Chapter 1

QCD Sum Rules

The theory of QCD was formulated to find one single framework that describes the many hadrons that exist. Unfortunately making use of *perturbative* QCD (PQCD) is limited. QCD predicts a large coupling constant for low energies. As a consequence we can only ever observe hadrons, but our theoretical foundation is ruled by the DOF of quarks and gluons. To extract QCD parameters (the six quark masses and the strong coupling) from hadrons we need to connect the quark-gluon picture with the hadron picture. To do so we will introduce the framework of QCDSR.

We will start by setting up the foundations of strong interaction with introducing the QCD Lagrangian. The QCD Lagrangian is ruled by the abelian gauge group $SU(3)$. The group implies an energy dependence of the coupling and thus limits the applicability of PT for low energies, where the coupling is large. Next we will focus on the two-point function, which plays a major role in the framework of QCDSR. The two-point function is defined as vacuum-expectation values of the time ordered product of two local fields

$$\Pi_{\mu\nu}(q^2) = \int \frac{d^4 q}{(2\pi)^4} e^{iqx} \langle \Omega | T\{J_\mu(x) J_\nu(0)\} | \Omega \rangle, \tag{1.0.1}$$

where J_μ is the Noether current. We can use it to theoretically describe processes, like τ decays into hadrons, by matching the quantum numbers of the fields, we choose in specifying the two-point function, to the outgoing hadrons. We will see, that the two-point function $\Pi(q^2)$ is related to hadronic states, by poles for $q^2 > 0$. Here NP effects become important and we need to introduce

the OPE, which handles NP parts through the QCD condensates. The condensates form part of the full physical vacuum and would not exist regarding the perturbative vacuum solely. Consequently the condensates are not accessible trough PT methods and have to be fitted from experiment or calculated with the help of NP tools, like LQCD. Finally we will combine a dispersion relation and Cauchy's theorem to finalise the discussion on the QCDSR with developing the *finite energy sum rules* (FESR), which we will apply to extract the strong coupling from τ decays into hadrons.

1.1 Quantum-Chromodynamics

Since the formulation of QED in the end of the forties it had been attempted to construct a QFT of the strong nuclear force, which has been achieved in the 70's as QCD [**GellMann1972**, **Fritzsch1973**, **Gross1973**, **Politzer1973**, **Weinberg1973**]. The fundamental fields of QCD are given by dirac spinors of spin-1/2, the so-called quarks, with a fractional electric charge of $\pm 1/3$ or $\pm 2/3$. The theory furthermore contains gauge fields of spin 1. These gauge fields are called gluons, do not carry electric charge and are massless. They are the force mediators, which interact with quarks and themselves, because they carry colour charge, in contrast to photons of QED, which interact only with fermions.

The corresponding gauge group of QCD is the non-abelian group $SU(3)$. Each of the quark flavours u, d, c, s, t and b belongs to the fundamental representation of $SU(3)$ and contains a triplet of fields Ψ.

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \tag{1.1.1}$$

The labels of the triplet are the colours red, green and blue, which play the role of *colour charge*, similar to the electric charge of QED. The gluons belong to the adjoint representation of $SU(3)$, contain an octet of fields and can be expressed using the Gell-Mann matrices λ_a

$$B_\mu = B_\mu^a \lambda_a \qquad a = 1, 2, \ldots 8 \tag{1.1.2}$$

Flavour	Mass
u	2.50(17) MeV
d	4.88(20) MeV
s	93.44(68) MeV
c	1.280(13) GeV
b	4.198(12) GeV
t	173.0(40) GeV

Table 1.1: List of quarks and their masses. The masses of the up, down and strange quark are quoted in the four-flavour theory ($N_f = 2+1+1$) at the scale $\mu = 2\,GeV$ in the $\overline{MS}$ scheme. The charm and bottom quark are also taken in the four-flavour theory and in the $\overline{MS}$ scheme, but at the scales $\mu = m_c$ and $\mu = m_b$ correspondingly. All quarks except for the top quark are taken from the FLAGFlavour Lattice Averaging Group [**FLAG2019**]. The mass of the top quark is not discussed in [**FLAG2019**] and has been taken from direct observations of top events [**PDG2018**].

The classical *Lagrange density* of QCD is given by [**Yndurain2006**, **Pascual1984**]:

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4}G^a_{\mu\nu}(x)G^{\mu\nu a}(x) + \sum_A \left[\frac{i}{2}\bar{q}^A(x)\gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m_A \bar{q}^A(x)a^A(x)\right], \tag{1.1.3}$$

with $q^A(x)$ representing the quark fields and $G^a_{\mu\nu}$ being the *gluon field strength tensor* given by:

$$G^a_{\mu\nu}(x) \equiv \partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x) + g f^{abc} B^b_\mu(x) B^c_\nu(x), \tag{1.1.4}$$

with f^{abc} as *structure constants* of the gauge group $SU(3)$ and $\overleftrightarrow{D}_\mu$ as covariant derivative acting to the left and to the right. Furthermore we have used $A, B, \ldots = 0, \ldots 5$ as flavour indices, $a, b, \ldots = 0, \ldots, 8$ as colour indices and

$\mu, \nu, \ldots = 0, \ldots 3$ as Lorentz indices. Explicitly the Lagrangian writes:

$$\begin{aligned}\mathcal{L}_0(x) = &-\frac{1}{4}\left[\partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x)\right]\left[\partial^\mu G^\nu_a(x) - \partial^\nu G^\mu_a(x)\right] \\ &+ \frac{i}{2}\overline{q}^A_\alpha(x)\gamma^\mu\partial_\mu q^A_\alpha(x) - \frac{i}{2}\left[\partial_\mu \overline{q}^A_\alpha(x)\right]\gamma^\mu q^A_\alpha(x) - m_A \overline{q}^A_\alpha(x) q^A_\alpha(x) \\ &+ \frac{g_s}{2}\overline{q}^A_\alpha(x)\lambda^a_{\alpha\beta}\gamma_\mu q^A_\beta(x) G^\mu_a(x) \\ &- \frac{g_s}{2} f_{abc}\left[\partial_\mu G^a_\nu(x) - \partial_\nu G^a_\mu(x)\right] G^\mu_b(x) G^\nu_c(x) \\ &- \frac{g_s^2}{4} f_{abc} f_{ade} G^b_\mu(x) G^c_\nu(x) G^\mu_d(x) G^\nu_e(x)\end{aligned} \tag{1.1.5}$$

The first term is the kinetic term for the massless gluons. The next three terms are the kinetic terms for the quark field with different masses for each flavour. The rest of the terms are the interaction terms. The fifth term represents the interaction between quarks and gluons and the last two terms the self interactions of gluon fields.

The corresponding Feynman rules have been displayed in fig. 1.1. The rules are based on PT, but can be enhanced with the QCD condensates, as we will see in the discussion of the OPE in section 1.3

Having derived the Lagrangian leaves us with its quantisation. The dirac-spinors can be quantised as in QED without any problems. The $\Psi(x)$ quantum field can be written as:

$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3 2E(\vec{p})} \sum_\lambda \left[u(\vec{p},\lambda) a(\vec{p},\lambda) e^{-ipx} + v(\vec{p},\lambda) b^\dagger(\vec{p},\lambda) e^{ipx}\right], \tag{1.1.6}$$

where the integration ranges over the positive sheet of the mass hyperboloid $\Omega_+(m) = \{p | p^2 = m^2, p^0 > 0\}$. The four spinors $u(\vec{p},\lambda)$ and $v(\vec{p},\lambda)$ are solutions to the dirac equations in momentum space

$$\begin{aligned}[\not{p} - m] u(\vec{p},\lambda) &= 0 \\ [\not{p} + m] v(\vec{p},\lambda) &= 0,\end{aligned} \tag{1.1.7}$$

with λ representing the helicity state of the spinors.

The quantisation of the gauge fields is more cumbersome. One is forced to introduce supplementary non-physical fields, the so-called Faddeev-Popov ghosts $c^a(x)$ **[Faddeev1967]**, to cancel unphysical helicity degrees of freedom of the gluon fields.

