Chapter 1

τ decays into hadrons

The τ-lepton is the only lepton heavy enough to decay into Hadrons. It permits one of the most precie determinations of the strong coupling α_s. The inclusive τ-decay ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{Hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} \tag{1.1}$$

can be precisly calculated and is sensitive to α_s. Due to the mass of the τ-lepton $m_\tau \approx 1.8\,\text{GeV}$ it is a good phenomen to perform low-energy QCD analysis. The theoretical expression of the hadronic τ-decay ratio was first derived by [**Tsai1971**] (using current algebra, a more recent derivation making use of the *optical theorem* can be taken from [**Schwab2002**]):

$$R_\tau = 12\pi \int_0^{m_\tau} = \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)\left[\left(1 + 2\frac{s}{m_\tau^2}\right)\operatorname{Im}\Pi^{(T)}(s) + \operatorname{Im}\Pi^{(L)}(s)\right]. \tag{1.2}$$

The τ-decay ratio depends on the two-point function

$$\Pi^{V/A}_{\mu\nu,ij}(s) \equiv i \int dx\, e^{ipx} \langle \Omega | T\{J^{V/A}_{\mu,ij}(x) J^{V/A}_{\nu,ij}(0)^\dagger\} | \Omega \rangle, \tag{1.3}$$

with $|\Omega\rangle$ being the physical vacuum. The vectorial and axial-vector currents are given by

$$J^V_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu q_i(x) \quad \text{and} \quad J^A_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu\gamma_5 q_i(x) \tag{1.4}$$

where i, j stand for the light quark flavours u, d and s.

The general correlator $\Pi^{\mu\nu}(q^2)$ can be decomposed into a vector/ axial-vector (V/A) and scalar/ pseudo-scalar (S/P) part containing a correction [**Broadhurst1975**]

$$\begin{aligned}\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{V,A}(q^2) + \frac{g^{\mu\nu}}{q^2}(m_i \mp m_j)\Pi^{S,P}(q^2) \\ + g^{\mu\nu}\frac{(m_i \mp m_j)}{q^2}[\langle \overline{q}_i q_i \rangle \mp \langle \overline{q}_j q_j \rangle],\end{aligned} \tag{1.5}$$

which is composed of a vector $\Pi^{V,A}$ and scalar $\Pi^{S,P}$ part. The third term are corrections arising due to the physical vacuum $|\Omega\rangle$.

The general correlator $\Pi^{\mu\nu}$ can also be decomposed into transversal and longitudinal components:

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T)}(q^2) + q^\mu q^\nu \Pi^{(L)}(q^2). \tag{1.6}$$

To relate the two different contributions we note, that only the scalar components of eq. (1.5) carry a mass term. Using the *Ward identity*

$$q_\mu \Pi^{\mu\nu}(q^2) = 0 \tag{1.7}$$

we can introduce two four-momenta into eq. (1.5)

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = (m_i \mp m_j)^2 \Pi^{S,P}(q^2) + (m_i \mp m_j)[\langle \bar{q}_i q_i \rangle \mp \langle \bar{q}_j q_j \rangle] \tag{1.8}$$

to relate the longitudinal of eq. (1.6)

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = q^4 \Pi^{(L)}(q^2) = s^2 \Pi^{(L)}(s), \tag{1.9}$$

where we defined $s \equiv q^2$. Thus

$$s^2 \Pi^{(L)}(s) = (m_i \mp m_j)^2 \Pi^{(S,P)}(s) + (m_i \mp m_j)[\langle \bar{q}_i q_i \rangle \mp \langle \bar{q}_j q_j \rangle]. \tag{1.10}$$

