Chapter 1

Measuring the Strong Coupling

Table 1.1: Timeline

1991	[**Braaten1991**]: Systematic description, including NP corrections to extract α_s from R_τ.
1992	[**LeDiberder1992**]: Introducing weights and fit methodology.
1993	[**Aleph1993**] ALEPH measures the strong coupling constant α_s.
1998	[**Opal1998**] OPAL measures the strong coupling constant α_s.
2005	[**Aleph2005**] ALEPH improves their data.
2011	[**Boito2011a**, **Boito2010**]: Include DV. Discover inconsistencies in ALEPH data.
2014	[**Davier2013**] ALEPH updates their data.

The strong coupling has been measured since many years from hadronic τ decays. The dashed line represents the mean value of 0.3212. An overview of the recently, but different, α_s values can be seen in fig. 1.1. Until today most of the applied QCDSR to τ decays are based on the methodology developed in the early nineties by Braaten, Pich and Narison [**Braaten1991**]. They gathered the at this time available perturbative and NP contributions to extract the strong coupling from comparing their theoretical results to the known inclusive hadronic τ decay ratio R_τ. Pich together with Le Diberder then formulated the fitting strategy of applying multiple moments of different weights to extract α_s parallel to Wilson coefficients of the OPE [**LeDiberder1992**], which later has been applied as standard in the ALEPH [**Aleph1993**] as well

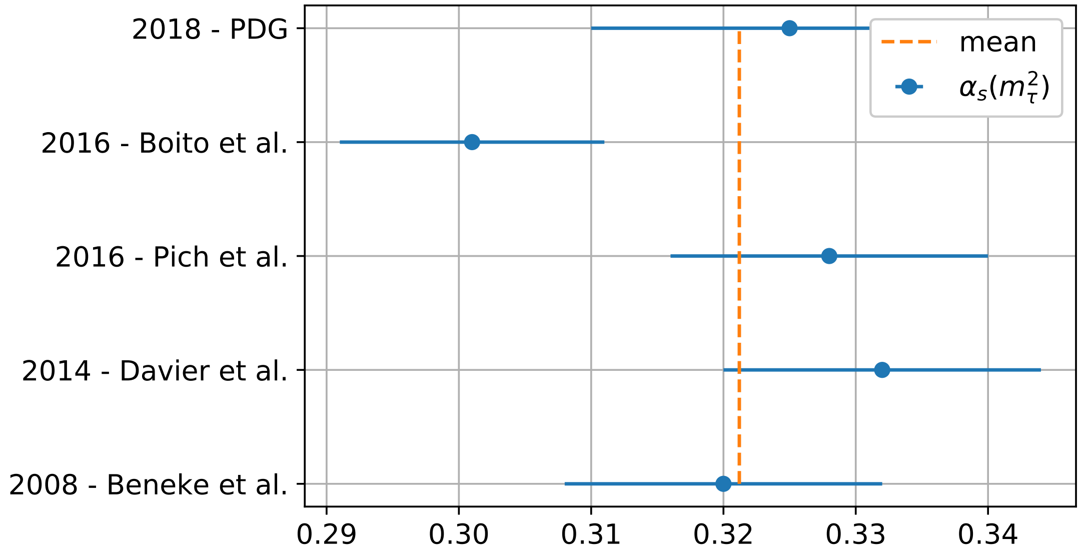

Figure 1.1: Recent values for $\alpha_s(m_\tau^2)$ from hadronic τ decays. The values are taken from [**PDG2018**, **Boito2016**, **Pich2016**, **Davier2013**, **Beneke2008**], from top to bottom.

as the OPAL [**Opal1998**] detectors. For the next ten years the methodology of extracting the strong coupling did not experience any major changes until 2011 when Boito, Cata, Golterman, Jamin, Osborn and Peris [**Boito2011a**] applied a duality model to include known DV effects to the QCD analysis of τ decays. The groups around Boito and Pich have different opinions on the importance of the newly introduced duality model [**Pich2016**, **Boito2016**] and consequently we want to deliver a third, opinion on the subject, favouring fits without the duality model.

1.1 Fit Strategy

The objective of this work is to extract α_s and argue about the importance of DV. Apart from the two main objectives we want to analyse the contribution of higher order OPE contributions up to dimension ten.

For performing the fits we had to restrict ourselves to the following approximations

- We used the PT expansion of the Adler function up to fifth order, including the educated guess of $c_{5,1}$.

