Chapter 1

Measuring the strong coupling

Table 1.1: Timeline

1991	[**Braaten1991**]: Systematic description, including NP corrections to extract α_s from R_τ.
1992	[**LeDiberder1992**]: Introducing weights and fit methodology later used by ALEPH [**Aleph1993**] and OPAL [**Opal1998**] collaborations
2011	[**Boito2011a**]: Include DV model to extract α_s from OPAL data.
2018	[**Boito2018**]: α_s from $e^+e^- \to$ hadrons up to 2 GeV.

The strong coupling has been measured since many years from hadronic τ decays. Until today most of the applied QCDSR to τ decays are based on the methodology developed in the early nineties by Braaten, Pich and Narison [**Braaten1991**]. They gathered the at this time available perturbative and NP contributions to extract the strong coupling from comparing their theoretical results to the known inclusive hadronic τ decay ratio R_τ. Pich together with Le Dibgerger then formulated the fitting strategy of fitting multiple moments of different weights to extract α_s parallel to Wilson coefficients of the OPE [**LeDiberger1992**], which later has been applied as standard in the ALEPH [**Aleph1993**] as well as the OPAL [**Opal1998**] detectors. For the next ten year years the methodology of extracting the strong coupling did not run through make major changes until in the year 2011 when Boito, Cata, Golterman, Jamin, Osborn and Peris [**Boito2011a**] applied a duality model to include known DV effects to the QCD analysis of τ decays. The group around Boito and Pich have different opin-

	Symbol	Term	Expansion	OPE contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	D6, D8
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	D6, D8, D10
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	D6, D8, D10, D12
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	D6
	ω_{M3}	$1-x^3$	$1-x^3$	D8
	ω_{M4}	$1-x^4$	$1-x^4$	D10
Pinched $+x$	ω_{X2}	$(1-x)^2$	$1-2x+x^2$	D4, D6
	ω_{X3}	$(1-x)^3$	$1-3x+3x^2-x^3$	D4, D6, D8
	ω_{X4}	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	D4, D6, D8, D10

Table 1.2: Displaying three categories of fits, each containing three weights with their corresponding mathematical expression and the OPE contributions the fitted integral momentum will be sensitive to.

ions on the importance of the newly introduced duality model [**Pich2016**, **Boito2016**] and consequently we want to deliver a third, opinion on the subject, favouring fits without the duality model. With new data becoming available from e^+e^- annihilation the extraction of α_s has recently been extended to analyses up to 2 GeV [**Boito2018**].

1.1 Fit Criteria

We are foremost interested in the extraction of α_s, but also want to analyse the importance of DV and higher order OPE corrections. We will reduce our analysis in the FOPT, but display our final results also in CIPT. Consequently to define a fit we have to choose a weight ω and momentum s_0. The only restriction from choosing a weight is, that the weight has to be analytic, leaving us with a variety of choices. For our strategy we have chosen three categories of weights, each of them containing fits with three different weights. A table with an overview of all used weights is given in table 1.2 To test for the stability of the fitted values we furthermore fit every weight for various momenta s_0.

The first category contains the *Pinched Weights without Monomial x*. As we do not want to implement a treatment of the DV we have to suppress them. This can be achieved with the help of pinched weights (see **??**). The higher the pinching the better the suppression of DV. In general it is said that DV are suppressed starting by double pinched weights, like the kinematic weight ω_τ. To probe the stability of the strong coupling and additionally analyse the higher order OPE contributions we also included a cubic ω_{cube} and quartic ω_{quartic} pinched weight, which do not carry a monomial term x. An x term would make the fits sensitive to the $D = 4$ OPE contribution, which has been shown to be problematic [**Beneke2012**] why?

