Chapter 1

Constants

In table 1.1 we collect all used constants that we have used in performing our fits.

Quantity	Value	Reference
V_{ud}	0.9742 ± 0.00021	?
S_{EW}	1.0198 ± 0.0006	?
B_e	17.815 ± 0.023	?
m_τ	1.77686 error?	?
$\langle aGG \rangle_I$	$0.012\,\mathrm{GeV}^2$	?
$\langle \overline{q}_{u/d} q_{u/d} \rangle_I$	$-0.020\,123\,648\,\mathrm{MeV}$	?
$\langle \overline{q}_s q_s \rangle_I$	$-0.016\,098\,918\,4\,\mathrm{MeV}$	?

Table 1.1: Numerical values of used constants in our fitting routine.

CHAPTER 2

Derivation of the used inverse covariance matrix from the Aleph data

While performing a **Generalized least squares** (GLS) we estimate our regression coefficients $\hat{\beta}$ as follows:

$$\hat{\beta} = \underset{b}{\operatorname{argmin}} (\mathbf{y} - \mathbf{X}\mathbf{b})^{T \cdot -1} (\mathbf{y} - \mathbf{X}\mathbf{b}), \tag{2.0.1}$$

with **b** being an candidate estimate of β, **X** being the design matrix, **y** being the response values and $^{\cdot -1}$ being the **inverse covariance matrix**.

The Aleph data includes the **standard error** (SE), which are equal to the **standard deviation** as per definition. Furthermore Aleph provides the **correlation coefficients** of the errors. We will use these two quantities in combination with **Gaussian error propagation** to derive derive an approximation of the covariance matrix.

2.1 Propagation of experimental errors and correlation

Let $\{f_k(x_1, x_2, \cdots x_n)\}$ be a set of m functions, which a linear combinations of n variables $x_1, x_2, \cdots x_n$ with combination coefficients $A_{k1}, A_{k2}, \cdots A_{kn}$, where

$k \in \{1, 2, \cdots, m\}$. Let the covariance matrix of x_n be denoted by

$$\Sigma^x = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.1.1}$$

Then the covariance matrix of the functions Σ^f is given by

$$\Sigma_{ij}^f = \sum_k^n \sum_l^n A_{ik} \sum_{kl}^x A_{jl}, \quad \Sigma^f = A\Sigma^x A^T. \tag{2.1.2}$$

In our case we are dealing with non-linear functions, which we will linearized with the help of the **Taylor expansion**

$$f_k \approx f_k^0 + \sum_i^n \frac{\partial f_k}{\partial x_i} x_i, \quad f \approx f^0 + Jx. \tag{2.1.3}$$

Therefore, the propagation of error follows from the linear case, replacing the Jacobian matrix with the combination coefficients ($J = A$)

Chapter 3

Coefficients

3.1 β function

There are several conventions for defining the β coefficients, depending on a minus sign and/or a factor of two (if one substitues $\mu \to \mu^2$) in the β-function **??**. We follow the convention from Pascual and Tarrach (except for the minus sign) and have taken the values from **[Boito2011]**

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \tag{3.1.1}$$

$$\beta_2 = \frac{1}{12}(17N_c^2 - 5N_cN_f - 3C_fN_f), \tag{3.1.2}$$

$$\beta_3 = \frac{1}{32}\left(\frac{2857}{54}N_c^3 - \frac{1415}{54}N_c^2N_f + \frac{79}{54}N_cN_f^2 - \frac{205}{18}N_cC_fN_f + \frac{11}{9}C_fN_f^2 + C_f^2N_f\right), \tag{3.1.3}$$

$$\beta_4 = \frac{140599}{2304} + \frac{445}{16}\zeta_3, \tag{3.1.4}$$

where we used $N_f = 3$ and $N_c = 3$ for β_4.

3.2 Anomalous mass dimension

$$\gamma_1 = \frac{3}{2} C_f, \tag{3.2.1}$$

$$\gamma_2 = \frac{C_f}{48}(97 N_c + 9 C_f - 10 N_f), \tag{3.2.2}$$

$$\gamma_3 = \frac{C_f}{32}\left[\frac{11413}{108} N_c^2 - \frac{129}{4} N_c C_f - \left(\frac{278}{27} + 24\zeta_3\right) N_c N_f + \frac{129}{2} C_f^2 - (23 - 24\zeta_3) C_f N_f - \frac{35}{27} N_f^2\right], \tag{3.2.3}$$

$$\gamma_4 = \frac{2977517}{20736} - \frac{9295}{216}\zeta_3 + \frac{135}{8}\zeta_4 - \frac{125}{6}\zeta_5, \tag{3.2.4}$$

where N_c is the number of colours, N_f the number of flavours and $C_f = (N_c^2 - 1)/2N_c$. We fixed furthermore fixed $N_f = 3$ and $N_c = 3$ for γ_4.

