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Anomalous dimensions of 4-quark operators and renormalon structure of mesonic 2-point correlators

Diogo Boito,[a] Dirk Hornung[b] and Matthias Jamin[c]

[a] *Instituto de Física, Universidade de São Paulo, Rua do Matão Travessa R, 187, 05508-090, São Paulo, SP, Brazil*

[b] *Institut de Física d'Altes Energies (IFAE), Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain.*

[c] *Institució Catalana de Recerca i Estudis Avançats (ICREA), IFAE, Theoretical Physics Group, UAB, E-08193 Bellaterra, Barcelona, Spain*

E-mail: dboito@if.usp.br, jamin@ifae.es

Abstract: In this work ...

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Contents

1 Introduction

The perturbative expansion in QCD is known to lead to a divergent series which is at best asymptotic. The asymptotic behaviour of the perturbative series manifests itself in the appearance of singularities for its Borel transform which lie on the negative or positive real axis in the Borel variable. Those singularities, connected with renormalisation of the theory, are termed *renormalons* [2, 3]. More specifically, the ones on the negative real axis are called ultraviolet (UV) renormalons and those on the positive real Borel axis infrared (IR) renormalons.

The presence of IR renormalon poles leads to ambiguities in the definition of the full function which is related to the perturbative series, because the Borel resummation (inverse Borel transform) entails to perform an integral over the positive real Borel axis which naively is not well defined. Associated with the ambiguities in the definition of the Borel integral is the appearance of higher-dimensional operator corrections, the so-called *QCD condensates*, such that the full function is unambiguous. The operators that display renormalon ambiguities are a subset of those that arise in the framework of the operator product expansion (OPE).

Limiting ourselves to correlation functions of vector or axialvector currents with respect to the QCD vacuum, the IR renormalon pole on the positive real axis closest to the origin of the Borel plane is associated to the vacuum matrix element of one dimension-4 operator, the *gluon consensate*. The next-closest singularity then is found to correspond to the dimension-6 *triple gluon condensate* and a set of dimension-6 *4-quark condensates*. It is these latter 4-quark condensates that we intend to investigate in more detail in the present work.

To be continued ...

2 Non-singlet vector and axialvector correlators

We begin by investigating the dimension-6 OPE contributions to the two-point correlation functions of non-singlet vector and axialvector currents $j_\mu^V(x) = (\bar{u}\gamma_\mu d)(x)$ and $j_\mu^A(x) = (\bar{u}\gamma_\mu\gamma_5 d)(x)$ which are relevant for QCD analyses of hadronic τ decays [1] and correspond to the charged ρ and A_1 mesons. For simplicity, massless light quarks will be assumed in which case the correlators take the form

$$\Pi_{\mu\nu}^{V/A}(q) \equiv i\int dx\, e^{iqx}\, \langle\Omega|T\{j_\mu^{V/A}(x)j_\nu^{V/A}(0)^\dagger\}|\Omega\rangle = \big(q_\mu q_\nu - g_{\mu\nu}q^2\big)\,\Pi^{V/A}(q^2)\,. \tag{2.1}$$

Here, $|\Omega\rangle$ denotes the full QCD vacuum and the second identity follows because in the massless limit non-singlet vector and axialvector currents are conserved.

In the framework of the OPE, the scalar functions $\Pi^{V/A}$ permit an expansion in powers of $1/Q^2$ with $Q^2 \equiv -q^2$ being in the Euclidean region,

$$\Pi^{V/A}(Q^2) = C_0(Q^2) + C_4(Q^2)\,\frac{\langle O_4\rangle}{Q^4} + C_6^{V/A}(Q^2)\,\frac{\langle O_6\rangle}{Q^6} + \ldots\,. \tag{2.2}$$

In writing eq. (2.2), for simplicity, we have suppressed the vacuum state. In the OPE, only the coefficient functions $C_i^{V/A}$ depend on the momentum, while the operators O_i are local. Both, coefficient functions and operators depend, however, on the renormalisation scale μ which is not shown explicitly. Furthermore, for flavour non-singlet currents the purely perturbative contribution $C_0(Q^2)$ is the same for vector and axialvector.[1] In the massless case, this also remains true for the dimension-4 contribution, which then only consists of the gluon condensate $\langle G_{\mu\nu}^a G^{a\,\mu\nu}\rangle$.