1.2 Fits

1.2.1 Pinched Weights without Monomial x

Kinematic weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

We have already encountered the kinematic weight in **??**. It is a polynomial weight function, defined as $\omega_\tau(x) = (1-x)^2(1+2x)$, double pinched, contains the unity and does not contain a term proportional to x, which makes it an optimal weight [**Beneke2012**]. As a doubled pinched weight it should have a good suppression of DV contributions and its polynomial contains terms proportional to x^2 and x^3, which makes it sensitive to the dimension six and eight OPE contributions. The fits have been performed within the framework of FOPT for different numbers of s_0. The momentum sets are characterised by its lowest energy s_{min}. We fitted values down to $1.5\,\text{GeV}$. Going to lower energies is questionable due to the coupling constant becoming to large, which implies a breakdown of PT and appearing DVs. Furthermore it bares the risk to be affected by the $\rho(770)$ and a_1 peaks in the vector and axial-vector spectral function, which we cannot model within the framework of the OPE. For the fitting-parameters α_s, c_6 and c_8 we have given the results in table 1.3 and graphically in fig. 1.1. We only display the fits for s_{min} larger than $2.1\,\text{GeV}$ as fits with higher s_{min} have either a too large χ^2 (larger than two) or have been excluded by us because the χ^2 value jumps from 0.19 to 1.3 between the $s_{min} = 2.1$ and $s_{min=2.2}$ fits. We experienced this behaviour with various of our

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_6	c_8	χ^2/dof
2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

Table 1.3: Table of our fitting values of $\alpha_s(m_\tau^2), c_6$ and c_8 for the kinematic weight $\omega(x) = (1-x)^2(1+2x)$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

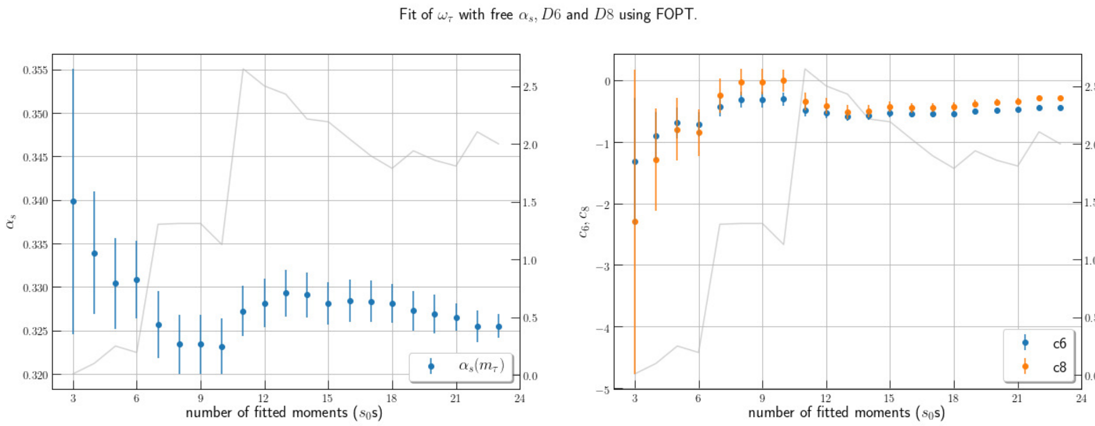


Figure 1.1: Fitting values of $\alpha_s(m_\tau^2), c_6$ and c_8 for the kinematic weight $\omega(x) = (1-x)^2(1+2x)$ using FOPT for different s_{min}. The left graph plots $\alpha_s(m_\tau^2)$ for different numbers of used s_0s. The right plot contains the dimension six and eight contributions to the OPE. Both plots have in grey the χ^2 per DOF.