- We ignore the dimension two OPE contribution as they are proportional to the quark masses and we work in the chiral limit.
- We neglect the logarithmic contributions to the dimension four OPE contributions
- For the higher dimensions of the OPE $D = 6, 8, \ldots$ we parametrise their contributions as a constant divided by the corresponding power in s

$$D^{(1+0)}_{V/A}\Big|_{D=d} \equiv \frac{\rho^{(d)}}{s^{d/2}}. \tag{1.1.1}$$

Our fitting strategy will be in choosing weights of lower and higher pinching. Lower pinched weights should be affected by DV, while higher pinched weights should be protected from DV. As a result in comparing different fits of lower and higher pinched weights it should be possible to argue about the strength of the DV that are (or are not) present.

Our hypothesis is that DV are small enough for fits of the combined vector and axial-vector channel in combination with pinched weights. Consequently we can extract parameters, like the strong coupling α_s from τ decays to high precision without a DV model.

We will perform our analysis of the DV in framework of FOPT for weights without a monomial term x. For weights including a monomial term x we will apply the BS. To define a fit we have to choose a weight ω and a momentum s_0. The only restriction from choosing a weight is, that it has to be analytic, leaving us with a variety of choices. For our strategy we have chosen three categories of weights, each of them containing fits with three or four different weights. A table with an overview of all used weights is given in table 1.2. To test the stability of the fitted values and have enough DOF to fit the higher OPE contributions we furthermore fit every weight for various momenta s_0.

To probe the validity of FOPT we apply the BS for weights without a monomial term x. If FOPT is valid the parameters obtained from the Borel model should be similar. As CIPT in general yields different results for the strong coupling and higher order contributions as FOPT, it should be the worse framework if BS and FOPT lead to similar parameter values.

	Symbol	Term	Expansion	OPE Contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	D6, D8
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	D6, D8, D10
	ω_{quartic}	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	D6, D8, D10, D12
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	D6
	ω_{M3}	$1-x^3$	$1-x^3$	D8
	ω_{M4}	$1-x^4$	$1-x^4$	D10
Pinched $+x$	$\omega_{1,0}$	$(1-x)$	$1-x$	D4
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	D4, D6
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	D4, D6, D8
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	D4, D6, D8, D10

Table 1.2: Displaying three categories of fits, each containing three weights with their corresponding mathematical expression and the OPE contributions the fitted integral momentum will be sensitive to.

1.2 Fits

In the following we will show the results of each of the three previously mentioned fit categories.

The first category contains the *pinched weights without a monomial term x*. The chosen weights are double (ω_τ), triple (ω_{cube}) and quadruple (ω_{quartic}) pinched and do not contain a monomial term x. An x term would make the fits sensitive to the $D = 4$ OPE contribution, which causes an unreliable perturbative expansion [**Beneke2012**]. The higher the pinching, the higher the suppression of DV. Consequently if we obtain stable values for α_s from the different pinched fits we should expect the DV to have no influence on the value of the strong coupling. The different weights imply an increasing number of active OPE contributions D_6, D_8, D_{10} and D_{12}, which can be used to probe the stability of higher order OPE contributions and to test for the convergence of the OPE.

The second category contains the *single pinched monomial weights*. In this case all of the weights are only single pinched and, as in the first category, do not carry a monomial term x. Consequently if DV affect the fits we should notice different fitting results in comparison to the fits of the first category. Furthermore the single pinched moments only carry two parameters, the strong coupling and an OPE Wilson coefficients. Thus we can further compare the $\rho^{(6)}$, $\rho^{(8)}$ and $\rho^{(10)}$ Wilson coefficients and argue about the stability of the fits.

The third and last category contains a similar pinching as the first category, but this time contains a monomial term in x. Consequently these fits are unreliable in the framework of FOPT and we have to apply the BS. Following the logic of the second and first category we then can compare the result to analyse the role of DV and compare the OPE contributions.