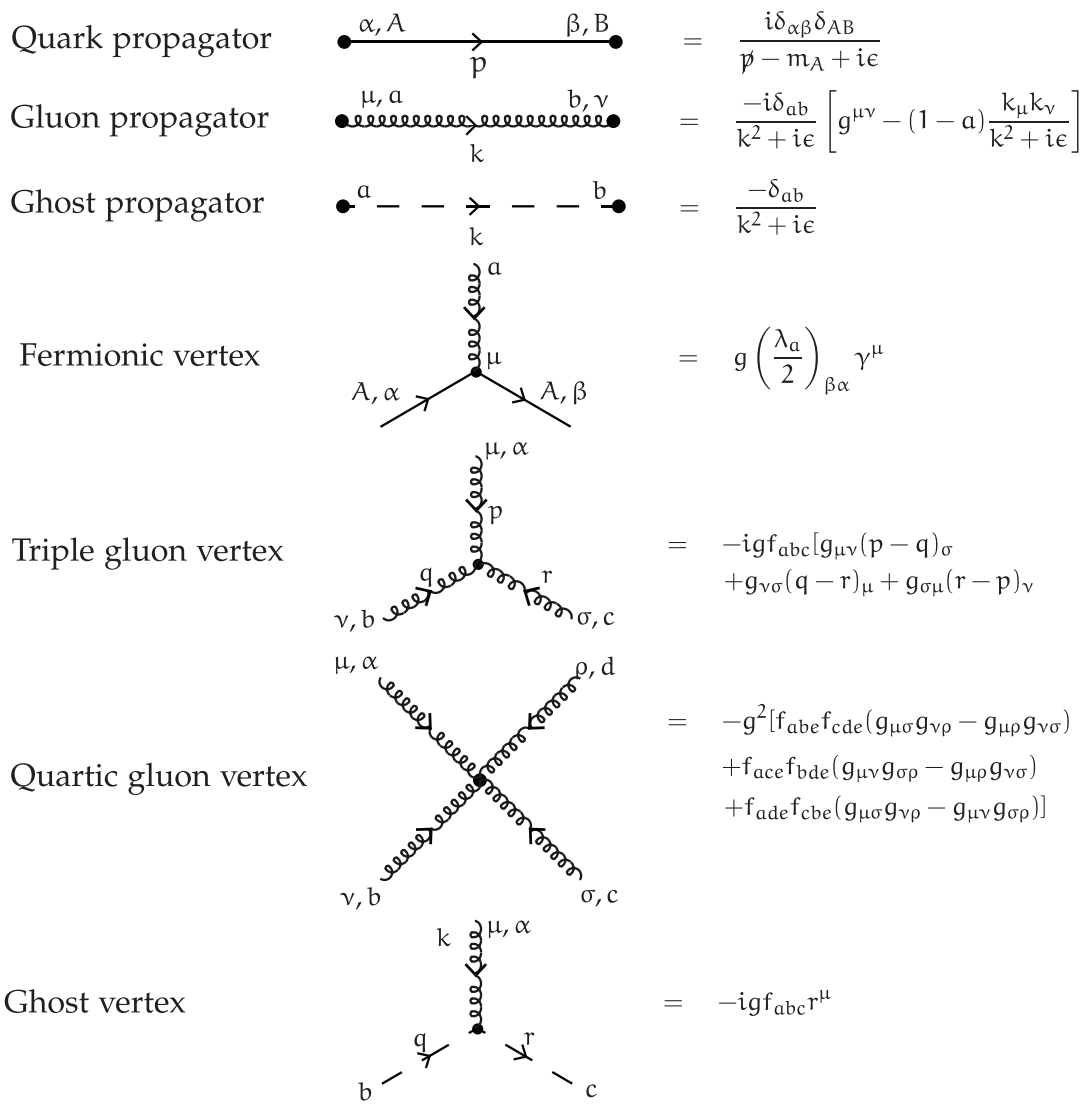

Figure 1.1: QCD Feynman rules.

The free propagators for the quark, the gluon and the ghost fields are then given by

$$\begin{aligned}
iS^{(0)AB}_{\alpha\beta}(x-y) &\equiv \overbrace{q^A_\alpha(x)\overline{q}^B_\beta}(y) \equiv \langle 0|T\{q^A_\alpha(x)\overline{q}^B_\beta\}|0\rangle = \delta_{AB}\delta_{\alpha\beta}iS^{(0)}(x-y) \\
&= i\delta_{AB}\delta_{\alpha\beta}\int \frac{d^4p}{(2\pi)^4}\frac{\not{p}+m}{(p^2-m^2+i\epsilon)} \\
iD^{(0)\mu\nu}_{ab}(x-y) &\equiv \overbrace{B^\mu_a(x)B^\nu_b}(y) \equiv \langle 0|T\{B^\mu_a(x)B^\nu_b(y)\}|0\rangle \equiv \delta_{ab}i\int \frac{d^4k}{(2\pi)^4}D^{(0)\mu\nu}(k)e^{-ik(x-y)} \\
&= i\delta_{ab}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\left[-g_{\mu\nu}+(1-a)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right]e^{-ik(x-y)} \\
i\widetilde{D}^{(0)}_{ab}(x-y) &\equiv \overbrace{\phi_a(x)\overline{\phi}_b}(y) \equiv \langle 0|T\{\phi_a(x)\overline{\phi}_b(y)\}|0\rangle = i\delta_{ab}\int \frac{d^4q}{(2\pi)^4}\frac{-1}{q^2+i\epsilon}e^{-q(x-y)} \\
&\equiv i\delta_{ab}\int \frac{d^4q}{(2\pi)^4}\widetilde{D}^{(0)}(q)e^{-iq(x-y)},
\end{aligned} \tag{1.1.8}$$

The previously introduced Feynman rules and propagators all make use of the perturbative vacuum $|0\rangle$ and thus count as tools of PT. Consequently they need a small coupling to approximate excitations of full QCD vacuum. We will see in the following section, that the strong coupling runs with energy and unfortunately is large for small energy scales.

1.1.1 Renormalisation Group

Computing observables with the QCD Lagrangian (eq. 1.1.3) lead to divergencies, which have to be *renormalised*. To render these divergent quantities finite we have to introduce a suitable parameter such that the "original divergent theory" corresponds to a certain value of that parameter. These procedure is referred to as *regularisation* and there are various approaches:

- **Cut-off regularisation:** In cut-off regularisation we limit the divergent momentum integrals by a cut-off $|\vec{p}| < \Lambda$. Here Λ has the dimension of mass. The cut-off regularisation breaks translational invariance, which can be guarded by making use of other regularisation methods.
- ***Pauli-Villars* (P-V) regularisation:** [**Pauli1949**] In P-V regularisation the propagator is forced to decrease faster than the divergence to appear. It

replaces the nominator by

$$(\vec{p}^2 + m^2)^{-1} \to (\vec{p}^2 + m^2)^{-1} - (\vec{p}^2 + M^2)^{-1}, \tag{1.1.9}$$

where M has the dimension acts similar as the previously presented cut-off, but conserves translational invariance.

- **Dimensional regularisation:** [**Bollini1972**, **tHooft1972**, **tHooft1973**] Dimensional regularisation has been introduced in the beginning of the seventies to regularise non-abelian gauge theories (like QCD), where Λ- and P-V-regularisation failed. In dimensional regularisation we expand the four space-time dimensions to arbitrary D dimensions. To compensate for the additional dimensions we introduce an additional scale μ^{D-4}. A typical Feynman integral then has the following appearance:

$$\int \frac{d^4 p}{(2\pi)^4} \frac{1}{\vec{p}^2 + m^2} \to \mu^{2\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{\vec{p}^2 + m^2}. \tag{1.1.10}$$

  Dimensional regularisation preserves all symmetries and allows an easy identification of divergences and naturally leads to the *minimal subtraction scheme* (MS) [**tHooft1973**, **Weinberg1973a**].

