Chapter 1

QCD Sum Rules

It is most remarkably that we can describe the properties of quarks and gluons via a local QFT based on the gauge group $SU(3)$. However QCD only applies to coloured particles and not to colourless particles like hadrons. Due to confinement we can only ever observe hadrons, but our theoretical foundation is ruled by the DOF of quarks and gluons. To extract QCD-parameters (the six quark-masses and the strong-coupling) from hadrons we need bridge the quark-gluon-with the hadron-picture. To do so we will introduce the theory of QCDSR. Starting with QCD we will discuss the QCD-Lagrangian. QCD is describing the quark-gluon picture solely and ruled by the abelian gauge group $SU(3)$, which has important implications on the running of the strong coupling and the applicability of PT. Next we will focus on the two-point function, which is the main physical object we are going to study within the QCDSR. It is vacuum-expectation values of quark currents. We can theoretically describe the processes, like τ-decays into hadrons, by choosing currents with matching quantum numbers to the outgoing particles. We will see, that the two-point function $\Pi(q^2)$ is related to hadronic states, by poles for $q^2 > 0$. Here NP-effects become important and we need to introduce the OPE, which handles NP-effects as QCD-condensates. These condensates are remainders of the QCD-vacuum Ω_{QCD}. In contrast to QED normal ordered products of fields do not vanish, but remain as parameters and have to be phenomenological fitted or calculated by other NP tools, like LQCD. ...

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have been discussed in many articles [**Narison1989**, **Rafael1997**, **Colangelo2000**, **Dominguez2013**] Duality first introduced Poggio, Quinn and Weinberg [**Poggio1975**]

1.1 Quantum-chromodynamics

Since the formulation of QED in the end of the 40's it has been attempted to describe the strong nuclear force as a QFT, which has been achieved in the 70's as QCD [**GellMann1972**, **Fritzsch1973**, **Gross1973**, **Politzer1973**, **Weinberg1973**]. QCD is a renormalisable QFT of the strong interaction, which fundamental fields are given by dirac spinors of spin-1/2, the so-called quarks, with a fractional electric charge of $\pm 1/3$ or $\pm 2/3$. The theory furthermore contains gauge-fields of spin 1, which are chargeless, massless and referred to as gluons. The gluons are the force-mediators, which interact with quarks and themselves, in contrast to photons of QED, which interact only with fermions (see fig. 1.1).

The corresponding gauge-group of QCD is the non-abelian group $SU(3)$. Each of the quark flavours u, d, c, s, t and b belongs to the fundamental representation of $SU(3)$ and contains a triplet of fields Ψ.

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} \text{red} \\ \text{green} \\ \text{blue} \end{pmatrix} \tag{1.1.1}$$

The components of the triplet are the so-called colours[1] red, green and blue, which is referred to as *colour-charge*. The gluons belong to the adjoint representation of $SU(3)$, contain an octet of fields and can be expressed using the Gell-Mann matrices λ_a

$$B_\mu = B_\mu^a \lambda_a \qquad a = 1, 2, \ldots 8 \tag{1.1.2}$$

The classical *Lagrange density* of QCD is given by [**Jamin2006**, **Pascual1984**]:

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G_{\mu\nu}^a(x) G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2} \bar{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A \bar{q}^A(x) a^A(x) \right], \tag{1.1.3}$$

with $q^A(x)$ representing the quark fields and $G_{\mu\nu}^a$ being the *gluon field strength tensor* given by:

$$G_{\mu\nu}^a(x) \equiv \partial_\mu B_\nu^a(x) - \partial_\nu^a(x) + g f^{abc} B_\mu^b(x) B_\nu^c(x), \tag{1.1.4}$$

with f^{abc} as *structure constants* of the gauge-group $SU(3)$ and $\mathrm{overleftrightarrow}D_\mu$ as covariant derivative acting to the left and to the right. Furthermore we have

[1] The colour denomination is not gauge-invariant. After a colour gauge transformation the new colours are a linear combination of the old colours, which breaks gauge-symmetry.

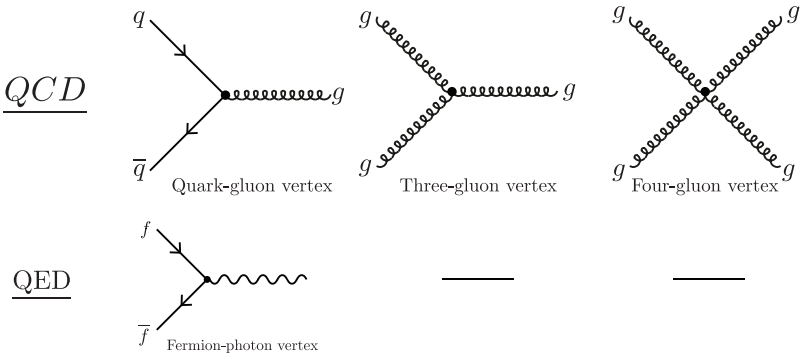


Figure 1.1: Feynman diagrams of the strong interactions with corresponding electromagnetic diagrams. We see that the gluons carry colour charge and thus couple to other gluons, which is not the case for the photons.

Flavour	Mass
u	3.48(24) MeV
d	6.80(29) MeV
s	130.0(18) MeV
c	1.523(18) GeV
b	6.936(57) GeV
t	173.0(40) GeV

Table 1.1: List of Quarks and their masses. The masses of the up, down, strange, charm and bottom quark are the renormalisation group invariant (RGI) quark masses and are quoted in the four-flavour theory ($N_f = 2 + 1$) at the scale $\mu = 2\,\mathrm{GeV}$ in the $\overline{MS}$ scheme and are taken from the *Flavour Lattice Averaging Group* [**FLAG2019**]. The mass of the top quark is not discussed in [**FLAG2019**] and has been taken from [**PDG2018**] from direct observations of top events.

