Chapter 1

QCD Sum Rules

It is most remarkably that we can describe the properties of quarks and gluons via a local QFT based on the gauge group $SU(3)$. However QCD only applies to coloured particles and not to colourless particles like hadrons. Due to confinement we can only ever observe hadrons, but our theoretical foundation is ruled by the DOF of quarks and gluons. To extract QCD-parameters (the six quark-masses and the strong-coupling) from hadrons we need bridge the quark-gluon picture with the hadron picture. To do so we will introduce the framework of QCDSR. Setting up the foundations of strong interaction we will discuss the standard tool of particle physics, namely the QCD-Lagrangian. The QCD-Lagrangian describes the quark-gluon picture solely and is ruled by the abelian gauge group $SU(3)$. The group has important implications on the strong coupling and the applicability of PT. Next we will focus on the two-point function, which plays a major role in the framework of QCDSR. The two-point function is defined as vacuum-expectation values of two local Noether-currents. We can use it to theoretically describe processes, like τ-decays into hadrons, by matching the quantum numbers of the Noether-current, we choose in specifying the two-point function, to the outgoing hadrons. We will see, that the two-point function $\Pi(q^2)$ is related to hadronic states, by poles for $q^2 > 0$. Here NP-effects become important and we need to introduce the OPE, which handles NP parts as QCD-condensates. These condensates are remainders of the QCD-vacuum Ω_{QCD}, which in contrast to the normal-ordered products of field in QED, do not vanish, but remain as parameters and have to be phenomenological fitted or calculated by other NP tools, like LQCD. ...

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have been discussed in many articles [**Narison1989**, **Rafael1997**, **Colangelo2000**, **Dominguez2013**] Duality first introduced Poggio, Quinn and Weinberg [**Poggio1975**]

1.1 Quantum-chromodynamics

Since the formulation of QED in the end of the 40's it has been attempted to describe the strong nuclear force as a QFT, which has been achieved in the 70's as QCD [**GellMann1972**, **Fritzsch1973**, **Gross1973**, **Politzer1973**, **Weinberg1973**]. QCD is a renormalisable QFT of the strong interaction, which fundamental fields are given by dirac spinors of spin-$1/2$, the so-called quarks, with a fractional electric charge of $\pm 1/3$ or $\pm 2/3$. The theory furthermore contains gauge-fields of spin 1, which are chargeless, massless and referred to as gluons. The gluons are the force-mediators, which interact with quarks and themselves, in contrast to photons of QED, which interact only with fermions (see fig. 1.1).

The corresponding gauge-group of QCD is the non-abelian group $SU(3)$. Each of the quark flavours u, d, c, s, t and b belongs to the fundamental representation of $SU(3)$ and contains a triplet of fields Ψ.

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} \text{red} \\ \text{green} \\ \text{blue} \end{pmatrix} \tag{1.1.1}$$

The components of the triplet are titled colours[1] red, green and blue, which play the role of *colour-charge*, similar to the electric charge of QED. The gluons belong to the adjoint representation of $SU(3)$, contain an octet of fields and can be expressed using the Gell-Mann matrices λ_a

$$B_\mu = B_\mu^a \lambda_a \qquad a = 1, 2, \ldots 8 \tag{1.1.2}$$

The classical *Lagrange density* of QCD is given by [**Jamin2006**, **Pascual1984**]:

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G_{\mu\nu}^a(x) G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2} \bar{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A \bar{q}^A(x) a^A(x) \right], \tag{1.1.3}$$

[1] The colour denomination is not gauge-invariant. After a colour gauge transformation the new colours are a linear combination of the old colours, which breaks gauge-symmetry.

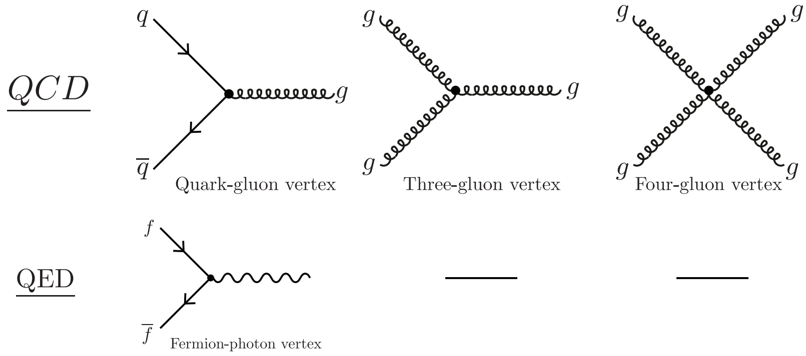


Figure 1.1: Feynman diagrams of the strong interactions with corresponding electromagnetic diagrams. We see that the gluons carry colour charge and thus couple to other gluons, which is not the case for the photons.

Flavour	Mass
u	3.48(24) MeV
d	6.80(29) MeV
s	130.0(18) MeV
c	1.523(18) GeV
b	6.936(57) GeV
t	173.0(40) GeV

Table 1.1: List of Quarks and their masses. The masses of the up, down, strange, charm and bottom quark are the renormalisation group invariant (RGI) quark masses and are quoted in the four-flavour theory ($N_f = 2 + 1$) at the scale $\mu = 2\,\text{GeV}$ in the $\overline{\text{MS}}$ scheme and are taken from the *Flavour Lattice Averaging Group* [**FLAG2019**]. The mass of the top quark is not discussed in [**FLAG2019**] and has been taken from [**PDG2018**] from direct observations of top events.