Our main concern in this work will be the dimension-6 term which receives contributions from the three-gluon condensate $\langle g^3 f_{abc} G_{\mu\nu}^a G^{b\,\nu}{}_\lambda G^{c\,\lambda\mu}\rangle$ and four-quark condensates. As the three-gluon condensate does not arise at leading order, below we concentrate only on the four-quark condensates. Their contribution to $\Pi^{V/A}(Q^2)$ has been computed at the next-to-leading order in refs. [4, 5]. For our following discussion, it will be convenient to present the corresponding results for $V-A$ and $V+A$ correlation functions, because in the former case the so-called penguin operator contributions cancel. For $N_f = 3$ light quark flavours and at $N_c = 3$, one then finds

$$C_6^{V-A}(Q^2)\,\langle O_6\rangle = 4\pi^2 a_s\left\{\left[2 + \Big(\tfrac{25}{6} - L\Big)a_s\right]\langle Q_-^o\rangle - \Big(\tfrac{11}{18} - \tfrac{2}{3}L\Big)a_s\langle Q_-^s\rangle\right\}, \tag{2.3}$$

and

$$\begin{aligned} C_6^{V+A}(Q^2)\,\langle O_6\rangle = -4\pi^2 a_s\bigg\{&\left[2 + \Big(\tfrac{155}{24} - \tfrac{7}{2}L\Big)a_s\right]\langle Q_+^o\rangle + \Big(\tfrac{11}{18} - \tfrac{2}{3}L\Big)a_s\langle Q_+^s\rangle + \\ &\left[\tfrac{4}{9} + \Big(\tfrac{37}{36} - \tfrac{95}{162}L\Big)a_s\right]\langle Q_3\rangle + \Big(\tfrac{35}{108} - \tfrac{5}{18}L\Big)a_s\langle Q_4\rangle + \\ &\Big(\tfrac{14}{81} - \tfrac{4}{27}L\Big)a_s\langle Q_6\rangle - \Big(\tfrac{2}{81} + \tfrac{4}{27}L\Big)a_s\langle Q_7\rangle\,. \end{aligned} \tag{2.4}$$

[1] For correlators of flavour-singlet currents, which will be discussed below, this is not the case.

Here, $a_s \equiv \alpha_s/\pi$, $L \equiv \ln Q^2/\mu^2$ and the constant terms of order a_s^2 correspond to the choice of an anti-commuting γ_5 in D space-time dimensions which can be made consistent as long as no traces with an odd number of γ_5's arise in the calculation [4].

The appearing four-quark operators are a subset which belong to the complete basis that below will be required for their one-loop renormalisation:

$$Q_V^o \;=\; (\bar{u}\gamma_\mu t^a d \bar{d}\gamma^\mu t^a u)\,, \quad Q_A^o \;=\; (\bar{u}\gamma_\mu\gamma_5 t^a d \bar{d}\gamma^\mu\gamma_5 t^a u)\,, \tag{2.5}$$

$$Q_V^s \;=\; (\bar{u}\gamma_\mu d \bar{d}\gamma^\mu u)\,, \quad Q_A^s \;=\; (\bar{u}\gamma_\mu\gamma_5 d \bar{d}\gamma^\mu\gamma_5 u)\,, \tag{2.6}$$

$$Q_3 \;\equiv\; (\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu t^a q)\,, \tag{2.7}$$

