

# Measuring the strong coupling

Table 1.1: Timeline

1991	•	[ <b>Braaten1991</b> ]: Systematic description, including NP corrections to extract $\alpha_s$ from $R_\tau$ .
1992	•	[ <b>LeDiberder1992</b> ]: Introducing weights and fit methodology later used by ALEPH [ <b>Aleph1993</b> ] and OPAL [ <b>Opal1998</b> ] collaborations
2011	•	[ <b>Boito2011a</b> ]: Include DV model to extract $\alpha_s$ from OPAL data.
2018	•	[ <b>Boito2018</b> ]: $\alpha_s$ from $e^+e^- \rightarrow$ hadrons up to 2 GeV.

The strong coupling has been measured since many years from hadronic  $\tau$  decays. Until today most of the applied QCDSR to  $\tau$  decays are based on the methodology developed in the early nineties by Braaten, Pich and Narison [**Braaten1991**]. They gathered the at this time available perturbative and NP contributions to extract the strong coupling from comparing their theoretical results to the known inclusive hadronic  $\tau$  decay ratio  $R_\tau$ . Pich together with Le Dibergerger then formulated the fitting strategy of fitting multiple moments of different weights to extract  $\alpha_s$  parallel to Wilson coefficients of the OPE [**LeDiberger1992**], which later has been applied as standard in the ALEPH [**Aleph1993**] as well

as the OPAL [**Opal1998**] detectors. For the next ten year years the methodology of extracting the strong coupling did not run through make major changes until in the year 2011 when Boito, Cata, Golterman, Jamin, Osborn and Peris [**Boito2011a**] applied a duality model to include known DV effects to the QCD analysis of  $\tau$  decays. The group around Boito and Pich have different opin-

ions on the importance of the newly introduced duality model [Pich2016, Boito2016] and consequently we want to deliver a third, opinion on the subject, favouring fits without the duality model.

## 1.1 Fit Strategy

The objective of this work is to extract  $s$  and argue about the importance of  $DV$ . Apart from the two main objectives we want to analyse the values of the OPE Wilson coefficients up to dimension ten.

Our fitting strategy will be in choosing weights of lower and higher pinching. Lower pinched weights should be affected by  $DV$ , while higher pinched weights should be protected from  $DV$ . As a result in comparing different fits of lower and higher pinched weights it should be possible to argue about strength of the  $DV$  that are (or are not) present.

Our hypothesis is that  $DV$  are small enough for fits of the combined vector and axial-vector channel in combination with pinched weights. Consequently we can extract parameters, like the strong coupling  $\alpha_s$  from  $\tau$  decays to high precision without a  $DV$  model.

We will perform our analysis in the framework of FOPT, but display our final results also in CIPT. Consequently to define a fit we have to choose a weight  $\omega$  and a momentum  $s_0$ . The only restriction from choosing a weight is, that the weight has to be analytic, leaving us with a variety of choices. For our strategy we have chosen three categories of weights, each of them containing fits with three or four different weights. A table with an overview of all used weights is given in [table 1.2](#) To test for the stability of the fitted values and have enough DOF to fit the higher OPE contributions we furthermore fit every weight for various momenta  $s_0$ .

## 1.2 Fits

In the following we will give the results of each of the three previously mentioned fit categories.

	Symbol	Term	Expansion	OPE Contributions
Pinched	$\omega_\tau$	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	D6, D8
	$\omega_{\text{cube}}$	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	D6, D8, D10
	$\omega_{\text{quartic}}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	D6, D8, D10, D12
Monomial	$\omega_{M2}$	$1-x^2$	$1-x^2$	D6
	$\omega_{M3}$	$1-x^3$	$1-x^3$	D8
	$\omega_{M4}$	$1-x^4$	$1-x^4$	D10
Pinched +x	$\omega_{X1}$	$(1-x)$	$1-x$	D4
	$\omega_{X2}$	$(1-x)^2$	$1-2x+x^2$	D4, D6
	$\omega_{X3}$	$(1-x)^3$	$1-3x+3x^2-x^3$	D4, D6, D8
	$\omega_{X4}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	D4, D6, D8, D10

Table 1.2: Displaying three categories of fits, each containing three weights with their corresponding mathematical expression and the OPE contributions the fitted integral momentum will be sensitive to.

