Chapter 1

Measuring the strong coupling

The strong coupling has been measured since many years from hadronic τ decays. Until today most of the applied QCDSR to τ decays are based on the method developed in the early nineties by Braaten, Pich and Narison [**Braaten1991**]. They gathered the at this time available perturbative and NP contributions to extract the strong coupling from comparing their theoretical results to the known inclusive hadronic τ decay ratio R_τ. the τ decay width using sdfaksdjfl sdfkal jsdflaksj ls sdflkjas;dfj ;asdfjka ;sdfklj a;sdlfj as;l skldfjsldfkjslkdfjslkdfjsldkfj slkdfjsldkfjsldfkjsdlkf skldfjsdlkfjsldkfjsldfkj slkdfjsldfkjslkdfj lsdkfjsldfjsldkfj sldkfjsdlkfjsdlkfjsdlkfj sldkfjsldfjsdlkfjsldkfjsd fklsdjflsdkjflksjdflksdj flskdjfkl

Table 1.1 Timeline

1991	[**Braaten1991**]: Systematic description, including NP corrections to extract α_s from R_τ.
2011	[**Boito2011a**]: Include DV model to extract α_s from OPAL data.
2015	[**Boito2014**]: Apply DV model to extract α_s from ALEPH data.
2016	[**Pich2016**]: Extract α_s from ALEPH τ data. Critic DV model.
2018	[**Boito2018**]: α_s from $e^+e^- \to$ hadrons up to 2 GeV.

1.1 Fit Criteria

1.2 Fits

1.2.1 Pinched Weights without monomial x

$$(1-x)^2(1+2x)$$

$$(1-x)^3(1+3x)$$

$$(1-x)^4(1+4x)$$

1.2.2 Monomial Weights

$$1-x^2$$

$$1-x^3$$

$$1-x^4$$

1.2.3 Pinched Weights with monomial x

$$(1-x)^2$$

$$(1-x)^3$$

$$(1-x)^4$$

1.3 Results

In the following we will perform fits to determine α_s at the m_τ^2-scale. The fits are separated corresponding to the used weight. Every weight contains multiple fits for different s_0-momenta. We will start with the kinematic weight, which appears naturally in the inclusive τ-decay ratio **??** and has the best fitting characteristics of all weights we have used.

1.3.1 Kinematic weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

The kinematic weight is defined as $\omega(x) = (1-x)^2(1+2x)$. It is a double pinched, polynomial weight-function that contains the unity and does not contain a term proportional to x, which makes it an optimal weight [**Beneke2012**]. As a doubled pinched weight it should have a good suppression of DV-contributions and its polynomial contains terms proportional to x^2 and x^3, which makes it sensitive to the dimension six and eight OPE contributions. The fits have been performed within the framework of FOPT for different numbers of s_0. The momentum sets are characterised by its lowest energy s_{min}. We fitted values down to 1.5 GeV. Going to lower energies is questionable due to the coupling constant becoming to large, which implies a breakdown of PT and appearing DVs. Furthermore it bares the risk to be affected by the $\rho(770)$ and a_1 peaks in the vector and axial-vector spectral function, which we cannot model within the framework of the OPE. For the fitting-parameters α_s, c_6 and c_8 we have given the results in table 1.2 and graphically in fig. 1.1.

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_6	c_8	χ^2/dof
1.950	10	0.3232(32)	-0.31(11)	-0.01(18)	1.13
2.000	9	0.3234(34)	-0.32(12)	-0.03(21)	1.31
2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

Table 1.2: Table of our fitting values of $\alpha_s(m_\tau^2), c_6$ and c_8 for the kinematic weight $\omega(x) = (1-x)^2(1+2x)$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

We only display the fits for s_{min} larger than 1.95 GeV as fits with higher s_{min} have a too large χ^2 (larger than two). We achieved six good fits with a χ^2 per DOF less or close to one, which we divided into two groups:

