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ANOMALOUS DIMENSIONS OF SPIN-ZERO FOUR-QUARK OPERATORS WITHOUT DERIVATIVES

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The anomalous dimensions of local spin-zero four-quark operators without derivatives are calculated for the case of three flavours. We also give the result in the approximation that no flavour mixing occurs, because this may be relevant for lattice calculations of four-quark condensates in the quenched approximation. We demonstrate the influence of the operator mixing in a specific example.

1. Introduction

The QCD sum rule approach [1] has been a powerful tool for the investigation of hadronic properties of QCD in recent years. Unfortunately, there are a number of limitations to this approach, the most stringent one being the inaccurate knowledge of the condensate parameters. In particular, for the evaluation of the four-quark condensates it was suggested in [1] to use the factorization hypothesis which was considered to be accurate up to approximately 10%. Recent investigations [2–4] have cast serious doubts on this claim.

One of the prerequisites for a more detailed analysis of these condensates – either on the lattice or in the continuum theory – is the knowledge of the anomalous dimension matrices of the four-quark operators. While special cases have already been discussed in the literature [1,5] to our knowledge there is no complete treatment of this problem available so far.

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2. Anomalous dimensions for three-quark flavours

In the following we will use the notation of Bjorken and Drell [6] for the γ-matrices and Γ will be from the set $\{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$. By means of Fierz transformations all four-quark operators, which contain one flavour only, can be expressed in terms of the five operators $(\bar{q}\Gamma q\bar{q}\Gamma q)$. Operators containing two different quark flavours, can be expressed in terms of $(\bar{q}\Gamma q\bar{Q}\Gamma Q)$ and $(\bar{q}\Gamma\frac{1}{2}\lambda^a q\bar{Q}\Gamma\frac{1}{2}\lambda^a Q)$ where λ^a are the Gell-Mann color matrices normalized by the condition $\mathrm{Tr}\,\lambda^a\lambda^b = 2\delta^{ab}$.

In order to obtain the anomalous dimensions one has to calculate the logarithmically divergent parts of the diagrams shown in fig. 1.

The evaluation of the conventional diagrams (fig. 1a, c) in the Feynman gauge is straightforward. For the calculation of the annihilation diagrams (fig. 1b) it is convenient to use the background field formalism. The possibility of choosing different gauges for the quantum field and background field is discussed in detail in [7]. Using the coordinate gauge condition $x_\mu A^\mu(x) = 0$ for the background field and the definition $\mathscr{D}_\mu = \partial_\mu - igA_\mu$ for the covariant derivative acting on the quark fields one obtains for the quark propagator

$$S(q) = \frac{\not{q}}{q^2} + \tfrac{1}{2}igF_{\alpha\beta}\frac{\not{q}\gamma^\alpha\not{q}\gamma^\beta\not{q}}{q^6} - \tfrac{1}{3}g\left(\mathscr{D}_\beta F_{\gamma\alpha} + \mathscr{D}_\gamma F_{\beta\alpha}\right)\frac{\not{q}\gamma^\alpha\not{q}\gamma^\beta\not{q}\gamma^\gamma\not{q}}{q^8} \dots \tag{1}$$

The second term of the expansion does not contribute to the anomalous dimensions of four-quark operators. After calculating the traces over γ-matrices, the last term is reduced to the form $\mathscr{D}_\alpha F^{\alpha\beta}$. Applying the equations of motion

$$\mathscr{D}_\alpha F^{\alpha\beta} = -\tfrac{1}{2}g\lambda^a \sum_{u,d,s} \left(\bar{q}\gamma^\beta\tfrac{1}{2}\lambda^a q\right); \tag{2}$$

this yields a contribution to the four-quark operator. One can also obtain the same result from a calculation in the Feynman gauge.

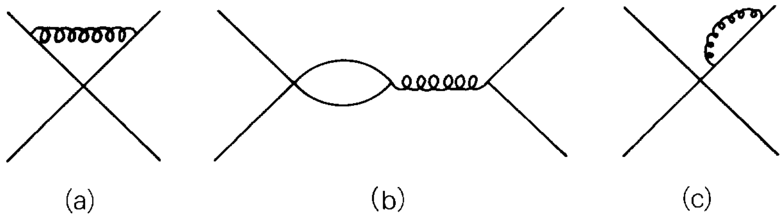

Fig. 1. Feynman diagrams contributing to the anomalous dimension matrix of the four-quark operators.