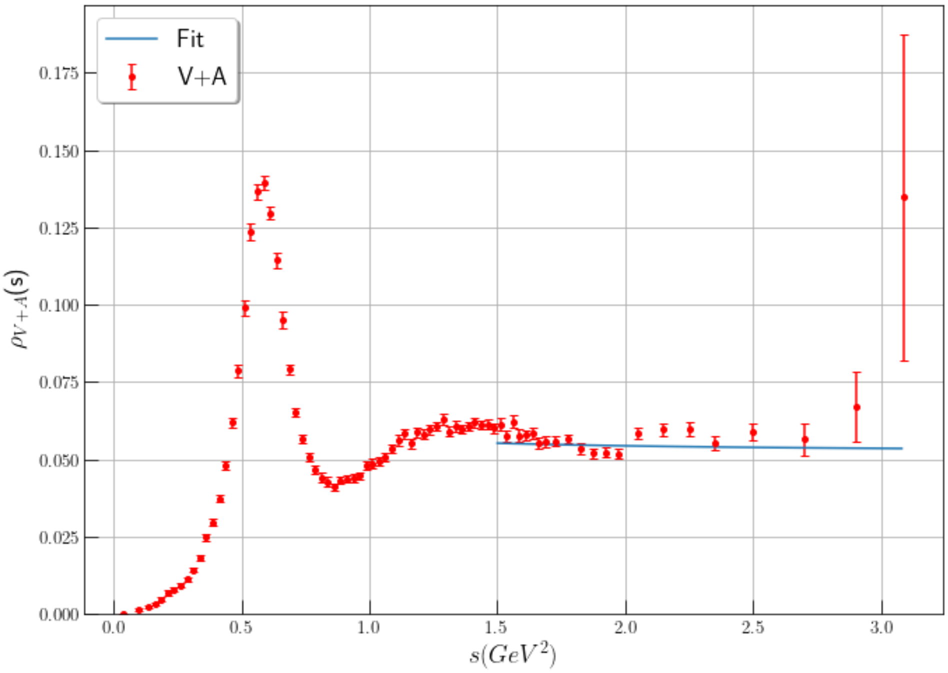


Figure 1.2: test

fits, which will be discussed later. Consequently we achieved achieved four good fits with a χ^2 per DOF less or close to one, which we divided into two groups:

- Fits with **5-7** momenta have χ^2 per DOF larger than one and means of $\alpha(m_\tau^2) = 0.3317(33), c_6 = -0.77(17)$ and $c_8 = -0.98(35)$, where we propagated the uncertainty. We have excluded the momentum containing four s_0s, because it χ^2 is too low and its errors are too large, which is because we have to fix three variables for only four data points.

- Fits with **8-10** momenta have small χ^2 per DOF values and lower means for the strong coupling $\alpha(m_\tau^2) = 0.3241(20)$ but the OPE contributions are higher $c_6 = -0.350(75)$ and $c_8 = -0.09(12)$.

The values for the less momenta are preferred by us due to two reasons. First below energies of 2.2 GeV we have to face the problematic influence of increasing resonances. Second, we will see, that the values obtained from the lower

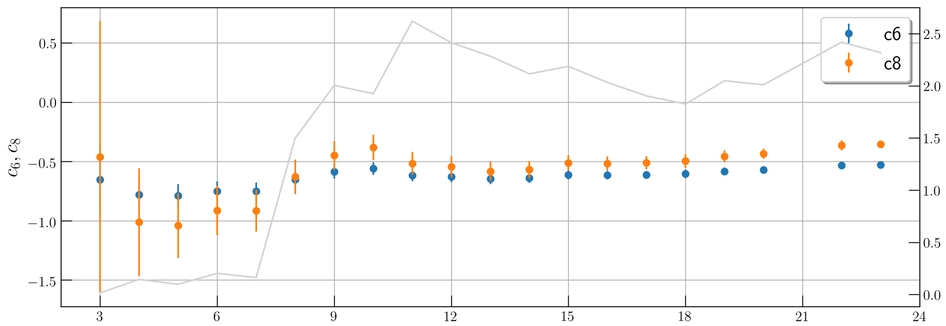


Figure 1.3

moment fits are more compatible with our other fits series. For both, the momenta sets, we see a good convergence of the OPE.

We further tested the stability of the dimension six and eight contributions to the OPE within the same fit series but for a fixed value of the strong coupling to our previous averaged result $\alpha_s(m_\tau^2) = 0.3179$. The fits have been plotted in fig. 1.3 and show good stability. The values for c_6 and c_8 are larger than the values given in our final results from table 1.3. This is explained with a smaller contribution from the strong coupling (α_s is smaller), which has to be compensated by larger OPE contributions.

