Chapter 1

Introduction

In particle physics we are concerned about small objects and their interactions. Since the 1970 the dynamics of these tiny pieces are best described by the Standard Model (SM).

The SM contains two groups of fermionic, Spin $1/2$ particles. The former group, the Leptons consist of: the electron (e), the muon (μ), the tau (τ) and their corresponding neutrinos ν_e, ν_μ and ν_τ. The latter group, the Quarks contain: u, d (up and down, the so called light quarks), s (strange), c (charm), b (beauty or beauty) and t (top or truth). The SM furthermore differenciates between three fundamental forces (and its carriers): the electromagnetic (γ photon), weak (Z- or W-Boson) and strong (g gluon) interactions. The before mentioned Leptons solely interact through the electromagnetic and the weak force (also refered to as electroweak interaction), whereas the quarks additionally interact through the strong force.

The strong force is denominated Quantumchromodynamics (QCD). As the name suggest[1] the force is characterized by the color charge. Every quark has next to its type one of the three colors blue, red or green. The color force is mediated through eight gluons, which each being bi-colored[2], interact with quarks and each other. The strength of the strong force is given by the coupling constant α_s, which we will determine within this work. The strong coupling constant $\alpha_s(E)$ is a function of energy E and increases with decreasing energies [3]. This is exclusive for QCD and leads to *asymptotic freedom* and *confinement*. The former phenomen describes the decreasing strong force between quarks and gluons for high energies (short distances), which become asymptotically free at large energies. The latter expresses the fact, that no isolated quark has been found until today. Quarks appear confined as *Hadrons*, the so called *Mesons*[4] and *Baryons*[5]. As we measure *Hadrons* in our experi-

[1] Chromo is the greek word for color.
[2] Each gluon carries a color and an anti-color.
[3] In contrast to the electromagnetic force, where $\alpha(E)$ decreases!
[4] Composite of a quark and an anti-quark.
[5] Composite of three quarks or three anti-quarks.

ments but calculate with quarks within our theoretical QCD model we have to assume *Quark-Hadron Duality*, which states that QCD, which is the theory of quarks and gluons, is still valid for Hadrons for energies sufficently heigh. For lower energies there are measurable *Duality Violations* (DV), which will be commented within this work.

In the following (section 1.1) we will describe the τ-decays, which play an essential role in our QCD analysis. Then in section 1.2 we want to give more details of QCD, especially about the coupling constant $\alpha_s(s)$ (which is not constant at all) and the *QCD sum rules*.

1.1 τ-Decays

The principal input to our QCD analysis are measurements of τ-decays, which represent an excellent tool to access low energy QCD.

The τ-particle is an elementary particle with negative electric charge and a spin of 1/2. Together with the lighter electron and muon it forms the *charged Leptons*[6]. Even though it is an elementary particle it decays via the *weak interaction* with a lifetime of $\tau_\tau = 2.9 \times 10^{-13}\ s$ and has a mass of $1776.86(12)\ MeV$**[PDG2018]**. It is furthermore the only lepton massive enough to decay into Hadrons. The final states of a decay are limited by *conservation laws*. In case of a τ-decay they must conserve the electric charge (-1) and *invariant mass* of the system. Thus, as we can see from the corresponding Feynman diagram (see fig. 1.1)[7] the τ decays by the emission of a *W boson* and a tau-neutrino ν_τ into pairs of $(e^-, \bar{\nu}_e), (\mu^-, \bar{\nu}_\mu)$ or $(q, \bar{q})$.

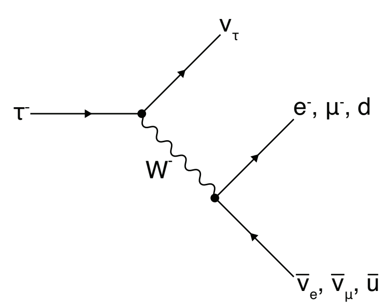


Figure 1.1: Feynman diagram of common decay of a τ-lepton into pairs of lepton-antineutrino or quark-antiquark by the emission of a *W boson*.

We are foremost interested into the hadronic decay channels, meaning τ-decays that have quarks in their final states. Unfortunately the quarks have never been measured isolated, but appear always in combination of *mesons*

[6] Leptons do not interact via the strong force.

[7] The τ-particle can also decay into strange quarks or charm quarks, but these decays are rather uncommon due to the heavy masses of s and c.

Name	Symbol	Quark content	Rest mass (MeV)
Pion	π^-	$\bar{u}d$	$139.570\,61(24)$ MeV
Pion	π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	$134.9770(5)$ MeV
Kaon	K^-	$\bar{u}s$	$493.677(16)$ MeV
Kaon	K^0	$d\bar{s}$	$497.611(13)$ MeV
Eta	η	$(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$	$547.862(17)$ MeV

Table 1.1: List of mesons produced by a τ-decay. Rare final states with branching Ratios smaller than 0.1 have been omitted. The list is taken from [**Davier2006**] with corresponding rest masses taken from [**PDG2018**].

Flavour	Mass	comment
u	$2.2^{+0.5}_{-0.4}$ MeV	$\overline{\text{MS}}$ at $\mu = 2$ GeV
d	$4.7^{+0.5}_{-0.3}$ MeV	$\overline{\text{MS}}$ at $\mu = 2$ GeV
s	95^{+9}_{-3} MeV	$\overline{\text{MS}}$ at $\mu = 2$ GeV
c	$1.275^{+0.025}_{-0.035}$ GeV	$\overline{\text{MS}}$ at $\mu = m_c$
b	$4.18^{+0.04}_{-0.03}$ GeV	$\overline{\text{MS}}$ at $\mu = m_b$
t	$173.0(40)$ GeV	direct observations of top events

Table 1.2: List of Quarks and their masses. Taken from [**PDG2018**].

and *baryons*. Due to its mass of $m_\tau \approx 1.8\,\text{GeV}$ the τ-particle decays into light mesons (pions-π, kaons-K, and eta-η, see table 1.1), which can be experimentally detected.

The hadronic $\tau - decay$ provides one of the most precise ways to determine the strong coupling [**Pich2016**] and can be calculated to high precision within the framework of QCD.

1.2 Quantumchromodynamics

QCD describe the strong interaction, which occur between *quarks* and are transmitted through *gluons*. A list of quarks can be found in table 1.2.

The QCD Lagrange density is similar to that of QED[**Jamin2006**],

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2} \bar{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A \bar{q}^A(x) a^A(x) \right], \tag{1.1}$$

where $q^A(x)$ represents the quark fields and $G^a_{\mu\nu}$ being the *gluon field strength tensor* given by:

$$G^a_{\mu\nu}(x) \equiv \partial_\mu B^a_\nu(x) - \partial^a_\nu(x) + g f^{abc} B^b_\mu(x) B^c_\nu(x), \tag{1.2}$$

where B^a_μ are the *gluon fields*, given in the *adjoint representation* of the SU(3) gauge group with f^{abc} as *structure constants*. Furthermore we have used

$A, B, \ldots = 0, \ldots 5$ as flavour indices, $a, b, \ldots = 0, \ldots, 8$ as color indices and $\mu, \nu, \ldots = 0, \ldots 3$ as lorentz indices.