The anomalous dimension matrix γ_Γ of a set of vertex functions Γ is defined as

$$\gamma_\Gamma = M \frac{\partial \log Z_\Gamma(g_0, \Lambda/M)}{\partial M}, \tag{3}$$

where g_0 is the bare coupling, Λ the ultraviolet cutoff, M the renormalization point defined by

$$\Gamma_0(p_i, g_0, \Lambda) = Z_\Gamma\left(g_0, \frac{\Lambda}{M}\right) \Gamma_{\text{ren}}(q_i, g, M)$$

and Z_Γ is the renormalization constant (mixing matrix).

To simplify the following presentation we will add the contribution of fig. 1c only at the end of the calculation. Using the notation

$$\bar{u}\Gamma u \bar{d}\Gamma d = \left(\bar{u}u\bar{d}d,\, \bar{u}\gamma^5 u\bar{d}\gamma^5 d,\, \bar{u}\gamma^\mu u\bar{d}\gamma_\mu d,\, \bar{u}\gamma^5\gamma^\mu u\bar{d}\gamma^5\gamma_\mu d,\, \bar{u}\sigma^{\mu\nu}u\bar{d}\sigma_{\mu\nu}d\right),$$

we choose the basis

$$\left(\bar{u}\Gamma u\bar{u}\Gamma u,\, \bar{d}\Gamma d\bar{d}\Gamma d,\, \bar{s}\Gamma s\bar{s}\Gamma s,\, \bar{u}\Gamma u\bar{d}\Gamma d,\, \bar{u}\Gamma u\bar{s}\Gamma s,\, \bar{d}\Gamma d\bar{s}\Gamma s,\right.$$
$$\left.\bar{u}\Gamma\tfrac{1}{2}\lambda^a u\bar{d}\Gamma\tfrac{1}{2}\lambda^a d,\, \bar{u}\Gamma\tfrac{1}{2}\lambda^a u\bar{s}\Gamma\tfrac{1}{2}\lambda^a s,\, \bar{d}\Gamma\tfrac{1}{2}\lambda^a d\bar{s}\Gamma\tfrac{1}{2}\lambda^a s\right). \tag{4}$$

In this basis we obtain the anomalous dimension matrix:

$$\gamma_\Gamma = \frac{\alpha_s}{\pi}\begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & B & B & 0 \\ 0 & A & 0 & 0 & 0 & 0 & B & 0 & B \\ 0 & 0 & A & 0 & 0 & 0 & 0 & B & B \\ 0 & 0 & 0 & C & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & C & 0 & 0 & D \\ E & E & 0 & F & 0 & 0 & G & H & H \\ E & 0 & E & 0 & F & 0 & H & G & H \\ 0 & E & E & 0 & 0 & F & H & H & G \end{pmatrix}. \tag{5}$$

The submatrices are:

$$A = \begin{pmatrix} -\frac{53}{12} & \frac{7}{12} & -\frac{1}{36} & -\frac{1}{12} & -\frac{1}{24} \\ \frac{7}{12} & -\frac{53}{12} & \frac{1}{36} & \frac{1}{12} & -\frac{1}{24} \\ \frac{7}{6} & -\frac{7}{6} & -\frac{19}{36} & \frac{5}{12} & 0 \\ -\frac{11}{6} & \frac{11}{6} & \frac{11}{36} & -\frac{5}{12} & 0 \\ 5 & 5 & 0 & 0 & \frac{3}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} -\frac{16}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{16}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D=\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ -12 & -12 & 0 & 0 & 0 \end{pmatrix},$$

$$E=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{36} & \frac{1}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F=\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 \\ -\frac{8}{3} & -\frac{8}{3} & 0 & 0 & 0 \end{pmatrix},$$

$$G=\begin{pmatrix} -\frac{5}{6} & 0 & 0 & 0 & -\frac{5}{24} \\ 0 & -\frac{5}{6} & 0 & 0 & -\frac{5}{24} \\ 0 & 0 & -\frac{35}{12} & \frac{5}{4} & 0 \\ 0 & 0 & \frac{5}{4} & -\frac{43}{12} & 0 \\ -5 & -5 & 0 & 0 & -\frac{9}{2} \end{pmatrix}, \qquad H=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5a}$$