1.2.1 Pinched Weights without a Monomial term x

Kinematic Weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

We previously encountered the kinematic weight in **??**. It is a polynomial weight function, defined as $\omega_\tau(x) = (1-x)^2(1+2x)$, double pinched, contains the unity and does not contain a term proportional to x. Consequently it is an optimal weight [**Beneke2012**]. As a doubled pinched weight it should have a good suppression of DV contributions and its polynomial contains terms proportional to x^2 and x^3, which makes it sensitive to the dimension six and eight OPE contributions. The fits have been performed within the framework of FOPT for different numbers of s_0. The momentum sets are characterised by its lowest energy s_{min}. We fitted values down to $1.5\,\mathrm{GeV}$. Going to lower energies is questionable due to the coupling constant becoming large, which implies a breakdown of PT. Furthermore it bares the risk to be affected by the $\rho(770)$ and a_1 peaks in the vector and axial-vector spectral function, which we cannot model within the framework of the OPE. For the three fitting parameters α_s, $\rho^{(6)}$ and $\rho^{(8)}$ we have given the results in table 1.3 and graphically in fig. 1.2.

We only display the fits for s_{min} larger than $2.1\,\mathrm{GeV}$. We noted a jump between the $s_{min} = 2.1\,\mathrm{GeV}$ and $s_{min} = 2.2\,\mathrm{GeV}$ of the χ^2/dof from 0.19 to 1.3. We consequently discarded fits with a $s_{min} < 2.2\,\mathrm{GeV}$, as fits lower s_{min} be-

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.200	7	0.3274(42)	-0.82(21)	-1.08(40)	0.21
FOPT	2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

Table 1.3: Table of our fitting values of $\alpha_s(m_\tau^2), \rho^{(6)}$ and $\rho^{(8)}$ for the kinematic weight $\omega(x) = (1-x)^2(1+2x)$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

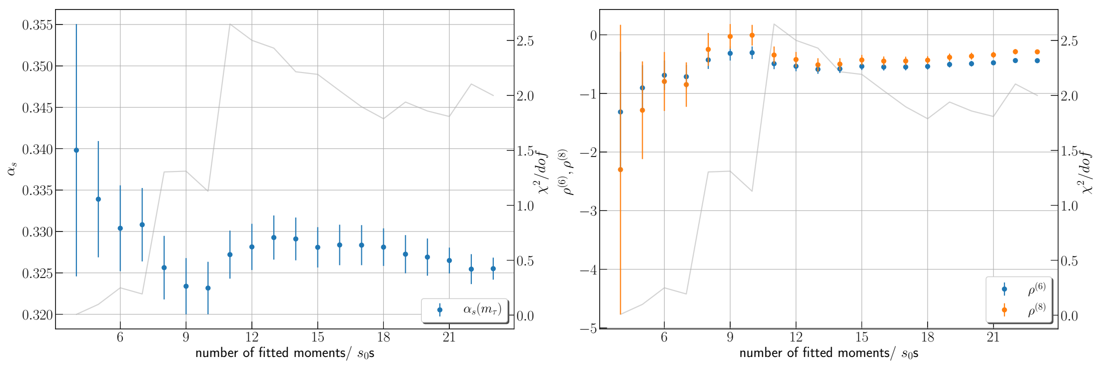

Figure 1.2: Fitting values of $\alpha_s(m_\tau^2), \rho^{(6)}$ and $\rho^{(8)}$ for the kinematic weight $\omega(x) = (1-x)^2(1+2x)$ using FOPT for different s_{min}. The left graph plots $\alpha_s(m_\tau^2)$ for different numbers of used s_0s. The right plot contains the dimension six and eight contributions to the OPE. Both plots have in grey the χ^2 per DOF.

have more stable[1]. The values with fewer momenta are preferred by us due to two reasons. First, below energies of 2.2 GeV we have to face the problematic influence of increasing resonances. Second, we will see, that the values obtained from the lower moment fits are more compatible with our other fits series. We further discarded the fit with four s_0s momenta, which has very small $\chi^2/\mathtt{dof} = 0.01$. This is due to the fact, that we have four s_0s momenta to fit three parameters, which leaves us with too few DOF.

The selected fits with 8-10 momenta have a small χ^2 per DOF. The fitted parameters, $\alpha_s, \rho^{(6)}$ and $\rho^{(8)}$ are in good agreement with each other. For all fits we have a good convergence of the OPE. For later comparisons we will focus on the fit right below the $\chi^2/\mathtt{dof}$ threshold. For the kinematic weight we get for the strong coupling, $D = 6$ and $D = 8$ contributions:

$$\alpha_s(m_\tau^2) = 0.3308(44), \quad \rho^{(6)} = -0.72(20) \quad \text{and} \quad \rho^{(8)} = -0.85(38). \qquad (1.2.1)$$

We further tested the stability of the dimension six and eight contributions to the OPE within the same fit series but for a fixed value of the strong coupling $\alpha_s(m_\tau^2) = 0.3179$. The values for $\rho^{(6)}$ and $\rho^{(8)}$ are larger than the values given in our final results from table 1.3. This is explained with a smaller contribution from the strong coupling, which has to be compensated by larger OPE contributions.