$$Q_4 \;\equiv\; (\bar{u}\gamma_\mu\gamma_5 t^a u + \bar{d}\gamma_\mu\gamma_5 t^a d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu\gamma_5 t^a q)\,, \tag{2.8}$$

$$Q_5 \;\equiv\; (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu q)\,, \tag{2.9}$$

$$Q_6 \;\equiv\; (\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d) \sum_{q=u,d,s} (\bar{q}\gamma^\mu\gamma_5 q)\,, \tag{2.10}$$

$$Q_7 \;\equiv\; \sum_{q=u,d,s} (\bar{q}\gamma_\mu t^a q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu t^a q')\,, \tag{2.11}$$

$$Q_8 \;\equiv\; \sum_{q=u,d,s} (\bar{q}\gamma_\mu\gamma_5 t^a q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu\gamma_5 t^a q')\,, \tag{2.12}$$

$$Q_9 \;\equiv\; \sum_{q=u,d,s} (\bar{q}\gamma_\mu q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu q')\,, \tag{2.13}$$

$$Q_{10} \;\equiv\; \sum_{q=u,d,s} (\bar{q}\gamma_\mu\gamma_5 q) \sum_{q'=u,d,s} (\bar{q}'\gamma^\mu\gamma_5 q')\,. \tag{2.14}$$

$Q_{V/A}^o$ and $Q_{V/A}^s$ are usually termed current-current operators and Q_3 to Q_{10} penguin operators. In addition, we have defined the four current-current operators which appear in eqs. (2.3) and (2.4).

$$Q_\pm^o \;\equiv\; Q_V^o \pm Q_A^o\,, \qquad Q_\pm^s \;\equiv\; Q_V^s \pm Q_A^s\,. \tag{2.15}$$

Next, we investigate the scale dependence of a general term R_O in the OPE, corresponding to a set of operators $\vec{O}$ with equal dimension,

$$R_O \;=\; \vec{C}^T(\mu)\,\langle\vec{O}(\mu)\rangle\,, \tag{2.16}$$

where now the renormalisation scale μ is displayed explicitly and the potential dependence on other dimensionful parameters is implicit. For vector and axialvector currents, the renormalisation scale dependence of the correlator only arises from the purely perturbative contribution. Hence, R_O should not depend on μ, and it immediately follows that

$$\left[\mu\,\frac{d}{d\mu}\,\vec{C}^T(\mu)\right]\langle\vec{O}(\mu)\rangle \;=\; -\,\vec{C}^T(\mu)\left[\mu\,\frac{d}{d\mu}\,\langle\vec{O}(\mu)\rangle\right]. \tag{2.17}$$

The anomalous dimension matrix $\hat{\gamma}_O$ of the operator matrix elements can be defined by

$$-\,\mu\,\frac{d}{d\mu}\,\langle\vec{O}(\mu)\rangle\;\equiv\;\hat{\gamma}_O(a_\mu)\,\langle\vec{O}(\mu)\rangle\,, \tag{2.18}$$

with $a_\mu \equiv a_s(\mu)$. If the bare and renormalised operator matrix elements are related by

$$\langle\vec{O}^B\rangle\;\equiv\;\hat{Z}_O(\mu)\,\langle\vec{O}(\mu)\rangle\,, \tag{2.19}$$

it follows that the anomalous dimension matrix can be computed from the renormalisation matrix $\hat{Z}_O(\mu)$ via

$$\hat{\gamma}_O(a_\mu)\;=\;\hat{Z}_O^{-1}(\mu)\,\mu\,\frac{d}{d\mu}\,\hat{Z}_O(\mu)\,. \tag{2.20}$$

Plugging eq. (2.18) into the RGE for R, eq. (2.17), one obtains an RGE that has to be satisfied by the coefficient functions $\vec{C}(\mu)$,

$$\mu\,\frac{d}{d\mu}\,\vec{C}(\mu)\;=\;\hat{\gamma}_O^T(a_\mu)\,\vec{C}(\mu)\,. \tag{2.21}$$