Due to the good results we will try to argue in favour of the values obtained by the lower momenta:

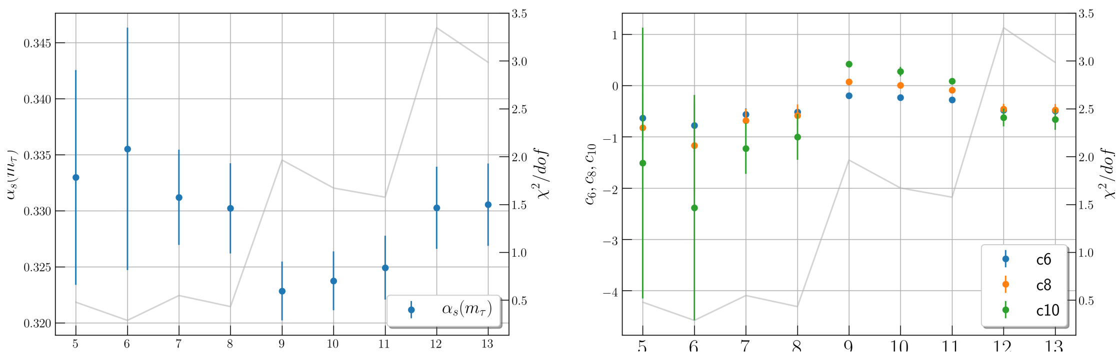
$$\alpha_s(m_\tau^2) = 0.3317(33), \quad c_6 = -0.77(17) \quad \text{and} \quad c_8 = -0.98(35). \tag{1.2.1}$$

1.2.2 Cubic weight: $\omega_{\text{cube}}(x) \equiv (1-x)^3(1+3x)$

To further consolidate the results from the kinematic weight, we test a weight of higher pinching, which is known to suppress DV more than a double pinched weight would do. Consequently, any differences to the previous fit could indicate a problem with the DV treatment. Our *cubic* weight will be triple pinched and optimal, as the kinematic weight is double pinched and we do not want any problematic contributions proportional to x. Thus we define the *cubic weight* as $\omega_{\text{cube}}(x) \equiv (1-x)^3(1+3x)$. It is due to its polynomial structure sensitive to the dimensions six, eight and ten contributions of the OPE, which

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_6	c_8	c_{10}	χ^2/dof
1.900	11	0.3249(29)	-0.280(20)	-0.088(21)	0.088(55)	1.58
1.950	10	0.3237(26)	-0.232(25)	0.005(42)	0.275(93)	1.67
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

Table 1.4: Table of our fitting values of $\alpha_s(m_\tau^2), c_6$, c_8 and c_{10} for the cubic weight $\omega(x) = (1-x)^3(1+3x)$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.



yields one more parameter to fit than with the kinematic weight ω_τ. The some good, selected fits, by χ^2 per DOF, can be seen in table 1.4 and graphically in section 1.2.2. As with the kinematic weight we get two different sets of value:

- The fits with **9, 10 and 11** momenta have a too high χ^2, but are comparable to our the upper entries of the kinematic weight table table 1.3. As with the kinematic weight the s_{min} seems to be affected by lower resonances.

- The fits with **5,6,7 and 8** have a better χ^2 per DOF value and are in good agreement with the corresponding fits of the kinematic weight. The averaged value with its propagated errors read: $\alpha_s(m_\tau^2) = 0.332478(61), c_6 =$

$-0.622(12), c_8 = -0.815(55)$ and $c_{10} = -1.5(3.1)$.

We furthermore found that the OPE is converging, but not as good as for the kinematic weight. The values of $\left|\delta^{(8)}\right|$ is only half as large as $\left|\delta^{(8)}\right|$. The values of the lower momentum count are in high agreement with the ones obtained from the kinematic weight. The conclusions that we take from the *cubic weight* are that the kinematic weight, with its double pinching, should sufficiently suppress any contributions from DVs. If DV would have an effect on the kinematic weight, we should have seen an improvement of the fits with the *cubic* weight, due to its triple pinching, which is not the case.

