The QCD Strong Coupling from Hadronic Tau Decays

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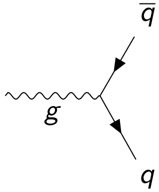
The Strong Coupling α_s

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu,a}(x) + \left[\sum_A \frac{i}{2} \overline{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m\overline{q}^A(x) q^A(x) \right], \tag{1}$$

where $D_\mu = \partial_\mu - ig\frac{\lambda^a}{2} B^a_\mu$

$$\mathcal{L}^{QG-Int}_{QCD}(x) = \sqrt{\pi\boldsymbol{\alpha_s}}\,\overline{q}(x)\lambda\gamma_\mu q(x) G(x) \quad \Rightarrow \tag{2}$$



- Focused on QCD, the theory of the strong interactions between quarks and gluons
- Quarks are the fundamental particles that make up Hadrons (proton, neutron, pion, ...)
- QCD is a non-abelian quantum field theory with symmetry group SU(3)
- Charge property is the colour, the force carriers are the gluons
- QCD Lagrangian, which contains the interaction terms
- The interaction term can be displayed as a Feynman diagram
- The strength of the interaction is given by the strong coupling constant α_s
- Our research focuses on measuring the strong coupling constant

$$\alpha_s(m_\tau^2) \approx 0.33$$
$$\alpha_s(m_Z^2) \approx 0.12 \quad (3)$$

$$m_\tau = 1776.86(12)\ \mathrm{MeV}^1$$
$$m_Z = 91.1876(21)\ \mathrm{GeV}^1 \quad (4)$$

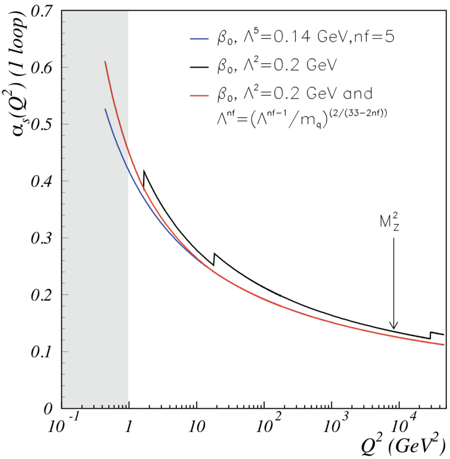
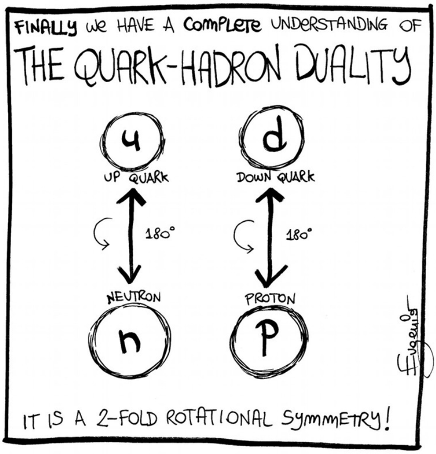


Figure: Taken from Deur, Brodsky, and Teramond, "The QCD Running Coupling", 2016

[1] Tanabashi et al., "Review of Particle Physics", 2018

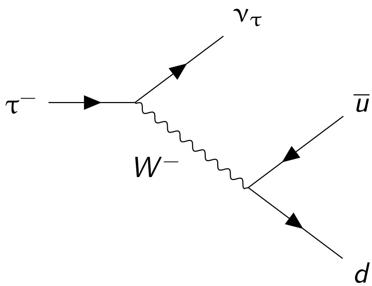
- Strong coupling constant is far from constant, but depends on the energy
- This is called as the "running of the strong coupling"
- E.g. at the for us interesting m_τ^2 scale $\alpha_s(m_\tau^2) \approx 0.33$
- In general the different values for α_s are compared at the m_Z^2 scale ($\alpha_s(m_Z^2) \approx 1.12$)
- On the right we can study the running of the strong coupling
- α_s decreases with increasing energy
- leads to asymptotic freedom: at high energies quarks and gluons interact weakly and can be treated perturbatively
- leads also to confinement: at low energies quarks are bound. An isolated quark has never been measured. They appear in hadrons, two or three quarks

Duality



- As we observe Hadrons, but our theory is based on quarks and gluons we assume Quark-Hadron Duality
- Which is known to exist, but in contrary to the image is not well understood
- In some cases there appear duality violations
- We want to show that Duality Violations do not have to be taken into account under certain circumstances

Hadronic τ decays



Tau decay ratio

$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \tag{5}$$

Name	Symbol	Quark content	Rest mass
Pion	π^-	$\overline{u}d$	$139.570\,61(24)$ MeV
Pion	π^0	$(u\overline{u} - d\overline{d})/\sqrt{2}$	$134.9770(5)$ MeV

- We measure the strong coupling constant from tau decays
- We are interested in the hadronic tau decay
- Here the tau lepton decays into W boson and a tau-neutrino
- the W^- boson then decays into an anti-up and a down quark
- Rarely it can decay into strange quarks, but we will neglect those cases
- The leftover quarks are not to be seen, as they appear as composite Hadrons, like the pions, given down below
- An important quantity is the hadronic tau decay ratio, which is the decay width of taus decaying into hadrons divided by the decay width of taus decaying into electrons
- We will use this quantity to perform our fits, as it is theoretically as experimentally accessible