used $A, B, \ldots = 0, \ldots 5$ as flavour indices, $a, b, \ldots = 0, \ldots, 8$ as colour indices and $\mu, \nu, \ldots = 0, \ldots 3$ as lorentz indices. Explicitly the Lagrangian writes:

$$\begin{aligned}\mathcal{L}_0(x) = &-\frac{1}{4}\left[\partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x)\right]\left[\partial^\mu G^\nu_a(x) - \partial^\nu G^\mu_a(x)\right] \\ &+ \frac{i}{2}\overline{q}^A_\alpha(x)\gamma^\mu\partial_\mu q^A_\alpha(x) - \frac{i}{2}\left[\partial_\mu \overline{q}^A_\alpha(x)\right]\gamma^\mu q^A_\alpha(x) - m_A\overline{q}^A_\alpha(x)q^A_\alpha(x) \\ &+ \frac{g_s}{2}\overline{q}^A_\alpha(x)\lambda^a_{\alpha\beta}\gamma_\mu q^A_\beta(x)G^\mu_a(x) \\ &- \frac{g_s}{2}f_{abc}\left[\partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x)\right]G^\mu_b(x)G^\nu_c(x) \\ &- \frac{g_s^2}{4}f_{abc}f_{ade}G^b_\mu(x)G^c_\nu(x)G^\mu_d(x)G^\nu_e(x)\end{aligned} \tag{1.1.5}$$

The first term is the kinetic term for the massless gluons. The next three terms are the kinetic terms for the quark field with different masses for each flavour. The rest of the terms are the interaction terms. The fifth term represents the interaction between quarks and gluons and the last two terms the self-interactions of gluon fields.

Having derived the Lagrangian leaves us with its quantisation. The dirac-spinors can be quantised as in QED without any problems. The $\Psi(x)$ quantum field can be written as:

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3 2E(\vec{p})}\sum_\lambda\left[u(\vec{p},\lambda)a(\vec{p},\lambda)e^{-ipx} + v(\vec{p},\lambda)b^\dagger(\vec{p},\lambda)e^{ipx}\right], \tag{1.1.6}$$

where the integration ranges over the positive sheet of the mass hyperboloid $\Omega_+(m) = \{p|p^2 = m^2, p^0 > 0\}$. The four spinors $u(\vec{p},\lambda)$ and $v(\vec{p},\lambda)$ are solutions to the dirac equations in momentum space

$$\begin{aligned}[\not{p} - m]u(\vec{p},\lambda) &= 0 \\ [\not{p} + m]v(\vec{p},\lambda) &= 0,\end{aligned} \tag{1.1.7}$$

with λ representing the helicity state of the spinors.

The quantisation of the gauge-fields are more cumbersome. One is forced to introduce supplementary non-physical fields, the so-called Faddeev-Popov ghosts $c^a(x)$ [**Faddeev1967**].

The free propagators for the quark-, the gluon-and the ghost-fields are then

given by

$$
\begin{aligned}
iS^{(0)AB}_{\alpha\beta}(x-y) &\equiv \overbrace{q^A_\alpha(x)\overline{q}^B_\beta}(y) \equiv \langle 0|T\{q^A_\alpha(x)\overline{q}^B_\beta\}|0\rangle = \delta_{AB}\delta_{\alpha\beta} iS^{(0)}(x-y) \\
&= i\delta_{AB}\delta_{\alpha\beta}\int \frac{d^4p}{(2\pi)^4}\frac{\not{p}+m}{(p^2-m^2+i\epsilon)} \\
iD^{(0)\mu\nu}_{ab}(x-y) &\equiv \overbrace{B^\mu_a(x)B^\nu_b}(y) \equiv \langle 0|T\{B^\mu_a(x)B^\nu_b(y)\}|0\rangle \equiv \delta_{ab} i\int \frac{d^4k}{(2\pi)^4} D^{(0)\mu\nu}(k)e^{-ik(x-y)} \\
&= i\delta_{ab}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\left[-g_{\mu\nu}+(1-a)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right]e^{-ik(x-y)} \\
i\widetilde{D}^{(0)}_{ab}(x-y) &\equiv \overbrace{\phi_a(x)\overline{\phi}_b}(y) \equiv \langle 0|T\{\phi_a(x)\overline{\phi}_b(y)\}|0\rangle = \frac{i}{(2\pi)^4}\delta_{ab}\int d^4q \frac{-1}{q^2+i\epsilon}e^{-q(x-y)} \\
&\equiv \frac{i}{(2\pi)^4}\delta_{ab}\int d^4q \widetilde{D}^{(0)}(q)e^{-iq(x-y)},
\end{aligned}
\tag{1.1.8}
$$

and the corresponding Feynman-rules have been displayed in fig. 1.2.

1.1.1 Renormalisation Group

The perturbations of the QCD Lagrangian in eq. 1.1.3 lead to divergencies, which have to be *renormalised*. Making these divergencies finite is referred to as *regularisation* and there are various approaches:

- **λ regularisation:** In Lambda regularisation we limit the divergent momentum integrals by a cutoff $|\vec{p}| < \Lambda$. Here Λ has the dimension of mass. The cutoff regularisation breaks translational invariance, which can be guarded by making use of other regularisation methods.

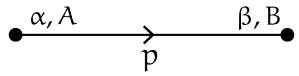
- ***P-I* (Pauli-Villars) regularisation:** [**Pauli1949**] In P-I regularisation the propagator is forced to decrease faster than the divergence to appear. It replaces the nominator by

$$
(\vec{p}^2+m^2)^{-1} \to (\vec{p}^2+m^2)^{-1} - (\vec{p}^2+M^2)^{-1}, \tag{1.1.9}
$$

  where M has the dimension acts similar as the previously presented cutoff, but conserves translational invariance.

