

# The QCD Strong Coupling from Hadronic $\tau$ decays

## A PhD Defense

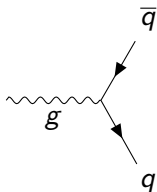
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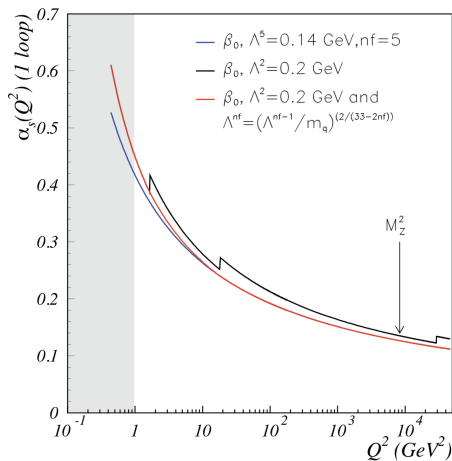


# The Strong Coupling $\alpha_s$



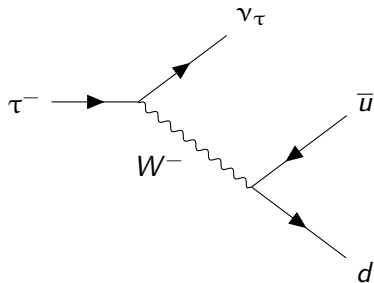
$$L_{QCD} = \dots + \sqrt{\pi\alpha_s} \bar{q}(x) \lambda \gamma_\mu q(x) G(x) + \dots (1)$$

$$\begin{aligned} \alpha_s(m_\tau^2) &\approx 0.33 \\ \alpha_s(m_Z^2) &\approx 1.12 \end{aligned} \quad (2)$$

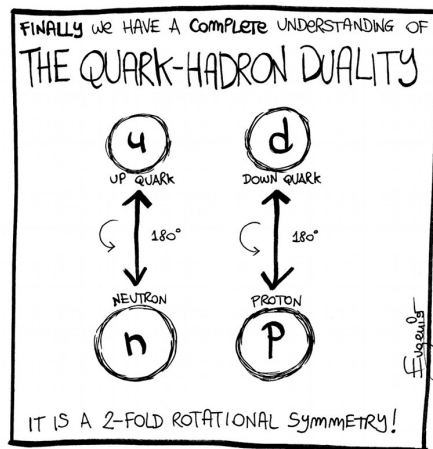


**Figure:** Taken from Deur, Brodsky, and Teramond, "The QCD Running Coupling"

# Exclusive Hadronic $\tau$ decays



$$R_\tau = \frac{\Gamma[\tau^- \rightarrow \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e]} \quad (3)$$



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## Inclusive Ratio

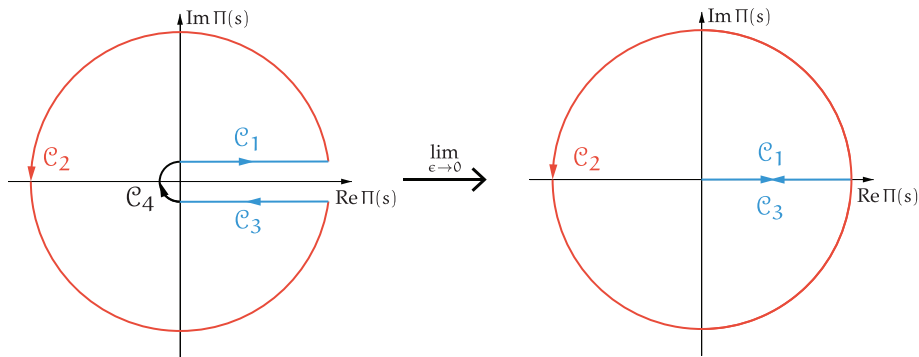
$$R_\tau = 12\pi |V_{ud}|^2 S_{EW} \int_0^{m_\tau} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \left[ \left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im} \Pi_{V/A}^{(1)}(s) + \text{Im} \Pi_{V/A}^{(0)}(s) \right] \quad (4)$$

