The QCD Strong Coupling from Hadronic Tau Decays

Dirk Hornung

Supervisor
Matthias Jamin

17th July 2019

Universitat Autònoma de Barcelona

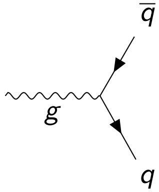
Institut de Física d'Altes Energies

The Strong Coupling α_s

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu,a}(x) + \left[\sum_A \frac{i}{2} \overline{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m \overline{q}^A(x) q^A(x) \right], \tag{1}$$

where $D_\mu = \partial_\mu - ig \frac{\lambda^a}{2} B^a_\mu$

$$\mathcal{L}^{QG-Int}_{QCD}(x) = \sqrt{\pi \boldsymbol{\alpha_s}}\, \overline{q}(x) \lambda \gamma_\mu q(x) G(x) \quad \Rightarrow \tag{2}$$



- Focused on QCD, the theory of the strong interactions between quarks and gluons
- Quarks are the fundamental particles that make up Hadrons (proton, neutron, pion, ...)
- QCD is a non-abelian quantum field theory with symmetry group SU(3)
- Charge property is the colour, the force carriers are the gluons
- QCD Lagrangian, which contains the interaction terms
- The interaction term can be displayed as a Feynman diagram
- The strength of the interaction is given by the strong coupling constant α_s
- Our research focuses on measuring the strong coupling constant

The Running of the Strong Coupling

$$\alpha_s(m_\tau^2) \approx 0.33$$
$$\alpha_s(m_Z^2) \approx 0.12 \quad (3)$$

$$m_\tau = 1776.86(12)\ \text{MeV}^1$$
$$m_Z = 91.1876(21)\ \text{GeV}^1 \quad (4)$$

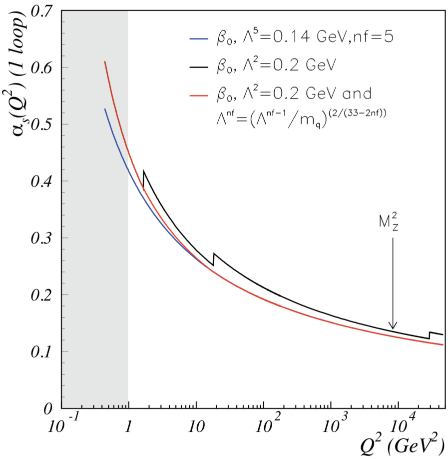
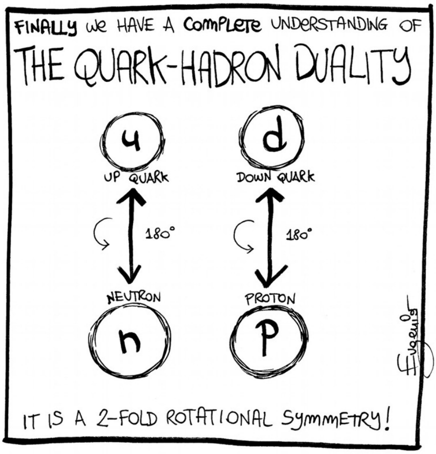


Figure: Taken from Deur, Brodsky, and Teramond, "The QCD Running Coupling", 2016

[1] Tanabashi et al., "Review of Particle Physics", 2018

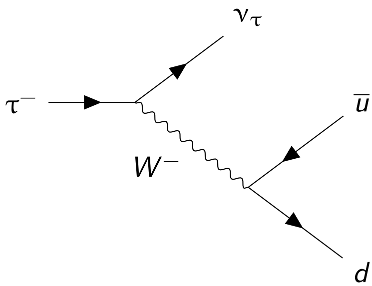
- Strong coupling constant is far from constant, but depends on the energy
- This is called as the "running of the strong coupling"
- E.g. at the for us interesting m_τ^2 scale $\alpha_s(m_\tau^2) \approx 0.33$
- In general the different values for α_s are compared at the m_Z^2 scale ($\alpha_s(m_Z^2) \approx 1.12$)
- On the right we can study the running of the strong coupling
- α_s decreases with increasing energy
- leads to asymptotic freedom: at high energies quarks and gluons interact weakly and can be treated perturbatively
- leads also to confinement: at low energies quarks are bound. An isolated quark has never been measured. They appear in hadrons, two or three quarks