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Two-Point Function:

$$\Pi^{\mu\nu}_{V/A}(q^2) \equiv i \int d^4 x e^{iqx} \langle 0| T \left\{ J^{\mu}_{V/A}(x) J^{\nu}_{V/A}(0) \right\} |0\rangle$$
$$= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1)}_{V/A}(q^2) + q^\mu q^\nu \Pi^{(0)}_{V/A}(q^2)$$
$$= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi^{(1+0)}_{V/A}(q^2) + q^2 g_{\mu\nu} \Pi^{(0)}(q^2) \quad (6)$$

where the current is given by

$$J^\mu_V = \overline{u}\gamma^\mu d \quad \text{and} \quad J^\mu_A = \overline{u}\gamma^\mu\gamma_5 d$$

and we redefined the correlator as follows

$$\Pi^{(1+0)}(q^2) = \Pi^{(1)}(q^2) + \Pi^{(0)}(q^2).$$

- The two-point function is defined as the vacuum expectation value of the time-ordered product of two currents
- We have given the expression in momentum space
- In our case the currents are non-strange V or A currents, distinguished by a γ^μ or $\gamma^\mu\gamma_5$ correspondingly
- We can lorentz decompose the two-point function, to obtain a scalar function Π
- The superscripts (0) and (1) label the transversal or longitudinal spin
- It is common to rewrite the newly introduced scalar function of the correlator to $\Pi^{(1+0)}$ and $\Pi^{(0)}$
- $\Pi^{(1+0)}(q^2)$ and $q^2\Pi^{(0)}$ are free of kinematic singularities

Inclusive Hadronic Tau Decay Ratio

Tau decay ratio

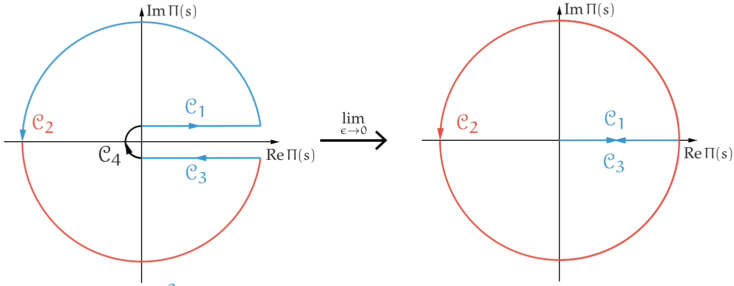
$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \tag{7}$$

Inclusive Hadronic Tau Decay Ratio $s \equiv -q^2$

$$R_\tau = 12\pi \left|V_{ud}\right|^2 S_{EW} \int_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2} \left(1 + 2\frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right] \tag{8}$$

- A central value is the inclusive hadronic tau decay ratio (i.e. all decays containing hadrons)
- The ratio can be calculated by using the optical theorem
- V_{ud} is the Cabbibo matrix element, S_{EW} the electroweak correction
- We have to integrate the two-point function from $0 \to m_\tau^2$
- The two-point function has poles on the positive real axis, on the remaining s plane the two-point function is analytic
- $\Pi^{(0)}$ will be neglected? There is no $J = 0$ vector contribution. The $J = 0$ axial-vector contribution is the pion pole. Which is missing in the experimental data.

Cauchy's Theorem



$$\int_0^{m_\tau^2} \mathrm{Im}\,\Pi(s)\,ds = \oint_{|s|=m_\tau^2} \Pi(s)\,ds$$

Inclusive Hadronic Tau Decay Ratio

$$R_\tau = 6\pi i \oint_{|s|=m_\tau} \frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2 \left[\left(1+2\frac{s}{m_\tau^2}\right)\Pi^{(1+0)}(s) - \left(\frac{2s}{m_\tau^2}\Pi^{(0)}\right)\right] \tag{9}$$

- We can avoid the positive real axis by making use of Cauchy's theorem
- A closed contour integral over an analytic function is zero
- Thus we get a line integral, which will represent the experimental value, equal to a circle contour, with radius of m_τ^2, which represents the theoretical value
- Applied to the inclusive hadronic tau decay ratio we get equation nine, where we also substituted $\Pi^{(1+0)}$
- Imaginary part of two-point function related to experimental accessible spectral function

Adler Function:

$$D(s) \equiv s \frac{\mathrm{d}}{\mathrm{d}s} \Pi(s) \tag{10}$$

$$D^{(1+0)}(s) \equiv -s \frac{\mathrm{d}}{\mathrm{d}s} \Pi^{(1+0)}(s), \qquad D^{(0)}(s) \equiv \frac{s}{m_\tau^2} \frac{\mathrm{d}}{\mathrm{d}s} \left(s\Pi^{(0)}(s) \right) \tag{11}$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{\mathrm{d}x}{x} (1-x)^3 \left[3(1+x) D^{(1+0)}(m_\tau^2 x) + 4 D^{(0)}(m_\tau^2 x) \right] \tag{12}$$

- It is common to rewrite the two-point function in terms of the Adler function.
- In case of vector correlator the derivative (Adler Function) is a physical quantity.
- Physical quantities are renormalisation scale invariant.
- The Adler function has different defintions for the $\Pi^{(1+0)}$ and $\Pi^{(0)}$.
- Our final expression for the inclusive hadronic tau decay ratio then is given in equation 12.