There exist a number of relations between these matrices which are essentially a consequence of the fact that the operators $(\bar{q}\Gamma\frac{1}{2}\lambda^a q\bar{q}\Gamma\frac{1}{2}\lambda^a q)$ can be expressed in terms of the operators $(\bar{q}\Gamma q\bar{q}\Gamma q)$ by Fierz transformations of the γ and λ-matrices. In particular, we have

$$\begin{aligned} A &= C+(B+D)\cdot T, \\ E &= H\cdot T, \\ F &= \tfrac{2}{9}\cdot D, \\ G &= T\cdot C\cdot T^{-1}+T\cdot D-F\cdot T^{-1}+2\cdot H, \end{aligned} \tag{6}$$

where T is the Fierz transformation matrix

$$T=\begin{pmatrix} -\frac{7}{24} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{16} \\ -\frac{1}{8} & -\frac{7}{24} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{12} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} & \frac{1}{12} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 0 & 0 & \frac{1}{12} \end{pmatrix}, \tag{7}$$

which is defined by the relation

$$\left(\bar{q}\Gamma\tfrac{1}{2}\lambda^a q\bar{q}\Gamma\tfrac{1}{2}\lambda^a q\right) = T\cdot\left(\bar{q}\Gamma q\bar{q}\Gamma q\right). \tag{8}$$

After diagonalization and taking into account the contribution of the diagrams of fig. 1c we obtain the anomalous dimensions from the matrix (5). The coefficients of the multiplicatively renormalizable four-quark operators in the basis (4) are obtained as the eigenvectors of the transposed anomalous dimension matrix. The resulting eigenvalues and their multiplicities are given in the appendix.

3. Influence of annihilation diagrams on the anomalous dimensions

The annihilation diagrams give rise to a mixing of operators of different flavour structure. As lattice calculations at present usually neglect the influence of internal quark loops, it is instructive to investigate how much the anomalous dimensions will change if the annihilation diagrams are neglected.

In this case no flavour mixing occurs and therefore we can confine ourselves to the submatrix for two flavours generated by the basis

$$\left(\bar{u}\Gamma u\bar{d}\Gamma d,\ \bar{u}\Gamma\tfrac{1}{2}\lambda^a u\bar{d}\Gamma\tfrac{1}{2}\lambda^a d\right). \tag{9}$$

The operators containing only fields of one flavour ($\bar{u}\Gamma u\bar{u}\Gamma u$) are contained in this calculation as a special case, as will be discussed at the end of this section. The anomalous dimension matrix occurs in the form

$$\gamma_\Gamma = \frac{\alpha_s}{\pi}\begin{pmatrix} C & D \\ F & G' \end{pmatrix}. \tag{10}$$

Here C, D, F are given in (5a) and G' is

$$G' = G - 2\cdot H = \begin{pmatrix} -\frac{5}{6} & 0 & 0 & 0 & -\frac{5}{24} \\ 0 & -\frac{5}{6} & 0 & 0 & -\frac{5}{24} \\ 0 & 0 & -\frac{43}{12} & \frac{5}{4} & 0 \\ 0 & 0 & \frac{5}{4} & -\frac{43}{12} & 0 \\ -5 & -5 & 0 & 0 & -\frac{9}{2} \end{pmatrix}.$$

After diagonalization and addition of the self-energy contribution one obtains eight different anomalous dimensions (γ_1 to γ_8), two of which (γ_2 and γ_6) are doubly degenerate. The anomalous dimensions and corresponding multiplicatively renormalizable operators in the basis (9) are:

$\gamma_1=-5.421\frac{\alpha_s}{\pi}$	$\gamma_2=-4\frac{\alpha_s}{\pi}$		$\gamma_3=-2.421\frac{\alpha_s}{\pi}$	$\gamma_4=-2\frac{\alpha_s}{\pi}$
0.603	0.000	0.707	0.410	0.000
0.603	0.000	−0.707	0.410	0.000
0.000	0.116	0.000	0.000	0.224
0.000	−0.116	0.000	0.000	0.224
0.010	0.000	0.000	−0.033	0.000
0.291	0.000	0.000	−0.549	0.000
0.291	0.000	0.000	−0.549	0.000
0.000	0.697	0.000	0.000	−0.671
0.000	−0.697	0.000	0.000	−0.671
0.321	0.000	0.000	−0.243	0.000

$\gamma_5=-0.246\frac{\alpha_s}{\pi}$	$\gamma_6=\frac{1}{2}\frac{\alpha_s}{\pi}$		$\gamma_7=\frac{\alpha_s}{\pi}$	$\gamma_8=2.754\frac{\alpha_s}{\pi}$
0.093	0.000	0.000	0.000	0.021
0.093	0.000	0.000	0.000	0.021
0.000	0.566	0.000	0.392	0.000
0.000	−0.566	0.000	0.392	0.000
0.097	0.000	0.000	0.000	−0.110
0.692	0.000	0.707	0.000	0.702
0.692	0.000	−0.707	0.000	0.702
0.000	−0.424	0.000	0.588	0.000
0.000	0.424	0.000	0.588	0.000
−0.130	0.000	0.000	0.000	−0.053

If both flavours are equal, the operators used as a basis in (9) are no longer linearly independent. The five linear equations (8) imply that the operators corresponding to γ_1, γ_4 and γ_5 are zero and that the pairs of operators corresponding to γ_2 and γ_6 coincide.

