

# Constants

In [table 1.1](#) we collect all used constants that we have used in performing our fits.

| Quantity  | Value                         | Reference      |
|---|-------------------------------|----------------|
| $V_{ud}$  | $0.9742 \pm 0.00021$          | [PDG2018]      |
| $S_{EW}$  | $1.0198 \pm 0.0006$           | [Marciano1988] |
| $B_e$   | $17.815 \pm 0.023$            | ?              |
| $m_\tau$  | $1.776\,86(12000)\text{ MeV}$ | [PDG2018]      |
| $\langle aGG \rangle_I$                         | $0.012\text{ GeV}^2$          | [Shifman1978a] |
| $\langle \bar{q}_{u/d} q_{u/d} \rangle(m_\tau)$ | $-272(15)\text{ MeV}$         | [Jamin2002]    |
| $\bar{s}s/\langle \bar{q}q \rangle$             | $0.8 \pm 0.3$                 | [Jamin2002]    |

Table 1.1: Numerical values of used constants in our fitting routine.



## Coefficients

### 2.1 $\beta$ function

There are several conventions for defining the  $\beta$  coefficients, depending on a minus sign and/or a factor of two (if one substitutes  $\mu \rightarrow \mu^2$ ) in the  $\beta$ -function ?? . We follow the convention from Pascual and Tarrach (except for the minus sign) and have taken the values from [Boito2011]

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \quad (2.1.1)$$

$$\beta_2 = \frac{1}{12}(17N_c^2 - 5N_cN_f - 3C_fN_f), \quad (2.1.2)$$

$$\beta_3 = \frac{1}{32} \left( \frac{2857}{54}N_c^3 - \frac{1415}{54}N_c^2N_f + \frac{79}{54}N_cN_f^2 - \frac{205}{18}N_cC_fN_f + \frac{11}{9}C_fN_f^2 + C_f^2N_f \right), \quad (2.1.3)$$

$$\beta_4 = \frac{140599}{2304} + \frac{445}{16}\zeta_3, \quad (2.1.4)$$

where we used  $N_f = 3$  and  $N_c = 3$  for  $\beta_4$ .

## 2.2 Anomalous mass dimension

$$\gamma_1 = \frac{3}{2}C_f, \quad (2.2.1)$$

$$\gamma_2 = \frac{C_f}{48}(97N_c + 9C_f - 10N_f), \quad (2.2.2)$$

$$\gamma_3 = \frac{C_f}{32} \left[ \frac{11413}{108}N_c^2 - \frac{129}{4}N_cC_f - \left( \frac{278}{27} + 24\zeta_3 \right) N_cN_f + \frac{129}{2}C_f^2 - (23 - 24\zeta_3)C_fN_f - \frac{35}{27}N_f^2 \right], \quad (2.2.3)$$

$$\gamma_4 = \frac{2977517}{20736} - \frac{9295}{216}\zeta_3 + \frac{135}{8}\zeta_4 - \frac{125}{6}\zeta_5, \quad (2.2.4)$$

where  $N_c$  is the number of colours,  $N_f$  the number of flavours and  $C_f = (N_c^2 - 1)/2N_c$ . We fixed furthermore fixed  $N_f = 3$  and  $N_c = 3$  for  $\gamma_4$ .

## 2.3 Adler function

The the derivative of the two-point function can be expressed as the Adler function, which can be written in terms of the Adler function coefficients

$$D_V^{(1+0)} = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}. \quad (2.3.1)$$

The coefficients are partly dependent on each other via the RGE

$$-\mu \frac{d}{d\mu} D_V^{(1+0)} = \left( 2 \frac{\partial}{\partial L} + \beta \frac{\partial}{\partial a_s} \right) D_V^{(1+0)} = 0, \quad (2.3.2)$$

which implies that per order there exists only one coefficient we have to know to describe the Adler function. For completeness we will mention the necessary coefficients up to order  $n = 5$  here once again

$$\begin{aligned} c_{1,1} &= 1 \\ c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left( \frac{11}{12} - \frac{2}{3}\zeta_3 \right) N_f \\ c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 \\ &\quad - \left( \frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5 \right) N_f + \left( \frac{151}{162} - \frac{19}{27}\zeta_3 \right) N_f^2 \\ c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7. \end{aligned} \quad (2.3.3)$$

The rest of the coefficients are given by

$$\begin{aligned}
 c_{2,2} &= -\frac{1}{4}\beta_1 c_{1,1} \\
 c_{3,2} &= (-\beta_2 c_{1,1} - 2\beta_1 c_{2,1}), \quad c_{3,3} = \frac{1}{12}\beta_1^2 c_{1,1} \\
 c_{4,2} &= \frac{1}{4}(-\beta_3 c_{1,1} - 2\beta_2 c_{2,1} - 3\beta_1 c_{3,1}), \\
 c_{4,3} &= \frac{1}{24}(6c_{2,1}\beta_1^2 + 5\beta_2\beta_1 c_{1,1}), \quad c_{4,4} = -\frac{1}{32}\beta_1^3 c_{1,1} \\
 c_{5,2} &= \frac{1}{4}(-\beta_4 c_{1,1} - 2\beta_3 c_{2,1} - 3\beta_2 c_{3,1} - 4\beta_1 c_{4,1}), \\
 c_{5,3} &= \frac{1}{24}(12c_{3,1}\beta_1^2 + 6\beta_1\beta_3 c_{1,1} + 14\beta_2\beta_1 c_{2,1} + 3\beta_2^2 c_{1,1}), \\
 c_{5,4} &= \frac{1}{96}(-12\beta_1^3 c_{2,1} - 13\beta_2\beta_1^2 c_{1,1}), \quad c_{5,5} = \frac{1}{80}\beta_1^4 c_{1,1}
 \end{aligned} \tag{2.3.4}$$

and all related to the previous stated Adler function coefficients  $c_{n,1}$ .



# Chapter 3

## Program

### 3.1 C++

C++ was created starting in 1979 by Bjarne Stroustrup [**Stroustrup1986**]. As the name indicates<sup>1</sup> it is an extension of the C programming language, also referred to as “C with Classes”.

---

<sup>1</sup>++ is the increment operator in C