Chapter 1

τ decays into hadrons

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{Hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} \tag{1.1}$$

in terms of V/A, S/P [**Broadhurst1975**]

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{V,A}(q^2) + \frac{g^{\mu\nu}}{q^2}(m_i \mp m_j)\Pi^{S,P}(q^2)$$
$$+ g^{\mu\nu}\frac{(m_i \mp m_j)}{q^2}[\langle \overline{q}_i q_i \rangle \mp \langle \overline{q}_j q_j \rangle] \tag{1.2}$$

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = (m_i \mp m_j)^2 \Pi^{S,P}(q^2) + (m_i \mp m_j)[\langle \overline{q}_i q_i \rangle \mp \langle \overline{q}_j q_j \rangle] \tag{1.3}$$

in terms of T and L

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(T)}(q^2) + q^\mu q^\nu \Pi^{(L)}(q^2) \tag{1.4}$$

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = q^4 \Pi^{(L)}(q^2) = s^2 \Pi^{(L)}(s), \tag{1.5}$$

where $s \equiv q^2$

relation L and S,P

$$s^2 \Pi^{(L)}(s) = (m_i \mp m_j)^2 \Pi^{(S,P)}(s) + (m_i \mp m_j)[\langle \overline{q}_i q_i \rangle \mp \langle \overline{q}_j q_j \rangle] \tag{1.6}$$

need relation T and V,A

$$\Pi^{\mu\nu}(s) = \underbrace{(q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(T)}(s) + (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(L)}(s)}_{=(q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(T+L)}(s)} + \frac{g^{\mu\nu}s^2}{q^2}\Pi^{(L)}(s)$$

$$\tag{1.7}$$

where $\Pi^{(T+L)}(s) \equiv \Pi^{(T)}(s) + \Pi^{(L)}(s)$

$$\Pi^{(V,A)}(s) = \Pi^{(T)}(s) + \Pi^{(L)} = \Pi^{(T+L)} \tag{1.8}$$

$$q_\mu q_n u \tag{1.9}$$

The theoretical expression of the hadronic τ-decay ratio was first derived by [**Tsai1971**] (using current algebra, a more recent derivation making use of the *optical theorem* can be taken from [**Schwab2002**]):

$$R_\tau = 12\pi \int_0^{m_\tau} = \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)\left[\left(1 + 2\frac{s}{m_\tau^2}\right)\operatorname{Im}\Pi^{(T)}(s) + \operatorname{Im}\Pi^{(L)}\right]. \tag{1.10}$$

R_τ introduces a problematic integral over the real axis of $\Pi(s)$ from 0 up to m_τ. The integral is problematic for two reasons:

- The *perturbative Quantum Chromodynamcs* (**pQCD**) and the OPE breaks down for low energies (over which we have to integrate).

- The positive euclidean axis of $\Pi(s)$ has a discontinuity cut and can theoretically not be evaluated.

To literally circunvent these issues we make use of *Cauchy's Theorem*

$$\int_{\mathcal{C}} f(z)\,dz = 0, \tag{1.11}$$

where $f(z)$ is an analytic function on a closed contour $\mathcal{C}$.

In our case we have to deal with the two-point correlator $\Pi(s)$, which is analytic except for the positive real axis (with which we will deal with to a later point[1]) Consequently, to rewrite we can rewrite the definite integral of eq. (1.10) into a contour integral over a closed circle with radius m_τ^2. The closed contour consists of four line integrals, which have been visualized in fig. 1.1. Summing over the four line integrals, performing a *analytic continuation* of the two-point correlator $\Pi(s) \to \Pi(s + i\epsilon)$ and finally taking the limit of $\epsilon \to 0$ gives us the needed relation between eq. (1.10) and the closed contour:

$$\begin{aligned}
\oint_{s=m_\tau} \Pi(s) &= \int_0^{m_\tau} \Pi(s+i\epsilon) + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{m_\tau}^0 \Pi(s-i\epsilon)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\
&= \int_0^{m_\tau} \Pi(s+i\epsilon) - \Pi(s-i\epsilon)\,ds + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\
&= \int_0^{m_\tau} \Pi(s+i\epsilon) - \overline{\Pi(s+i\epsilon)} + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\
&\overset{\lim \epsilon \to 0}{=} 2i \int_0^{m_\tau} \operatorname{Im}\Pi(s)\,ds + \oint_{s=m_\tau} \Pi(s)\,ds
\end{aligned} \tag{1.12}$$

where we made use of $\Pi(z) = \overline{\Pi(\bar{z})}$ (due to $\Pi(s)$ is analytic) and $\Pi(z) - \overline{\Pi(z)} = 2i \operatorname{Im}\Pi(z)$. The result can be rewritten in a more intuitive form, which we also