1.2.1 Renormalisation Group

The perurbations of the QCD Lagrangian 1.1 lead to divergencies, which have to be *renormalized*. There are different aproaches to 'make' these divergencies finite. The most popular one is **dimensional regularisation**.

In *dimensional regularisation* we expand the four space-time dimensions to arbitrary dimensions. Consequently the in QCD calculations appearing *Feyman integrals* have to be continued to D-dimensions like

$$\mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 - m^2 + i0][(q-p)^2 = m^2 + i0]}, \tag{1.3}$$

where we introduced the scale parameter μ to account for the extra dimensions and conserve the mass dimension of the non continued integral.

In addition *physical quantities*[8] cannot depend on the renormalisation scale μ. Thus the derivative by μ of a general *physical quantity* $R(q, a_s, m)$ that depends on the external momentum q, the renormalised coupling $a_s \equiv \alpha_s/\pi$ and the renormalized quark mass m has to yield zero

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{d a_s}{d\mu} \frac{\partial}{\partial m} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m}\right] R(q, a_s, m) = 0. \tag{1.4}$$

eq. (1.4) is referred to as **renormalization group equation** and is the basis for defining the *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu \frac{d a_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \ldots \qquad \beta - \text{function} \tag{1.5}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m} \frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \ldots \qquad \text{anomalous mass dimension.} \tag{1.6}$$

Running gauge coupling

The β-function and the anomalous mass dimension are responsible for the running of the strong coupling and the running of the quark mass respektively. In this section we will shortly review the β-function and its implications on the strong coupling, whereas in the following section we will discuss the anomalous-mass dimension.

Regarding the β-function we notice, that $a_s(\mu)$ is not a constant, but *runs* by varying its scale μ. Lets observe the running of the strong coupling constant by integrating the β-function

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{d a_s}{\beta(a_s)} = -\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \log \frac{\mu_1}{\mu_2}. \tag{1.7}$$

[8] Observables that can be measured.

To analytically evaluate the above integral we can approximate the β-function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \tag{1.8}$$

yielding

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log\frac{\mu_1}{\mu_2}\right)}. \tag{1.9}$$

As we have three colours $N_c = 3$ and six flavours $N_f = 6$ the first β-function 1.5 is positive. Thus for $\mu_2 > \mu_1$ $a_s(\mu_2)$ decreases logarithmically and vanishes for $\mu_2 \to \infty$. This behaviour is known as *asymptotic freedom*. The coefficients of the β-function are currently known up to the 5th order and listed in the appendix 4.1.

Running quark mass

Not only the coupling but also the masses carry an energy dependencies, which is governed by the *anomalou mass dimension* $\gamma(a_s)$.

The properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* 1.6

$$\log\frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \tag{1.10}$$

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically **[Schwab2002]**

$$m(\mu_2) = m(\mu_1)\left(\frac{a(\mu_2)}{a(\mu_1)}\right)^{\frac{\gamma_1}{\beta_1}}\left(1 + \mathcal{O}(\beta_2, \gamma_2)\right). \tag{1.11}$$

As β_1 and γ_1 (see 4.2) are positive the quark mass decreases with increasing μ. The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1)\exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}\right) \tag{1.12}$$

and can be solved numerically to run the quark mass to the needed scale μ_2.

To theoretically describe the strong interaction regime of τ-decays we have to introduce the **QCD sum rules** for which we will devote the following section.

1.2.2 Two-Point function

The vacuum expectation value of the product of the conserved noether current $J_\mu(x)$ at different space-times points x and y is known as the **two-point function** (or simply **correlator**)

$$\Pi_{\mu\nu}(q^2) = \langle 0|J_\mu(x)J_\nu(y)|0\rangle, \tag{1.13}$$

where the noether current is given by

$$J_\mu(x) = \overline{q}(x)\Gamma q(y) \tag{1.14}$$

, where Γ stands for one of the dirac matrices $\Gamma \in \{1, i\gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$, specifying the quantum number of the current (S: *scalar*, P: *pseudo-Scalar*, V: *vectorial*, A: *axial-vectorial*, respectively).

The correlator tensor $\Pi_{\mu\nu}(q^2)$ can be lorentz decomposed to a scalar function $\Pi(q^2)$. There are only two possible terms that can reproduce the second order tensor $q_\mu q_\nu$ and $q^2 g_{\mu\nu}$. The sum of both multiplied with two arbitrary functions $A(q^2)$ and $B(q^2)$ yields

$$\Pi_{\mu\nu}(q^2) = q_\mu q_\nu A(q^2) + q^2 g_{\mu\nu} B(q^2). \tag{1.15}$$

By making use of the **Ward-identity [Peskin1995]**

$$q^\mu \Pi_{\mu\nu}(q^2) = q^\nu \Pi_{\mu\nu} = 0 \tag{1.16}$$

we can demonstrate, that the two arbitrary functions are related

$$\begin{aligned} q^\mu q^\nu \Pi_{\mu\nu} &= q^4 A(q^2) + q^4 B(q^2) = 0 \\ &\implies A(q^2) = -B(q^2). \end{aligned} \tag{1.17}$$

Thus redefining $A(q^2) \equiv \Pi(q^2)$ we expressed the correlator as a scalar function

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2). \tag{1.18}$$

The scalar QCD two point function can then be related to the spectrum of hadronic states. The correlator is then related to an integral over the **spectral function** $\rho(s)$ via the *Källén-Lehmann spectral representation* [**Kallen1952**, **Lehmann1954**], which is known since the early fities

$$\Pi(q^2) = \int_0^\infty ds \frac{\rho(s)}{s - q^2 - i\epsilon}. \tag{1.19}$$

Equation 1.19 is refered to as **dispersion relation** analogous to similar relations which arise for example in electrodynamics and defines the **spectral function** (a derivation can be found in [**Rafael1997**])

$$\rho(s) = \frac{1}{\pi} \operatorname{Im} \Pi(s). \tag{1.20}$$

Until know we connected theoretical correlators with the measurable hadronic spectrum. Nevertheless the analytic properties of the correlators have to be discussed as the function has discontinuities.

