

The QCD Strong Coupling from Hadronic Tau Decays

A PhD Defense

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$$\begin{aligned}\mathcal{L}_{QCD}(x) = & -\frac{1}{4}G_{\mu\nu}^a(x)G^{\mu\nu,a}(x) \\ & + \left[\sum_A \frac{i}{2}\bar{q}^A(x)\gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m\bar{q}^A(x)q^A(x) \right],\end{aligned}\tag{1}$$

with $D_\mu = \partial_\mu - ig\frac{\lambda^a}{2}B_\mu^a$

say hello now

The Strong Coupling α_s

$$\mathcal{L}_{QCD}(x) = -\frac{1}{4} G_{\mu\nu}^a(x) G^{\mu\nu,a}(x) + \left[\sum_A \frac{i}{2} \bar{q}^A(x) \gamma^\mu \overleftrightarrow{D}_\mu q^A(x) - m \bar{q}^A(x) q^A(x) \right], \quad (1)$$

with $D_\mu = \partial_\mu - ig \frac{\lambda^a}{2} B_\mu^a$

$$\mathcal{L}_{QCD}^{QG-Int}(x) = \sqrt{\pi\alpha_s} \bar{q}(x) \lambda \gamma_\mu q(x) G(x) \quad \Rightarrow \quad \text{Feynman diagram: a wavy line labeled } g \text{ connects to a vertex from which two fermion lines emerge, labeled } \bar{q} \text{ and } q. \quad (2)$$

say hello now

The Running of the Strong Coupling

$$\alpha_s(m_\tau^2) \approx 0.33$$

$$\alpha_s(m_Z^2) \approx 1.12 \quad (3)$$

$$m_\tau = 1776.86(12) \text{ MeV}^1$$

$$m_Z = 91.1876(21) \text{ GeV}^1 \quad (4)$$

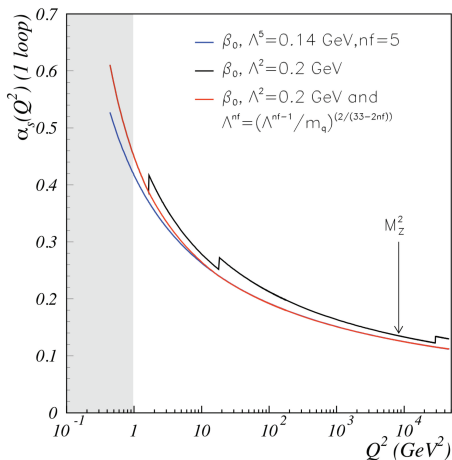
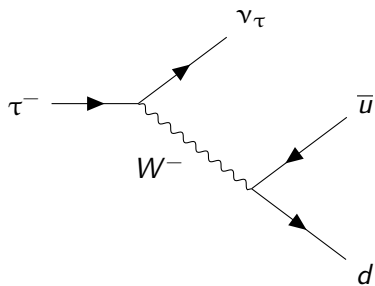
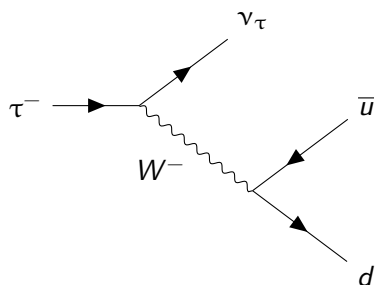


Figure: Taken from Deur, Brodsky, and Teramond, "The QCD Running Coupling", 2016

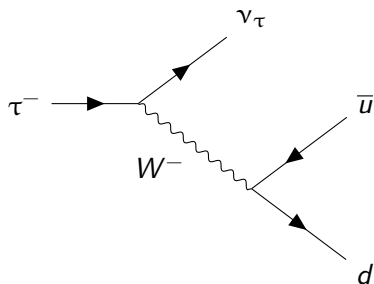
¹Tanabashi et al., "Review of Particle Physics", 2018

Hadronic τ decays





$$R_\tau = \frac{\Gamma[\tau^- \rightarrow \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e]} \quad (5)$$



$$R_\tau = \frac{\Gamma[\tau^- \rightarrow \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e]} \tag{5}$$