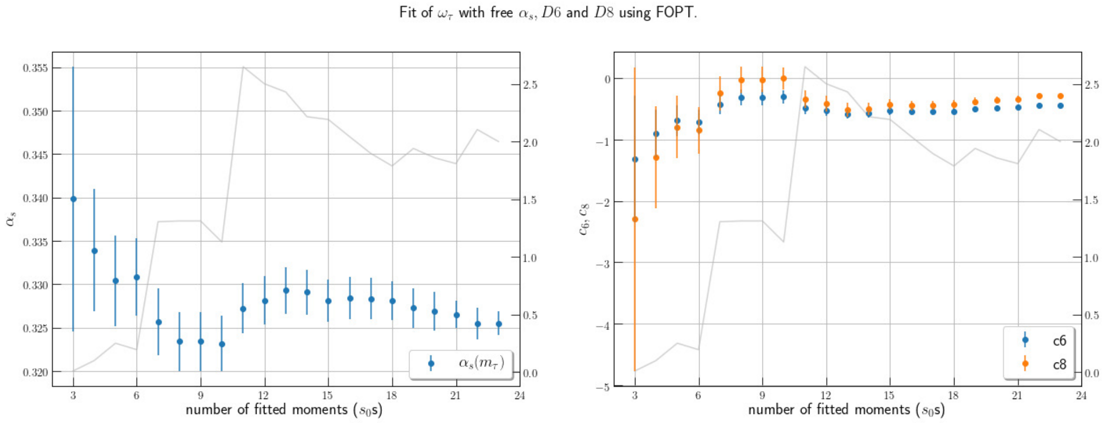


Figure 1.1: Fitting values of $\alpha_s(m_\tau^2), c_6$ and c_8 for the kinematic weight $\omega(x) = (1-x)^2(1+2x)$ using FOPT for different s_{min}. The left graph plots $\alpha_s(m_\tau^2)$ for different numbers of used s_0s. The right plot contains the dimension six and eight contributions to the OPE. Both plots have in grey the χ^2 per degree of freedom (dof).