This equation shall be checked for the coefficient functions of the dimension-6 operators in eqs. (2.3) and (2.4). Furthermore, below it will be convenient to consider the anomalous dimension matrix in a linearly transformed basis. If the transformed basis $\langle\vec{O}'(\mu)\rangle$ of operator matrix elements is defined by

$$\langle\vec{O}'(\mu)\rangle\;\equiv\;\hat{T}\,\langle\vec{O}(\mu)\rangle\,, \tag{2.22}$$

the corresponding transformed anomalous dimension matrix takes the form

$$\hat{\gamma}_{O'}\;=\;\hat{T}\,\hat{\gamma}_O\,\hat{T}^{-1}\,. \tag{2.23}$$

The calculation of one-loop anomalous dimension matrices for dimension-6 four-quark operators is fairly standard and details can for example be found in ref. [6]. We have performed the actual computation in two ways: firstly, by explicit calculation of the diagrams of figure 1, and secondly, by relating the appearing operators to the complete basis of dimension-6 four-quark operators without derivatives in the case of three quark flavours that had been employed in ref. [7]. Further details on the second approach can be found in appendix A.

Expanding the anomalous dimension matrix in a power series in a_s,

$$\hat{\gamma}_O(a_s)\;=\;a_s\,\hat{\gamma}_O^{(1)}+a_s^2\,\hat{\gamma}_O^{(2)}+\ldots\,, \tag{2.24}$$

the leading-order anomalous dimension matrix corresponding to the operators $Q_- \equiv (Q_-^o,\, Q_-^s)$ appearing in eq. (2.3) is found to be

$$\hat{\gamma}_{Q_-}^{(1)}\;=\;\begin{pmatrix} -\frac{3N_c}{2}+\frac{3}{N_c} & -\frac{3C_F}{2N_c} \\ -\,3 & 0 \end{pmatrix}. \tag{2.25}$$

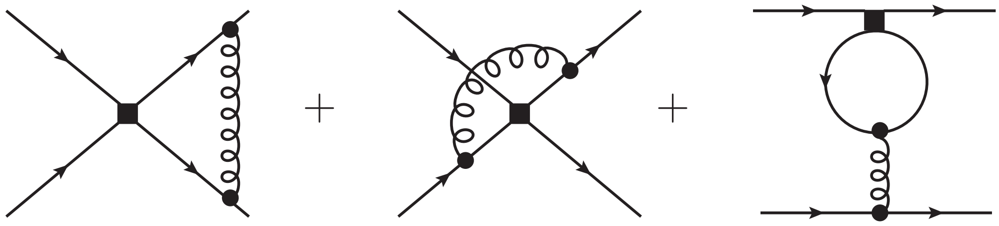

Figure 1. Exemplary one-loop current-current and penguin diagrams that have to be calculated in the process of obtaining the anomalous dimension matrix of four-quark operators. Quark self-energy diagrams are not displayed.

At this order, the set of two operators is closed under renormalisation, meaning that no additional operators are generated through the diagrams that have to be computed.[2] Employing $\hat{\gamma}^{(1)}_{Q_-}$ and the coefficient function $C_6^{V-A}(Q^2)$ of eq. (2.3), it is a simple matter to confirm that the RGE (2.21) is indeed satisfied.