Name	Symbol	Quark content	Rest mass
Pion	π^-	$\bar{u}d$	139.570 61(24) MeV
Pion	π^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	134.9770(5) MeV

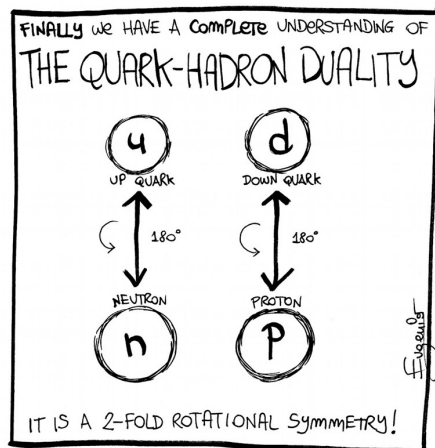


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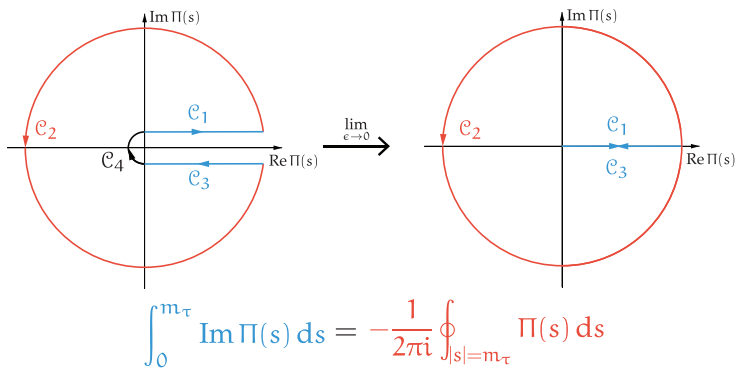
Two-Point Function:

$$\begin{aligned}\Pi_{V/A}^{\mu\nu}(q^2) &\equiv i \int d^4x e^{iqx} \langle 0 | T \left\{ J_{V/A}^\mu(x) J_{V/A}^\nu(0) \right\} | 0 \rangle \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{V/A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{V/A}^{(0)}(q^2) \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi_{V/A}^{(1+0)}(q^2) + q^2 g_{\mu\nu} \Pi^{(0)}(q^2)\end{aligned}\tag{6}$$

$$J_V^\mu = \bar{u} \gamma^\mu d \quad \text{and} \quad J_A^\mu = \bar{u} \gamma^\mu \gamma_5 d$$

- Two-point function
- $J_{V/A}^\mu$ is the non-strange V or A current
- superscripts (0) and (1) label spin
- Lorentz decomposed
- $\Pi^{(1+0)}(q^2) q^2 \Pi^{(0)}(q^2)$ free of kinematic singularities

Cauchy's Theorem



- Experimental data only accessible on positive real axis
- Imaginary part of two-point function related to experimental accessible spectral function
- integrate from 0 to m_τ^2 to reproduce $R_\tau(m_\tau^2)$
- Theoretically the positive real axis is not accessible
- Two-point function has poles on positive real axis
- \Rightarrow Cauchy's theorem
- Identify

Spectral Function

$$\rho^{(1+0)}(s) = \frac{1}{\pi} \text{Im} \Pi^{(1+0)}(s) \quad (7)$$

Spectral Moment

$$I_{V/A}^{\omega}(s_0) \equiv \frac{1}{s_0} \int_0^{s_0} ds \, \omega \left(\frac{s}{s_0} \right) \rho(s) = \frac{-1}{2\pi i s_0} \oint_{|s|=s_0} ds \, \omega \left(\frac{s}{s_0} \right) \Pi(s) \quad (8)$$

- Spectral function equal to the imaginary part of the correlator
- Spectral function is given by the experiment
- Experiment only valid to certain energy s_0
- \Rightarrow *Finite Energy* Sum Rule
- Define spectral moment
- Introduce weights
- Weights ω have to be analytic