The first category contains the *Pinched Weights without Monomial  $x$* . The chosen weights are double ( $\omega_\tau$ ), triple ( $\omega_{\text{cube}}$ ) and quadruple ( $\omega_{\text{quartic}}$ ) pinched and do not contain a monomial term  $x$ . An  $x$  term would make the fits sensitive to the  $D = 4$  OPE contribution, which causes an unreliable perturbative expansion [Beneke2012]. The higher the pinching, the higher the suppression of  $\text{dV}$ . Consequently if we obtain stable values for  $\alpha_s$  from the different pinched fits we should expect the  $\text{dV}$  to have no influence on the value of the strong coupling. The different weights imply an increasing number of active OPE contributions  $D_6, D_8, D_{10}$  and  $D_{12}$ , which can be used to compare to the stability of higher order OPE contributions and to test for the convergence of the OPE.

The second category contains the *Single Pinched Monomial weights*. In this case all of the weights are only single pinched and, as in the first category, do not carry a monomial in  $x$ . Consequently if  $\text{dV}$  affect the fits we should notice different fitting results in comparison to the fits of the first category. Furthermore the single pinched moments only carry two parameters, the strong coupling and an OPE Wilson coefficients. Thus we can further compare the  $C_6, C_8$  and  $C_{10}$  Wilson coefficients and argue about the stability of the fits.

The third and last category contains a similar pinching as the first category, but this time contains a monomial term in  $x$ . Consequently these fits are unreliable in the framework of FOPT and we have to apply the *Borel sum* (BS). Following the logic of the second and first category we then can compare the result to analyse the role of DV and compare the Wilson coefficients.

### 1.2.1 Pinched Weights without a Monomial $x$

$$\text{Kinematic weight: } \omega_\tau(x) \equiv (1-x)^2(1+2x)$$

We previously encountered the kinematic weight in ???. It is a polynomial weight function, defined as  $\omega_\tau(x) = (1-x)^2(1+2x)$ , double pinched, contains the unity and does not contain a term proportional to  $x$ . Consequently it is an optimal weight [Beneke2012]. As a doubled pinched weight it should have a good suppression of DV contributions and its polynomial contains terms proportional to  $x^2$  and  $x^3$ , which makes it sensitive to the dimension six and eight OPE contributions. The fits have been performed within the framework of FOPT for different numbers of  $s_0$ . The momentum sets are characterised by its lowest energy  $s_{\min}$ . We fitted values down to 1.5 GeV. Going to lower energies is questionable due to the coupling constant becoming large, which implies a breakdown of PT. Furthermore it bares the risk to be affected by the  $\rho(770)$  and  $a_1$  peaks in the vector and axial-vector spectral function, which we cannot model within the framework of the OPE. For the three fitting parameters  $\alpha_s$ ,  $C_6$  and  $C_8$  we have given the results in table 1.3 and graphically in fig. 1.1.

We only display the fits for  $s_{\min}$  larger than 2.1 GeV. We noted a jump between the  $s_{\min} = 2.1$  GeV and  $s_{\min} = 2.2$  GeV of the  $\chi^2/\text{dof}$  from 0.19 to 1.3. We consequently discarded fits with a  $s_{\min} < 2.2$  GeV, as fits lower  $s_{\min}$  behave more stable<sup>1</sup>. The values for the less momenta are preferred by us due to two reasons. First below energies of 2.2 GeV we have to face the problematic influence of increasing resonances. Second, we will see, that the values obtained from the lower moment fits are more compatible with our other fits series. We further discarded the fit with four  $s_0$ s momenta, which has very small  $\chi^2/\text{dof} = 0.01$ . This is due to the fact, that we have four  $s_0$ s momenta to