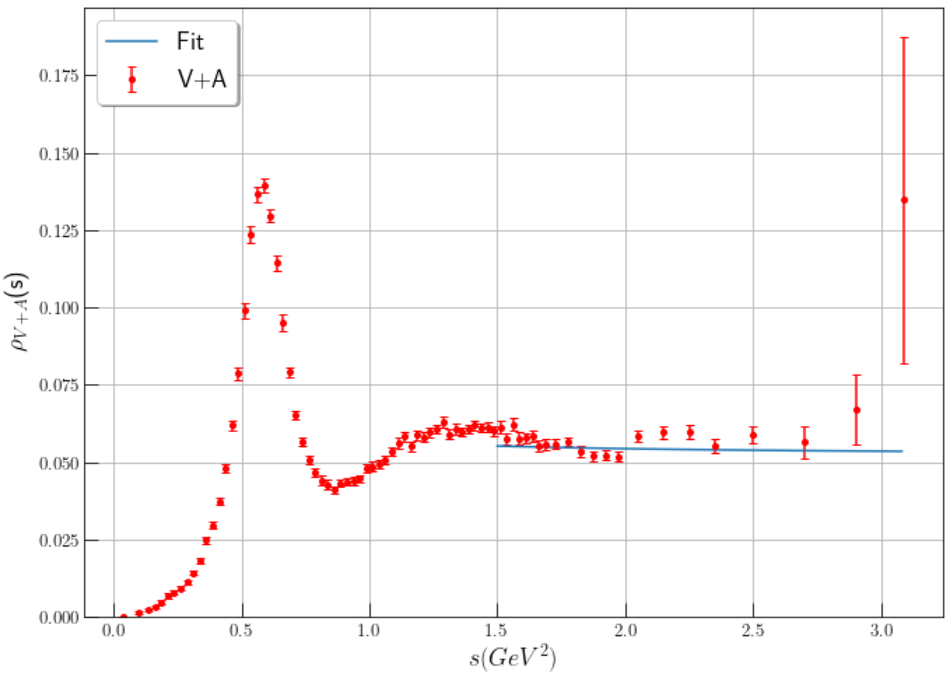


Figure 1.2: test

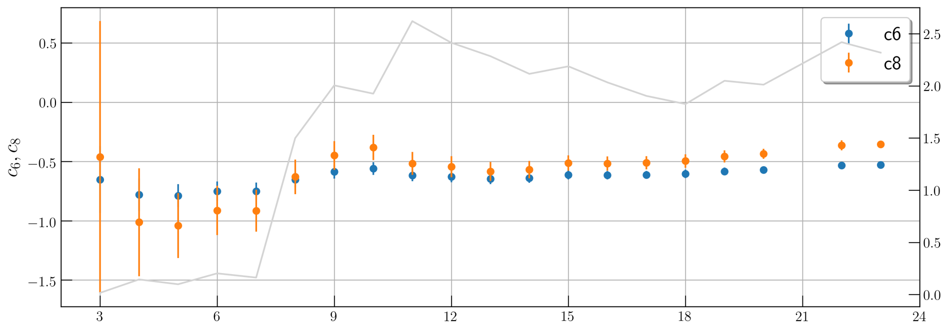


Figure 1.3

- Fits with **5-7** momenta have χ^2 per DOF larger than one and means of $\alpha(m_\tau^2) = 0.3317(33), c_6 = -0.77(17)$ and $c_8 = -0.98(35)$, where we propagated the uncertainty. We have excluded the momentum containing four s_0s, because it χ^2 is too low and its errors are too large, which is because we have to fix three variables for only four data points.
- Fits with **8-10** momenta have small χ^2 per DOF values and lower means for the strong coupling $\alpha(m_\tau^2) = 0.3241(20)$ but the OPE contributions are higher $c_6 = -0.350(75)$ and $c_8 = -0.09(12)$.

The values for the less momenta are preferred by us due to two reasons. First below energies of 2.2 GeV we have to face the problematic influence of increasing resonances. Second, we will see, that the values obtained from the lower moment fits are more compatible with our other fits series. For both, the momenta sets, we see a good convergence of the OPE.

We further tested the stability of the dimension six and eight contributions to the OPE within the same fit series but for a fixed value of the strong coupling to our previous averaged result $\alpha_s(m_\tau^2) = 0.3179$. The fits have been plotted in fig. 1.3 and show good stability. The values for c_6 and c_8 are larger than the values given in our final results from table 1.2. This is explained with a smaller contribution from the strong coupling (α_s is smaller), which has to be compensated by larger OPE contributions.

Due to the good results we will try to argue in favour of the values obtained by the lower momenta:

$$\alpha_s(m_\tau^2) = 0.3317(33), \quad c_6 = -0.77(17) \quad \text{and} \quad c_8 = -0.98(35). \tag{1.3.1}$$

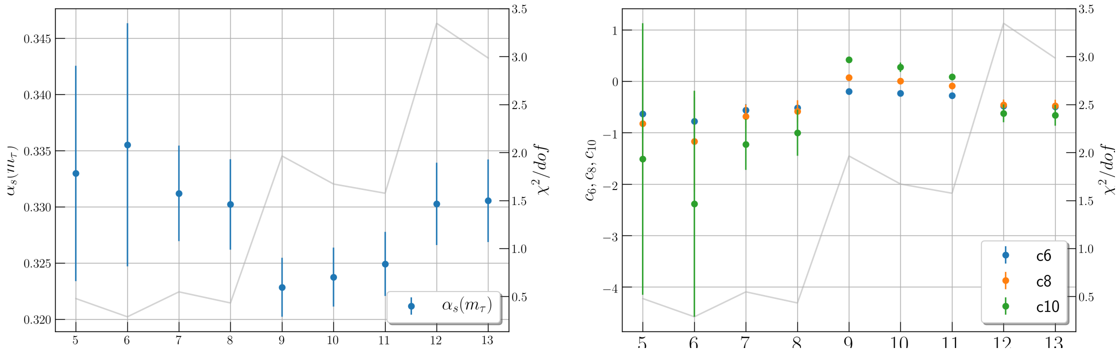
s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_6	c_8	c_{10}	χ^2/dof
1.900	11	0.3249(29)	-0.280(20)	-0.088(21)	0.088(55)	1.58
1.950	10	0.3237(26)	-0.232(25)	0.005(42)	0.275(93)	1.67
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