---

[1]To not evaluate $\Pi(s)$ at the positive real axis we have to introduce *pinched weights*. The *pinched weights* vanish for $s \to m_\tau$.

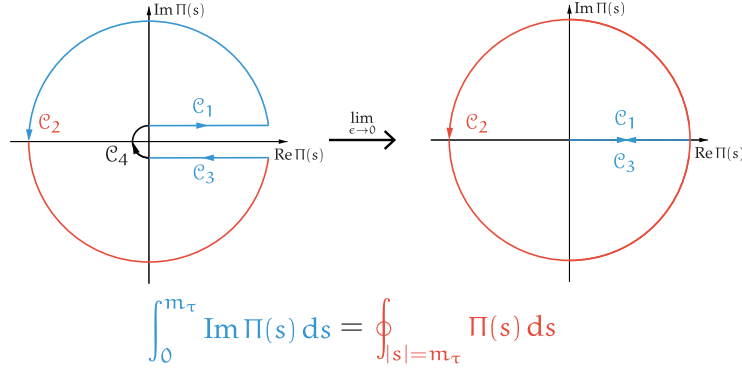$$\int_0^{m_\tau} \operatorname{Im}\Pi(s)\,ds = \oint_{|s|=m_\tau} \Pi(s)\,ds$$

Figure 1.1: Visualization of the usage of Cauchy's theorem to transform eq. (1.10) into a closed contour integral over a circle of radius m_τ^2.

visualized in fig. 1.1

$$\int_0^{m_\tau} \Pi(s)\,ds = \frac{i}{2}\oint_{s=m_\tau} \Pi(s)\,ds \tag{1.13}$$

Finally combining eq. (1.13) with eq. (1.10) we get

$$R_\tau = 6\pi i \oint_{s=m_\tau} \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)\left[\left(1 + 2\frac{s}{m_\tau^2}\right)\Pi^{(T)}(s) + \Pi^{(L)}(s)\right] \tag{1.14}$$

for the hadronic τ-decay ratio.

The contour integral obtained is an import result as we can now theoretically evaluate the hadronic τ-decay ratio sufficiently large energy scales ($m_\tau \approx 1.78\,MeV$) at which $\alpha_s(m_\tau) \approx 0.33$ [Pich2016] is tolerable heigh for applying perturbation theory and the OPE. Obviously we would benefit from a contour integral over a bigger circunference, but τ-decays are limited by the m_τ. Nevertheless there are promising e^+e^- annihilation data, which yields valuable R-ratio values up to $2\,GeV$ [Boito2018][Keshavarzi2018].

It is convenient to rewrite the

$$\Pi^{(L+T)} = \Pi^{(L)} + \Pi^{(T)} \tag{1.15}$$

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2\left[\left(1 + 2\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s) - \left(\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s)\right] \tag{1.16}$$

$$D^{(L+T)}(s) \equiv -s\frac{d}{ds}\Pi^{(L+T)}(s), \qquad D^{(L)}(s) \equiv \frac{s}{m_\tau^2}\frac{d}{ds}(s\Pi^{(L)}(s)) \tag{1.17}$$

Integration by parts

$$\int_a^b u(x)V(x)\,dx = \big[U(x)V(x)\big]_a^b - \int_a^b U(x)v(x)\,dx \tag{1.18}$$

$$R_\tau^{(1)} = \frac{6\pi i}{m_\tau^2} \oint_{|s|=m_\tau^2} \underbrace{\left(1-\frac{s}{m_\tau^2}\right)^2 \left(1+2\frac{s}{m_\tau^2}\right)}_{=u(x)} \underbrace{\Pi^{(L+T)}(s)}_{=V(x)}$$

$$= \frac{6\pi i}{m_\tau^2} \Bigg\{ \left[-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3 \left(1+\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)\right]_{|s|=m_\tau^2}$$