with $q^A(x)$ representing the quark fields and $G^a_{\mu\nu}$ being the *gluon field strength tensor* given by:

$$G^a_{\mu\nu}(x) \equiv \partial_\mu B^a_\nu(x) - \partial^a_\nu(x) + g f^{abc} B^b_\mu(x) B^c_\nu(x), \qquad (1.1.4)$$

with f^{abc} as *structure constants* of the gauge-group $SU(3)$ and $\overleftrightarrow{D}_\mu$ as covariant derivative acting to the left and to the right. Furthermore we have used $A, B, \ldots = 0, \ldots 5$ as flavour indices, $a, b, \ldots = 0, \ldots, 8$ as colour indices and $\mu, \nu, \ldots = 0, \ldots 3$ as lorentz indices. Explicitly the Lagrangian writes:

$$\begin{aligned}
\mathcal{L}_0(x) = &-\frac{1}{4}\left[\partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x)\right]\left[\partial^\mu G^\nu_a(x) - \partial^\nu G^\mu_a(x)\right] \\
&+ \frac{i}{2}\overline{q}^{\,A}_{\,\alpha}(x)\gamma^\mu \partial_\mu q^A_\alpha(x) - \frac{i}{2}\left[\partial_\mu \overline{q}^{\,A}_{\,\alpha}(x)\right]\gamma^\mu q^A_\alpha(x) - m_A \overline{q}^{\,A}_{\,\alpha}(x) q^A_\alpha(x) \\
&+ \frac{g_s}{2}\overline{q}^{\,A}_{\,\alpha}(x)\lambda^a_{\alpha\beta}\gamma_\mu q^A_\beta(x) G^\mu_a(x) \\
&- \frac{g_s}{2} f_{abc}\left[\partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x)\right] G^\mu_b(x) G^\nu_c(x) \\
&- \frac{g_s^2}{4} f_{abc} f_{ade} G^b_\mu(x) G^c_\nu(x) G^\mu_d(x) G^\nu_e(x)
\end{aligned} \qquad (1.1.5)$$

The first term is the kinetic term for the massless gluons. The next three terms are the kinetic terms for the quark field with different masses for each flavour. The rest of the terms are the interaction terms. The fifth term represents the interaction between quarks and gluons and the last two terms the self-interactions of gluon fields.

Having derived the Lagrangian leaves us with its quantisation. The dirac-spinors can be quantised as in QED without any problems. The $\Psi(x)$ quantum field can be written as:

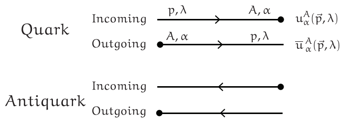
$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3 2E(\vec{p})} \sum_\lambda \left[u(\vec{p},\lambda) a(\vec{p},\lambda) e^{-ipx} + v(\vec{p},\lambda) b^\dagger(\vec{p},\lambda) e^{ipx}\right], \qquad (1.1.6)$$

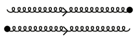
where the integration ranges over the positive sheet of the mass hyperboloid $\Omega_+(m) = \{p|p^2 = m^2, p^0 > 0\}$. The four spinors $u(\vec{p},\lambda)$ and $v(\vec{p},\lambda)$ are solutions to the dirac equations in momentum space

$$\begin{aligned}
[\not{p} - m]u(\vec{p},\lambda) &= 0 \\
[\not{p} + m]v(\vec{p},\lambda) &= 0,
\end{aligned} \qquad (1.1.7)$$

with λ representing the helicity state of the spinors.

The quantisation of the gauge-fields are more cumbersome. One is forced to introduce supplementary non-physical fields, the so-called Faddeev-Popov ghosts $c^a(x)$ **[Faddeev1967]**.





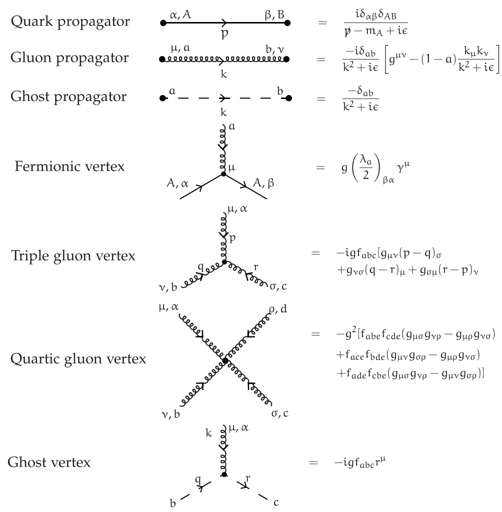


Figure 1.2: QCD Feynman rules.

The free propagators for the quark-, the gluon-and the ghost-fields are then given by

$$
\begin{aligned}
iS^{(0)AB}_{\alpha\beta}(x-y) &\equiv \langle q^A_\alpha(x)\overline{q}^B_\beta(y)\rangle \equiv \langle 0|T\{q^A_\alpha(x)\overline{q}^B_\beta\}|0\rangle = \delta_{AB}\delta_{\alpha\beta} iS^{(0)}(x-y) \\
&= i\delta_{AB}\delta_{\alpha\beta}\int \frac{d^4p}{(2\pi)^4}\frac{\not{p}+m}{(p^2-m^2+i\epsilon)} \\
iD^{(0)\mu\nu}_{ab}(x-y) &\equiv \langle B^\mu_a(x)B^\nu_b(y)\rangle \equiv \langle 0|T\{B^\mu_a(x)B^\nu_b(y)\}|0\rangle \equiv \delta_{ab} i\int\frac{d^4k}{(2\pi)^4}D^{(0)\mu\nu}(k)e^{-ik(x-y)} \\
&= i\delta_{ab}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\left[-g_{\mu\nu}+(1-a)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right]e^{-ik(x-y)} \\
i\widetilde{D}^{(0)}_{ab}(x-y) &\equiv \langle \phi_a(x)\overline{\phi}_b(y)\rangle \equiv \langle 0|T\{\phi_a(x)\overline{\phi}_b(y)\}|0\rangle = \frac{i}{(2\pi)^4}\delta_{ab}\int d^4q\frac{-1}{q^2+i\epsilon}e^{-q(x-y)} \\
&\equiv \frac{i}{(2\pi)^4}\delta_{ab}\int d^4q\,\widetilde{D}^{(0)}(q)e^{-iq(x-y)},
\end{aligned}
\tag{1.1.8}
$$

and the corresponding Feynman-rules have been displayed in fig. 1.2.