Duality



- As we observe Hadrons, but our theory is based on quarks and gluons we assume Quark-Hadron Duality
- Which is known to exist, but in contrary to the image is not well understood
- In some cases there appear duality violations
- We want to show that Duality Violations do not have to be taken into account under certain circumstances

Hadronic τ decays



Tau decay ratio

$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \tag{5}$$

Name	Symbol	Quark content	Rest mass
Pion	π^-	$\overline{u}d$	$139.570\,61(24)$ MeV
Pion	π^0	$(u\overline{u} - d\overline{d})/\sqrt{2}$	$134.9770(5)$ MeV

- We measure the strong coupling constant from tau decays
- We are interested in the hadronic tau decay
- Here the tau lepton decays into W boson and a tau-neutrino
- the W^- boson then decays into an anti-up and a down quark
- Rarely it can decay into strange quarks, but we will neglect those cases
- The leftover quarks are not to be seen, as they appear as composite Hadrons, like the pions, given down below
- An important quantity is the hadronic tau decay ratio, which is the decay width of taus decaying into hadrons divided by the decay width of taus decaying into electrons
- We will use this quantity to perform our fits, as it is theoretically as experimentally accessible

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Two-Point Function:

$$
\begin{aligned}
\Pi^{\mu\nu}_{V/A}(q^2) &\equiv i\int d^4\, x e^{iqx}\langle 0|T\left\{J^{\mu}_{V/A}(x)J^{\nu}_{V/A}(0)\right\}|0\rangle \\
&= (q^{\mu}q^{\nu}-q^2g^{\mu\nu})\Pi^{(1)}_{V/A}(q^2)+q^{\mu}q^{\nu}\Pi^{(0)}_{V/A}(q^2) \\
&= (q^{\mu}q^{\nu}-q^2g^{\mu\nu})\Pi^{(1+0)}_{V/A}(q^2)+q^2g_{\mu\nu}\Pi^{(0)}(q^2)
\end{aligned}
\tag{6}
$$

where the current is given by

$J^{\mu}_{V}=\overline{u}\gamma^{\mu}d$ and $J^{\mu}_{A}=\overline{u}\gamma^{\mu}\gamma_5 d$

and we redefined the correlator as follows

$$\Pi^{(1+0)}(q^2)=\Pi^{(1)}(q^2)+\Pi^{(0)}(q^2).$$

- The two-point function is defined as the vacuum expectation value of the time-ordered product of two currents
- We have given the expression in momentum space
- In our case the currents are non-strange V or A currents, distinguished by a γ^{μ} or $\gamma^{\mu}\gamma_5$ correspondingly
- We can lorentz decompose the two-point function, to obtain a scalar function Π
- The superscripts (0) and (1) label the transversal or longitudinal spin
- It is common to rewrite the newly introduced scalar function of the correlator to $\Pi^{(1+0)}$ and $\Pi^{(0)}$
- $\Pi^{(1+0)}(q^2)$ and $q^2\Pi^{(0)}$ are free of kinematic singularities

Inclusive Hadronic Tau Decay Ratio

Tau decay ratio

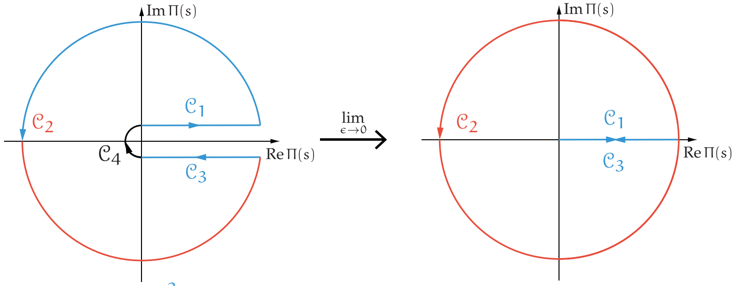
$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \tag{7}$$

