CHAPTER 1

Tau Decays into Hadrons

Building on the previously presented QCDSR we will elaborate the needed theory to extract α_s from the process of hadronic tau decays. ... complete

1.1 Tau Decays into hadrons

The tau lepton is the only lepton heavy enough to decay into hadrons. It permits one of the most precise determinations of the strong coupling α_s. The inclusive tau decay ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} \tag{1.1.1}$$

can be precisely calculated and is sensitive to α_s. Due to the small mass of the tau lepton $m_\tau \approx 1.776\,\text{GeV}$ the tau decays are excellent for performing a low-energy QCD analysis. The theoretical expression of the hadronic tau decay ratio was first derived by [**Tsai1971**], using current algebra, a more recent derivation making use of the *optical theorem*, as already mentioned in **??** can be taken from [**Schwab2002**].

1.1.1 The Inclusive Decay Ratio

The inclusive ratio is given by:

$$R_\tau(s) = 12\pi \int_0^{m_\tau} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \operatorname{Im} \Pi^{(T)}(s) + \operatorname{Im} \Pi^{(L)}(s) \right], \tag{1.1.2}$$

where $\operatorname{Im}\Pi$ is imaginary part of the two-point function we introduced in **??**. Applying Cauchy's theorem, as seen in **??**, to the eq. 1.1.2 we get

$$R_\tau = 6\pi i \oint_{s=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(T)}(s)+\Pi^{(L)}(s)\right]. \tag{1.1.3}$$

It is furthermore convenient to work with $\Pi^{(T+L)}$, which has been defined in **??**. As a result we can further rewrite the hadronic tau decay ratio into

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)-\left(\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s)\right]. \tag{1.1.4}$$

In the case of tau decays we only have to consider vector and axial-vector contributions of decays into up, down and strange quarks. Thus taking i, j as the flavour indices for the light quarks (u, d and s) we can express the two-point function as

$$\Pi^{V/A}_{\mu\nu,ij}(s) \equiv i\int dx\, e^{ipx}\langle\Omega|T\{J^{V/A}_{\mu,ij}(x)J^{V/A}_{\nu,ij}(0)^\dagger\}|\Omega\rangle, \tag{1.1.5}$$

with $|\Omega\rangle$ being the physical vacuum. The vector and axial-vector currents are then distinguished by the corresponding dirac-matrices ($\gamma_\mu \text{and} \gamma_\mu\gamma_5$) given by

$$J^V_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu q_i(x) \quad \text{and} \quad J^A_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu\gamma_5 q_i(x). \tag{1.1.6}$$

With eq. 1.1.4 we have a suitable physical quantity that can be theoretically as experimentally obtained. As the circle contour integral we used is has a radius of s_0 we successfully avoided low energies at which the application of PT would be questionable. For example if we would choose a radius with the size of the tau mass $m_\tau \approx 1.78\,\text{MeV}$ the strong coupling would have a perturbatively safe value of $\alpha_s(m_\tau) \approx 0.33$ [**Pich2016**]. Obviously we would benefit even more from a contour integral over a bigger circumference, but tau decays are limited by their mass. Nevertheless there are promising e^+e^- annihilation data, which yields valuable R-ratio values up to $2\,\text{GeV}$ [**Boito2018**][**Keshavarzi2018**].

1.1.2 Renormalisation Group Invariance

We have seen in **??**, that the two-point function is not a physical quantity. From the dispersion relation (**??**) we saw that it contains a unphysical polynom. Luckily for the vector correlator we are using in hadronic tau decays the polynom is just a constant. Consequently by taking the derivative with respect to the momentum s we can derive a physical quantity from the two-point function:

$$D(s) \equiv -s\frac{d}{ds}\Pi(s). \tag{1.1.7}$$

$D(s)$ is called the *Adler function* and fulfils the RGE (**??**). The Adler function commonly has separte definitions for the longitudinal plus transversal and the solely longitudinal part contributions:

$$D^{(T+L)}(s) \equiv -s\frac{d}{ds}\Pi^{(T+L)}(s), \qquad D^{(L)}(s) \equiv \frac{s}{m_\tau^2}\frac{d}{ds}(s\Pi^{(L)}(s)). \tag{1.1.8}$$