Operator Product Expansion

$$\lim_{x^\mu \to 0} A(x)B(0) = \sum_n C_n(x)\mathcal{O}_n(0) \tag{13}$$

$$\begin{aligned} \Pi_{OPE}(s) &= \sum_{n=0,2,4,\dots} \frac{C_n \langle \Omega | \mathcal{O}_n(x) | \Omega \rangle}{(s)^{n/2}} \\ &= \underbrace{C_0}_{PT} + \underbrace{\sum_{k=1}^{\infty} \frac{C_{2k}(s)}{s^k}}_{NPT} \end{aligned} \tag{14}$$

- The QCD vacuum cannot be solely described perturbatively and we have to take non-perturbative effects into account
- To do so we will describe the two-point function in terms of the operator product expansion
- Here $A(x)$ and $B(0)$ are local operators and $C_n(x)$ is a c-number function and $\mathcal{O}_n(0)$ are higher dimensional operators
- The OPE separates short distances (high energies/ PT) from long distances (NPT)
- Short distances are given by the Wilson coefficients $C_n(x-y)$, whereas the long distances are given by higher order operators $\langle \Omega | \mathcal{O}_n(x) \rangle$.
- The two-point function can then be written has a series of Wilson coefficients multiplied by operators of dimension $0, 2, \dots$.
- The Wilson coefficients can be calculated from Feynman diagrams, but the higher dimensional contributions have to be taken from NPT tools like lattice qcd or from our fits. We will determine values for the dimension six and eight operators
- The dimension zero contribution is the perturbative contribution, whereas the higher dimensional contributions are non-perturbative. We will deal with the PT contributions first before coming back to the NPT ones.

Perturbative Contributions

$$\Pi_V^{(1+0)}(s) = -\frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a(\mu^2)^n \sum_{k=0}^{n+1} c_{n,k} L^k \quad \text{with} \quad L \equiv \log\frac{-s}{\mu^2} \tag{15}$$

$$D_V^{(1+0)}(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a(\mu^2)^n \sum_{k=1}^{n+1} k\, c_{n,k} L^{k-1} \tag{16}$$

- The two-point function can be expanded as a sum over different powers of the strong coupling
- We are working in the chiral limit. Consequently the two-point function of the V and A are identical. Thus from now on we will consider only the vector two-point function Π_V
- We can apply the derivative with respect to s to obtain the expansion of the Adler function
- The coefficients $c_{n,k}$ are referred to as Adler function coefficients and can be calculated from Feynman diagrams
- We will see now that we do not have to calculate every single coefficient, but that the coefficients are internally related due to the RGE

Adler Function Coefficients

Renormalisation Group Equation ($a_s \equiv \alpha(s)/\pi$)

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{d a_s}{d\mu} \frac{\partial}{\partial a_s} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} \right] R(q, a_s, m) = 0 \quad (17)$$

Beta function

$$\beta(a_s) - \mu \frac{d a_s}{d\mu} = \beta_1 a_s^2 + \beta_2 a_s^3 + \dots \quad (18)$$

Adler RGE:

$$\left(2 \frac{\partial}{\partial L} + \beta \frac{\partial}{\partial a_s} \right) D_V^{(1+0)}(s) = 0 \quad (19)$$

$$\begin{aligned} c_{0,0} &= -\frac{5}{3}, \quad c_{0,1} = 1, \\ c_{1,1} &= 1 \\ c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f \\ &\dots \end{aligned} \quad (20)$$

$$\begin{aligned} c_{2,2} &= -\frac{1}{4}\beta_1 c_{1,1}, \\ c_{3,2} &= \frac{1}{4}(-\beta_2 c_{1,1} - 2\beta_1 c_{2,1}), \\ c_{3,3} &= \frac{1}{12}\beta_1 c_{1,1} \\ &\dots \end{aligned} \quad (21)$$

- Let $R(q, a_s, m)$ be a physical quantity, which is renormalisation scale invariant, then the derivative with respect to the renormalisation scale is zero
- From the RGE we can also define the β function, which is responsible for the running of the coupling
- For the expanded Adler function $D_V^{(1+0)}(s)$ the RGE yields equation 19
- If we plug in the expansion of the Adler function in equation 19 we can relate the Adler coefficients
- Thus we only have to calculate Adler coefficients on the left side like $c_{0,0}, c_{1,1}, \dots$ and Adler coefficients of higher indices are related to those
- The full series of the expanded Adler function is renormalisation scale invariant, but the Adler coefficients are only known up to fifth order. Consequently we have to cut off the series, which leaves us with a choice of fixing the invariant scale.

FOPT vs CIPT

Perturbative Contribution ($\omega_J(x) \equiv \frac{1}{2\pi i}(1-x)^3(1+x)$):

$$\delta_{pt} \equiv \oint_{|x|=1} \frac{\mathrm{d}x}{x} \sum_{n=1}^{\infty} a_\mu^n \sum_{k=1}^{n} k\, c_{n,k} \omega_J(x) \log\left(\frac{-m_\tau^2 x}{\mu^2}\right)^{k-1} \tag{22}$$

Fixed-Order Perturbation Theory (FOPT)

$$\mu \equiv m_\tau^2$$

$$\delta_{FOPT}^{(0)} = \sum_{n=1}^{\infty} a(m_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k} J_{k-1} \tag{23}$$

$$J_l \equiv \oint_{|x|=1} \frac{\mathrm{d}x}{x} \omega_J(x) \log^l(-x) \tag{24}$$