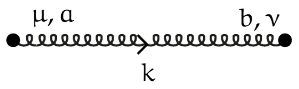
- **Dimensional regularisation:** [**Bollini1972**, **tHooft1972**, **tHooft1973**] Dimensional regularisation has been introduced in the beginning of the

Quark propagator



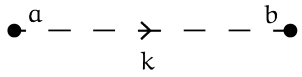
$$= \frac{i\delta_{\alpha\beta}\delta_{AB}}{\not{p} - m_A + i\epsilon}$$

Gluon propagator



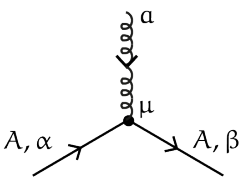
$$= \frac{-i\delta_{ab}}{k^2 + i\epsilon}\left[g^{\mu\nu} - (1-a)\frac{k_\mu k_\nu}{k^2 + i\epsilon}\right]$$

Ghost propagator



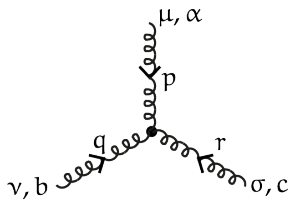
$$= \frac{-\delta_{ab}}{k^2 + i\epsilon}$$

Fermionic vertex



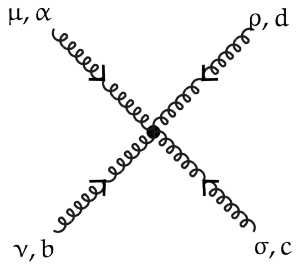
$$= g\left(\frac{\lambda_a}{2}\right)_{\beta\alpha}\gamma^\mu$$

Triple gluon vertex



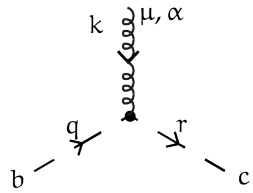
$$= -igf_{abc}[g_{\mu\nu}(p-q)_\sigma + g_{\nu\sigma}(q-r)_\mu + g_{\sigma\mu}(r-p)_\nu$$

Quartic gluon vertex



$$= -g^2[f_{abe}f_{cde}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) + f_{ace}f_{bde}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\nu\sigma}) + f_{ade}f_{cbe}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\sigma\rho})]$$

Ghost vertex



$$= -igf_{abc}r^\mu$$

Figure 1.2: QCD Feynman rules.

seventies to regularise non-abelian gauge theories (like QCD), where Λ- and P-V-regularisation failed. In dimensional regularisation we expand the four space-time dimensions to arbitrary D-dimensions. To compensate for the additional dimensions we introduce an additional scale μ^{D-4}. A typical Feynman-integral then has the following appearance:

$$\int \frac{d^4p}{(2\pi)^4}\frac{1}{\vec{p}^2+m^2} \to \mu^{2\epsilon}\int \frac{d^Dp}{(2\pi)^D}\frac{1}{\vec{p}^2+m^2}, \tag{1.1.10}$$

Dimensional regularisation preserves all symmetries, it allows an easy identification of divergences and naturally leads to the *minimal subtraction scheme* (MS).

In all of the three regularisation schemes we introduced an arbitrary parameter to regularise the divergence. This parameter causes an scale dependence of the strong coupling and the quark masses. As we are mainly concerned with the non-abelian gauge theory QCD we will focus on dimensional regularisation, which introduced the parameter μ. Measurable observables (*Physical quantities*) cannot depend on the renormalisation scale μ. In the Therefore the derivative by μ of a general physical quantity has to yield zero. The physical quantity $R(q, a_s, m)$, that depends on the external momentum q, the renormalised coupling $a_s \equiv \alpha_s/\pi$ and the renormalised quark mass m can then be expressed as

$$\mu\frac{d}{d\mu}R(q,a_s,m) = \left[\mu\tfrac{\partial}{\partial\mu} + \mu\tfrac{da_s}{d\mu}\tfrac{\partial}{\partial m} + \mu\tfrac{dm}{d\mu}\tfrac{\partial}{\partial m}\right]R(q,a_s,m) = 0. \tag{1.1.11}$$

Equation 1.1.11 is referred to as **renormalisation group equation** and is the basis for defining the two *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu\tfrac{da_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \ldots \qquad \beta-\text{function} \tag{1.1.12}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m}\tfrac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \ldots \qquad \text{anomalous mass dimension.} \tag{1.1.13}$$

The coefficients of the β-function and the mass anomalous dimension are currently known up to the 5^{th} order and listed in the appendix **??**.

Running gauge coupling

The β-function and the anomalous mass dimension are responsible for the running of the strong coupling and the running of the quark mass respectively. In this section we will shortly review the β-function and its implications

on the strong coupling, whereas in the following section we will discuss the anomalous-mass dimension.

Regarding the β-function we notice, that $a_s(\mu)$ is not a constant, but *runs* by varying its scale μ. Lets observe the running of the strong coupling constant by integrating the β-function

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta(a_s)} = -\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \log \frac{\mu_1}{\mu_2}. \tag{1.1.14}$$

To analytically evaluate the above integral we can approximate the β-function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \tag{1.1.15}$$

yielding

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log \frac{\mu_1}{\mu_2}\right)}. \tag{1.1.16}$$

As we have three colours ($N_c = 3$) and six flavours ($N_f = 6$) the first β-function 1.1.12 is positive. Thus for $\mu_2 > \mu_1$ $a_s(\mu_2)$ decreases logarithmically and vanishes for $\mu_2 \to \infty$. This behaviour is known as *asymptotic freedom* and leads to *confinement*.

Asymptotic freedom states, that for high energies (small distances), the strong coupling becomes diminishing small and quarks and gluons do not interact. Thus in isolated baryons and mesons the quarks are separated by small distances, move freely and do not interact. On the other hand we are not able to separate the quarks in a meson or baryon. No quark has been detected as single particle yet. This is qualitatively explained with the gluon field carrying colour charge. These gluons form so-called *flux-tubes* between quarks, which cause a constant strong force between particles regardless of their separation. Consequently the energy needed to separate quarks is proportional to the distance between them and at some point there is enough energy to favour the creation of a new quark pair. Thus before separating two quarks we create a quark-antiquark pair. As a result we will probably never be able to observe an isolated quark. This phenomenon is referred to as colour confinement or simply confinement.

Running quark mass

Not only the coupling but also the masses carry an energy dependencies, which is governed by the *anomalous mass dimension* $\gamma(a_s)$.

The properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* eq. 1.1.13

$$\log \frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \tag{1.1.17}$$

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically [**Schwab2002**]

$$m(\mu_2) = m(\mu_1) \left(\frac{a(\mu_2)}{a(\mu_1)}\right)^{\frac{\gamma_1}{\beta_1}} \left(1 + \mathcal{O}(\beta_2, \gamma_2)\right). \tag{1.1.18}$$

As β_1 and γ_1 (see **??**) are positive the quark mass decreases with increasing μ. The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1) \exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}\right) \tag{1.1.19}$$

and can be solved numerically to run the quark mass to the needed scale μ_2.

QCD in general has a precision problem caused by uncertainties and largeness of the strong coupling constant α_s. The fine-structure constant (the coupling QED) is known to eleven digits, whereas the strong coupling is only known to about four. Furthermore for low energies the strong coupling constant is much larger than the fine-structure constant. E.g. at the Z-mass, the standard mass to compare the strong coupling, we have an α_s of 0.11, whereas the fine structure constant would be around 0.007. Consequently to use PT we have to calculate our results to much higher orders, including tens of thousands of Feynman diagrams, in QCD to achieve a precision equal to QED. For even lower energies, around 1 GeV, the strong coupling reaches a critical value of ≈ 0.5 leading to a break down of PT.

In this work we try to achieve a higher precision in the value of α_s. Our method to measure the strong coupling are the QCDSR, which in our case make use of the *two-point function* for which we will devote the following section

1.2 Two-Point function

The *two-point* function (or *correlator*) is the vacuum expectation value of two local fields. In the simplest case, for a scalar field ϕ, it is defined as

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle. \tag{1.2.1}$$

Here $|\Omega\rangle$ is the physical vacuum and the fields operators $\phi(x)$ and $\phi(y)$ are time dependent Heisenberg operators, which can be translated to its origin. The two-point function can be interpreted physically as the amplitude of propagating single-or multi-particles and their excitations. The correlator can be expressed as a *spectral decomposition*, which has been discovered early on by **[Kallen1952, Lehmann1954]** and is named *Källén-Lehmann spectral representation*. This spectral representation is interesting to us for two main reasons. First it is experimentally measurable and second it carries a problematic "branch cut", which we want to discuss now.

1.2.1 Källén-Lehmann spectral representation

In the second quantisation we applied latter operators $a_o^\dagger$ on the free vacuum $|0\rangle$ to obtain single particle states, carrying a momentum p

$$a_p^\dagger |0\rangle = \frac{1}{\sqrt{2E_p}} |\vec{p}\rangle, \tag{1.2.2}$$

where the factor $\sqrt{2E_p}$ is a convention resulting from the harmonic oscillator. The identity operator for one-particle state, which selects only single particle states is then given by

$$\mathbb{1} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} |\vec{p}\rangle \langle \vec{p}|. \tag{1.2.3}$$

The above complete set of one-particle states can be enhanced to include multiple-particle states, where we have to sum not only over all possible momentum states $|\vec{p}\rangle$, but over all possible multi-particle states and their enhanced states as well. If we define these state vectors as $\vec{X}$ we can express the complete set of one-and multiple-particle states as

$$\mathbb{1} = \sum_{\vec{X}} \int d \prod_X |X\rangle \langle X|, \tag{1.2.4}$$

where we have to sum over all possible $|X\rangle$ single- or multi-particle states and integrate over the momentum $\vec{p}$ via

$$d\Pi_X \equiv \prod_{i\in X} \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_j}. \tag{1.2.5}$$

To get to the desired spectral decomposition we have to translate the Heisenberg operator $\phi(x)$ to its origin, like

$$\begin{aligned} \langle \Omega|\phi(x)|X\rangle &= \langle \Omega|e^{i\hat{P}x} e^{-i\hat{P}x}\phi(x)e^{i\hat{P}x}e^{-i\hat{P}x}|X\rangle \\ &= e^{-ip_X x}\langle \Omega|e^{-i\hat{P}x}\phi(x)e^{i\hat{P}x}|X\rangle \\ &= e^{-ip_X x}\langle \Omega|\phi(0)|X\rangle. \end{aligned} \tag{1.2.6}$$

Equally we can we can express $\phi(y)$ as

$$\langle X|\phi(y)|\Omega\rangle = e^{ipy}\langle \Omega|\phi(0)|\Omega\rangle. \tag{1.2.7}$$

With these expressions in hand we can spectral decompose the two-point function (eq. 1.2.1)

$$\begin{aligned} \langle \Omega|\phi(x)\phi(y)|0\rangle =& \sum_X \int d\Pi_X \langle \Omega|\phi(x)|X\rangle\langle X|\phi(y)|\Omega\rangle \\ =& \sum_X \int d\Pi_X e^{-ip_X(x-y)} \left|\langle \Omega|\phi(0)|X\rangle\right|^2 \\ =& \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} (2\pi)^3 \left[\sum_X \int d\Pi_X 2\pi\delta^{(4)}(p-p_X)\left|\langle \Omega|\phi(0)|X\rangle\right|^2\right], \end{aligned} \tag{1.2.8}$$

where the term in the bracket is Lorentz invariant, implying that it depends only on the Lorentz-invariant measure p^2. The multi-particle states $|X\rangle$ have to have positive energy eigenstates $p^0 > 0$. Consequently we can define the *spectral function* as

$$\theta(p^0)\rho(p^2) \equiv \sum_X \int d\Pi_X 2\pi\delta^{(4)}(p-p_X)\left|\langle \Omega|\phi(0)|X\rangle\right|^2, \tag{1.2.9}$$

with $\rho(p^2)$ being the *spectral density*. The simple two-point function (eq. 1.2.1) can then be rewritten to

$$\begin{aligned} \langle \Omega|\phi(x)\phi(y)|\Omega\rangle &= \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)}\theta(p^0)\rho(p^2) \\ &= \int_0^\infty dq^2 \rho(q^2) D(x,y,q^2) \end{aligned} \tag{1.2.10}$$

with

$$\begin{aligned} D(x,y,m^2) &= \int \frac{d^3}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^3}e^{-ip(x-y)}\theta(p_0)\delta(p^2-m^2). \end{aligned} \tag{1.2.11}$$