## Two-Point Function

$$\begin{aligned} \Pi_{V/A}^{\mu\nu}(q^2) &\equiv i \int d^4x e^{iqx} \langle 0 | T \left\{ J_{V/A}^\mu(x) J_{V/A}^\nu(0) \right\} | 0 \rangle \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{V/A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{V/A}^{(0)}(q^2) \end{aligned} \quad (5)$$

where

$$J_V^\mu = \bar{u} \gamma^\mu d \quad \text{and} \quad J_A^\mu = \bar{u} \gamma^\mu \gamma_5 d$$



$$\int_0^{m_\tau} \text{Im } \Pi(s) \, ds = -\frac{1}{2\pi i} \oint_{|s|=m_\tau} \Pi(s) \, ds$$

$$\text{with } s \equiv -q^2$$

$$\int_0^{s_0} \omega(s) \rho(s) \, ds = \frac{-1}{2\pi i} \oint_{|s|=s_0} \omega(s) \Pi_{OPE}(s) \, ds \quad (6)$$

$$R_{\tau}(s) = 12\pi \int_0^{m_{\tau}} \frac{ds}{m_{\tau}^2} \left(1 - \frac{s}{m_{\tau}^2}\right) \left[ \left(1 + 2\frac{s}{m_{\tau}^2} \operatorname{Im} \Pi^{(1)}(s)\right) + \operatorname{Im} \Pi^{(0)}(s) \right] \quad (7)$$

$$R_{\tau,V/A} = \frac{N_c}{2} (1 + \delta_{pt} + \delta_{npt}) \quad (8)$$



$$D(s) \equiv s \frac{d}{ds} \Pi(s) \quad (9)$$

$$D^{(1+0)}(s) \equiv -s \frac{d}{ds} \Pi^{(1+0)}(s), \quad D^{(0)}(s) \equiv \frac{s}{m_\tau^2} \frac{d}{ds} \left( s \Pi^{(0)}(s) \right) \quad (10)$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{dx}{x} (1-x)^3 \left[ 3(1+x) D^{(1+0)}(m_\tau^2 x) + 4 D^{(0)}(m_\tau^2 x) \right] \quad (11)$$

$$x \equiv \frac{s}{m_\tau^2} \quad (12)$$

$$D_V^{(1+0)} = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_{\mu}^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1} \quad \text{with} \quad L \equiv \log \frac{-s}{\mu^2} \quad (13)$$

$$c_{0,0} = -\frac{5}{3}, \quad c_{0,1} = 1 \quad (14)$$

$$c_{2,1} = \frac{365}{24} - 11\zeta_3 - \left( \frac{11}{12} - \frac{2}{3}\zeta_3 \right) N_f, \quad (15)$$

$$c_{3,1} = \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5, \quad (16)$$

$$- \left( \frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5 \right) N_f + \left( \frac{151}{162} - \frac{19}{27}\zeta_3 \right) N_f^2, \quad (17)$$

$$c_{4,1} = \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7, \quad (18)$$

$$c_{5,1} = 283 \quad (19)$$

$$\delta_{pt} = \sum_{n=1}^{\infty} a_{\mu}^n \sum_{k=1}^n k c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log \left( \frac{-m_{\tau}^2 x}{\mu^2} \right)^{k-1} \quad (20)$$

$$\delta_{FOPT}^{(0)} = \sum_{n=1}^{\infty} a(m_{\tau}^2)^n \sum_{k=1}^n k c_{n,k} J_{k-1} \quad (21)$$

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log^l(-x) \quad (22)$$

$$\delta_{CIPT}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2) \quad (23)$$

$$J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-m_\tau^2 x) \quad (24)$$

$$\delta_{FOPT}^{(0)} = \alpha_s^2 + \alpha_s^2 + \alpha_s^3 + \alpha_s^4 + \alpha_s^5 = 0.1082 + 0.0609 + 0.0334 + 0.0174(+0.0088) = 0.2200(0.2288) \quad (25)$$

$$\delta_{CIPT}^{(0)} = 0.1479 + 0.0297 + 0.0122 + 0.0086(+0.0038) = 0.1984(0.2021). \quad (26)$$

Borel integral

$$A \equiv \int_0^\infty dt e^{-t} \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n, \quad (27)$$