The two-point functions in **??** can now be replaced with the help of partial integration

$$\int_a^b u(x)V(x)\,dx = \big[U(x)V(x)\big]_a^b - \int_a^b U(x)v(x)\,dx. \tag{1.1.9}$$

We will do the computation for each of the two cases $(T+L)$ and (L) separate. Starting by the transversal plus longitudinal contribution we get:

$$\begin{aligned}
R_\tau^{(1)} &= \frac{6\pi i}{m_\tau^2}\oint_{|s|=m_\tau^2} \underbrace{\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)}_{=u(x)}\underbrace{\Pi^{(L+T)}(s)}_{=V(x)} \\
&= \frac{6\pi i}{m_\tau^2}\left\{\left[-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)\right]_{|s|=m_\tau^2}\right. \\
&\quad \left. +\oint_{|s|=m_\tau^2}\underbrace{-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)}_{=U(x)}\underbrace{\tfrac{d}{ds}\Pi^{(L+T)}(s)}_{=v(x)}\right\} \\
&= -3\pi i\oint_{|s|=m_\tau^2 s}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\tfrac{d}{ds}D^{(L+T)}
\end{aligned} \tag{1.1.10}$$

where we fixed the integration constant to $C = -\frac{m_\tau^2}{2}$ in the second line and left the antiderivatives contained in the squared brackets untouched. If we parameterizing the integral appearing in the expression in the squared brackets we can see that it vanishes:

$$\left[-\frac{m_\tau^2}{2}\left(1-e^{-i\phi}\right)^3\left(1+e^{-i\phi}\right)\Pi^{(L+T)}(m_\tau^2 e^{-i\phi})\right]_0^{2\pi} = 0 \tag{1.1.11}$$

where $s \to m_\tau^2 e^{-i\phi}$ and $(1-e^{-i\cdot 0}) = (1-e^{-i\cdot 2\pi}) = 0$. Repeating the same calculation for the longitudinal part yields

$$\begin{aligned} R_\tau^{(L)} &= \oint_{|s|=m_\tau^2} ds \left(1-\frac{s}{m_\tau^2}\right)^2 \left(-\frac{2s}{m_\tau^2}\right) \Pi^{(L)}(s) \\ &= -4\pi i \oint \frac{ds}{s} \left(1-\frac{s}{m_\tau^2}\right)^3 D^{(L)}(s) \end{aligned} \tag{1.1.12}$$

Consequently combining the two parts results in

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(1-\frac{s}{m_\tau^2}\right)^3 \left[3\left(1+\frac{s}{m_\tau^2} D^{(L+T)}(s) + 4D^{(L)}(s)\right)\right]. \tag{1.1.13}$$

It is convenient to define $x = s/m_\tau^2$ such that we can rewrite the inclusive ratio as

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{dx}{x} (1-x)^3 \left[3(1+x) D^{(L+T)}(m_\tau^2 x) + 4D^{(L)}(m_\tau^2 x)\right], \tag{1.1.14}$$

which will be the final expression we will be using to express the inclusive tau decay ratio.

