1 General remarks

Though below our central interest will be in the OPE of the correlation function $\Pi^{(1+0)}_{ud,V-A}$, there is a subtlety, because the spectral functions measured in hadronic τ decays correspond to the combination

$$\rho_{ud,V/A}(s) \;\equiv\; \rho^{(1+0)}_{ud,V/A}(s) - \frac{2s}{(s_\tau + 2s)}\, \rho^{(0)}_{ud,V/A}(s)\,, \tag{1}$$

including a contribution from the longitudinal correlation function, and where we have defined $s_\tau \equiv M_\tau^2$. Generally, this contribution is suppressed by two powers of the quark mass, but since we are considering $V-A$, it should also be estimated.

Next, it is interesting to investigate moments of the $V-A$ spectral function,

$$R^{(w)}_{ud,V-A}(s_0) \;\equiv\; \int\limits_0^{s_0} ds\, w(s)\, [\, \rho_{ud,V}(s) - \rho_{ud,A}(s)\,]\;. \tag{2}$$

The cases $w(s) = 1$ and $w(s) = s$ correspond to the first and second Weinberg sum rule respectively. Assuming that a factor $|V_{ud}|^2 S_{EW}$ has been removed from the experimental data, $\rho_{ud,V/A}(s)$ denote pure QCD quantities.[1] Furthermore, the hadronic τ decay data do not include the contribution of the pion pole in $\rho_{ud,A}(s)$ which has to be added by hand. The pseudoscalar pion-pole spectral function is given by

$$\rho^{(0)}_\pi(s) \;=\; 2 f_\pi^2\, \delta(s - s_\pi)\,, \tag{3}$$

with $s_\pi \equiv M_\pi^2$, leading to

$$R^{(w)}_{ud,V-A}(s_0,\pi) \;=\; 2 f_\pi^2\, w(s_\pi) \left[1 - \frac{2 s_\pi}{(s_\tau + 2 s_\pi)} \right]. \tag{4}$$

The second term in eq. (4) is independent of the weight function and amounts to a 1.2 % correction which might be included in the numerics. (See, however, the remarks at the end of the next section.)

2 D=2 contribution

The $D = 2$ contribution to $\Pi^{(1+0)}_{ud,V-A}$ can be inferred from the result of eq. (4.2) of ref. [1], giving[2]

$$\Pi^{(1+0)}_{ud,V-A}(s) \;=\; \frac{m_u(\mu)\, m_d(\mu)}{\pi^2\, s} \Big\{ a_s(\mu) + \big(-\tfrac{17}{4} L + \tfrac{154}{9} - \tfrac{55}{18}\zeta_3 - \tfrac{5}{18}\zeta_5 \big)\; a_s^2(\mu)$$
$$+ \Big[\tfrac{221}{16} L^2 + \big(-\tfrac{4421}{36} + \tfrac{715}{36}\zeta_3 + \tfrac{65}{36}\zeta_5 \big) L + \tfrac{3}{2}\, e^{(1+0)}_{3,\Pi} \Big]\, a_s^3(\mu) + \mathcal{O}(a_s^4) \Big\}\,, \tag{5}$$

[1] Since S_{EW} has a non-factorisable component, it makes more sense to include it on the theoretical side, but we ignore this additional subtlety.

[2] The computation is done in the file `delD2FOnew.m`.

where $a_s(\mu) \equiv \alpha_s(\mu)/\pi$, $L \equiv \ln(-s/\mu^2)$, and $e_{3,\Pi}^{(1+0)}$ is a currently unknown constant.

Next, the contribution of $\Pi_{ud,V-A}^{(1+0)}(s)$ to the first and second Weinberg sum rules shall be calculated. This amounts to calculating the integral

$$W_{\rm SR}^{(w)}(s_0) \,=\, \frac{i}{2\pi} \oint\limits_{|s|=s_0} ds\, w(s)\, \Pi_{ud,V-A}^{(1+0)}(s)\,, \tag{6}$$

where $w(s) = 1$ (first WSR) or $w(s) = s$ (second WSR). To this end, the following types of integrals have to be computed:

$$\begin{aligned} & \frac{i}{2\pi} \oint\limits_{|s|=s_0} ds\, s^{n-1} \ln^k\Big(\frac{-s}{\mu^2}\Big) \\ = \; & \frac{i}{2\pi}\, (-)^{n-1} s_0^n \oint\limits_{|x|=1} dx\, (-x)^{n-1} \ln^k(-x) \,=\, (-)^{n-1} s_0^n\, \widehat{I}_{k,n}\,, \end{aligned} \tag{7}$$

where the logarithm has been resummed with the choice $\mu^2 = s_0$ and the integrals $\widehat{I}_{k,n} \equiv I_{k,n}/(2\pi)$ with $I_{k,n}$ being given in eq. (3.4) of ref. [2]. In particular, the integrals needed for the first and second Weinberg sum rules are

$$\widehat{I}_{0,0} \,=\, 1\,, \qquad \widehat{I}_{1,0} \,=\, 0\,, \qquad \widehat{I}_{1,1} \,=\, -1\,, \qquad \widehat{I}_{2,1} \,=\, 2\,. \tag{8}$$

Employing these integrals, one obtains:

$$W_{\rm SR}^{(w=1)}(s_0) = \frac{m_u(s_0)\, m_d(s_0)}{\pi^2}\, a_s(s_0) \Big[\, 1 + \big(\tfrac{154}{9} - \tfrac{55}{18}\zeta_3 - \tfrac{5}{18}\zeta_5\big)\, a_s(s_0) + \ldots \,\Big], \tag{9}$$

$$W_{\rm SR}^{(w=s)}(s_0) = \frac{17}{4\pi^2}\, m_u(s_0)\, m_d(s_0)\, a_s(s_0)^2\, s_0 \Big[\, 1 + \big(\tfrac{10831}{306} - \tfrac{715}{153}\zeta_3 - \tfrac{65}{153}\zeta_5\big)\, a_s(s_0) + \ldots \,\Big] \tag{10}$$

Evaluating the 1st Weinberg sum rule at $s_0 = M_\tau^2$, the leading-order term gives $1.4 \cdot 10^{-7}\,{\rm GeV}^2$. This should be compared to $2f_\pi^2 = 1.7 \cdot 10^{-2}\,{\rm GeV}^2$, so that the relative contribution of the LO m_q^2 correction is $8.5 \cdot 10^{-6}$. Numerically, the next-to-leading order correction is $13.2\, a_s(s_0)$ and thus bigger than the LO term for $s_0 \approx M_\tau^2$, pointing to the very bad perturbative behaviour of the m_q^2 series. Therefore, the precise value of the m_q^2 contribution cannot be predicted reliably, but with "a few" times 10^{-5} it can still be considered negligible.

The situation for the 2nd Weinberg sum rule at $s_0 = M_\tau^2$ is very similar. At LO the m_q^2 term amounts to $2.0 \cdot 10^{-7}\,{\rm GeV}^4$, which should be compared to $2f_\pi^2\, M_\pi^2 = 3.3 \cdot 10^{-4}\,{\rm GeV}^4$. Hence, the relative correction is $5.9 \cdot 10^{-4}$. Now the NLO correction is huge which may be due to the fact that in addition to the already bad behaviour of the m_q^2 series we have a moment proportional to s. Numerically, the NLO correction is $29.3\, a_s(s_0)$, about three times the leading order. Again, a reliable prediction cannot be made, but presumably the relative m_q^2 correction should not be larger than "a few" times 10^{-3}.

Let us now move to the contribution of the longitudinal correlation function to the $D = 2$ term by collecting the necessary equations. The central one, which derives from a Ward identity for the V/A correlators reads

$$\Pi^{(0)}_{ud,V/A}(s) \;=\; \frac{1}{s^2}\,\Big[\,\Psi_{ud,V/A}(s) - \Psi_{ud,V/A}(0)\,\Big]\;, \tag{11}$$

where $\Psi_{ud,V/A}(s)$ are the correlators for the divergencies of V and A, and

$$\Psi_{ud,V/A}(s) \;=\; (m_u \mp m_d)^2\,\Pi_{ud,S/P}(s) \tag{12}$$

their relation to the scalar and pseudoscalar correlators $\Pi_{ud,S/P}$. The general expansion of the purely perturbative part of $\Psi_{ud,V/A}(s)$ takes the form