1.2.3 Quartic weight: ω(x) $\equiv$(1-x)$^4(1+4x)$

To include an even higher pinching of four and to compare the previously obtained value for the dimension ten OPE contribution we performed fits with the *quartic weight* defined as ω(x) $\equiv$(1-x)$^4(1+4x)$, which also fulfils the definition of an optimal weight [**Beneke2012**]. Unfortunately the fits only converged for $s_{min} = 2\,\mathrm{GeV}$ (nine s_0s moment combination). The results for , with a χ^2 per DOF of 0.67 are given by:

$$\begin{aligned} \alpha_s(m_\tau^2) = 0.3290(11), \quad c_6 = -0.3030(46), \quad c_8 = -0.1874(28), \\ c_{10} = 0.3678(45) \quad \text{and} \quad c_{12} = -0.4071(77) \end{aligned} \tag{1.2.2}$$

Due to the problematic of the fitting routing, which is caused by too many OPE contributions fitted simultaneously, we will discard the fitting results for the quartic weight.

1.2.4 Monomial Weights

$$1 - x^2$$

$$1 - x^3$$

$$1 - x^4$$

1.2.5 Pinched Weights with monomial x

$$(1 - x)^2$$

$$(1 - x)^3$$

$$(1 - x)^4$$

1.3 Results

In the following we will perform fits to determine α_s at the m_τ^2-scale. The fits are separated corresponding to the used weight. Every weight contains multiple fits for different s_0-momenta. We will start with the kinematic weight, which appears naturally in the inclusive τ-decay ratio **??** and has the best fitting characteristics of all weights we have used.

1.3.1 Third power monomial: $\omega_{m3}(x) \equiv 1 - x^3$

To study the behaviour of the DV and the higher order OPE contributions of dimension eight and ten we further included two optimal, single pinched weights. The first one is defined as $\omega_{m3}(x) \equiv 1 - x^3$ and contains a single third power monomial and is consequently sensitive to dimension eight contributions from the OPE. Our fitting results can be taken from table 1.5. The χ^2 per DOF is like in the ω_τ and ω_{cubic} fits good for $s_{min} \leqslant 2.2\,\mathrm{GeV}$, but jumps to values $\chi^2/\mathrm{dof} > 1.4$ for smaller s_{min}. This is, as before, explained through resonances that appear in lower energies. Due to the good χ^2 and the internally compatible fitting values we averaged over all rows except the last one of table 1.5. The last row, at $s_{min} = 2.8\,\mathrm{GeV}$ has only one DOF and thus high

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_8	χ^2/dof
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Table 1.5: Table of our fitting values of $\alpha_s(m_\tau^2)$, and c_8 for the single pinched third power monomial weight $\omega(x) = 1 - x^3$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

errors. The averaged values are thus given by

$$\alpha(m_\tau^2) = 0.32382(42) \qquad \text{and} \qquad c_8 = -1.33(67). \tag{1.3.1}$$

We note that the strong coupling is smaller as our expected values from the kinematic weight **??**, but the dimension eight contribution is in good agreement. The strong coupling from the monomial weight to third order seems to be in better agreement with the 8-10 momenta used in the kinematic fits, whereas the dimension eight contributions agrees more with the 4-7 momenta fits.

We have made use of a single pinched weight and discovered that the fitting result is not completely compatible with our previous fitting results. Consequently weights with a pinching less than two are affected by DV and should not be used to determine the strong coupling.

1.3.2 Fourth power monomial: $\omega_{m4}(x) \equiv 1 - x^4$

We already analysed the cubic and quartic weights, which depend on the dimension ten OPE contribution, in section 1.2.2 and section 1.2.3 correspondingly. Now, even with the visible DV for fourth power monomial $\omega_{m4} \equiv 1 - x^4$ to study another single pinched moment and the dimension ten OPE contribution. The results of the are given in **??**. The fitting behaviour is very similar to

s_{min}	$\#s_0 s$	$\alpha_s(m_\tau^2)$	c_{10}	χ^2/dof
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

Table 1.6: Table of our fitting values of $\alpha_s(m_\tau^2)$ and c_{10} for the single pinched fourth power monomial weight $\omega(x)=1-x^4$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

the third power monomial (**??**) and we will directly cite our obtained results:

$$\alpha_s(m_\tau^2) = 0.32277(40) \qquad \text{and} \qquad c_{10} = -2.4(3.6). \tag{1.3.2}$$

As before the values for the strong coupling are lower than the ones obtained by the kinematic weight fit. Furthermore the error on the tenth dimension contribution of the OPE are too huge, although the huge errors makes it compatible with all previous results. All in all the usage of the single pinched fourth power monomial weight is questionable and does not deliver any additional insights.