3.3 Adler function

Chapter 4

Källén-Lehmann spectral representation

In the second quantisation we applied latter operators $a_0^\dagger$ on the free vacuum $|0\rangle$ to obtain single particle states, carrying a momentum p

$$a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|\vec{p}\rangle, \tag{4.0.1}$$

where the factor $\sqrt{2E_p}$ is a convention resulting from the harmonic oscillator. The identity operator for one-particle state, which selects only single particle states is then given by

$$\mathbb{1} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} |\vec{p}\rangle\langle\vec{p}|. \tag{4.0.2}$$

The above complete set of one-particle states can be enhanced to include multiple-particle states, where we have to sum not only over all possible momentum states $|\vec{p}\rangle$, but over all possible multi-particle states and their enhanced states as well. If we define these state vectors as $\vec{X}$ we can express the complete set of one-and multiple-particle states as

$$\mathbb{1} = \sum\!\!\!\!\!\!\!\int_X d\prod_X |X\rangle\langle X|, \tag{4.0.3}$$

where we have to sum over all possible $|X\rangle$ single- or multi-particle states and integrate over the momentum $\vec{p}$ via

$$d\Pi_X \equiv \prod_{i\in X} \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_j}. \tag{4.0.4}$$

To get to the desired spectral decomposition we have to translate the Heisenberg operator $\phi(x)$ to its origin, like

$$\begin{aligned}\langle\Omega|\phi(x)|X\rangle &= \langle\Omega|e^{i\hat{P}x}e^{-i\hat{P}x}\phi(x)e^{i\hat{P}x}e^{-i\hat{P}x}|X\rangle \\ &= e^{-ip_X x}\langle\Omega|e^{-i\hat{P}x}\phi(x)e^{i\hat{P}x}|X\rangle \\ &= e^{-ip_X x}\langle\Omega|\phi(0)|X\rangle.\end{aligned} \tag{4.0.5}$$

Equally we can we can express $\phi(y)$ as

$$\langle X|\phi(y)|\Omega\rangle = e^{ipy}\langle\Omega|\phi(0)|\Omega\rangle. \tag{4.0.6}$$

With these expressions in hand we can spectral decompose the two-point function (**??**)

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|0\rangle =& \sum\!\!\!\!\!\!\int_X d\Pi_X\langle\Omega|\phi(x)|X\rangle\langle X|\phi(y)|\Omega\rangle \\ =& \sum\!\!\!\!\!\!\int_X d\Pi_X e^{-ip_X(x-y)}\left|\langle\Omega|\phi(0)|X\rangle\right|^2 \\ =& \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}(2\pi)^3\left[\sum\!\!\!\!\!\!\int_X d\Pi_X 2\pi\delta^{(4)}(p-p_X)\left|\langle\Omega|\phi(0)|X\rangle\right|^2\right],\end{aligned} \tag{4.0.7}$$

where the term in the bracket is Lorentz invariant, implying that it depends only on the Lorentz-invariant measure p^2. The multi-particle states $|X\rangle$ have to have positive energy eigenstates $p^0 > 0$. Consequently we can define the *spectral function* as

$$\theta(p^0)\rho(p^2) \equiv \sum\!\!\!\!\!\!\int_X d\Pi_X 2\pi\delta^{(4)}(p-p_X)\left|\langle\Omega|\phi(0)|X\rangle\right|^2, \tag{4.0.8}$$

with $\rho(p^2)$ being the *spectral density*. The simple two-point function (**??**) can then be rewritten to

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\theta(p^0)\rho(p^2) \\ &= \int_0^\infty dq^2\rho(q^2)D(x,y,q^2)\end{aligned} \tag{4.0.9}$$

with

$$\begin{aligned}D(x,y,m^2) &= \int\frac{d^3}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)} \\ &= \int\frac{d^4p}{(2\pi)^3}e^{-ip(x-y)}\theta(p_0)\delta(p^2-m^2).\end{aligned} \tag{4.0.10}$$