Inclusive Hadronic Tau Decay Ratio $s \equiv -q^2$

$$R_\tau = 12\pi \left| V_{ud} \right|^2 S_{EW} \int_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2} \left(1 + 2\frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \operatorname{Im}\Pi^{(1)}(s) + \operatorname{Im}\Pi^{(0)}(s) \right] \tag{8}$$

- A central value is the inclusive hadronic tau decay ratio (i.e. all decays containing hadrons)
- The ratio can be calculated by using the optical theorem
- V_{ud} is the Cabbibo matrix element, S_{EW} the electroweak correction
- We have to integrate the two-point function from $0 \to m_\tau^2$
- The two-point function has poles on the positive real axis
- $\Pi^{(0)}$ will be neglected? There is no $J = 0$ vector contribution. The $J = 0$ axial-vector contribution is the pion pole. Which is missing in the experimental data.

Cauchy's Theorem



$$\int_0^{m_\tau^2} \mathrm{Im}\,\Pi(s)\,ds = \oint_{|s|=m_\tau^2} \Pi(s)\,ds$$

Can be rewritten into

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(1+0)}(s)-\left(\frac{2s}{m_\tau^2}\Pi^{(0)}\right)\right] \tag{9}$$

- Experimental data only accessible on positive real axis
- Imaginary part of two-point function related to experimental accessible spectral function
- integrate from 0 to m_τ^2 to reproduce $R_\tau(m_\tau^2)$
- Theoretically the positive real axis is not accessible
- Two-point function has poles on positive real axis
- $\Rightarrow$ Cauchy's theorem
- Identify

Adler Function:

$$D(s) \equiv s\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s) \tag{10}$$

$$D^{(1+0)}(s) \equiv -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi^{(1+0)}(s), \qquad D^{(0)}(s) \equiv \frac{s}{m_\tau^2}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\Pi^{(0)}(s)\right) \tag{11}$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{\mathrm{d}x}{x}(1-x)^3\left[3(1+x)D^{(1+0)}(m_\tau^2 x) + 4D^{(0)}(m_\tau^2 x)\right] \tag{12}$$

$\mathrm{d}\omega x$

- Adler Function for convenience
- In case of vector correlator the derivative (Adler Function) is a physical quantity
- Physical quantities are renormalisation scale invariant
- For the $(1+0)$ and (0) we use a different definition

Finite Energy Sum Rule (FESR)

Spectral Function

$$\rho^{(1+0)}(s) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi^{(1+0)}(s) \tag{13}$$

Spectral Moment

$$I^{\omega}_{V/A}(s_0) \equiv \frac{1}{s_0}\int_0^{s_0} \mathrm{d}s\,\omega\left(\frac{s}{s_0}\right)\rho(s) = \frac{-1}{2\pi i s_0}\oint_{|s|=s_0} \mathrm{d}s\,\omega\left(\frac{s}{s_0}\right)\Pi(s) \tag{14}$$

- Spectral function equal to the imaginary part of the correlator
- Spectral function is given by the experiment
- Experiment only valid to certain energy s_0
- $\Rightarrow$ *Finite Energy* Sum Rule
- Define spectral moment
- Introduce weights
- Weights ω have to be analytic

$$\Pi_{OPE}(q^2) = -\frac{1}{3q^2}\sum_n \langle\Omega|\mathcal{O}_n(0)|\Omega\rangle \int d^4\; e^{iqx} C_n(x) \tag{15}$$

$$\begin{aligned}\Pi_{OPE,V/A}(s) &= \sum_{D=0,2,4,\dots} \frac{C^{(D)}\langle\Omega|\mathcal{O}^{(D)}(x)|\Omega\rangle}{(s)^{D/2}} \\ &= \underbrace{C_0}_{PT} + \underbrace{\sum_{k=1}^{\infty}\frac{C_{2k}(s)}{s^k}}_{NPT}\end{aligned} \tag{16}$$

Perturbative Contributions

$$\Pi_V^{(1+0)}(s) = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{n,k} L^k \quad \text{with} \quad L \equiv \log \frac{-s}{\mu^2} \tag{17}$$

$$D_V^{(1+0)} = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k} L^{k-1} \tag{18}$$

