Chapter 1

τ decays into hadrons

The τ-lepton is the only lepton heavy enough to decay into Hadrons. It permits one of the most precise determinations of the strong coupling α_s. The inclusive τ-decay ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{Hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} \tag{1.1}$$

can be precisly calculated and is sensitive to α_s. Due to low the mass of the τ-lepton $m_\tau \approx 1.8\,\text{GeV}$ τ-decays are excellent for performing a low-energy QCD analysis. The theoretical expression of the hadronic τ-decay ratio was first derived by [**Tsai1971**], using current algebra, a more recent derivation making use of the *optical theorem* can be taken from [**Schwab2002**]. The inclusive ratio is then given by:

$$R_\tau(s) = 12\pi \int_0^{m_\tau} = \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)\left[\left(1 + 2\frac{s}{m_\tau^2}\right) \operatorname{Im}\Pi^{(T)}(s) + \operatorname{Im}\Pi^{(L)}(s)\right], \tag{1.2}$$

where $\operatorname{Im}\Pi$ is the two-point function (see **??**). In the case of τ-decays we only have to consider vector (V) and axial-vector contributions (A) of decays into up, down and strange quarks. Thus taking i, j as the flavour indices for the light quarks (u, d and s) we can express the correlator as

$$\Pi^{V/A}_{\mu\nu,ij}(s) \equiv i \int dx\, e^{ipx} \langle \Omega | T\{J^{V/A}_{\mu,ij}(x) J^{V/A}_{\nu,ij}(0)^\dagger\} | \Omega \rangle, \tag{1.3}$$

with $|\Omega\rangle$ being the physical vacuum. The vector and axial-vector currents are then distinguished by the corresponding dirac-matrices ($\gamma_\mu \text{and} \gamma_\mu \gamma_5$) given by

$$J^V_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu q_i(x) \quad \text{and} \quad J^A_{\mu,ij}(x) = \overline{q}_j(x)\gamma_\mu\gamma_5 q_i(x). \tag{1.4}$$

The two-point function can be decomposed into its vector and axial-vector contributions, but also into transversal and longitudinal components. We will give now both of these decompositions and relate them, which has some implications for a common used approximation: the **chiral limit**, where the quark masses are taken to 0 ($m_q \to 0$).

Starting with the decomposition into vector, axial-vector, scalar (S) and pseudo-scalar (P) components we can write [**Broadhurst1981**, **Jamin1992**]

$$\begin{aligned}\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi^{V,A}(q^2) + \frac{g^{\mu\nu}}{q^2}(m_i \mp m_j)\Pi^{S,P}(q^2) \\ + g^{\mu\nu}\frac{(m_i \mp m_j)}{q^2}[\langle \overline{q}_i q_i\rangle \mp \langle \overline{q}_j q_j\rangle],\end{aligned} \tag{1.5}$$

which is composed of a vector $\Pi^{V,A}$ and scalar $\Pi^{S,P}$ part. The third term are corrections arising due to the physical vacuum $|\Omega\rangle$. The latter decomposition rewrites the correlator $\Pi^{\mu\nu}(q^2)$ into transversal and longitudinal components:

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T)}(q^2) + q^\mu q^\nu \Pi^{(L)}(q^2). \tag{1.6}$$

With the two decompositions eq. (1.5) and eq. (1.6) we can now identify the longitudinal components of the correlator as being purely scalar, by multiplying eq. (1.5) by two four-momenta and making use of the Ward-idendity **??** we can write

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = (m_i \mp m_j)^2 \Pi^{S,P}(q^2) + (m_i \mp m_j)[\langle \overline{q}_i q_i\rangle \mp \langle \overline{q}_j q_j\rangle], \tag{1.7}$$

which then can be related to the longitudinal component of eq. (1.6) by comparisson of the two equations

$$q_\mu q_\nu \Pi^{\mu\nu}(q^2) = q^4 \Pi^{(L)}(q^2) = s^2 \Pi^{(L)}(s) \quad \text{with} \quad s \equiv q^2. \tag{1.8}$$