$$\Psi^{\rm PT}_{ud,V/A}(s) \;=\; \frac{N_c}{8\pi^2}\,[m_u(\mu) \mp m_d(\mu)]^2\, s \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} d_{n,k} L^k\,, \tag{13}$$

where the independent coefficients $d_{n,1}$ are know up to order α_s^4 and all other $d_{n,k}$ can be determined from RGE's for the spectral function or the second derivative of $\Psi_{ud,V/A}(s)$. Eqs. (11) and (13) then lead to the longitudinal $V - A$ correlator

$$\Pi^{(0){\rm PT}}_{ud,V-A}(s) \;=\; -\,\frac{N_c}{2\pi^2}\,\frac{m_u(\mu)\,m_d(\mu)}{s} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} d_{n,k} L^k\,. \tag{14}$$

Note that the terms proportional to $d_{0,k}$ depend on the renormalisation and should drop out for a physical quantity. This is obvious for the observables based on $\Psi_{ud,V/A}$, that is the spectral function, or the second derivative.

Next, we inspect the contribution of the longitudinal correlators to the $V - A$ FESR's, which is given by

$$-\,\frac{2s}{(s_\tau+2s)}\,\Pi^{(0){\rm PT}}_{ud,V-A}(s) \;=\; \frac{N_c}{\pi}\,\frac{m_u(\mu)\,m_d(\mu)}{(s_\tau+2s)} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} d_{n,k} L^k\,. \tag{15}$$

Dividing the total τ spectral function by the kinematic weight has created an artificial pole at $s = -s_\tau/2$. In the FESR's, this pole would also lead to a contribution from the unphysical terms proportional to $d_{0,k}$, which cannot be the correct procedure. This problem points to the correct way of dealing with the longitudinal contribution in a τ FESR with general weight function. First, this contribution should be subtracted from the data, for example with a phenomenological parametrisation, as the perturbative behaviour of the scalar/pseudoscalar correlators is rather bad, and only then one should divide out the kinematic weight to arrive at $\Pi^{(1+0)}_{ud,V/A}$. Since the pion pole is not present in the Aleph and Opal data, it is thus justified to only add the "$(1+0)$" part of eq. (4) to a particular τ moment. Furthermore, the contributions from higher scalar and pseudoscalar resonances to $\Pi^{(0)}_{ud,V/A}$ should be tiny and to a very good approximation can be neglected. (This is different in the (us)-channel.)

3 D=6 contribution

3.1 General OPE notation

Consider a physical quantity R with an OPE expansion (summation convention understood)

$$R \,=\, C_i(\mu)\,\langle O_i(\mu)\rangle \,, \tag{16}$$

where the renormalisation scale μ is displayed explicitly and the potential dependence on other dimensionful parameters is implicit. Since R should not depend on μ, it immediately follows that

$$\left[\mu\,\frac{d}{d\mu}\,C_i(\mu)\right]\langle O_i(\mu)\rangle \,=\, -\,C_i(\mu)\left[\mu\,\frac{d}{d\mu}\,\langle O_i(\mu)\rangle\right]. \tag{17}$$

Next, the anomalous dimension matrix $\hat\gamma_O$ of the operator matrix elements can be defined as

$$-\,\mu\,\frac{d}{d\mu}\,\langle \vec O(\mu)\rangle \,\equiv\, \hat\gamma_O(a_\mu)\,\langle \vec O(\mu)\rangle\,. \tag{18}$$

If the bare and renormalised operator matrix elements are related by

$$\langle \vec O^B\rangle \,\equiv\, \hat Z_O(\mu)\,\langle \vec O(\mu)\rangle\,, \tag{19}$$

it follows that the anomalous dimension matrix can be computed from the renormalisation matrix $\hat Z_O(\mu)$ via

$$\hat\gamma_O(a_\mu) \,=\, \hat Z_O^{-1}(\mu)\,\mu\,\frac{d}{d\mu}\,\hat Z_O(\mu)\,. \tag{20}$$

In a transformed basis $\langle \vec Q(\mu)\rangle$ of operator matrix elements, defined by

$$\langle \vec Q(\mu)\rangle \,\equiv\, \hat T\,\langle \vec O(\mu)\rangle\,, \tag{21}$$

the corresponding anomalous dimension matrix takes the form

$$\hat\gamma_Q \,=\, \hat T\,\hat\gamma_O\,\hat T^{-1}\,. \tag{22}$$

If a matrix $\hat T = \hat D$ diagonalises $\hat\gamma_O$, the vector $\langle \vec Q(\mu)\rangle$ contains the RGI combinations of operator matrix elements.