1.1.1 Renormalisation Group

The perturbations of the QCD Lagrangian in eq. 1.1.3 lead to divergencies, which have to be *renormalised*. Making these divergencies finite is referred to as *regularisation* and there are various approaches:

- **λ regularisation:** In Lambda regularisation we limit the divergent momentum integrals by a cutoff $|\vec{p}| < \Lambda$. Here Λ has the dimension of mass. In QCD Λ marks the separation between short-and long-distance effects. For momenta smaller than the cutoff ($|\vec{p}| < \Lambda$) we probe short-distances. On the contrary for large momenta $|\vec{p}| > \Lambda$ we have to include long-distance effects. We will see that we can make use of PT only for high-momenta (short-distances). The cutoff regularisation breaks translational invariance, which can be guarded by making use of other regularisation methods.

- ***P-I* (Pauli-Villars) regularisation: [Pauli1949]** In P-I regularisation the propagator is forced to decrease faster than the divergence to appear. It replaces the nominator by

$$(\vec{p}^2 + m^2)^{-1} \to (\vec{p}^2 + m^2)^{-1} - (\vec{p}^2 + M^2)^{-1}, \tag{1.1.9}$$

  where M has the dimension acts similar as the previously presented cutoff, but conserves translational invariance.

- **Dimensional regularisation: [Bollini1972, tHooft1972, tHooft1973]** Dimensional regularisation has been introduced in the beginning of the seventies to regularise non-abelian gauge theories (like QCD), where Λ- and P-V-regularisation failed. In dimensional regularisation we expand the four space-time dimensions to arbitrary D-dimensions. To compensate for the additional dimensions we introduce an additional scale μ^{D-4}. A typical Feynman-integral then has the following appearance:

$$\int \frac{d^4 p}{(2\pi)^4} \frac{1}{\vec{p}^2 + m^2} \to \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{\vec{p}^2 + m^2}, \tag{1.1.10}$$

  Dimensional regularisation preserves all symmetries, it allows an easy identification of divergences and naturally leads to the *minimal subtraction scheme* (MS) **[tHooft1973, Weinberg1973a]**.

In all of the three regularisation schemes we introduced an arbitrary parameter to regularise the divergence. This parameter causes an scale dependence of the strong coupling and the quark masses. As we are mainly concerned with the non-abelian gauge theory QCD we will focus on dimensional regularisation, which introduced the parameter μ. Measurable observables (*Physical quantities*) cannot depend on the renormalisation scale μ. In the Therefore the derivative by μ of a general physical quantity has to yield zero. The physical quantity $R(q, a_s, m)$, that depends on the external momentum q, the renormalised coupling $a_s \equiv \alpha_s/\pi$ and the renormalised quark mass m can then be expressed as

$$\mu\frac{d}{d\mu}R(q,a_s,m) = \left[\mu\tfrac{\partial}{\partial\mu} + \mu\tfrac{da_s}{d\mu}\tfrac{\partial}{\partial m} + \mu\tfrac{dm}{d\mu}\tfrac{\partial}{\partial m}\right]R(q,a_s,m) = 0. \tag{1.1.11}$$

Equation 1.1.11 is referred to as **renormalisation group equation** and is the basis for defining the two *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu\tfrac{da_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \qquad \beta-\text{function} \tag{1.1.12}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m}\frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \dots \qquad \text{anomalous mass dimension.} \tag{1.1.13}$$

The coefficients of the β-function and the mass anomalous dimension are currently known up to the 5^{th} order and listed in the appendix **??**.

Running gauge coupling

The β-function and the anomalous mass dimension are responsible for the running of the strong coupling and the running of the quark mass respectively. In this section we will shortly review the β-function and its implications on the strong coupling, whereas in the following section we will discuss the anomalous-mass dimension.

Regarding the β-function we notice, that $a_s(\mu)$ is not a constant, but *runs* by varying its scale μ. Lets observe the running of the strong coupling constant by integrating the β-function

$$\int_{a_s(\mu_1)}^{a_s(\mu_2)} \frac{da_s}{\beta(a_s)} = -\int_{\mu_1}^{\mu_2}\frac{d\mu}{\mu} = \log\frac{\mu_1}{\mu_2}. \tag{1.1.14}$$

To analytically evaluate the above integral we can approximate the β-function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \tag{1.1.15}$$

yielding

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log\frac{\mu_1}{\mu_2}\right)}. \tag{1.1.16}$$

As we have three colours ($N_c = 3$) and six flavours ($N_f = 6$) the first β-function 1.1.12 is positive. Thus for $\mu_2 > \mu_1$ $a_s(\mu_2)$ decreases logarithmically and vanishes for $\mu_2 \to \infty$. This behaviour is known as *asymptotic freedom* and leads to *confinement*.

Asymptotic freedom states, that for high energies (small distances), the strong coupling becomes diminishing small and quarks and gluons do not interact. Thus in isolated baryons and mesons the quarks are separated by small distances, move freely and do not interact. On the other hand we are not able to separate the quarks in a meson or baryon. No quark has been detected as single particle yet. This is qualitatively explained with the gluon field carrying colour charge. These gluons form so-called *flux-tubes* between quarks, which cause a constant strong force between particles regardless of their separation. Consequently the energy needed to separate quarks is proportional to the distance between them and at some point there is enough energy to favour the creation of a new quark pair. Thus before separating two quarks we create a quark-antiquark pair. As a result we will probably never be able to observe an isolated quark. This phenomenon is referred to as colour confinement or simply confinement.

