The QCD Strong Coupling from Hadronic Tau Decays

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The Running of the Strong Coupling

$$\alpha_s(m_\tau^2) \approx 0.33$$
$$\alpha_s(m_Z^2) \approx 0.12 \qquad (1)$$

$$m_\tau = 1776.86(12)\,\mathrm{MeV}^1$$
$$m_Z = 91.1876(21)\,\mathrm{GeV}^1 \qquad (2)$$

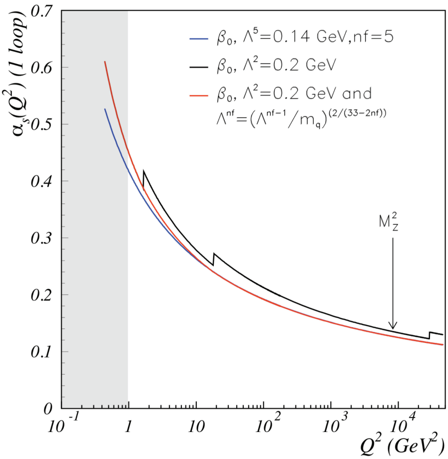


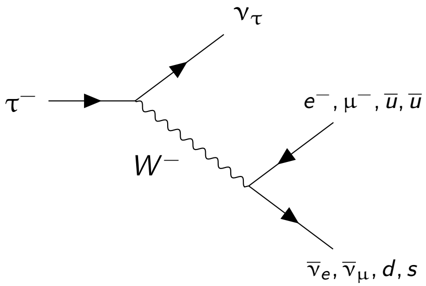
Figure: Taken from Deur, Brodsky, and Teramond, "The QCD Running Coupling", 2016

[1] Tanabashi et al., "Review of Particle Physics", 2018

- Strong coupling constant is far from constant, but depends on the energy
- This is called as the "running of the strong coupling"
- E.g. at the for us interesting m_τ^2 scale $\alpha_s(m_\tau^2) \approx 0.33$
- In general the different values for α_s are compared at the m_Z^2 scale ($\alpha_s(m_Z^2) \approx 1.12$)
- On the right we can study the running of the strong coupling
- α_s decreases with increasing energy
- leads to asymptotic freedom: at high energies quarks and gluons interact weakly and can be treated perturbatively
- leads also to confinement: at low energies quarks are bound. An isolated quark has never been measured. They appear in hadrons, two or three quarks

Tau decays

- Feynman diagram of the tau decay



- Mesons produced by tau decays

Symbol	Quark content	Rest mass
π^-	$\overline{u}d$	139.570 61(24) MeV
π^0	$(u\overline{u} - d\overline{d})/\sqrt{2}$	134.9770(5) MeV
K^-	$\overline{u}s$	493.677(16) MeV
K^0	$d\overline{s}$	497.611(13) MeV
η	$(u\overline{u} + d\overline{d} - 2s\overline{s})/\sqrt{6}$	547.862(17) MeV

- We measure the strong coupling constant from tau decays
- We are interested in the hadronic tau decay
- Here the tau lepton decays into W boson and a tau-neutrino
- the W^- boson then decays into an anti-up and a down quark
- Rarely it can decay into strange quarks, but we will neglect those cases
- The leftover quarks are not to be seen, as they appear as composite Hadrons, like the pions, given down below
- An important quantity is the hadronic tau decay ratio, which is the decay width of taus decaying into hadrons divided by the decay width of taus decaying into electrons
- We will use this quantity to perform our fits, as it is theoretically as experimentally accessible

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Two-Point Function

Two-Point Function:

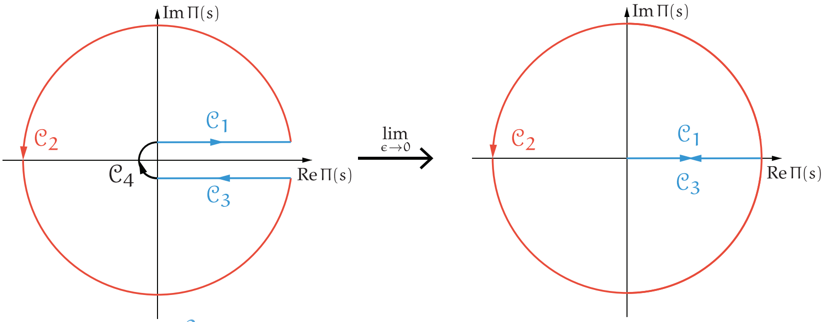
$$\begin{aligned}\Pi^{\mu\nu}_{V/A}(q^2) &\equiv i\int d^4x\, e^{iqx}\langle 0|T\left\{J^{\mu}_{V/A}(x)J^{\nu}_{V/A}(0)\right\}|0\rangle \\ &= (q^{\mu}q^{\nu}-q^2g^{\mu\nu})\Pi^{(1)}_{V/A}(q^2)+q^{\mu}q^{\nu}\Pi^{(0)}_{V/A}(q^2)\end{aligned} \tag{3}$$

where the current is given by

$$J^{\mu}_{V}=\bar{u}\gamma^{\mu}d \quad \text{and} \quad J^{\mu}_{A}=\bar{u}\gamma^{\mu}\gamma_5 d$$

