Chapter 1

Introduction

In particle physics we are concerned about small objects and their interactions. Their dynamics are currently best described by the Standard Model (SM).

The SM contains two groups of fermionic, Spin 1/2 particles. The former group, the Leptons consist of: the electron (e), the muon (μ), the tau (τ) and their corresponding neutrinos ν_e, ν_μ and ν_τ. The latter group, the Quarks contain: u, d (up and down, the so called light quarks), s (strange), c (charm), b (beauty or beauty) and t (top or truth). The SM furthermore differenciates between three fundamental forces (and its carriers): the electromagnetic (γ photon), weak (Z- or W-Boson) and strong (g gluon) interactions. The before mentioned Leptons solely interact through the electromagnetic and the weak force (also refered to as electroweak interaction), whereas the quarks additionally interact through the strong force.

The strong force is also refered to as Quantumchromodynamics (QCD). As the name suggest[1] the force is characterized by the color charge. Every quark has next to its type one of the three colors blue, red or green. The color force is mediated through eight gluons, which each being bi-colored[2], interact with quarks and each other. The strength of the strong force is given by the coupling constant α_s. The coupling constants are a function of energy E and $\alpha_s(E)$ increases with energy[3]. This is exclusive for QCD and leads to *asymptotic freedom* an *confinement*. The former phenomen describes the decreasing strong force between quarks and gluons, which become asymptotically free at large energies. The latter expresses the fact, that no isolated quark has been found until today. Quarks appear confined as *Hadrons*, the so called *Mesons*[4] and *Baryons*[5]. As we measure *Hadrons* in our experiments but calculate with quarks within our theoretical QCD model we have to assume *Quark-Hadron Duality*, which states that QCD is still valid for Hadrons for energies suffi-

[1] Chromo is the greek word for color.
[2] Each gluon carries a color and an anti-color.
[3] In contrast to the electromagnetic force, where $\alpha(E)$ decreases!
[4] Composite of a quark and an anti-quark.
[5] Composite of three quarks or three anti-quarks.

Flavour	Mass	comment
u	$2.2^{+0.5}_{-0.4}$ MeV	$\overline{\text{MS}}$
d	$4.7^{+0.5}_{-0.3}$ MeV	
s	95^{+9}_{-3} MeV	
c	$1.275^{+0.025}_{-0.035}$ GeV	
b	$4.18^{+0.04}_{-0.03}$ GeV	
t	$173.0(40)$ GeV	

Table 1.1: List of Quarks and their masses[4].

cently heigh energies. There exist *Duality Violations* (DV), which will be investigated within this work.

1.1 τ-Decays

1.2 Quantumchromodynamics

QCD describes the strong interaction, which occur between *quarks* and are transmitted through *gluons*. A list of quarks can be found in 1.2.

The QCD Lagrange density is similar to that of QED[2],

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2} \bar{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A \bar{q}^A(x) a^A(x) \right], \tag{1.1}$$

where $q^A(x)$ represents the quark fields and $G^a_{\mu\nu}$ being the *gluon field strength tensor* given by:

$$G^a_{\mu\nu}(x) \equiv \partial_\mu B^a_\nu(x) - \partial^a_\nu(x) + g f^{abc} B^b_\mu(x) B^c_\nu(x), \tag{1.2}$$

where B^a_μ are the *gluon fields*, given in the *adjoint representation* of the SU(3) gauge group with f^{abc} as *structure constants*. Furthermore we have used $A, B, \ldots$ as flavour indeces, $a, b, \ldots$ as color indeces and $\mu, \nu, \ldots$ as lorentz indeces.

1.2.1 Renormalisation Group

The perurbations of the QCD Lagrangian 1.1 lead to divergencies, which have to be *renormalized*. There are different aproaches to 'make' these divergencies finite. The most popular one is *dimensional regularisation*. In *Dimensional regularisation* we expand the four space-time dimensions to arbitrary dimensions. Consequently the in QCD calculations appearing *Feyman integrals* have to be continued to D-dimensions like

$$\mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 - m^2 + i0][(q-p)^2 = m^2 + i0]}, \tag{1.3}$$

where we introduced the scale parameter μ to account for the extra dimensions and conserve the mass dimension of the non continued integral.