Running quark mass

Not only the coupling but also the masses carry an energy dependencies, which is governed by the *anomalous mass dimension* $\gamma(a_s)$.

The properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* eq. 1.1.13

$$\log\frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \tag{1.1.17}$$

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically **[Schwab2002]**

$$m(\mu_2) = m(\mu_1)\left(\frac{a(\mu_2)}{a(\mu_1)}\right)^{\frac{\gamma_1}{\beta_1}} \left(1 + \mathcal{O}(\beta_2, \gamma_2)\right). \tag{1.1.18}$$

As β_1 and γ_1 (see **??**) are positive the quark mass decreases with increasing μ. The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1)\exp\left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)}\right) \tag{1.1.19}$$

and can be solved numerically to run the quark mass to the needed scale μ_2.

QCD in general has a precision problem caused by uncertainties and largeness of the strong coupling constant α_s. The fine-structure constant (the coupling QED) is known to eleven digits, whereas the strong coupling is only known to about four. Furthermore for low energies the strong coupling constant is much larger than the fine-structure constant. E.g. at the Z-mass, the standard mass to compare the strong coupling, we have an α_s of 0.11, whereas the fine structure constant would be around 0.007. Consequently to use PT we have to calculate our results to much higher orders, including tens of thousands of Feynman diagrams, in QCD to achieve a precision equal to QED. For even lower energies, around 1 GeV, the strong coupling reaches a critical value of ≈ 0.5 leading to a break down of PT.

In this work we try to achieve a higher precision in the value of α_s. The framework we use to measure the strong coupling constant are the QCDSR. A central object needed to describe hadronic states with the help of QCD is the *two-point function* for which we will devote the following section.

1.2 Two-Point function

A lot of particle physics is dedicated of calculating the *S-matrix*, which contains all the information about how initial states evolve in time. One important tool for obtaining the S-matrix is the *LSZ (Lehmann-Symanzik-Zimmermann)-reduction formula* **[Lehmann1954a, Schwartz2013]**

$$\begin{aligned}\langle f|S|i\rangle = \left[i\int_0^\infty \frac{d^4x_1}{(2\pi)^4} e^{-ip_1x_1}(\Box^2+m^2)\right]\cdots\left[i\int_0^\infty \frac{d^4x_n}{(2\pi)^4} e^{ip_nx_n}(\Box^2+m^2)\right] \\ \times\langle\Omega|T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle,\end{aligned} \tag{1.2.1}$$

with the $-i$ in the exponent applying for initial states and the $+i$ for final states. The LSZ-reduction formula relates the S-matrix to the *correlator* (also referred to as *n-point function*)

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|\Omega\rangle, \tag{1.2.2}$$

where $T\{\cdots\}$ is the time-ordered product and $|\Omega\rangle$ is the ground state/ vacuum of the interacting theory. Note that the fields are in general given in the Heisenberg picture, which implies translational invariance.

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \langle\Omega|\phi(x)e^{i\hat{P}y}e^{-i\hat{P}y}\phi(y)e^{i\hat{P}y}e^{-i\hat{P}y}|\Omega\rangle \\ &= \langle\Omega|\phi(x-y)\phi(0)|\Omega\rangle,\end{aligned} \tag{1.2.3}$$

where we made use of the translation operator $\hat{T}(x) = e^{-i\hat{P}x}$.

In this work we are solely concerned about the *two-point function*, especially in the vacuum expectation value of the Fourier transform of two time-ordered QCD quark Noether-currents

$$\Pi_{\mu\nu}(q^2) \equiv \int \frac{d^4 q}{(2\pi)^4} e^{iqx}\langle\Omega|J_\mu(x)J_\nu(0)|\Omega\rangle, \tag{1.2.4}$$

where the Noether current is given by

$$J_\mu(x) = \overline{q}\,(x)\Gamma q(x). \tag{1.2.5}$$

Here, Γ can be any of the following dirac matrices $\Gamma \in \{1, i\gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$, specifying the quantum number of the current (S: *scalar*, P: *pseudo-Scalar*, V: *vectorial*, A: *axial-vectorial*, respectively). By choosing the right quantum numbers we can theoretically represent the processes we want to study, which will be important when we want to match the hadrons produced in τ-decays.

From a Feynman diagram point of view we can illustrate the two-point function as quark-antiquark pair, which is produced by an external source, e.g. the virtual *W*-boson of $\tau\overline{\tau}$-annihilation as seen in fig. 1.3. Here the quarks are propagating at *short-distances*, which implies that we can make use of PT, thus avoiding *long-distance* (NPT-) effects, that would appear if the initial and final states where given by hadrons **[Colangelo2000]**.

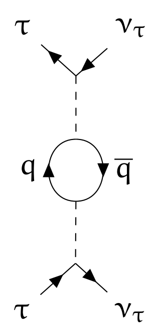


Figure 1.3: $\tau\overline{\tau}$-annihilation with a quark-antiquark pair.

1.2.1 Short-Distances vs. Long-Distances

If we want to calculate the two-point function in QCD we have to differentiate short-and long-distances or large or small momenta. In general when we talk about small distances we refer to large momenta. Large momenta implies a small strong coupling constant. Consequently we can use PT for short-distances. On the contrary long distances involve small momenta, which implies a large coupling constant. Thus for long distances the perturbation theory has broken down and cannot be used. We can make use of Λ-regularisation (section 1.1.1) to define the differentiation of short-and long-distances. Thus for $q < \Lambda$[2] we have short-distances and for $q > \Lambda$ we have long-distances. In our case we need the quark-antiquark pair of **??** to be highly virtual [3]. To separate long-distances from short-distances using a length scale we can say that the length scale should be smaller than the radius of a hadron.