Table 1.3: Table of our fitting values of $\alpha_s(m_\tau^2), c_6$, c_8 and c_{10} for the cubic weight $\omega(x) = (1-x)^3(1+3x)$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

1.3.2 Cubic weight: $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$

To further consolidate the results from the kinematic weight, we test a weight of higher pinching, which is known to suppress DV more than a double pinched weight would do. Consequently, any differences to the previous fit could indicate a problem with the DV treatment. Our *cubic* weight will be triple pinched and optimal, as the kinematic weight is double pinched and we do not want any problematic contributions proportional to x. Thus we define the *cubic weight* as $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$. It is due to its polynomial structure sensitive to the dimensions six, eight and ten contributions of the OPE, which yields one more parameter to fit than with the kinematic weight ω_τ. The some good, selected fits, by χ^2 per DOF, can be seen in table 1.3 and graphically in section 1.3.2. As with the kinematic weight we get two different sets of value:

- The fits with **9, 10 and 11** momenta have a too high χ^2, but are comparable to our the upper entries of the kinematic weight table table 1.2. As with the kinematic weight the s_{min} seems to be affected by lower resonances.
- The fits with **5,6,7 and 8** have a better χ^2 per DOF value and are in good agreement with the corresponding fits of the kinematic weight. The av-



eraged value with its propagated errors read: $\alpha_s(m_\tau^2) = 0.332478(61), c_6 = -0.622(12), c_8 = -0.815(55)$ and $c_{10} = -1.5(3.1)$.

We furthermore found that the OPE is converging, but not as good as for the kinematic weight. The values of $\left|\delta^{(8)}\right|$ is only half as large as $\left|\delta^{(8)}\right|$. The values of the lower momentum count are in high agreement with the ones obtained from the kinematic weight. The conclusions that we take from the *cubic weight* are that the kinematic weight, with its double pinching, should sufficiently suppress any contributions from DVs. If DV would have an effect on the kinematic weight, we should have seen an improvement of the fits with the *cubic* weight, due to its triple pinching, which is not the case.

1.3.3 Quartic weight: $\omega(x) \equiv (1-x)^4(1+4x)$

To include an even higher pinching of four and to compare the previously obtained value for the dimension ten OPE contribution we performed fits with the *quartic weight* defined as $\omega(x) \equiv (1-x)^4(1+4x)$, which also fulfils the definition of an optimal weight [**Beneke2012**]. Unfortunately the fits only converged for $s_{min} = 2\,\text{GeV}$ (nine s_0s moment combination). The results for , with a χ^2 per DOF of 0.67 are given by:

$$\begin{aligned} \alpha_s(m_\tau^2) = 0.3290(11), \quad c_6 = -0.3030(46), \quad c_8 = -0.1874(28), \\ c_{10} = 0.3678(45) \quad \text{and} \quad c_{12} = -0.4071(77) \end{aligned} \tag{1.3.2}$$

Due to the problematic of the fitting routing, which is caused by too many OPE contributions fitted simultaneously, we will discard the fitting results for the quartic weight.

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_8	χ^2/dof
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Table 1.4: Table of our fitting values of $\alpha_s(m_\tau^2)$, and c_8 for the single pinched third power monomial weight $\omega(x) = 1 - x^3$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