<sup>1</sup>As will be seen by comparing the kinematic weight with the cubic and quartic weight

$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$C_6$	$C_8$	$\chi^2/\text{dof}$
2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

Table 1.3: Table of our fitting values of  $\alpha_s(m_\tau^2)$ ,  $C_6$  and  $C_8$  for the kinematic weight  $\omega(x) = (1-x)^2(1+2x)$  using FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

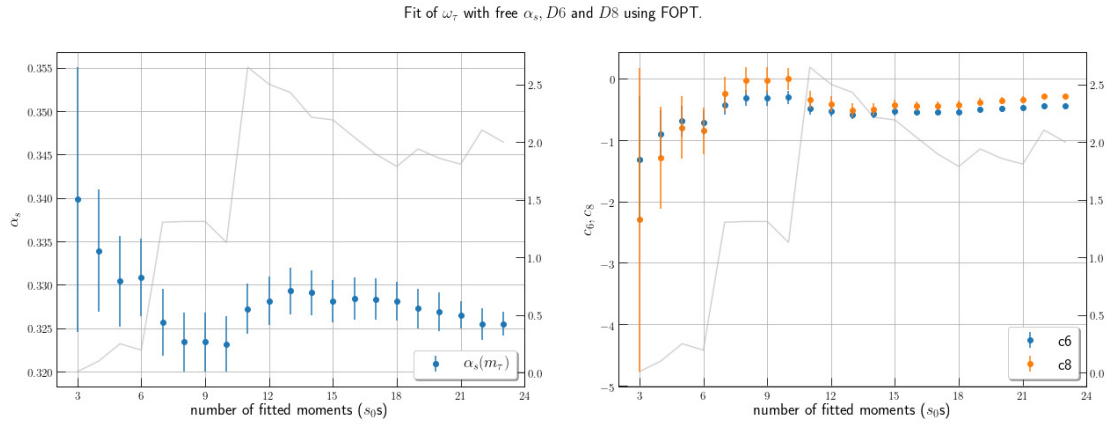


Figure 1.1: Fitting values of  $\alpha_s(m_\tau^2)$ ,  $C_6$  and  $C_8$  for the kinematic weight  $\omega(x) = (1-x)^2(1+2x)$  using FOPT for different  $s_{\min}$ . The left graph plots  $\alpha_s(m_\tau^2)$  for different numbers of used  $s_0s$ . The right plot contains the dimension six and eight contributions to the OPE. Both plots have in grey the  $\chi^2$  per dof.

fit three parameters, which leaves us with too few DOF.

The selected fits with 8-10 momenta have a small  $\chi^2$  per DOF. The fitted parameters,  $\alpha_s$ ,  $c_6$  and  $c_8$  are in good agreement with each other. For all fits we have a good convergence of the OPE. For later comparisons we will give the means for the strong coupling,  $D = 6$  and  $D = 8$  contributions:

$$\alpha_s(m_\tau^2) = 0.3317(33), \quad C_6 = -0.77(17) \quad \text{and} \quad C_8 = -0.98(35). \quad (1.2.1)$$

We further tested the stability of the dimension six and eight contributions to the OPE within the same fit series but for a fixed value of the strong coupling to our previous averaged result  $\alpha_s(m_\tau^2) = 0.3179$ . The values for  $C_6$  and  $C_8$  are larger than the values given in our final results from [table 1.3](#). This is explained with a smaller contribution from the strong coupling, which has to be compensated by larger OPE contributions.

$$\textbf{Cubic weight: } \omega_{\text{cube}}(x) \equiv (1 - x)^3(1 + 3x)$$

To further consolidate the results from the kinematic weight, we tested a weight of higher pinching, which should suppress DV more than a double pinched weight. Consequently, if we obtain similar results to our previous fits we could exclude DV effects for the kinematic weight. On the other hand, any differences to the previous fit would indicate present DV in the kinematic weight. Our *cubic* weight will be triple pinched and optimal, as it does not contain a  $x$  monomial. It is due to its polynomial structure sensitive to the dimensions six, eight and ten contributions of the OPE, which yields one more parameter to fit than with the kinematic weight  $\omega_\tau$ . The fitting results can be seen in [table 1.4](#) and graphically in [fig. 1.2](#).