Furthermore we can relate the transversal and vectorial components via

$$\Pi^{\mu\nu}(s) = \underbrace{(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T)}(s) + (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(L)}(s)}_{=(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T+L)}(s)} + \frac{g^{\mu\nu} s^2}{q^2}\Pi^{(L)}(s) \tag{1.11}$$

where $\Pi^{(T+L)}(s) \equiv \Pi^{(T)}(s) + \Pi^{(L)}(s)$, such that

$$\Pi^{(V,A)}(s) = \Pi^{(T)}(s) + \Pi^{(L)} = \Pi^{(T+L)}. \tag{1.12}$$

Inspecting the inclusive ratio R_τ in eq. (1.1) introduces a problematic integral over the real axis of $\Pi(s)$ from 0 up to m_τ. The integral is problematic for two reasons:

- The *perturbative Quantum Chromodynamcs* (**pQCD**) and the OPE breaks down for low energies (over which we have to integrate).
- The positive euclidean axis of $\Pi(s)$ has a discontinuity cut and can theoretically not be evaluated.

To literally circunvent these issues we make use of *Cauchy's Theorem*

$$\int_{\mathcal{C}} f(z)\, dz = 0, \tag{1.13}$$

where $f(z)$ is an analytic function on a closed contour $\mathcal{C}$.

In our case we have to deal with the two-point correlator $\Pi(s)$, which is analytic except for the positive real axis (with which we will deal with to a later point[1]) Consequently, to rewrite we can rewrite the definite integral of eq. (1.2) into a contour integral over a closed circle with radius m_τ^2. The closed contour consists of four line integrals, which have been visualized in fig. 1.1. Summing over the four line integrals, performing a *analytic continuation* of the two-point correlator $\Pi(s) \to \Pi(s+i\epsilon)$ and finally taking the limit of $\epsilon \to 0$ gives us the needed relation between eq. (1.2) and the closed contour:

$$\begin{aligned}
\oint_{s=m_\tau} \Pi(s) &= \int_0^{m_\tau} \Pi(s+i\epsilon) + \int_{\mathcal{C}_2} \Pi(s)\, ds + \int_{m_\tau}^0 \Pi(s-i\epsilon)\, ds + \int_{\mathcal{C}_4} \Pi(s)\, ds \\
&= \int_0^{m_\tau} \Pi(s+i\epsilon) - \Pi(s-i\epsilon)\, ds + \int_{\mathcal{C}_2} \Pi(s)\, ds + \int_{\mathcal{C}_4} \Pi(s)\, ds \\
&= \int_0^{m_\tau} \Pi(s+i\epsilon) - \overline{\Pi(s+i\epsilon)} + \int_{\mathcal{C}_2} \Pi(s)\, ds + \int_{\mathcal{C}_4} \Pi(s)\, ds \\
&\overset{\lim \epsilon\to 0}{=} 2i \int_0^{m_\tau} \operatorname{Im}\Pi(s)\, ds + \oint_{s=m_\tau} \Pi(s)\, ds
\end{aligned} \tag{1.14}$$

where we made use of $\Pi(z) = \overline{\Pi(\overline{z})}$ (due to $\Pi(s)$ is analytic) and $\Pi(z) - \overline{\Pi(z)} = 2i \operatorname{Im}\Pi(z)$. The result can be rewritten in a more intuitive form, which we also visualized in fig. 1.1

$$\int_0^{m_\tau} \Pi(s)\, ds = \frac{i}{2} \oint_{s=m_\tau} \Pi(s)\, ds \tag{1.15}$$

Finally combining eq. (1.15) with eq. (1.2) we get

$$R_\tau = 6\pi i \oint_{s=m_\tau} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(T)}(s) + \Pi^{(L)} \right] \tag{1.16}$$

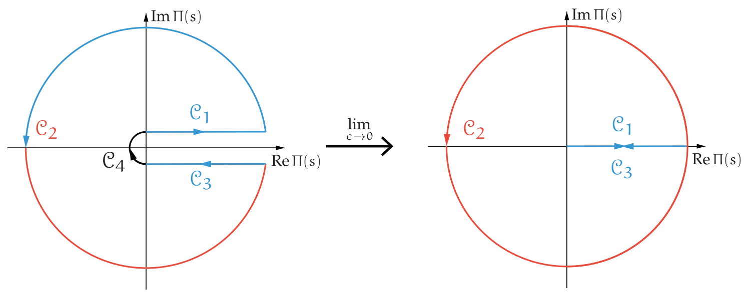
for the hadronic τ-decay ratio.