$\rho(p^2)$ *spectral density*, $\rho(p^2) > 0$. The spectral function is a non-negative function, describing physically measurable particle states. Until now we have given the spectral function for our "simple" two-point function (eq. 1.2.1), but we can generalise the decomposition to the *Fourier transform* (FT) of the *time-ordered two-point function*

$$\Pi(p^2) = i\int \frac{d^4x}{(2\pi)^4}e^{-ixp}\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle, \tag{1.2.12}$$

where we made use of the translation invariance of the correlator[2]. The time-ordered scalar correlator can then be expressed as

$$\begin{aligned} \Pi(p^2) &= i\int \frac{d^4x}{(2\pi)^4}e^{-ixp}\left[\theta(x^0-0)\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle + \theta(y^0-0)\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle\right] \\ &= i\int \frac{d^4x}{(2\pi)^4}e^{-ixp}\int_0^\infty dq^2\,\rho(q^2)\left[\theta(x^0-0)D(x,0,q^2) + \theta(y^0-0)D(0,x,q^2)\right] \\ &= i\int \frac{d^4x}{(2\pi)^4}e^{-ixp}\int_0^\infty dq^2\,\rho(q^2)\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-q^2-i\epsilon}e^{ipx} \end{aligned} \tag{1.2.13}$$

by making use of the mathematical identity[3]

$$\theta(x^0-y^0)D(x,y,q^2) + \theta(y^0-x^0)D(y,x,q^2) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-q^2-i\epsilon}e^{ip(x-y)}. \tag{1.2.14}$$

The last line of is given by two cancelling Fourier transforms. Thus we recognise the Källén-Lehmann spectral decomposition of the scalar time-ordered two-point function

$$\Pi(p^2) = \int_0^\infty dq^2\frac{i\rho(q^2)}{p^2-q^2-i\epsilon}. \tag{1.2.15}$$

Also notice that when applying the following mathematical identity

$$\mathrm{Im}\,\frac{1}{p^2-q^2+i\epsilon} = -\pi\delta(p^2-q^2) \tag{1.2.16}$$

[2] $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|e^{i\hat{P}y}\underbrace{e^{-i\hat{P}y}\phi(x)e^{i\hat{P}y}}_{x-y}\underbrace{e^{-i\hat{P}y}\phi(y)e^{i\hat{P}y}}_{=\phi(0)}e^{-i\hat{P}y}|\Omega\rangle = \langle\Omega|\phi(x-y)\phi(0)|\Omega\rangle$

[3] The identity should be known to the reader from the derivation of the Feynman propagator.

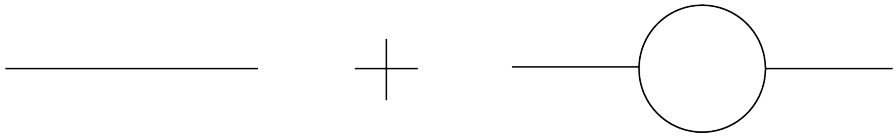


Figure 1.3: Self energy

that the spectral function can be given by the imaginary part of the correlator

$$\rho(p^2) = -\frac{1}{\pi}\,\mathrm{Im}\,\Pi(p^2). \tag{1.2.17}$$

The spectral function usually has a pole at one-particle states and a branch cut above the multi-particle state threshold. The branch cut is of high importance to us as the experimental detected spectral function is only accessible at the possible real axis, where the branch cut exists. We will further study these singularities with the following example of a toy-Lagrangian of two interacting scalar fields.

1.2.2 Analytic Structure of the Spectral Function

To analyse the singularities contained in the spectral function $\rho(s)$ we have a look at the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\partial_\mu\partial^\mu + M^2)\phi - \frac{1}{2}\pi(\partial_\mu\partial^\mu + M^2)\pi + \frac{\lambda}{2}\phi\pi^2. \tag{1.2.18}$$

The Lagrangian contains two fields ϕ and π, which interact via the interaction term $\lambda/2\phi\pi^2$. We now want to calculate the self-interaction of the ϕ-field, as shown in fig. 1.3, to first order. To do so we need an expression for the free propagator of the ϕ field

$$G_\phi = \frac{1}{p^2 - M^2 + i\epsilon}. \tag{1.2.19}$$

Furthermore we need the result of the Feynman loop-diagram given in fig. 1.3

$$\mathrm{Im}\,\mathcal{M}^{\mathrm{Loop}} = \frac{\lambda^2}{32\pi}\sqrt{1 - 4\frac{m^2}{M^2}}\,\theta(M - 2m) \tag{1.2.20}$$

Summing over all possible self-energy graphs we get a geometric series

$$iG(p^2) = \frac{i}{p^2 - m_R^2 + \Sigma(p^2) + i\epsilon}, \tag{1.2.21}$$

where we can plugin the free propagator and the result of the loop diagram of our toy-Lagrangian to get a typical spectral function

$$\begin{aligned}\rho(q^2) &= -\frac{1}{\pi}\operatorname{Im}\Pi(q^2) \\ &= \frac{1}{\pi}\operatorname{Im}\left[p^2 - M^2 + i\epsilon + \frac{\lambda^2}{32\pi}\sqrt{1-4\frac{m^2}{M^2}}\theta(M-2m)\right]^{-1} \\ &= \delta(q^2-M^2) + \theta(q^2-4m^2)\frac{\lambda^2}{32\pi^2}\frac{1}{(q^2-M^2)^2}\sqrt{\frac{q^2-4m^2}{q^2}},\end{aligned} \tag{1.2.22}$$

where we have used eq. 1.2.16.

correct sum up What have we done so far: The scalar QCD two point function can then be described by the spectrum of hadronic states. The correlator is then related to an integral over the *spectral function* $\rho(s)$ via the *Källén-Lehmann spectral representation* **[Kallen1952, Lehmann1954]**, which is known since the early fifties

$$\Pi(q^2) = \int_0^\infty ds \frac{\rho(s)}{s - q^2 - i\epsilon}. \tag{1.2.23}$$

Equation 1.2.23 is referred to as *dispersion relation* analogous to similar relations which arise for example in electrodynamics and defines the *spectral function* (a derivation can be found in **[Rafael1997]**)

$$\rho(s) = \frac{1}{\pi}\operatorname{Im}\Pi(s). \tag{1.2.24}$$

Until know we connected theoretical correlator with the measurable hadronic spectrum. Nevertheless the analytic properties of the correlators have to be discussed as the function has discontinuities.