1.3.4 Third power monomial: $\omega_{m3}(x) \equiv 1 - x^3$

To study the behaviour of the DV and the higher order OPE contributions of dimension eight and ten we further included two optimal, single pinched weights. The first one is defined as $\omega_{m3}(x) \equiv 1 - x^3$ and contains a single third power monomial and is consequently sensitive to dimension eight contributions from the OPE. Our fitting results can be taken from table 1.4. The χ^2 per DOF is like in the ω_τ and ω_{cubic} fits good for $s_{min} \leqslant 2.2\,\mathrm{GeV}$, but jumps to values $\chi^2/\mathrm{dof} > 1.4$ for smaller s_{min}. This is, as before, explained through resonances that appear in lower energies. Due to the good χ^2 and the internally compatible fitting values we averaged over all rows except the last one of table 1.4. The last row, at $s_{min} = 2.8\,\mathrm{GeV}$ has only one DOF and thus high errors. The averaged values are thus given by

$$\alpha(m_\tau^2) = 0.32382(42) \qquad \text{and} \qquad c_8 = -1.33(67). \tag{1.3.3}$$

We note that the strong coupling is smaller as our expected values from the kinematic weight **??**, but the dimension eight contribution is in good agreement. The strong coupling from the monomial weight to third order seems to be in better agreement with the 8-10 momenta used in the kinematic fits, whereas the dimension eight contributions agrees more with the 4-7 momenta fits.

We have made use of a single pinched weight and discovered that the fitting

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	c_{10}	χ^2/dof
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

Table 1.5: Table of our fitting values of $\alpha_s(m_\tau^2)$ and c_{10} for the single pinched fourth power monomial weight $\omega(x) = 1 - x^4$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

result is not completely compatible with our previous fitting results. Consequently weights with a pinching less than two are affected by DV and should not be used to determine the strong coupling.

1.3.5 Fourth power monomial: $\omega_{m4}(x) \equiv 1 - x^4$

We already analysed the cubic and quartic weights, which depend on the dimension ten OPE contribution, in section 1.3.2 and section 1.3.3 correspondingly. Now, even with the visible DV for fourth power monomial $\omega_{m4} \equiv 1 - x^4$ to study another single pinched moment and the dimension ten OPE contribution. The results of the are given in **??**. The fitting behaviour is very similar to the third power monomial (**??**) and we will directly cite our obtained results:

$$\alpha_s(m_\tau^2) = 0.32277(40) \qquad \text{and} \qquad c_{10} = -2.4(3.6). \tag{1.3.4}$$

As before the values for the strong coupling are lower than the ones obtained by the kinematic weight fit. Furthermore the error on the tenth dimension contribution of the OPE are too huge, although the huge errors makes it compatible with all previous results. All in all the usage of the single pinched fourth power monomial weight is questionable and does not deliver any additional insights.

1.3.6 Pich's Optimal Moments [Pich2016]

Next to the previously mentioned *optimal weights* from Beneke and Jamin [**Beneke2012**] there are *optimal moments* introduced by Pich [**LeDiberder1992**]. Combinations of these optimal moments have been widely used by the ALEPH collaboration to perform QCD analysis on the Large electron-positron collider (LEP). These moments include the for FOPT problematic proportional term in x [**Beneke2012**], thus we will perform additional fits in the Borel-sum.

$$\omega_{(n,m)}(x) = (1-x)^n \sum_{k=0}^{m} (k+1)x^k \tag{1.3.5}$$

$$\omega(x) = (1-x)^2$$

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$aGGInv$	c_6	χ^2/dof
2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89
2.400	5	0.3450(93)	-0.0243(99)	0.10(17)	0.71
2.600	4	0.337(16)	-0.014(18)	0.36(45)	0.98

Table 1.6: Table of our fitting values of $\alpha_s(m_\tau^2)$, $aGGInv$ and c_6 for the triple pinched optimal weight $\omega^{(2,0)}(x) = (1-x)^2$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

$$\omega(x) = (1-x)^3$$

s_{min}	$\#s_0 s$	$\alpha_s(m_\tau^2)$	$aGGInv$	c_6	c_8	χ^2/dof
1.900	11	0.34281(92)	-0.01473(73)	-0.103(22)	-0.534(46)	1.52
1.950	10	0.34154(99)	-0.01304(61)	-0.050(17)	-0.389(44)	1.42
2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41
2.300	6	0.3522(64)	-0.0249(62)	-0.42(18)	-1.51(57)	0.29
2.400	5	0.3480(89)	-0.0199(100)	-0.25(33)	-0.96(10)	0.39