Additionally we applied the BS for the fit below the χ^2 threshold containing seven s_0s. Even though we used a different framework than FOPT the results are compatible. This further underlines the good results of the kinematic weight fit and can be seen as an indicator for FOPT being the superior framework as compared to CIPT, which already has been argued in [**Beneke2008**].

Cubic Weight: $\omega_{\mathrm{cube}}(x) \equiv (1-x)^3(1+3x)$

To further consolidate the results from the kinematic weight, we tested a weight of higher pinching, which should suppress DV more than a double pinched weight. Consequently, if we obtain similar results to our previous fits we could exclude DV effects for the kinematic weight. On the other hand,

[1] As will be seen by comparing the kinematic weight with the cubic and quartic weight

s_{min}	$\#s_0 s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

Table 1.4: Table of our fitting values of $\alpha_s(m_\tau^2), \rho^{(6)}, \rho^{(8)}$ and $\rho^{(10)}$ for the cubic weight $\omega(x) = (1-x)^3(1+3x)$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

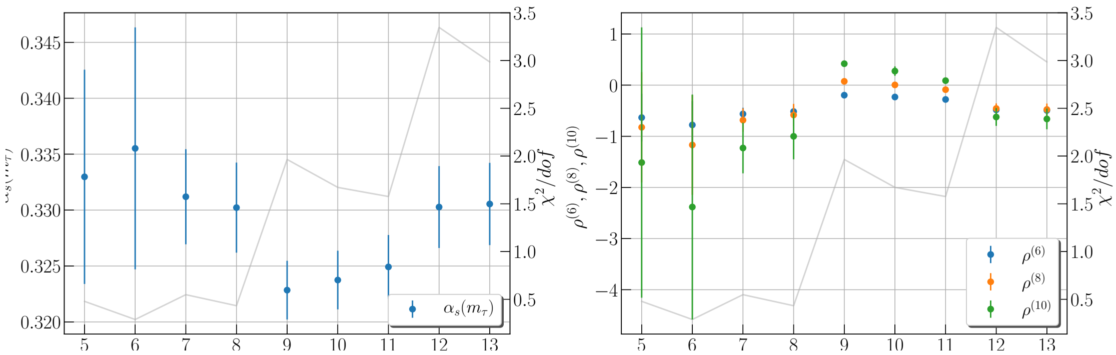

Figure 1.3: Graphic representation of the fitting values of $\alpha_s(m_\tau^2)$ in the left and $\rho^{(6)}, \rho^{(8)}$ and $\rho^{(10)}$ in the right plot for the cubic weight $\omega(x) = (1-x)^3(1+3x)$. The fits have been performed in the FOPT scheme and the data points are given with error bars and are ordered by increasing s_{min}. The grey line in displays the χ^2 function.

any differences to the previous fit would indicate present DV in the kinematic weight. Our *cubic* weight will be triple pinched and optimal, as it does not contain a monomial term x. It is due to its polynomial structure sensitive to the dimensions six, eight and ten contributions of the OPE, which yields one more parameter to fit than with the kinematic weight ω_τ. The fitting results can be seen in table 1.4 and graphically in fig. 1.3.

We performed fits for $s_0 \leqslant 1.5\,\text{GeV}$, but could only reach convergence for fits with energies larger or equal than $1.8\,\text{GeV}$. As before the χ^2 makes a jump at $s_0 = 2.1\,\text{GeV}$ to values per DOF of almost 2. Consequently we excluded theses fits and focused on fits from five to eight s_0s momenta.

The selected fits have a good χ^2/dof and the fitted parameters, $\alpha_s, \rho^{(6)}, \rho^{(8)}$ and $\rho^{(10)}$ are in agreement with each other, except for the fit with six momenta. The fit with a $s_{min} = 2.3\,\mathrm{GeV}$ has the lowest $\chi^2 = 0.29$ and error on α_s, but takes slightly different values for the OPE Wilson coefficients in comparison to the other selected fits. The fit right below the χ^2/dof threshold for the strong coupling, $D = 6$ and $D = 8$ contributions yields

$$\alpha_s(m_\tau^2) = 0.3302(40), \quad \rho^{(6)} = -0.52(11), \quad \rho^{(8)} = -0.58(22) \quad \text{and} \quad \rho^{(10)} = -1.00(45). \tag{1.2.2}$$

We furthermore found that the OPE is converging, but not as fast as for the kinematic weight. The values of $\left|\delta^{(8)}\right|$ is only half as large as $\left|\delta^{(6)}\right|$. The values of the lower momentum count are in high agreement with the ones obtained from the kinematic weight. The conclusions we take from the *cubic weight* are, that the kinematic weight, with its double pinching, should sufficiently suppress any contributions from DVs. If DV would have an effect on the kinematic weight, we should have seen an improvement of the fits with the *cubic* weight, due to its triple pinching, which is not the case.

