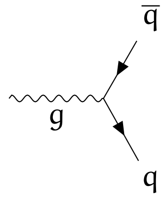
0.1 Slide 1

$$\alpha_s(m_\tau^2) \approx 0.33$$

$$\mathcal{L}_{\mathrm{QCD}} = \dots + \sqrt{\pi \boldsymbol{\alpha_s}}\overline{q}(x)\lambda\gamma_\mu q(x)G(x) + \dots$$



$$\mu\frac{d}{d\mu}R(q, a_s, m) = \left[\mu\frac{\partial}{\partial\mu} + \mu\frac{da_s}{d\mu}\frac{\partial}{\partial a_s} + \mu\frac{dm}{d\mu}\frac{\partial}{\partial m}\right]R(q, a_s, m)$$

$$\beta(a_s) \equiv -\mu\frac{da_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots$$

$$\beta_1 = \frac{1}{6}(11N_c - 2N_f)$$

$$a_s(\mu_2) = \frac{a_s(\mu_1)}{\left(1 - a_s(\mu_1)\beta_1 \log\frac{\mu_1}{\mu_2}\right)}$$

1 Theoretical Background

1.1 Framework 1

$$R_\tau = 12\pi |V_{ud}|^2 S_{EW} \int_0^{m_\tau} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}_{V/A}(s) + \mathrm{Im}\,\Pi^{(0)}_{V/A}(s)\right]$$

$$\Pi^{\mu\nu}_{V/A}(q^2) \equiv i \int d^4 x e^{iqx} \langle 0| T \left\{ J^\mu_{V/A}(x) J^\nu_{V/A}(0) \right\} |0\rangle$$

$$= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}_{V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{V/A}(q^2)$$

$$J^\mu_V = \overline{u}\gamma^\mu d$$

$$J^\mu_A = \overline{u}\gamma^\mu \gamma_5 d$$

$$\text{with} \quad s \equiv -q^2$$

$$\int_0^{s_0} \frac{ds}{s_0} \omega(s)\, \mathrm{Im}\,\Pi_{V/A}(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{d}{s_0} \omega(s) \Pi_{V/A}(s)$$

1.2 Framework 2

$$R_{\tau,V/A} = \frac{N_c}{2}(1 + \delta_{pt} + \delta_{npt})$$

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$$D(s) \equiv -s \frac{d}{ds} \Pi(s)$$

$$D^{(1+0)}(s) \equiv -s \frac{d}{ds} \Pi^{(1+0)}(s), \qquad D^{(0)}(s) \equiv \frac{s}{m_\tau^2} \frac{d}{ds} (s \Pi^{(0)}(s))$$

$$x \equiv \frac{s}{m_\tau^2}$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{dx}{x} (1-x)^3 \left[3(1+x) D^{(1+0)}(m_\tau^2 x) + 4 D^0(m_\tau^2 x)\right]$$

$$D^{(1+0)}_V = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}$$

$$a_\mu \equiv \frac{\alpha_s(\mu^2)}{\pi}$$

$$
\begin{aligned}
c_{0,0} &= -\frac{5}{3}, \quad c_{0,1} = 1 \\
c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f, \\
c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5, \\
&\quad - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2, \\
c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7, \\
c_{5,1} &= 283
\end{aligned}
$$

1.3 FOPT vs CIPT

$$
\delta_{pt} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k\, c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x) \log\left(\frac{-m_\tau^2 x}{\mu^2}\right)^{k-1}
$$

$$
\mu^2 = m_\tau^2
$$

$$
\delta_{FOPT}^{(0)} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k} J_{k-1}
$$

$$
J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x) \log^l(-x)
$$

$$
\mu^2 = -m_\tau^2 x
$$

$$
\delta_{CIPT}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2)
$$

$$
J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x}(1-x)^3(1+x) a^n(-m_\tau^2 x)
$$