Plugging eq. (18) into the RGE for R, eq. (17), one obtains an RGE that has to be satisfied by the coefficient functions $\vec{C}(\mu)$,

$$\mu \frac{d}{d\mu} \vec{C}(\mu) \,=\, \hat{\gamma}_O^T(a_\mu)\, \vec{C}(\mu)\,. \tag{23}$$

This equation shall be checked for the coefficient functions of the dimension-6 operators.

3.2 $\Pi_{ud,V-A}^{(1+0),D=6}(s)$

Defining the combinations of 4-quark operator matrix elements

$$\langle Q_-^o\rangle \equiv \langle Q_V^o\rangle - \langle Q_A^o\rangle \,=\, \langle \bar{u}\gamma_\mu t^a d\bar{d}\gamma^\mu t^a u\rangle - \langle \bar{u}\gamma_\mu\gamma_5 t^a d\bar{d}\gamma^\mu\gamma_5 t^a u\rangle\,, \tag{24}$$

$$\langle Q_-^s\rangle \equiv \langle Q_V^s\rangle - \langle Q_A^s\rangle \,=\, \langle \bar{u}\gamma_\mu d\bar{d}\gamma^\mu u\rangle - \langle \bar{u}\gamma_\mu\gamma_5 d\bar{d}\gamma^\mu\gamma_5 u\rangle\,, \tag{25}$$

$$\tag{26}$$

at NLO the dimension-6 contribution to the $V-A$ correlation function $\Pi_{ud,V-A}^{(1+0)}$ can be written as [3] (see also ref. [4])

$$Q^6\,\Pi_{ud,V-A}^{(1+0),D=6}(Q^2) = \pi^2\Big[\,8 + \Big(\tfrac{119}{3} - 4L\Big)a_\mu\,\Big]a_\mu\langle Q_-^o(\mu)\rangle + \pi^2\Big(2 + \tfrac{8}{3}L\Big)a_\mu^2\langle Q_-^s(\mu)\rangle \tag{27}$$

$$\equiv \pi^2\Big[c_{10}^{(1)} + \Big(c_{20}^{(1)} + c_{21}^{(1)}L\Big)a_\mu\Big]a_\mu\langle Q_-^o(\mu)\rangle + \pi^2\Big(c_{20}^{(2)} + c_{21}^{(2)}L\Big)a_\mu^2\langle Q_-^s(\mu)\rangle\,,$$

where $L = \ln(Q^2/\mu^2)$. Given the anomalous dimension matrix for $\langle \vec{Q}_-(\mu)\rangle$,

$$\hat{\gamma}_{Q_-}(a_\mu) \,\equiv\, \begin{pmatrix} \gamma_{11}^{(1)} & \gamma_{12}^{(1)} \\ \gamma_{21}^{(1)} & \gamma_{22}^{(1)} \end{pmatrix} a_\mu + \mathcal{O}(a_\mu^2)\,, \tag{28}$$

from the RGE for the coefficient function (23) it follows that

$$\pi^2 \begin{pmatrix} -\,2c_{21}^{(1)} - \beta_1\, c_{10}^{(1)} \\ -\,2c_{21}^{(2)} \end{pmatrix} a_\mu^2 \,=\, \pi^2 \begin{pmatrix} \gamma_{11}^{(1)}\, c_{10}^{(1)} \\ \gamma_{12}^{(1)}\, c_{10}^{(1)} \end{pmatrix} a_\mu^2\,. \tag{29}$$