4. Factorization at low μ^2

According to SVZ [1] it is expected that the four-quark condensates can be estimated at small μ^2 by the assumption of the dominance of the vacuum intermediate state. In this section we will discuss the Q^2 evolution for the four-quark condensates assuming that factorization occurs at a low normalization point μ. For definiteness we choose a factorization point $\mu = 200$ MeV as suggested in [1],

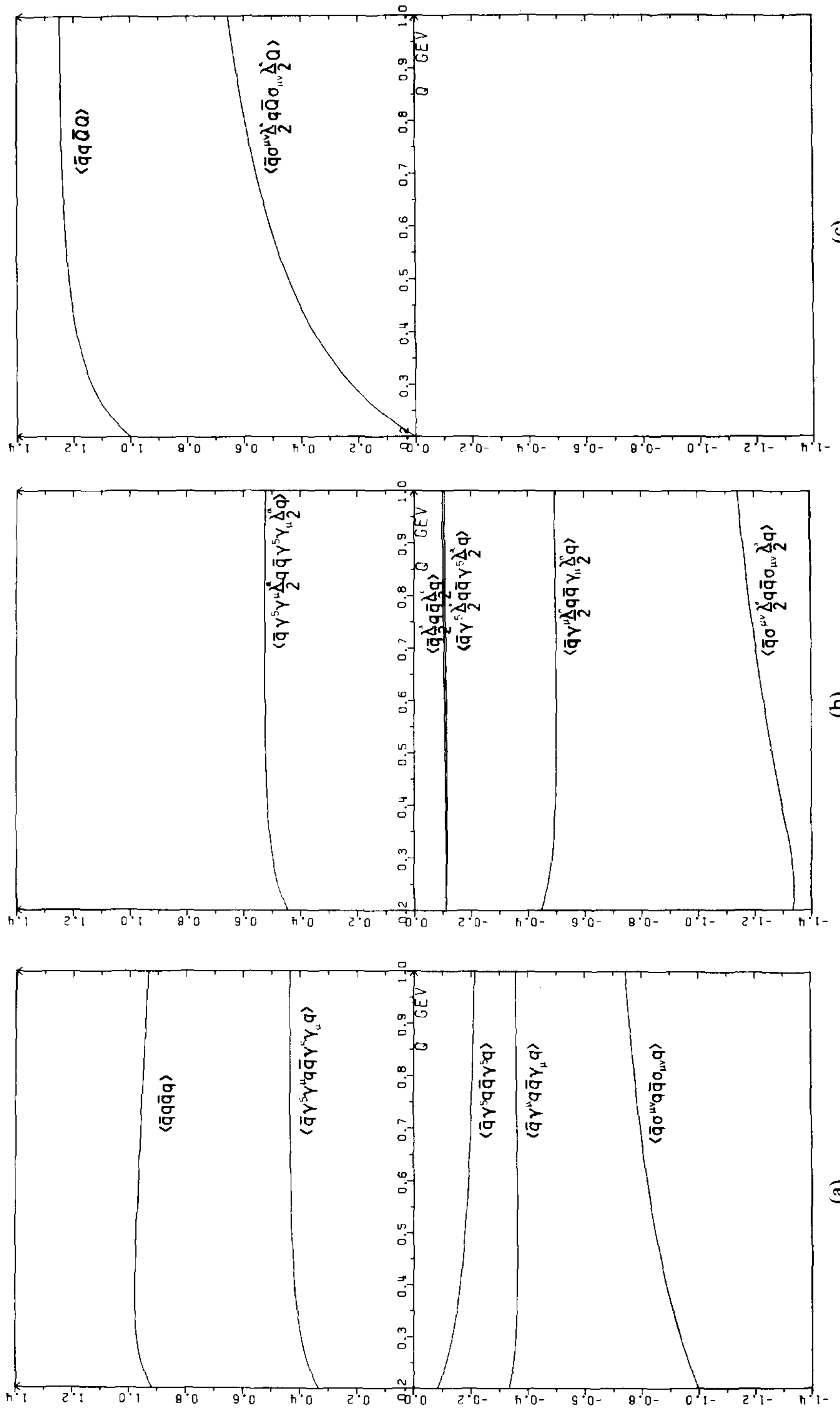

Fig. 2. Q-dependence of the four-quark condensates in units of $\langle qq \rangle^2|_Q$ assuming factorization at $\mu = 200$ MeV. q and Q denote different quark flavours. The case of mixed flavours is shown in (c). All other condensates containing quarks of different flavours yield zero at the factorization point and are below 0.1 at $Q = 1$ GeV.