Adler Function:

$$D(s) \equiv s \frac{d}{ds} \Pi(s) \quad (9)$$

$$\begin{aligned} R_{\tau,V/A}^{\omega}(s_0) &\equiv \frac{12\pi^2}{s_0} \int_0^{s_0} ds \, \omega\left(\frac{s}{s_0}\right) \rho(s) \\ &= \frac{6\pi i}{s_0} \oint_{|s|=s_0} ds \, \omega\left(\frac{s}{s_0}\right) \Pi(s) \\ &= -3\pi i \oint_{|x|=1} \frac{dx}{x} \omega_D(x) D(xs_0), \end{aligned} \quad (10)$$

where $x \equiv \frac{s}{s_0}$ and $\omega_D \equiv 2 \int_x^1 d\omega x$

- Adler Function for convenience
- In case of vector correlator the derivative (Adler Function) is a physical quantity
- Physical quantities are renormalisation scale invariant
- For the $(1+0)$ and (0) we use a different definition

$$\Pi_{OPE}(q^2) = -\frac{1}{3q^2} \sum_n \langle \Omega | \mathcal{O}_n(0) | \Omega \rangle \int d^4x e^{iqx} C_n(x) \quad (11)$$

$$\begin{aligned} \Pi_{OPE,V/A}(s) &= \sum_{D=0,2,4,\dots} \frac{C^{(D)} \langle \Omega | \mathcal{O}^{(D)}(x) | \Omega \rangle}{(s)^{D/2}} \\ &= \underbrace{C_0}_{PT} + \underbrace{\sum_{k=1}^{\infty} \frac{C_{2k}(s)}{s^k}}_{NPT} \end{aligned} \quad (12)$$

$$\Pi_V^{(1+0)}(s) = -\frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_{\mu}^n \sum_{k=0}^{n+1} c_{n,k} L^k \quad \text{with} \quad L \equiv \log \frac{-s}{\mu^2} \quad (13)$$

$$D_V^{(1+0)} = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_{\mu}^n \sum_{k=1}^{n+1} k c_{n,k} L^{k-1} \quad (14)$$

Renormalisation Group Equation

$$\mu \frac{d}{d\mu} R(q, a_s, m) = \left[\mu \frac{\partial}{\partial \mu} + \mu \frac{da_s}{d\mu} \frac{\partial}{\partial a_s} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} \right] R(q, a_s, m) = 0 \tag{15}$$

$$\left(2 \frac{\partial}{\partial L} + \beta \frac{\partial}{\partial a_s} \right) D_V^{(1+0)} = 0 \tag{16}$$

$$\begin{aligned} c_{0,0} &= -\frac{5}{3}, & c_{0,1} &= 1, \\ c_{1,1} &= 1 \\ c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right)N_f \\ &\dots \end{aligned} \tag{17}$$

$$\begin{aligned} c_{2,2} &= -\frac{1}{4}\beta_1 c_{1,1}, \\ c_{3,2} &= \frac{1}{4}(-\beta_2 c_{1,1} - 2\beta_1 c_{2,1}), \\ c_{3,3} &= \frac{1}{12}\beta_1 c_{1,1} \\ &\dots \end{aligned} \tag{18}$$

- Beta function definition missing

$$\delta_{pt} = \sum_{n=1}^{\infty} a_{\mu}^n \sum_{k=1}^n k c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log \left(\frac{-m_{\tau}^2 x}{\mu^2} \right)^{k-1} \quad (19)$$

Fixed-Order Perturbation Theory
(FOPT)

$$\mu \equiv m_{\tau}^2$$

Contour-Improved Perturbation Theory
(CIPT)

$$\mu \equiv -m_{\tau}^2 x \quad (20)$$

$$\delta_{FOPT}^{(0)} = \sum_{n=1}^{\infty} a(m_{\tau}^2)^n \sum_{k=1}^n k c_{n,k} J_{k-1} \quad (21)$$

$$J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \log^l(-x) \quad (22)$$

$$\delta_{CIPT}^{(0)} = \sum_{n=1}^{\infty} c_{n,1} J_n^a(m_\tau^2) \quad (23)$$

$$J_n^a(m_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) a^n(-m_\tau^2 x) \quad (24)$$

$$\delta_{FOPT}^{(0)} = 0.1082 + 0.0609 + 0.0334 + 0.0174(+0.0088) = 0.2200(0.2288) \quad (25)$$

$$\delta_{CIPT}^{(0)} = 0.1479 + 0.0297 + 0.0122 + 0.0086(+0.0038) = 0.1984(0.2021) \quad (26)$$