Likewise, the anomalous dimension matrix for the operators appearing in the $V+A$ case of eq. (2.4) can be calculated. Here, three additional operators have to be added in the course of renormalisation, and at the leading order a closed set can be chosen as $Q_+ \equiv (Q_+^o, Q_+^s, Q_3, Q_4, Q_6, Q_7, Q_8, Q_9, Q_{10})$. In general, also the operator Q_5 of the basis presented above arises. However, in four dimensions, one operator in the full set is redundant and can be expressed through the others by means of Fierz transformations. Since Q_5 does not appear in the OPE expression (2.4), we have rewritten it through the remaining operators. The anomalous dimension matrix then takes the form

$$\hat{\gamma}^{(1)}_{Q_+} = \left(\begin{array}{cccc} -\frac{3}{N_c} & \frac{3C_F}{2N_c} & -\frac{1}{3N_c} & 0 \\ 3 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{N_f}{3} - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} \\ \frac{3}{2} + \frac{3}{2N_c} & -\frac{3C_F}{2N_c} & \frac{3N_c}{4} + \frac{3}{2} - \frac{11}{6N_c} & -\frac{3N_c}{4} + \frac{3}{2} + \frac{3}{2Nc} \\ 0 & 0 & \frac{11}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right.$$

[2]This is only true when working with a strictly anti-commuting γ_5, and projecting out evanescent operators, which however is admissible at the leading order. Further discussion can be found in ref. [4].

$$
\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{3C_F}{2N_c} & \frac{2}{3} & 0 & 0 & 0 \\
-\frac{3C_F}{2N_c} & -\frac{3}{4}-\frac{3}{4N_c} & -\frac{3}{4}-\frac{3}{4N_c} & \frac{3C_F}{4N_c} & \frac{3C_F}{4N_c} \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{2N_f}{3}-\frac{3N_c}{4}-\frac{1}{3N_c} & \frac{3N_c}{4}-\frac{3}{N_c} & 0 & \frac{3C_F}{2N_c} \\
0 & \frac{3N_c}{4}-\frac{10}{3N_c} & -\frac{3N_c}{4} & \frac{3C_F}{2N_c} & 0 \\
0 & \frac{2}{3} & 3 & 0 & 0 \\
0 & \frac{11}{3} & 0 & 0 & 0
\end{matrix}\right) . \tag{2.26}
$$

The 4×4 sub-matrix corresponding to the operators Q_7 to Q_{10} agrees with the result presented in section 3.2.3 of ref. [3]. Again, it is straightforward to verify that by using $\hat{\gamma}^{(1)}_{Q_+}$ and the coefficient function of eq. (2.4), the RGE (2.21) is satisfied to leading order.

For the subsequent discussion, it will be convenient to consider a basis of four-quark operators in which the leading-order anomalous dimension matrix is diagonal. From linear algebra it is well known that the anomalous dimension matrix $\hat{\gamma}^{(1)}_O$ can be diagonalised by a matrix $\hat{V}$, which as columns contains the eigenvectors of $\hat{\gamma}^{(1)}_O$, in the following fashion:

$$
\hat{\gamma}^{(1)}_D \,=\, \hat{V}^{-1}\,\hat{\gamma}^{(1)}_O\,\hat{V}\,. \tag{2.27}
$$

The diagonal entries of $\hat{\gamma}^{(1)}_D$ then correspond to the eigenvalues of $\hat{\gamma}^{(1)}_O$. Furthermore, the operator basis with $\hat{\gamma}^{(1)}_D$ as the leading-order anomalous dimension matrix is given by $\hat{V}^{-1}\vec{O}$. Rewriting the term R_O of (2.16) in the OPE,

$$
R_O \,=\, \vec{C}^{\,T}(\mu)\,\hat{V}\,\hat{V}^{-1}\langle\vec{O}(\mu)\rangle\,, \tag{2.28}
$$

the logarithms in $\vec{C}(\mu)$ can be resummed to leading order by solving the RGE (2.21), leading to

$$
R_O \,=\, \vec{C}^{\,T}(Q)\,\hat{V}\left[\left(\frac{a_Q}{a_\mu}\right)^{\vec{\gamma}^{(1)}_D/\beta_1}\right]_D V^{-1}\langle\vec{O}(\mu)\rangle\,. \tag{2.29}
$$

The somewhat condensed notation in (2.29) should be read as follows: $\vec{\gamma}^{(1)}_D$ is a vector containing the eigenvalues of $\hat{\gamma}^{(1)}_O$ ordered according to the eigenvectors in $\hat{V}$. Then $[..]_D$ is the diagonal matrix which contains as diagonal entries the ratios of a_s to the power of elements in $\vec{\gamma}^{(1)}_D/\beta_1$. The generalisation of (2.29) to next-to-leading order is slightly non-trivial because anomalous dimension matrices at different couplings do not commute, but it can for example be found in refs. [8, 9].