In a more eloquent way this can be expressed as

$$s^2 \Pi^{(L)}(s) = (m_i \mp m_j)^2 \Pi^{(S,P)}(s) + (m_i \mp m_j)[\langle \overline{q}_i q_i\rangle \mp \langle \overline{q}_j q_j\rangle], \tag{1.9}$$

where we can see, that all mass terms are related to the longitudinal component of the correlator. By defining a combination of the transversal and longitudinal correlator

$$\Pi^{(T+L)}(s) \equiv \Pi^{(T)}(s) + \Pi^{(L)}(s) \tag{1.10}$$

we can additionaly relate the transversal and vectorial components via

$$\Pi^{\mu\nu}(s) = \underbrace{(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T)}(s) + (q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(L)}(s)}_{=(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi^{(T+L)}(s)} + \frac{g^{\mu\nu} s^2}{q^2}\Pi^{(L)}(s), \tag{1.11}$$

such that

$$\Pi^{(V,A)}(s) = \Pi^{(T)}(s) + \Pi^{(L)} = \Pi^{(T+L)}, \tag{1.12}$$

where the vector/ axial-vector component of the correlator is now related to the newly defined transversal and longitudinal combination of the correlator. As the τ-decays, with the limiting factor of the τ-mass, can only decay into light quarks we will often neglect the quark masses and work in the so called chiral limit. In the chiral limit the longitudinal component, which is proportional to the quark masses, of the correlator vanishes.

Examining the inclusive ratio R_τ in eq. (1.1), we note that we have to deal with a problematic integral over the real axis of $\Pi(s)$ from 0 up to m_τ. The integral is problematic for two reasons:

- **Perturbative Quantum Chromodynamcs** (pQCD) and the OPE breaks down for low energies (over which we have to integrate).

- The positive euclidean axis of $\Pi(s)$ has a discontinuity cut and can theoretically not be evaluated (see **??**).

To literally circunvent the former issue we make use of *Cauchy's Theorem* **??**. Consequently we rewrite the definite integral of eq. (1.2) into a contour integral over a closed circle with radius m_τ^2. The closed contour consists of four line integrals, which have been visualized in fig. 1.1. Summing over the four line integrals, performing a *analytic continuation* of the two-point correlator $\Pi(s) \to \Pi(s+i\epsilon)$ and finally taking the limit of $\epsilon \to 0$ gives us the needed relation between eq. (1.2) and the closed contour:

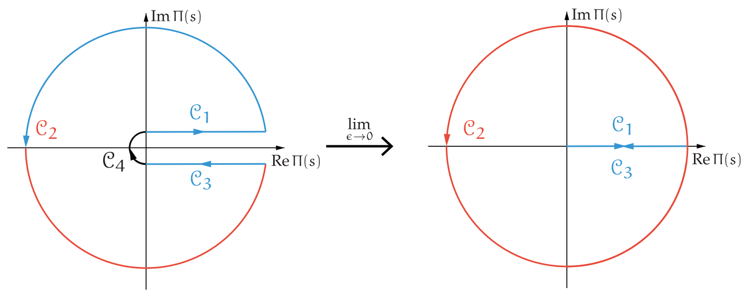
$$\begin{aligned}\oint_{s=m_\tau} \Pi(s) &= \int_0^{m_\tau} \Pi(s+i\epsilon) + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{m_\tau}^{0} \Pi(s-i\epsilon)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\ &= \int_0^{m_\tau} \Pi(s+i\epsilon) - \Pi(s-i\epsilon)\,ds + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\ &= \int_0^{m_\tau} \Pi(s+i\epsilon) - \overline{\Pi(s+i\epsilon)} + \int_{\mathcal{C}_2} \Pi(s)\,ds + \int_{\mathcal{C}_4} \Pi(s)\,ds \\ &\overset{\lim \epsilon\to 0}{=} 2i \int_0^{m_\tau} \operatorname{Im}\Pi(s)\,ds + \oint_{s=m_\tau} \Pi(s)\,ds\end{aligned} \tag{1.13}$$

where we made use of $\Pi(z) = \overline{\Pi(\overline{z})}$ (due to $\Pi(s)$ is analytic) and $\Pi(z) - \overline{\Pi(z)} = 2i \operatorname{Im}\Pi(z)$. The result can be rewritten in a more intuitive form, which we also visualized in fig. 1.1

$$\int_0^{m_\tau} \Pi(s)\,ds = \frac{i}{2} \oint_{s=m_\tau} \Pi(s)\,ds \tag{1.14}$$

Due to the circle-contour we can avoid low energies at which pQCD would break down.