In all of the three regularisation schemes we introduced an arbitrary parameter to regularise the divergence. This parameter causes scale dependence of the strong coupling and the quark masses. As we are mainly concerned with the non-abelian gauge theory QCD we will focus on dimensional regularisation, which introduced the parameter μ. Measurable observables (*physical quantities*) cannot depend on the renormalisation scale μ. Therefore the derivative by μ of a general physical quantity has to yield zero. A physical quantity $R(q, a_s, m)$, that depends on the external momentum q, the renormalised coupling $a_s \equiv \alpha_s/\pi$ and the renormalised quark mass m can then be expressed as

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{da_s}{d\mu} \frac{\partial}{\partial a_s} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m}\right] R(q, a_s, m) = 0. \tag{1.1.11}$$

Equation 1.1.11 is referred to as a *renormalisation group equation* (RGE) and is the basis for defining the two *renormalisation group functions*:

$$\beta(a_s) \equiv -\mu \frac{da_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \qquad \beta\text{-function} \tag{1.1.12}$$

$$\gamma(a_s) \equiv -\frac{\mu}{m} \frac{dm}{d\mu} = \gamma_1 a_s + \gamma_2 a_s^2 + \dots \qquad \text{anomalous mass dimension.} \tag{1.1.13}$$

The β-function dictates the running of the strong coupling, whereas the anomalous mass dimension describes the running of the quark masses. We have a special interest in the running of the strong coupling, but will also shortly sum up the running of the quark masses.

Running gauge coupling

Regarding the β-function we notice, that $\alpha_s(\mu)$ is not a constant, but that it *runs* by varying its scale μ. To better understand the running of the strong coupling we integrate the β-function

$$\int_{\alpha_s(\mu_1)}^{\alpha_s(\mu_2)} \frac{d\alpha_s}{\beta(\alpha_s)} = -\int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} = \log \frac{\mu_1}{\mu_2}. \tag{1.1.14}$$

We analytically evaluate the above integral by approximating the β-function to first order, with the known coefficient

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f), \tag{1.1.15}$$

which yields

$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{\left(1 - \alpha_s(\mu_1)\beta_1 \log \frac{\mu_1}{\mu_2}\right)}. \tag{1.1.16}$$

Equation 1.1.16 has some important implications for the strong coupling:

- The coupling at a scale μ_2 depends on $\alpha_s(\mu_1)$. Thus we have to take care of the scale μ, while comparing different values of α_s. In the literature (e.g. **[PDG2018]**) α_s is commonly compared at the Z boson scale of around 91 GeV. As we are extracting the strong coupling at the mass of the τ lepton, around 1.776 GeV we need to run the strong coupling up to the desired scale. While running the coupling, we have to take care of the quark thresholds. Each quark gets active at a certain energy scale, which leads to a running of α_s as shown in fig. 1.2. Typically one runs the coupling with the aid of software packages like *RunDec* **[Chetyrkin2000, Herren2017]**, which has also been ported to support C (*CRunDec*, **[Schmidt2012]**) and Python **[Straub2016]**.
- As we have three colours ($N_c = 3$) and six flavours ($N_f = 6$) the β_1 coefficient 1.1.12 is positive. Thus for the two scales $\mu_2 < \mu_1$ the strong

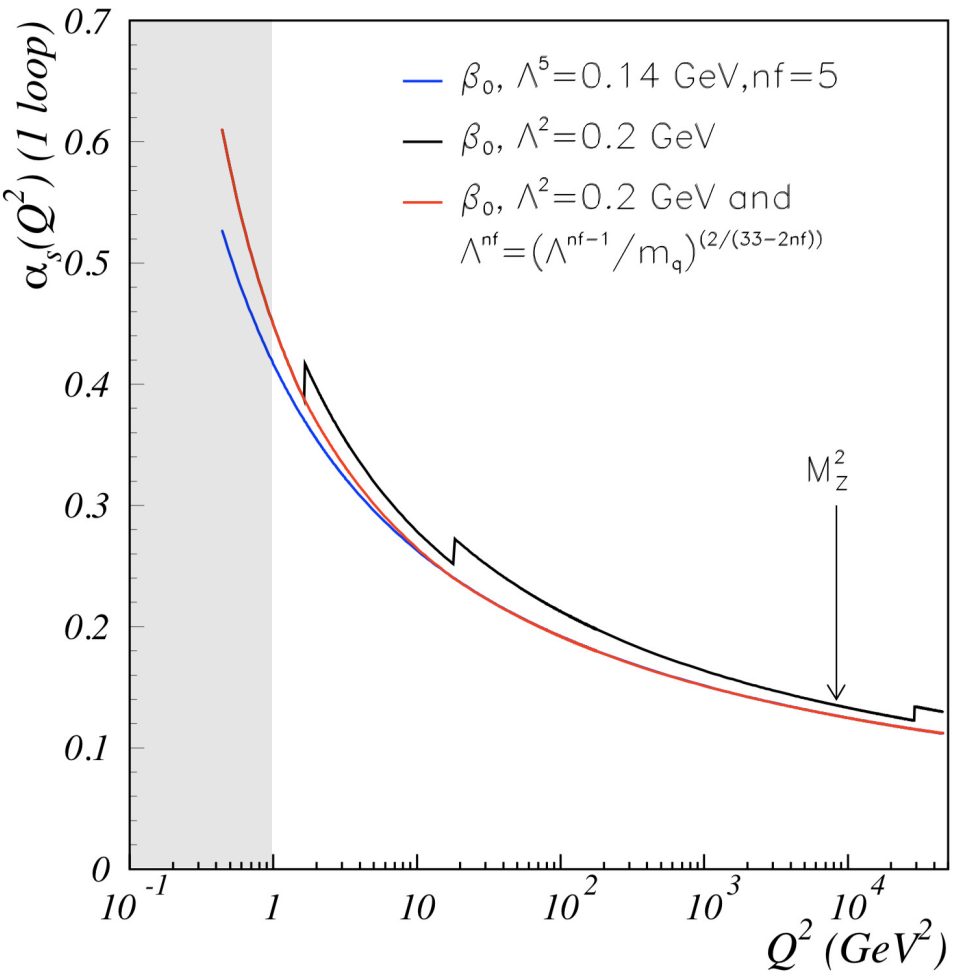

Figure 1.2: Running of the strong coupling $\alpha_s(Q^2)$ at first order. The blue line represents the uncorrected coupling constant, with an $\Lambda^{nf=5}$ chosen to match an experimental value of the coupling at $Q^2 = M_Z^2$. The quark-thresholds are shown by the black line and the corrected running is given by the red line. We additionally marked the breakdown of PT with a grey background for $Q^2 < 1$. The image is taken from an recent review of the strong coupling [**Deur2016**].

coupling $a_s(\mu_2)$ increases logarithmically and at a scale of $\mu_2 = 1\,GeV$ reaches a value of

$$\alpha_s(1\,GeV) \approx 0.5, \tag{1.1.17}$$

which questions the applicability of PT for energies lower than $1\,GeV$ (as seen from the grey zone in fig. 1.2).

- A large coupling for small scales implies confinement. We are not able to separate quarks in a meson or baryon. No quark has been detected as single particle yet. This is qualitatively explained with the gluon field carrying colour charge. These gluons form so-called *flux-tubes* between quarks, which cause a constant strong force between particles regardless of their separation. Consequently the energy needed to separate quarks is proportional to the distance between them and at some point there is enough energy to favour the creation of a new quark pair. Thus before separating two quarks we create a quark-antiquark pair. We will probably never be able to observe an isolated quark. This phenomenon is referred to as colour confinement or simply confinement.
- With the first β coefficient being positive we notice that for increasing scales ($\mu_2 > \mu_1$) the coupling decreases logarithmically. This leads to asymptotic freedom, which states, that for high energies (small distances), the strong coupling becomes diminishing small and quarks and gluons do not interact. Thus in isolated baryons and mesons the quarks are separated by small distances, move freely and do not interact.