This constraint allows to fix two elements of the anomalous dimension matrix,

$$\gamma_{11}^{(1)} \,=\, -\,2\,\frac{c_{21}^{(1)}}{c_{10}^{(1)}} - \beta_1 \,=\, -\,\frac{7}{2}\,, \tag{30}$$

$$\gamma_{12}^{(1)} \,=\, -\,2\,\frac{c_{21}^{(2)}}{c_{10}^{(1)}} \,=\, -\,\frac{2}{3}\,, \tag{31}$$

which can be checked against a direct calculation. For dimensional reasons, the OPE terms of $\Pi_{ud,V/A}^{(1+0)}$ themselves are physical quantities that satisfy a homogeneous RGE, not just the

Adler function contributions. The only Q^2-independent, but renormalisation dependent constant appears in the purely perturbative part. (Results for the RGE constraints at NLO can be found in the file `D6rge.m`.)

Below, the anomalous dimensions of the 4-quark operators shall be extracted from the work of ref. [5]. Because of the basis of operators employed in [5], Fierz transformations need to be applied. Strictly speaking they are only valid in four dimensions, but since we are only interested in the leading-order anomalous dimension matrix, this will be sufficient. To prepare the necessary formulae, next the Fierz transformation will be reviewed. Consider the 4-quark operator matrix element

$$\langle \bar{u}^i \Gamma d^j \bar{d}^k \Gamma u^l \rangle \;=\; -\,\Gamma_{\alpha\beta}\Gamma_{\gamma\delta} \langle \bar{u}^i_\alpha u^l_\delta \bar{d}^k_\gamma d^j_\beta \rangle \,, \tag{32}$$

where the matrices Γ are one of the set of Dirac matrices $\{\, \mathbb{1},\, \gamma_5,\, \gamma_\mu,\, \gamma_\mu\gamma_5,\, \sigma_{\mu\nu}\}$. The i, j, k, l are colour indices, and the colour structure will be discussed further below. Regarding the γ-structure, we have to decompose $\Gamma_{\alpha\beta}\Gamma_{\gamma\delta}$ into the set of γ-matrices according to

$$\begin{aligned} \Gamma_{\alpha\beta}\Gamma_{\gamma\delta} \,=\, & C_1\, \mathbb{1}_{\alpha\delta}\mathbb{1}_{\gamma\beta} + C_5\, (\gamma_5)_{\alpha\delta}(\gamma_5)_{\gamma\beta} + C_\mu\, (\gamma_\mu)_{\alpha\delta}(\gamma^\mu)_{\gamma\beta} + \\ & C_{\mu 5}\, (\gamma_\mu\gamma_5)_{\alpha\delta}(\gamma^\mu\gamma_5)_{\gamma\beta} + C_\sigma\, (\sigma_{\mu\nu})_{\alpha\delta}(\sigma^{\mu\nu})_{\gamma\beta} \,. \end{aligned} \tag{33}$$

The corresponding coefficients C_Γ are collected in table 1.

$\Gamma_{\alpha\beta}\Gamma_{\gamma\delta}$	C_1	C_5	C_μ	$C_{\mu 5}$	C_σ
$\mathbb{1}_{\alpha\beta}\mathbb{1}_{\gamma\delta}$	$1/4$	$1/4$	$1/4$	$-\,1/4$	$1/8$
$(\gamma_5)_{\alpha\beta}(\gamma_5)_{\gamma\delta}$	$1/4$	$1/4$	$-\,1/4$	$1/4$	$1/8$
$(\gamma_\mu)_{\alpha\beta}(\gamma^\mu)_{\gamma\delta}$	1	$-\,1$	$-\,1/2$	$-\,1/2$	0
$(\gamma_\mu\gamma_5)_{\alpha\beta}(\gamma^\mu\gamma_5)_{\gamma\delta}$	$-\,1$	1	$-\,1/2$	$-\,1/2$	0
$(\sigma_{\mu\nu})_{\alpha\beta}(\sigma^{\mu\nu})_{\gamma\delta}$	3	3	0	0	$-1/2$

Table 1: Coefficients C_Γ in the expansion of eq. (33).