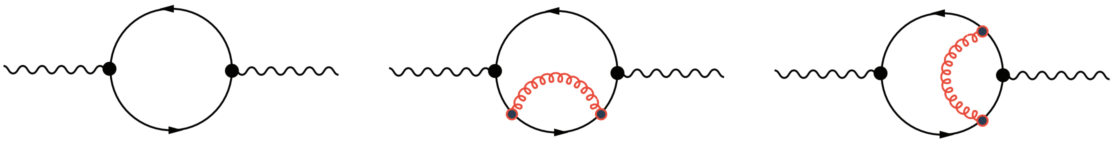
1.2 Computation of R_τ

The previously derived expression for the tau decay ratio can be further organised in different δ contributions

$$R_{\tau,V/A}^{\omega} = \frac{N_c}{2} S_{EW} |V_{ud}|^2 \left(1 + \delta_\omega^{(0)} + \delta_\omega^{EW} + \delta_\omega^{DVs} + \sum_{D\leqslant 2} \delta_{ud,\omega}^{(D)}\right), \tag{1.2.1}$$

where $\delta^{(0)}$ is the perturbative contribution, $\delta^{(EW)}$ is a electroweak correction, $\delta^{(DV)}$ is the duality violation correction and $\delta^{(D)}$ are the higher dimensional contributions from the OPE. Note that in this work we will assume $\mathtt{delta}^{(DV)}$ to be negligible small for the weights we will apply.

We now want to derive the theoretical expressions needed to calculate every contribution to eq. 1.2.1 starting with the perturbative contribution.



(a) (b) (c)

1.2.1 The perturbative contribution

We will treat the correlator in the chiral limit for which the longitudinal components $\Pi^L(s)$ of **??** vanish. Thus the axial and vectorial contributions are equal. In the massless case we then can write the vector correlation function $\Pi(s)$ as a sum over different orders of α [**Beneke2008**]:

$$\Pi_V^{T+L}(s) = -\frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{n,k} L^k \quad \text{with} \quad L \equiv \ln\frac{-s}{\mu^2}. \tag{1.2.2}$$

The coefficient $c_{n,k}$ up to two-loop order can be obtained by Feynman diagram calculations. With the diagrams of section 1.2.1 can calculate the zero-loop result of the correlator [**Jamin2006**]

$$\Pi^B_{\mu\nu}(q^2)\Big|^{1-\text{loop}} = \frac{N_c}{12\pi^2}\left(\frac{1}{\hat{\epsilon}} - \log\frac{(-q^2 - i0)}{\mu^2} + \frac{5}{3} + \mathcal{O}(\epsilon)\right), \tag{1.2.3}$$

where $\Pi^B_{\mu\nu}(q^2)$ is the bare two-point function and is not renormalised[1] This result can then be used to extract the first two coefficients of the correlator expansion given in eq. 1.2.2

$$c_{00} = -\frac{5}{3} \qquad \text{and} \qquad c_{01} = 1. \tag{1.2.4}$$

The second loop can also be calculated by diagram techniques resulting in [**Boito2011**]

$$\Pi_V^{(1+0)}(s)\Big|^{2-\text{loop}} = -\frac{N_c}{12\pi^2} a_\mu \log(\frac{-s}{\mu^2}) + \cdots \tag{1.2.5}$$

yielding $c_{11} = 1$.

Beginning from three loop diagrams the algebra becomes exhausting and one has to use dedicated algorithms to compute the higher loops. The third loop calculations have been done in the late seventies by [**Chetyrkin1979**, **Dine1979**, **Celmaster1979**]. The four loop evaluation have been completed a little more

[1] The term $1/\hat{\epsilon}$, which is of order zero in α_s, will be cancelled by renormalisation.

than ten years later by [**Gorishnii1990**, **Surguladze1990**]. The highest loop published, that amounts to α_s^4, was published in 2008 [**Baikov2008**] almost 20 years later.

Fixing the number of colors to $N_c = 3$ the missing coefficients up to order four in α_s read:

$$\begin{aligned}
c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f \\
c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 \\
&\quad - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2 \\
c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7,
\end{aligned} \tag{1.2.6}$$

where used the flavor number $N_f = 3$ for the last line.

