

Chapter 1

Introduction

In particle physics we are concerned about small objects and their interactions. Their dynamics are currently best described by the Standard Model (SM).

The SM contains two groups of fermionic, Spin $1/2$ particles. The former group, the Leptons consist of: the electron (e), the muon (μ), the tau (τ) and their corresponding neutrinos ν_e , ν_μ and ν_τ . The latter group, the Quarks contain: u , d (up and down, the so called light quarks), s (strange), c (charm), b (beauty or beauty) and t (top or truth). The SM furthermore differentiates between three fundamental forces (and its carriers): the electromagnetic (γ photon), weak (Z- or W-Boson) and strong (g gluon) interactions. The before mentioned Leptons solely interact through the electromagnetic and the weak force (also referred to as electroweak interaction), whereas the quarks additionally interact through the strong force.

The strong force is also referred to as Quantumchromodynamics (QCD). As the name suggests¹ the force is characterized by the color charge. Every quark has next to its type one of the three colors blue, red or green. The color force is mediated through eight gluons, which each being bi-colored², interact with quarks and each other. The strength of the strong force is given by the coupling constant α_s . The coupling constants are a function of energy E and $\alpha_s(E)$ increases with energy³. This is exclusive for QCD and leads to *asymptotic freedom* and *confinement*. The former phenomenon describes the decreasing strong force between quarks and gluons, which become asymptotically free at large energies. The latter expresses the fact, that no isolated quark has been found until today. Quarks appear confined as *Hadrons*, the so called *Mesons*⁴ and *Baryons*⁵. As we measure *Hadrons* in our experiments but calculate with quarks within our theoretical QCD model we have to assume *Quark-Hadron Duality*, which states that QCD is still valid for Hadrons for energies suffi-

¹Chromo is the greek word for color.

²Each gluon carries a color and an anti-color.

³In contrast to the electromagnetic force, where $\alpha(E)$ decreases!

⁴Composite of a quark and an anti-quark.

⁵Composite of three quarks or three anti-quarks.

cently heigh energies. There exist *Duality Violations* (DV), which will be investigated within this work.

1.1 τ -Decays

The τ -particle is an elementary particle with negative electric charge and a spin of $1/2$. Together with the lighter electron and muon it forms the *charged Leptons*⁶. Even though it is an elementary particle it decays via the *weak interaction* with a lifetime of $\tau_\tau = 2.9 \times 10^{-13}$ s and a mass of 1776.86(12) MeV[8]. It is the only lepton massive enough to decay into Hadrons. The final states of a decay are limited by *conservation laws*. In case of a τ -decay they must conserve the electric charge (-1) and *invariant mass* of the system must be conserved. Thus, as we can see from the corresponding Feynman diagram 1.1⁷ the τ decays by the emission of a W boson and a tau-neutrino ν_τ into different pairs of $(e^-, \bar{\nu}_e)$, $(\mu^-, \bar{\nu}_\mu)$ or (q, \bar{q}) . We are foremost interested into the hadronic decay channels, meaning τ -decays that have quarks in their final states. Unfortunately the quarks have never been measured isolated, but appear always in combination of *mesons* and *baryons*. Due to its mass of $\tau \approx 1.8$ GeV the τ -particle decays into light mesons (pions- π , kaons-K, and eta- η , see 1.1).

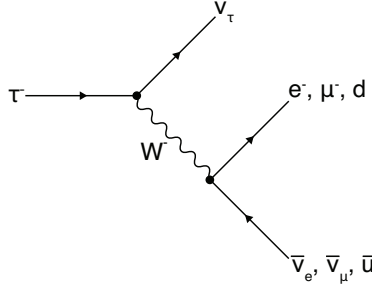


Figure 1.1: Feynman diagram of common decay of a τ -lepton into pairs of lepton-antineutrino or quark-antiquark by the emission of a W boson.

1.2 Quantumchromodynamics

QCD describes the strong interaction, which occur between *quarks* and are transmitted through *gluons*. A list of quarks can be found in 1.2.

⁶Leptons do not interact via the strong force.

⁷The τ -particle can also decay into strange quarks or charm quarks, but these decays are rather uncommon due to the heavy masses of s and c.

Name	Symbol	Quark content	Rest mass ()
Pion	π^-	$\bar{u}d$	139.570 61(24) MeV
Pion	π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	134.9770(5) MeV
Kaon	K^-	$\bar{u}s$	493.677(16) MeV
Kaon	K^0	$d\bar{s}$	497.611(13) MeV
Eta	η	$(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$	547.862(17) MeV

Table 1.1: List of mesons produced by a τ -decay. Rare final states with branching Ratios smaller than 0.1 have been omitted. The list is taken from [2] with corresponding rest masses taken from [8].

Flavour	Mass	comment
u	$2.2^{+0.5}_{-0.4}$ MeV	\overline{MS}
d	$4.7^{+0.5}_{-0.3}$ MeV	
s	95^{+9}_{-3} MeV	
c	$1.275^{+0.025}_{-0.035}$ GeV	
b	$4.18^{+0.04}_{-0.03}$ GeV	
t	173.0(40) GeV	

Table 1.2: List of Quarks and their masses[8].