The main contribution from the spectral function given in eq. 1.2.23 are the hadronic final states

$$2\pi\rho(m^2) = \sum_n \langle 0|J_\mu(x)|n\rangle\langle n|J_\nu(y)\rangle(2\pi^2)^4\delta^{(4)}(p-p_n), \tag{1.2.25}$$

which lead to a series of continuous poles on the positive real axis for the two-point function (see Fig. fig. 1.4).

The vacuum expectation value of the product of the conserved Noether current $J_\mu(x)$ at different space-time points x and y is known as the **two-point function** (or simply **correlator**)

$$\Pi_{\mu\nu}(q^2) = \int dx\, e^{iqx}\langle 0|J_\mu(x)J_\nu(0)|0\rangle, \tag{1.2.26}$$

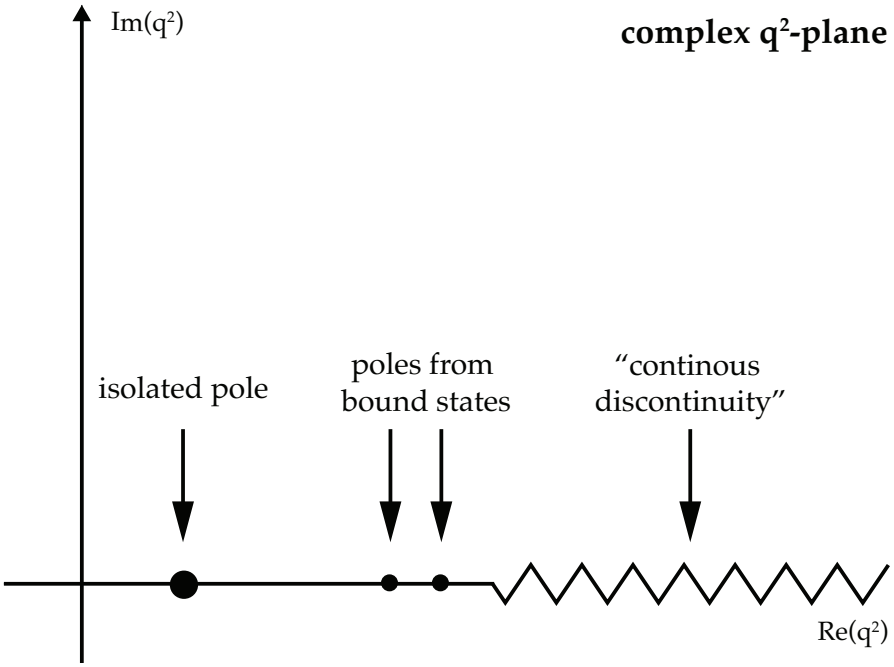


Figure 1.4: Analytic structure in the complex q^2-plane of the Fourier transform of the two-point function. The hadronic final states are responsible for poles appearing on the real-axis. The one-particle states contribute as isolated pole and the multi-particle states contribute as bound-states poles or a continues "discontinuity cut" [**Peskin1995**].

where the Noether current is given by

$$J_\mu(x) = \overline{q}(x)\Gamma q(x), \tag{1.2.27}$$

where Γ can be any of the following dirac matrices $\Gamma \in \{1, i\gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$, specifying the quantum number of the current (S: *scalar*, P: *pseudo-Scalar*, V: *vectorial*, A: *axial-vectorial*, respectively).

The vector correlator $\Pi_{\mu\nu}(q^2)$ can be Lorentz decomposed to a scalar function $\Pi(q^2)$. There are only two possible terms that can reproduce the second order tensor $q_\mu q_\nu$ and $q^2 g_{\mu\nu}$. The sum of both multiplied with two arbitrary functions $A(q^2)$ and $B(q^2)$ yields

$$\Pi_{\mu\nu}(q^2) = q_\mu q_\nu A(q^2) + q^2 g_{\mu\nu} B(q^2). \tag{1.2.28}$$

By making use of the *Ward-identity*

$$\begin{aligned} q^\mu \Pi_{\mu\nu} &= \int dx q^\mu e^{iqx} \langle 0|J_\mu(x) J_\nu(0)|0\rangle \\ &= -i \int dx i q^\mu e^{iq^\nu x_\nu} \langle 0|J_\mu(x) J_\nu(0)|0\rangle \\ &= i \int dx e^{iqx} \langle 0|\partial_\mu [J_\mu(x)] J_\nu(0)|0\rangle \\ &= 0, \quad \text{with} \quad \partial_\mu J_\mu(x) = 0, \end{aligned} \tag{1.2.29}$$

where we used $iq^\mu e^{iq^\nu x_\nu} = \partial_\mu e^{iq^\nu x_\nu}$ in the second and integration by parts in the third line. The Ward identity is dependent on the conserved Noether-current J_μ and thus only holds for same flavour quarks. With the Ward-identity we are able to demonstrate, that the two arbitrary functions are related

$$\begin{aligned} q^\mu q^\nu \Pi_{\mu\nu} = q^4 A(q^2) + q^4 B(q^2) = 0 \\ \implies A(q^2) = -B(q^2). \end{aligned} \tag{1.2.30}$$

Thus redefining $A(q^2) \equiv \Pi(q^2)$ we expressed the correlator as a scalar function

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2). \tag{1.2.31}$$

These discontinuities can be tackled with *Cauchy's theorem*, which we will apply in section 1.2.4.