From the RGE we have seen, that not only the coupling but also the masses carry an energy dependencies.

Running quark mass

The mass dependence on energy is governed by the *anomalous mass dimension* $\gamma(a_s)$. Its properties of the running quark mass can be derived similar to the gauge coupling. Starting from integrating the *anomalous mass dimension* eq. 1.1.13

$$\log \frac{m(\mu_2)}{m(\mu_1)} = \int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \tag{1.1.18}$$

we can approximate the *anomalous mass dimension* to first order and solve the integral analytically [**Schwab2002**]

$$m(\mu_2) = m(\mu_1) \left(\frac{a(\mu_2)}{a(\mu_1)} \right)^{\frac{\gamma_1}{\beta_1}} \left(1 + \mathcal{O}(\beta_2, \gamma_2) \right). \tag{1.1.19}$$

As β_1 and γ_1 (see **??**) are positive the quark mass decreases with increasing μ. The general relation between different scales is given by

$$m(\mu_2) = m(\mu_1) \exp \left(\int_{a_s(\mu_1)}^{a_s(\mu_2)} da_s \frac{\gamma(a_s)}{\beta(a_s)} \right) \tag{1.1.20}$$

and can be solved numerically to run the quark mass to the needed scale μ_2. Both, the β-function and the anomalous mass dimension are currently known up to the 5^{th} order and listed in the appendix **??**.

We will make use of the anomalous dimension while running the quark masses for NP contributions, which include the quark masses at different energy scales.

QCD in general has a precision problem caused by uncertainties and largeness of the strong coupling constant α_s. The fine-structure constant (the coupling of QED) is known to eleven digits, whereas the strong coupling is only known to about four. Furthermore for low energies the strong coupling constant is much larger than the fine-structure constant. E.g. at the Z mass, the standard mass to compare the strong coupling, we have an α_s of 0.11, whereas the fine structure constant would be around 0.007. Consequently to use PT we have to calculate our results to much higher orders, including tens of thousands of Feynman diagrams, in QCD to achieve a precision equal to QED. For even lower energies, around $1\,\text{GeV}$, the strong coupling reaches a critical value of around 0.5 leading to a break down of PT.

In this work we try to achieve a higher precision in the value of α_s. The framework we use to measure the strong coupling constant are the QCDSR. A central object needed to describe hadronic states with the help of QCD is the *two-point function* for which we will devote the following section.

1.2 Two-Point Function

In analogy to the Green's function for elemental fields we can define a propagator for composite currents, referred to as *two-point function*

$$\Pi(x) = \langle\Omega|T\{J(x)J(y)\}|\Omega\rangle, \tag{1.2.1}$$

where $T\{\cdots\}$ is the time-ordered product and $|\Omega\rangle$ is the ground state/vacuum of the interacting theory. Note that the fields are in general given in the Heisenberg picture, which implies translational invariance.

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \langle\Omega|\phi(x)e^{i\hat{P}y}e^{-i\hat{P}y}\phi(y)e^{i\hat{P}y}e^{-i\hat{P}y}|\Omega\rangle \\ &= \langle\Omega|\phi(x-y)\phi(0)|\Omega\rangle,\end{aligned} \tag{1.2.2}$$

where we made use of the translation operator $\hat{T}(x) = e^{-i\hat{P}x}$.

In this work we are especially concerned about the vacuum expectation value of the Fourier transform of two time-ordered QCD quark Noether currents

$$\Pi_\Gamma(p^2) \equiv \int \frac{d^4x}{(2\pi)^4} e^{ipx}\langle\Omega|T\left\{J_\Gamma(x)J_\Gamma(0)\right\}|\Omega\rangle, \tag{1.2.3}$$

where the Noether current is given by

$$J_\Gamma(x) = \overline{q}(x)\Gamma q(x). \tag{1.2.4}$$

Here, Γ can be any of the following dirac matrices $\Gamma \in \{1, i\gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$, specifying the quantum number of the current (*scalar* (S), *pseudo-scalar* (P), *vector* (V) and *axial-vector* (A), respectively). By choosing the right quantum numbers we can theoretically represent the processes we want to study, which will be important when we want to theoretically describe the hadrons produced in τ decays.

From a Feynman diagram point of view we can illustrate the two-point function as quark-antiquark pair, which is produced by an external source, e.g. the virtual W boson of $\tau\overline{\tau}$ annihilation as seen in fig. 1.3. Here the quarks are propagating at *short distances*, which implies that we can make use of PT, thus avoiding *long-distance* (NPT) effects, that would appear if the initial and

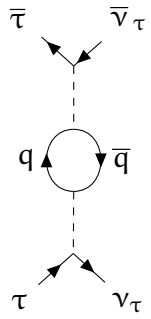

Figure 1.3: $\tau\overline{\tau}$-annihilation with a quark-antiquark pair.

final states where given by hadrons [**Colangelo2000**]. It is interesting to note, that the same process with the help of the *optical theorem* can be used to derive the total decay width of hadronic tau decays.

1.2.1 Short Distances vs. Long Distances

If we want to calculate the two-point function in QCD we have to differentiate short and long distances (large or small momenta). In general when we talk about small distances we refer to large momenta. Large momenta implies a small strong coupling. Consequently we can use PT for short distances without problems. On the contrary long distances involve small momenta, which implies a large coupling constant. Thus for long distances the NP effects become important and have to be dealt with. To apply PT to the case of the $\tau\overline{\tau}$ annihilation we need the quark-antiquark pair of fig. 1.3 to be highly virtual[1]. To roughly separate long distances from short distances using a length scale we can say that the length scale should be smaller than the radius of a hadron.

1.2.2 Relating the Two-Point Function to Hadrons

The two-point function can be interpreted physically as the amplitude of propagating single- or multi-particle states and their excitations. The possible states, in our case, the hadrons we describe through the correlator, are fixed by the quantum numbers of the current, we define for the vacuum expectation value. For example the neutral ρ meson is a spin-1 vector meson with a quark content of $(u\overline{u} - d\overline{d})/\sqrt{2}$. Consequently by choosing a current

$$J_\mu(x) = \frac{1}{2}(\overline{u}(x)\gamma_\mu u(x) - \overline{d}(x)\gamma_\mu d(x)) \tag{1.2.5}$$

the two-point function contains the same quantum numbers as the ρ meson and is said to materialise to it. A list of some ground-state mesons for combinations of the light quarks u, d and s is given in table 1.2.

[1] Which is the same of saying, that the quark-antiquark pair needs a high external momentum q.

Symbol	Quark content	Isospin	J	Current
π^+	$u\bar{d}$	1	0	$:\bar{u}\gamma_\mu\gamma_5 d:$
π^0	$\frac{u\bar{u}-d\bar{d}}{2}$	1	0	$:\bar{u}\gamma_\mu\gamma_5 u+\bar{d}\gamma_\mu\gamma_5 d:$
η	$\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}$	0	0	$:\bar{u}\gamma_\mu\gamma_5 u+\bar{d}\gamma_\mu\gamma_5 d-2\bar{s}\gamma_\mu\gamma_5 s:$
η'	$\frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}}$	0	0	$:\bar{u}\gamma_\mu\gamma_5 u+\bar{d}\gamma_\mu\gamma_5 d+\bar{s}\gamma_\mu\gamma_5 s:$
ρ^0	$\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}$	1	1	$:\bar{u}\gamma_\mu u-\bar{d}\gamma_\mu d:$
ω	$\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$	0	1	$:\bar{u}\gamma_\mu u+\bar{d}\gamma_\mu d:$
ϕ	$s\bar{s}$	0	1	$:\bar{s}\gamma_\mu\gamma_5 s:$
K^+	$u\bar{s}$	$\frac{1}{2}$	0	$:\bar{u}\gamma_\mu\gamma_5 s:$
K^0	$d\bar{s}$	$\frac{1}{2}$	0	$:\bar{d}\gamma_\mu\gamma_5 s:$

Table 1.2: Ground-state vector and pseudoscalar mesons for the light quarks u, d and s with their corresponding currents in the two-point function. Note that we use γ_μ for vector and $\gamma_\mu\gamma_5$ for the pseudoscalar mesons.