As before we performed fits for  $s_0 \leq 1.5$  GeV, but could only reach convergence for fits with energies larger or equal than 1.8 GeV. As before the  $\chi^2$  makes a jump at  $s_0 = 2.1$  GeV to values per DOF of almost 2. Consequently we excluded these fits and focused on fits from five to eight  $s_0$ s momenta.

The selected fits have a good  $\chi^2/\text{dof}$  and the fitted parameters,  $\alpha_s$ ,  $C_6$ ,  $C_8$  and  $C_{10}$  are in agreement with each other, except for the fit with six momenta. The fit with a  $s_{\min} = 2.3$  GeV has the lowest  $\chi^2 = 0.29$  and error on  $\alpha_s$ , but takes slightly different values for the OPE Wilson coefficients in comparison to

$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$C_6$	$C_8$	$C_{10}$	$\chi^2/\text{dof}$
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

Table 1.4: Table of our fitting values of  $\alpha_s(m_\tau^2)$ ,  $C_6$ ,  $C_8$  and  $C_{10}$  for the cubic weight  $\omega(x) = (1-x)^3(1+3x)$  using FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

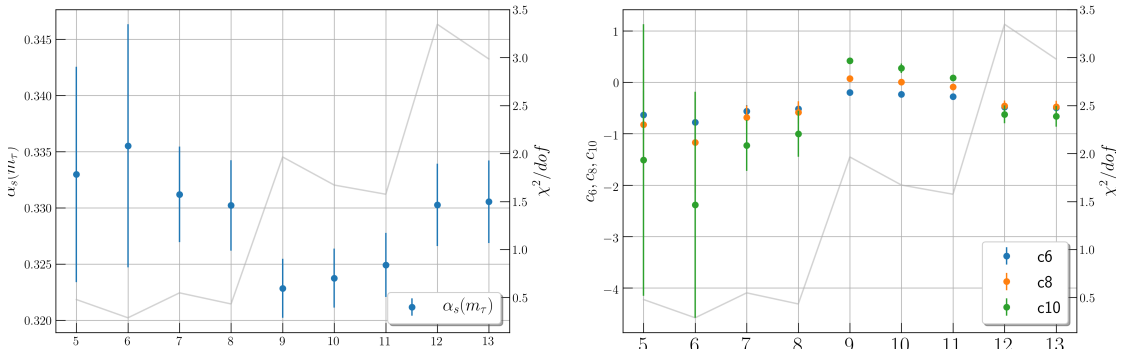


Figure 1.2: Graphic representation of the fitting values of  $\alpha_s(m_\tau^2)$  in the left and  $C_6$ ,  $C_8$  and  $C_{10}$  in the right plot for the cubic weight  $\omega(x) = (1-x)^3(1+3x)$ . The fits have been performed in the FOPT scheme and the data points are given with error bars and are ordered by increasing  $s_{\min}$ . The grey line in displays the  $\chi^2$  function.

the other selected fits. The means for the strong coupling,  $D = 6$  and  $D = 8$  contributions:

$$\alpha_s(m_\tau^2) = 0.33, \quad C_6 = -0.62, \quad C_8 = -0.81 \quad \text{and} \quad C_{10} = -1.5. \quad (1.2.2)$$

We furthermore found that the OPE is converging, but not as fast as for the kinematic weight. The values of  $|\delta^{(8)}|$  is only half as large as  $|\delta^{(8)}|$ . The values of the lower momentum count are in high agreement with the ones obtained from the kinematic weight. The conclusions that we take from the *cubic weight* are that the kinematic weight, with its double pinching, should sufficiently suppress any contributions from DVs. If DV would have an effect on the kinematic weight, we should have seen an improvement of the fits with the *cubic weight*, due to its triple pinching, which is not the case.