Quartic Weight: $\omega(x) \equiv (1-x)^4(1+4x)$

The last fits of the pinched weights without a monomial term in x uses the *quartic weight* defined as $\omega(x) \equiv (1-x)^4(1+4x)$. It contains five fitting parameters $(\alpha_s, \rho^{(6)}, \rho^{(8)}, \rho^{(10)}, \rho^{(12)})$ and did only converge for $s_{min} = 2\,\mathrm{GeV}$ (nine s_0s momenta). The results for , with a χ^2 per DOF of 0.67 are given by:

$$\alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28), \quad \rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77). \tag{1.2.3}$$

Due to the problematic of the fitting routing, which is caused by too many OPE contributions fitted simultaneously, we will discard the fitting results for the quartic weight.

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	χ^2/dof
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

Table 1.5: Table of our fitting values of $\alpha_s(m_\tau^2)$, and $\rho^{(6)}$ for the single pinched double power monomial weight $\omega_{M2}(x) = 1 - x^2$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

1.2.2 Single Pinched Monomial Weights

To further argue in favour of our hypothesis we want to probe some weights with a single pinching. If DV play a role then we should note deviating results to fits with higher pinchings. The advantage of these weights is that they only let one OPE dimension contribute, thus leaving us with only two parameters per fit.

Second Power Monomial: $\omega_{M2}(x) \equiv 1 - x^2$

The first weight is defined as $\omega_{M2}(x) \equiv 1 - x^2$. We only have to fit two parameters, the strong coupling α_s and the dimension six OPE contribution. The results can be seen in table 1.5. For this fit we only obtained two fits without converging problems for a $s_{min} \leqslant 2.2\,\mathrm{GeV}$. Like in the ω_τ and ω_{cubic} we obtain stable fits for $s_{min} \leqslant 2.2\,\mathrm{GeV}$, but the χ^2/dof jumps to values $\chi^2/\mathrm{dof} > 1.6$ for smaller s_{min}. The values obtained for fitting six and seven s_0s moments are in good agreement with each other and furthermore carry a acceptable χ^2 per DOF. The best fit, chosen to be closest to the χ^2/dof threshold, then yields the following parameter values

$$\alpha_s(m_\tau^2) = 0.3248(52) \qquad \text{and} \qquad \rho^{(6)} = -0.77(22). \tag{1.2.4}$$

We note that the strong coupling obtained from the single pinched weight is similar to the one of the previous fits (≈ 3.33) which indicates, that even single pinched weights have sufficiently suppressed DV.

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	χ^2/dof
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Table 1.6: Table of our fitting values of $\alpha_s(m_\tau^2)$, and $\rho^{(8)}$ for the single pinched third power monomial weight $\omega_{M3}(x) = 1 - x^3$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

Third Power Monomial: $\omega_{M3}(x) \equiv 1 - x^3$

The second weight is defined as $\omega_{M3}(x) \equiv 1 - x^3$ and contains a single third power monomial. Consequently it is sensitive to dimension eight contribution of the OPE. Our fitting results can be taken from table 1.6. We note a good χ^2/dof except for the last row. The last row, at $s_{min} = 2.8\,\mathrm{GeV}$ has only one DOF and consequently high errors. The fit closest to the χ^2/dof threshold then yields the following parameter values

$$\alpha(m_\tau^2) = 0.3214(49) \qquad \text{and} \qquad \rho^{(8)} = -1.01(39). \tag{1.2.5}$$

Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

We already analysed the cubic and quartic weights, which depend on the dimension ten OPE contribution, in section 1.2.1 and section 1.2.1 correspondingly. Now, even with the visible DV for fourth power monomial $\omega_{M4} \equiv 1 - x^4$ to study another single pinched moment and the dimension ten OPE contribution. The results of the are given in table 1.7. The fitting behaviour is very similar to the third power monomial (table 1.6) and we will directly cite the fits closest to the χ^2/dof threshold:

$$\alpha_s(m_\tau^2) = 0.3203(48) \qquad \text{and} \qquad \rho^{(10)} = -1.64(77). \tag{1.2.6}$$

s_{min}	$\#s_0 s$	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	χ^2/dof
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