1.3.3 Pich's Optimal Moments [Pich2016]

Next to the previously mentioned *optimal weights* from Beneke and Jamin [**Beneke2012**] there are *optimal moments* introduced by Pich [**LeDiberder1992**]. Combinations of these optimal moments have been widely used by the ALEPH collaboration to perform QCD analysis on the Large electron-positron collider (LEP). These moments include the for FOPT problematic proportional term in x [**Beneke2012**], thus we will perform additional fits in the Borel-sum.

$$\omega_{(n,m)}(x) = (1-x)^n \sum_{k=0}^{m} (k+1)x^k \tag{1.3.3}$$

ω(x) = (1-x)^2

s_{min}	#s_0s	$\alpha_s(m_\tau^2)$	aGGInv	c_6	χ^2/dof
2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89
2.400	5	0.3450(93)	-0.0243(99)	0.10(17)	0.71
2.600	4	0.337(16)	-0.014(18)	0.36(45)	0.98

Table 1.7: Table of our fitting values of $\alpha_s(m_\tau^2)$, $aGGInv$ and c_6 for the triple pinched optimal weight $\omega^{(2,0)}(x) = (1-x)^2$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

ω(x) = (1-x)^3

s_{min}	#s_0s	$\alpha_s(m_\tau^2)$	aGGInv	c_6	c_8	χ^2/dof
1.900	11	0.34281(92)	-0.01473(73)	-0.103(22)	-0.534(46)	1.52
1.950	10	0.34154(99)	-0.01304(61)	-0.050(17)	-0.389(44)	1.42
2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41
2.300	6	0.3522(64)	-0.0249(62)	-0.42(18)	-1.51(57)	0.29
2.400	5	0.3480(89)	-0.0199(100)	-0.25(33)	-0.96(10)	0.39

Table 1.8: Table of our fitting values of $\alpha_s(m_\tau^2)$, $aGGInv$, c_6 and c_8 for the optimal weight $\omega^{(3,0)}(x) = (1-x)^3$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

weight	s_{min}	$\alpha_s(m_\tau^2)$	aGGInv	c_6	c_8	c_{10}	χ^2/dof
ω_{kin}	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
ω_{cube}	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
$\omega_{3,0}^*$	2.1	0.3239(30)	-0.2125(26)	-0.627(87)	-0.74(17)	-	0.46
$\omega_{quartic}$	2.0	0.3290(11)	-	-0.3030(46)	-0.1874(28)	0.3678(45)	0.67
ω_{m3}	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
ω_{m4}	2.2	0.3203(48)	-	-	-	-1.64(77)	0.42
$\omega_{2,0}$	2.2	0.3401(57)	-0.0185(52)	0.220(88)	-	-	0.73
$\omega_{3,0}$	2.1	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	-	0.31

Table 1.9: Table of the best fits (selected by χ^2/dof and compatibility of the fitting values) for each weight including at least the strong coupling $\alpha_s(m_\tau^2)$ as a fitting variable. All fits have been performed using FOPT, except weights marked with a star ω^*, which have been fitted using the *Borel sum*.

1.3.4 Comparison

To create an overview of our previous results we have gathered the most compatible rows by hand. These are shown in table 1.9, which is composed of two parts:

- The upper three rows represent fits we found to have good properties for determining the strong coupling.
- The lower five rows are problematic fits due to too many OPE contributions, too low pinching or to terms proportional to x.

We have found that the kinematic weight is in excellent agreement with the cubic ω_{cube} and Pich's optimal weight $\omega_{3,0}$, fitted using the borel model. The fitted parameters from the kinematic weight (α_s, c_6 and c_8) are all within error ranges and thus compatible. One fact that has to be investigated is the negative appearing sign for the gluon-condensate from the borel-sum of $\omega_{3,0}$.