The main contribution from the spectral function given in eq. (1.19) are the hadronic final states

$$2\pi\rho(m^2) = \sum_n \langle 0|J_\mu(x)|n\rangle\langle n|J_\nu(y)\rangle(2\pi^2)^4\delta^{(4)}(p - p_n), \tag{1.21}$$

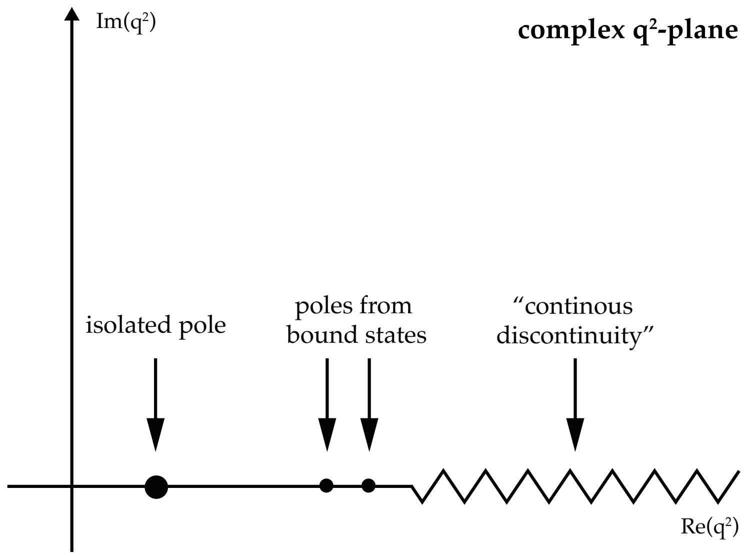


Figure 1.2: Analytic structure in the complex q^2-plane of the Fourier transform of the two-point function. The hadronic final states are responsible for poles appearing on the real-axis. The one-particle states contribute as isolated pole and the multi-particle states contribute as bound-states poles or a continues "discontinuity cut" [**Peskin1995**].

which lead to a series of continuous poles on the positive real axis for the two-point function, see Fig. 1.2. These discontinuities can be tackled with *Cauchy's theorem*, which we will apply in **??**.

Until now we exclusively dealt with the perturbative (PT) part of the theory, but QCD is known to have not negligible non-perturbative (NPT) contributions. Thus before continuing with the *Sum Rules* we need a final ingredient the operator product expansion, which implements NPT cotributions to our theory.

1.2.3 Operator Product Expansion

The **Operator Product Expansion** (OPE) was introduced by Wilson in 1969 [**Wilson1969**]. The expansion states that non-local operators can be rewritten into a sum of composite local operators and their corresponding coefficients:

$$\lim_{x \to y} \mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_n C_n(x-y)\mathcal{O}_n(x), \tag{1.22}$$

where $C_n(x-y)$ are the so-called *Wilson-coefficients*.

The OPE lets us separate *short-distance* from *long-distance* effects. In perturbation theory (PT) we can only amount for *short-distances*, which are equal to hight energies, where the strong-coupling α_s is small. Consequently the OPE decodes the long-distance effects in the higher dimensionsional operators.

The form of the composite operators are dictated by Gauge- and Lorentz symmetry. Thus we can only make use of operators of even dimension. The operators up to dimension six are given by **[Pascual1984]**

$$\begin{aligned}
&\text{Dimension 0:} && \mathbb{1} \\
&\text{Dimension 4:} && :m_i \overline{q}\, q: \\
& && :G_a^{\mu\nu}(x) G^a_{\mu\nu}(x): \\
&\text{Dimension 6:} && :\overline{q}\,\Gamma q \overline{q}\,\Gamma q: \\
& && :\overline{q}\,\Gamma \tfrac{\lambda^a}{2} q_\beta(x) \overline{q}\,\Gamma \tfrac{\lambda^a}{2} q: \\
& && :m_i \overline{q}\, \tfrac{\lambda^a}{2} \sigma_{\mu\nu} q G_a^{\mu\nu}: \\
& && :f_{abc} G_a^{\mu\nu} G_b^{\nu\delta} G_c^{\delta\mu}:,
\end{aligned} \tag{1.23}$$

where Γ stands for one of the dirac matrices $\Gamma \in \{1, i\gamma_5, \gamma^\mu, \gamma^\mu\gamma_5\}$, specifying the quantum number of the current (S, P, A, respectively). As all the operators appear normal ordered they vanish by definition in PT. Consequently they appear as **Condensates** in Non-perturbative (NPT) QCD like quark-condensate $\langle \overline{q}\, q\rangle$ or the gluon-condensate $\langle aGG\rangle$ (both of dimension four). These non-vanishing condensats characterize the QCD-vacuum.

As we work with dimensionless functions (e.g. Π) in Sum Rules, the r.h.s. of **??** has to be dimensionless. Consequently the Wilson-coefficients have to cancel the dimension of the operator with their inverse mass dimension. To account for the dimensions we can make the inverse momenta explicit

$$\Pi^{OPE}_{V/A}(s) = \sum_{D=0,2,4\dots} \frac{c^{(D)} \langle \mathcal{O}^{(D)}(x)\rangle}{-s^{D/2}}, \tag{1.24}$$

where we used $C^{(D)} = c/(-s)^{D/2}$ with D being the dimension. Consequently the OPE should converge with increasing dimension for suficienty large momenta s.

Let's show how the OPE contributions are calculated with a the "standard example" (following **[Pascual1986]**), where we compute the perturbative and quark-condensate Wilson-coefficients for the ρ-meson. For the ρ-meson, which is composed of u and d quarks, the current of eq. (1.13) takes the following form

$$j^\mu(x) = \frac{1}{2}\left(:[\overline{u}\,\gamma^\mu u](x) - \overline{d}\,\gamma^\mu d](x)\right). \tag{1.25}$$

In fig. 1.3 we draw the Feynman-diagram, from which we can take the uncontracted mathematical expression for the scalar correlator

$$\begin{aligned}\Pi(q^2) = &-\frac{i}{4q^2(D-1)} \int d^D x e^{iqx} \langle \Omega | T\{: \overline{u}\,(x)\gamma^\mu u(x) - \overline{d}\,(x)\gamma^{mu} d(x): \\ &\times : \overline{u}\,(0\gamma_\mu u(0) - \overline{d}\,(0)\gamma_\mu d(0):\}\rangle.\end{aligned} \tag{1.26}$$

Using Wick's theorem we can contract all of the fields and calculate the first term of the OPE ($\mathbb{1}$), which represents the perturbative contribution of the

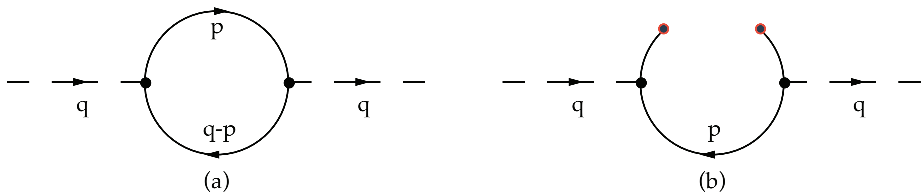


Figure 1.3: Feynman diagrams of the perturbative (a) and the quark-condensate (b) contribution. The upper part of the right diagram is not wick-contracted and responsible for the condensate.