Table 1.7: Table of our fitting values of $\alpha_s(m_\tau^2)$ and $\rho^{(10)}$ for the single pinched fourth power monomial weight $\omega_{M4}(x) = 1 - x^4$ using FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

The values for the strong coupling are a little bit lower than the ones obtained by the kinematic and cubic weight fits. Furthermore the error on the tenth dimension contribution of the OPE are large. All in all the usage of the single pinched fourth power monomial weight is questionable and does not deliver any additional insights.

1.2.3 Pinched Weights with monomial x

Next to the previously mentioned *optimal weights* from Beneke et al. **[Beneke2012]**, which are weights without a monomial term in x, there exist another type of *optimal' weights*[2] introduced by Pich **[LeDiberder1992]**

$$\omega_{(n,m)}(x) = (1-x)^n \sum_{k=0}^{m} (k+1)x^k. \tag{1.2.7}$$

Combinations of these optimal moments have been widely used by the ALEPH collaboration to perform QCD analysis on the LEP data. To keep our study as simple as possible we will only use weights without the sum and set $m = 0$. The resulting weights $\omega_{n,0}$ are n-pinched, but do not contain higher dimensional OPE contributions. The moments fitted in this section include the for

[2]Pich has a different definition of "optimal" moments than Beneke, Boito and Jamin. To differentiate the two definition we marked Pich's optimal' moments with an apostrophe.

FOPT problematic proportional term in x, thus we will perform additional fits using the BS.

$$\omega_{1,0} \equiv (1-x)$$

The first weight is single pinched with only two fitting parameters: α_s and $\langle aGG\rangle_I$. The results for BS and FOPT fits have been displayed in table 1.8. We note that the α_s values of the two frameworks differ, which is due to the problematic of the monomial term x, of the weight function. The BS produces similar values for the parameters as the previous fits. The values obtained from the FOPT framework differ from the previous fits. In general we trust the results of the BS more than those of the FOPT for weights containing a monomial term x. This is further underlined while regarding the higher χ^2/dof values of the FOPT fits. Moreover the values of the BS fits agree, within the different used s_0s moments for this particular weight, whereas the fits of the FOPT yield inconsistent values. Regarding explicitly the fits from the BS we note that the fits have good χ^2/dof values, although they jump from 0.2 to 0.95 between the first two fitted moments. Also note that we had to fit the invariant gluon condensate for the first time. In the literature $\langle aGG\rangle_I$ should be around 2.1, but here we obtain a smaller, negative value, which could be connected to problems in the fit.

$$\omega_{2,0} \equiv (1-x)^2$$

The next weight is double pinched. Additional to the strong coupling and the invariant gluon-condensate we also had to fit the dimension six OPE contribution. Our fits can be seen in table 1.9. If we compare the BS with the FOPT fits we note, next to the before mentioned incompatibilities, a sign difference for the $D = 6$ contribution. From now we will skip the FOPT discussion for weights containing a monomial term x term, and trust in the BS fits. In comparison to the previous fit with the single pinched weight we have higher χ^2/dof values, a lower α_s value and an $\langle aGG\rangle_I$ numeric value similar to the value from the literature around 0.21, but with opposite sign. We observe a high agreement between the double pinched and single pinched weight containing a monomial

	s_{min}	$\#s_0 s$	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	χ^2/dof
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	0.38
	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24

Table 1.8: Table of our fitting values of $\alpha_s(m_\tau^2)$ and $\langle aGG\rangle_I$ for the single pinched optimal weight $\omega_{1,0}(x) = (1-x)$ using the FOPT and BS ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

term x by applying the BS, which further stresses the even for single pinched weights DV are sufficiently suppressed.

$$\omega_{3,0} \equiv (1-x)^3 \textbf{ and } \omega_{4,0} \equiv (1-x)^4$$

The fits with a triple and quadruple pinched weight do not give any further insights. We give the results in table 1.10 and table 1.11. Both of the weights include similar values to the double pinched weights, which affirms validity of the fits and the sufficiently suppressed DV. The quadruple pinched weight contains five fitting parameters and as a result has notable convergence problems.