Table 1.7: Table of our fitting values of $\alpha_s(m_\tau^2), aGGInv, c_6$ and c_8 for the optimal weight $\omega^{(3,0)}(x) = (1-x)^3$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

weight	s_{min}	$\alpha_s(m_\tau^2)$	aGGInv	c_6	c_8	c_{10}	χ^2/dof
ω_{kin}	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
ω_{cube}	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
$\omega^*_{3,0}$	2.1	0.3239(30)	-0.2125(26)	-0.627(87)	-0.74(17)	-	0.46
$\omega_{quartic}$	2.0	0.3290(11)	-	-0.3030(46)	-0.1874(28)	0.3678(45)	0.67
ω_{m3}	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
ω_{m4}	2.2	0.3203(48)	-	-	-	-1.64(77)	0.42
$\omega_{2,0}$	2.2	0.3401(57)	-0.0185(52)	0.220(88)	-	-	0.73
$\omega_{3,0}$	2.1	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	-	0.31

Table 1.8: Table of the best fits (selected by χ^2/dof and compatibility of the fitting values) for each weight including at least the strong coupling $\alpha_s(m_\tau^2)$ as a fitting variable. All fits have been performed using FOPT, except weights marked with a star ω^*, which have been fitted using the *Borel sum*.

1.3.7 Comparison

To create an overview of our previous results we have gathered the most compatible rows by hand. These are shown in table 1.8, which is composed of two parts:

- The upper three rows represent fits we found to have good properties for determining the strong coupling.
- The lower five rows are problematic fits due to too many OPE contributions, too low pinching or to terms proportional to x.

We have found that the kinematic weight is in excellent agreement with the cubic ω_{cube} and Pich's optimal weight $\omega_{3,0}$, fitted using the borel model. The fitted parameters from the kinematic weight (α_s, c_6 and c_8) are all within error ranges and thus compatible. One fact that has to be investigated is the negative appearing sign for the gluon-condensate from the borel-sum of $\omega_{3,0}$.

1.3.8 Toni Pich 2006

4. ALEPH determination

Toni built moments with five different weights:

$$\omega_{kl(x)} = (1-x)^{2+k}x^{l}(1+x) \quad \text{with} \quad (k,l) = (0,0),(1,0),(1,1),(1,2),(1,3) \tag{1.3.6}$$

He always fitted weight combinations, which we do not include.

5. Optimal moments

Used single moments

$$\omega^{(n,m)}(x) = (1-x)^{n}\sum_{k=0}^{n}(k+1)x^{k} \quad \text{with} \quad (n,m) = (1,0),(1,1),(1,2),(1,3),(1,4),(1,5),(2,0),(2,1 \tag{1.3.7}$$

but omitted NPT corrections! He fitted the kinematic weight with free α_s for $\omega(x)^{(2,1)}$. Later on he uses combined fits which is not in our interest. It is called optimal moments, because n stands for the pinching factor, which suppresses DV!

6. Including information from the s_0 dependence

Pich fits $A^{(2,0)}, A^{(2,1)}$ and $A^{(2,2)}$ separately for $s_{min} = 2\,\text{GeV}$. The corresponding weights with fitted OPE dimensions are given by:

$$\omega^{(2,0)} = (1-x)^2 \quad c_4, c_6 \tag{1.3.8}$$

$$\omega^{(2,1)} = \omega_\tau \quad c_6, c_8 \tag{1.3.9}$$

$$\omega^{(2,2)} = (1-x)^2(1+2x+x^2) = (x^2-1)^2 \quad c_8, c_{10} \tag{1.3.10}$$

Thus we can compare our results from the kinematic weight with his results and furthermore add $(1-x)^2$ to our fitting list?

Alpha is comparable, which just have a bigger error. For D6 and D8 we have to compare our definition of c_6, c_8 with his.