OPE ($\mathbb{1}$)

$$\begin{aligned}\Pi(q^2) &= \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D\, x e^{iqx} \\ &\times \quad \left[u_{j\alpha}(x)\overline{u}_{k\beta}(0)\cdot u_{l\beta}(0)\overline{u}_{i\alpha}(x) + (u\to d)\right] \\ &= \frac{3}{8\pi^2}\left[\frac{5}{3} - \log\left(-\frac{q^2}{\nu^2}\right)\right].\end{aligned} \tag{1.27}$$

To calculate the higher dimensional contributions of the OPE we use the same techniques as before, but leave some of the fields uncontracted. For the quark-condensate, which we want to derive for tree-level, we leave two fields uncontracted

$$\begin{aligned}\Pi(q^2) = \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D\, x e^{iqx}\Big[\\ + u_{j\alpha}(x)\overline{u}_{k\beta}(0)\cdot\langle\Omega| : \overline{u}_{i\alpha}(x)u_{l\beta}(0) : |\Omega\rangle \\ + u_{l\beta}(0)\overline{u}_{i\alpha}(x)\cdot\langle\Omega| : \overline{u}_{k\beta}(0)u_{j\alpha}(x) : |\Omega\rangle + (u\to d)\Big].\end{aligned} \tag{1.28}$$

The non contracted fields can then be expanded in x

$$\begin{aligned}\langle\Omega| : \overline{q}\,(x)q(0) : |\Omega\rangle = \langle\Omega| : \overline{q}\,(0)q(0) : |\Omega\rangle \\ + \langle\Omega| : \left[\partial_\mu\overline{q}\,(0)\right]q(0) : |\Omega\rangle x^\mu + \ldots\end{aligned} \tag{1.29}$$

and redefined to a more elegant notation

$$\langle\overline{q}\,q\rangle \equiv \langle\Omega| : \overline{q}\,(0)q(0) : |\Omega\rangle. \tag{1.30}$$

The finally result can be taken from **[Pascual1984]** and yields

$$\Pi_{(\rho)}(q^2) = \frac{1}{2}\frac{1}{\left(-q^2\right)^2}\left[m_u\langle\overline{u}\,u + m_d\langle\overline{d}\rangle\right]. \tag{1.31}$$

The usage of the OPE and its validity is far from obvious. We are deriving the OPE from matching the Wilson-coefficients to Feynman-graph analyses. These Feynman-graphs are calculated perturbatively but the coefficients with dimension $D > 0$ correspond to NPT condensates!

Having gathered all of the necessary concepts we can close the gap between the theory and experiment in the last section of the introduction: QCD Sum Rules.

1.2.4 Sum Rules

To relate the measurable hadronic final states of a QCD process (e.g. τ-decays into Hadrons) to a theoretical calculable **QCD sum rules** have been empliyed by Shifman in the late sevent [**Shifman1978**].

The sum rules are a combination of the two-point function and its analyticity, the OPE, a dispersion relation, the optical theorem and quark hadron duality.

The previously introduced two-point function eq. (1.13) is generally descriped by the OPE to account for NPT effects.

$$\Pi(q^2) = \Pi^{OPE}(q^2). \tag{1.32}$$

Furthermore it is related to the theoretical spectral function $\rho(s)$ via a dispersion relation **??**eq:dispersionRelation). Using QCD we are computing interactions based on quarks and gluons, but due to confinement, we are only able to observe Hadrons. Consequently to connect the theory to the experiment we have to assume **quark-hadron duality**[9], which implies that physical quantities can be described equally good in the hadronic or in the quark-gluon picture. Thus we can rewrite the dispersion relation eq. (1.19) as

$$\Pi^{OPE}_{th}(q^2) = \int_0^\infty \frac{\rho_{exp}(q^2)}{(s - q^2 - i\epsilon)}, \tag{1.33}$$

where we connected the theoretical correlator Π_{th} with the experimental measurable spectral function ρ_{exp}.

We have seen that the theoretical description of the correlator Π_{th} contains poles on the real axis, but the experimental data ρ_{exp} is solely accesible on the positive real axis. Thus we have to make use of Cauchy's theorem to access the theoretical values of the two-point function close to the postive real axis (see section 1.2.4).

The final ingredient of the QCD sum rules is the *optical theorem*, relating experimental data with the imaginary part of the correlator (the spectral function $\rho(s)$).

In total, with the help Cauchy's theorem, the QCD sum rules can be sumed up in the following expression

$$\frac{1}{\pi}\int_0^\infty \frac{\rho_{exp}(t)}{t-s}\,dt = \frac{1}{\pi}\oint_{\mathcal{C}} \frac{\operatorname{Im}\Pi_{OPE}(t)}{t-s}\,dt, \tag{1.34}$$

[9] Or simply duality.

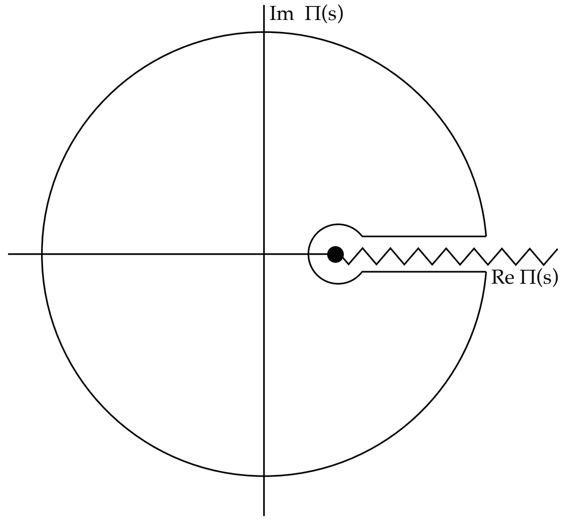


Figure 1.4: Analytical structure of $\Pi(s)$ with the used contour $\mathcal{C}$ for the final QCD Sum Rule expression eq. (1.34).

where the l.h.s. is given by the experiment and the r.h.s. can be theoretically evaluated with by applying the OPE of the correlator $\Pi_{OPE}(s)$.

Chapter 2

τ decays into hadrons

The τ-lepton is the only lepton heavy enough to decay into Hadrons. It permits one of the most precie determinations of the strong coupling α_s. The inclusive τ-decay ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{Hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} \tag{2.1}$$

can be precisly calculated and is sensitive to α_s. Due to the mass of the τ-lepton $m_\tau \approx 1.8\,\text{GeV}$ it is a good phenomen to perform low-energy QCD analysis. The theoretical expression of the hadronic τ-decay ratio was first derived by **[Tsai1971]** (using current algebra, a more recent derivation making use of the *optical theorem* can be taken from **[Schwab2002]**):

$$R_\tau = 12\pi \int_0^{m_\tau} = \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)\left[\left(1 + 2\frac{s}{m_\tau^2}\right)\operatorname{Im}\Pi^{(T)}(s) + \operatorname{Im}\Pi^{(L)}(s)\right]. \tag{2.2}$$

The τ-decay ratio depends on the two-point function

$$\Pi^{V/A}_{\mu\nu,ij}(s) \equiv i \int dx\, e^{ipx} \langle \Omega | T\{J^{V/A}_{\mu,ij}(x) J^{V/A}_{\nu,ij}(0)^\dagger\} | \Omega \rangle, \tag{2.3}$$

with $|\Omega\rangle$ being the physical vacuum. The vectorial and axial-vector currents are given by

$$J^V_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu q_i(x) \quad \text{and} \quad J^A_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu\gamma_5 q_i(x) \tag{2.4}$$

where i, j stand for the light quark flavours u, d and s.