1.2.2 Relating Two-Point Function and Hadrons

The two-point function can be interpreted physically as the amplitude of propagating single- or multi-particle states and their excitations. The possible states, in our case hadrons we describe through the correlator is fixed by the quantum numbers of the current we set for the vacuum expectation value. For example the neutral ρ-meson has a quark content of $(u\overline{u} - d\overline{d})/\sqrt{2}$ and is a spin-1 vector meson. Consequently by choosing a current

$$J_\mu(x) = \frac{1}{2}(\overline{u}(x)\gamma_\mu u(x) - \overline{d}(x)\gamma_\mu d(x)) \tag{1.2.6}$$

the two-point function contains the same quantum numbers as the ρ-meson and is said to materialise to it. A list of some ground-state mesons for combinations of the light-quarks u, d and s is given in table 1.2.

The correlator is materialising into a spectrum of hadrons. Thus if we insert a complete set of states of hadrons we can make use of the unitary relation

$$\langle\Omega|J_\mu(x)_\nu(0)|\Omega\rangle = \sum_X \langle\Omega|J_\mu(x)|X\rangle\langle X|J_\nu(0)|\Omega\rangle. \tag{1.2.7}$$

[2] Λ is a momentum-cutoff, so we have to compare momenta and not distances.

[3] Which is the same of saying, that the quark-antiquark pair needs a high external momentum q.

Symbol	Quark content	Isospin	J	Current
π^0	$\frac{u\bar{u}-d\bar{d}}{2}$	1	0	$\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d$
η	$\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}$	0	0	$\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d - 2\bar{s}\gamma_\mu\gamma_5 s$
$\eta\prime$	$\frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}}$	0	0	$\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d + \bar{s}\gamma_\mu\gamma_5 s$
ρ^0	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$	1	1	$\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d$
ω	$\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$	0	1	$\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d$
ϕ	$s\bar{s}$	0	1	$\bar{s}\gamma_\mu\gamma_5 s$

Table 1.2: Ground-state vector and pseudoscalar mesons for the light-quarks u, d and s with their corresponding currents in the two-point function. Note that we use γ_μ for vector and $\gamma_\mu\gamma_5$ for the pseudoscalar mesons.

to represent the two-point correlator via a spectral function $\rho(t)$

$$\Pi(q^2) = \int_0^\infty ds \frac{\rho(s)}{s - p^2 - i\epsilon}. \tag{1.2.8}$$

The above relation is referred to as *Källén-Lehmann spectral representation* and has been discovered early on by [**Kallen1952**, **Lehmann1954**]. It relates the two-point function to the spectral function ρ, which can be represented as sum over all possible hadronic states

$$\rho(s) = (2\pi)^3 \sum\!\!\!\!\!\!\int_X \left|\langle\Omega|J_\mu(0)|X\rangle\right|^2 \delta^4(s - p_X). \tag{1.2.9}$$

Note that the analytic properties of the two-point are in one-to-one correspondence with the newly introduced spectral function and thus determined by the possible hadrons states, which only form on positive real axis. A full derivation of the *Källén-Lehmann spectral representation* can be found in the appendix The spectral function is interesting to us for two reasons. First it is experimentally measurable and second it carries a problematic "branch cut", which we want to discuss now.

1.2.3 Analytic Structure of the Two-Point Function

The general two-point function $\rho(s)$ has some interesting, but problematic analytic properties. It has poles for single-particle states and a continuous branch

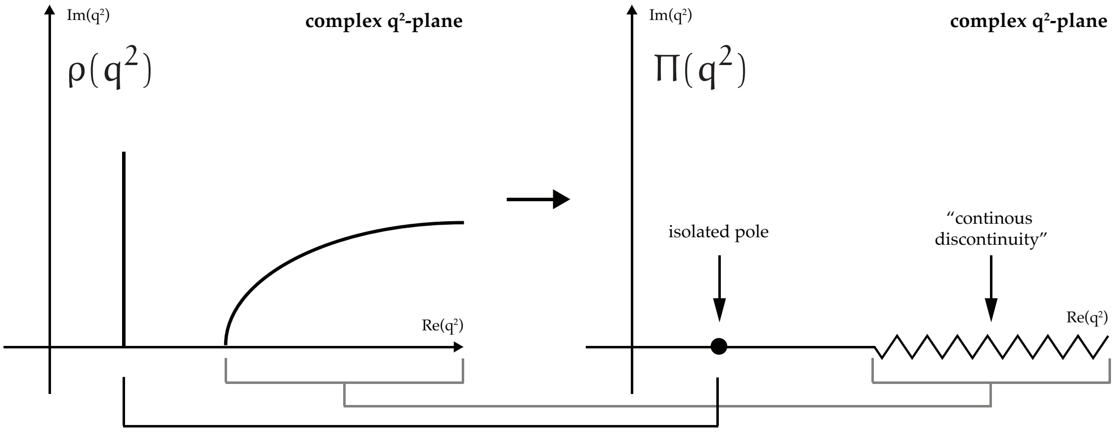


Figure 1.4: Analytic structure in the complex q^2-plane of the Fourier transform of the two-point function. The hadronic final states are responsible for poles appearing on the real-axis. The one-particle states contribute as isolated pole and the multi-particle states contribute as bound-states poles or a continues "discontinuity cut" [**Peskin1995**, **Zwicky2016**].

