Chapter 1

Introduction

In particle physics we are concerned about small objects and their interactions. Their dynamics are currently best described by the Standard Model (SM).

The SM contains two groups of fermionic, Spin 1/2 particles. The former group, the Leptons consist of: the electron (e), the muon (μ), the tau (τ) and their corresponding neutrinos ν_e, ν_μ and ν_τ. The latter group, the Quarks contain: u, d (up and down, the so called light quarks), s (strange), c (charm), b (beauty or beauty) and t (top or truth). The SM furthermore differenciates between three fundamental forces (and its carriers): the electromagnetic (γ photon), weak (Z- or W-Boson) and strong (g gluon) interactions. The before mentioned Leptons solely interact through the electromagnetic and the weak force (also refered to as electroweak interaction), whereas the quarks additionally interact through the strong force.

The strong force is denominated Quantumchromodynamics (QCD). As the name suggest[1] the force is characterized by the color charge. Every quark has next to its type one of the three colors blue, red or green. The color force is mediated through eight gluons, which each being bi-colored[2], interact with quarks and each other. The strength of the strong force is given by the coupling constant α_s. The coupling constants are a function of energy E and $\alpha_s(E)$ increases with energy[3]. This is exclusive for QCD and leads to *asymptotic freedom* an *confinement*. The former phenomen describes the decreasing strong force between quarks and gluons, which become asymptotically free at large energies. The latter expresses the fact, that no isolated quark has been found until today. Quarks appear confined as *Hadrons*, the so called *Mesons*[4] and *Baryons*[5]. As we measure *Hadrons* in our experiments but calculate with quarks within our theoretical QCD model we have to assume *Quark-Hadron Duality*, which states that QCD is still valid for Hadrons for energies suffi-

[1] Chromo is the greek word for color.
[2] Each gluon carries a color and an anti-color.
[3] In contrast to the electromagnetic force, where $\alpha(E)$ decreases!
[4] Composite of a quark and an anti-quark.
[5] Composite of three quarks or three anti-quarks.

cently heigh energies. There exist *Duality Violations* (DV), which will be investigated within this work.

In the following (section 1.1) we will describe the τ-decays, which play an essential role in our QCD analysis. Then (section 1.2) we want to explain some more details of QCD, especially about the coupling constant $\alpha_s(s)$ (which is not constant at all) and the *QCD sum rules*.

1.1 τ-Decays

The τ-particle is an elementary particle with negative electric charge and a spin of $1/2$. Together with the lighter electron and muon it forms the *charged Leptons*[6]. Even though it is an elementary particle it decays via the *weak interaction* with a lifetime of $\tau_\tau = 2.9 \times 10^{-13}$ s and a mass of $1776.86(12)\,\mathrm{MeV}$[**PDG2018**]. It is the only lepton massive enough to decay into Hadrons. The final states of a decay are limited by *conservation laws*. In case of a τ-decay they must conserve the electric charge (-1) and *invariant mass* of the system. Thus, as we can see from the corresponding Feynman diagram (see section 1.1)[7] the τ decays by the emission of a *W boson* and a tau-neutrino ν_τ into different pairs of $(e^-, \bar{\nu}_e), (\mu^-, \bar{\nu}_\mu)$ or $(q, \bar{q})$.

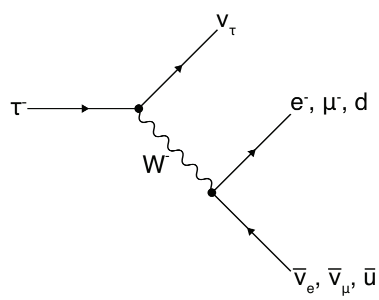


Figure 1.1: Feynman diagram of common decay of a τ-lepton into pairs of lepton-antineutrino or quark-antiquark by the emission of a *W boson*.

We are foremost interested into the hadronic decay channels, meaning τ-decays that have quarks in their final states. Unfortunately the quarks have never been measured isolated, but appear always in combination of *mesons* and *baryons*. Due to its mass of $m_\tau \approx 1.8\,\mathrm{GeV}$ the τ-particle decays into light mesons (pions-π, kaons-K, and eta-η, see section 1.1), which can be experimentally detected.

The hadronic $\tau - decay$ provides one of the most precise ways to determine the strong coupling [**Pich2016**] and can be calculated to high precision within the framework of QCD.

[6] Leptons do not interact via the strong force.