1.3 Comparison

To create an overview of our previous results we have gathered the most compatible rows by hand. The fits have been selected by regarding the χ^2/dof threshold. For every weight, which has not been excluded by being problematic, we have chosen a fit closest, but below the χ^2/dof threshold. They are shown in table 1.12, which is composed of two parts. The upper four rows are fits using FOPT and the lower two rows are fits using BS. For the weights

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	χ^2/dof
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

Table 1.9: Table of our fitting values of $\alpha_s(m_\tau^2), \langle aGG\rangle_I$ and $\rho^{(6)}$ for the double pinched optimal weight $\omega_{2,0}(x) = (1-x)^2$ using the BS or FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

Table 1.10: Table of our fitting values of $\alpha_s(m_\tau^2), \langle aGG\rangle_I, \rho^{(6)}$ and $\rho^{(8)}$ for the optimal weight $\omega_{3,0}(x) = (1-x)^3$ using the BS or FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	aGGInv	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

Table 1.11: Table of our fitting values of $\alpha_s(m_\tau^2), \langle aGG\rangle_I, \rho^{(6)}, \rho^{(8)}$ and $\rho^{(10)}$ for the optimal weight $\omega_{4,0}(x) = (1-x)^4$ using the BS or FOPT ordered by increasing s_{min}. The errors are given in parenthesis after the observed value.

	weight	s_{min}	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
FOPT	ω_τ	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
	ω_{cube}	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
	ω_{M2}	2.2	0.3248(52)	-	-0.77(22)	-	-	0.38
	ω_{M3}	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
BS	$\omega_{1,0}$	2.2	0.3246(52)	-0.2262(59)	-	-	-	0.38
	$\omega_{2,0}$	2.2	0.3270(54)	-0.0254(61)	-0.77(21)	-	-	0.74
	$\omega_{3,0}$	2.1	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	-	0.46

Table 1.12: Table of the best fits. The fits have been selected as being closest to the previously discussed χ^2/dof jump. Each weight includes the strong coupling $\alpha_s(m_\tau^2)$ as a fitting variable. The first four fits have been performed using FOPT and the last two have been performed using BS. They are visually distinguished in the table by a horizontal line.

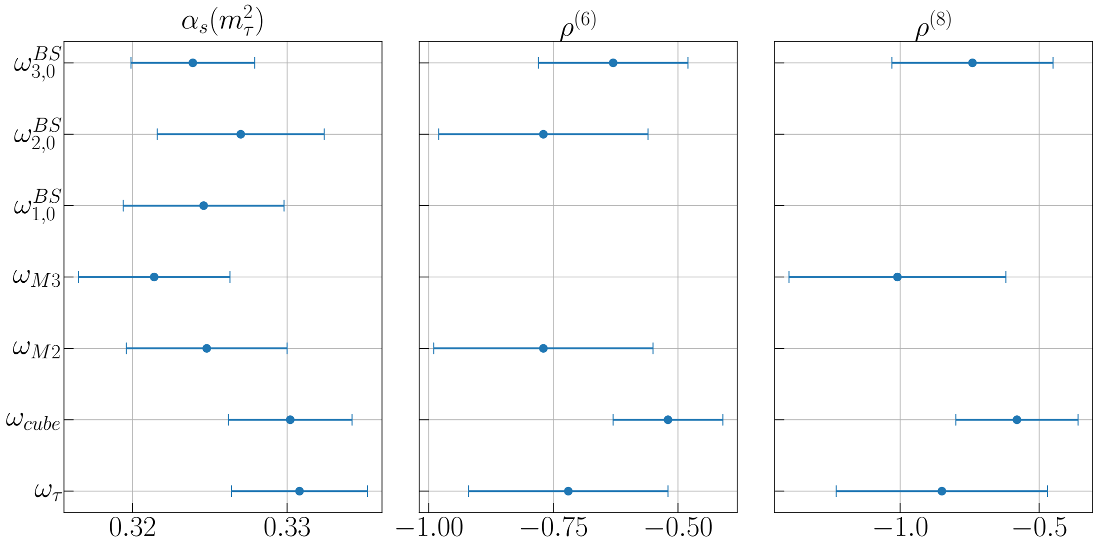

Figure 1.4: Visual comparison of the fitted parameters $\alpha_s(m_\tau^2)$, $\rho^{(6)}$ and $\rho^{(8)}$ for the results of table 1.12. The values of the different fits and models values are very comparable to each other. For the weights including a monomial term in x display the results obtained from the BS as can be seen from the upper index of $\omega_{n,0}^{BS}$.

with a monomial term in x we only included fits, which make use of the BS. Fits applying FOPT result in deviating parameter values in comparison with fit results from weights without a monomial term in x. This behaviour has already been illustrated in [**Beneke2012**] and is further supported by this work. Consequently for fits including weights without a monomial term in x we can apply FOPT, but for fits including weights with a monomial term in x the BS is needed.