The correlator is materialising into a spectrum of hadrons. Thus if we insert a complete set of states of hadrons we can make use of the unitary relation

$$\langle\Omega|J_\mu(x)J_\nu(0)|\Omega\rangle = \sum_X \langle\Omega|J_\mu(x)|X\rangle\langle X|J_\nu(0)|\Omega\rangle \tag{1.2.6}$$

to represent the two-point correlator via a spectral function $\rho(t)$

$$\Pi(p^2) = \int_0^\infty ds \frac{\rho(s)}{s-p^2-i\epsilon}. \tag{1.2.7}$$

The above relation is referred to as *Källén-Lehmann spectral representation* [**Kallen1952**, **Lehmann1954**] or *dispersion relation*. It relates the two-point function to the spectral function ρ, which can be represented as sum over all possible hadronic states

$$\rho(s) = (2\pi)^3 \sum_X \int d\Pi_X \left|\langle\Omega|J_\mu(0)|X\rangle\right|^2 \delta^4(s-p_X). \tag{1.2.8}$$

Note that the analytic properties of the two-point function are in one-to-one correspondence with the newly introduced spectral function and thus determined by the possible hadrons states, which only form on the positive real axis.

A full derivation of the *Källén-Lehmann spectral representation* can be found in [**Rafael1997**]. The spectral function is interesting to us for two reasons. First, it is experimentally measurable and second it carries a "branch cut", which we want to discuss now.

1.2.3 Analytic Structure of the Two-Point Function

The general two-point function $\rho(s)$ has some interesting analytic properties. It has poles for single particle states and a continuous branch cut for multi particle states. The single and multi particle states, for a general correlator, can mathematically be separated by

$$\rho(s) = Z\delta(s - m^2) + \theta(s - s_0)\sigma(s), \tag{1.2.9}$$

where the second term is the contribution from multi particle states. $\sigma(s)$ is zero till we reach the threshold, where we have sufficient energy to form multi particle states. The analytic structure is depicted by fig. 1.4 and we can see that the spectral function has δ spikes for single particle states and a continuous contribution for $s \geqslant 4m$ resulting from multi particle states. These lead to poles and a continuous branch cut of the two-point function.

1.2.4 Decompositions

Apart the spectral decomposition we can also Lorentz decompose the two-point function or write it in terms of V, A, S and P contributions.

Lorentz decomposition

Due to the Lorentz invariance of the two-point function, and by assuming the conservation of the Noether current, we can apply the Ward identity to decompose the correlator $\Pi_{\mu\nu}$ into its scalar contribution Π.

There exist only two possible terms that can guard the structure of the second order tensor: $q_\mu q_\nu$ and $q^2 g_{\mu\nu}$. The sum of both multiplied with two arbitrary functions $A(q^2)$ and $B(q^2)$ yields

$$\Pi_{\mu\nu}(q^2) = q_\mu q_\nu A(q^2) + q^2 g_{\mu\nu} B(q^2). \tag{1.2.10}$$

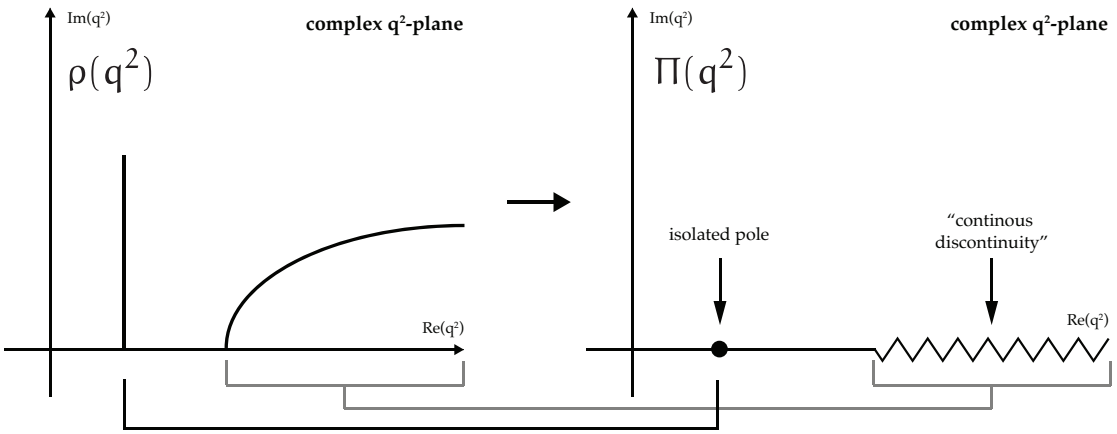

Figure 1.4: Analytic structure in the complex q^2-plane of the Fourier transform of the two-point function. The hadronic final states are responsible for poles appearing on the real-axis. The single particle states contribute as isolated pole and the multi particle states contribute as bound states poles or a continues "discontinuity cut" [**Peskin1995**, **Zwicky2016**].

By assuming that we deal with equal quark flavours and that the vector current is conserved, i.e. $\partial^\mu j_\mu(x) = 0$, we can make use of the *Ward identity*

$$q^\mu \Pi_{\mu\nu} = 0 \tag{1.2.11}$$

to demonstrate, that the two arbitrary functions are related

$$\begin{aligned} q^\mu q^\nu \Pi_{\mu\nu} = q^4 A(q^2) + q^4 B(q^2) = 0 \\ \implies A(q^2) = -B(q^2). \end{aligned} \tag{1.2.12}$$

Thus redefining $A(q^2) \equiv \Pi(q^2)$ we expressed the correlator as a scalar function of spin 1

$$\Pi_{\mu\nu}(q^2) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^{(1)}(q^2). \tag{1.2.13}$$

In case of a current of different quark flavours, the current will not be conserved and we cannot apply the Ward identity. Consequently the standard Lorentz decomposition into transversal and longitudinal components reads

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(1)}(q^2) + q^\mu q^\nu \Pi^{(0)}(q^2). \tag{1.2.14}$$

Transversal and Longitudinal Relations

By comparing the standard Lorentz decomposition (eq. 1.2.14) with the decomposition into V/A and S/P parts we can identify the longitudinal components of the correlator as being purely scalar. The latter decomposition can be written as [**Broadhurst1981, Jamin1992**]

$$\begin{aligned} q^2\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{V,A}(q^2) + g^{\mu\nu}(m_i \mp m_j)\Pi^{S,P}(q^2) \\ + g^{\mu\nu}(m_i \mp m_j)\left[\langle\Omega|\overline{q}_i q_i|\Omega\rangle \mp \langle\Omega|\overline{q}_j q_j|\Omega\rangle\right], \end{aligned} \tag{1.2.15}$$

where the third term is a correction arising due to the physical vacuum $|\Omega\rangle$. By multiplying eq. 1.2.15 by two four-momenta and making use of the Ward identity eq. 1.2.11 we can write

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = (m_i \mp m_j)^2\Pi^{S,P}(q^2) + (m_i \mp m_j)[\langle\overline{q}_i q_i\rangle \mp \langle\overline{q}_j q_j\rangle], \tag{1.2.16}$$

which then can be related to the longitudinal component of eq. 1.2.14 by comparison with

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = q^4\Pi^{(0)}(q^2) = s^2\Pi^{(0)}(s) \quad \text{with} \quad s \equiv q^2, \tag{1.2.17}$$

leading to

$$s^2\Pi^{(0)}(s) = (m_i \mp m_j)^2\Pi^{(S,P)}(s) + (m_i \mp m_j)[\langle\overline{q}_i q_i\rangle \mp \langle\overline{q}_j q_j\rangle]. \tag{1.2.18}$$

Note that all appearing mass terms are related to the longitudinal component.

As the τ decays, with the limiting factor of the tau mass, can only decay into light quarks we will often neglect the quark masses and work in the so-called *chiral limit* ($m_q \to 0$), in which the longitudinal component is going to vanish.