cut for multi-particle states. The single and multi-particle states, for a general correlator, can be mathematically separated by

$$\rho(s) = Z\delta(s - m^2) + \theta(s - s_0)\sigma(s) + \sigma(s), \tag{1.2.10}$$

where the second term is the contribution from multi-particle states. $\sigma(s)$ is zero till we reach the threshold, where we have sufficient energy to form multi-particle states. The analytic structure is depicted by fig. 1.4 and we can see that the spectral function has δ-spikes for single-particle states and a continuous contribution for $s \geqslant 4m$ resulting from multi-particle states. These lead to poles and a continuous branch cut of the two-point function.

correct sum up is referred to as *dispersion relation* analogous to similar relations which arise for example in electrodynamics and defines the *spectral function*

$$\rho(s) = \frac{1}{\pi}\operatorname{Im}\Pi(s). \tag{1.2.11}$$

1.2.4 Lorentz Decomposition

Apart the spectral decomposition we can Lorentz decompose the correlator to a scalar function $\Pi(q^2)$. There are only two possible terms that can reproduce the second order tensor $q_\mu q_\nu$ and $q^2 g_{\mu\nu}$. The sum of both multiplied with two arbitrary functions $A(q^2)$ and $B(q^2)$ yields

$$\Pi_{\mu\nu}(q^2) = q_\mu q_\nu A(q^2) + q^2 g_{\mu\nu} B(q^2). \tag{1.2.12}$$

By making use of the *Ward-identity*

$$\begin{aligned} q^\mu \Pi_{\mu\nu} &= \int dx q^\mu e^{iqx} \langle 0|J_\mu(x) J_\nu(0)|0\rangle \\ &= -i \int dx i q^\mu e^{iq^\gamma x_\nu} \langle 0|J_\mu(x) J_\nu(0)|0\rangle \\ &= i \int dx e^{iqx} \langle 0|\partial_\mu [J_\mu(x)] J_\nu(0)|0\rangle \\ &= 0, \quad \text{with} \quad \partial_\mu J_\mu(x) = 0, \end{aligned} \tag{1.2.13}$$

where we used $iq^\mu e^{iq^\gamma x_\nu} = \partial_\mu e^{iq^\gamma x_\nu}$ in the second and integration by parts in the third line. The Ward identity is dependent on the conserved Noether-current J_μ and thus only holds for same flavour quarks. With the Ward-identity we are able to demonstrate, that the two arbitrary functions are related

$$\begin{aligned} q^\mu q^\gamma \Pi_{\mu\nu} = q^4 A(q^2) + q^4 B(q^2) = 0 \\ \implies A(q^2) = -B(q^2). \end{aligned} \tag{1.2.14}$$

Thus redefining $A(q^2) \equiv \Pi(q^2)$ we expressed the correlator as a scalar function

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2). \tag{1.2.15}$$

These discontinuities can be tackled with *Cauchy's theorem*, which we will apply in section 1.4.

Having dealt exclusively with the perturbative part of the theory, we have to discuss non-perturbative contributions, as QCD is known to have non-negligible contributions. Thus before continuing with the *Sum Rules* we need a final ingredient the *Operator Product Expansion* (OPE), which treats the non-perturbative contributions of our theory.

1.3 Operator Product Expansion

The OPE was introduced by Wilson in 1969 [**Wilson1969**] as an alternative to the in this time commonly used current-algebra. The expansion states that non-local operators can be rewritten into a sum of composite local operators and their corresponding coefficients:

$$\lim_{x\to y} A(x)B(y) = \sum_n C_n(x-y)\mathcal{O}_n(x), \tag{1.3.1}$$

where $C_n(x-y)$ are the so-called *Wilson-coefficients* and A, B and $\mathcal{O}_n$ are operators.

The OPE lets us separate short-distance from long-distance effects. In PT we can only amount for short-distances, which are equal to high energies, where the strong-coupling α_s is small. Consequently the OPE decodes the long-distance effects in the higher dimensional operators.

The form of the composite operators are dictated by gauge- and Lorentz symmetry. Thus we can only make use of operators of even dimension. The scalar operators up to dimension six are given by **[Pascual1984]**

$$\begin{aligned}
\text{Dimension 0:}\quad & \mathbb{1} \\
\text{Dimension 4:}\quad & : m_i\bar{q}\,q : \\
& : G_a^{\mu\nu}(x)G^a_{\mu\nu}(x) : \\
\text{Dimension 6:}\quad & : \bar{q}\,\Gamma q\bar{q}\,\Gamma q : \\
& : \bar{q}\,\Gamma\tfrac{\lambda^a}{2}q_\beta(x)\bar{q}\,\Gamma\tfrac{\lambda^a}{2}q : \\
& : m_i\bar{q}\,\tfrac{\lambda^a}{2}\sigma_{\mu\nu}qG_a^{\mu\nu} : \\
& : f_{abc}G_a^{\mu\nu}G_b^{\nu\delta}G_c^{\delta\mu} :,
\end{aligned} \tag{1.3.2}$$

where Γ stands for one of possible dirac matrices (as seen eq. 1.2.5). The operator of dimension zero is the identity and its Wilson-coefficient is solely PT. The higher dimension operators appear as normal ordered products of fields and vanish by definition in PT. On the contrary, in NPT QCD they appear as *condensates*. Condensates are remainders of the QCD vacuum, which contribute to all strong processes. For example the condensates of dimension four are the quark-condensate $\langle\bar{q}\,q\rangle$ and the gluon-condensate $\langle aGG\rangle$.