Borel integral

$$A \equiv \int_0^\infty dt e^{-t} \sum_{n=0}^\infty \frac{a_n}{n!} t^n, \quad (27)$$

Borel transform

$$B[A](t) = \sum_{n=0}^\infty \frac{a_n}{n!} t^n. \quad (28)$$

$$\frac{12\pi^2}{N_c} D_V^{1+0}(s) \equiv 1 + \hat{D}(s) \equiv 1 + \sum_{n=0}^\infty r_n \alpha_s(\sqrt{s})^{n+1}. \quad (29)$$

$$B[\widehat{D}](u) = B[\widehat{D}_1^{UV}](u) + B[\widehat{D}_2^{IR}](u) + B[\widehat{D}_3^{IR}](u) + d_0^{PO} + d_1^{PO} u, \quad (30)$$

$$B[\widehat{D}_p^{IR}](u) \equiv \frac{d_p^{IR}}{(p-u)^{1+\widetilde{\gamma}}} \left[1 + \widetilde{b}_1(p-u) + \widetilde{b}_2(p-u)^2 + \dots \right] \quad (31)$$

$$B[\widehat{D}_p^{UV}](u) \equiv \frac{d_p^{UV}}{(p+u)^{1+\overline{\gamma}}} \left[1 + \overline{b}_1(p+u) + \overline{b}_2(p+u)^2 \right], \quad (32)$$

Beneke and Jamin, “ α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory”, 2008

$$\Pi_{NPT,V/A}^{OPE}(s) = \sum_{D=2,4,\dots} \frac{C^{(D)} \langle \Omega | \mathcal{O}^{(D)}(x) | \Omega \rangle}{(-q^2)^{D/2}} \quad (33)$$

Dimension 0: $\mathbb{1}$

Dimension 4: $: m; q q :$

$: G_a^{\mu\nu}(x) G_{\mu\nu}^a(x) :$

Dimension 6: $: q \Gamma q q \Gamma q :$

$: q \Gamma \frac{\lambda^a}{2} q_\beta(x) q \Gamma \frac{\lambda^a}{2} q :$

$: m; q \frac{\lambda^a}{2} \sigma_{\mu\nu} q G_a^{\mu\nu} :$

$: f_{abc} G_a^{\mu\nu} G_b^{\nu\delta} G_c^{\delta\mu} :$

...

...

(34)

$$D_{ij}^{(1+0)}(s) \Big|_{D=4} = \frac{1}{s^2} \sum_n \Omega^{(1+0)}(s/\mu^2) a^n, \quad (35)$$

$$\begin{aligned} \Omega_n^{(1+0)}(s/\mu^2) = & \frac{1}{6} \langle aGG \rangle p_n^{(1+0)}(s/\mu^2) + \sum_k m_k \langle \bar{q}_k q_k \rangle r_n^{(1+0)}(s/\mu^2) \\ & + 2 \langle m_i \bar{q}_i q_i + m_j \bar{q}_j q_j \rangle q_n^{(1+0)}(s/\mu^2) \pm \frac{8}{3} \langle m_j \bar{q}_i q_i + m_i \bar{q}_j q_j \rangle t_n^{(1+0)} \\ & - \frac{3}{\pi^2} (m_i^4 + m_j^4) h_n^{(1+0)}(s/\mu^2) \mp \frac{5}{\pi^2} m_i m_j (m_i^2 + m_j^2) k_n^{(1+0)}(s/\mu^2) \\ & + \frac{3}{\pi^2} m_i^2 m_j^2 g_n^{(1+0)}(s/\mu^2) + \sum_k m_{kln}^4 (s/\mu^2) + 2 \sum_{k \neq l} m_k^2 m_l^2 u_n^{(1+0)}(s/\mu^2). \end{aligned} \quad (36)$$