To deal with the latter issue we have to suppress the contributions of the correlator close to the positive real axis, which can be achieved by introducing so called **pinched weights**.



$$\int_0^{m_\tau} \operatorname{Im}\Pi(s)\,ds = \oint_{|s|=m_\tau} \Pi(s)\,ds$$

Figure 1.1: Visualization of the usage of Cauchy's theorem to transform eq. (1.2) into a closed contour integral over a circle of radius m_τ^2.

1.1 Pinched weights

We furthermore evade regions of the two-point correlator on the positive real axis (with which we will deal with to a later point[1]) Finally combining eq. (1.14) with eq. (1.2) we get

$$R_\tau = 6\pi i \oint_{s=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(T)}(s)+\Pi^{(L)}\right] \tag{1.15}$$

for the hadronic τ-decay ratio.

The contour integral obtained is an import result as we can now theoretically evaluate the hadronic τ-decay ratio sufficiently large energy scales ($m_\tau \approx 1.78\,\mathrm{MeV}$) at which $\alpha_s(m_\tau) \approx 0.33$ **[Pich2016]** is tolerable heigh for applying perturbation theory and the OPE. Obviously we would benefit from a contour integral over a bigger circumference, but τ-decays are limited by the m_τ. Nevertheless there are promising e^+e^- annihilation data, which yields valuable R-ratio values up to $2\,\mathrm{GeV}$ **[Boito2018][Keshavarzi2018]**.

It is convenient to rewrite the

$$\Pi^{(L+T)} = \Pi^{(L)} + \Pi^{(T)} \tag{1.16}$$

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)-\left(\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s)\right] \tag{1.17}$$

$$D^{(L+T)}(s) \equiv -s\frac{d}{ds}\Pi^{(L+T)}(s), \qquad D^{(L)}(s) \equiv \frac{s}{m_\tau^2}\frac{d}{ds}(s\Pi^{(L)}(s)) \tag{1.18}$$

[1] To not evaluate $\Pi(s)$ at the positive real axis we have to introduce *pinched weights*. The *pinched weights* vanish for $s \to m_\tau$.

Integration by parts

$$\int_a^b u(x)V(x)\,dx = \big[U(x)V(x)\big]_a^b - \int_a^b U(x)v(x)\,dx \tag{1.19}$$

$$\begin{aligned}
R_\tau^{(1)} &= \frac{6\pi i}{m_\tau^2}\oint_{|s|=m_\tau^2} \underbrace{\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)}_{=u(x)}\underbrace{\Pi^{(L+T)}(s)}_{=V(x)} \\
&= \frac{6\pi i}{m_\tau^2}\left\{\left[-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\Pi^{(L+T)}(s)\right]_{|s|=m_\tau^2}\right. \\
&\quad \left. +\oint_{|s|=m_\tau^2} \underbrace{-\frac{m_\tau^2}{2}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)}_{=U(x)}\underbrace{\tfrac{d}{ds}\Pi^{(L+T)}(s)}_{=v(x)}\right\} \\
&= -3\pi i\oint_{|s|=m_\tau^2 s}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau^2}\right)\tfrac{d}{ds}D^{(L+T)}
\end{aligned} \tag{1.20}$$

where we fixed the integration constant to $C = -\frac{m_\tau^2}{2}$ in the second line and left the antiderivatives contained in the squared brackets untouched. Parametrizing the expression in the squared brackets

$$\left[-\frac{m_\tau^2}{2}\left(1-e^{-i\phi}\right)^3\left(1+e^{-i\phi}\right)\Pi^{(L+T)}(m_\tau^2 e^{-i\phi})\right]_0^{2\pi} = 0 \tag{1.21}$$