Contour-Improved Perturbation Theory (CIPT)

$$\mu \equiv -m_\tau^2 x \tag{25}$$

$$\delta_{CIPT}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2) \tag{26}$$

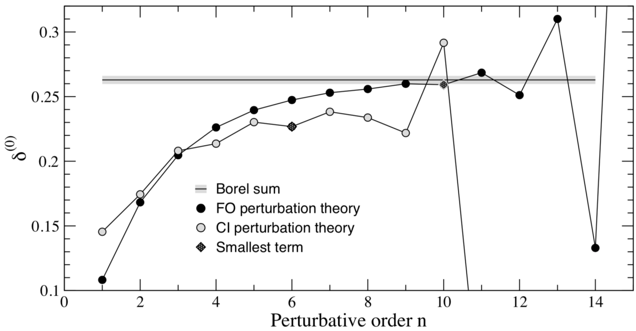
$$J_n^a(m_\tau^2) \equiv \oint_{|x|=1} \frac{\mathrm{d}x}{x} \omega_J(x) a^n(-m_\tau^2 x) \tag{27}$$

- The general perturbative contribution δ_{pt} is defined in equation 22, where we plugged in the expanded Adler function in to the tau decay ratio and factorised $12\pi^2$
- Having the freedom to fix μ leads to two different treatments of the PT contributions
- FOPT where we fix $\mu \equiv m_\tau^2$
- This leads to a constant a_μ, so we do not have to run the strong coupling. We are left with the integration of the logarithms $\log(-x)$
- On the other hand CIPT fixed $\mu \equiv -m_\tau^2 x$, which sums up the logarithms, but leaves us with a running coupling
- Both approaches lead to different results

FOPT vs CIPT

Perturbative FOPT and CIPT contributions ($\alpha(m_\tau^2) = 0.34$):

$$\begin{aligned} & \qquad\quad \alpha_s^2 \qquad \alpha_s^2 \qquad \alpha_s^3 \qquad \alpha_s^4 \qquad\quad \alpha_s^5 \\ \delta^{(0)}_{FOPT} &= 0.1082 + 0.0609 + 0.0334 + 0.0174(+0.0088) = 0.2200(0.2288) \qquad (28) \\ \delta^{(0)}_{CIPT} &= 0.1479 + 0.0297 + 0.0122 + 0.0086(+0.0038) = 0.1984(0.2021) \qquad (29) \end{aligned}$$



Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008

- E.g. here we display the FOPT and CIPT contribution up to fifth order.
- From the table we can conclude that CIPT converges faster, but has a smaller contribution as FOPT, which leads to larger values of α_s
- The graph below has been taken from a paper of Beneke and Jamin who invested the topic
- here we see as the black dots the FOPT contribution, as the gray dots the CIPT contribution and as a straight line the Borel sum to which we will come in a minute to which we will come in a minute to which is used to sum asymptotic series like in this case
- Note that FOPT converges in line with the Borel sum, but CIPT does not
- We will make the same observation while performing our fits

Borel Summation

Borel integral:

$$A \equiv \int_0^\infty \mathrm{d}t e^{-t} \sum_{n=0}^{\infty} \frac{a_k}{n!} t^n, \tag{30}$$

Borel transform:

$$B[A](t) = \sum_{n=0}^{\infty} \frac{a_k}{n!} t^n. \tag{31}$$

$$\frac{12\pi^2}{N_c} D_V^{1+0}(s) \equiv 1 + \widehat{D}(s) \equiv 1 + \sum_{n=0}^{\infty} r_n \alpha_s(\sqrt{(s)})^{n+1}. \tag{32}$$

- The Borel summation is a summation method for divergent asymptotic series and should give us the best possible sum
- It consists of the Borel integral and the Borel transform, which we apply to the expansion of the Adler function
- We will follow the notation of Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008, which redefined the Adler function expansion as $1 + D(s)$

Borel Model

$$B[\widehat{D}](u) = B[\widehat{D}_1^{UV}](u) + B[\widehat{D}_2^{IR}](u) + B[\widehat{D}_3^{IR}](u) + d_0^{PO} + d_1^{PO} u, \quad (33)$$

$$B[\widehat{D}_p^{IR}](u) \equiv \frac{d_p^{IR}}{(p-u)^{1+\widetilde{\gamma}}} \left[1 + \widetilde{b}_1(p-u) + \widetilde{b}_2(p-u)^2 + \dots\right] \quad (34)$$

$$B[\widehat{D}_p^{UV}](u) \equiv \frac{d_p^{UV}}{(p+u)^{1+\overline{\gamma}}} \left[1 + \overline{b}_1(p+u) + \overline{b}_2(p+u)^2\right], \quad (35)$$

Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008

- Beneke and Jamin created a created the Borel model
- The model depends on five parameters
- The Borel model should reproduce the exactly known result up to order fourth order of the Adler function
- Consequently the five parameters are obtained by matching the result
- This model is then used to sum the Adler function expansion