As we work with dimensionless functions (e.g. the correlator Π), the r.h.s. of eq. 1.3.1 has to be dimensionless. As a result the Wilson-coefficients have to cancel the dimension of the operator with their inverse mass dimension. To

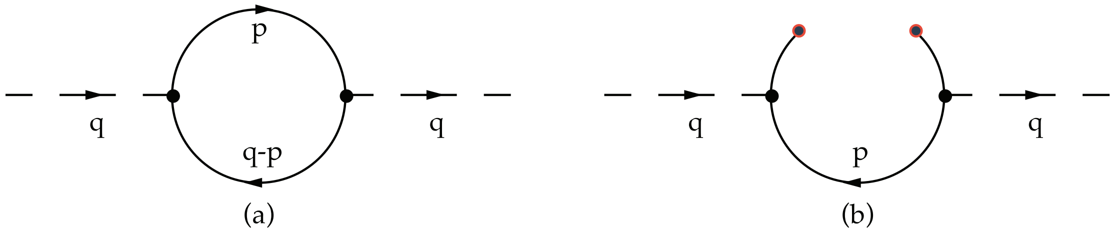


Figure 1.5: Feynman diagrams of the perturbative (a) and the quark-condensate (b) contribution. The upper part of the right diagram is not wick-contracted and responsible for the condensate.

account for the dimensions we can make the inverse momenta explicit

$$\Pi^{\mathrm{OPE}}_{V/A}(s) = \sum_{D=0,2,4\ldots} \frac{c^{(D)}\langle \mathcal{O}^{(D)}(x)\rangle}{(-q^2)^{D/2}}, \tag{1.3.3}$$

where we used $C^{(D)} = c/(-s)^{D/2}$ with D being the dimension. Thus the OPE should converge with increasing dimension for sufficiently large momenta s.

1.3.1 A practical example

Let's show how the OPE contributions are calculated [**Shifman1978**, **Pascual1984**]. We will compute the perturbative and quark-condensate Wilson-coefficients for the rho meson. To do that we have to evaluate Feynman diagrams using standard PT.

The rho meson is a vector meson of isospin one composed of u and d quarks. As a result (see. table 1.2) we can match its quantum numbers with the current

$$J^{\mu}(x) = \frac{1}{2}\Big(: [\bar{u}\gamma^{\mu}u](x) - [\bar{d}\gamma^{\mu}d](x) : \Big) . \tag{1.3.4}$$

Pictorial the dimension zero contribution is given by the quark-antiquark loop Feynman diagram fig. 1.5. The higher dimension contributions are given by the same Feynman diagram, but with non contracted fields. These non contracted fields are yielding the condensates. Thus not contracting the quark-antiquark field (see. fig. 1.5 b) will give us access to the Wilson coefficient of the dimension four quark-condensate $\langle \bar{q}\, q\rangle$.

The perturbative part (the Wilson coefficient of dimension zero) can than be taken from the mathematical expression for the scalar correlator

$$\begin{aligned}\Pi(q^2) = &-\frac{i}{4q^2(D-1)}\int d^D x e^{iqx}\langle\Omega|T\{: \overline{u}(x)\gamma^\mu u(x) - \overline{d}(x)\gamma^\mu d(x) : \\ &\times : \overline{u}(0\gamma_\mu u(0) - \overline{d}(0)\gamma_\mu d(0) :\}\rangle.\end{aligned} \tag{1.3.5}$$

To extract the dimension zero Wilson coefficient we apply Wick's theorem to contract all of the fields, which represents the lowest order of the perturbative contribution. The calculation is only using standard PT and we will restrict ourselves in displaying the result and omitting the calculation[4].

$$\begin{aligned}\Pi(q^2) &= \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D x e^{iqx} \\ &\times \left[u_{j\alpha}(x)\overline{u}_{k\beta}(0)\cdot u_{l\beta}(0)\overline{u}_{i\alpha}(x) + (u\to d)\right] \\ &= \frac{3}{8\pi^2}\left[\frac{5}{3} - \log\left(-\frac{q^2}{\nu^2}\right)\right].\end{aligned} \tag{1.3.6}$$

To calculate the higher dimensional contributions of the OPE we use the same techniques as before, but leave some of the fields uncontracted. Thus instead of applying Wick's theorem for all possible contractions fields, we leave some fields uncontracted. For leaving the quark field uncontracted in eq. 1.3.5 we get

$$\begin{aligned}\Pi(q^2) = &\frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D x e^{iqx}\Big[\\ &+ u_{j\alpha}(x)\overline{u}_{k\beta}(0)\cdot\langle\Omega| : \overline{u}_{i\alpha}(x)u_{l\beta}(0) : |\Omega\rangle \\ &+ u_{l\beta}(0)\overline{u}_{i\alpha}(x)\cdot\langle\Omega| : \overline{u}_{k\beta}(0)u_{j\alpha}(x) : |\Omega\rangle + (u\to d)\Big].\end{aligned} \tag{1.3.7}$$

Here we can observe the condensates as non-vanishing vacuum values of normal ordered product of fields:

$$\langle\Omega_{QCD}|\overline{q}(x)q(0)|\Omega_{QCD}\rangle \neq 0. \tag{1.3.8}$$

We emphasised the QCD vacuum Ω_{QCD}, which is responsible for vacuum expectation values different than zero. E.g. for a vacuum of QED this contributions would vanish by definition. Pictorial the condensates take form of unconnected propagators, sometimes marked with an x, as seen in fig. 1.5.

[4] The interested reader can follow **[Pascual1984]** for a detailed calculation.