Adler Function Coefficients

Renormalisation Group Equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}R(q,a_s,m)=\left[\mu\frac{\partial}{\partial\mu}+\mu\frac{\mathrm{d}a_s}{\mathrm{d}\mu}\frac{\partial}{\partial a_s}+\mu\frac{\mathrm{d}m}{\mathrm{d}\mu}\frac{\partial}{\partial m}\right]R(q,a_s,m)=0 \tag{19}$$

$$\left(2\frac{\partial}{\partial L}+\beta\frac{\partial}{\partial a_s}\right)D_V^{(1+0)}=0 \tag{20}$$

$$\begin{aligned} c_{0,0}&=-\frac{5}{3},\quad c_{0,1}=1,\\ c_{1,1}&=1\\ c_{2,1}&=\frac{365}{24}-11\zeta_3-(\frac{11}{12}-\frac{2}{3}\zeta_3)N_f\\ &\dots \end{aligned} \tag{21}$$

$$\begin{aligned} c_{2,2}&=-\frac{1}{4}\beta_1c_{1,1},\\ c_{3,2}&=\frac{1}{4}(-\beta_2c_{1,1}-2\beta_1c_{2,1}),\\ c_{3,3}&=\frac{1}{12}\beta_1c_{1,1}\\ &\dots \end{aligned} \tag{22}$$

- Beta function definition missing

Perturbative Contribution

$$\delta_{pt} = \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k\, c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log\left(\frac{-m_\tau^2 x}{\mu^2}\right)^{k-1} \tag{23}$$

Fixed-Order Perturbation Theory (FOPT)

Contour-Improved Perturbation Theory (CIPT)

$$\mu \equiv m_\tau^2 \qquad\qquad \mu \equiv -m_\tau^2 x \tag{24}$$

Fixed-Order Perturbation Theory

$$\delta^{(0)}_{FOPT} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k} J_{k-1} \tag{25}$$

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (1-x)^3 (1+x) \log^l(-x) \tag{26}$$

CIPT

$$\delta^{(0)}_{CIPT} = \sum_{n=1}^{\infty} c_{n,1}\, J^a_n(m_\tau^2) \tag{27}$$

$$J^a_n(m_\tau^2) \equiv \frac{1}{2\pi i}\oint_{|x|=1} \frac{\mathrm{d}x}{x}(1-x)^3(1+x)\,a^n(-m_\tau^2 x) \tag{28}$$

FOPT vs CIPT

$$
\begin{array}{rl}
 & \quad\;\; \alpha_s^2 \qquad\; \alpha_s^2 \qquad\; \alpha_s^3 \qquad\; \alpha_s^4 \qquad\qquad\; \alpha_s^5 \\
\delta^{(0)}_{FOPT} & = 0.1082 + 0.0609 + 0.0334 + 0.0174 (+0.0088) = 0.2200 (0.2288) \qquad (29) \\
\delta^{(0)}_{CIPT} & = 0.1479 + 0.0297 + 0.0122 + 0.0086 (+0.0038) = 0.1984 (0.2021) \qquad (30)
\end{array}
$$

Borel Summation

Borel integral

$$A \equiv \int_0^\infty \mathrm{d}t e^{-t} \sum_{n=0}^{\infty} \frac{a_k}{n!} t^n, \tag{31}$$

Borel transform

$$B[A](t) = \sum_{n=0}^{\infty} \frac{a_k}{n!} t^n. \tag{32}$$

$$\frac{12\pi^2}{N_c} D_V^{1+0}(s) \equiv 1 + \widehat{D}(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \alpha_s(\sqrt{(s)})^{n+1}. \tag{33}$$

Borel Model

$$B[\widehat{D}](u) = B[\widehat{D}_1^{UV}](u) + B[\widehat{D}_2^{IR}](u) + B[\widehat{D}_3^{IR}](u) + d_0^{PO} + d_1^{PO} u, \qquad (34)$$

$$B[\widehat{D}_p^{IR}](u) \equiv \frac{d_p^{IR}}{(p-u)^{1+\widetilde{\gamma}}} \left[1 + \widetilde{b}_1 (p-u) + \widetilde{b}_2 (p-u)^2 + \ldots\right] \qquad (35)$$

$$B[\widehat{D}_p^{UV}](u) \equiv \frac{d_p^{UV}}{(p+u)^{1+\overline{\gamma}}} \left[1 + \overline{b}_1 (p+u) + \overline{b}_2 (p+u)^2\right], \qquad (36)$$

Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008

Non-Perturbative Contributions

$$\Pi^{OPE}_{NPT,V/A}(s) = \sum_{D=2,4,\ldots} \frac{C^{(D)}\langle\Omega|\mathcal{O}^{(D)}(x)|\Omega\rangle}{(-q^2)^{D/2}} \tag{37}$$

$$\begin{array}{ll}
\text{Dimension 0:} & \mathbb{1} \\
\text{Dimension 4:} & : m_i qq : \\
 & : G_a^{\mu\nu}(x) G^a_{\mu\nu}(x) : \\
\text{Dimension 6:} & : q\Gamma qq\Gamma q : \\
 & : q\Gamma \frac{\lambda^a}{2} q_\beta(x) q\Gamma \frac{\lambda^a}{2} q : \\
 & : m_i q \frac{\lambda^a}{2} \sigma_{\mu\nu} q G_a^{\mu\nu} : \\
 & : f_{abc} G_a^{\mu\nu} G_b^{\nu\delta} G_c^{\delta\mu} : \\
\ldots & \ldots
\end{array} \tag{38}$$

Dimension Four Corrections

$$D_{ij}^{(1+0)}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_n \Omega^{(1+0)}(s/\mu^2)a^n, \tag{39}$$

$$\begin{aligned}\Omega_n^{(1+0)}(s/\mu^2) \ = \ & \frac{1}{6}\langle aGG\rangle p_n^{(1+0)}(s/\mu^2) + \sum_k m_k\langle\overline{q}_k q_k\rangle r_n^{(1+0)}(s/\mu^2) \\ & + 2\langle m_i\overline{q}_i q_i + m_j\overline{q}_j q_j\rangle q_n^{(1+0)}(s/\mu^2) \pm \frac{8}{3}\langle m_j\overline{q}_i q_i + m_i\overline{q}_j q_j\rangle t_n^{(1+0)} \\ & - \frac{3}{\pi^2}(m_i^4 + m_j^4)h_n^{(1+0)}(s/\mu^2) \mp \frac{5}{\pi^2}m_i m_j(m_i^2 + m_j^2)k_n^{(1+0)}(s/\mu^2) \\ & + \frac{3}{\pi^2}m_i^2 m_j^2 g_n^{(1+0)}(s/\mu^2) + \sum_k m_k^4 j_n^{(1+0)}(s/\mu^2) + 2\sum_{k\neq l} m_k^2 m_l^2 u_n^{(1+0)}(s/\mu^2).\end{aligned} \tag{40}$$

Dimension Six and Eight Corrections

$$
\begin{aligned}
D^{(1+0)}_{ij,V/A}\Big|_{D=8} &= 4\frac{\rho^{(8)}_{V/A}}{s^4} \\
D^{(1+0)}_{ij,V/A}\Big|_{D=10} &= 5\frac{\rho^{(10)}_{V/A}}{s^5} \\
D^{(1+0)}_{ij,V/A}\Big|_{D=12} &= 6\frac{\rho^{(12)}_{V/A}}{s^6}
\end{aligned}
\tag{41}
$$

Duality Violations

Duality Violations (DV):

$$\Delta(s) \equiv \Pi(s) - \Pi_{OPE}(s) \tag{42}$$

DV Model:

$$\rho^{DV}_{V/A}(s) = e^{-(\delta_{V/A}+\gamma_{V/A}s)} \sin(\alpha_{V/A} + \beta_{V/A}s) \tag{43}$$

DV Contribution:

$$D_\omega(s_0) = -12\pi^2 S_{EW}|V_{ud}|^2 \int_{s_0}^{\infty} \frac{ds}{s_0} \omega(s) \rho^{DV}_{V/A} \tag{44}$$