Numerically, at $N_c = N_f = 3$ the eigenvalues of the anomalous dimension matrices $\hat{\gamma}^{(1)}_{Q_-}$ and $\hat{\gamma}^{(1)}_{Q_+}$, ordered in increasing value, are found to be

$$
\vec{\gamma}^{(1)}_{D,Q_-} \,=\, (-4,\, 0.5)\,, \tag{2.30}
$$

$$
\vec{\gamma}^{(1)}_{D,Q_+} \,=\, (-3.611,\, -3.387,\, -1.878,\, -1.494,\, 0.538,\, 0.567,\, 1,\, 1.340,\, 1.703)\,. \tag{2.31}
$$

Besides the eigenvalue 1, the entries in $\vec{\gamma}^{(1)}_{D,Q_+}$ are found as the roots of the two quartic polynomials $176 - 316z - 101z^2 + 130z^3 + 36z^4$ and $88 - 122z - 91z^2 + 47z^3 + 18z^4$. The corresponding eigenvectors have been collected in appendix B. Regarding the operator combinations $\hat{V}^{-1}_{Q_+}\vec{Q}_+$, it is found that four of them just include the operators Q_7 to Q_{10}, and a further combination misses the operator Q_6, while the remaining ones contain all contributing operators.

To conclude this section, we investigate the case of flavour SU(2). Then, the closed $(V + A)$ operator basis is given by $\overline{Q}_+ \equiv (Q^o_+, Q^s_+, \overline{Q}_3, \overline{Q}_4, \overline{Q}_6)$, where in the penguin operators the strange quark is removed and hence the operators $\overline{Q}_3$ to $\overline{Q}_6$ are analogous to Q_7 to Q_{10} with the sums just running over up and down quarks. The corresponding anomalous dimension matrix is found to be

$$\hat{\gamma}^{(1)}_{\overline{Q}_+} = \begin{pmatrix} -\frac{3}{N_c} & \frac{3C_F}{2N_c} & -\frac{1}{3N_c} & 0 & 0 \\ 3 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2N_f}{3} - \frac{3N_c}{4} - \frac{1}{3N_c} & \frac{3N_c}{4} - \frac{3}{N_c} & \frac{3C_F}{2N_c} \\ 3 + \frac{3}{N_c} & -\frac{3C_F}{2N_c} & \frac{3N_c}{4} + \frac{3}{2} - \frac{11}{6N_c} & -\frac{3N_c}{4} + \frac{3}{2} + \frac{3}{2Nc} & -\frac{3C_F}{2N_c} \\ 0 & 0 & \frac{11}{3} & 0 & 0 \end{pmatrix}, \tag{2.32}$$

with the eigenvalues

$$\vec{\gamma}^{(1)}_{D,\overline{Q}_+} = (-3.521,\, -1.751,\, 0.549,\, 1,\, 1.445) \tag{2.33}$$

at $N_c = 3$ and $N_f = 2$. It is observed that the eigenvalues are slightly shifted with respect to the SU(3) case (2.31), but span approximately the same range.