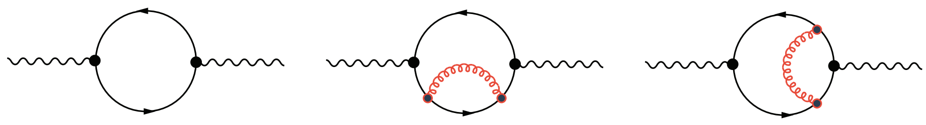
where $s \to m_\tau^2 e^{-i\phi}$ and $(1-e^{-i\cdot 0}) = (1-e^{-i\cdot 2\pi}) = 0$.

$$\begin{aligned}
R_\tau^{(2)} &= \oint_{|s|=m_\tau^2} ds\left(1-\frac{s}{m_\tau^2}\right)^2\left(-\frac{2s}{m_\tau^2}\right)\Pi^{(L)}(s) \\
&= -4\pi i\oint\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3 D^{(L)}(s)
\end{aligned} \tag{1.22}$$

$$R_\tau = -\pi i\oint_{|s|=m_\tau^2}\frac{ds}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left[3\left(1+\frac{s}{m_\tau^2}D^{(L+T)}(s)+4D^{(L)}(s)\right)\right] \tag{1.23}$$

$$R_\tau = -\pi i\oint_{|s|=m_\tau^2}\frac{dx}{x}(1-x)^3\left[3(1+x)D^{(L+T)}(m_\tau^2 x)+4D^{(L)}(m_\tau^2 x)\right], \tag{1.24}$$

where $x = s/m_\tau^2$.



$$R^{\omega}_{\tau,V/A} = \frac{N_c}{2} S_{EW} |V_{ud}|^2 \left(1 + \delta^{(0)}_{\omega} + \delta^{EW}_{\omega} + \delta^{DVs}_{\omega} + \sum_{D\leqslant 2} \delta^{(D)}_{ud,\omega}\right) \tag{1.25}$$

1.2 The perturbative expansion

We will treat the correlator in the chiral limit for which the longitudinal components $\Pi^L(s)$ vanish (see eq. (1.11)) and the axial and vectorial contributions are equal. Consequently **[Beneke2008]** we can write the vector correlation function $\Pi(s)$ as:

$$\Pi^{T+L}_V(s) = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a^n_{\mu} \sum_{k=0}^{n+1} c_{n,k} L^k \quad \text{with} \quad L \equiv \ln \frac{-s}{\mu^2}. \tag{1.26}$$

The coeffiecient $c_{n,k}$ up to two-loop order can be obtained by Feynman-diagram calculations. add complete calculation E.g. we can compare the zero-loop result of the correlator **[Jamin2006]**

$$\Pi^B_{\mu\nu}(q^2)\Big|^{1-\text{loop}} = \frac{N_c}{12\pi^2}\left(\frac{1}{\hat{\epsilon}} - \log\frac{(-q^2 - i0)}{\mu^2} + \frac{5}{3} + \mathcal{O}(\epsilon)\right) \tag{1.27}$$

with eq. (1.26) and extract the first two coefficients

$$c_{00} = -\frac{5}{3} \qquad \text{and} \qquad c_{01} = 1, \tag{1.28}$$

where $\Pi^B_{\mu\nu}(q^2)$ is not renormalized[2]

The second loop can also be calculated by diagram techniques resulting in **[Boito2011]**

$$\Pi^{(1+0)}_V(s)\Big|^{2-\text{loop}} = -\frac{N_c}{12\pi^2} a_{\mu} \log(\frac{-s}{\mu^2}) + \cdots \tag{1.29}$$

yielding $c_{11} = 1$.

Beginning from three loop diagrams the algebra becomes exausting and one has to use dedicated algorithms to compute the heigher loops. The third loop calculations have been done in the late seventies by **[Chetyrkin1979, Dine1979, Celmaster1979]**. The four loop evaluation have been completed

[2] The term $1/\hat{\epsilon}$, which is of order 0 in α_s, will be cancelled by renormalization.

a little more than ten years later by [**Gorishnii1990**, **Surguladze1990**]. The heighest loop published, that amounts to α_s^4, was published in 2008 [**Baikov2008**] almost 20 years later.