- The two-point function is defined as the vacuum expectation value of the time-ordered product of two currents
- We have given the expression in momentum space
- In our case the currents are non-strange V or A currents, distinguished by a γ^{μ} or $\gamma^{\mu}\gamma_5$ correspondingly
- We can lorentz decompose the two-point function, to obtain a scalar function Π
- The superscripts (0) and (1) label the transversal or longitudinal spin
- It is common to rewrite the newly introduced scalar function of the correlator to $\Pi^{(1+0)}$ and $\Pi^{(0)}$
- $\Pi^{(1+0)}(q^2)$ and $q^2\Pi^{(0)}$ are free of kinematic singularities

Cauchy's Theorem



$$\int_0^{m_\tau^2} \mathrm{Im}\,\Pi(s)\,\mathrm{d}s = \oint_{|s|=m_\tau^2} \Pi(s)\,\mathrm{d}s$$

- We can avoid the positive real axis by making use of Cauchy's theorem
- A closed contour integral over an analytic function is zero
- Thus we get a line integral, which will represent the experimental value, equal to a circle contour, with radius of m_τ^2, which represents the theoretical value
- Applied to the inclusive hadronic tau decay ratio we get equation nine, where we also substituted $\Pi^{(1+0)}$
- Imaginary part of two-point function related to experimental accessible spectral function

- Spectral Function:

$$\rho(s) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi(s) \tag{4}$$

Integral Moment

$$I^{(\omega)}_{V/A}(s_0) \equiv \frac{12\pi^2}{s_0}\int_0^{s_0} ds\,\omega\left(\frac{s}{s_0}\right)\rho_{V/A}(s) = \frac{6\pi i}{s_0}\oint_{|s|=s_0} ds\,\omega\left(\frac{s}{s_0}\right)\Pi_{V/A}(s) \tag{5}$$

The lhs can be experimentally measured, whereas the rhs has to be theoretically calculated.

The Theoretical Computation

$$I^{th}(s_0) \equiv -\frac{1}{2\pi i s_0}\oint_{|s|=s_0} \mathrm{d}s\,\omega\left(\frac{s}{s_0}\right)\Pi_{V/A}(s) \tag{6}$$

- The two-point function is predicted by the operator product expansion

$$\Pi \to \Pi_{OPE}(s) = \sum_{D} \frac{1}{(-s)^{D/2}} \sum_{dim\mathcal{O}=D} C(-s,\mu)\langle \mathcal{O}(\mu)\rangle \equiv \sum_{k=0}^{\infty} \frac{C_{2k}(s)}{(-s)^k} \tag{7}$$

- The term with $D = 0$ corresponds to the perturbative contribution

- The QCD vacuum cannot be solely described perturbatively and we have to take non-perturbative effects into account
- To do so we will describe the two-point function in terms of the operator product expansion
- Here $A(x)$ and $B(0)$ are local operators and $C_n(x)$ is a c-number function and $\mathcal{O}_n(0)$ are higher dimensional operators
- The OPE separates short distances (high energies/ PT) from long distances (NPT)
- Short distances are given by the Wilson coefficients $C_n(x-y)$, whereas the long distances are given by higher order operators $\langle\Omega|\mathcal{O}_n(x)\rangle$.
- The two-point function can then be written has a series of Wilson coefficients multiplied by operators of dimension $0, 2, \ldots$.
- The Wilson coefficients can be calculated from Feynman diagrams, but the higher dimensional contributions have to be taken from NPT tools like lattice qcd or from our fits. We will determine values for the dimension six and eight operators
- The dimension zero contribution is the perturbative contribution, whereas the higher dimensional contributions are non-perturbative. We will deal with the PT contributions first before coming back to the NPT ones.