The 6-loop calculation has until today not been achieved, but Beneke and Jamin [**Beneke2008**] used and educated guess to estimate the coefficient

$$c_{5,1} \approx 283 \pm 283. \tag{1.2.7}$$

In stating the coefficients $c_{n,k}$ of the correlator expansion we have restricted ourselves to k-indices equal to one. This is due to the RGE, which relates coefficients with k different than one to the already stated coefficients $c_{n,1}$. To relate the coefficients we have to make use of the RGE. Consequently the correlator $\Pi_V^{T+L}(s)$ needs to be a physical quantity, which we can be achieved with the previously defined Adler function (eq. 1.1.8). The correct expression for the correlator expansion in eq. 1.2.2 is then given by

$$D_V^{(T+L)} = -s\frac{d\Pi_V^{(T+L)}(s)}{ds} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}, \tag{1.2.8}$$

where we used $dL^k/ds = k\ln(-s/\mu^2)^{k-1}(-1/\mu^2)$. Applying the RGE (**??**) to the scale-invariant Adler function yields

$$-\mu\frac{d}{d\mu}D_V^{(T+L)} = -\mu\frac{d}{d\mu}\left(\tfrac{\partial}{\partial L}\,dL + \tfrac{\partial}{\partial a_s}\,da_s\right) D_V^{T+L} = \left(2\tfrac{\partial}{\partial L} + \beta\tfrac{\partial}{\partial a_s}\right) D_V^{T+L} = 0, \tag{1.2.9}$$

where we made use of the β function, which is defined in **??**, and of the expression $dL/d\mu = -2/\mu$.

The relation between the correlator expansion coefficients can then be taken by calculating the Adler function for a desired order and plugging it into the RGE. For example the Adler function to the second order in α_s

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + a_\mu(c_{11} + 2c_{12}L) + a_\mu^2(c_{21} + 2c_{22}L + 3c_{23}L^2)\right], \qquad (1.2.10)$$

can be inserted into RGEAdler

$$4a_\mu c_{12} + 2a_\mu^2(2c_{22} + 6c_{23}L) + \beta_1 a_\mu^2(c_{11} + 2c_{12}L) + \mathcal{O}(a_\mu^3) = 0 \qquad (1.2.11)$$

to compare the coefficients order by order in α_s. At order α_μ only the c_{12} term is present and has consequently to be zero. For $\mathcal{O}(a_\mu^2 L)$ only c_{23} exists as $c_{12} = 0$ and thus also has to vanish. Finally at $\mathcal{O}(a)$ we can relate c_{22} with c_{11} resulting in:

$$c_{12} = 0, \quad c_{22} = \frac{\beta_1 c_{11}}{4} \quad \text{and} \quad c_{23} = 0. \qquad (1.2.12)$$

Implementing the newly obtained Adler coefficients we can write out the Adler function to the first order:

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + c_{11}a_\mu\left(c_{21} - \frac{1}{2}\beta_1 c_{11}L\right)a_\mu^2\right] + \mathcal{O}(a_\mu^3). \qquad (1.2.13)$$

We have used the RGE to relate Adler-function coefficients and thus reduce its numbers. But as we will see in the following section the RGE gives us two different choices in the order of the computation of the perturbative contribution to the inclusive tau decay ratio.

1.2.2 Renormalization group summation

By making use of the RGE we have to decide about the order of mathematical operations we perform. As the perturbative contribution $\delta^{(0)}$ is independent on the scale μ we are confronted with two choices **fixed-order perturbation theory** (FOPT) or **contour-improved perturbation theory**. Each of them yields a different result and is the main source of error in extracting the strong coupling from τ-decays.