Having dealt exclusively with the perturbative part of the theory, we have to discuss non-perturbative contributions, as QCD is known to have non-negligible contributions. Thus before continuing with the *Sum Rules* we need a final ingredient the *Operator Product Expansion* (OPE), which treats the non-perturbative contributions of our theory.

1.2.3 Operator Product Expansion

The OPE was introduced by Wilson in 1969 **[Wilson1969]**. The expansion states that non-local operators can be rewritten into a sum of composite local operators and their corresponding coefficients:

$$\lim_{x\to y} \mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_n C_n(x-y)\mathcal{O}_n(x), \tag{1.2.32}$$

where $C_n(x-y)$ are the so-called *Wilson-coefficients*.

The OPE lets us separate short-distance from long-distance effects. In PT we can only amount for short-distances, which are equal to height energies, where the strong-coupling α_s is small. Consequently the OPE decodes the long-distance effects in the higher dimensional operators.

The form of the composite operators are dictated by Gauge- and Lorentz symmetry. Thus we can only make use of operators of even dimension. The operators up to dimension six are given by **[Pascual1984]**

$$\begin{aligned}
&\text{Dimension 0:} && \mathbb{1}\\
&\text{Dimension 4:} && :m_i\overline{q}\,q:\\
&&& :G_a^{\mu\nu}(x)G^a_{\mu\nu}(x):\\
&\text{Dimension 6:} && :\overline{q}\,\Gamma q\overline{q}\,\Gamma q:\\
&&& :\overline{q}\,\Gamma\tfrac{\lambda^a}{2}q_\beta(x)\overline{q}\,\Gamma\tfrac{\lambda^a}{2}q:\\
&&& :m_i\overline{q}\,\tfrac{\lambda^a}{2}\sigma_{\mu\nu}qG_a^{\mu\nu}:\\
&&& :f_{abc}G_a^{\mu\nu}G_b^{\nu\delta}G_c^{\delta\mu}:,
\end{aligned} \tag{1.2.33}$$

where Γ stands for one of possible dirac matrices (as seen eq. 1.2.27). As all the operators appear normal ordered they vanish by definition in PT. Consequently they appear as *condensates* in NPT QCD like quark-condensate $\langle\overline{q}\,q\rangle$ or the gluon-condensate $\langle aGG\rangle$ (both of dimension four). These non-vanishing condensates characterise the QCD-vacuum.

As we work with dimensionless functions (e.g. the correlator Π) in Sum Rules, the r.h.s. of eq. 1.2.32 has to be dimensionless. Consequently the Wilson-coefficients have to cancel the dimension of the operator with their inverse mass dimension. To account for the dimensions we can make the inverse momenta explicit

$$\Pi^{OPE}_{V/A}(s) = \sum_{D=0,2,4\ldots} \frac{c^{(D)}\langle\mathcal{O}^{(D)}(x)\rangle}{-s^{D/2}}, \tag{1.2.34}$$

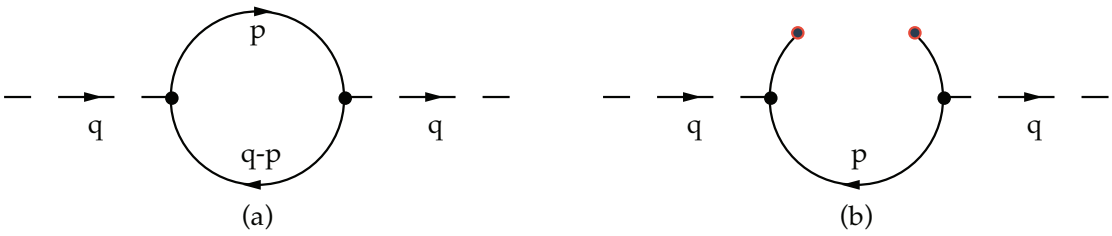


Figure 1.5: Feynman diagrams of the perturbative (a) and the quark-condensate (b) contribution. The upper part of the right diagram is not wick-contracted and responsible for the condensate.

where we used $C^{(D)} = c/(-s)^{D/2}$ with D being the dimension. Consequently the OPE should converge with increasing dimension for sufficiently large momenta s.

Let's show how the OPE contributions are calculated with a the "standard example" (following **[Pascual1984]**), where we compute the perturbative and quark-condensate Wilson-coefficients for the ρ-meson. For the ρ-meson, which is composed of u and d quarks, the current of eq. 1.2.26 takes the following form

$$j^{\mu}(x) = \frac{1}{2}\left(:[\overline{u}\,\gamma^{\mu}u](x) - \overline{d}\,\gamma^{\mu}d](x)\right). \tag{1.2.35}$$

In fig. 1.5 we draw the Feynman-diagram, from which we can take the uncontracted mathematical expression for the scalar correlator

$$\begin{aligned}\Pi(q^2) = -\frac{i}{4q^2(D-1)}\int d^D x e^{iqx}\langle\Omega|T\{:\overline{u}\,(x)\gamma^{\mu}u(x) - \overline{d}\,(x)\gamma^{\mu}d(x):\\ \times :\overline{u}\,(0\gamma_{\mu}u(0) - \overline{d}\,(0)\gamma_{\mu}d(0):\}\rangle.\end{aligned} \tag{1.2.36}$$

Using Wick's theorem we can contract all of the fields and calculate the first term of the OPE (𝟙), which represents the perturbative contribution of the OPE (𝟙)

$$\begin{aligned}\Pi(q^2) &= \frac{i}{4q^2(D-1)}(\gamma^{\mu})_{ij}(\gamma_{\mu})_{kl}\int d^D x e^{iqx}\\ &\times \quad \left[\overbrace{u_{j\alpha}(x)\overline{u}}_{k\beta}(0)\cdot\overbrace{u_{l\beta}(0)\overline{u}}_{i\alpha}(x) + (u\to d)\right]\\ &= \frac{3}{8\pi^2}\left[\frac{5}{3} - \log\left(-\frac{q^2}{\nu^2}\right)\right].\end{aligned} \tag{1.2.37}$$