By defining a combination of the transversal and longitudinal correlator

$$\Pi^{(1+0)}(s) \equiv \Pi^{(1)}(s) + \Pi^{(0)}(s) \tag{1.2.19}$$

we can additionally relate the transversal and vectorial components via

$$\Pi^{\mu\nu}(s) = \underbrace{(q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(1)}(s) + (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(1)}(s)}_{=(q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(1+0)}(s)} + \frac{g^{\mu\nu}s^2}{q^2}\Pi^{(0)}(s), \tag{1.2.20}$$

such that

$$\Pi^{(V,A)}(s) = \Pi^{(1)}(s) + \Pi^{(0)}(s) = \Pi^{(1+0)}(s), \tag{1.2.21}$$

where the vector/axial-vector component of the correlator is now related to the newly defined transversal and longitudinal combination of the correlator.

Having dealt exclusively with the perturbative part of the theory, we have to discuss NP contributions. These arise due to non negligible long distance effects. Thus to complete the needed ingredients for the QCDSR we need a final ingredient the OPE, which treats the NP contributions of our theory.

1.3 Operator Product Expansion

The OPE was introduced by Wilson in 1969 **[Wilson1969]** as an alternative to the in this time commonly used current algebra. The expansion states that products of operators at different space-time points can be rewritten into a sum of composite local operators and their corresponding coefficients:

$$\lim_{x\to y} A(x)B(y) = \sum_n C_n(x-y)\mathcal{O}_n(x), \tag{1.3.1}$$

where $C_n(x-y)$ are the so-called *Wilson coefficients* and A, B and $\mathcal{O}_n$ are operators.

The OPE lets us separate short distances from long distances. In pure PT we can only amount for short distances, which are equal to high energies, where the strong coupling α_s is small. The OPE on the other hand accounts for long-distance effects with higher dimensional operators. Applying the OPE to the two-point function we get a sum over the vacuum expectation values

$$\Pi_{OPE}(q^2) = -\frac{1}{3q^2}\sum_n \langle\Omega|\mathcal{O}_n(0)|\Omega\rangle \int d^4x e^{iqx} C_n(x). \tag{1.3.2}$$

The form of the composite operators are dictated by gauge and Lorentz symmetry. For the two-point function in eq. 1.3.2 we only have to consider operators $\mathcal{O}_n$ of dimension

$$d(\mathcal{O}_n) \leqslant (D-4) + 2N. \tag{1.3.3}$$

The scalar operators up to dimension six are then given by [**Pascual1984**]

$$\begin{aligned}
&\text{Dimension 0:} && \mathbb{1} \\
&\text{Dimension 2:} && m_i^2 \\
&\text{Dimension 4:} && :m_i\overline{q}\,q: \\
& && :G_a^{\mu\nu}(x)G_{\mu\nu}^a(x): \\
& && m_i^4 \\
&\text{Dimension 6:} && :\overline{q}\,\Gamma q\overline{q}\,\Gamma q: \\
& && :\overline{q}\,\Gamma\tfrac{\lambda^a}{2}q_\beta(x)\overline{q}\,\Gamma\tfrac{\lambda^a}{2}q: \\
& && :m_i\overline{q}\,\tfrac{\lambda^a}{2}\sigma_{\mu\nu}qG_a^{\mu\nu}: \\
& && :f_{abc}G_a^{\mu\nu}G_b^{\nu\delta}G_c^{\delta\mu}:,
\end{aligned} \tag{1.3.4}$$

where Γ stands for one of the possible dirac matrices (as seen eq. 1.2.4). Note, that the $D = 2$ operator violates gauge symmetry and is consequently excluded from our list. Within PT only the unit operator would exist, as the higher dimensional operators would appear as normal ordered products of fields and vanish by being sandwiched into the perturbative vacuum. On the contrary, in NP QCD they appear as *condensates*. Condensates are the vacuum expectation values of non-vanishing normal ordered fields by applying the full QCD vacuum, which contribute to all strong processes. For example the condensates of dimension four are the quark-condensate $m_i\langle\overline{q}\,q\rangle$ and the gluon-condensate $\langle GG\rangle$.

As long as working with dimensionless functions (e.g. the correlator Π in eq. 1.3.2), the *right-hand side* (RHS) of eq. 1.3.1 has to be dimensionless. As a result the Wilson coefficients have to cancel the dimension of the operator with their inverse mass dimension. To account for the dimensions we can make the inverse momenta explicit

$$\Pi_{V/A}^{OPE}(s) = \sum_{D=0,2,4\ldots} \frac{C^{(D)}\langle\Omega|\mathcal{O}^{(D)}(x)|\Omega\rangle}{(-q^2)^{D/2}}, \tag{1.3.5}$$

where we used $c^{(D)} = C^{(D)}/(-s)^{D/2}$ with D being the dimension. Thus the OPE should converge with increasing dimension for sufficiently large momenta s.

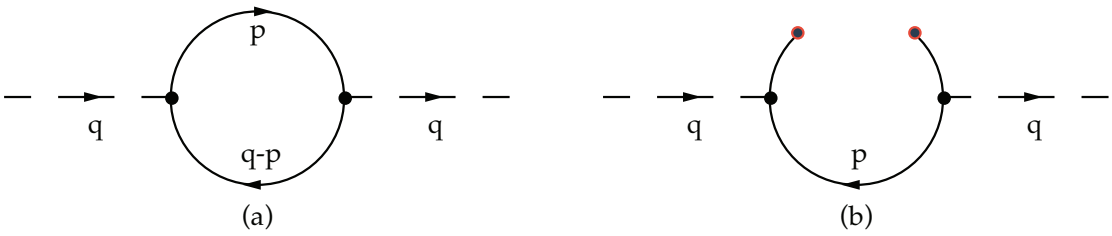

Figure 1.5: Feynman diagrams of the perturbative (a) and the quark-condensate (b) contribution. The upper part of the right diagram is not Wick contracted and responsible for the condensate.

1.3.1 A practical example

Let us show how the OPE contributions are calculated with a standard example [**Shifman1978**, **Pascual1984**]. We will compute the perturbative and quark condensate Wilson coefficients for the ρ meson. To do so we have to evaluate Feynman diagrams using standard PT.

The ρ meson is a vector meson of isospin one composed of u and d quarks. As a result (see. table 1.2) we can match its quantum numbers with the current

$$J^{\mu}(x) = \frac{1}{2}\left(:[\overline{u}\gamma^{\mu}u](x) - [\overline{d}\gamma^{\mu}d](x):\right). \tag{1.3.6}$$

Pictorial the dimension zero contribution is given by the quark-antiquark loop Feynman diagram in fig. 1.5. The higher dimension contributions are given by the same Feynman diagram, but with non contracted fields. These non contracted fields contain the condensates. Thus not contracting the quark-antiquark field (see. fig. 1.5 b) will give us access to the Wilson coefficient of the dimension four quark condensate $m_i\langle\overline{q}\,q\rangle$.