Borel model:

$$\Pi^{OPE}_{NPT,V/A}(s) = \sum_{n=2,4,\ldots} \frac{C_n\langle\Omega|\mathcal{O}_n(x)|\Omega\rangle}{(s)^{n/2}} \tag{36}$$

$$\begin{array}{ll}
\text{Dimension 0:} & \mathbb{1} \\
\text{Dimension 4:} & :m_i\overline{q}q: \\
 & :G_a^{\mu\nu}(x)G^a_{\mu\nu}(x): \\
\text{Dimension 6:} & :\overline{q}\Gamma q\overline{q}\Gamma q: \\
 & :\overline{q}\Gamma\frac{\lambda^a}{2}q_\beta(x)\overline{q}\Gamma\frac{\lambda^a}{2}q: \\
 & :m_i\overline{q}\frac{\lambda^a}{2}\sigma_{\mu\nu}qG_a^{\mu\nu}: \\
 & :f_{abc}G_a^{\mu\nu}G_b^{\nu\delta}G_c^{\delta\mu}: \\
\ldots & \ldots
\end{array} \tag{37}$$

- Next to the PT contribution we have to implement the NPT contributions from the OPE
- We can see that the OPE series is suppressed by powers of s thus we can approximate the series by a cutoff
- The lowest dimensional operators are given in equation 37
- In our analysis we will neglect the dimension two contributions as we work in the chiral limit and their contributions are proportional to the quark masses

Dimension Four Contributions:

$$D_{ij}^{(1+0)}(s)\Big|_{D=4} = \frac{1}{s^2}\sum_n \Omega^{(1+0)}(s/\mu^2)a^n, \tag{38}$$

$$\begin{aligned}\Omega_n^{(1+0)}(s/\mu^2) \;=\; & \frac{1}{6}\langle aGG\rangle p_n^{(1+0)}(s/\mu^2) + \sum_k m_k\langle\overline{q}_k q_k\rangle r_n^{(1+0)}(s/\mu^2)\\ & + 2\langle m_i\overline{q}_i q_i + m_j\overline{q}_j q_j\rangle q_n^{(1+0)}(s/\mu^2) \pm \frac{8}{3}\langle m_j\overline{q}_i q_i + m_i\overline{q}_j q_j\rangle t_n^{(1+0)}\\ & - \frac{3}{\pi^2}(m_i^4 + m_j^4)h_n^{(1+0)}(s/\mu^2) \mp \frac{5}{\pi^2}m_i m_j(m_i^2 + m_j^2)k_n^{(1+0)}(s/\mu^2)\\ & + \frac{3}{\pi^2}m_i^2 m_j^2 g_n^{(1+0)}(s/\mu^2) + \sum_k m_k^4 j_n^{(1+0)}(s/\mu^2) + 2\sum_{k\neq l} m_k^2 m_l^2 u_n^{(1+0)}(s/\mu^2).\end{aligned} \tag{39}$$

Pich and Prades, "Strange quark mass determination from Cabibbo suppressed tau decays", 1999

- Here is the dimension four OPE contribution, which has been formalised in an article by Pich
- A lot of coefficient, but what you should take from this slide is the Gluon condensate, which we will fit

Dimension Six and Eight Corrections

Higher Dimensional Contributions:

$$
\begin{aligned}
D^{(1+0)}_{ij,V/A}\Big|_{D=8} &= 4\frac{\rho^{(8)}_{V/A}}{s^4} \\
D^{(1+0)}_{ij,V/A}\Big|_{D=10} &= 5\frac{\rho^{(10)}_{V/A}}{s^5} \\
D^{(1+0)}_{ij,V/A}\Big|_{D=12} &= 6\frac{\rho^{(12)}_{V/A}}{s^6}
\end{aligned}
\tag{40}
$$

- Higher order OPE contributions are problematic to parametrise
- Starting from dimension six we will apply the simplest approach possible to parametrise the higher dimensional OPE contributions.
- We will define a constant ρ for every dimension, which represents the contribution
- needs backup slide! why simplest approach

Duality

- Duality:

$$\Pi(s) \to \Pi_{OPE}(s) \tag{41}$$

- Duality Violations (DV):

$$\Delta(s) \equiv \Pi(s) - \Pi_{OPE}(s) \tag{42}$$

- DV Model:

$$\rho^{DV}_{V/A}(s) = e^{-(\delta_{V/A} + \gamma_{V/A} s)} \sin(\alpha_{V/A} + \beta_{V/A} s) \tag{43}$$

Boito et al., "A new determination of α_s from hadronic τ decays", 2011

- DV Contribution:

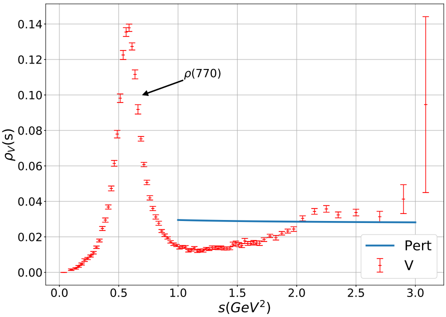
$$D_\omega(s_0) = -12\pi^2 S_{EW} |V_{ud}|^2 \int_{s_0}^{\infty} \frac{ds}{s_0} \omega(s) \rho^{DV}_{V/A} \tag{44}$$

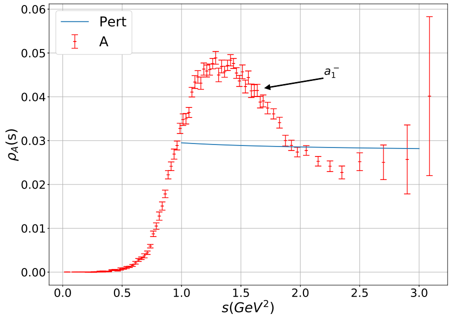
- We can represent duality as $\Pi(s) \to \Pi_{OPE}(s)$
- The difference $\Delta(s)$ defines the duality violating contribution to Π
- DV can be parametrised via a model
- The model has four parameters for the vector and four parameters for the axial channel
- Too many parameters: e.g. α_s, ρ_6, ρ_8 three parameters vs eight!
- We will further research the necessity of including DV

