Chapter 1

τ decays into hadrons

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{Hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} \tag{1.1}$$

The theoretical expression of the hadronic τ-decay ratio was first derived by [**Tsai1971**] (using current algebra, a more recent derivation making use of the *optical theorem* can be taken from [**Schwab2002**]):

$$R_\tau = 12\pi \int_0^{m_\tau} = \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \operatorname{Im} \Pi^{(T)}(s) + \operatorname{Im} \Pi^{(L)} \right]. \tag{1.2}$$

R_τ introduces a problematic integral over the real axis of $\Pi(s)$ from 0 up to m_τ. The integral is problematic for two reasons:

- The *perturbative Quantum Chromodynamcs* (**pQCD**) and the OPE breaks down for low energies (over which we have to integrate).
- The positive euclidean axis of $\Pi(s)$ has a discontinuity cut and can theoretically not be evaluated.

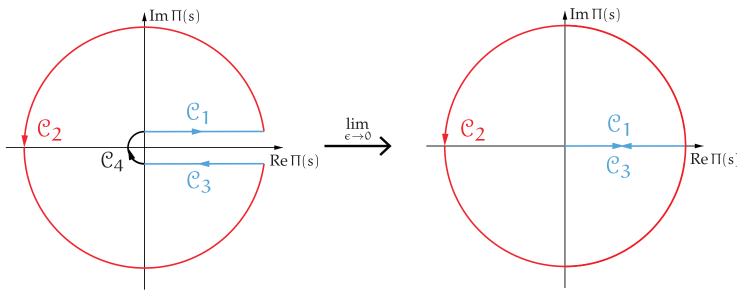
To literally circunvent these issues we make use of *Cauchy's Theorem*

$$\int_{\mathcal{C}} f(z)\, dz = 0, \tag{1.3}$$

where $f(z)$ is an analytic function on a closed contour $\mathcal{C}$.

In our case we have to deal with the two-point correlator $\Pi(s)$, which is analytic except for the positive real axis (with which we will deal with to a later point[1]) Consequently, to rewrite we can rewrite the definite integral of eq. (1.2) into a contour integral over a closed circle with radius m_τ^2. The closed contour consists of four line integrals, which have been visualized in fig. 1.1.

[1] To not evaluate $\Pi(s)$ at the positive real axis we have to introduce *pinched weights*. The *pinched weights* vanish for $s \to m_\tau$.



$$\int_0^{m_\tau} \mathrm{Im}\,\Pi(s)\,\mathrm{d}s \quad = \quad \oint_{s=m_\tau} \Pi(s)\,\mathrm{d}s$$

Figure 1.1: Visualization of the usage of Cauchy's theorem to transform eq. (1.2) into a closed contour integral over a circle of radius m_τ^2.

Summing over the four line integrals, performing a *analytic continuation* of the two-point correlator $\Pi(s) \to \Pi(s+i\epsilon)$ and finally taking the limit of $\epsilon \to 0$ gives us the needed relation between eq. (1.2) and the closed contour:

$$\begin{aligned}
\oint_{s=m_\tau} \Pi(s) &= \int_0^{m_\tau} \Pi(s+i\epsilon) + \int_{\mathcal{C}_2} \Pi(s)\,\mathrm{d}s + \int_{m_\tau}^{0} \Pi(s-i\epsilon)\,\mathrm{d}s + \int_{\mathcal{C}_4} \Pi(s)\,\mathrm{d}s \\
&= \int_0^{m_\tau} \Pi(s+i\epsilon) - \Pi(s-i\epsilon)\,\mathrm{d}s + \int_{\mathcal{C}_2} \Pi(s)\,\mathrm{d}s + \int_{\mathcal{C}_4} \Pi(s)\,\mathrm{d}s \\
&= \int_0^{m_\tau} \Pi(s+i\epsilon) - \overline{\Pi(s+i\epsilon)} + \int_{\mathcal{C}_2} \Pi(s)\,\mathrm{d}s + \int_{\mathcal{C}_4} \Pi(s)\,\mathrm{d}s \\
&\overset{\lim \epsilon\to 0}{=} 2i \int_0^{m_\tau} \mathrm{Im}\,\Pi(s)\,\mathrm{d}s + \oint_{s=m_\tau} \Pi(s)\,\mathrm{d}s
\end{aligned} \tag{1.4}$$