3 Singlet vector and axialvector correlators

4 Renormalon structure of dimension-6 four-quark operators

In this section, the perturbative ambiguities that are connected to the dimension-6 four-quark OPE contributions, shall be investigated. The discussion closely follows section 3.3 of ref. [3] and section 5 of ref. [10].

$$Q^o_- = -\frac{4C_F}{N_c}\big(\bar{u}_L u_R \bar{d}_R d_L + (L \leftrightarrow R)\big) + \frac{4}{N_c}\big(\bar{u}_L t^a u_R \bar{d}_R t^a d_L + (L \leftrightarrow R)\big), \tag{4.1}$$

$$Q^s_- = -\frac{4}{N_c}\big(\bar{u}_L u_R \bar{d}_R d_L + (L \leftrightarrow R)\big) - 8\big(\bar{u}_L t^a u_R \bar{d}_R t^a d_L + (L \leftrightarrow R)\big). \tag{4.2}$$

$$Q^o_+ = \frac{2C_F}{N_c}\big(\bar{u}_L \gamma_\mu u_L \bar{d}_L \gamma^\mu d_L + (L \to R)\big) - \frac{2}{N_c}\big(\bar{u}_L \gamma_\mu t^a u_L \bar{d}_L \gamma^\mu t^a d_L + (L \to R)\big), \tag{4.3}$$

$$Q^s_+ = \frac{2}{N_c}\big(\bar{u}_L \gamma_\mu u_L \bar{d}_L \gamma^\mu d_L + (L \to R)\big) + 4\big(\bar{u}_L \gamma_\mu t^a u_L \bar{d}_L \gamma^\mu t^a d_L + (L \to R)\big). \tag{4.4}$$

5 Conclusions

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A Anomalous dimensions of 4-quark operators

In this Appendix, we present a generalisation of the results of reference [7] to an arbitrary number N_c of colour degrees of freedom. In [7], the leading order anomalous dimension matrix of a complete set of local spin-zero four-quark operators without derivatives was calculated in the case of three quark flavours.

The complete basis consists of 45 four-quark operators which in reference [7] were chosen as follows: with respect to the Dirac-structure, there are five types of operators, namely, scalar, pseudoscalar, vector, axialvector and tensor. They can be expressed as

$$\bar{u}\Gamma u\bar{d}\Gamma d = \left(\bar{u}u\bar{d}d,\, \bar{u}\gamma_5 u\bar{d}\gamma_5 d,\, \bar{u}\gamma_\mu u\bar{d}\gamma^\mu d,\, \bar{u}\gamma_\mu\gamma_5 u\bar{d}\gamma^\mu\gamma_5 d,\, \bar{u}\sigma_{\mu\nu}u\bar{d}\sigma^{\mu\nu}d\right) \tag{A.1}$$

in the $\bar{u}u\bar{d}d$ flavour case. Employing this notation, the complete basis O of operaptors can be chosen to be:

$$\begin{aligned} O \;\equiv\; \big(&\bar{u}\Gamma u\bar{u}\Gamma u,\, \bar{d}\Gamma d\bar{d}\Gamma d,\, \bar{s}\Gamma s\bar{s}\Gamma s,\, \bar{u}\Gamma u\bar{d}\Gamma d,\, \bar{u}\Gamma u\bar{s}\Gamma s,\, \bar{d}\Gamma d\bar{s}\Gamma s, \\ &\bar{u}\Gamma t^a u\bar{d}\Gamma t^a d,\, \bar{u}\Gamma t^a u\bar{s}\Gamma t^a s,\, \bar{d}\Gamma t^a d\bar{s}\Gamma t^a s\big)\,. \end{aligned} \tag{A.2}$$

In this basis, the leading order anomalous dimension matrix takes the form

$$\gamma_O^{(1)} = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & B & B & 0 \\ 0 & A & 0 & 0 & 0 & 0 & B & 0 & B \\ 0 & 0 & A & 0 & 0 & 0 & 0 & B & B \\ 0 & 0 & 0 & C & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & C & 0 & 0 & D \\ E & E & 0 & F & 0 & 0 & G & H & H \\ E & 0 & E & 0 & F & 0 & H & G & H \\ 0 & E & E & 0 & 0 & F & H & H & G \end{pmatrix}. \tag{A.3}$$