In addition *physical quantities*[6] cannot depend on the renormalisation scale μ. Thus examining a *physical quantity* $R(q, a_s, m)$ that depends on the external momentum q, the renormalised coupling $a_s = \alpha_s/\pi$ and the renormalized quark mass m

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \frac{\partial}{\partial\mu} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m}\right] R(q, a_s, m) = 0 \tag{1.4}$$

we can define the *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu \frac{d a_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \qquad \beta-\text{function} \tag{1.5}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m} \frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \dots \qquad \text{anomalous mass dimension.} \tag{1.6}$$

Running gauge coupling

Regarding the β-function we notice, that $a_s(\mu)$ is not a constant, but *runs* by varying the scale μ. Integrating the β-function yields

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta(a_s)} = -\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \log \frac{\mu_1}{\mu_2}. \tag{1.7}$$

To analytically evaluate the above integral we can approximate the β-function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \tag{1.8}$$

yielding

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log \frac{\mu_1}{\mu_2}\right)}. \tag{1.9}$$

As we have three colours $N_c = 3$ and six flavours $N_f = 6$ the first β-function 1.5 is positive. Thus for $\mu_2 > \mu_1$ $a_s(\mu_2)$ decreases logarithmically and vanishes for $\mu_2 \to \infty$. This behaviour is known as *asymptotic freedom*. The coefficients of the β-function are currently known up to the 5th order, which are displayed in the appendix 3.1.

Running quark mass

The properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* 1.6

$$\log \frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \tag{1.10}$$

[6]Observables that can be measured.

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically [3]

$$m(\mu_2) = m(\mu_1)\left(\frac{a(\mu_2)}{a(\mu_1)}\right)^{\frac{\gamma_1}{\beta_1}}\left(1+\mathcal{O}(\beta_2,\gamma_2)\right). \tag{1.11}$$

As β_1 and γ_1 (see 3.2) are positive the quark mass decreases with increasing μ. The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1)\exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}\right) \tag{1.12}$$

and can be solved numerically to run the quark mass to the needed scale μ_2.

Chapter 2

Derivation of the used inverse covariance matrix from the Aleph data

While performing a **Generalized least squares** (GLS) we estimate our regression coefficients $\hat{\beta}$ as follows:

$$\hat{\beta} = \underset{b}{\operatorname{argmin}}(\mathbf{y} - \mathbf{X}\mathbf{b})^T \mathbf{\Omega}^{-1}(\mathbf{y} - \mathbf{X}\mathbf{b}), \tag{2.1}$$

with **b** being an candidate estimate of β, **X** being the design matrix, **y** being the response values and $\mathbf{\Omega}^{-1}$ being the **inverse covariance matrix**.

The Aleph data includes the **standard error** (SE), which are equal to the **standard deviation** as per definition. Furthermore Aleph provides the **correlation coefficients** of the errors. We will use these two quantities in combination with **Gaussian error propagation** to derive derive an approximation of the covariance matrix.

2.1 Propagation of experimental errors and correlation

Let $\{f_k(x_1, x_2, \cdots x_n)\}$ be a set of m functions, which a linear combinations of n variables $x_1, x_2, \cdots x_n$ with combination coefficients $A_{k1}, A_{k2}, \cdots A_{kn}$, where $k \in \{1, 2, \cdots, m\}$. Let the covariance matrix of x_n be denoted by

$$\Sigma^x = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \cdots \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.2}$$

Then the covariance matrix of the functions Σ^f is given by

$$\Sigma^f_{ij} = \sum_k^n \sum_l^n A_{ik} \sum_{kl}^x A_{jl}, \quad \Sigma^f = A\Sigma^x A^T. \tag{2.3}$$

In our case we are dealing with non-linear functions, which we will linearized with the help of the **Taylor expansion**

$$f_k \approx f_k^0 + \sum_i^n \frac{\partial f_k}{\partial x_i} x_i, \quad f \approx f^0 + Jx. \tag{2.4}$$

Therefore, the propagation of error follows from the linear case, replacing the Jacobian matrix with the combination coefficients ($J = A$)

Chapter 3

Coefficients

3.1 β function

There are several conventions for defining the β coefficients, depending on a minus sign and/or a factor of two (if one substitues $\mu \to \mu^2$) in the β-function 1.5. We follow the convention from Pascual and Tarrach (except for the minus sign) and have taken the values from [1]

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f) \tag{3.1}$$

$$\beta_2 = \frac{1}{12}(17N_c^2 - 5N_c N_f - 3C_f N_f) \tag{3.2}$$

$$\beta_3 = \frac{1}{32}\left(\frac{2857}{54}N_c^3 - \frac{1415}{54}N_c^2 N_f + \frac{79}{54}N_c N_f^2 - \frac{205}{18}N_c C_f N_f + \frac{11}{9}C_f N_f^2 + C_f^2 N_f\right) \tag{3.3}$$

$$\beta_4 = \frac{140599}{2304} + \frac{445}{16}\zeta_3, \tag{3.4}$$

where we used $N_f = 6$ and $N_c = 3$ for β_4.

3.2 Anomalous mass dimension

3.3 Adler function

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