The contour integral obtained is an import result as we can now theoretically evaluate the hadronic τ-decay ratio sufficiently large energy scales ($m_\tau \approx 1.78\,\text{MeV}$) at which $\alpha_s(m_\tau) \approx 0.33$ **[Pich2016]** is tolerable heigh for applying perturbation theory and the OPE. Obviously we would benefit from a contour integral over a bigger circumference, but τ-decays are limited by the m_τ. Nevertheless there are promising e^+e^- annihilation data, which yields valuable R-ratio values up to $2\,\text{GeV}$ **[Boito2018][Keshavarzi2018]**.

It is convenient to rewrite the

$$\Pi^{(L+T)} = \Pi^{(L)} + \Pi^{(T)} \tag{1.17}$$

[1] To not evaluate $\Pi(s)$ at the positive real axis we have to introduce *pinched weights*. The *pinched weights* vanish for $s \to m_\tau$.



$$\int_0^{m_\tau} \mathrm{Im}\,\Pi(s)\,\mathrm{d}s = \oint_{|s|=m_\tau} \Pi(s)\,\mathrm{d}s$$

Figure 1.1: Visualization of the usage of Cauchy's theorem to transform eq. (1.2) into a closed contour integral over a circle of radius m_τ^2.

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{\mathrm{d}s}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)-\left(\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s)\right] \tag{1.18}$$

$$D^{(L+T)}(s) \equiv -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi^{(L+T)}(s), \qquad D^{(L)}(s) \equiv \frac{s}{m_\tau^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\Pi^{(L)}(s)\right) \tag{1.19}$$

Integration by parts

$$\int_a^b u(x)V(x)\,\mathrm{d}x = \left[U(x)V(x)\right]_a^b - \int_a^b U(x)v(x)\,\mathrm{d}x \tag{1.20}$$

$$\begin{aligned}
R_\tau^{(1)} &= \frac{6\pi i}{m_\tau^2}\oint_{|s|=m_\tau^2} \underbrace{\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)}_{=u(x)}\underbrace{\Pi^{(L+T)}(s)}_{=V(x)} \\
&= \frac{6\pi i}{m_\tau^2}\left\{\left[-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)\right]_{|s|=m_\tau^2}\right. \\
&\qquad \left. +\oint_{|s|=m_\tau^2}\underbrace{-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)}_{=U(x)}\underbrace{\tfrac{\mathrm{d}}{\mathrm{d}s}\Pi^{(L+T)}(s)}_{=v(x)}\right\} \\
&= -3\pi i \oint_{|s|=m_\tau^2 s}\frac{\mathrm{d}s}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\tfrac{\mathrm{d}}{\mathrm{d}s}D^{(L+T)}
\end{aligned} \tag{1.21}$$

where we fixed the integration constant to $C = -\frac{m_\tau^2}{2}$ in the second line and left the antiderivatives contained in the squared brackets untouched. Parametrizing the expression in the squared brackets

$$\left[-\frac{m_\tau^2}{2}\left(1-e^{-i\phi}\right)^3\left(1+e^{-i\phi}\right)\Pi^{(L+T)}(m_\tau^2 e^{-i\phi})\right]_0^{2\pi}=0 \tag{1.22}$$

where $s \to m_\tau^2 e^{-i\phi}$ and $(1-e^{-i\cdot 0})=(1-e^{-i\cdot 2\pi})=0$.