- The difference $\Delta(s)$ defines the duality violating contribution to Π
- DV can be parametrised via a model
- The model has four parameters for the vector and four parameters for the axial channel
- Too many parameters: e.g. α_s, ρ_6, ρ_8 three parameters vs eight!
- why is the DV integration from $s_0 \to \infty$ and not $0 \to s_0$

Experimental Spectral Moment:

$$\rho^{(1)}_{V/A} = \frac{m_\tau^2}{12\pi^2 |V_{ud}|^2 \, S_{EW}} \frac{B_{V/A}}{\mathcal{B}_e} \frac{\mathrm{d}N_{V/A}}{N_{V/A}\,\mathrm{d}s} \frac{1}{\omega_\tau} \tag{45}$$

Integral Moment:

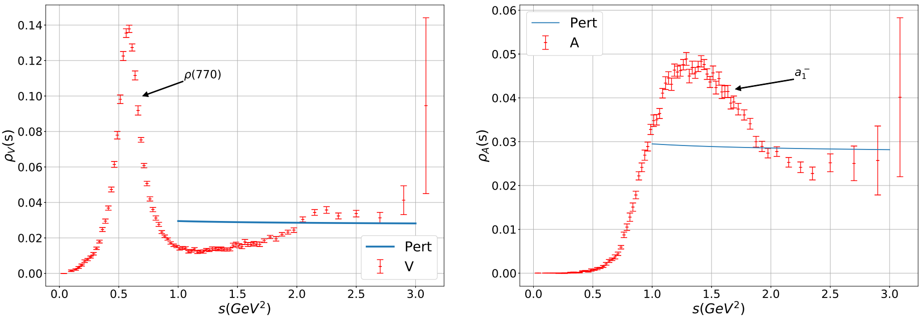
$$R^{\omega}_{\tau,V/A} \equiv 12\pi^2 \int_0^{s_0} \frac{\mathrm{d}s}{s_0} \omega\left(\frac{s}{s_0}\right) \rho_{V/A}(s) \tag{46}$$

Experimental Moment:

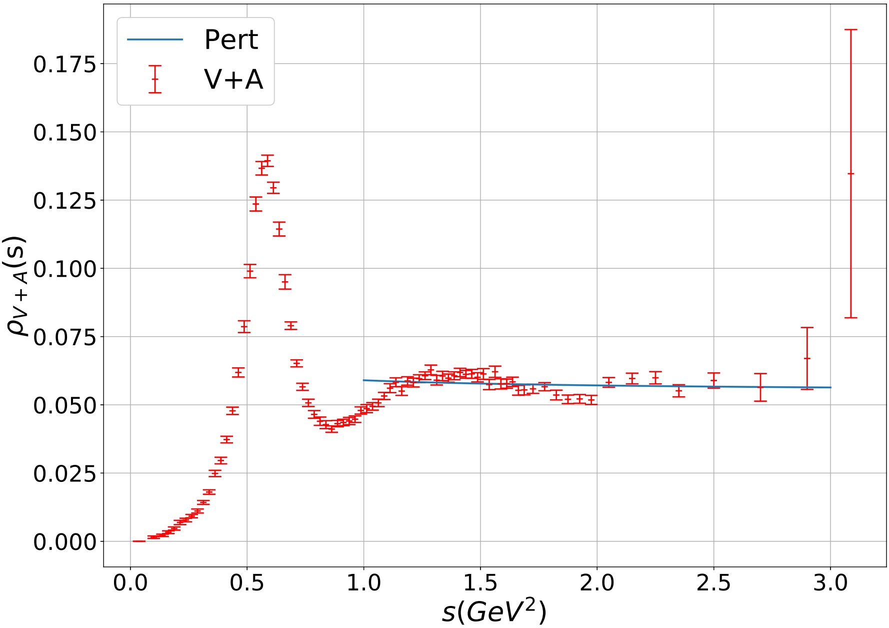
$$I^{\omega}_{exp,V/A}(s_0) = \frac{m_\tau^2}{\mathcal{B}_e s_0} \sum_{i=1}^{N(s_0)} \frac{\omega(s_i/s_0)}{\omega_\tau(s_i/s_0)} \,\mathrm{sfm2}_{V/A}(s_i) \tag{47}$$

- $\rho^{(0)}_V$ does not exist
- $\rho^{(0)}_A$ is the pion pole, which is not included in the data?
- $\mathrm{sfm2}_{V/A} \equiv B_{V/A} \frac{\mathrm{d}N_{V/A}}{N_{V/A}\,\mathrm{d}s}$
- $|V_{ud}|^2 \, and S_{EW}$?