ALEPH Data

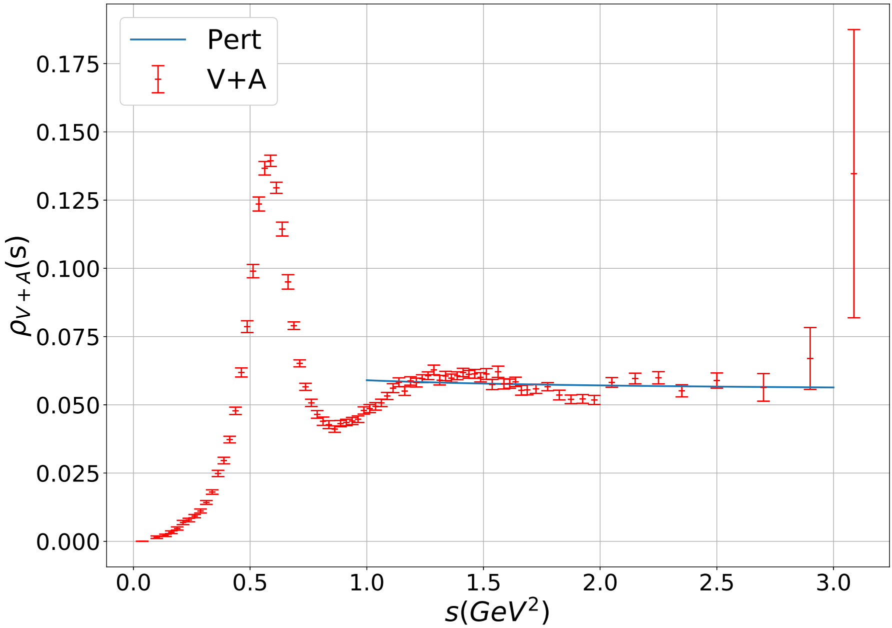
- Experimental Spectral Moment:

$$\rho^{(1)}_{V/A}(s) = \frac{m_\tau^2}{12\pi^2 |V_{ud}|^2 S_{EW}} \frac{\mathcal{B}_{V/A}}{\mathcal{B}_e} \frac{dN_{V/A}}{N_{V/A}\, ds} \frac{1}{\omega_\tau} \quad (45)$$





- The data we use is given by the ALEPH group
- ALEPH was a particle detector on the Large Electron-Positron collider in the nineties
- The data is given as a the normalised invariant mass squared distribution $dN/N/ds$ for each channel V, A and $V+A$
- In the two graphs we see the contribution of the V channel (left) and the A channel (right)
- In the vector channel we see the $\rho(770)$ resonance
- In the axial channel we see the a_1^- resonance
- We also plotted the Perturbative contribution, which cannot reproduce the experimental data, especially for lower energies



- Here we see the experimental spectral function of the $V + A$ channel
- Note that for higher energies the perturbative contribution matches the spectral function far better
- Also note that we still see a wavy behaviour of the spectral function in the data, which is connected to Duality Violations
- We assume that in the $V + A$ channel DV are sufficiently suppressed to avoid modelling their contributions

Integral Moments

- Integral Moment:

$$I^{\omega}_{V/A}(s_0) \equiv 12\pi^2 \int_0^{s_0} \frac{\mathrm{d}s}{s_0}\,\omega\left(\frac{s}{s_0}\right)\rho^{exp}_{V/A}(s) = \frac{3\pi}{i}\oint_{|s|=s_0} \frac{\mathrm{d}s}{s_0}\,\omega\left(\frac{s}{s_0}\right) D^{th}(s) \quad (46)$$

- Experimental Moment:

$$I^{\omega}_{exp,V/A}(s_0) = \frac{m_\tau^2}{\mathcal{B}_e s_0}\sum_{i=1}^{N(s_0)} \frac{\omega(s_i/s_0)}{\omega_\tau(s_i/s_0)}\,\mathrm{sfm2}_{V/A}(s_i) \quad (47)$$

- We defined the so-called integral moments, which we will use to define our chi-squared function
- the experimental moment is then a sum given by equation 47
- sfm2 is given in the binned data of the aleph group
- $\mathrm{sfm2}_{V/A} \equiv B_{V/A}\frac{\mathrm{d}N_{V/A}}{N_{V/A}\,\mathrm{d}s}$
- $\rho_V^{(0)}$ does not exist
- $\rho_A^{(0)}$ is the pion pole, which is not included in the data?