$$\begin{aligned} R_\tau^{(2)} &= \oint_{|s|=m_\tau^2} ds\left(1-\frac{s}{m_\tau^2}\right)^2\left(-\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s) \\ &= -4\pi i\oint\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 D^{(L)}(s) \end{aligned} \tag{1.23}$$

$$R_\tau=-\pi i\oint_{|s|=m_\tau^2}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left[3\left(1+\frac{s}{m_\tau^2}D^{(L+T)}(s)+4D^{(L)}(s)\right)\right] \tag{1.24}$$

$$R_\tau=-\pi i\oint_{|s|=m_\tau^2}\frac{dx}{x}(1-x)^3\left[3(1+x)D^{(L+T)}(m_\tau^2 x)+4D^{(L)}(m_\tau^2 x)\right], \tag{1.25}$$

where $x=s/m_\tau^2$.

$$R^{\omega}_{\tau,V/A}=\frac{N_c}{2}S_{EW}|V_{ud}|^2\left(1+\delta_\omega^{(0)}+\delta_\omega^{EW}+\delta_\omega^{DVs}+\sum_{D\leqslant 2}\delta^{(D)}_{ud,\omega}\right) \tag{1.26}$$

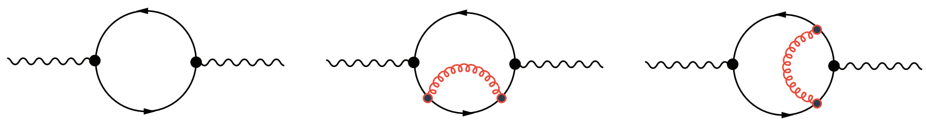
1.1 The perturbative expansion

We will treat the correlator in the chiral limit for which the longitudinal components $\Pi^L(s)$ vanish (see eq. (1.11)) and the axial and vectorial contributions are equal. Consequently **[Beneke2008]** we can write the vector correlation function $\Pi(s)$ as:

$$\Pi_V^{T+L}(s)=-\frac{N_c}{12\pi^2}\sum_{n=0}^{\infty}a_\mu^n\sum_{k=0}^{n+1}c_{n,k}L^k \quad \text{with} \quad L\equiv\ln\frac{-s}{\mu^2}. \tag{1.27}$$

The coeffiecient $c_{n,k}$ up to two-loop order can be obtained by Feynman-diagram calculations. add complete calculation E.g. we can compare the zero-loop result of the correlator **[Jamin2006]**

$$\Pi^B_{\mu\nu}(q^2)\Big|^{1-\text{loop}}=\frac{N_c}{12\pi^2}\left(\frac{1}{\hat{\epsilon}}-\log\frac{(-q^2-i0)}{\mu^2}+\frac{5}{3}+\mathcal{O}(\epsilon)\right) \tag{1.28}$$



with eq. (1.27) and extract the first two coefficients

$$c_{00} = -\frac{5}{3} \qquad \text{and} \qquad c_{01} = 1, \tag{1.29}$$

where $\Pi^{B}_{\mu\nu}(q^2)$ is not renormalized[2]

The second loop can also be calculated by diagram techniques resulting in **[Boito2011]**

$$\Pi_V^{(1+0)}(s)\Big|^{2-\mathrm{loop}} = -\frac{N_c}{12\pi^2} a_\mu \log(\frac{-s}{\mu^2}) + \cdots \tag{1.30}$$

yielding $c_{11} = 1$.

Beginning from three loop diagrams the algebra becomes exausting and one has to use dedicated algorithms to compute the heigher loops. The third loop calculations have been done in the late seventies by [**Chetyrkin1979**, **Dine1979**, **Celmaster1979**]. The four loop evaluation have been completed a little more than ten years later by [**Gorishnii1990**, **Surguladze1990**]. The heighest loop published, that amounts to α_s^4, was published in 2008 [**Baikov2008**] almost 20 years later.

Fixing the number of colors to $N_c = 3$ the missing coefficients up to order four in α_s read:

$$\begin{aligned}
c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f \\
c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 \\
&\quad - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2 \\
c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7,
\end{aligned} \tag{1.31}$$

where used the flavour number $N_f = 3$ for the last line.