The general correlator $\Pi^{\mu\nu}(q^2)$ can be decomposed into a vector/ axial-vector (V/A) and scalar/ pseudo-scalar (S/P) part containing a correction **[Broadhurst1975]**

$$\begin{aligned}\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{V,A}(q^2) + \frac{g^{\mu\nu}}{q^2}(m_i \mp m_j)\Pi^{S,P}(q^2) \\ + g^{\mu\nu}\frac{(m_i \mp m_j)}{q^2}[\langle \overline{q}_i q_i \rangle \mp \langle \overline{q}_j q_j \rangle],\end{aligned} \tag{2.5}$$

which is composed of a vector $\Pi^{V,A}$ and scalar $\Pi^{S,P}$ part. The third term are corrections arising due to the physical vacuum $|\Omega\rangle$.

The general correlator $\Pi^{\mu\nu}$ can also be decomposed into transversal and longitudinal components:

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T)}(q^2) + q^\mu q^\nu \Pi^{(L)}(q^2). \tag{2.6}$$

To relate the two different contributions we note, that only the scalar components of eq. (2.5) carry a mass term. Using the *Ward identity*

$$q_\mu \Pi^{\mu\nu}(q^2) = 0 \tag{2.7}$$

we can introduce two four-momenta into eq. (2.5)

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = (m_i \mp m_j)^2 \Pi^{S,P}(q^2) + (m_i \mp m_j)[\langle \bar{q}_i q_i\rangle \mp \langle \bar{q}_j q_j\rangle] \tag{2.8}$$

to relate the longitudinal of eq. (2.6)

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = q^4 \Pi^{(L)}(q^2) = s^2 \Pi^{(L)}(s), \tag{2.9}$$

where we defined $s \equiv q^2$. Thus

$$s^2 \Pi^{(L)}(s) = (m_i \mp m_j)^2 \Pi^{(S,P)}(s) + (m_i \mp m_j)[\langle \bar{q}_i q_i\rangle \mp \langle \bar{q}_j q_j\rangle]. \tag{2.10}$$

Furthermore we can relate the transversal and vectorial components via

$$\Pi^{\mu\nu}(s) = \underbrace{(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T)}(s) + (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(L)}(s)}_{=(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T+L)}(s)} + \frac{g^{\mu\nu} s^2}{q^2}\Pi^{(L)}(s) \tag{2.11}$$

where $\Pi^{(T+L)}(s) \equiv \Pi^{(T)}(s) + \Pi^{(L)}(s)$, such that

$$\Pi^{(V,A)}(s) = \Pi^{(T)}(s) + \Pi^{(L)} = \Pi^{(T+L)}. \tag{2.12}$$

Inspecting the inclusive ratio R_τ in eq. (2.1) introduces a problematic integral over the real axis of $\Pi(s)$ from 0 up to m_τ. The integral is problematic for two reasons:

- The *perturbative Quantum Chromodynamcs* (**pQCD**) and the OPE breaks down for low energies (over which we have to integrate).
- The positive euclidean axis of $\Pi(s)$ has a discontinuity cut and can theoretically not be evaluated.

To literally circunvent these issues we make use of *Cauchy's Theorem*

$$\int_{\mathcal{C}} f(z)\, dz = 0, \tag{2.13}$$

where $f(z)$ is an analytic function on a closed contour $\mathcal{C}$.

In our case we have to deal with the two-point correlator $\Pi(s)$, which is analytic except for the positive real axis (with which we will deal with to a later point[1]) Consequently, to rewrite we can rewrite the definite integral of eq. (2.2) into a contour integral over a closed circle with radius m_τ^2. The closed contour consists of four line integrals, which have been visualized in fig. 2.1. Summing over the four line integrals, performing a *analytic continuation* of the two-point correlator $\Pi(s) \to \Pi(s+i\epsilon)$ and finally taking the limit of $\epsilon \to 0$ gives us the needed relation between eq. (2.2) and the closed contour:

$$\begin{aligned}\oint_{s=m_\tau} \Pi(s) &= \int_0^{m_\tau} \Pi(s+i\epsilon) + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{m_\tau}^0 \Pi(s-i\epsilon)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\ &= \int_0^{m_\tau} \Pi(s+i\epsilon) - \Pi(s-i\epsilon)\,ds + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\ &= \int_0^{m_\tau} \Pi(s+i\epsilon) - \overline{\Pi(s+i\epsilon)} + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\ &\overset{\lim \epsilon\to 0}{=} 2i\int_0^{m_\tau} \operatorname{Im}\Pi(s)\,ds + \oint_{s=m_\tau} \Pi(s)\,ds\end{aligned} \tag{2.14}$$

where we made use of $\Pi(z) = \overline{\Pi(\overline{z})}$ (due to $\Pi(s)$ is analytic) and $\Pi(z) - \overline{\Pi(z)} = 2i\operatorname{Im}\Pi(z)$. The result can be rewritten in a more intuitive form, which we also visualized in fig. 2.1

$$\int_0^{m_\tau} \Pi(s)\,ds = \frac{i}{2}\oint_{s=m_\tau} \Pi(s)\,ds \tag{2.15}$$

Finally combining eq. (2.15) with eq. (2.2) we get

$$R_\tau = 6\pi i \oint_{s=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(T)}(s) + \Pi^{(L)}\right] \tag{2.16}$$

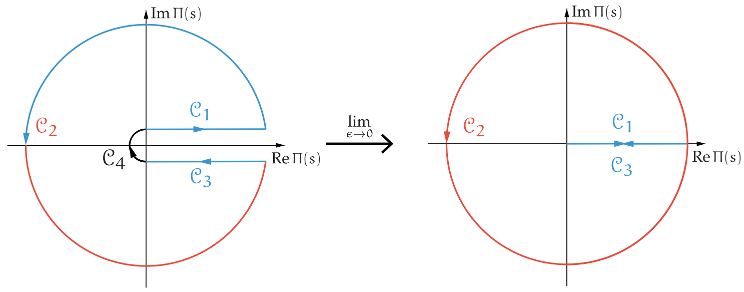
for the hadronic τ-decay ratio.

The contour integral obtained is an import result as we can now theoretically evaluate the hadronic τ-decay ratio sufficiently large energy scales ($m_\tau \approx 1.78\,\text{MeV}$) at which $\alpha_s(m_\tau) \approx 0.33$ **[Pich2016]** is tolerable heigh for applying perturbation theory and the OPE. Obviously we would benefit from a contour integral over a bigger circumference, but τ-decays are limited by the m_τ. Nevertheless there are promising e^+e^- annihilation data, which yields valuable R-ratio values up to $2\,\text{GeV}$ **[Boito2018][Keshavarzi2018]**.

It is convenient to rewrite the

$$\Pi^{(L+T)} = \Pi^{(L)} + \Pi^{(T)} \tag{2.17}$$

[1] To not evaluate $\Pi(s)$ at the positive real axis we have to introduce *pinched weights*. The *pinched weights* vanish for $s \to m_\tau$.



$$\int_0^{m_\tau} \operatorname{Im}\Pi(s)\,ds = \oint_{|s|=m_\tau} \Pi(s)\,ds$$

Figure 2.1: Visualization of the usage of Cauchy's theorem to transform eq. (2.2) into a closed contour integral over a circle of radius m_τ^2.