The perturbative part (the Wilson coefficient of dimension zero) can than be taken from the mathematical expression for the scalar correlator

$$\begin{aligned}\Pi(q^2) = -\frac{i}{4q^2(D-1)}\int d^D x e^{iqx}\langle\Omega|T\{:\overline{u}(x)\gamma^{\mu}u(x) - \overline{d}(x)\gamma^{\mu}d(x):\\ \times :\overline{u}(0\gamma_{\mu}u(0) - \overline{d}(0)\gamma_{\mu}d(0):\}\rangle.\end{aligned} \tag{1.3.7}$$

To extract the dimension zero Wilson coefficient we apply Wick's theorem to contract all of the fields, which represents the lowest order of the perturbative contribution. The calculation is solely using standard PT and we will restrict

ourselves in displaying the result and omitting the calculation[2]

$$\begin{aligned}\Pi(q^2) &= \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D x e^{iqx} \\ &\times \quad \left[\overbrace{u_{j\alpha}(x)\overline{u}}_{k\beta}(0)\cdot\overbrace{u_{l\beta}(0)\overline{u}}_{i\alpha}(x) + (u\to d)\right] \\ &= \frac{3}{8\pi^2}\left[\frac{5}{3} - \log\left(-\frac{q^2}{\nu^2}\right)\right].\end{aligned} \tag{1.3.8}$$

To calculate the higher dimensional contributions of the OPE we use the same techniques as before. We apply Wick's theorem, but in this case, due to the NP vacuum, we have non-vanishing vacuum expectation value of normal ordered products of fields. Thus some of the fields are left uncontracted, as can be graphically seen in fig. 1.5. For leaving the quark field uncontracted in eq. 1.3.7 we get

$$\begin{aligned}\Pi(q^2) = \frac{i}{4q^2(D-1)}(\gamma^\mu)_{ij}(\gamma_\mu)_{kl}\int d^D x e^{iqx}\Big[\\ + \overbrace{u_{j\alpha}(x)\overline{u}}_{k\beta}(0)\cdot\langle\Omega|:\overline{u}_{i\alpha}(x)u_{l\beta}(0):|\Omega\rangle \\ +\overbrace{u_{l\beta}(0)\overline{u}}_{i\alpha}(x)\cdot\langle\Omega|:\overline{u}_{k\beta}(0)u_{j\alpha}(x):|\Omega\rangle + (u\to d)\Big],\end{aligned} \tag{1.3.9}$$

where $(u \to d)$ is representing the previous expressions with u and d interchanged. Here we can observe the condensates as non-vanishing vacuum values of normal ordered product of fields:

$$\langle\Omega_{QCD}|:\overline{q}(x)q(0):|\Omega_{QCD}\rangle \neq 0. \tag{1.3.10}$$

We emphasised the QCD vacuum Ω_{QCD}, which is responsible for vacuum expectation values different than zero. E.g. for a vacuum of QED this contributions would vanish by definition. Pictorial the condensates take form of unconnected propagators, sometimes marked with an $\times$, as seen in fig. 1.5.

To make the non-contracted fields local, we can expanded them in x

$$\begin{aligned}\langle\Omega|:\overline{q}(x)q(0):|\Omega\rangle = \langle\Omega|:\overline{q}(0)q(0):|\Omega\rangle \\ + \langle\Omega|:\left[\partial_\mu\overline{q}(0)\right]q(0):|\Omega\rangle x^\mu + \ldots,\end{aligned} \tag{1.3.11}$$

[2]The interested reader can follow **[Pascual1984]** for a detailed calculation.

where terms with derivatives lead to higher dimensional operators, which can be seen by applying the equation of motions. We then can focus on the first term and introduce a standard notation for the localised condensate

$$\langle \overline{q}\, q \rangle \equiv \langle \Omega | : \overline{q}\,(0) q(0) : | \Omega \rangle. \tag{1.3.12}$$

Finally, the contribution to the ρ scalar correlator is then given by the following expression

$$\Pi_{(\rho)}(q^2) = \frac{1}{2} \frac{1}{(Q)^2} \left[m_u \langle \overline{u}\, u \rangle + m_d \langle \overline{d}\, d \rangle \right], \tag{1.3.13}$$

where we defined $Q \equiv -q^2$. Here we can clearly see that for dimension four we get a factor of $1/(Q)^2$, which is responsible for the suppression of the series. The condensates $\langle \overline{u}\, u \rangle$ and $\langle \overline{d}\, d \rangle$ are numbers, that have to be derived by phenomenological fits or computed from LQCD. Fortunately once found, the value of the condensate can be used for any process.

In summary we note that the usage of the OPE and its validity is far from obvious. Until today there is no analytic proof of the OPE. Furthermore we are deriving the OPE from matching the Wilson coefficients to Feynman graph analyses. These Feynman graphs are calculated perturbatively but the coefficients with dimension $D > 0$ correspond to NP condensates! The condensates by themselves have to be gathered from external, NP methods.

Now that we have a tool to deal with the QCD vacuum and NPT effects we are left with two problems. First, we still do not know how to deal with hadronic states in the quark-gluon picture. This will be tackled by duality. Secondly, we have seen that we can access the two-point function theoretically on the physical sheet except for the positive real axis, due to its analytic properties. Unfortunately the experimental measurable spectral function is solely be defined on this positive real axis, which is theoretically not accessible. To match the theory with the experiment we will have to apply Cauchy's theorem. In the final section of this chapter we will combine the two-point function, the OPE, duality and Cauchy's theorem to formulate the QCDSR.

1.4 Sum Rules

The QCDSR are a method to connect the *degrees of freedom* (DOF) of QCD, the quarks and gluon fields, to the DOF of the vacuum spectrum of hadrons, thereby allowing for the determination of the strong coupling. To do so we have to treat the in section 1.2 introduced two-point function NP with the help of the OPE

$$\Pi(s) \to \Pi_{OPE}(s). \tag{1.4.1}$$

QCDSR furthermore introduce an ad hoc assumption, namely *quark-hadron duality*, stating that the observable hadron picture can be equally described by the QCD quark-gluon picture and that both pictures are equally valid. As the experimentally measured hadronic states are represented in poles and cuts on the positive real axis of the two-point function, which we have encountered in the analytic properties of its spectral decomposition, we will follow the prescription of QCDSR to apply *Cauchy's theorem* and weight functions to take care of perturbative complications close to the positive real axis.

1.4.1 The Dispersion Relation

We have already seen the Källén-Lehmann spectral representation in eq. 1.4.2. The general dispersion relation is defined to have an additional polynomial function $P(s)$

$$\Pi(s) = \int_0^\infty \frac{\rho(s')}{s' - s - i\epsilon} + P(s), \tag{1.4.2}$$

which accounts for the fact, that the two-point function increases for large s, but the integral on the RHS cannot reproduce this behaviour. For example the vector correlator carries only a constant and the scalar correlator a linear polynom. The two-point function is in general an unphysical quantity, whereas the spectral function $\rho(s)$ is a physical quantity. As a result the polynomial accounts carries the unphysical scale dependency of the two-point function.

1.4.2 Duality

QCD treats quarks and gluon as its fundamental DOF, but due to confinement we are only ever able to observe hadrons. The mechanism that connects the two worlds is the *quark-hadron duality* (or simply duality), which implies that physical quantities can be described equally good in the hadronic as in the quark-gluon picture. Thus we can connect experimental detected with theoretically calculated values from the two-point function in the dispersion relation eq. 1.4.2 as

$$\Pi_{th}(s) = \int_0^\infty \frac{\rho(s')_{exp}}{s' - s - i\epsilon} + P(s), \tag{1.4.3}$$

where we connected the theoretical correlator Π_{th} with the experimental measurable spectral function ρ_{exp}. We can represent duality as, substituting the two-point function [**Cata2005**]

$$\Pi(s) \to \Pi_{OPE}(s). \tag{1.4.4}$$

If this approximation carries no error, we would say that the experimental spectral function is dual to the OPE. On the contrary if the substitution is not exact we are missing contributions, which are represented by so-called DV.

Duality Violations

There exist situations where we cannot make use of duality as an assumption. These situation are referred to as DV and belong to the NP part of the theory. It is often assumed that by applying the OPE to all orders we account for all NP effects, including DV. Unfortunately this assumption is only partly right. Even if we could compute the OPE to all orders, we would still experience discrepancies to our theoretical results. In general it is said, that if we have deviations beyond the natural uncertainty of the OPE we call them DV [**Shifman2000**]. E.g. if we compute $\Pi(s)$ to orders of α^2 and $\frac{1}{Q^4}$, while we cutoff higher orders (α^3 and $\frac{1}{Q^6}$) we get a natural error, because we have not calculated the full series. Values of the hadronic spectral density, out of range of the natural error, are then referred to as DV.

A detailed discussion of duality has been given by the Shifman in [**Shifman2000**].