To make the non-contracted fields local, we can expanded them in x

$$\begin{aligned}\langle\Omega| : \overline{q}\,(x) q(0) : |\Omega\rangle = \langle\Omega| : \overline{q}\,(0) q(0) : |\Omega\rangle \\ + \langle\Omega| : \left[\partial_\mu \overline{q}\,(0)\right] q(0) : |\Omega\rangle x^\mu + \ldots .\end{aligned} \tag{1.3.9}$$

and introduce a standard notation for the localised condensate

$$\langle\overline{q}\,q\rangle \equiv \langle\Omega| : \overline{q}\,(0) q(0) : |\Omega\rangle. \tag{1.3.10}$$

Finally, the contribution to the rho scalar correlator is then given by the following expression

$$\Pi_{(\rho)}(q^2) = \frac{1}{2}\frac{1}{\left(-q^2\right)^2}\left[m_u\langle\overline{u}\,u\rangle + m_d\langle\overline{d}\,d\rangle\right]. \tag{1.3.11}$$

Here we can clearly see that for dimension four we get a factor of $1/(-q^2)^2$, which is responsible for the suppression of the series. The condensates $\langle\overline{u}\,u\rangle$ and $\langle\overline{d}\,d\rangle$ are numbers, that have to be derived by phenomenological fits or LQCD. Fortunately once found, the value of the condensate can be used for any process.

In summary we note that the usage of the OPE and its validity is far from obvious. We are deriving the OPE from matching the Wilson-coefficients to Feynman-graph analyses. These Feynman-graphs are calculated perturbatively but the coefficients with dimension $D > 0$ correspond to NPT condensates! The condensates by themselves have to be gathered from external, NPT methods.

Now that we have a tool to deal with the problematic QCD vacuum and NPT-effects we are left with two problems. First we still do not know how to deal with hadronic states in the quark-gluon picture. This will be tackled by Duality. Secondly we have seen that we can access the two-point function theoretically on the physical sheet except for the positive real axis, due to its analytic properties, but that the experimental measurable spectral function is solely be defined on the positive real axis. Thus we need to make use of Cauchy's Theorem. In total we will bring together the two-point function, the OPE, Duality and Cauchy's theorem to formulate the QCDSR.

1.4 Sum Rules

To relate the measurable hadronic final states of a QCD process (e.g. τ-decays into hadrons) to a theoretical calculable QCDSR have been employed by Shifman in the late seventies [**Shifman1978**].

The sum rules are a combination of the two-point function, its analyticity, the OPE, a dispersion relation, the optical theorem and quark hadron duality.

The previously introduced two-point function **??** is generally described by the OPE to account for NPT effects.

$$\Pi(q^2) = \Pi^{\text{OPE}}(q^2). \tag{1.4.1}$$

Furthermore it is related to the theoretical spectral function $\rho(s)$ via a dispersion relation (**??**). Using QCD we are computing interactions based on quarks and gluons, but as we have seen (confinement), we are only able to observe hadrons. Consequently to connect the theory to the experiment we have to assume **quark-hadron duality**[5], which implies that physical quantities can be described equally good in the hadronic or in the quark-gluon picture. Thus we rewrite the dispersion relation **??** as

$$\Pi_{\text{th}}^{\text{OPE}}(q^2) = \int_0^\infty \frac{\rho_{\text{exp}}(q^2)}{(s - q^2 - i\epsilon)}, \tag{1.4.2}$$

where we connected the theoretical correlator Π_{th} with the experimental measurable spectral function ρ_{exp}.

We have seen, that the theoretical description of the correlator Π_{th} contains poles on the real axis. Unfortunately the experimental data ρ_{exp} is solely accessible on the positive real axis. As a result we have to make use of Cauchy's theorem to access the theoretical values of the two-point function close to the positive real axis (see section 1.4), which is given by

$$\int_{\mathcal{C}} f(z)\, dz = 0, \tag{1.4.3}$$

where $f(z)$ is an analytic function on a closed contour $\mathcal{C}$.

The final ingredient of the QCD sum rules is the *optical theorem*, relating experimental data with the imaginary part of the correlator (the spectral function $\rho(s)$).

[5]Or simply duality.

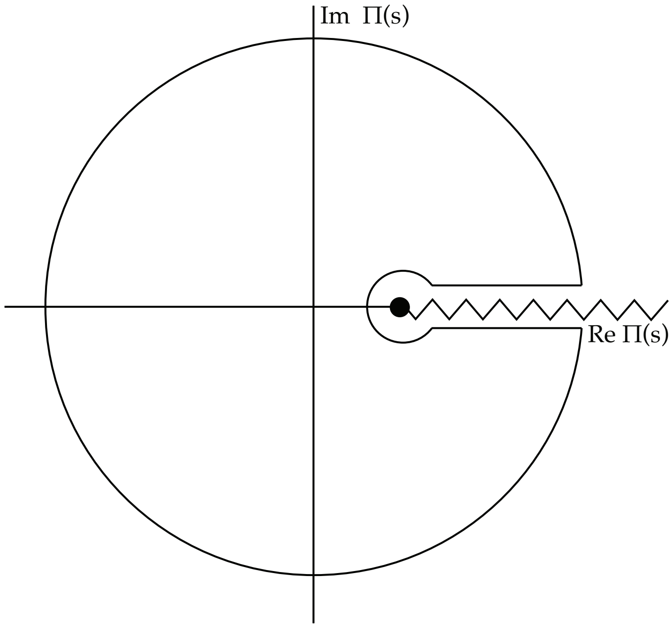


Figure 1.6: Analytical structure of $\Pi(s)$ with the used contour $\mathcal{C}$ for the final QCD Sum Rule expression eq. 1.4.4.

In total, with the help Cauchy's theorem, the QCD sum rules can be summed up in the following expression

$$\frac{1}{\pi}\int_0^\infty \frac{\rho_{exp}(t)}{t-s}\,dt = \frac{1}{\pi}\oint_{\mathcal{C}} \frac{\mathrm{Im}\,\Pi_{OPE}(t)}{t-s}\,dt, \tag{1.4.4}$$

where the l.h.s. is given by the experiment and the r.h.s. can be theoretically evaluated by applying the OPE of the correlator $\Pi_{OPE}(s)$.