$\rho(p^2)$ *spectral density*, $\rho(p^2) > 0$. The spectral function is a non-negative function, describing physically measurable particle states. Until now we have given the spectral function for our "simple" two-point function (**??**), but we can generalise the decomposition to the *Fourier transform* (FT) of the *time-ordered two-point function*

$$\Pi(p^2) = i\int \frac{d^4x}{(2\pi)^4} e^{-ixp} \langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle, \tag{4.0.11}$$

where we made use of the translation invariance of the correlator[1]. The time-ordered scalar correlator can then be expressed as

$$\begin{aligned}
\Pi(p^2) &= i\int \frac{d^4x}{(2\pi)^4} e^{-ixp} \Big[\theta(x^0-0)\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle + \theta(y^0-0)\langle\Omega|T\{\phi(x)\phi(0)\}|\Omega\rangle\Big] \\
&= i\int \frac{d^4x}{(2\pi)^4} e^{-ixp} \int_0^\infty dq^2\, \rho(q^2) \Big[\theta(x^0-0)D(x,0,q^2) + \theta(y^0-0)D(0,x,q^2)\Big] \\
&= i\int \frac{d^4x}{(2\pi)^4} e^{-ixp} \int_0^\infty dq^2\, \rho(q^2) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-q^2-i\epsilon} e^{ipx}
\end{aligned} \tag{4.0.12}$$

by making use of the mathematical identity[2]

$$\theta(x^0-y^0)D(x,y,q^2) + \theta(y^0-x^0)D(y,x,q^2) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-q^2-i\epsilon} e^{ip(x-y)}. \tag{4.0.13}$$

The last line of is given by two cancelling Fourier transforms. Thus we recognise the Källén-Lehmann spectral decomposition of the scalar time-ordered two-point function

$$\Pi(p^2) = \int_0^\infty dq^2 \frac{i\rho(q^2)}{p^2-q^2-i\epsilon}. \tag{4.0.14}$$

Also notice that when applying the following mathematical identity

$$\mathrm{Im}\, \frac{1}{p^2-q^2+i\epsilon} = -\pi\delta(p^2-q^2) \tag{4.0.15}$$

that the spectral function can be given by the imaginary part of the correlator

$$\rho(p^2) = -\frac{1}{\pi}\,\mathrm{Im}\,\Pi(p^2). \tag{4.0.16}$$

The spectral function usually has a pole at one-particle states and a branch cut above the multi-particle state threshold. The branch cut is of high importance

[1] $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|e^{i\hat{P}y}\underbrace{e^{-i\hat{P}y}\phi(x)e^{i\hat{P}y}}_{x-y}\underbrace{e^{-i\hat{P}y}\phi(y)e^{i\hat{P}y}}_{=\phi(0)}e^{-i\hat{P}y}|\Omega\rangle = \langle\Omega|\phi(x-y)\phi(0)|\Omega\rangle$

[2] The identity should be known to the reader from the derivation of the Feynman propagator.

to us as the experimental detected spectral function is only accessible at the possible real axis, where the branch cut exists. We will further study these singularities with the following example of a toy-Lagrangian of two interacting scalar fields.

Chapter 5

Analytic Structure of the Spectral Function ϕ^4-theory Example

To analyse the singularities contained in the spectral function $\rho(s)$ we have a look at the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\partial_\mu\partial^\mu + M^2)\phi - \frac{1}{2}\pi(\partial_\mu\partial^\mu + M^2)\pi + \frac{\lambda}{2}\phi\pi^2. \tag{5.0.1}$$

The Lagrangian contains two fields ϕ and π, which interact via the interaction term $\lambda/2\phi\pi^2$. We now want to calculate the self-interaction of the ϕ-field, as shown in fig. 5.1, to first order. To do so we need an expression for the free propagator of the ϕ field

$$G_\phi = \frac{1}{p^2 - M^2 + i\epsilon}. \tag{5.0.2}$$

Furthermore we need the result of the Feynman loop-diagram given in fig. 5.1

$$\operatorname{Im}\mathcal{M}^{\text{Loop}} = \frac{\lambda^2}{32\pi}\sqrt{1 - 4\frac{m^2}{M^2}}\theta(M - 2m) \tag{5.0.3}$$

Summing over all possible self-energy graphs we get a geometric series

$$iG(p^2) = \frac{i}{p^2 - m_R^2 + \Sigma(p^2) + i\epsilon}, \tag{5.0.4}$$

Figure 5.1: Self energy

where we can plugin the free propagator and the result of the loop diagram of our toy-Lagrangian to get a typical spectral function

$$\begin{aligned}
\rho(q^2) &= -\frac{1}{\pi}\operatorname{Im}\Pi(q^2) \\
&= \frac{1}{\pi}\operatorname{Im}\left[p^2 - M^2 + i\epsilon + \frac{\lambda^2}{32\pi}\sqrt{1-4\frac{m^2}{M^2}}\theta(M-2m)\right]^{-1} \\
&= \delta(q^2-M^2) + \theta(q^2-4m^2)\frac{\lambda^2}{32\pi^2}\frac{1}{(q^2-M^2)^2}\sqrt{\frac{q^2-4m^2}{q^2}},
\end{aligned} \tag{5.0.5}$$

where we have used eq. 4.0.15.