We can write the perturbative contribution $\delta^{(0)}$ to R_τ (see eq. 1.2.1) in the chiral limit, such that $D^{(L)}$ vanishes as

$$\delta^{(0)} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k c_{n,k} \frac{1}{2\pi i}\oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x)\log\left(\frac{-M_\tau^2 x}{\mu^2}\right)^{k-1}, \qquad (1.2.14)$$

where we inserted the expansion of $D_V^{(T+L)}$ eq. 1.1.8 into R_τ eq. 1.1.14. Keep in mind that the contributions from the vector and axial-vector correlator are identical in the massless case:

$$D^{(T+L)} = D_V^{(T+L)} + D_A^{(T+L)} = 2D_V^{(T+L)}. \tag{1.2.15}$$

In the following we will explain both the descriptions, starting by FOPT. By using the FOPT prescription we fix $\mu^2 = m_\tau^2$ leading to

$$\delta_{FO}^{(0)} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k c_{n,k} J_{k-1} \tag{1.2.16}$$

where the contour integrals J_l are defined by

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x)\log^l(-x). \tag{1.2.17}$$

The integrals J_l up to order α_s^4 are given by [**Beneke2008**]:

$$J_0 = 1, \quad J_1 = -\frac{19}{12} \quad J_2 = \frac{265}{72} - \frac{1}{3}\pi^2, \quad J_3 = -\frac{3355}{288} + \frac{19}{12}\pi^2. \tag{1.2.18}$$

Using FOPT the strong coupling $a(\mu)$, which runs with the scale μ, is fixed at $a(m_\tau^2)$ and can be taken out of the closed-contour integral. Thus we solely to integrate over the logarithms $\log(-s/m_\tau^2)$.

Using CIPT we can sum the logarithms by setting the scale to $\mu^2 = -m_\tau^2 x$ in eq. 1.2.14, resulting in:

$$\delta_{CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2), \tag{1.2.19}$$

where the contour integrals J_l are defined by

$$J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x)a^n(-m_\tau^2 x). \tag{1.2.20}$$

All logarithms vanish except the ones for $k = 1$:

$$\log(1)^{k-1} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & k \neq 1 \end{cases} \tag{1.2.21}$$

which selects adler function coefficients $c_{n,1}$ with a fixed $k = 1$. Handling the logarithms left us with the integration of $\alpha_s(-m_\tau^2 x)$ over the closed-contour $\oint_{|x|=1}$, which now depends on the integration variable x. In general we have to decide if we want to perform a contour integration with a constant coupling constant and variable logarithms (FOPT) or "constant logarithms" and a running coupling (CIPT).

To emphasize the differences in both approaches we can calculate the perturbative contribution $\delta^{(0)}$ to R_τ for the two different prescriptions yielding [**Beneke2008**]

$$
\begin{aligned}
&\qquad\quad\ \alpha_s^2 \qquad\ \alpha_s^2 \qquad\ \alpha_s^3 \qquad\ \alpha_s^4 \qquad\qquad \alpha_s^5 \\
\delta^{(0)}_{FO} &= 0.1082 + 0.0609 + 0.0334 + 0.0174(+0.0088) = 0.2200(0.2288) \qquad (1.2.22)\\
\delta^{(0)}_{CI} &= 0.1479 + 0.0297 + 0.0122 + 0.0086(+0.0038) = 0.1984(0.2021). \qquad (1.2.23)
\end{aligned}
$$

The series indicate, that CIPT converges faster and that both series approach a different value. This discrepancy represents currently the biggest theoretical uncertainty while extracting the strong coupling α_s.

As today we do not know if FOPT or CIPT is the correct approach of measuring α_s. Therefore there are currently three ways of stating results:

- Quoting the average of both results.
- Quoting the CIPT result.
- Quoting the FOPT result.

We follow the approach of Beneke and Jamin [**Beneke2008**] who have shown advantages of FOPT over CIPT.

1.3 Non-Perturbative OPE Contribution

The perturbative contribution to the Sum-Rule, that we have seen so far, is the dominant one. With

$$
\begin{aligned}
R_\tau^{FOPT} &= \\
R_\tau^{CIPT} &=
\end{aligned}
\qquad (1.3.1)
$$

The NP vs perturbative contributions can be varied by chosen different weights than ω_τ.