The QCD Lagrange density is similar to that of QED[3],

$$\mathcal{L}_{\text{QCD}}(x) = -\frac{1}{4}G_{\mu\nu}^a(x)G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2}\bar{q}^A(x)\gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A \bar{q}^A(x)q^A(x) \right], \quad (1.1)$$

where $q^A(x)$ represents the quark fields and $G_{\mu\nu}^a$ being the *gluon field strength tensor* given by:

$$G_{\mu\nu}^a(x) \equiv \partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x) + gf^{abc}B_\mu^b(x)B_\nu^c(x), \quad (1.2)$$

where B_μ^a are the *gluon fields*, given in the *adjoint representation* of the $SU(3)$ gauge group with f^{abc} as *structure constants*. Furthermore we have used A, B, \dots as flavour indices, a, b, \dots as color indices and μ, ν, \dots as lorentz indices.

1.2.1 Renormalisation Group

The perturbations of the QCD Lagrangian 1.1 lead to divergencies, which have to be *renormalized*. There are different approaches to 'make' these divergencies finite. The most popular one is *dimensional regularisation*. In *Dimensional regularisation* we expand the four space-time dimensions to arbitrary dimensions. Consequently the in QCD calculations appearing *Feynman integrals* have to be

continued to D-dimensions like

$$\mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 - m^2 + i0][(q - p)^2 - m^2 + i0]}, \quad (1.3)$$

where we introduced the scale parameter μ to account for the extra dimensions and conserve the mass dimension of the non continued integral.

In addition *physical quantities*⁸ cannot depend on the renormalisation scale μ . Thus examining a *physical quantity* $R(q, a_s, m)$ that depends on the external momentum q , the renormalised coupling $a_s = \alpha_s/\pi$ and the renormalized quark mass m

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{da_s}{d\mu} \frac{\partial}{\partial a_s} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} \right] R(q, a_s, m) = 0 \quad (1.4)$$

we can define the *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu \frac{da_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \quad \beta - \text{function} \quad (1.5)$$

$$\gamma(a_s) \equiv -\frac{\mu}{m} \frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \dots \quad \text{anomalous mass dimension.} \quad (1.6)$$

Running gauge coupling

Regarding the β -function we notice, that $a_s(\mu)$ is not a constant, but *runs* by varying the scale μ . Integrating the β -function yields

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta(a_s)} = - \int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \log \frac{\mu_1}{\mu_2}. \quad (1.7)$$

To analytically evaluate the above integral we can approximate the β -function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \quad (1.8)$$

yielding

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log \frac{\mu_1}{\mu_2}\right)}. \quad (1.9)$$

As we have three colours $N_c = 3$ and six flavours $N_f = 6$ the first β -function 1.5 is positive. Thus for $\mu_2 > \mu_1$ $a_s(\mu_2)$ decreases logarithmically and vanishes for $\mu_2 \rightarrow \infty$. This behaviour is known as *asymptotic freedom*. The coefficients of the β -function are currently known up to the 5th order, which are displayed in the appendix 3.1.

⁸Observables that can be measured.

Running quark mass

The properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* 1.6

$$\log \frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \quad (1.10)$$

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically [7]

$$m(\mu_2) = m(\mu_1) \left(\frac{a(\mu_2)}{a(\mu_1)} \right)^{\frac{\gamma_1}{\beta_1}} (1 + \mathcal{O}(\beta_2, \gamma_2)). \quad (1.11)$$

As β_1 and γ_1 (see 3.2) are positive the quark mass decreases with increasing μ . The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1) \exp \left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \right) \quad (1.12)$$

and can be solved numerically to run the quark mass to the needed scale μ_2 .

1.2.2 Operator Product Expansion

1.2.3 Sum Rules

We need to relate the measurable hadronic final states of a QCD process (e.g. τ -decays into Hadrons) to a theoretical calculable value. To do this we have to combine a *dispersion relation* and the *optical theorem*. We will start with the *Two-point function* (Correlator):

$$\Pi(q^2) = \langle 0 | J_\mu(x) J_\nu(y) | 0 \rangle, \quad (1.13)$$

where $J_\mu(x)$ is the conserved noether current

$$J_\mu(x) = \Psi^\dagger(x) \gamma_\mu \Psi(x). \quad (1.14)$$

We can rewrite the Correlator into the so called *Källén-Lehmann spectral representation* [4][5]

$$\Pi(q^2) = \int_0^\infty s \frac{\rho(s)}{s - q^2 - i\epsilon}, \quad (1.15)$$

which is a *dispersion relation*. The *spectral function* $\rho(s)$ is given by the sum over all Hadronic states squared

$$2\pi\rho(m^2) = \sum_n \langle 0 | J_\mu(x) | n \rangle \langle n | J_\nu(y) \rangle (2\pi^2)^4 \delta^{(4)}(p - p_n) \quad (1.16)$$

and has poles on the positive real axis, see Fig. 1.2.3.