[7] The τ-particle can also decay into strange quarks or charm quarks, but these decays are rather uncommon due to the heavy masses of s and c.

Name	Symbol	Quark content	Rest mass (MeV)
Pion	π^-	$\bar{u}d$	$139.570\,61(24)$ MeV
Pion	π^0	$(u\bar{u}-d\bar{d})/\sqrt{2}$	$134.9770(5)$ MeV
Kaon	K^-	$\bar{u}s$	$493.677(16)$ MeV
Kaon	K^0	$d\bar{s}$	$497.611(13)$ MeV
Eta	η	$(u\bar{u}+d\bar{d}-2s\bar{s})/\sqrt{6}$	$547.862(17)$ MeV

Table 1.1: List of mesons produced by a τ-decay. Rare final states with branching Ratios smaller than 0.1 have been omitted. The list is taken from [**Davier2006**] with corresponding rest masses taken from [**PDG2018**].

Flavour	Mass	comment
u	$2.2^{+0.5}_{-0.4}$ MeV	$\overline{\text{MS}}$
d	$4.7^{+0.5}_{-0.3}$ MeV	
s	95^{+9}_{-3} MeV	
c	$1.275^{+0.025}_{-0.035}$ GeV	
b	$4.18^{+0.04}_{-0.03}$ GeV	
t	$173.0(40)$ GeV	

Table 1.2: List of Quarks and their masses[**PDG2018**].

1.2 Quantumchromodynamics

QCD describes the strong interaction, which occur between *quarks* and are transmitted through *gluons*. A list of quarks can be found in 1.2.

The QCD Lagrange density is similar to that of QED[**Jamin2006**],

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4}G^a_{\mu\nu}(x)G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2}\bar{q}^A(x)\gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A\bar{q}^A(x)a^A(x)\right], \tag{1.1}$$

where $q^A(x)$ represents the quark fields and $G^a_{\mu\nu}$ being the *gluon field strength tensor* given by:

$$G^a_{\mu\nu}(x) \equiv \partial_\mu B^a_\nu(x) - \partial^a_\nu(x) + gf^{abc}B^b_\mu(x)B^c_\nu(x), \tag{1.2}$$

where B^a_μ are the *gluon fields*, given in the *adjoint representation* of the SU(3) gauge group with f^{abc} as *structure constants*. Furthermore we have used $A, B, \cdots = 0, \ldots 5$ as flavour indices, $a, b, \cdots = 0, \ldots, 8$ as color indices and $\mu, \nu, \cdots = 0, \ldots 3$ as lorentz indices.

1.2.1 Renormalisation Group

The perurbations of the QCD Lagrangian 1.1 lead to divergencies, which have to be *renormalized*. There are different aproaches to 'make' these divergencies finite. The most popular one is *dimensional regularisation*. In *Dimensional regularisation* we expand the four space-time dimensions to arbitrary dimensions. Consequently the in QCD calculations appearing *Feyman integrals* have to be continued to D-dimensions like

$$\mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 - m^2 + i0][(q-p)^2 = m^2 + i0]}, \tag{1.3}$$

where we introduced the scale parameter μ to account for the extra dimensions and conserve the mass dimension of the non continued integral.

In addition *physical quantities*[8] cannot depend on the renormalisation scale μ. Thus examining a *physical quantity* $R(q, a_s, m)$ that depends on the external momentum q, the renormalised coupling $a_s = \alpha_s/\pi$ and the renormalized quark mass m

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \tfrac{\partial}{\partial \mu} + \mu \tfrac{dm}{d\mu} \tfrac{\partial}{\partial m}\right] R(q, a_s, m) = 0 \tag{1.4}$$

we can define the *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu \tfrac{d a_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \qquad \beta - \text{function} \tag{1.5}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m} \frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \dots \qquad \text{anomalous mass dimension.} \tag{1.6}$$

Running gauge coupling

Regarding the β-function we notice, that $a_s(\mu)$ is not a constant, but *runs* by varying the scale μ. Integrating the β-function yields

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta(a_s)} = -\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \log \frac{\mu_1}{\mu_2}. \tag{1.7}$$

To analytically evaluate the above integral we can approximate the β-function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11 N_c - 2 N_f), \tag{1.8}$$

yielding

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log \frac{\mu_1}{\mu_2}\right)}. \tag{1.9}$$

As we have three colours $N_c = 3$ and six flavours $N_f = 6$ the first β-function 1.5 is positive. Thus for $\mu_2 > \mu_1$ $a_s(\mu_2)$ decreases logarithmically and vanishes for $\mu_2 \to \infty$. This behaviour is known as *asymptotic freedom*. The coefficients of the β-function are currently known up to the 5th order, which are displayed in the appendix **??**.