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)-\left(\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s)\right] \tag{2.18}$$

$$D^{(L+T)}(s) \equiv -s\frac{d}{ds}\Pi^{(L+T)}(s), \qquad D^{(L)}(s) \equiv \frac{s}{m_\tau^2}\frac{d}{ds}\left(s\Pi^{(L)}(s)\right) \tag{2.19}$$

Integration by parts

$$\int_a^b u(x)V(x)\,dx = \left[U(x)V(x)\right]_a^b - \int_a^b U(x)v(x)\,dx \tag{2.20}$$

$$\begin{aligned}
R_\tau^{(1)} &= \frac{6\pi i}{m_\tau^2}\oint_{|s|=m_\tau^2} \underbrace{\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)}_{=u(x)}\underbrace{\Pi^{(L+T)}(s)}_{=V(x)} \\
&= \frac{6\pi i}{m_\tau^2}\left\{\left[-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)\right]_{|s|=m_\tau^2}\right. \\
&\qquad \left. +\oint_{|s|=m_\tau^2}\underbrace{-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)}_{=U(x)}\underbrace{\frac{d}{ds}\Pi^{(L+T)}(s)}_{=v(x)}\right\} \\
&= -3\pi i\oint_{|s|=m_\tau^2 s}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\frac{d}{ds}D^{(L+T)}
\end{aligned} \tag{2.21}$$

where we fixed the integration constant to $C = -\frac{m_\tau^2}{2}$ in the second line and left the antiderivatives contained in the squared brackets untouched. Parametrizing the expression in the squared brackets

$$\left[-\frac{m_\tau^2}{2}\left(1-e^{-i\phi}\right)^3\left(1+e^{-i\phi}\right)\Pi^{(L+T)}(m_\tau^2 e^{-i\phi})\right]_0^{2\pi}=0 \tag{2.22}$$

where $s \to m_\tau^2 e^{-i\phi}$ and $(1-e^{-i\cdot 0})=(1-e^{-i\cdot 2\pi})=0$.

$$\begin{aligned} R_\tau^{(2)} &= \oint_{|s|=m_\tau^2} ds\left(1-\frac{s}{m_\tau^2}\right)^2\left(-\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s) \\ &= -4\pi i\oint\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 D^{(L)}(s) \end{aligned} \tag{2.23}$$

$$R_\tau=-\pi i\oint_{|s|=m_\tau^2}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left[3\left(1+\frac{s}{m_\tau^2}D^{(L+T)}(s)+4D^{(L)}(s)\right)\right] \tag{2.24}$$

$$R_\tau=-\pi i\oint_{|s|=m_\tau^2}\frac{dx}{x}(1-x)^3\left[3(1+x)D^{(L+T)}(m_\tau^2x)+4D^{(L)}(m_\tau^2x)\right], \tag{2.25}$$

where $x=s/m_\tau^2$.

$$R^{\omega}_{\tau,V/A}=\frac{N_c}{2}S_{EW}|V_{ud}|^2\left(1+\delta^{(0)}_\omega+\delta^{EW}_\omega+\delta^{DVs}_\omega+\sum_{D\leqslant 2}\delta^{(D)}_{ud,\omega}\right) \tag{2.26}$$

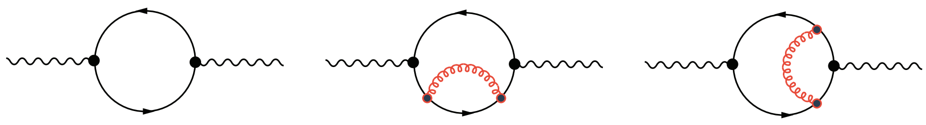
2.1 The perturbative expansion

We will treat the correlator in the chiral limit for which the longitudinal components $\Pi^L(s)$ vanish (see eq. (2.11)) and the axial and vectorial contributions are equal. Consequently [**Beneke2008**] we can write the vector correlation function $\Pi(s)$ as:

$$\Pi_V^{T+L}(s)=-\frac{N_c}{12\pi^2}\sum_{n=0}^{\infty}a_\mu^n\sum_{k=0}^{n+1}c_{n,k}L^k \quad \text{with} \quad L\equiv\ln\frac{-s}{\mu^2}. \tag{2.27}$$

The coeffiecient $c_{n,k}$ up to two-loop order can be obtained by Feynman-diagram calculations. add complete calculation E.g. we can compare the zero-loop result of the correlator [**Jamin2006**]

$$\Pi^B_{\mu\nu}(q^2)\Big|^{1-\text{loop}}=\frac{N_c}{12\pi^2}\left(\frac{1}{\hat\epsilon}-\log\frac{(-q^2-i0)}{\mu^2}+\frac{5}{3}+\mathcal{O}(\epsilon)\right) \tag{2.28}$$



with eq. (2.27) and extract the first two coefficients

$$c_{00} = -\frac{5}{3} \qquad \text{and} \qquad c_{01} = 1, \tag{2.29}$$

where $\Pi^{B}_{\mu\nu}(q^2)$ is not renormalized[2]

The second loop can also be calculated by diagram techniques resulting in **[Boito2011]**

$$\Pi_V^{(1+0)}(s)\Big|^{2-\text{loop}} = -\frac{N_c}{12\pi^2} a_\mu \log(\frac{-s}{\mu^2}) + \cdots \tag{2.30}$$

yielding $c_{11} = 1$.

Beginning from three loop diagrams the algebra becomes exausting and one has to use dedicated algorithms to compute the heigher loops. The third loop calculations have been done in the late seventies by [**Chetyrkin1979**, **Dine1979**, **Celmaster1979**]. The four loop evaluation have been completed a little more than ten years later by [**Gorishnii1990**, **Surguladze1990**]. The heighest loop published, that amounts to α_s^4, was published in 2008 [**Baikov2008**] almost 20 years later.

Fixing the number of colors to $N_c = 3$ the missing coefficients up to order four in α_s read:

$$\begin{aligned} c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f \\ c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 \\ &\quad - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2 \\ c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7, \end{aligned} \tag{2.31}$$

where used the flavour number $N_f = 3$ for the last line.