Borel transform

$$B[A](t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n. \quad (28)$$

$$\frac{12\pi^2}{N_c} D_V^{1+0}(s) \equiv 1 + \hat{D}(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \alpha_s (\sqrt{s})^{n+1}. \quad (29)$$

$$B[\hat{D}](u) = B[\hat{D}_1^{UV}](u) + B[\hat{D}_2^{IR}](u) + B[\hat{D}_3^{IR}](u) + d_0^{PO} + d_1^{PO} u, \quad (30)$$

$$B[\hat{D}_p^{IR}](u) \equiv \frac{d_p^{IR}}{(p-u)^{1+\gamma}} \left[ 1 + b_1(p-u) + b_2(p-u)^2 + \dots \right] \quad (31)$$

$$B[\hat{D}_p^{UV}](u) \equiv \frac{d_p^{UV}}{(p+u)^{1+\gamma}} \left[ 1 + b_1(p+u) + b_2(p+u)^2 \right], \quad (32)$$

Beneke and Jamin, “ $\alpha_s$  and the  $\tau$  hadronic width: fixed-order, contour-improved and higher-order perturbation theory”

## OPE

$$\lim_{x \rightarrow y} A(x)B(y) = \sum_n C_n(x-y) \mathcal{O}_n(x) \quad (33)$$

$$\Pi_{OPE}(q^2) = -\frac{1}{3q^2} \sum_n \langle \Omega | \mathcal{O}_n(0) | \Omega \rangle \int d^4 x e^{iqx} C_n(x) \quad (34)$$

$$\Pi_{V/A}^{OPE}(s) = \sum_{D=0,2,4,\dots} \frac{C^{(D)} \langle \Omega | \mathcal{O}^{(D)}(x) | \Omega \rangle}{(-q^2)^{D/2}} \quad (35)$$



$$D_{ij}^{(1+0)}(s) \Big|_{D=4} = \frac{1}{s^2} \sum_n \Omega^{(1+0)}(s/\mu^2) a^n, \quad (36)$$

where the  $\Omega^{(1+0)}(s/\mu^2)$  is given by

$$\begin{aligned} \Omega_n^{(1+0)}(s/\mu^2) = & \frac{1}{6} \langle aGG \rangle p_n^{(1+0)}(s/\mu^2) + \sum_k m_k \langle q_k q_k \rangle r_n^{(1+0)}(s/\mu^2) \\ & + 2 \langle m_i q_i q_i + m_j q_j q_j \rangle q_n^{(1+0)}(s/\mu^2) \pm \frac{8}{3} \langle m_j q_i q_i + m_i q_j q_j \rangle t_n^{(1+0)} \\ & - \frac{3}{\pi^2} (m_i^4 + m_j^4) h_n^{(1+0)}(s/\mu^2) \mp \frac{5}{\pi^2} m_i m_j (m_i^2 + m_j^2) k_n^{(1+0)}(s/\mu^2) \\ & + \frac{3}{\pi^2} m_i^2 m_j^2 g_n^{(1+0)}(s/\mu^2) + \sum_k m_k^4 j_n^{(1+0)}(s/\mu^2) + 2 \sum_{k \neq l} m_k^2 m_l^2 u_n^{(1+0)} \end{aligned} \quad (37)$$

$$\begin{aligned} D_{ij,V/A}^{(1+0)} \Big|_{D=8} &= 4 \frac{\rho_{V/A}^{(8)}}{s^4} \\ D_{ij,V/A}^{(1+0)} \Big|_{D=10} &= 5 \frac{\rho_{V/A}^{(10)}}{s^5} \\ D_{ij,V/A}^{(1+0)} \Big|_{D=12} &= 6 \frac{\rho_{V/A}^{(12)}}{s^6} \end{aligned} \tag{38}$$

$$R_{\tau,V/A}^{\omega} = \frac{N_c}{2} S_{EW} |V_{ud}|^2 (1 + \delta_{pt}^{\omega} + \delta_{npt}^{\omega} + \delta_{DV}^{\omega}) \quad (39)$$

$$\rho_{V/A}^{DV}(s) = e^{-(\delta_{V/A} + \gamma_{V/A}s)} \sin(\alpha_{V/A} + \beta_{V/A}s) \quad (40)$$