Weights

- Weight function:

$$\omega(x) \equiv \sum_i a_i x^i \tag{48}$$

- E.g.
  - double pinched
  - no monomial
  - D6 and D8

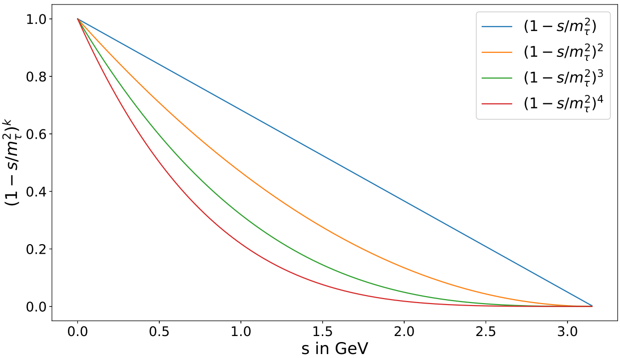
$$\begin{aligned} \omega_\tau &\equiv (1-x)^2(1+2x) \\ &= 1 - 3x^2 + 2x^3 \end{aligned} \tag{49}$$

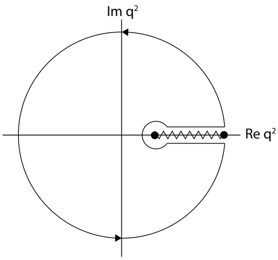
- The weight is an analytic function
- Thus we can define it as an arbitrary polynomial
- As an example we can take the natural appearing kinetic weight ω_τ
- It is double pinched, does not contain a monomial and as we will see has active D6 and D8 contributions

Pinched Weights

- Pinched weight:

$$\omega(x) = (1-x)^k \tag{50}$$





- The theoretical two-point function contains DV close to the positive real axis
- To suppress DV contributions we introduce pinched weights
- The order of the pinching is given by the exponent k in equation 50
- The higher the pinching the fewer the contributions close to the positive real axis. This can be seen by plotting the weights. Blue is single pinched and decreases linear. Higher pinched weights decrease faster.
- Thus implementing a sufficient pinching should avoid DV

Weighting OPE Contributions

$$\oint_C x^k \, \mathrm{d}x = i \int_0^{2\pi} \left(e^{i\theta}\right)^{k+1} \mathrm{d}\theta = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise} \end{cases} . \tag{51}$$

$$R(x)\Big|_{D=0,2,4,\ldots} = \oint_{|x|=1} \mathrm{d}x \, x^{k-D/2} C^{(D)} \tag{52}$$

Active Dimensions:

$$D = 2(k+1) \tag{53}$$

monomial:	x^0	x^1	x^2	x^3	x^5	x^6	x^7
dimension:	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

- The weights are also used to “activate” different OPE contributions
- If we regard a closed contour integral over a x^k, we can see that the contour integral is different than zero only if the exponent is equal to -1
- Similarly regarding the integral moment, we see that only certain dimensions contribute, while other dimensions are strongly suppressed
- Thus the active dimensions are given by $D = 2(k+1)$
- In the table we can
- (The OPE contributions have logarithmic x dependence. In general we approximated them as constants.)

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Chi-Squared

- Chi-Squared function:

$$\chi^2 = (I_i^{exp} - I_i^{th}(\vec{\alpha}))\, C_{ij}^{-1} (I_j^{exp} - I_j^{th}(\vec{\alpha})) \tag{54}$$

- Covariance Matrix:

$$C_{ij} = \mathrm{cov}(I_i^{exp}, I_j^{exp}) \tag{55}$$

- Chi-Squared per Degrees of Freedom:

$$\frac{\chi^2}{dof} \approx 1 \tag{56}$$

- The chi-squared function is constructed from the theoretical and experimental moments
- The indices i and j represent the dependency of the moments on the chosen weight and s_0
- The fits are highly correlated.
- The correlation matrix is given with the data.
- A good fit is characterised by a $\chi^2/dof \approx 1$
- As we have to deal with missing correlations, we will also interpret fits with a χ^2/dof smaller than 1 as good

Parameters and Momenta

- 3 Moments
- max three parameters
- e.g. $\alpha_s, \rho^{(6)}_{V/A}, \rho^{(8)}_{V/A}$ (fully determined)

#	3 Moments	
1	s_1	ω
2	s_2	ω
3	s_3	ω

#	9 Moments	
1	s_1	ω
2	s_2	ω
$\vdots$	$\vdots$	$\vdots$
9	s_9	ω

- 9 Moments
- max nine parameters

- Our “data points” are the integral momenta. We will construct multiple integral momenta for every fit by varying the number of s_0.
- In principle we could also vary the weights, but have not done this
- E.g. if we want to fit three parameters as

Strategy

- Extract α_s
- Probe Duality Violations
- FOPT vs CIPT

- To extract α_s at the m_τ^2 scale, we perform fits with multiple s_0 moments.
- We check isolated weights for stability for different s_0 moments
- Check stability for different weights and pinchings. If we obtain similar weights DV should not be present.
- Perform additional fits with the BS. If parameters are similar to FOPT, then FOPT should be the preferred framework.

Chosen Weights

	Symbol	Term	Expansion	OPE Contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	$D6$, $D8$
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	$D6$, $D8$, $D10$
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	$D6$, $D8$, $D10$, $D12$
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	$D6$
	ω_{M3}	$1-x^3$	$1-x^3$	$D8$
	ω_{M4}	$1-x^4$	$1-x^4$	$D10$
Pinched $+x$	$\omega_{1,0}$	$(1-x)$	$1-x$	$D4$
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	$D4$, $D6$
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	$D4$, $D6$, $D8$
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	$D4$, $D6$, $D8$, $D10$

- To apply the strategy we have to choose several weights
- We selected three categories:
  - Pinched weights without a monomial term x, these are double, triple or quadruple pinched,
  - Monomial weights, these weights are single pinched and do not contain a monomial term x
  - "Pichs optimal" weights, these weights are single up to quadruple pinched and contain a term monomial in x
- We cannot apply FOPT to weights with a monomial term $x \Rightarrow$ BS