ALEPH Data



0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
$\rho_V(s)$
$\rho(770)$
Pert
V
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$s(GeV^2)$
0.06
0.05
0.04
0.03
0.02
0.01
0.00
$\rho_A(s)$
Pert
A
a_1^-
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$s(GeV^2)$



Pert
V+A
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
$\rho_{V+A}(s)$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$s(GeV^2)$

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Chi-Squared

Chi-Squared function:

$$\chi^2 = (I_i^{exp} - I_i^{th}(\vec{\alpha}))C_{ij}^{-1}(I_j^{exp} - I_j^{th}(\vec{\alpha})) \tag{48}$$

Covariance Matrix:

$$C_{ij} = \text{cov}(I_i^{exp}, I_j^{exp}) \tag{49}$$

Chi-Squared per Degrees of Freedom:

$$\frac{\chi^2}{dof} \approx 1 \tag{50}$$

Weights

Weight function:

$$\omega(x) \equiv \sum_i a_i x^i \tag{51}$$

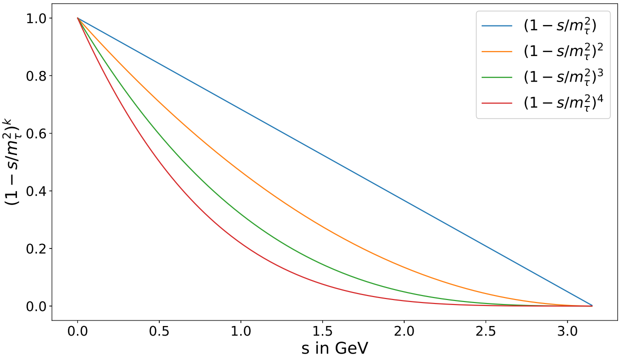
E.g.

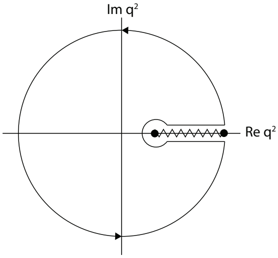
- double pinched
- no monomial
- D6 and D8

$$\begin{aligned} \omega_\tau &\equiv (1-x)^2(1+2x) \\ &= 1 - 3x^2 + 2x^3 \end{aligned} \tag{52}$$

Pinched Weights

$$\omega(s) = \left(1 - \frac{s}{m_\tau^2}\right)^k \tag{53}$$





Weighting OPE Contributions

$$\oint_C x^k \, \mathrm{d}x = i \int_0^{2\pi} \left(e^{i\theta}\right)^{k+1} \mathrm{d}\theta = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise} \end{cases}. \tag{54}$$

$$R(x)\big|_{D=0,2,4,\ldots} = \oint_{|x|=1} \mathrm{d}x \, x^{k-D/2} C^{(D)} \tag{55}$$

Active Dimensions:

$$D = 2(k+1) \tag{56}$$

monomial:	x^0	x^1	x^2	x^3	x^5	x^6	x^7
dimension:	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

Parameters and Momenta

- 3 Moments
- max three parameters
- e.g. α_s, $D6$, $D8$ (fully determined)

#	3 Moments	
1	s_1	ω
2	s_2	ω
3	s_3	ω

#	9 Moments	
1	s_1	ω
2	s_2	ω
$\vdots$	$\vdots$	$\vdots$
9	s_9	ω

- 9 Moments
- max nine parameters

Strategy

- Extract α_s
- Probe Duality Violations
- FOPT vs CIPT

- Check isolated weights of stability for different s_0 moments
- Check stability for different weights and pinchings. If we obtain similar weights DV should not be present.
- Perform additional fits with the BS. If parameters are similar to FOPT, then FOPT should be the preferred framework.