[8] Observables that can be measured.

Running quark mass

The properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* 1.6

$$\log \frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \tag{1.10}$$

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically **[Schwab2002]**

$$m(\mu_2) = m(\mu_1) \left(\frac{a(\mu_2)}{a(\mu_1)} \right)^{\frac{\gamma_1}{\beta_1}} \left(1 + \mathcal{O}(\beta_2, \gamma_2)\right). \tag{1.11}$$

As β_1 and γ_1 (see **??**) are positive the quark mass decreases with increasing μ. The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1) \exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \right) \tag{1.12}$$

and can be solved numerically to run the quark mass to the needed scale μ_2.

1.2.2 Operator Product Expansion

The **Operator Product Expansion** (OPE) was introduced by Wilson in 1969 **[Wilson1969]**. The expansion states that non-local operators can be rewritten into a sum of composite local operators and their corresponding coefficients:

$$\lim_{x \to y} \mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_n C_n(x-y)\mathcal{O}_n(x), \tag{1.13}$$

where $C_n(x-y)$ are the so-called *Wilson-coefficients*.

The OPE lets us separate *short-distance* from *long-distance* effects. In perturbation theory (PT) we can only amount for *short-distances*, which are equal to hight energies, where the strong-coupling α_s is small. Consequently the OPE decodes the long-distance effects in the higher dimensionsional operators.

The form of the composite operators are dictated by Gauge- and Lorentz symmetry. Thus we can only make use of operators of even dimension. The operators up to dimension six are given by **[Pascual1984]**

$$\begin{aligned}
&\text{Dimension 0:} && \mathbb{1} \\
&\text{Dimension 4:} && :m_i \bar{q}\, q: \\
& && :G^{\mu\nu}_a(x) G^a_{\mu\nu}(x): \\
&\text{Dimension 6:} && :\bar{q}\,\Gamma q \bar{q}\,\Gamma q: \\
& && :\bar{q}\,\Gamma \tfrac{\lambda^a}{2} q_\beta(x) \bar{q}\,\Gamma \tfrac{\lambda^a}{2} q: \\
& && :m_i \bar{q}\, \tfrac{\lambda^a}{2} \sigma_{\mu\nu} q G^{\mu\nu}_a: \\
& && :f_{abc} G^{\mu\nu}_a G^{\nu\delta}_b G^{\delta\mu}_c:,
\end{aligned} \tag{1.14}$$

where Γ stands for one of the dirac matrices $\Gamma \in \{1, i\gamma_5, \gamma^\mu, \gamma^\mu\gamma_5\}$, specifying the quantum number of the current (S, P, A, respectively). As all the operators appear normal ordered they vanish by definition in PT. Consequently they appear as **Condensates** in Non-perturbative (NPT) QCD like quark-condensate $\langle \overline{q}\, q \rangle$ or the gluon-condensate $\langle aGG \rangle$ (both of dimension four).

As we work with dimensionless functions (e.g. Π) in Sum Rules, the r.h.s. of **??** has to be dimensionless. Consequently the Wilson-coefficients have to cancel the dimension of the operator with their inverse mass dimension. To account for the dimensions we can make the inverse momenta explicit

$$\Pi^{OPE}_{V/A}(s) = \sum_{D=0,2,4\ldots} \frac{c^{(D)} \langle \mathcal{O}^{(D)}(x) \rangle}{-s^{D/2}}, \tag{1.15}$$

where we used $C^{(D)} = c/(-s)^{D/2}$ with D being the dimension. Consequently the OPE should converge with increasing dimension for suficienty large momenta s.

The usage of the OPE and its validity is far from obvious. We are deriving the OPE from matching the Wilson-coefficients to Feynman-graph analyses. These Feynman-graphs are calculated perturbatively but the coefficients with dimension $D > 0$ correspond to NPT condensates!

$$\Pi^{\mu\nu}(q) = i \int d^4\, x e^{iqx} \langle \Omega | T\{ j^\mu(x) j^\nu(0) \} \rangle \tag{1.16}$$

$$j^\mu(x) = [\overline{q}\,_i \Gamma q_j](x) \tag{1.17}$$

Standard example (following [**Pascual1986**])