where we made use of $\Pi(z) = \overline{\Pi(\overline{z})}$ (due to $\Pi(s)$ is analytic) and $\Pi(z) - \overline{\Pi(z)} = 2i\,\mathrm{Im}\,\Pi(z)$. The result can be rewritten in a more intuitive form, which we also visualized in fig. 1.1

$$\int_0^{m_\tau} \Pi(s)\,\mathrm{d}s = \frac{i}{2} \oint_{s=m_\tau} \Pi(s)\,\mathrm{d}s \tag{1.5}$$

$$R_\tau = 6\pi i \oint_{s=m_\tau} \frac{\mathrm{d}s}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(T)}(s) + \Pi^{(L)}\right] \tag{1.6}$$

$$\Pi^{(L+T)} = \Pi^{(L)} + \Pi^{(T)} \tag{1.7}$$

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(L+T)}(s) - \left(\frac{2s}{m_\tau^2}\right) \Pi^{(L)}(s)\right] \tag{1.8}$$

$$D^{(L+T)}(s) \equiv -s\frac{d}{ds}\Pi^{(L+T)}(s), \qquad D^{(L)}(s) \equiv \frac{s}{m_\tau^2}\frac{d}{ds}(s\Pi^{(L)}(s)) \tag{1.9}$$

Integration by parts

$$\int_a^b u(x)V(x)\,dx = \left[U(x)V(x)\right]_a^b - \int_a^b U(x)v(x)\,dx \tag{1.10}$$

$$\begin{aligned}
R_\tau^{(1)} &= \frac{6\pi i}{m_\tau^2}\oint_{|s|=m_\tau^2} \underbrace{\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)}_{=u(x)}\underbrace{\Pi^{(L+T)}(s)}_{=V(x)} \\
&= \frac{6\pi i}{m_\tau^2}\Bigg\{\left[-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)\right]_{|s|=m_\tau^2} \\
&\quad + \oint_{|s|=m_\tau^2}\underbrace{-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)}_{=U(x)}\underbrace{\tfrac{d}{ds}\Pi^{(L+T)}(s)}_{=v(x)}\Bigg\} \\
&= -3\pi i\oint_{|s|=m_\tau^2 s}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\tfrac{d}{ds}D^{(L+T)}
\end{aligned} \tag{1.11}$$

where we fixed the integration constant to $C = -\frac{m_\tau^2}{2}$ in the second line and left the antiderivatives contained in the squared brackets untouched. Parametrizing the expression in the squared brackets

$$\left[-\frac{m_\tau^2}{2}\left(1-e^{-i\phi}\right)^3\left(1+e^{-i\phi}\right)\Pi^{(L+T)}(m_\tau^2 e^{-i\phi})\right]_0^{2\pi} = 0 \tag{1.12}$$

where $s \to m_\tau^2 e^{-i\phi}$ and $(1-e^{-i\cdot 0}) = (1-e^{-i\cdot 2\pi}) = 0$.

$$\begin{aligned}
R_\tau^{(2)} &= \oint_{|s|=m_\tau^2} ds\left(1-\frac{s}{m_\tau^2}\right)^2\left(-\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s) \\
&= -4\pi i\oint\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 D^{(L)}(s)
\end{aligned} \tag{1.13}$$

$$R_\tau = -\pi i\oint_{|s|=m_\tau^2}\frac{d}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left[3\left(1+\frac{s}{m_\tau^2}D^{(L+T)}(s)+4D^{(L)}(s)\right)\right] \tag{1.14}$$

$$R_\tau = -\pi i\oint_{|s|=m_\tau^2}\frac{d}{x}(1-x)^3\left(3(1+x)D^{(L+T)}(s)+4D^{(L)}(s)\right), \tag{1.15}$$

where $x = s/m_\tau^2$.