Kinematic Weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.200	7	0.3274(42)	-0.82(21)	-1.08(41)	0.21
FOPT	2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

- Starting with the kinematic weight
- appears naturally in the inclusive hadronic tau decay ratio
- is double pinched $\Rightarrow$ should suppress DV sufficiently
- Has two active OPE dimensions, namely dimension six and eight
- Leaves us with three fitting parameters: α_s, $\rho^{(6)}$ and $\rho^{(8)}$
- s_{min} is the smallest invariant mass squared value that is included in the fit
- One has to imagine that the data is binned and that we construct our moments starting from the highest available energy
- We then perform fits with an increasing number of s_0s, including more and more bins and thus include lower and lower energies
- beginning from $2.2\,\mathrm{GeV}^2$ the fits get problematic due to the appearing resonances
- Lets regard the two first lines of the FOPT table, we also applied the BS for the best fit
- Regarding the χ^2/dof we se a jump in its value, which we noted for every weight. If we go to too low energies the fits become unreliable, which is also notable from the deviating values for the parameters.
- We decided to take the fits above, but closest to this threshold to be the best fit
- For the fits above the threshold we note a great stability between the values obtained for α_s

Cubic Weight: $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

- The cubic weight is triple pinched
- Has three active OPE contributions, $D6$, $D8$, and $D10$
- Consequently we fitted four paremters
- Shows very similar behaviour to the kinematice weight (threshold, low χ^2/dof)
- Has also very stable values for α_s

Quartic Weight: $\omega_{quartic}(x) \equiv (1-x)^4(1+4x)$

$$\begin{aligned} \alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28), \\ \rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77). \end{aligned} \tag{57}$$

- Too many parameters. Only one fit converged

Comparison

PT	weight	$\#s_0$'s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
FOPT	$(1-x)^2(1+2x)$	7	0.3308(44)	2.1*	-0.72(20)	-0.85(38)	0.19
	$(1-x)^3(1+2x)$	8	0.3302(40)	2.1*	-0.52(11)	-0.58(22)	0.43
	$1-x^2$	7	0.3248(52)	2.1*	-0.77(22)	0*	0.38
	$1-x^3$	7	0.3214(49)	2.1*	0*	-1.01(39)	0.41
BS	$(1-x)^2(1+2x)$	7	0.3274(42)	2.1*	-0.82(21)	-1.08(41)	0.21
	$1-x$	7	0.3246(52)	-0.2262(59)	0*	0*	0.38
	$(1-x)^2$	7	0.3270(54)	-0.0254(61)	-0.77(21)	0*	0.74
	$(1-x)^3$	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46

- Here we gathered the "best" fits, which are fits with the highest $\#s_0$'s, but being above the threshold of unstable fits
- We left also out the problematic fourth pinched weights, which include too high dimensions of the OPE
- We can clearly see that all the values obtained for α_s are very similar
- values obtained for $\rho^{(6)}$ and $\rho^{(8)}$ are within error boundaries
- Even though we used different pinchings, aka different amounts of suppression for DV
- Note that even a single pinched weights like in the second row of the BS we achieve comparable results
- Comparing the parameters obtained from FOPT, we also see that they are very similar to parameters obtained from the BS

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Conclusions

- $\alpha_s(m_\tau^2) = 0.3261 \pm 0.0050$
- $\rho^{(6)} = -0.68 \pm 0.2$
- $\rho^{(8)} = -0.80 \pm 0.38$

- DV not present if using single pinched weights in the V+A channel
- FOPT more valid than CIPT
- $\alpha_s(m_Z^2) = 0.11940(60)$

Questions

Constants

Quantity	Value
V_{ud}	0.9742 ± 0.00021
S_{EW}	1.0198 ± 0.0006
B_e	17.818 ± 0.023
m_τ	$1.776\,86(12000)\ \mathrm{MeV}$
$\langle aGG \rangle_I$	$0.012\ \mathrm{GeV}^2$
$\langle q_{u/d} q_{u/d} \rangle (m_\tau)$	$-272(15)\ \mathrm{MeV}$
$ss/\langle qq \rangle$	$0.8\ \pm 0.3$

DV-model

$$-\frac{1}{2\pi i}\oint_{|s|=s_0}\frac{\mathrm{d}s}{s_0}\omega(s/s_0)\Delta_{V/A}(s) = -\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s_0}\omega(s/s_0)\frac{1}{\pi}\,\mathrm{Im}\,\Delta_{V/A}(s) \quad (58)$$

Pion Pole

$$R_{\tau,A}^{\omega}(s_0,\pi) = 24\pi^2 |V_{ud}|^2 \, S_{EW} \frac{f_\pi^2}{s_0} \omega\left(\frac{s_\pi}{s_0}\right) \left[1 - \frac{2s_\pi}{s_\tau + 2s_\pi}\right] \tag{59}$$

$\omega_{M2}(x) \equiv 1 - x^2$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	χ^2/dof
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

$\omega_{M3}(x) \equiv 1 - x^3$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	χ^2/dof
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	χ^2/dof
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

$\omega_{1,0} \equiv (1-x)$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	χ^2/dof
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	0.38
	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24

$\omega_{2,0} \equiv (1-x)^2$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	χ^2/dof
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

$\omega_{3,0} \equiv (1-x)^3$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

$\omega_{4,0} \equiv (1-x)^4$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$aGGInv$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

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