Fixing the number of colors to $N_c = 3$ the missing coefficients up to order four in α_s read:

$$\begin{aligned}
c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f \\
c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 \\
&\quad - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2 \\
c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7,
\end{aligned} \tag{1.30}$$

where used the flavour number $N_f = 3$ for the last line.

The 6-loop calculation has until today not been achieved, but Beneke und Jamin [**Beneke2008**] used and educated guess to estimate the coefficient

$$c_{5,1} \approx 283 \pm 283. \tag{1.31}$$

Until know we have mentioned the coefficients $c_{n,k}$ with a fixed $k = 1$. This is due to the RGE, which relates coefficients with a different k to the coefficients mentioned above. To make usage of the RGE $\Pi_V^{T+L}(s)$ needs to be a physical quantity, which can be achieved by rewriting eq. (1.18) to:

$$D_V^{(T+L)} = -s\frac{d\Pi_V^{(T+L)}(s)}{ds} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}, \tag{1.32}$$

where we used $dL^k/ds = k\ln(-s/\mu^2)^{k-1}(-1/\mu^2)$. D_V^{1+0} being a physical quantity has to fulfill the RGE **??**

$$-\mu\frac{d}{d\mu}D_V^{(T+L)} = -\mu\frac{d}{d\mu}\left(\tfrac{\partial}{\partial L}dL + \tfrac{\partial}{\partial a_s}da_s\right)D_V^{T+L} = \left(2\tfrac{\partial}{\partial L} + \beta\tfrac{\partial}{\partial a_s}\right)D_V^{T+L} = 0, \tag{1.33}$$

where we defined the β-function in **??** and used $dL/d\mu = -2/\mu$. The RGE puts constraints on the $c_{n,k}$-coefficients, ... not independent

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + a_\mu(c_{11} + 2c_{12}L) + a_\mu^2(c_{21} + 2c_{22}L + 3c_{23}L^2)\right] \tag{1.34}$$

inserting into RGE

$$4a_\mu c_{12} + 2a_\mu^2(2c_{22} + 6c_{23}L) + \beta_1 a_\mu^2(c_{11} + 2c_{12}L) + \mathcal{O}(a_\mu^3) = 0 \tag{1.35}$$

Thus

$$c_{12} = 0 \quad c_{22} = \frac{\beta_1 c_{11}}{4} \quad c_{23} = 0 \tag{1.36}$$

or $D(s)$ to the first order in α_s

$$D(s) = \frac{N_c}{12\pi^2}\left[c_{01} + c_{11}a_\mu\left(c_{21} - \frac{1}{2}\beta_1 c_{11}L\right)a_\mu^2\right] + \mathcal{O}(a_\mu^3) \tag{1.37}$$

1.2.1 Renormalisation group summation

We can express the perturbative contribution $\delta^{(0)}$ to R_τ (see eq. (1.25)) as

$$\delta^{(0)} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log\left(\frac{-M_\tau^2 x}{\mu^2}\right)^{k-1}, \quad (1.38)$$

where we inserted the expansion of $D_V^{(T+L)}$ eq. (1.18) into R_τ eq. (1.24). Keep in mind that we are working in the chiral limit, such that $D^L = 0$ vanishes and the contributions from the vector and axialvector correlator are identical

$$D^{(T+L)} = D_V^{(T+L)} + D_A^{(T+L)} = 2D_V^{(T+L)}. \quad (1.39)$$

The perturbative contribution $\delta^{(0)}$ is a physical quantity and satisfies the homogeneous RGE, thus is independent on the scale μ. Consequently we have the freedom to choose μ, which leads to two main descriptions **fixed-order perturbation theory** (FOPT) and **contour-improved perturbation theory** (CIPT). The two resulting series should converge to equal values, but differ notably.