Duality

- The equality of the quark-gluon picture and the hadronic picture is called duality
- Differences between the physical spectral function and its OPE approximation are referred to as duality violations
- DV are connected to the behaviour of the correlator close to the positive real axis

- DV can be modelled with the following ansatz:

$$\rho^{DV}_{V/A}(s) = e^{-(\delta_{V/A}+\gamma_{V/A}s)} \sin(\alpha_{V/A} + \beta_{V/A}s) \tag{8}$$

Boito et al., "A new determination of α_s from hadronic τ decays", 2011

- The Model is theoretically well motivated, but cannot be derived from first principles

- We can represent duality as $\Pi(s) \to \Pi_{OPE}(s)$
- The difference $\Delta(s)$ defines the duality violating contribution to Π
- DV can be parametrised via a model
- The model has four parameters for the vector and four parameters for the axial channel
- Too many parameters: e.g. α_s, ρ_6, ρ_8 three parameters vs eight!
- We will further research the necessity of including DV

Perturbative Contribution

- In the chiral limit the vector and axial-vector contributions are equal
- The renormalisation-scale-invariant Adler function:

$$D_{OPE}^{D=0}(s) \equiv -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s) = \frac{1}{4\pi^2}\sum_{n=0}^{\infty} a^n(\mu^2)\sum_{k=1}^{n+1} k\, c_{n,k}\log\left(\frac{-s}{\mu^2}\right)^{k-1} \quad (9)$$

where

$$a(\mu^2) \equiv \frac{\alpha(\mu^2)}{\pi} \quad (10)$$

- The Adler function only depends on the coefficients $c_{n,1}$. All other $c_{n,k}$ can be expressed in terms of the $c_{n,1}$ through the RGE.

$$c_{0,1} = c_{1,1} = 1, \quad c_{2,1} = 1.63982, \quad c_{3,1} = 6.37101, \quad c_{4,1} = 49.07570, \quad (11)$$

- It is common to rewrite the two-point function in terms of the Adler function.
- In case of vector correlator the derivative (Adler Function) is a physical quantity.
- Physical quantities are renormalisation scale invariant.
- The Adler function has different defintions for the $\Pi^{(1+0)}$ and $\Pi^{(0)}$.
- Our final expression for the inclusive hadronic tau decay ratio then is given in equation 12.

- Perturbative Integral Moment:

$$
\begin{aligned}
I^{th,PT} &\equiv \frac{6\pi i}{s_0} \oint_{|s|=s_0} ds\, \omega\left(\frac{s}{s_0}\right) \Pi_{OPE}^{D=0}(s) \\
&= \frac{6\pi i}{s_0} \oint_{|s|=s_0} \frac{ds}{s} \omega_D \left(\frac{s}{s_0}\right) D_{OPE}^{D=0}(s) \\
&= \frac{3i}{2\pi s_0} \oint_{|s|=s_0} \frac{ds}{s} \omega_D \left(\frac{s}{s_0}\right) \sum_{n=0}^{\infty} a^n(\mu^2) \sum_{k=1}^{n+1} k\, c_{n,k} \log\left(\frac{-s}{\mu^2}\right)^{k-1}
\end{aligned} \tag{12}
$$

where

$$
\omega_D \equiv \int_0^{s_0} \omega(s')\, ds \tag{13}
$$

FOPT vs CIPT

- Perturbative Moment $(x \equiv s/s_0)$

$$I^{th,PT} = \frac{3i}{2\pi s_0} \oint_{|x|=1} \frac{\mathrm{d}x}{x} \omega_D(x) \sum_{n=0}^{\infty} a^n(\mu^2) \sum_{k=1}^{n+1} k\, c_{n,k} \log\left(\frac{-xs_0}{\mu^2}\right)^{k-1} \quad (14)$$

Fixed-Order Perturbation Theory (FOPT)

$\mu^2 \equiv s_0$

- Constant $a(s_0)$

Contour-Improved Perturbation Theory (CIPT)

$\mu^2 \equiv -xs_0$

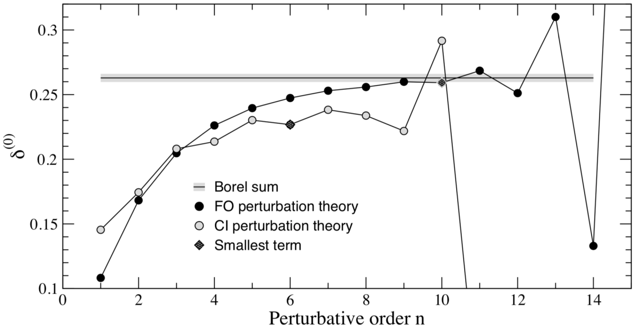
- Resums the logarithms
- Variable $a(-xs_0)$