In addition, a rewriting of the colour structure is required. The left-hand side of eq. (32) either has a "singlet" or "octet" colour structure. Hence, the following two relations are required:

$$\delta_{ij}\,\delta_{kl} \,=\, 2\,(t^a)_{il}(t^a)_{kj} + \frac{1}{N_c}\,\delta_{il}\,\delta_{kj} \tag{34}$$

$$(t^a)_{ij}(t^a)_{kl} \,=\, \frac{C_F}{N_c}\,\delta_{il}\,\delta_{kj} - \frac{1}{N_c}\,(t^a)_{il}(t^a)_{kj} \,. \tag{35}$$

Since quite a few operator matrix elements play a role, let us introduce a short-hand notation for them. First of all, Q shall denote operators with $(\bar{u}d\bar{d}u)$ and O with $(\bar{u}u\bar{d}d)$ flavour content, and for simplicity, the vacuum expectation values will be dropped. Next, the lower index indicates the γ-structure, $\{S,\, P,\, V,\, A,\, T\}$ for $\{\, \mathbb{1},\, \gamma_5,\, \gamma_\mu,\, \gamma_\mu\gamma_5,\, \sigma_{\mu\nu}\}$ respectively, and the upper index signifies the colour structure, "octet" or "singlet". In addition, to make contact to the results of ref. [5], the operators in the basis of eq. (4) of that work will be given. Employing the Fierz transformation, as well as the colour relations, the following expressions for the octet operators Q_V^o and Q_A^o are obtained:

$$\begin{aligned} Q_V^o &= -\tfrac{4}{9}\, O_S^s + \tfrac{4}{9}\, O_P^s + \tfrac{2}{9}\, O_V^s + \tfrac{2}{9}\, O_A^s + \tfrac{1}{3}\, O_S^o - \tfrac{1}{3}\, O_P^o - \tfrac{1}{6}\, O_V^o - \tfrac{1}{6}\, O_A^o \\ &= -\tfrac{4}{9}\, O_{16} + \tfrac{4}{9}\, O_{17} + \tfrac{2}{9}\, O_{18} + \tfrac{2}{9}\, O_{19} + \tfrac{1}{3}\, O_{31} - \tfrac{1}{3}\, O_{32} - \tfrac{1}{6}\, O_{33} - \tfrac{1}{6}\, O_{34}\,, \end{aligned} \tag{36}$$

$$\begin{aligned} Q_A^o &= +\tfrac{4}{9}\, O_S^s - \tfrac{4}{9}\, O_P^s + \tfrac{2}{9}\, O_V^s + \tfrac{2}{9}\, O_A^s - \tfrac{1}{3}\, O_S^o + \tfrac{1}{3}\, O_P^o - \tfrac{1}{6}\, O_V^o - \tfrac{1}{6}\, O_A^o \\ &= +\tfrac{4}{9}\, O_{16} - \tfrac{4}{9}\, O_{17} + \tfrac{2}{9}\, O_{18} + \tfrac{2}{9}\, O_{19} - \tfrac{1}{3}\, O_{31} + \tfrac{1}{3}\, O_{32} - \tfrac{1}{6}\, O_{33} - \tfrac{1}{6}\, O_{34}\,. \end{aligned} \tag{37}$$

This leads to the required relation for Q_-^o:

$$Q_-^o \;=\; -\tfrac{8}{9}\, O_S^s + \tfrac{8}{9}\, O_P^s + \tfrac{2}{3}\, O_S^o - \tfrac{2}{3}\, O_P^o \;=\; -\tfrac{8}{9}\, O_{16} + \tfrac{8}{9}\, O_{17} + \tfrac{2}{3}\, O_{31} - \tfrac{2}{3}\, O_{32}\,. \tag{38}$$

The needed relations for the singlet operators Q_V^s and Q_A^s read:

$$\begin{aligned} Q_V^s &= -\tfrac{1}{3}\, O_S^s + \tfrac{1}{3}\, O_P^s + \tfrac{1}{6}\, O_V^s + \tfrac{1}{6}\, O_A^s - 2\, O_S^o + 2\, O_P^o + O_V^o + O_A^o \\ &= -\tfrac{1}{3}\, O_{16} + \tfrac{1}{3}\, O_{17} + \tfrac{1}{6}\, O_{18} + \tfrac{1}{6}\, O_{19} - 2\, O_{31} + 2\, O_{32} + O_{33} + O_{34}\,, \end{aligned} \tag{39}$$

$$\begin{aligned} Q_A^s &= +\tfrac{1}{3}\, O_S^s - \tfrac{1}{3}\, O_P^s + \tfrac{1}{6}\, O_V^s + \tfrac{1}{6}\, O_A^s + 2\, O_S^o - 2\, O_P^o + O_V^o + O_A^o \\ &= +\tfrac{1}{3}\, O_{16} - \tfrac{1}{3}\, O_{17} + \tfrac{1}{6}\, O_{18} + \tfrac{1}{6}\, O_{19} + 2\, O_{31} - 2\, O_{32} + O_{33} + O_{34}\,. \end{aligned} \tag{40}$$