$$\textbf{Quartic weight: } \omega(x) \equiv (1-x)^4(1+4x)$$

The last fits of the the pinched weights without a monomial term in  $x$  uses the *quartic weight* defined as  $\omega(x) \equiv (1-x)^4(1+4x)$ . It contains five fitting parameters ( $\alpha_s, C_6, C_8, C_{10}, C_{12}$ ) and did only converge for  $s_{\min} = 2 \text{ GeV}$  (nine  $s_0$ s momenta). The results for , with a  $\chi^2$  per DOF of 0.67 are given by:

$$\begin{aligned} \alpha_s(m_\tau^2) = 0.3290(11), \quad C_6 = -0.3030(46), \quad C_8 = -0.1874(28), \\ C_{10} = 0.3678(45) \quad \text{and} \quad C_{12} = -0.4071(77) \end{aligned} \quad (1.2.3)$$

Due to the problematic of the fitting routing, which is caused by too many OPE contributions fitted simultaneously, we will discard the fitting results for the quartic weight.

### 1.2.2 Single Pinched Monomial Weights

To further solidate our hypothesis we want to probe some weights with a single pinching. If DV play a role then we should note deviating results to fits with higher pinchings. The advantage of these weights is that they only depend on one additional OPE Wilson coefficients, thus leaving us with only two parameters per fit.



$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$c_6$	$\chi^2/\text{dof}$
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

Table 1.5: Table of our fitting values of  $\alpha_s(m_\tau^2)$ , and  $C_6$  for the single pinched double power monomial weight  $\omega_{M2}(x) = 1 - x^2$  using FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$c_8$	$\chi^2/\text{dof}$
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Table 1.6: Table of our fitting values of  $\alpha_s(m_\tau^2)$ , and  $C_8$  for the single pinched third power monomial weight  $\omega_{M3}(x) = 1 - x^3$  using FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

**Second power monomial:**  $\omega_{M2}(x) \equiv 1 - x^2$

The first weight is defined as  $\omega_{M2}(x) \equiv 1 - x^2$ . We only have to fit two parameters, the strong coupling  $\alpha_s$  and the dimension six OPE contribution. The results can be seen in [table 1.5](#). Note that we have again a jump of the  $\chi^2/\text{dof}$  value between if we fit with a too low  $s_{\min}$ . The values obtained for fitting six and seven  $s_0s$  moments are in agreement with good  $\chi^2$  values.

**Third power monomial:**  $\omega_{M3}(x) \equiv 1 - x^3$

The second weight is defined as  $\omega_{M3}(x) \equiv 1 - x^3$  and contains a single third power monomial. Consequently it is sensitive to dimension eight contributions from the OPE. Our fitting results can be taken from [table 1.6](#). The  $\chi^2$  per dof is like in the  $\omega_\tau$  and  $\omega_{\text{cubic}}$  fits good for  $s_{\min} \leq 2.2 \text{ GeV}$ , but jumps to values

$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$c_{10}$	$\chi^2/\text{dof}$
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

Table 1.7: Table of our fitting values of  $\alpha_s(m_\tau^2)$  and  $C_{10}$  for the single pinched fourth power monomial weight  $\omega_{M4}(x) = 1 - x^4$  using FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

$\chi^2/\text{dof} > 1.4$  for smaller  $s_{\min}$ . This is explained through resonances that appear in lower energies. Due to the good  $\chi^2$  and the internally compatible fitting values we averaged over all rows except the last one of table 1.6. The last row, at  $s_{\min} = 2.8 \text{ GeV}$  has only one DOF and thus high errors. The averaged values are given by

$$\alpha(m_\tau^2) = 0.32382(42) \quad \text{and} \quad C_8 = -1.33(67). \quad (1.2.4)$$

We note that the strong coupling is smaller as our expected values from the previous fits of around 3.33, but the dimension eight contribution is in good agreement. This is a sign of appearing DV, although the parameters of the different fits do not vary by huge numbers.