To calculate the higher dimensional contributions of the OPE we use the same techniques as before, but leave some of the fields uncontracted. For the quark-

condensate, which we want to derive for tree-level, we leave two fields uncontracted

$$\begin{aligned}\Pi(q^2) = \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D x e^{iqx}\Big[\\ + \overbrace{u_{j\alpha}(x)\bar{u}}_{k\beta}(0)\cdot\langle\Omega|:\bar{u}_{i\alpha}(x)u_{l\beta}(0):|\Omega\rangle \\ + \overbrace{u_{l\beta}(0)\bar{u}}_{i\alpha}(x)\cdot\langle\Omega|:\bar{u}_{k\beta}(0)u_{j\alpha}(x):|\Omega\rangle + (u\to d)\Big].\end{aligned} \tag{1.2.38}$$

The non contracted fields can then be expanded in x

$$\begin{aligned}\langle\Omega|:\bar{q}(x)q(0):|\Omega\rangle = \langle\Omega|:\bar{q}(0)q(0):|\Omega\rangle \\ + \langle\Omega|:\left[\partial_\mu\bar{q}(0)\right]q(0):|\Omega\rangle x^\mu + \dots\end{aligned} \tag{1.2.39}$$

and redefined to a more elegant notation

$$\langle\bar{q}\,q\rangle \equiv \langle\Omega|:\bar{q}(0)q(0):|\Omega\rangle. \tag{1.2.40}$$

The finally result can be expressed as

$$\Pi_{(\rho)}(q^2) = \frac{1}{2}\frac{1}{(-q^2)^2}\left[m_u\langle\bar{u}\,u\rangle + m_d\langle\bar{d}\,d\rangle\right]. \tag{1.2.41}$$

The usage of the OPE and its validity is far from obvious. We are deriving the OPE from matching the Wilson-coefficients to Feynman-graph analyses. These Feynman-graphs are calculated perturbatively but the coefficients with dimension $D > 0$ correspond to NPT condensates!

Having gathered all of the necessary concepts we can close the gap between the theory and experiment in the last section of the introduction: QCDSR.

1.2.4 Sum Rules

To relate the measurable hadronic final states of a QCD process (e.g. τ-decays into hadrons) to a theoretical calculable QCDSR have been employed by Shifman in the late seventies [**Shifman1978**].

The sum rules are a combination of the two-point function, its analyticity, the OPE, a dispersion relation, the optical theorem and quark hadron duality.

The previously introduced two-point function eq. 1.2.26 is generally described by the OPE to account for NPT effects.

$$\Pi(q^2) = \Pi^{OPE}(q^2). \tag{1.2.42}$$

Furthermore it is related to the theoretical spectral function $\rho(s)$ via a dispersion relation (eq. 1.2.23). Using QCD we are computing interactions based on quarks and gluons, but as we have seen (confinement), we are only able to observe hadrons. Consequently to connect the theory to the experiment we have to assume **quark-hadron duality**[4], which implies that physical quantities can be described equally good in the hadronic or in the quark-gluon picture. Thus we rewrite the dispersion relation eq. 1.2.23 as

$$\Pi_{th}^{OPE}(q^2) = \int_0^\infty \frac{\rho_{exp}(q^2)}{(s - q^2 - i\epsilon)}, \tag{1.2.43}$$

where we connected the theoretical correlator Π_{th} with the experimental measurable spectral function ρ_{exp}.

We have seen, that the theoretical description of the correlator Π_{th} contains poles on the real axis. Unfortunately the experimental data ρ_{exp} is solely accessible on the positive real axis. As a result we have to make use of Cauchy's theorem to access the theoretical values of the two-point function close to the positive real axis (see section 1.2.4), which is given by

$$\int_{\mathcal{C}} f(z)\, dz = 0, \tag{1.2.44}$$

where $f(z)$ is an analytic function on a closed contour $\mathcal{C}$.

The final ingredient of the QCD sum rules is the *optical theorem*, relating experimental data with the imaginary part of the correlator (the spectral function $\rho(s)$).

In total, with the help Cauchy's theorem, the QCD sum rules can be summed up in the following expression

$$\frac{1}{\pi}\int_0^\infty \frac{\rho_{exp}(t)}{t - s}\, dt = \frac{1}{\pi}\oint_{\mathcal{C}} \frac{\operatorname{Im}\Pi_{OPE}(t)}{t - s}\, dt, \tag{1.2.45}$$

where the l.h.s. is given by the experiment and the r.h.s. can be theoretically evaluated by applying the OPE of the correlator $\Pi_{OPE}(s)$.

[4] Or simply duality.

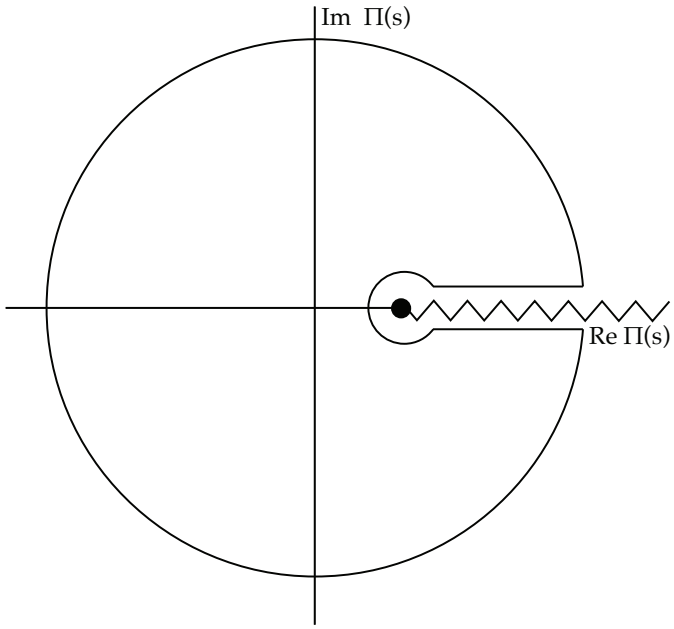


Figure 1.6: Analytical structure of $\Pi(s)$ with the used contour $\mathcal{C}$ for the final QCD Sum Rule expression eq. 1.2.45.