$$\begin{aligned} D_{ij,V/A}^{(1+0)} \Big|_{D=8} &= 4 \frac{\rho_{V/A}^{(8)}}{s^4} \\ D_{ij,V/A}^{(1+0)} \Big|_{D=10} &= 5 \frac{\rho_{V/A}^{(10)}}{s^5} \\ D_{ij,V/A}^{(1+0)} \Big|_{D=12} &= 6 \frac{\rho_{V/A}^{(12)}}{s^6} \end{aligned} \tag{37}$$

Duality Violations (DV):

$$\Delta(s) \equiv \Pi(s) - \Pi_{OPE}(s) \quad (38)$$

DV Model:

$$\rho_{V/A}^{DV}(s) = e^{-(\delta_{V/A} + \gamma_{V/A}s)} \sin(\alpha_{V/A} + \beta_{V/A}s) \quad (39)$$

DV Contribution:

$$D_\omega(s_0) = -12\pi^2 S_{EW} |V_{ud}|^2 \int_{s_0}^{\infty} \frac{ds}{s_0} \omega(s) \rho_{V/A}^{DV} \quad (40)$$

- The difference $\Delta(s)$ defines the duality violating contribution to Π
- DV can be parametrised via a model
- The model has four parameters for the vector and four parameters for the axial channel
- Too many parameters: e.g. α_s, ρ_6, ρ_8 three parameters vs eight!
- why is the DV integration from $s_0 \rightarrow \infty$ and not $0 \rightarrow s_0$

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Inclusive Tau Decay Ratio

$$R_\tau = 12\pi |V_{ud}|^2 S_{EW} \int_0^{s_0} \frac{ds}{m_\tau^2} \left(1 + 2\frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \text{Im} \Pi^{(1)}(s) + \Pi^{(0)}(s) \right] \quad (41)$$

$$\Pi^{(1+0)}(s) = \Pi^{(1)}(s) + \Pi^{(0)}(s) \quad (42)$$

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(1+0)}(s) - \left(\frac{2s}{m_\tau^2} \Pi^{(0)}\right) \right] \quad (43)$$

$$D^{(1+0)}(s) \equiv -s \frac{d}{ds} \Pi^{(1+0)}(s), \quad D^{(0)}(s) \equiv \frac{s}{m_\tau^2} \frac{d}{ds} (s \Pi^{(0)}(s)) \quad (44)$$

$$R_\tau = -\pi i \oint_{|s|=m_\tau^2} \frac{dx}{x} (1-x)^3 \left[3(1+x) D^{(1+0)}(m_\tau^2 x) + 4 D^{(0)}(m_\tau^2 x) \right] \quad (45)$$

- $\Pi^{(1+0)}$ free of kinematic singularities
- $\Pi^{(0)}$ will be neglected?

Experimental Spectral Moment:

$$\rho_{V/A}^{(1)} = \frac{m_\tau^2}{12\pi^2 |V_{ud}|^2 S_{EW}} \frac{B_{V/A}}{\mathcal{B}_e} \frac{dN_{V/A}}{N_{V/A} ds} \frac{1}{\omega_\tau} \quad (46)$$

Integral Moment:

$$R_{\tau,V/A}^\omega \equiv 12\pi^2 \int_0^{s_0} \frac{ds}{s_0} \omega\left(\frac{s}{s_0}\right) \rho_{V/A}(s) \quad (47)$$

Experimental Moment:

$$I_{exp,V/A}^\omega(s_0) = \frac{m_\tau^2}{\mathcal{B}_e s_0} \sum_{i=1}^{N(s_0)} \frac{\omega(s_i/s_0)}{\omega_\tau(s_i/s_0)} \text{sfm2}_{V/A}(s_i) \quad (48)$$