#### Fourth power monomial: $\omega_{m4}(x) \equiv 1 - x^4$

We already analysed the cubic and quartic weights, which depend on the dimension ten OPE contribution, in section 1.2.1 and section 1.2.1 correspondingly. Now, even with the visible DV for fourth power monomial  $\omega_{m4} \equiv 1 - x^4$  to study another single pinched moment and the dimension ten OPE contribution. The results of the are given in ???. The fitting behaviour is very similar to the third power monomial (??) and we will directly cite our obtained results:

$$\alpha_s(m_\tau^2) = 0.32277(40) \quad \text{and} \quad c_{10} = -2.4(3.6). \quad (1.2.5)$$

As before the values for the strong coupling are lower than the ones obtained by the fit kinematic and cubic weight fits. Furthermore the error on the tenth dimension contribution of the OPE are large. All in all the usage of the single pinched fourth power monomial weight is questionable and does not deliver any additional insights.

### 1.2.3 Pinched Weights with monomial $x$

Next to the previously mentioned *optimal weights* from Beneke and Jamin [Beneke2012] there exist another type of *optimal' weights*<sup>2</sup> introduced by Pich [LeDiberder1992]

$$\omega_{(n,m)}(x) = (1-x)^n \sum_{k=0}^m (k+1)x^k, \quad (1.2.6)$$

Combinations of these optimal moments have been widely used by the ALEPH collaboration to perform QCD analysis on the LEP data. The moments fitted in this section include the for FOPT problematic proportional term in  $x$ , thus we will perform additional fits using the Borel-sum.

$$\omega_{x1} \equiv (1-x)$$

The first weight is single pinched with only two fitting parameters:  $\alpha_s$  and  $\chi^2/\text{dof}$ . The results for BS and FOPT fits have been displayed in [table 1.8](#). We note that the  $\alpha_s$  values of the two frameworks differ, which is most probably due to the problematic of the monomial in  $x$ , appearing in the weight function. In general we trust the results of the BS more than those of FOPT. This is further underlined while regarding the higher  $\chi^2/\text{dof}$  values of the FOPT fits. Furthermore the values of the BS fits agree with each other, whereas the fits of the FOPT yield inconsistent values. Regarding explicitly the fits from the BS we note that the fits have good  $\chi^2/\text{dof}$  values, although a jump from 0.2 to 0.95 between the first two fitted moments. Also note that we had to fit the invariant gluon-condensate for the first time. In the literature  $\langle \alpha G G \rangle_1$  should be around 2.1, but here we obtain a smaller, negative value, which could be connected to problems in the fit.

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<sup>2</sup>Pich has a different definition of “optimal” moments than Beneke and Jamin. We to differentiate the two definition we marked Pich’s optimal’ moments with an apostrophe.

	$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\chi^2/\text{dof}$
BS	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24
FOPT	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	1.91
	2.300	6	0.3260(60)	-0.2453(73)	1.71

Table 1.8: Table of our fitting values of  $\alpha_s(m_\tau^2)$  and  $\langle aGG \rangle_I$  for the single pinched optimal weight  $\omega_{\chi 1}(x) = (1 - x)$  using the FOPT and BS ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

$$\omega_{\chi 2} \equiv (1 - x)^2$$

The next weight is double pinched. Additionally to the strong coupling and the invariant gluon-condensate we also had to fit the dimension six OPE contribution. Our fits can be seen in [table 1.9](#). If we compare the BS with the FOPT fits we note, next to the before mentioned incompatibilities, a sign difference for the  $D = 6$  contribution. From now we will skip the FOPT discussion for weights containing monomial  $x$ , and trust in the BS fits. In comparison to the previous fit with the single pinched weight we have higher  $\chi^2/\text{dof}$  values, a lower  $\alpha_s$  value and an  $\langle aGG \rangle_I$  numeric value similar to the value from the literature around 0.21, but with opposite sign. Consequently we note some tension between the single pinched weight and the double pinched weight, which could be caused by DV being not sufficiently suppressed by a single pinched weight containing a monomial  $x$ .