1.3.1 Dimension four

For the OPE contributions of dimension four we have to take into account the terms with masses to the fourth power m^4, the quark condensate multiplied by a mass $m\langle\overline{q}\,q\rangle$ and the gluon condensate $\langle GG\rangle$. The resulting expression can be taken from the appendix of [**Pich1999**], yielding:

$$D_{ij}^{(L+T)}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_n \Omega^{(1+0)}(s/\mu^2)a^n, \tag{1.3.2}$$

where

$$\begin{aligned}\Omega_n^{(1+0)}(s/\mu^2) \;=\; & \frac{1}{6}\langle aGG\rangle p_n^{(L+T)}(s/\mu^2) + \sum_k m_k\langle\overline{q}_k q_k\rangle r_n^{(L+T)}(s/\mu^2)\\ & + 2\langle m_i\overline{q}_i q_i + m_j\overline{q}_j q_j\rangle q_n^{(L+T)}(s/\mu^2) \pm \frac{8}{3}\langle m_j\overline{q}_i q_i + m_i\overline{q}_j q_j\rangle t_n^{(L+T)}\\ & - \frac{3}{\pi^2}(m_i^4 + m_j^4)h_n^{(L+T)}(s/\mu^2) \mp \frac{5}{\pi^2}m_i m_j(m_i^2 + m_j^2)k_n^{(L+T)}(s/\mu^2)\\ & + \frac{3}{\pi^2}m_i^2 m_j^2 g_n^{(L+T)}(s/\mu^2) + \sum_k m_k^4 j_n^{(L+T)}(s/\mu^2) + 2\sum_{k\neq l} m_k^2 m_l^2 u_n^{(L+T)}(s/\mu^2)\end{aligned} \tag{1.3.3}$$

The perturbative expansion coefficients are known to $\mathcal{O}(a^2)$ for the condensate contributions,

$$\begin{aligned} &p_0^{(L+T)} = 0, && p_1^{(L+T)} = 1, && p_2^{(L+T)} = \tfrac{7}{6},\\ &r_0^{(L+T)} = 0, && r_1^{(L+T)} = 0, && r_2^{(L+T)} = -\tfrac{5}{3} + \tfrac{8}{3}\zeta_3 - \tfrac{2}{3}\log(s/\mu^2),\\ &q_0^{(L+T)} = 1, && q_1^{(L+T)} = -1, && q_2^{(L+T)} = -\tfrac{131}{24} + \tfrac{9}{4}\log(s/\mu^2)\\ &t_0^{(L+T)} = 0 && t_1^{(L+T)} = 1, && t_2^{(L+T)} = \tfrac{17}{2} + \tfrac{9}{2}\log(s/\mu^2). \end{aligned} \tag{1.3.4}$$

while the m^4 terms have been only computed to $\mathcal{O}(a)$

$$\begin{aligned} &h_0^{(L+T)} = 1 - 1/2\log(s/\mu^2), && h_1^{(L+T)} = \tfrac{25}{4} - 2\zeta_3 - \tfrac{25}{6}\log(s/\mu^2) - 2\log(s/\mu^2)^2,\\ &k_0^{(L+T)} = 0, && k_1^{(L+T)} = 1 - \tfrac{2}{5}\log(s/\mu^2),\\ &g_0^{(L+T)} = 1, && g_1^{(L+T)} = \tfrac{94}{9} - \tfrac{4}{3}\zeta_3 - 4\log(s/\mu^2),\\ &j_0^{(L+T)} = 0, && j_1^{(L+T)} = 0,\\ &u_0^{(L+T)} = 0, && u_2^{(L+T)} = 0. \end{aligned} \tag{1.3.5}$$