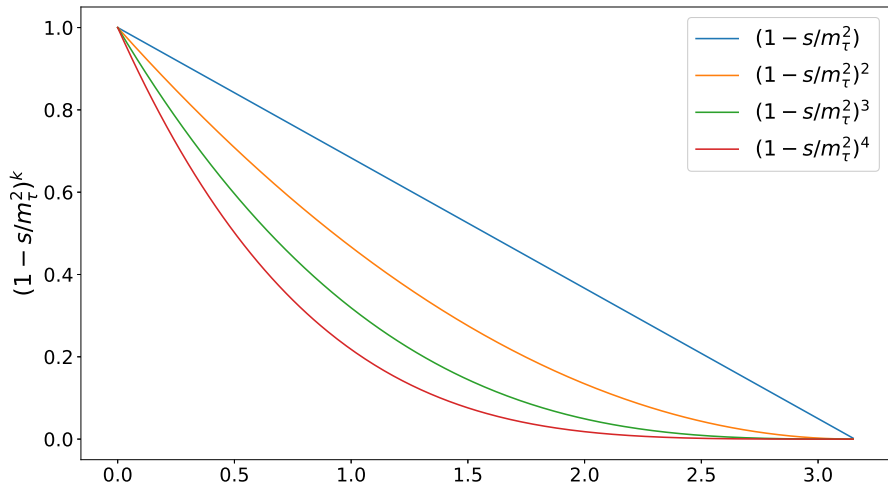
$$D_{\omega}(m_{\tau}^2) = -12\pi^2 \int_{m_{\tau}^2}^{\infty} \frac{ds}{m_{\tau}^2} \omega(s) \rho_{V/A}^{DV} \quad (41)$$

$$\omega(x) \equiv \sum_i a_i x^i \quad (42)$$

kinematic weights

$$\omega_\tau \equiv \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2 \frac{s}{m_\tau^2}\right) \quad (43)$$

$$\omega(s) = \left(1 - \frac{s}{m_\tau^2}\right)^k \quad (44)$$



# Weighting OPE Contributions

$$\oint_C x^k dx = i \int_0^{2\pi} (e^{i\theta})^{k+1} d\theta = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise} \end{cases}. \quad (45)$$

$$R(x)|_{D=0,2,4,\dots} = \oint_{|x|=1} dx x^{k-D/2} C^{(D)} \quad (46)$$

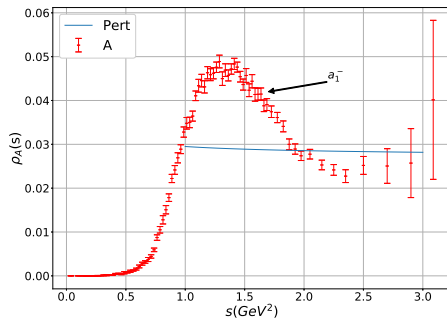
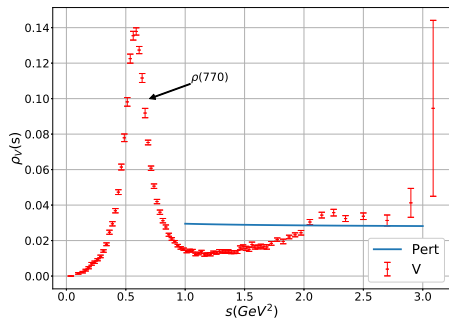
active dimension

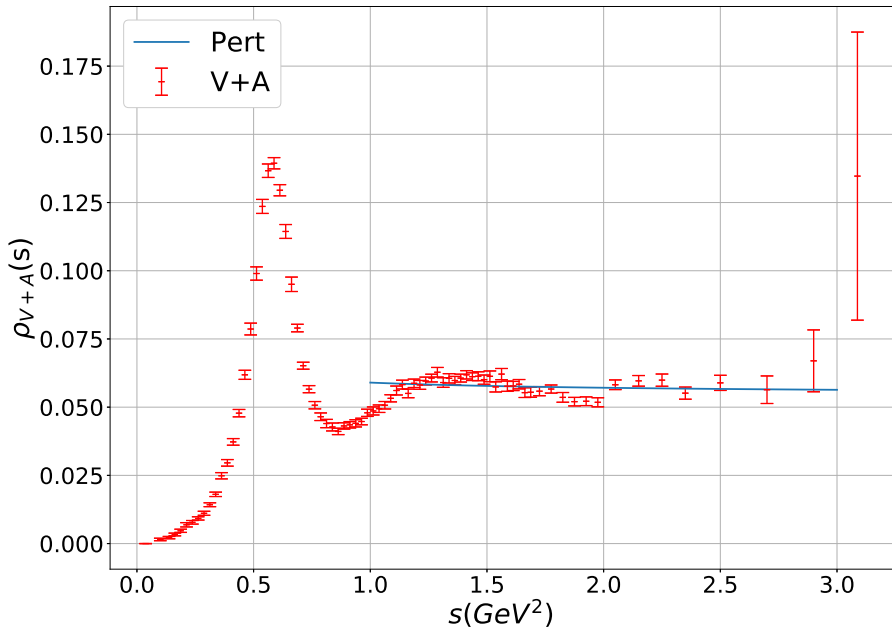
$$D = 2(k + 1) \quad (47)$$

<b>monomial:</b>	$x^0$	$x^1$	$x^2$	$x^3$	$x^5$	$x^6$	$x^7$
<b>dimension:</b>	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

**Table:** List of monomial and their corresponding “active” dimensions in the OPE. Note that the perturbative contributions of the OPE are always present.