$$+ \oint_{|s|=m_\tau^2} \underbrace{-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3 \left(1+\frac{s}{m_\tau^2}\right)}_{=U(x)} \underbrace{\frac{d}{ds}\Pi^{(L+T)}(s)}_{=v(x)} \Bigg\} \tag{1.19}$$

$$= -3\pi i \oint_{|s|=m_\tau^2} \frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 \left(1+\frac{s}{m_\tau^2}\right)\frac{d}{ds}D^{(L+T)}$$

where we fixed the integration constant to $C = -\frac{m_\tau^2}{2}$ in the second line and left the antiderivatives contained in the squared brackets untouched. Parametrizing the expression in the squared brackets

$$\left[-\frac{m_\tau^2}{2}\left(1-e^{-i\phi}\right)^3\left(1+e^{-i\phi}\right)\Pi^{(L+T)}(m_\tau^2 e^{-i\phi})\right]_0^{2\pi} = 0 \tag{1.20}$$

where $s \to m_\tau^2 e^{-i\phi}$ and $(1-e^{-i\cdot 0}) = (1-e^{-i\cdot 2\pi}) = 0$.

$$R_\tau^{(2)} = \oint_{|s|=m_\tau^2} ds \left(1-\frac{s}{m_\tau^2}\right)^2 \left(-\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s)$$

$$= -4\pi i \oint \frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 D^{(L)}(s) \tag{1.21}$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{d}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 \left[3\left(1+\frac{s}{m_\tau^2}D^{(L+T)}(s)+4D^{(L)}(s)\right)\right] \tag{1.22}$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{d}{x}(1-x)^3 \left(3(1+x)D^{(L+T)}(s)+4D^{(L)}(s)\right), \tag{1.23}$$

where $x = s/m_\tau^2$.

4

1.1 The perturbative expansion

We will treat the correlator in the chiral limit for which the longitudinal components $\Pi^L(s)$ vanish (see eq. (1.7)) and the axial and vectorial contributions are equal. Consequently [**Beneke2008**] we can write the vector correlation function $\Pi(s)$ can be written as follows:

$$\Pi_V^{T+L} = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1} \quad \text{with} \quad L \equiv \ln\frac{-s}{\mu^2}. \tag{1.24}$$

$\Pi_V^{T+L}(s)$ is not a physical quantity but we can use eq. (1.17) to rewrite it into one:

$$D_V^{T+L} = -s\frac{d\Pi_V^{1+0}(s)}{ds} = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}, \tag{1.25}$$

where we used $dL^k/ds = k\ln(-s/\mu^2)^{k-1}(-1/\mu^2)$. D_V^{1+0} being a physical quantity has to fulfill the RGE **??**

$$-\mu\frac{d}{d\mu}D_V^{T+L} = -\mu\frac{d}{d\mu}\left(\tfrac{\partial}{\partial L}\,dL + \tfrac{\partial}{\partial a_s}\,da_s\right)D_V^{T+L} = \left(2\tfrac{\partial}{\partial L} + \beta\tfrac{\partial}{\partial a_s}\right)D_V^{T+L} = 0, \tag{1.26}$$

where we defined the β-function in **??** and used $dL/d\mu = -2/\mu$. The RGE puts constraints on the $c_{n,k}$-coefficients, ... not independent

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + a_\mu(c_{11} + 2c_{12}L) + a_\mu^2(c_{21} + 2c_{22}L + 3c_{23}L^2)\right] \tag{1.27}$$

inserting into RGE

$$4a_\mu c_{12} + 2a_\mu^2(2c_{22} + 6c_{23}L) + \beta_1 a_\mu^2(c_{11} + 2c_{12}L) + \mathcal{O}(a_\mu^3) = 0 \tag{1.28}$$

Thus

$$c_{12} = 0 \quad c_{22} = \frac{\beta_1 c_{11}}{4} \quad c_{23} = 0 \tag{1.29}$$

or $D(s)$ to the first order in α_s

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + c_{11}a_\mu\left(c_{21} - \frac{1}{2}\beta_1 c_{11}L\right)a_\mu^2\right] + \mathcal{O}(a_\mu^3) \tag{1.30}$$