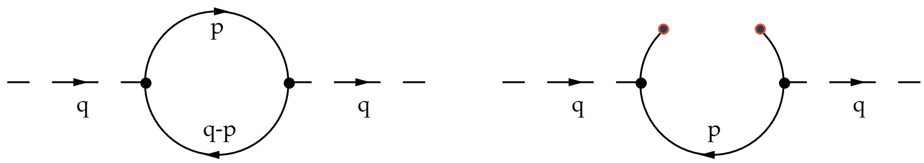
$$j^\mu(x) = \frac{1}{2} \Big(:[\overline{u}\,\gamma^\mu u](x) - \overline{d}\,\gamma^\mu d](x) \Big) \tag{1.18}$$

$$\Pi^{\mu\nu}(s) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi(s) \tag{1.19}$$

$$\begin{aligned} \Pi(q^2) = &-\frac{i}{4q^2(D-1)} \int d^D\, x e^{iqx} \langle \Omega | T\{ : \overline{u}\,(x) \gamma^\mu u(x) - \overline{d}\,(x) \gamma^{mu} d(x) : \\ &\times : \overline{u}\,(0 \gamma_\mu u(0) - \overline{d}\,(0) \gamma_\mu d(0) : \} \rangle \end{aligned} \tag{1.20}$$

Using Wick's theorem

$$\Pi(q^2) = \frac{i}{4q^2(D-1)} (\gamma^\mu)_{ij} (\gamma_\mu)_{kl} \int d^D\, x e^{iqx} \left[\overline{u_{j\alpha}(x) \overline{u}}\,_{k\beta}(0) \cdot \overline{u_{l\beta}(0) \overline{u}}\,_{i\alpha}(x) + (u \to d) \right] \tag{1.21}$$



(a) (b)

Condensate contraction

$$
\begin{aligned}
\Pi(q^2) = \frac{i}{4q^2(D-1)}(\gamma^{\mu})_{ij}(\gamma_{\mu})_{kl}\int d^D\, x e^{iqx} \Big[\\
+ u_{j\alpha}(x)\overline{u}_{k\beta}(0)\cdot\langle\Omega| : \overline{u}_{i\alpha}(x)u_{l\beta}(0) : |\Omega\rangle \\
+ u_{l\beta}(0)\overline{u}_{i\alpha}(x)\cdot\langle\Omega| : \overline{u}_{k\beta}(0)u_{j\alpha}(x) : |\rangle + (u \to d)\Big]
\end{aligned} \tag{1.22}
$$

1.2.3 Sum Rules

We need to relate the measurable hadronic final states of a QCD process (e.g. τ-decays into Hadrons) to a theoretical calculable value. Consequently we will employ **QCD Sum Rules[Shifman1978]**, which is a combination of the **operator product expansion** (OPE), the **optical theorem**, a **dispersion relation** the analyticity of the **two-poin function** and the **quark hadron duality**.

Starting from the the vacuum expectation value of the product of the conserved noether current $J_\mu(x)$ at different space-times points x and y, which is known as the *two-point function* (or simply correlator)

$$\Pi(q^2) = \langle 0|J_\mu(x)J_\nu(y)|0\rangle, \tag{1.23}$$

where the noether current is given by

$$J_\mu(x) = \Psi^\dagger(x)\gamma_\mu(\gamma_5)\Psi(x). \tag{1.24}$$

The two-point function, within the framework of QCD sum rules, is improved by the OPE expansion

$$\Pi_{OPE}(s) = \sum_n C_{2n}(s,\mu)\frac{\langle\hat{O}(\mu)\rangle}{s^n}, \tag{1.25}$$

where we used $q^2 = s$. It is furthermore related to the hadronic **spectral function** $\rho(q^2)$ through the *Källén-Lehmann spectral representation* **[Kallen1952][Lehmann1954]**

$$\Pi(q^2) = \int_0^\infty ds\frac{\rho(s)}{s - q^2 - i\epsilon}, \tag{1.26}$$

where the spectral function $\rho(s)$ is defined as

$$\rho(s) \equiv \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s). \tag{1.27}$$

Equation 1.26 is refered to as **dispersion relation** analogous to similar realtions which arise for example in electrodynamics. The the main contribution from the spectral function given in eq. (1.26) are the hadronic final states

$$2\pi\rho(m^2) = \sum_n \langle 0|J_\mu(x)|n\rangle\langle n|J_\nu(y)\rangle(2\pi^2)^4\delta^{(4)}(p-p_n), \tag{1.28}$$

which lead to a series of continuos poles on the positive real axis for the two-point function, see Fig. 1.2.3. As the experimental data, which solely