The 6-loop calculation has until today not been achieved, but Beneke und Jamin [**Beneke2008**] used and educated guess to estimate the coefficient

$$c_{5,1} \approx 283 \pm 283. \tag{1.32}$$

Until know we have mentioned the coefficients $c_{n,k}$ with a fixed $k = 1$. This is due to the RGE, which relates coefficients with a different k to the

[2] The term $1/\hat{\epsilon}$, which is of order 0 in α_s, will be cancelled by renormalization.

coefficients mentioned above. To make usage of the RGE $\Pi_V^{T+L}(s)$ needs to be a physical quantity, which can be achieved by rewriting eq. (1.19) to:

$$D_V^{(T+L)} = -s\frac{d\Pi_V^{(T+L)}(s)}{ds} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}, \tag{1.33}$$

where we used $dL^k/ds = k\ln(-s/\mu^2)^{k-1}(-1/\mu^2)$. D_V^{1+0} being a physical quantity has to fulfill the RGE **??**

$$-\mu\frac{d}{d\mu}D_V^{(T+L)} = -\mu\frac{d}{d\mu}\left(\frac{\partial}{\partial L}dL + \frac{\partial}{\partial a_s}da_s\right)D_V^{T+L} = \left(2\frac{\partial}{\partial L} + \beta\frac{\partial}{\partial a_s}\right)D_V^{T+L} = 0, \tag{1.34}$$

where we defined the β-function in **??** and used $dL/d\mu = -2/\mu$. The RGE puts constraints on the $c_{n,k}$-coefficients, ... not independent

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + a_\mu(c_{11} + 2c_{12}L) + a_\mu^2(c_{21} + 2c_{22}L + 3c_{23}L^2)\right] \tag{1.35}$$

inserting into RGE

$$4a_\mu c_{12} + 2a_\mu^2(2c_{22} + 6c_{23}L) + \beta_1 a_\mu^2(c_{11} + 2c_{12}L) + \mathcal{O}(a_\mu^3) = 0 \tag{1.36}$$

Thus

$$c_{12} = 0 \quad c_{22} = \frac{\beta_1 c_{11}}{4} \quad c_{23} = 0 \tag{1.37}$$

or $D(s)$ to the first order in α_s

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + c_{11}a_\mu\left(c_{21} - \frac{1}{2}\beta_1 c_{11}L\right)a_\mu^2\right] + \mathcal{O}(a_\mu^3) \tag{1.38}$$

1.1.1 Renormalisation group summation

We can express the perturbative contribution $\delta^{(0)}$ to R_τ (see eq. (1.26)) as

$$\delta^{(0)} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k c_{n,k}\frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}(1-x)^3(1+x)\log\left(\frac{-M_\tau^2 x}{\mu^2}\right)^{k-1}, \tag{1.39}$$

where we inserted the expansion of $D_V^{(T+L)}$ eq. (1.19) into R_τ eq. (1.25). Keep in mind that we are working in the chiral limit, such that $D^L = 0$ vanishes and the contributions from the vector and axialvector correlator are identical

$$D^{(T+L)} = D_V^{(T+L)} + D_A^{(T+L)} = 2D_V^{(T+L)}. \tag{1.40}$$

The perturbative contribution $\delta^{(0)}$ is a physical quantity and satisfies the homogeneous RGE, thus is independent on the scale μ. Consequently we have the freedom to choose μ, which leads to two main descriptions **fixed-order perturbation theory** (FOPT) and **contour-improved perturbation theory** (CIPT). The two resulting series should converge to equal values, but differ notably.