These results lead to the relation for Q_-^s:

$$Q_-^s \;=\; -\tfrac{2}{3}\, O_S^s + \tfrac{2}{3}\, O_P^s - 4\, O_S^o + 4\, O_P^o \;=\; -\tfrac{2}{3}\, O_{16} + \tfrac{2}{3}\, O_{17} - 4\, O_{31} + 4\, O_{32}\,. \tag{41}$$

From the anomalous dimension matrix presented in eq. (5) of ref. [5], we are now in a position to derive the anomalous dimension matrix for the set of operators Q_-^o and Q_-^s. (The computation is performed in `gam4q.m`.) The result reads

$$\hat{\gamma}_{Q_-}(a_\mu) \;=\; \begin{pmatrix} -\frac{7}{2} & -\frac{2}{3} \\ -3 & 0 \end{pmatrix} a_\mu + \mathcal{O}(a_\mu^2)\,. \tag{42}$$

As expected, the matrix elements $\gamma_{11}^{(1)}$ and $\gamma_{12}^{(1)}$ are in agreement with the findings of eq. (30).

From the eigenvectors of the anomalous dimension matrix (42) it is an easy matter to derive the RGI operator combinations. They are found to be

$$\widehat{Q}_- = \begin{pmatrix} \frac{2}{3}\,Q^o_- + \frac{1}{9}\,Q^s_- \\ -\frac{2}{3}\,Q^o_- + \frac{8}{9}\,Q^s_- \end{pmatrix}, \tag{43}$$

with the corresponding anomalous dimension matrix

$$\hat{\gamma}_{\widehat{Q}_-}(a_\mu) = \begin{pmatrix} -4 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} a_\mu + \mathcal{O}(a_\mu^2)\,. \tag{44}$$

3.3 $\Pi^{(1+0),D=6}_{ud,V+A}(s)$

In the case of the $V+A$ correlator, the $D=6$ OPE corrections receive contributions from a larger set of operators, now also including penguin type diagrams. At next-to-leading order, the operators can be found in refs. [3] or [4]. However, as we shall see below, this set is not yet complete and has to be enlarged. In total, nine operators will be needed. Again, in order to use the results of [5] for the anomalous dimensions, the occuring operators have to be expressed in the corresponding basis. The relations for the first two current-current type operators can be inferred from eqs. (36) and (37) as well as (39) and (40). They are found to be:

$$Q^o_+ \equiv Q^o_V + Q^o_A = \tfrac{4}{9}\,O^s_V + \tfrac{4}{9}\,O^s_A - \tfrac{1}{3}\,O^o_V - \tfrac{1}{3}\,O^o_A = \tfrac{4}{9}\,O_{18} + \tfrac{4}{9}\,O_{19} - \tfrac{1}{3}\,O_{33} - \tfrac{1}{3}\,O_{34}\,, \tag{45}$$

$$Q^s_+ \equiv Q^s_V + Q^s_A = \tfrac{1}{3}\,O^s_V + \tfrac{1}{3}\,O^s_A + 2\,O^o_V + 2\,O^o_A = \tfrac{1}{3}\,O_{18} + \tfrac{1}{3}\,O_{19} + 2\,O_{33} + 2\,O_{34}\,. \tag{46}$$