The 6-loop calculation has until today not been achieved, but Beneke und Jamin [**Beneke2008**] used and educated guess to estimate the coefficient

$$c_{5,1} \approx 283 \pm 283. \tag{2.32}$$

Until know we have mentioned the coefficients $c_{n,k}$ with a fixed $k = 1$. This is due to the RGE, which relates coefficients with a different k to the

[2] The term $1/\hat{\epsilon}$, which is of order 0 in α_s, will be cancelled by renormalization.

coefficients mentioned above. To make usage of the RGE $\Pi_V^{T+L}(s)$ needs to be a physical quantity, which can be achieved by rewriting eq. (2.19) to:

$$D_V^{(T+L)} = -s\frac{d\Pi_V^{(T+L)}(s)}{ds} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}, \tag{2.33}$$

where we used $dL^k/ds = k\ln(-s/\mu^2)^{k-1}(-1/\mu^2)$. D_V^{1+0} being a physical quantity has to fulfill the RGE eq. (1.4)

$$-\mu\frac{d}{d\mu}D_V^{(T+L)} = -\mu\frac{d}{d\mu}\left(\frac{\partial}{\partial L}dL + \frac{\partial}{\partial a_s}da_s\right)D_V^{T+L} = \left(2\frac{\partial}{\partial L} + \beta\frac{\partial}{\partial a_s}\right)D_V^{T+L} = 0, \tag{2.34}$$

where we defined the β-function in eq. (1.5) and used $dL/d\mu = -2/\mu$. The RGE puts constraints on the $c_{n,k}$-coefficients, ... not independent

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + a_\mu(c_{11} + 2c_{12}L) + a_\mu^2(c_{21} + 2c_{22}L + 3c_{23}L^2)\right] \tag{2.35}$$

inserting into RGE

$$4a_\mu c_{12} + 2a_\mu^2(2c_{22} + 6c_{23}L) + \beta_1 a_\mu^2(c_{11} + 2c_{12}L) + \mathcal{O}(a_\mu^3) = 0 \tag{2.36}$$

Thus

$$c_{12} = 0 \quad c_{22} = \frac{\beta_1 c_{11}}{4} \quad c_{23} = 0 \tag{2.37}$$

or $D(s)$ to the first order in α_s

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + c_{11}a_\mu\left(c_{21} - \frac{1}{2}\beta_1 c_{11}L\right)a_\mu^2\right] + \mathcal{O}(a_\mu^3) \tag{2.38}$$

2.1.1 Renormalisation group summation

We can express the perturbative contribution $\delta^{(0)}$ to R_τ (see eq. (2.26)) as

$$\delta^{(0)} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k c_{n,k} \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-x)^3(1+x)\log\left(\frac{-M_\tau^2 x}{\mu^2}\right)^{k-1}, \tag{2.39}$$

where we inserted the expansion of $D_V^{(T+L)}$ eq. (2.19) into R_τ eq. (2.25). Keep in mind that we are working in the chiral limit, such that $D^L = 0$ vanishes and the contributions from the vector and axialvector correlator are identical

$$D^{(T+L)} = D_V^{(T+L)} + D_A^{(T+L)} = 2D_V^{(T+L)}. \tag{2.40}$$

The perturbative contribution $\delta^{(0)}$ is a physical quantity and satisfies the homogeneous RGE, thus is independent on the scale μ. Consequently we have the freedom to choose μ, which leads to two main descriptions **fixed-order perturbation theory** (FOPT) and **contour-improved perturbation theory** (CIPT). The two resulting series should converge to equal values, but differ notably.

By using the FOPT prescription we fix $\mu^2 = m_\tau^2$ leading to

$$\delta^{(0)}_{FO} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k c_{n,k} J_{k-1} \tag{2.41}$$

where the contour integrals J_l are defined by

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log^l(-x). \tag{2.42}$$

The integrals J_l up to order α_s^4 are given by [**Beneke2008**]:

$$J_0 = 1, \quad J_1 = -\frac{19}{12} \quad J_2 = \frac{265}{72} - \frac{1}{3}\pi^2, \quad J_3 = -\frac{3355}{288} + \frac{19}{12}\pi^2. \tag{2.43}$$

Using FOPT the strong coupling $a(\mu)$, which runs with the scale μ, is fixed at $a(m_\tau^2)$ and can be taken out of the closed-contour integral. We still have to integrate over the logarithms $\log(-s/m_\tau^2)$.

Using CIPT we can sum the logarithms by setting the scale to $\mu^2 = -m_\tau^2 x$ in eq. (2.39), resulting in:

$$\delta^{(0)}_{CI} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2), \tag{2.44}$$

where the contour integrals J_l are defined by

$$J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-m_\tau^2 x). \tag{2.45}$$

All logarithms vanish except the ones for $k = 1$:

$$\log(1)^{k-1} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & k \neq 1 \end{cases} \tag{2.46}$$

which selectes adler function coefficients $c_{n,1}$ with a fixed $k = 1$. Handling the logarithms left us with the integration of $\alpha_s(-m_\tau^2 x)$ over the closed-contour $\oint_{|x|=1}$, which now depends on the integration variable x.

Calculating the perturbative contribution $\delta^{(0)}$ to R_τ for the two different prescriptions yields [**Beneke2008**]

$$\begin{aligned} &\qquad\quad \alpha_s^2 \qquad \alpha_s^2 \qquad \alpha_s^3 \qquad \alpha_s^4 \qquad\quad \alpha_s^5 \\ \delta^{(0)}_{FO} &= 0.1082 + 0.0609 + 0.0334 + 0.0174 (+0.0088) = 0.2200 (0.2288) \qquad (2.47) \\ \delta^{(0)}_{CI} &= 0.1479 + 0.0297 + 0.0122 + 0.0086 (+0.0038) = 0.1984 (0.2021). \qquad (2.48) \end{aligned}$$

The series indicate, that CIPT converges faster and that both series aproach a different value. This discrepancy represents currently the biggest theoretical uncertainity while extracting the strong coupling α_s.

As today we do not know which if FOPT or CIPT is the correct approach of measuring α_s. Therefore there are currently three ways of stating result:

- Quoting the average of both results.
- Quoting the CIPT result.
- Quoting the FOPT result.

We follow the approach of Beneke and Jamin [**Benke2008**] who prefere FOPT.

2.2 Non-Perturbative OPE Contribution

The perturbative contribution to the Sum-Rule, that we have seen so far, is the dominant one. With

$$\begin{aligned} R_\tau^{FOPT} &= \\ R_\tau^{CIPT} &= \end{aligned} \tag{2.49}$$

The NP vs perturbative contributions can be varied by choosen different weights than ω_τ.

2.2.1 Dimension four

For the OPE contributions of dimension four we have to take into account the terms with masses to the fourth power m^4, the quark condensate multiplied by a mass $m\langle \overline{q}\, q\rangle$ and the glucon condensate $\langle GG\rangle$. The resulting expression can be taken from the appendix of [**Pich1999**], yielding:

$$D_{ij}^{(L+T)}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_n \Omega^{(1+0)}(s/\mu^2)a^n, \tag{2.50}$$

where

$$\begin{aligned} \Omega_n^{(1+0)}(s/\mu^2) \;=\; & \frac{1}{6}\langle aGG\rangle p_n^{(L+T)}(s/\mu^2) + \sum_k m_k\langle \overline{q}_k q_k\rangle r_n^{(L+T)}(s/\mu^2) \\ & + 2\langle m_i\overline{q}_i q_i + m_j\overline{q}_j q_j\rangle q_n^{(L+T)}(s/\mu^2) \pm \frac{8}{3}\langle m_j\overline{q}_i q_i + m_i\overline{q}_j q_j\rangle t_n^{(L+T)} \\ & - \frac{3}{\pi^2}(m_i^4+m_j^4)h_n^{(L+T)}(s/\mu^2) \mp \frac{5}{\pi^2}m_i m_j(m_i^2+m_j^2)k_n^{(L+T)}(s/\mu^2) \\ & + \frac{3}{\pi^2}m_i^2 m_j^2 g_n^{(L+T)}(s/\mu^2) + \sum_k m_k^4 j_n^{(L+T)}(s/\mu^2) + 2\sum_{k\neq l} m_k^2 m_l^2 u_n^{(L+T)}(s/\mu^2) \end{aligned} \tag{2.51}$$