$\Lambda_{\rm QCD} = 150$ MeV and assume that $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$ for this calculation. The use of the lowest order anomalous dimensions at such a small value of μ^2 may, of course, be questioned. On the other hand it is known from lattice calculations [8] that the transition from strong coupling behaviour (where factorization is expected to be a good approximation) to weak coupling behaviour occurs very rapidly, so that – apart from the uncertainty in the appropriate choice of μ – the lowest order renormalization group analysis may still be a reasonable approximation. This assumption is also inherent in the work of SVZ [1].

Using the results of sect. 2 the calculation of the Q^2 dependence proceeds essentially as discussed in [1]. In fig. 2 we display the ratios

$$\frac{\langle O_i \rangle (Q)}{\langle \bar{q}q \rangle^2 (Q)},$$

which should be almost constant, if the factorization hypothesis is well satisfied. Fig. 2 shows that under the above assumptions deviations from the factorization hypothesis by 20 to 30% must generally be expected and that in specific cases ($\langle \bar{q}\gamma^5 q \bar{q}\gamma^5 q \rangle, \langle \bar{q}\sigma^{\mu\nu}\frac{1}{2}\lambda^a q \bar{Q}\sigma^{\mu\nu}\frac{1}{2}\lambda^a Q \rangle$) much more dramatic deviations occur.

5. Summary

We have calculated the eigenvalues of the anomalous dimension matrix and the corresponding multiplicatively renormalizable four-quark operators which, according to their quantum numbers, can have nonvanishing expectation values in the QCD vacuum.

Our calculation shows that most of the operators are left unchanged by the annihilation diagrams. Only the 12 operators corresponding to the anomalous dimensions $\gamma_3, \gamma_4, \gamma_6, \gamma_7, \gamma_{10}, \gamma_{11}, \gamma_{13}, \gamma_{14}$ are affected by the mixing due to the annihilation diagrams. The most dramatic change occurs in the case of γ_{14} which is $1.703\alpha_s/\pi$ as compared to $1 \cdot \alpha_s/\pi$ without flavour mixing. In most other cases the changes of the anomalous dimensions are of the order of 10–25%. Therefore, it is not useful to perform a two-loop evaluation of the anomalous dimensions for the analysis of lattice calculations as long as the quenched approximation is used on the lattice.

In a specific example we have shown the importance of the operator mixing, which is most dramatic in the case of the condensate $\langle \bar{q}\sigma_{\mu\nu}\frac{1}{2}\lambda^a q \bar{Q}\sigma^{\mu\nu}\frac{1}{2}\lambda^a q \rangle$ which vanishes at the factorization point, but cannot be neglected at larger Q^2.

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Note added in proof

After completion of this work we learnt of a related work by Barfoot and Broadhurst [9], who calculated the anomalous dimensions of the four-quark flavour singlet operators for an arbitrary number of flavours. The results of [9] for $N_F = 3$ are contained in our results and the numbers agree.

Appendix

In this appendix we give a list of the eigenvalues of the anomalous dimension matrix and their multiplicities.

eigenvalue	multiplicity
$\gamma_1 = -5.421 \frac{\alpha_s}{\pi}$	3
$\gamma_2 = -4 \frac{\alpha_s}{\pi}$	6
$\gamma_3 = -3.611 \frac{\alpha_s}{\pi}$	2
$\gamma_4 = -3.387 \frac{\alpha_s}{\pi}$	1
$\gamma_5 = -2.421 \frac{\alpha_s}{\pi}$	6
$\gamma_6 = -1.878 \frac{\alpha_s}{\pi}$	2
$\gamma_7 = -1.494 \frac{\alpha_s}{\pi}$	1
$\gamma_8 = -0.246 \frac{\alpha_s}{\pi}$	3
$\gamma_9 = 0.5 \frac{\alpha_s}{\pi}$	6
$\gamma_{10} = 0.538 \frac{\alpha_s}{\pi}$	2
$\gamma_{11} = 0.567 \frac{\alpha_s}{\pi}$	1
$\gamma_{12} = 1 \cdot \frac{\alpha_s}{\pi}$	3
$\gamma_{13} = 1.340 \frac{\alpha_s}{\pi}$	2
$\gamma_{14} = 1.704 \frac{\alpha_s}{\pi}$	1
$\gamma_{15} = 2.754 \frac{\alpha_s}{\pi}$	6

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