The next, penguin type, operator matrix element that already appears at leading order reads:

$$\begin{aligned} Q_3^{V+A} &\equiv \langle(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d)\sum_{q=u,d,s}\bar{q}\gamma^\mu t^a q\rangle \\ &= -\tfrac{1}{2}\,O_1 + \tfrac{1}{2}\,O_2 + \tfrac{1}{12}\,O_3 + \tfrac{1}{4}\,O_4 - \tfrac{1}{2}\,O_6 + \tfrac{1}{2}\,O_7 + \tfrac{1}{12}\,O_8 + \tfrac{1}{4}\,O_9 \\ &\quad + 2\,O_{33} + O_{38} + O_{43}\,, \end{aligned} \tag{47}$$

where the rewriting in the basis of [5] is already given. The next three operators appear at the next-to-leading order, and their rewriting will also be given immediately together with the definition. We have

$$\begin{aligned} Q_4^{V+A} &\equiv \langle(\bar{u}\gamma_\mu\gamma_5 t^a u + \bar{d}\gamma_\mu\gamma_5 t^a d)\sum_{q=u,d,s}\bar{q}\gamma^\mu\gamma_5 t^a q\rangle \\ &= \tfrac{1}{2}\,O_1 - \tfrac{1}{2}\,O_2 + \tfrac{1}{4}\,O_3 + \tfrac{1}{12}\,O_4 + \tfrac{1}{2}\,O_6 - \tfrac{1}{2}\,O_7 + \tfrac{1}{4}\,O_8 + \tfrac{1}{12}\,O_9 \\ &\quad + 2\,O_{34} + O_{39} + O_{44}\,, \end{aligned} \tag{48}$$

$$Q_5^{V+A} \equiv \langle(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d)\sum_{q=u,d,s}\bar{q}\gamma^\mu\gamma_5 q\rangle$$

$$= O_4 + O_9 + 2\,O_{19} + O_{24} + O_{29}\,, \tag{49}$$

$$Q_6^{V+A} \equiv \langle\bar{s}\gamma_\mu t^a s\sum_{q=u,d,s}\bar{q}\gamma^\mu t^a q\rangle$$

$$= -\tfrac{1}{2}\,O_{11} + \tfrac{1}{2}\,O_{12} + \tfrac{1}{12}\,O_{13} + \tfrac{1}{4}\,O_{14} + O_{38} + O_{43}\,. \tag{50}$$

The last operator matrix element Q_6^{V+A} has been defined differently than in ref. [4] in order to facilitate the rewriting. The operator of [4] is given by $Q_3^{V+A} + Q_6^{V+A}$.

In order to account for the mixing of the operators Q_4^{V+A} and Q_6^{V+A}, three further operators have to be included in the set. They read:

$$Q_7^{V+A} \equiv \langle(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)\sum_{q=u,d,s}\bar{q}\gamma^\mu q\rangle$$

$$= O_3 + O_8 + 2\,O_{18} + O_{23} + O_{28}\,, \tag{51}$$

$$Q_8^{V+A} \equiv \langle\bar{s}\gamma_\mu s\sum_{q=u,d,s}\bar{q}\gamma^\mu q\rangle = O_{13} + O_{23} + O_{28}\,, \tag{52}$$

$$Q_9^{V+A} \equiv \langle\bar{s}\gamma_\mu\gamma_5 s\sum_{q=u,d,s}\bar{q}\gamma^\mu\gamma_5 q\rangle = O_{14} + O_{24} + O_{29}\,, \tag{53}$$

Plugging the anomalous dimensions found in ref. [5] for all operators, the following anomalous dimension matrix for the set $Q_+ \equiv (Q_+^o,\, Q_+^s,\, Q_{3-9}^{V+A})$ is obtained:

$$\hat{\gamma}_{Q_+}(a_\mu) = \begin{pmatrix} -1 & \frac{2}{3} & -\frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{25}{36} & \frac{5}{4} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{41}{36} & -\frac{9}{4} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{2} & -\frac{5}{6} & \frac{19}{12} & \frac{5}{4} & -\frac{5}{12} & -\frac{41}{18} & -\frac{5}{12} & \frac{5}{12} & \frac{13}{12} \\ 0 & 0 & \frac{2}{3} & 3 & 0 & 0 & 0 & 0 & 0 \\ 6 & -2 & 3 & 3 & -1 & -\frac{7}{3} & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{11}{3} & 0 & 0 & 0 \end{pmatrix} a_\mu + \mathcal{O}(a_\mu^2)\,. \tag{54}$$

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