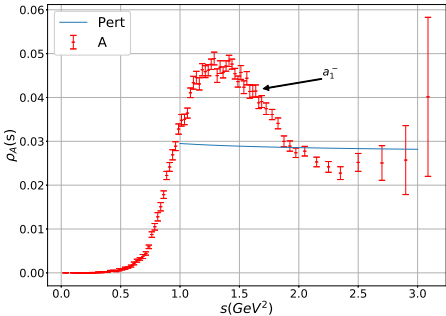
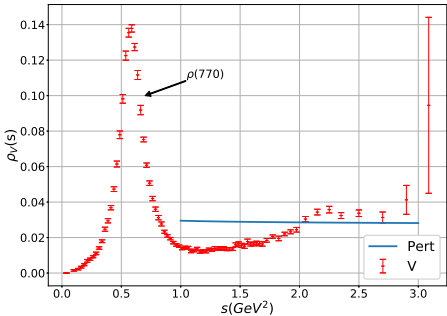
Experiment

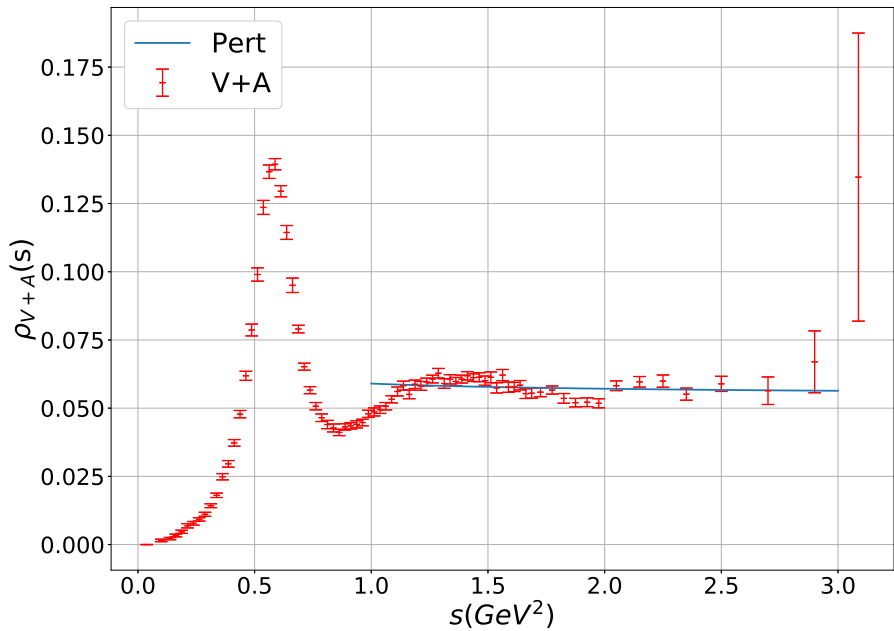
ALEPH Data

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- $\rho_V^{(0)}$ does not exist
- $\rho_A^{(0)}$ is the pion pole, which is not included in the data?
- $\text{sfm2}_{V/A} \equiv B_{V/A} \frac{dN_{V/A}}{N_{V/A} ds}$
- $|V_{ud}|^2$ and S_{EW} ?





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Chi-Squared function:

$$\chi^2 = (I_i^{exp} - I_i^{th}(\vec{\alpha})) C_{ij}^{-1} (I_j^{exp} - I_j^{th}(\vec{\alpha})) \quad (49)$$

Covariance Matrix:

$$C_{ij} = \text{cov}(I_i^{exp}, I_j^{exp}) \quad (50)$$

Chi-Squared per Degrees of Freedom:

$$\frac{\chi^2}{dof} \approx 1 \quad (51)$$

Weight function:

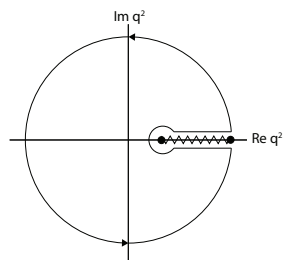
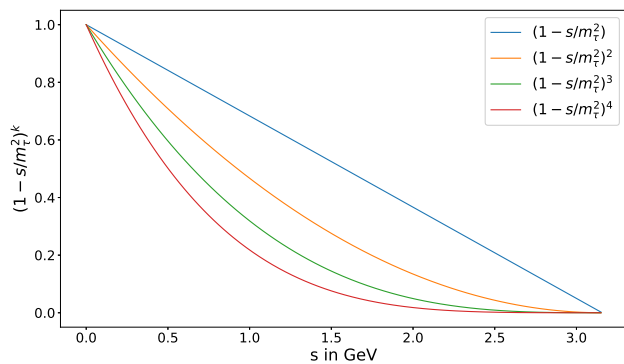
$$\omega(x) \equiv \sum_i a_i x^i \quad (52)$$

E.g.

- double pinched
- no monomial
- D6 and D8

$$\begin{aligned} \omega_\tau &\equiv (1-x)^2(1+2x) \\ &= 1 - 3x^2 + 2x^3 \end{aligned} \quad (53)$$

$$\omega(s) = \left(1 - \frac{s}{m_\tau^2}\right)^k \quad (54)$$



$$\oint_C x^k dx = i \int_0^{2\pi} \left(e^{i\theta}\right)^{k+1} d\theta = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise} \end{cases} . \tag{55}$$

$$R(x)|_{D=0,2,4,\dots} = \oint_{|x|=1} dx\, x^{k-D/2} C^{(D)} \tag{56}$$

Active Dimensions:

$$D = 2(k + 1) \tag{57}$$

monomial:	x^0	x^1	x^2	x^3	x^5	x^6	x^7
dimension:	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

- 3 Moments
- max three parameters
- e.g. $\alpha_s, D6, D8$ (fully determined)

#	3 Moments	
1	s_1	ω
2	s_2	ω
3	s_3	ω

#	9 Moments	
1	s_1	ω
2	s_2	ω
\vdots	\vdots	\vdots
9	s_9	ω

- 9 Moments
- max nine parameters

- Extract α_s
- Probe Duality Violations
- FOPT vs CIPT

- Check isolated weights of stability for different s_0 moments
- Check stability for different weights and pinchings. If we obtain similar weights DV should not be present.
- Perform additional fits with the BS. If parameters are similar to FOPT, then FOPT should be the preferred framework.

	Symbol	Term	Expansion	OPE Contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	$D6, D8$
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	$D6, D8, D10$
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	$D6, D8, D10, D12$
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	$D6$
	ω_{M3}	$1-x^3$	$1-x^3$	$D8$
	ω_{M4}	$1-x^4$	$1-x^4$	$D10$
Pinched +x	$\omega_{1,0}$	$(1-x)$	$1-x$	$D4$
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	$D4, D6$
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	$D4, D6, D8$
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	$D4, D6, D8, D10$

	s_{min}	$\#s_0s$	$\alpha_s(m_{\tau}^2)$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.200	7	0.3274(42)	-0.82(21)	-1.08(40)	0.21
FOPT	2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

$$\begin{aligned}\alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28), \\ \rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77).\end{aligned}\tag{58}$$

$\omega_{M2}(x) \equiv 1 - x^2$

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	χ^2/dof
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

$$\omega_{M3}(x) \equiv 1 - x^3$$

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	χ^2/dof
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	χ^2/dof
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

$\omega_{1,0} \equiv (1 - x)$

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	χ^2/dof
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	0.38
	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24

$\omega_{2,0} \equiv (1-x)^2$

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	χ^2/dof
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

$\omega_{3,0} \equiv (1-x)^3$

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

$$\omega_{4,0} \equiv (1-x)^4$$

	s_{min}	$\#s_0s$	$\alpha_s(m_\tau^2)$	$aGGInv$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

	weight	s_{min}	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
FOPT	ω_τ	2.2	0.3308(44)	-	-0.72(20)	-0.85(38)	-	0.19
	ω_{cube}	2.1	0.3302(40)	-	-0.52(11)	-0.58(22)	-1.00(45)	0.43
	ω_{M2}	2.2	0.3248(52)	-	-0.77(22)	-	-	0.38
	ω_{M3}	2.2	0.3214(49)	-	-	-1.01(39)	-	0.41
BS	$\omega_{1,0}$	2.2	0.3246(52)	-0.2262(59)	-	-	-	0.38
	$\omega_{2,0}$	2.2	0.3270(54)	-0.0254(61)	-0.77(21)	-	-	0.74
	$\omega_{3,0}$	2.1	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	-	0.46