By using the FOPT prescription we fix $\mu^2 = m_\tau^2$ leading to

$$\delta_{FO}^{(0)} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k c_{n,k} J_{k-1} \quad (1.40)$$

where the contour integrals J_l are defined by

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log^l(-x). \quad (1.41)$$

The integrals J_l up to order α_s^4 are given by [**Beneke2008**]:

$$J_0 = 1, \quad J_1 = -\frac{19}{12} \quad J_2 = \frac{265}{72} - \frac{1}{3}\pi^2, \quad J_3 = -\frac{3355}{288} + \frac{19}{12}\pi^2. \quad (1.42)$$

Using FOPT the strong coupling $a(\mu)$, which runs with the scale μ, is fixed at $a(m_\tau^2)$ and can be taken out of the closed-contour integral. We still have to integrate over the logarithms $\log(-s/m_\tau^2)$.

Using CIPT we can sum the logarithms by setting the scale to $\mu^2 = -m_\tau^2 x$ in eq. (1.38), resulting in:

$$\delta_{CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2), \quad (1.43)$$

where the contour integrals J_l are defined by

$$J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-m_\tau^2 x). \quad (1.44)$$

All logarithms vanish except the ones for $k = 1$:

$$\log(1)^{k-1} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & k \neq 1 \end{cases} \tag{1.45}$$

which selectes adler function coefficients $c_{n,1}$ with a fixed $k = 1$. Handling the logarithms left us with the integration of $\alpha_s(-m_\tau^2 x)$ over the closed-contour $\oint_{|x|=1}$, which now depends on the integration variable x.

Calculating the perturbative contribution $\delta^{(0)}$ to R_τ for the two different prescriptions yields [**Beneke2008**]

$$\begin{aligned} & \qquad\quad \alpha_s^2 \qquad\; \alpha_s^2 \qquad\; \alpha_s^3 \qquad\; \alpha_s^4 \qquad\qquad \alpha_s^5 \\ \delta_{FO}^{(0)} &= 0.1082 + 0.0609 + 0.0334 + 0.0174(+0.0088) = 0.2200(0.2288) \qquad (1.46) \\ \delta_{CI}^{(0)} &= 0.1479 + 0.0297 + 0.0122 + 0.0086(+0.0038) = 0.1984(0.2021). \qquad (1.47) \end{aligned}$$

The series indicate, that CIPT converges faster and that both series aproach a different value. This discrepancy represents currently the biggest theoretical uncertainity while extracting the strong coupling α_s.

As today we do not know which if FOPT or CIPT is the correct approach of measuring α_s. Therefore there are currently three ways of stating result:

- Quoting the average of both results.
- Quoting the CIPT result.
- Quoting the FOPT result.

We follow the approach of Beneke and Jamin [**Benke2008**] who prefere FOPT.

1.3 Non-Perturbative OPE Contribution

The perturbative contribution to the Sum-Rule, that we have seen so far, is the dominant one. With

$$\begin{aligned} R_\tau^{FOPT} &= \\ R_\tau^{CIPT} &= \end{aligned} \tag{1.48}$$

The NP vs perturbative contributions can be varied by choosen different weights than ω_τ.

1.3.1 Dimension four

For the OPE contributions of dimension four we have to take into account the terms with masses to the fourth power m^4, the quark condensate multiplied by a mass $m\langle \overline{q}\, q\rangle$ and the glucon condensate $\langle GG\rangle$. The resulting expression can be taken from the appendix of [**Pich1999**], yielding:

$$D_{ij}^{(L+T)}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_n \Omega^{(1+0)}(s/\mu^2)a^n, \tag{1.49}$$