The fits of the comparison table 1.12 are in great agreement with each other. The strong coupling as the OPE contributions up to dimension eight are compatible within small error ranges. To underline their agreement we have visualised the different values and errors in fig. 1.4. The fits furthermore all have a good χ^2/dof.

The high agreement between all fits shows that no DV have to be taken into account, while implementing fits in the $V + A$ channel and the corresponding framework, FOPT or BS depending if the weight does not or does include a monomial term in x, for at least single pinched weights. At least single and not double pinched, because the weights ω_{M2} and ω_{M3} are only single pinched, but still in high agreement with the other higher pinched weights! We consequently do not see any need to include DV terms in an analysis of α_s from τ decay data.

1.4 Final Results

Here we will state our final results for the strong coupling and the dimensions six and eight OPE contributions. We have decided to average over all values of the selected fits seen in table 1.12, but focus on the error given by the FOPT fit of the kinematic weight. We did not want to use the value for the strong coupling of the kinematic weight solely, because it is representing a rather large value in comparison to the other selected fits. We further did not want to average over the errors of all compared fits, as we do not know their correlations. Averaging over all errors would most probably lead to an underestimated error.

1.4.1 Theoretical error

The values have an statistical error given by the fitting routine MINUIT and an theoretical error. The main contribution from the theoretical error comes from the fifth Adler function coefficient $c_{5,1}$, which has not been calculated yet. Though estimated in **[Beneke2008]** with a value of $c_{5,1} = 283$ we will assume a relative error of 100% to its value. Consequently we performed two additional fits with $c_{5,1} = 0$ and $c_{5,1} = 566$ and via comparison extracted the error to the central value of the corresponding parameter. We have further decided instead of stating the resulting asymmetric errors the larger value of the upper and lower error as symmetric error. The results of the additional fits with varying the fifth Adler function coefficient can be seen in table 1.13. The symmetrical theoretical errors are then given by $0.0025, 0.03$ and 0.03 for $\alpha_s, \rho^{(6)}$ and $\rho^{(8)}$ correspondingly.

	$c_{5,1} = 0$	$c_{5,1} = 283$	$c_{5,1} = 566$	Δ
$\alpha_s(m_\tau^2)$	0.3333	0.3308	0.3285	0.0025
$\rho^{(6)}$	-0.74	-0.72	-0.69	0.03
$\rho^{(8)}$	-0.87	-0.85	0.82	0.03

Table 1.13: Values of $\alpha_s, \rho^{(6)}$ and $\rho^{(8)}$ for varying $c_{5,1}$ and the corresponding, symmetric error Δ for each parameter to the central value of $c_{5,1} = 283$.

1.4.2 Parameter Values

We will average over the values of table 1.12 leading to

$$\alpha_s(m_\tau^2) = 0.3261 \pm (0.0044)_{\text{MINUIT}} \pm (0.0025)_{c_{5,1}} = 0.3261 \pm 0.0051 \qquad (1.4.1)$$

for the strong coupling at the τ^2 scale. The dimension six and eight OPE contributions are very stable in the fits we compared. Consequently we will state their averaged numerical values

$$\rho^{(6)} = -0.68 \pm (0.2)_{\text{MINUIT}} \pm (0.03)_{c_{5,1}} = -0.68 \pm 0.20 \qquad (1.4.2)$$

$$\rho^{(8)} = -0.80 \pm (0.38)_{\text{MINUIT}} \pm (0.03)_{c_{5,1}} = -0.80 \pm 0.38. \qquad (1.4.3)$$

Note that the $\rho^{(6)}$ values from the cubic weight are slightly different. This is due to the fact, that the cubic weight includes a fourth fitting parameter, which contribution needs to be compensated by the other parameters.

The value of higher dimension OPE parameters are still compatible, but have a higher variation than the previous two parameters. Beginning from the $D = 10$ contribution we do not have enough good fits to evaluate their contribution. Consequently we do not state a single value for OPE parameters of dimension eight and higher.