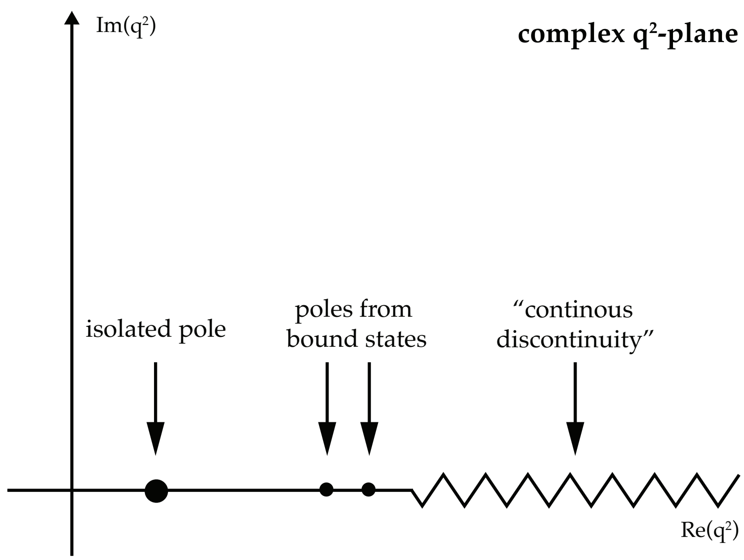


Figure 1.2: Analytic structure in the complex q^2-plane of the Fourier transform of the two-point function. The hadronic final states are responsible for poles appearing on the real-axis. The one-particle states contribute as isolated pole and the multi-particle states contribute as bound-states poles or a continues "discontinuity cut" (see **[Peskin1995]**).

contributes to the spectral function $\rho(q^2)$, is only accesible on the postive real axis, we have to use Cauchy's theorem to access the theoretical values of the two-point function close to the postive real axis (see section 1.2.3).

The final ingredient of the QCD sum rules is the *optical theorem*, relating experimental data with the imaginary part of the correlator. E.g. taking the the total e^+e^- cross section scattering into hadrons

$$R_q(s) \equiv \frac{\sigma(e^+e^- \to \gamma^* \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 12\pi \operatorname{Im}\Pi_{Had}(s). \tag{1.29}$$

Due to asymptotic freedom[9] experiments can only detect Hadrons (note the exp-indice in $\operatorname{Im}\Pi_{exp}(s)$), but on the theory side we are calculating with

[9]There are no free quarks. They are bound in pairs of two or three.

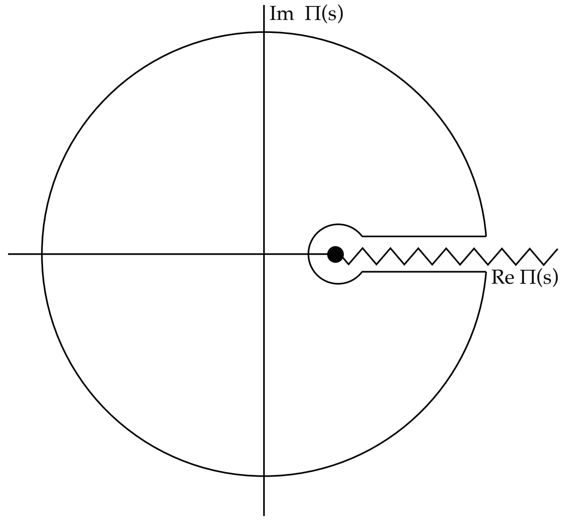


Figure 1.3: Analytical structure of $\Pi(s)$ with the used contour $\mathcal{C}$ for the final QCD Sum Rule expression eq. (1.30).

quarks as degrees of freedom. Consequently we assume that Π_{Had} can be set equal to Π_{OPE}, which is referred to as *quark hadron duality*.

In total, with the help Cauchy's theorem, the QCD sum rules can be sumed up in the following expression

$$\frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\Pi_{\mathrm{Had}}(t)}{t-s}\,dt = \frac{1}{\pi}\oint_{\mathcal{C}} \frac{\operatorname{Im}\Pi_{\mathrm{OPE}}(t)}{t-s}\,dt, \tag{1.30}$$

where the l.h.s. is given by the experiment and the r.h.s. can be theoretically evaluated with by applying the OPE of the correlator $\Pi_{\mathrm{OPE}}(s)$.