	Symbol	Term	Expansion	OPE Contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	$D6$, $D8$
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	$D6$, $D8$, $D10$
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	$D6$, $D8$, $D10$, $D12$
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	$D6$
	ω_{M3}	$1-x^3$	$1-x^3$	$D8$
	ω_{M4}	$1-x^4$	$1-x^4$	$D10$
Pinched $+x$	$\omega_{1,0}$	$(1-x)$	$1-x$	$D4$
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	$D4$, $D6$
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	$D4$, $D6$, $D8$
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	$D4$, $D6$, $D8$, $D10$

Kinematic Weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.200	7	0.3274(42)	-0.82(21)	-1.08(40)	0.21
FOPT	2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

Cubic Weight: $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

Quartic Weight: $\omega_{quartic}(x) \equiv (1-x)^4(1+4x)$

$$\alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28),$$
$$\rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77). \tag{57}$$

$\omega_{M2}(x) \equiv 1 - x^2$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	χ^2/dof
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

$\omega_{M3}(x) \equiv 1 - x^3$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	χ^2/dof
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	χ^2/dof
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

$\omega_{1,0} \equiv (1-x)$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	χ^2/dof
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	0.38
	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24

$\omega_{2,0} \equiv (1-x)^2$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	χ^2/dof
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

$\omega_{3,0} \equiv (1-x)^3$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

$\omega_{4,0} \equiv (1-x)^4$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$aGGInv$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

Comparison

	weight	s_{min}	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
FOPT	ω_τ	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
	ω_{cube}	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
	ω_{M2}	2.2	0.3248(52)	-	-0.77(22)	-	-	0.38
	ω_{M3}	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
BS	$\omega_{1,0}$	2.2	0.3246(52)	-0.2262(59)	-	-	-	0.38
	$\omega_{2,0}$	2.2	0.3270(54)	-0.0254(61)	-0.77(21)	-	-	0.74
	$\omega_{3,0}$	2.1	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	-	0.46

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Conclusions

- $\alpha_s(m_\tau^2) = 0.3261 \pm 0.0050$
- $\rho^{(6)} = -0.68 \pm 0.2$
- $\rho^{(8)} = -0.80 \pm 0.38$

Conclusions

- $\alpha_s(m_\tau^2) = 0.3261 \pm 0.0050$
- $\rho^{(6)} = -0.68 \pm 0.2$
- $\rho^{(8)} = -0.80 \pm 0.38$

- DV not present if using single pinched weights in the V+A channel
- FOPT more valid than CIPT
- $\alpha_s(m_Z^2) = 0.11940(60)$

Questions

Constants

Quantity	Value	Reference
V_{ud}	0.9742 ± 0.00021	Tanabashi et al., “Review of Partic
S_{EW}	1.0198 ± 0.0006	Marciano and Sirlin, “Electroweak
B_e	17.818 ± 0.023	Davier et al., “The Determination
m_τ	$1.776\,86(12000)\,\mathrm{MeV}$	Tanabashi et al., “Review of Partic
$\langle aGG \rangle_I$	$0.012\,\mathrm{GeV}^2$	Shifman, Vainshtein, and Zakharov
$\langle q_{u/d} q_{u/d} \rangle (m_\tau)$	$-272(15)\,\mathrm{MeV}$	Jamin, “Flavor symmetry breaking
$ss/\langle qq \rangle$	$0.8\ \pm 0.3$	Jamin, “Flavor symmetry breaking

DV-model

$$-\frac{1}{2\pi i}\oint_{|s|=s_0}\frac{\mathrm{d}s}{s_0}\omega(s/s_0)\Delta_{V/A}(s)=-\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s_0}\omega(s/s_0)\frac{1}{\pi}\,\mathrm{Im}\,\Delta_{V/A}(s) \quad (58)$$

Pion Pole

$$R^{\omega}_{\tau,A}(s_0,\pi) = 24\pi^2 |V_{ud}|^2 \, S_{EW} \frac{f_\pi^2}{s_0} \omega\left(\frac{s_\pi}{s_0}\right)\left[1 - \frac{2s_\pi}{s_\tau + 2s_\pi}\right] \tag{59}$$

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