The submatrices are given by:

$$A = \begin{pmatrix} \frac{11}{12} - 3C_F & \frac{7}{12} & -\frac{1}{12} + \frac{1}{6N_c} & -\frac{1}{12} & -\frac{1}{8} + \frac{1}{4N_c} \\ \frac{7}{12} & \frac{11}{12} - 3C_F & \frac{1}{12} - \frac{1}{6N_c} & \frac{1}{12} & -\frac{1}{8} + \frac{1}{4N_c} \\ \frac{7}{6} & -\frac{7}{6} & \frac{11}{12} - \frac{1}{3N_c} & \frac{11}{12} - \frac{3}{2N_c} & 0 \\ -\frac{11}{6} & \frac{11}{6} & \frac{11}{12} - \frac{11}{6N_c} & \frac{11}{12} & 0 \\ 3 + \frac{6}{N_c} & 3 + \frac{6}{N_c} & 0 & 0 & \frac{3}{2} + C_F \end{pmatrix}, \tag{A.4}$$

$$B = \begin{pmatrix} 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} -3C_F & 0 & 0 & 0 & 0 \\ 0 & -3C_F & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_F \end{pmatrix}, \tag{A.5}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -12 & -12 & 0 & 0 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{12} - \frac{1}{6N_c} & \frac{1}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A.6}$$

$$G = \begin{pmatrix} \frac{3}{2N_c} & 0 & 0 & 0 & -\frac{N_c}{8} + \frac{1}{2N_c} \\ 0 & \frac{3}{2N_c} & 0 & 0 & -\frac{N_c}{8} + \frac{1}{2N_c} \\ 0 & 0 & -\frac{3N_c}{4} + \frac{2}{3} & \frac{3N_c}{4} - \frac{3}{N_c} & 0 \\ 0 & 0 & \frac{3N_c}{4} - \frac{3}{N_c} & -\frac{3N_c}{4} & 0 \\ -3N_c + \frac{12}{N_c} & -3N_c + \frac{12}{N_c} & 0 & 0 & C_F - \frac{3N_c}{2} \end{pmatrix}, \tag{A.7}$$

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.8}$$

In contrast to ref. [7], the matrices A, C and G already include the quark self-energy contributions depicted in figure 1c) of [7], such that they are gauge independent. (The corresponding matrices of [7] were given in the Feynman gauge without self-energy contribution.)

B Eigenvectors of anomalous dimension matrices

$$\hat{V}_{Q_-} = \begin{pmatrix} \frac{4}{3} & -\frac{1}{6} \\ 1 & 1 \end{pmatrix} . \tag{B.1}$$

$$\hat{V}_{Q_+} = \begin{pmatrix} -0.059 & -0.028 & 0.472 & -0.093 & 0.045 & -0.015 & 0.316 & 0.081 & -0.020 \\ 0.145 & 0.066 & -0.664 & 0.100 & 0.082 & -0.027 & 0.949 & 0.338 & -0.117 \\ -0.519 & -0.209 & -0.254 & 0.191 & -0.138 & 0.043 & 0 & 0.313 & -0.208 \\ 0.654 & 0.386 & -0.159 & 0.077 & 0.292 & -0.115 & 0 & 0.219 & -0.105 \\ 0.527 & 0.226 & 0.496 & -0.469 & -0.942 & 0.275 & 0 & 0.856 & -0.447 \\ 0 & -0.314 & 0 & 0.287 & 0 & 0.064 & 0 & 0 & -0.312 \\ 0 & 0.578 & 0 & 0.115 & 0 & -0.173 & 0 & 0 & -0.157 \\ 0 & -0.450 & 0 & -0.359 & 0 & -0.839 & 0 & 0 & -0.399 \\ 0 & 0.340 & 0 & -0.703 & 0 & 0.413 & 0 & 0 & -0.671 \end{pmatrix} . \tag{B.2}$$

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