1. Introduction
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- $\alpha_s(m_\tau^2) = 0.3261 \pm 0.0050$
- $\rho^{(6)} = -0.68 \pm 0.2$
- $\rho^{(8)} = -0.80 \pm 0.38$

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- $\rho^{(6)} = -0.68 \pm 0.2$
- $\rho^{(8)} = -0.80 \pm 0.38$
- DV not present if using single pinched weights in the V+A channel
- FOPT more valid than CIPT
- $\alpha_s(m_Z^2) = 0.11940(60)$

Questions

Quantity	Value	Reference
V_{ud}	0.9742 ± 0.00021	Tanabashi et al., "Review of Parti
S_{EW}	1.0198 ± 0.0006	Marciano and Sirlin, "Electroweak
B_e	17.818 ± 0.023	Davier et al., "The Determination
m_τ	$1.776\,86(12000)\,\text{MeV}$	Tanabashi et al., "Review of Parti
$\langle aGG \rangle_I$	$0.012\,\text{GeV}^2$	Shifman, Vainshtein, and Zakharo
$\langle q_{u/d}q_{u/d} \rangle(m_\tau)$	$-272(15)\,\text{MeV}$	Jamin, "Flavor symmetry breaking
$ss/\langle qq \rangle$	0.8 ± 0.3	Jamin, "Flavor symmetry breaking

$$-\frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s_0} \omega(s/s_0) \Delta_{V/A}(s) = - \int_{s_0}^{\infty} \frac{ds}{s_0} \omega(s/s_0) \frac{1}{\pi} \operatorname{Im} \Delta_{V/A}(s) \quad (59)$$

$$R_{\tau,A}^{\omega}(s_0, \pi) = 24\pi^2 |V_{ud}|^2 S_{EW} \frac{f_{\pi}^2}{s_0} \omega \left(\frac{s_{\pi}}{s_0} \right) \left[1 - \frac{2s_{\pi}}{s_{\tau} + 2s_{\pi}} \right] \quad (60)$$

- Beneke, Martin and Matthias Jamin. " α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory". In: *JHEP* 09 (2008), p. 044. DOI: 10.1088/1126-6708/2008/09/044. arXiv: 0806.3156 [hep-ph].
- Davier, M. et al. "The Determination of $\alpha(s)$ from Tau Decays Revisited". In: *Eur. Phys. J. C* 56 (2008), pp. 305–322. DOI: 10.1140/epjc/s10052-008-0666-7. arXiv: 0803.0979 [hep-ph].
- Deur, Alexandre, Stanley J. Brodsky, and Guy F. de Teramond. "The QCD Running Coupling". In: *Prog. Part. Nucl. Phys.* 90 (2016), pp. 1–74. DOI: 10.1016/j.ppnp.2016.04.003. arXiv: 1604.08082 [hep-ph].
- Jamin, Matthias. "Flavor symmetry breaking of the quark condensate and chiral corrections to the Gell-Mann-Oakes-Renner relation". In: *Phys. Lett. B* 538 (2002), pp. 71–76. DOI: 10.1016/S0370-2693(02)01951-2. arXiv: hep-ph/0201174 [hep-ph].
- Marciano, W. J. and A. Sirlin. "Electroweak Radiative Corrections to tau Decay". In: *Phys. Rev. Lett.* 61 (1988), pp. 1815–1818. DOI: 10.1103/PhysRevLett.61.1815.
- Shifman, Mikhail A., A. I. Vainshtein, and Valentin I. Zakharov. "QCD pp. 448–518. DOI: 10.1016/0550-3213(79)90023-3.
- Tanabashi, M. et al. "Review of Particle Physics". In: *Phys. Rev. D* 98.3 (2018), p. 030001. DOI: 10.1103/PhysRevD.98.030001.