\, [\gamma^\mu S(q-k)S(q-k)\gamma^\nu] D_{\mu\nu}(k) \\
&=\; -4\pi i\,\alpha_s C_F(D-1+a)\,\mu^{2\varepsilon}\!\int \frac{d^Dk}{(2\pi)^D}\,\frac{1}{k^2(q-k)^2} \\
&=\; C_F a_s\left\{\,(3+a)\left(\frac{1}{\hat\varepsilon} - \ln\frac{-q^2}{\mu^2}\right) + 4 + 2a\,\right\}
\end{aligned}$$

Now, the renormalised operator in terms of renormalised quark fields is given by:

$$:\!\bar q q\!: \;\equiv\; Z_{\bar q q}\,(:\!\bar q^{\mathrm{R}} q^{\mathrm{R}}\!:)^{\mathrm{R}}\,.$$

The dependence of $\Sigma^{(1)}_{\bar{q}q}(q)$ on the gauge parameter a exactly cancels against the corresponding dependence of the propagator renormalisation. Since the propagator renormalisation is proportional to a, the full result corresponds to the one in the Landau gauge $a = 0$. The operator renormalisation constant $Z_{\bar{q}q}$ is thus found to be

$$Z_{\bar{q}q} \;=\; 1 + \frac{3}{4}\, C_F\, a_s\, \frac{1}{\hat{\varepsilon}} + \mathcal{O}(a_s^2)\,,$$

independent of the gauge parameter at this order.