$$\omega_{\chi 3} \equiv (1 - x)^3 \text{ and } \omega_{\chi 4} \equiv (1 - x)^4$$

The fits with a triple and quadruple pinched weight do not give any further insights. We give the results in [table 1.10](#) and [table 1.11](#). Both of the weights include similar values to the double pinched weights, which affirms the problems of the single pinched weights. The quadruple pinched weight contains five fitting parameters and has notable convergence problems.

	$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$c_6$	$\chi^2/\text{dof}$
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

Table 1.9: Table of our fitting values of  $\alpha_s(m_\tau^2)$ ,  $\langle aGG \rangle_I$  and  $C_6$  for the double pinched optimal weight  $\omega_{\chi^2}(x) = (1-x)^2$  using the BS or FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

	$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$c_6$	$c_8$	$\chi^2/\text{dof}$
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3240(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

Table 1.10: Table of our fitting values of  $\alpha_s(m_\tau^2)$ ,  $\langle aGG \rangle_I$ ,  $c_6$  and  $c_8$  for the optimal weight  $\omega_{\chi^3}(x) = (1-x)^3$  using the BS or FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

	$s_{\min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$aGG_{\text{Inv}}$	$C_6$	$C_8$	$C_{10}$	$\chi^2/\text{dof}$
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

Table 1.11: Table of our fitting values of  $\alpha_s(m_\tau^2)$ ,  $\langle aGG \rangle_I$ ,  $c_6$ ,  $c_8$  and  $c_{10}$  for the optimal weight  $\omega_{\chi^4}(x) = (1-x)^4$  using the BS or FOPT ordered by increasing  $s_{\min}$ . The errors are given in parenthesis after the observed value.

weight	$s_{\min}$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$c_6$	$c_8$	$c_{10}$	$\chi^2/\text{dof}$
$\omega_{\text{kin}}$	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
$\omega_{\text{cube}}$	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
$\omega_{3,0}^*$	2.1	0.3239(30)	-0.2125(26)	-0.627(87)	-0.74(17)	-	0.46
$\omega_{\text{quartic}}$	2.0	0.3290(11)	-	-0.3030(46)	-0.1874(28)	0.3678(45)	0.67
$\omega_{m3}$	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
$\omega_{m4}$	2.2	0.3203(48)	-	-	-	-1.64(77)	0.42
$\omega_{2,0}$	2.2	0.3401(57)	-0.0185(52)	0.220(88)	-	-	0.73
$\omega_{3,0}$	2.1	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	-	0.31

Table 1.12: Table of the best fits (selected by  $\chi^2/\text{dof}$  and compatibility of the fitting values) for each weight including at least the strong coupling  $\alpha_s(m_\tau^2)$  as a fitting variable. All fits have been performed using `FOPT`, except weights marked with a star  $\omega^*$ , which have been fitted using the *Borel sum*.

### 1.3 Comparison

To create an overview of our previous results we have gathered the most compatible rows by hand. These are shown in [table 1.12](#), which is composed of two parts:

- The upper three rows represent fits we found to have good properties for determining the strong coupling.
- The lower five rows are problematic fits due to too many OPE contributions, too low pinching or to terms proportional to  $x$ .

We have found that the kinematic weight is in excellent agreement with the cubic  $\omega_{\text{cube}}$  and Pich’s optimal weight  $\omega_{3,0}$ , fitted using the borel model. The fitted parameters from the kinematic weight ( $\alpha_s, c_6$  and  $c_8$ ) are all within error ranges and thus compatible. One fact that has to be investigated is the negative appearing sign for the gluon-condensate from the borel-sum of  $\omega_{3,0}$ .