# ALEPH data







$$R_{\tau,V/A} = \frac{\mathcal{B}_{V/A}}{\mathcal{B}_e} = \int_0^{m_\tau^2} ds \frac{\text{sfm2}_{V/A}(s)}{100\mathcal{B}_e} \quad (48)$$

$$I_{exp,V/A}^\omega(s_0) = \frac{s_\tau}{100\mathcal{B}_e s_0} \sum_{i=1}^{N(s_0)} \frac{\omega\left(\frac{s_i}{s_0}\right)}{\omega_\tau\left(\text{sfm2}_{V/A}(s_i)\right)} \quad (49)$$

$$\chi^2 = (I_i^{\text{exp}} - I_i^{\text{th}}(\vec{\alpha})) C_{ij}^{-1} (I_j^{\text{exp}} - I_j^{\text{th}}(\vec{\alpha})) \quad (50)$$

$$C_{ij} = \text{cov}(I_i^{\text{exp}}, I_j^{\text{exp}}) \quad (51)$$

$$\chi^2 \approx 1 \quad (52)$$

	Symbol	Term	Expansion	OPE Contributions
Pinched	$\omega_\tau$	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	$D6, D8$
	$\omega_{cube}$	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	$D6, D8, D10$
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	$D6, D8, D10, D12$
Monomial	$\omega_{M2}$	$1-x^2$	$1-x^2$	$D6$
	$\omega_{M3}$	$1-x^3$	$1-x^3$	$D8$
	$\omega_{M4}$	$1-x^4$	$1-x^4$	$D10$
Pinched +x	$\omega_{1,0}$	$(1-x)$	$1-x$	$D4$
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	$D4, D6$
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	$D4, D6, D8$
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	$D4, D6, D8, D10$

# Kinematic Weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

	$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\chi^2/dof$
BS	2.200	7	0.3274(42)	-0.82(21)	-1.08(40)	0.21
FOPT	2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

Cubic Weight:  $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$

$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	$\chi^2/dof$
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

$$\alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28),$$

$$\rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77). \quad (5)$$

$$\omega_{M2}(x) \equiv 1 - x^2$$

$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\chi^2/dof$
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

$$\omega_{M3}(x) \equiv 1 - x^3$$

$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	$\chi^2/dof$
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17



## Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	$\chi^2/dof$
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

$$\omega_{1,0} \equiv (1 - x)$$

	$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\chi^2/dof$
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	0.38
	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24

$$\omega_{2,0} \equiv (1-x)^2$$

	$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\chi^2/dof$
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

$$\omega_{3,0} \equiv (1-x)^3$$

	$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	$\chi^2/dof$
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

$$\omega_{4,0} \equiv (1-x)^4$$

	$s_{min}$	$\#s_0s$	$\alpha_s(m_\tau^2)$	$aG\bar{G}lnv$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	$\chi^2/dof$
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

# Comparison

	weight	$s_{min}$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	$\chi^2/dof$
FOPT	$\omega_\tau$	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
	$\omega_{cube}$	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
	$\omega_{M2}$	2.2	0.3248(52)	-	-0.77(22)	-	-	0.38
	$\omega_{M3}$	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
BS	$\omega_{1,0}$	2.2	0.3246(52)	-0.2262(59)	-	-	-	0.38
	$\omega_{2,0}$	2.2	0.3270(54)	-0.0254(61)	-0.77(21)	-	-	0.74
	$\omega_{3,0}$	2.1	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	-	0.46

- $\alpha_s(m_\tau^2) = 0.3261 \pm 0.0050$
- $\rho^{(6)} = -0.68 \pm 0.2$
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- DV not present if using single pinched weights in the V+A channel
- FOPT more valid than CIPT
- $\alpha_s(m_Z^2) = 0.11940(60)$



Thank you

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