By using the FOPT prescription we fix $\mu^2 = m_\tau^2$ leading to

$$\delta_{FO}^{(0)} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k c_{n,k} J_{k-1} \tag{1.41}$$

where the contour integrals J_l are defined by

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log^l(-x). \tag{1.42}$$

The integrals J_l up to order α_s^4 are given by [**Beneke2008**]:

$$J_0 = 1, \quad J_1 = -\frac{19}{12} \quad J_2 = \frac{265}{72} - \frac{1}{3}\pi^2, \quad J_3 = -\frac{3355}{288} + \frac{19}{12}\pi^2. \tag{1.43}$$

Using FOPT the strong coupling $a(\mu)$, which runs with the scale μ, is fixed at $a(m_\tau^2)$ and can be taken out of the closed-contour integral. We still have to integrate over the logarithms $\log(-s/m_\tau^2)$.

Using CIPT we can sum the logarithms by setting the scale to $\mu^2 = -m_\tau^2 x$ in eq. (1.39), resulting in:

$$\delta_{CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2), \tag{1.44}$$

where the contour integrals J_l are defined by

$$J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-m_\tau^2 x). \tag{1.45}$$

All logarithms vanish except the ones for $k = 1$:

$$\log(1)^{k-1} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & k \neq 1 \end{cases} \tag{1.46}$$

which selectes adler function coefficients $c_{n,1}$ with a fixed $k = 1$. Handling the logarithms left us with the integration of $\alpha_s(-m_\tau^2 x)$ over the closed-contour $\oint_{|x|=1}$, which now depends on the integration variable x.

Calculating the perturbative contribution $\delta^{(0)}$ to R_τ for the two different prescriptions yields [**Beneke2008**]

$$\begin{aligned} & \qquad\; \alpha_s^2 \qquad\;\; \alpha_s^2 \qquad\;\; \alpha_s^3 \qquad\;\; \alpha_s^4 \qquad\qquad \alpha_s^5 \\ \delta_{FO}^{(0)} &= 0.1082 + 0.0609 + 0.0334 + 0.0174 (+0.0088) = 0.2200 (0.2288) \qquad (1.47) \\ \delta_{CI}^{(0)} &= 0.1479 + 0.0297 + 0.0122 + 0.0086 (+0.0038) = 0.1984 (0.2021). \qquad (1.48) \end{aligned}$$

The series indicate, that CIPT converges faster and that both series aproach a different value. This discrepancy represents currently the biggest theoretical uncertainity while extracting the strong coupling α_s.

As today we do not know which if FOPT or CIPT is the correct approach of measuring α_s. Therefore there are currently three ways of stating result:

- Quoting the average of both results.
- Quoting the CIPT result.
- Quoting the FOPT result.

We follow the approach of Beneke and Jamin [**Benke2008**] who prefere FOPT.

1.2 Non-Perturbative OPE Contribution

The perturbative contribution to the Sum-Rule, that we have seen so far, is the dominant one. With

$$\begin{aligned} R_\tau^{FOPT} &= \\ R_\tau^{CIPT} &= \end{aligned} \tag{1.49}$$

The NP vs perturbative contributions can be varied by choosen different weights than ω_τ.

1.2.1 Dimension four

For the OPE contributions of dimension four we have to take into account the terms with masses to the fourth power m^4, the quark condensate multiplied by a mass $m\langle\overline{q}q\rangle$ and the glucon condensate $\langle GG\rangle$. The resulting expression can be taken from the appendix of [**Pich1999**], yielding:

$$D_{ij}^{(L+T)}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_n \Omega^{(1+0)}(s/\mu^2)a^n, \tag{1.50}$$