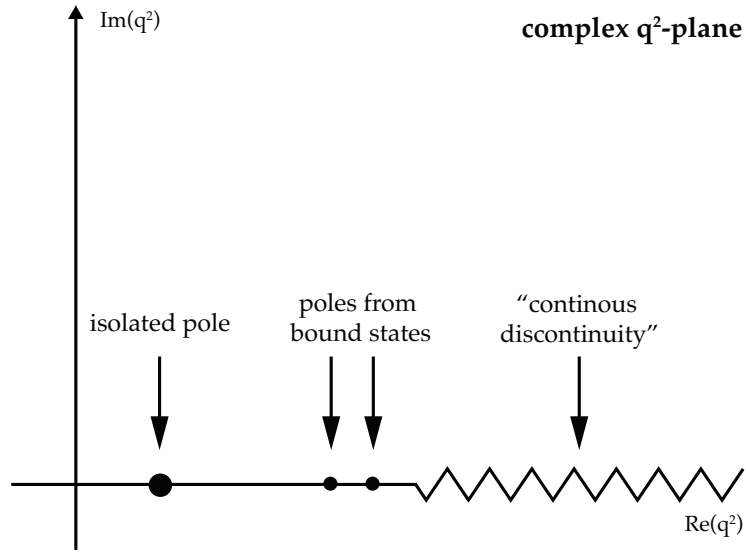


Figure 1.2: Analytic structure of the [6]

$$f(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} dz \frac{f(z)}{z-t} \quad (1.17)$$

$$\frac{1}{t-s-i\epsilon} = \mathcal{P} \frac{1}{t-s} + i\pi\delta(t-s) \quad (1.18)$$

$$\rho(s) = \frac{1}{\pi} \mathcal{I}\Pi(s) \quad (1.19)$$

Chapter 2

Derivation of the used inverse covariance matrix from the Aleph data

While performing a **Generalized least squares** (GLS) we estimate our regression coefficients $\hat{\beta}$ as follows:

$$\hat{\beta} = \underset{\mathbf{b}}{\operatorname{argmin}} (\mathbf{y} - \mathbf{X}\mathbf{b})^T \mathbf{\Omega}^{-1} (\mathbf{y} - \mathbf{X}\mathbf{b}), \quad (2.1)$$

with \mathbf{b} being an candidate estimate of β , \mathbf{X} being the design matrix, \mathbf{y} being the response values and $\mathbf{\Omega}^{-1}$ being the **inverse covariance matrix**.

The Aleph data includes the **standard error** (SE), which are equal to the **standard deviation** as per definition. Furthermore Aleph provides the **correlation coefficients** of the errors. We will use these two quantities in combination with **Gaussian error propagation** to derive an approximation of the covariance matrix.

2.1 Propagation of experimental errors and correlation

Let $\{f_k(x_1, x_2, \dots, x_n)\}$ be a set of m functions, which are linear combinations of n variables x_1, x_2, \dots, x_n with combination coefficients $A_{k1}, A_{k2}, \dots, A_{kn}$, where $k \in \{1, 2, \dots, m\}$. Let the covariance matrix of x_n be denoted by

$$\Sigma^x = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (2.2)$$

Then the covariance matrix of the functions Σ^f is given by

$$\Sigma_{ij}^f = \sum_k^n \sum_l^n A_{ik} \sum_{kl}^x A_{jl}, \quad \Sigma^f = A \Sigma^x A^T. \quad (2.3)$$

In our case we are dealing with non-linear functions, which we will linearized with the help of the **Taylor expansion**

$$f_k \approx f_k^0 + \sum_i^n \frac{\partial f_k}{\partial x_i} x_i, \quad f \approx f^0 + Jx. \quad (2.4)$$

Therefore, the propagation of error follows from the linear case, replacing the Jacobian matrix with the combination coefficients ($J = A$)

Chapter 3

Coefficients

3.1 β function

There are several conventions for defining the β coefficients, depending on a minus sign and/or a factor of two (if one substitutes $\mu \rightarrow \mu^2$) in the β -function 1.5. We follow the convention from Pascual and Tarrach (except for the minus sign) and have taken the values from [1]

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f) \quad (3.1)$$

$$\beta_2 = \frac{1}{12}(17N_c^2 - 5N_cN_f - 3C_fN_f) \quad (3.2)$$

$$\beta_3 = \frac{1}{32} \left(\frac{2857}{54}N_c^3 - \frac{1415}{54}N_c^2N_f + \frac{79}{54}N_cN_f^2 - \frac{205}{18}N_cC_fN_f + \frac{11}{9}C_fN_f^2 + C_f^2N_f \right) \quad (3.3)$$

$$\beta_4 = \frac{140599}{2304} + \frac{445}{16}\zeta_3, \quad (3.4)$$

where we used $N_f = 6$ and $N_c = 3$ for β_4 .

3.2 Anomalous mass dimension

3.3 Adler function

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