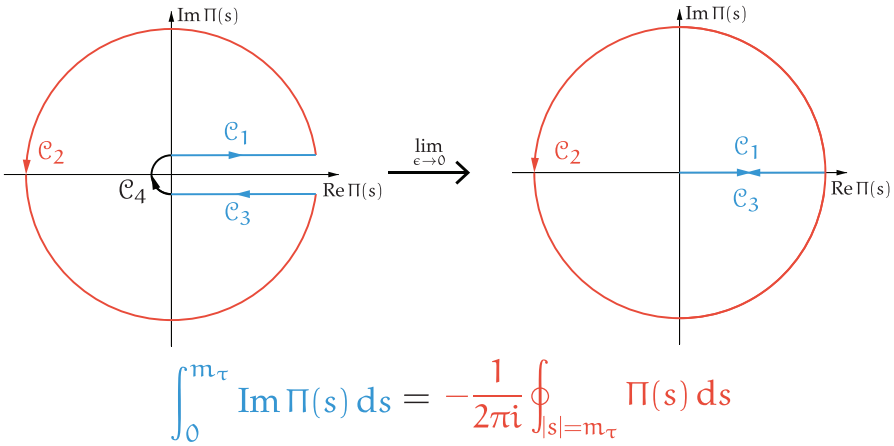

Figure 1.6: Visualisation of the usage of Cauchy's theorem to transform eq. 1.4.2 into a closed contour integral over a circle of radius s_0.

1.4.3 Finite Energy Sum Rules

To theoretically calculate the two-point function we have to integrate the experimental data $\rho_{exp}(s)$ from zero to infinity. No experiment will ever take data for an infinite momentum s. For τ decays we are limited to energies around the τ mass of $1.776\,GeV$. To deal with the upper integration limit several approaches have been made. One of them, the *Borel transform*, is to exponentially suppress higher energy contributions (see [**Weinberg1996**, **Rafael1997**]). The technique we are focusing on is called *finite energy sum rules* (FESR) and introduces a energy cut-off. We thus integrate the experimental data $\rho(s)$ only to a certain energy s_0. Furthermore we have to theoretically evaluate the integral over the spectral function of the dispersion relation (eq. 1.4.2), which includes singularities caused by the hadronic spectrum. As a result we have to apply Cauchy's theorem

$$\oint_{\mathcal{C}} f(z) = 0, \tag{1.4.5}$$

which states that any integral over an analytic function $f(z)$ on a closed contour $\mathcal{C}$ has to be zero. Thus we can construct a contour to avoid the positive problematic real axis. Pictorial the contour is drawn in fig. 1.6 and mathematically we can express it as

$$\oint \Pi(s) = \int_0^{s_0} \Pi(s+i\epsilon) - \Pi(s-i\epsilon)\,ds + \int_{0+\alpha(\epsilon)}^{2\pi-\alpha(\epsilon)} \Pi(s_0 e^{i\theta})\,d\theta + \int_{3\pi/2}^{\pi/2} \Pi(\epsilon e^{i\theta})\,d\theta. \tag{1.4.6}$$

If we make to use of *Schwartz reflection principle*:

$$f(\bar{z}) = \overline{f(z)}, \tag{1.4.7}$$

which can be applied if $f(z)$ is analytic and maps only to real values on the positive real axis, we can express the integrand of the first integral of eq. 1.4.6 as the imaginary part of the two-point function

$$\Pi(s + i\epsilon) - \Pi(s - i\epsilon) = \Pi(s + i\epsilon) - \Pi^*(s + i\epsilon) = 2i \operatorname{Im} \Pi(s + i\epsilon), \tag{1.4.8}$$

which is by definition equal to the spectral function

$$\rho(s) \equiv \frac{\operatorname{Im} \Pi(s)}{\pi}. \tag{1.4.9}$$

After taking the limit of small ϵ we can relate the line integral with the lower limit zero and the upper limit s_0 and the experimental spectral function as integrand to a theoretical accessible circular contour integral of radius s_0

$$\int_0^{s_0} \rho(s) = \frac{-1}{2\pi i} \oint_{|s|=s_0} \Pi(s)\, ds, \quad \text{where we applied} \quad \epsilon \to 0. \tag{1.4.10}$$

Note that the unphysical contribution of the polynomial in eq. 1.4.2 cancel in the contour integral.

We are free to multiply the upper equation with an analytic function $\omega(s)$, which completes the FESR

$$\int_0^{s_0} \omega(s)\rho(s) = \frac{-1}{2\pi i} \oint_{|s|=s_0} \omega(s)\Pi_{OPE}(s)\, ds \tag{1.4.11}$$

where the *left-hand side* (LHS) can be taken from experiment and the RHS by the theoretically evaluated correlator $\Pi_{OPE}(s)$. The analytic function $\omega(s)$ plays the role of a weight. It can be used to further suppress the non-perturbative contributions coming from DV and also enhance or suppress different contributions of the OPE as we will see.

1.4.4 Weighting OPE dimensions

We have seen that the perturbative part of the two-point function carries a discontinuity on the positive real axis. Consequently we applied Cauchy's

theorem to avoid the non-analytic part of the two-point function. This left us with non-closed contour integral for the perturbative part of the OPE, which will always contribute. On the other hand, the strength of the higher dimension contributions of the OPE can be modified. We can use different weights to control the dimensions of the OPE that contribute. The weights we are using have to be analytic, so that we can make use of Cauchy's theorem. Thus they can be represented as polynomials

$$\omega(x) = \sum_i a_i x^i, \tag{1.4.12}$$

every contributing monomial is responsible for a dimension of the OPE. Dimensions that are not represented in the weight polynomial do not contribute at all or are very suppressed as we will demonstrate now.

The residue of a monomial x^k is only different from zero if its power $k = -1$:

$$\oint_C x^k \, dx = i \int_0^{2\pi} \left(e^{i\theta}\right)^{k+1} d\theta = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise} \end{cases}. \tag{1.4.13}$$

We will see in discussing the total τ decay ratio, that the integrand of the closed-contour integral in eq. 1.4.11 for the different OPE contributions is the weight function divided by a term proportional to $x^{D/2}$, where D is the dimension of the contributing OPE operator. If we regard solely a monomial as weight and neglect all terms of no interest to us we can write

$$\begin{aligned} R'(x)\big|_{D=0,2,4\ldots} &= \oint_{|x|=1} dx \frac{x^k}{x^{\frac{D}{2}}} C^D \\ &= \oint_{|x|=1} dx \, x^{k-D/2} \, C^D, \end{aligned} \tag{1.4.14}$$

where C^D are the D dimensional Wilson coefficients. We can make use of eq. 1.4.14 as we neglect the logarithmic corrections of dimension four ope contributions and use an simple ansatz for the higher order OPE dimension, which treat the Wilson coefficients as a constant. Thus combining eq. 1.4.13 with eq. 1.4.14 we see that only Dimension which fulfil

$$k - D/2 = -1 \quad \Longrightarrow \quad D = 2(k+1) \tag{1.4.15}$$

contribute to the OPE. For example the polynomial of the kinematic weight

$$\omega_\tau(s) \equiv \left(1 - \frac{s}{m_\tau^2}\right)\left(1 + 2\frac{s}{m_\tau^2}\right) \tag{1.4.16}$$

monomial:	x^0	x^1	x^2	x^3	x^5	x^6	x^7
dimension:	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

Table 1.3: List of monomial and their corresponding "active" dimensions in the OPE. Note that the perturbative contributions of the OPE are always present.

, which will appear naturally in the context of the total τ decay ratio, is given by

$$(1-x)^2(1+2x) = \underbrace{1}_{D=2} - 3\underbrace{x^2}_{D=6} + 2\underbrace{x^3}_{D=8}, \tag{1.4.17}$$

where the underbraced monomials express the active dimensions. A list of monomials and their corresponding dimensions up to dimension 14 can be found in table 1.3. This behaviour enables us to bring out different dimensions of the OPE and suppress contributions of higher order ($D \geqslant 10$) for which less is known.

For the interested reader we gathered several introduction texts to the QCDSR, which where of great use to us [**Narison1989**, **Rafael1997**, **Colangelo2000**, **Dominguez2013**].