where

$$\begin{aligned} \Omega_n^{(1+0)}(s/\mu^2) \;=\; & \frac{1}{6}\langle aGG\rangle p_n^{(L+T)}(s/\mu^2) + \sum_k m_k\langle\overline{q}_k q_k\rangle r_n^{(L+T)}(s/\mu^2) \\ & + 2\langle m_i\overline{q}_i q_i + m_j\overline{q}_j q_j\rangle q_n^{(L+T)}(s/\mu^2) \pm \frac{8}{3}\langle m_j\overline{q}_i q_i + m_i\overline{q}_j q_j\rangle t_n^{(L+T)} \\ & - \frac{3}{\pi^2}(m_i^4 + m_j^4)h_n^{(L+T)}(s/\mu^2) \mp \frac{5}{\pi^2}m_i m_j(m_i^2+m_j^2)k_n^{(L+T)}(s/\mu^2) \\ & + \frac{3}{\pi^2}m_i^2 m_j^2 g_n^{(L+T)}(s/\mu^2) + \sum_k m_k^4 j_n^{(L+T)}(s/\mu^2) + 2\sum_{k\neq l} m_k^2 m_l^2 u_n^{(L+T)}(s/\mu^2) \end{aligned} \tag{1.51}$$

The perturbative expansion coefficients are known to $\mathcal{O}(a^2)$ for the condensate contributions,

$$\begin{array}{lll} p_0^{(L+T)} = 0, & p_1^{(L+T)} = 1, & p_2^{(L+T)} = \frac{7}{6}, \\ r_0^{(L+T)} = 0, & r_1^{(L+T)} = 0, & r_2^{(L+T)} = -\frac{5}{3} + \frac{8}{3}\zeta_3 - \frac{2}{3}\log(s/\mu^2), \\ q_0^{(L+T)} = 1, & q_1^{(L+T)} = -1, & q_2^{(L+T)} = -\frac{131}{24} + \frac{9}{4}\log(s/\mu^2) \\ t_0^{(L+T)} = 0 & t_1^{(L+T)} = 1, & t_2^{(L+T)} = \frac{17}{2} + \frac{9}{2}\log(s/\mu^2). \end{array} \tag{1.52}$$

while the m^4 terms have been only computed to $\mathcal{O}(a)$

$$\begin{array}{ll} h_0^{(L+T)} = 1 - 1/2 \log(s/\mu^2), & h_1^{(L+T)} = \frac{25}{4} - 2\zeta_3 - \frac{25}{6}\log(s/\mu^2) - 2\log(s/\mu^2)^2, \\ k_0^{(L+T)} = 0, & k_1^{(L+T)} = 1 - \frac{2}{5}\log(s/\mu^2), \\ g_0^{(L+T)} = 1, & g_1^{(L+T)} = \frac{94}{9} - \frac{4}{3}\zeta_3 - 4\log(s/\mu^2), \\ j_0^{(L+T)} = 0, & j_1^{(L+T)} = 0, \\ u_0^{(L+T)} = 0, & u_2^{(L+T)} = 0. \end{array} \tag{1.53}$$

1.2.2 Dimension six and eight

Our application of dimension six contributions is founded in [**Braaten1991**] and has previously been calculated beyond leading order by [**Lanin1986**]. The operators appearing are the masses to the power six m^6, the four-quark condensates $\langle \overline{q}q\overline{q}q \rangle$, the three-gluon condensates $\langle g^3 G^3 \rangle$ and lower dimensional condensates multiplies by the corresponding masses, such that in total the mass dimension of the operator will be six. As there are too many parameters to be fitted with experimental data we have to omit some of them, starting with the three-gluon condensate, which does not contribute at leading order. The four-quark condensates known up to $\mathcal{O}(a^2)$, but we will make use of the *vacuum saturation approach* [**Beneke2008**, **Braaten1991**, **Shifman1978**] to express them in quark, anti-quark condensates $\langle q\overline{q} \rangle$. In our work we take the simplest approach possible: Introducing an effective dimension six coefficient $\rho^{(6)}_{V/A}$ divided by the appropiate power in s

$$D^{(1+0)}_{ij,V/A}\Big|_{D=6} = 0.03 \frac{\rho^{(6)}_{V/A}}{s^3} \tag{1.54}$$

As for the dimension eigth contribution the situation is not better than the dimension six one we keep the simplest approach, leading to

$$D^{(1+0)}_{ij,V/A}\Big|_{D=8} = 0.04 \frac{\rho^{(8)}_{V/A}}{s^4}. \tag{1.55}$$