The perturbative expansion coefficients are known to $\mathcal{O}(a^2)$ for the condensate contributions,

$$\begin{array}{lll} p_0^{(L+T)} = 0, & p_1^{(L+T)} = 1, & p_2^{(L+T)} = \frac{7}{6}, \\ r_0^{(L+T)} = 0, & r_1^{(L+T)} = 0, & r_2^{(L+T)} = -\frac{5}{3}+\frac{8}{3}\zeta_3-\frac{2}{3}\log(s/\mu^2), \\ q_0^{(L+T)} = 1, & q_1^{(L+T)} = -1, & q_2^{(L+T)} = -\frac{131}{24}+\frac{9}{4}\log(s/\mu^2) \\ t_0^{(L+T)} = 0 & t_1^{(L+T)} = 1, & t_2^{(L+T)} = \frac{17}{2}+\frac{9}{2}\log(s/\mu^2). \end{array} \tag{2.52}$$

while the m^4 terms have been only computed to $\mathcal{O}(a)$

$$
\begin{aligned}
&h_0^{(L+T)} = 1 - 1/2\log(s/\mu^2), && h_1^{(L+T)} = \tfrac{25}{4} - 2\zeta_3 - \tfrac{25}{6}\log(s/\mu^2) - 2\log(s/\mu^2)^2,\\
&k_0^{(L+T)} = 0, && k_1^{(L+T)} = 1 - \tfrac{2}{5}\log(s/\mu^2),\\
&g_0^{(L+T)} = 1, && g_1^{(L+T)} = \tfrac{94}{9} - \tfrac{4}{3}\zeta_3 - 4\log(s/\mu^2),\\
&j_0^{(L+T)} = 0, && j_1^{(L+T)} = 0,\\
&u_0^{(L+T)} = 0, && u_2^{(L+T)} = 0.
\end{aligned}
\tag{2.53}
$$

2.2.2 Dimension six and eight

Our application of dimension six contributions is founded in [**Braaten1991**] and has previously been calculated beyond leading order by [**Lanin1986**]. The operators appearing are the masses to the power six m^6, the four-quark condensates $\langle \overline{q}\, q \overline{q}\, q\rangle$, the three-gluon condensates $\langle g^3 G^3\rangle$ and lower dimensional condensates multiplies by the corresponding masses, such that in total the mass dimension of the operator will be six. As there are too many parameters to be fitted with experimental data we have to omit some of them, starting with the three-gluon condensate, which does not contribute at leading order. The four-quark condensates known up to $\mathcal{O}(a^2)$, but we will make use of the *vacuum saturation approach* [**Beneke2008**, **Braaten1991**, **Shifman1978**] to express them in quark, anti-quark condensates $\langle q\overline{q}\,\rangle$. In our work we take the simplest approach possible: Introducing an effective dimension six coefficient $\rho^{(6)}_{V/A}$ divided by the appropiate power in s

$$
D^{(1+0)}_{ij,V/A}\Big|_{D=6} = 0.03\frac{\rho^{(6)}_{V/A}}{s^3}
\tag{2.54}
$$

As for the dimension eigth contribution the situation is not better than the dimension six one we keep the simplest approach, leading to

$$
D^{(1+0)}_{ij,V/A}\Big|_{D=8} = 0.04\frac{\rho^{(8)}_{V/A}}{s^4}.
\tag{2.55}
$$

Chapter 3

Derivation of the used inverse covariance matrix from the Aleph data

While performing a **Generalized least squares** (GLS) we estimate our regression coefficients $\hat{\beta}$ as follows:

$$\hat{\beta} = \underset{b}{\operatorname{argmin}} (\mathbf{y} - \mathbf{X}\mathbf{b})^T \mathbf{\Omega}^{-1} (\mathbf{y} - \mathbf{X}\mathbf{b}), \tag{3.1}$$

with $\mathbf{b}$ being an candidate estimate of β, $\mathbf{X}$ being the design matrix, $\mathbf{y}$ being the response values and $\mathbf{\Omega}^{-1}$ being the **inverse covariance matrix**.

The Aleph data includes the **standard error** (SE), which are equal to the **standard deviation** as per definition. Furthermore Aleph provides the **correlation coefficients** of the errors. We will use these two quantities in combination with **Gaussian error propagation** to derive derive an approximation of the covariance matrix.

3.1 Propagation of experimental errors and correlation

Let $\{f_k(x_1, x_2, \cdots x_n)\}$ be a set of m functions, which a linear combinations of n variables $x_1, x_2, \cdots x_n$ with combination coefficients $A_{k1}, A_{k2}, \cdots A_{kn}$, where $k \in \{1, 2, \cdots, m\}$. Let the covariance matrix of x_n be denoted by

$$\Sigma^x = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.2}$$

Then the covariance matrix of the functions Σ^f is given by

$$\Sigma^f_{ij} = \sum_k^n \sum_l^n A_{ik} \sum_{kl}^x A_{jl}, \quad \Sigma^f = A\Sigma^x A^T. \tag{3.3}$$

In our case we are dealing with non-linear functions, which we will linearized with the help of the **Taylor expansion**

$$f_k \approx f_k^0 + \sum_i^n \frac{\partial f_k}{\partial x_i} x_i, \quad f \approx f^0 + Jx. \tag{3.4}$$

Therefore, the propagation of error follows from the linear case, replacing the Jacobian matrix with the combination coefficients ($J = A$)

Chapter 4

Coefficients

4.1 β function

There are several conventions for defining the β coefficients, depending on a minus sign and/or a factor of two (if one substitues $\mu \to \mu^2$) in the β-function 1.5. We follow the convention from Pascual and Tarrach (except for the minus sign) and have taken the values from **[Boito2011]**

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f) \tag{4.1}$$

$$\beta_2 = \frac{1}{12}(17N_c^2 - 5N_cN_f - 3C_fN_f) \tag{4.2}$$

$$\beta_3 = \frac{1}{32}\left(\frac{2857}{54}N_c^3 - \frac{1415}{54}N_c^2N_f + \frac{79}{54}N_cN_f^2 - \frac{205}{18}N_cC_fN_f + \frac{11}{9}C_fN_f^2 + C_f^2N_f\right) \tag{4.3}$$

$$\beta_4 = \frac{140599}{2304} + \frac{445}{16}\zeta_3, \tag{4.4}$$

where we used $N_f = 6$ and $N_c = 3$ for β_4.

4.2 Anomalous mass dimension

4.3 Adler function