1.3.2 Dimension six and eight

Our application of dimension six contributions is founded in [**Braaten1991**] and has previously been calculated beyond leading order by [**Lanin1986**]. The operators appearing are the masses to the power six m^6, the four-quark condensates $\langle \overline{q}\, q \overline{q}\, q \rangle$, the three-gluon condensates $\langle g^3 G^3 \rangle$ and lower dimensional condensates multiplies by the corresponding masses, such that in total the mass dimension of the operator will be six. As there are too many parameters to be fitted with experimental data we have to omit some of them, starting with the three-gluon condensate, which does not contribute at leading order. The four-quark condensates known up to $\mathcal{O}(\alpha^2)$, but we will make use of the *vacuum saturation approach* [**Beneke2008**, **Braaten1991**, **Shifman1978**] to express them in quark, anti-quark condensates $\langle q \overline{q} \rangle$. In our work we take the simplest approach possible: Introducing an effective dimension six coefficient $\rho^{(6)}_{V/A}$ divided by the appropriate power in s

$$D^{(1+0)}_{ij,V/A}\Big|_{D=6} = 0.03 \frac{\rho^{(6)}_{V/A}}{s^3} \tag{1.3.6}$$

As for the dimension eight contribution the situation is not better than the dimension six one we keep the simplest approach, leading to

$$D^{(1+0)}_{ij,V/A}\Big|_{D=8} = 0.04 \frac{\rho^{(8)}_{V/A}}{s^4}. \tag{1.3.7}$$

1.3.3 Duality Violations

1.4 Experiment

The τ-decay data we use to perform our QCD-analysis is from the **ALEPH** experiment. The ALEPH experiment was located at the large-electron-positron (LEP) collider at CERN laboratory in Geneva. LEP started producing particles in 1989 and was replaced in the late 90s by the large-hadron-collider, which makes use of the same tunnel of 27km circumference. The data produced within the experiment is still maintained by former ALEPH group members under led by M. Davier, which have performed regular updates on the datasets [**Davier2013**, **Davier2008**, **Aleph2005**].

The measured spectral functions for the Aleph data are defined in [**Davier2007**] and given for the transverse and longitudinal components separately:

$$
\begin{aligned}
\rho^{(T)}_{V/A}(s) &= \frac{m_\tau^2}{12\left|V_{ud}^2\right| S_{EW}} \frac{\mathcal{B}(\tau^- \to V^-/A^-\nu_\tau)}{\mathcal{B}(\tau^- \to e^-\overline{\nu}_e\nu_\tau)} \\
&\quad \times \frac{dN_{V/A}}{N_{V/A}\, ds}\left[\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+\frac{2s}{m_\tau^2}\right)\right]^{-1} \qquad (1.4.1) \\
\rho^{(L)}_{A}(s) &= \frac{m_\tau^2}{12\left|V_{ud}^2 S_{EW}\right|} \frac{\mathcal{B}(\tau^- \to \pi^-(K^-)\nu_\tau)}{\mathcal{B}(\tau^- \to e^-\overline{\nu}_e\nu_\tau)} \times \frac{dN_A}{N_A\, ds}\left(1-\frac{s}{m_\tau^2}\right)^{-2}.
\end{aligned}
$$

$$\mathcal{B}_e = \ldots \tag{1.4.2}$$

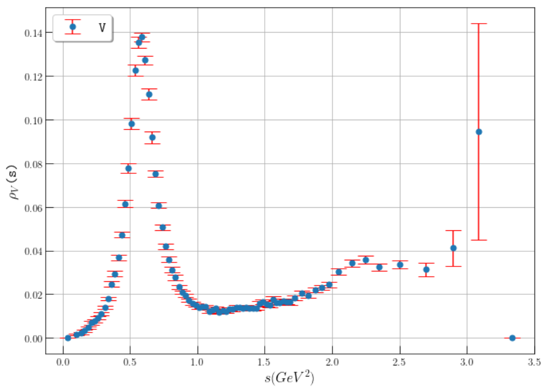
$$R_{\tau,V/A} = \frac{B_{V/A,\tau}}{B_e} \tag{1.4.3}$$

The data relies on a separation into vector and axial-vector channels. In the case of the Pions this can be achieved via counting. The vector channel is characterised by a negative parity, whereas the axial-vector channel has positive parity. A quark has by definition positive parity, thus an anti-quark has a negative parity. A meson, like the Pion particle, is a composite particle consisting of an quark an anti-quark. Consequently a single Pion carries negative parity, an even number of Pions carries positive parity and an odd number of Pions carries negative parity:

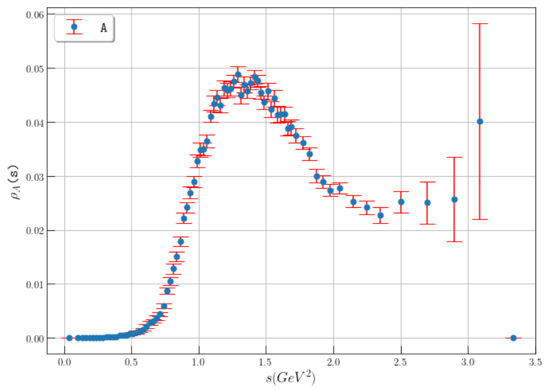
$$n \times \pi = \begin{cases} \text{vector} & \text{if } n \text{ is even,} \\ \text{axial-vector} & \text{otherwise} \end{cases}. \tag{1.4.4}$$

The contributions to the vector and axial channel can be seen in figure. The dominant modes in the vector case are **[Davier2006]** $\tau^- \to \pi^-\pi^0\nu_\tau$ and the $\tau^- \to \pi^-\pi^-\pi^+\pi^0\nu_\tau$. The first of these is produced by the $\rho(770)$ meson, which in contrary to the pions carries angular momentum of one, which is also clearly visible as peak around 770 GeV in figure vector. The dominant modes in the axial-vector case are $\tau^- \to \pi^-\nu_\tau$, $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$ and $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$. Here the three pion final states stem from the a_1^--meson, which is also clearly visible as a peak in figure.

wavy => DV OPE cannot reproduce suppressed in VpA regions below 1.5 GeV can still not be applied



(a) A gull



(b) A mouse

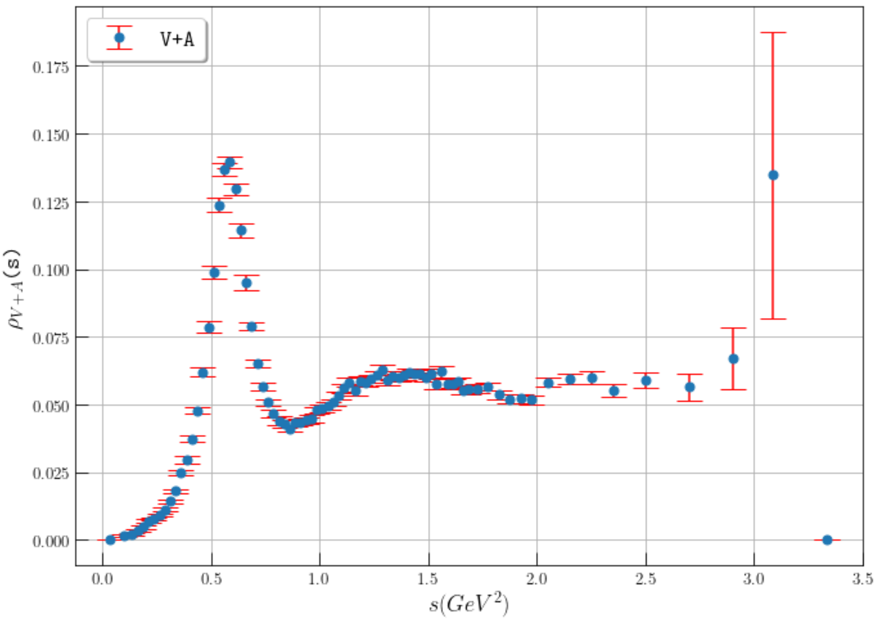


Figure 1.1: Pictures of animals

The different inclusive τ-decay ratios are then given by

$$R_{\tau,V} = \ldots \tag{1.4.5}$$