where

$$\begin{aligned}\Omega_n^{(1+0)}(s/\mu^2) \;=\; & \frac{1}{6}\langle aGG\rangle p_n^{(L+T)}(s/\mu^2) + \sum_k m_k\langle \overline{q}_k q_k\rangle r_n^{(L+T)}(s/\mu^2) \\ & + 2\langle m_i\overline{q}_i q_i + m_j\overline{q}_j q_j\rangle q_n^{(L+T)}(s/\mu^2) \pm \frac{8}{3}\langle m_j\overline{q}_i q_i + m_i\overline{q}_j q_j\rangle t_n^{(L+T)} \\ & - \frac{3}{\pi^2}(m_i^4+m_j^4)h_n^{(L+T)}(s/\mu^2) \mp \frac{5}{\pi^2}m_i m_j(m_i^2+m_j^2)k_n^{(L+T)}(s/\mu^2) \\ & + \frac{3}{\pi^2}m_i^2 m_j^2 g_n^{(L+T)}(s/\mu^2) + \sum_k m_k^4 j_n^{(L+T)}(s/\mu^2) + 2\sum_{k\neq l} m_k^2 m_l^2 u_n^{(L+T)}(s/\mu^2)\end{aligned} \tag{1.50}$$

The perturbative expansion coefficients are known to $\mathcal{O}(a^2)$ for the condensate contributions,

$$\begin{array}{lll} p_0^{(L+T)} = 0, & p_1^{(L+T)} = 1, & p_2^{(L+T)} = \frac{7}{6}, \\ r_0^{(L+T)} = 0, & r_1^{(L+T)} = 0, & r_2^{(L+T)} = -\frac{5}{3}+\frac{8}{3}\zeta_3-\frac{2}{3}\log(s/\mu^2), \\ q_0^{(L+T)} = 1, & q_1^{(L+T)} = -1, & q_2^{(L+T)} = -\frac{131}{24}+\frac{9}{4}\log(s/\mu^2) \\ t_0^{(L+T)} = 0 & t_1^{(L+T)} = 1, & t_2^{(L+T)} = \frac{17}{2}+\frac{9}{2}\log(s/\mu^2). \end{array} \tag{1.51}$$

while the m^4 terms have been only computed to $\mathcal{O}(a)$

$$\begin{array}{ll} h_0^{(L+T)} = 1-1/2\log(s/\mu^2), & h_1^{(L+T)} = \frac{25}{4}-2\zeta_3-\frac{25}{6}\log(s/\mu^2)-2\log(s/\mu^2)^2, \\ k_0^{(L+T)} = 0, & k_1^{(L+T)} = 1-\frac{2}{5}\log(s/\mu^2), \\ g_0^{(L+T)} = 1, & g_1^{(L+T)} = \frac{94}{9}-\frac{4}{3}\zeta_3-4\log(s/\mu^2), \\ j_0^{(L+T)} = 0, & j_1^{(L+T)} = 0, \\ u_0^{(L+T)} = 0, & u_2^{(L+T)} = 0. \end{array} \tag{1.52}$$

1.3.2 Dimension six and eight

Our application of dimension six contributions is founded in [**Braaten1991**] and has previously been calculated beyond leading order by [**Lanin1986**]. The operators appearing are the masses to the power six m^6, the four-quark condensates $\langle \overline{q}\, q\overline{q}\, q\rangle$, the three-gluon condensates $\langle g^3G^3\rangle$ and lower dimensional condensates multiplies by the corresponding masses, such that in total the mass dimension of the operator will be six. As there are too many parameters to be fitted with experimental data we have to omit some of them, starting with the three-gluon condensate, which does not contribute at leading order. The four-quark condensates known up to $\mathcal{O}(a^2)$, but we will make use of the *vacuum saturation approach* [**Beneke2008**, **Braaten1991**, **Shifman1978**] to express them in quark, anti-quark condensates $\langle q\overline{q}\rangle$. In our work we take the simplest approach possible: Introducing an effective dimension six coefficient $\rho^{(6)}_{V/A}$ divided by the appropiate power in s

$$D^{(1+0)}_{ij,V/A}\Big|_{D=6} = 0.03\frac{\rho^{(6)}_{V/A}}{s^3} \tag{1.53}$$

As for the dimension eigth contribution the situation is not better than the dimension six one we keep the simplest approach, leading to

$$D^{(1+0)}_{ij,V/A}\Big|_{D=8} = 0.04\frac{\rho^{(8)}_{V/A}}{s^4}. \tag{1.54}$$