- The general perturbative contribution δ_{pt} is defined in equation 22, where we plugged in the expanded Adler function in to the tau decay ratio and factorised $12\pi^2$
- Having the freedom to fix μ leads to two different treatments of the PT contributions
- FOPT where we fix $\mu \equiv m_\tau^2$
- This leads to a constant a_μ, so we do not have to run the strong coupling. We are left with the integration of the logarithms $\log(-x)$
- On the other hand CIPT fixed $\mu \equiv -m_\tau^2 x$, which sums up the logarithms, but leaves us with a running coupling
- Both approaches lead to different results

FOPT vs CIPT

Perturbative FOPT and CIPT contributions ($\alpha(m_\tau^2) = 0.34$):

$$\begin{aligned} & \qquad\quad \alpha_s^2 \qquad \alpha_s^2 \qquad \alpha_s^3 \qquad \alpha_s^4 \qquad\qquad \alpha_s^5 \\ \delta^{(0)}_{FOPT} &= 0.1082 + 0.0609 + 0.0334 + 0.0174(+0.0088) = 0.2200(0.2288) \qquad (15) \\ \delta^{(0)}_{CIPT} &= 0.1479 + 0.0297 + 0.0122 + 0.0086(+0.0038) = 0.1984(0.2021) \qquad (16) \end{aligned}$$



Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008

- E.g. here we display the FOPT and CIPT contribution up to fifth order.
- From the table we can conclude that CIPT converges faster, but has a smaller contribution as FOPT, which leads to larger values of α_s
- The graph below has been taken from a paper of Beneke and Jamin who invested the topic
- here we see as the black dots the FOPT contribution, as the gray dots the CIPT contribution and as a straight line the Borel sum to which we will come in a minute to which we will come in a minute to which is used to sum asymptotic series like in this case
- Note that FOPT converges in line with the Borel sum, but CIPT does not
- We will make the same observation while performing our fits

- Borel summation is a summation divergent asymptotic series, like the Adler function
- Borel transform and Borel integral:

$$A \equiv \int_0^\infty \mathrm{d}t e^{-t/a} B[A](t) \quad \text{with} \quad B[A](t) = \sum_{n=0}^{\infty} \frac{a_k}{n!} t^n. \tag{17}$$

- Borel model[2] fixed to the to the known coefficients $c_{n,1}$:

$$D(\alpha) \equiv \int_0^\infty \mathrm{d}t e^{-t/\alpha} B[D](t) \tag{18}$$

$$B[\widehat{D}](u) = B[\widehat{D}_1^{UV}](u) + B[\widehat{D}_2^{IR}](u) + B[\widehat{D}_3^{IR}](u) + d_0^{PO} + d_1^{PO} u, \tag{19}$$

[2] Beneke and Jamin, “α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory”, 2008.

- The Borel summation is a summation method for divergent asymptotic series and should give us the best possible sum
- It consists of the Borel integral and the Borel transform, which we apply to the expansion of the Adler function
- We will follow the notation of Beneke and Jamin, “α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory”, 2008, which redefined the Adler function expansion as $1 + D(s)$

Non-Perturbative Contributions

- Dimension four vacuum condensate contributions:

$$\mathcal{O}_{4,V/A} = \frac{1}{12}\left[1 - \frac{11}{18}a_s\right]\langle a_s GG\rangle + \left[1 + \frac{\pm 36 - 23}{27}a_s\right]\langle (m_u + m_d)\overline{q}q\rangle \tag{20}$$

- The condensates has to be determined from NPT methods, e.g. the gluon condensate

$$\langle a_s GG\rangle \approx 0.021\,\mathrm{GeV}^4 \tag{21}$$

- Higher dimensional contributions are approximated by simplest possible approach:

$$D^{(6)}_{V/A} = 3\frac{\rho^{(6)}_{V/A}}{s^3}, \quad D^{(8)}_{V/A} = 4\frac{\rho^{(8)}_{V/A}}{s^4}, \quad D^{(10)}_{V/A} = 5\frac{\rho^{(10)}_{V/A}}{s^5}, \quad D^{(12)}_{V/A} = 6\frac{\rho^{(12)}_{V/A}}{s^6} \tag{22}$$

- Next to the PT contribution we have to implement the NPT contributions from the OPE
- We can see that the OPE series is suppressed by powers of s thus we can approximate the series by a cutoff
- The lowest dimensional operators are given in equation 37
- In our analysis we will neglect the dimension two contributions as we work in the chiral limit and their contributions are proportional to the quark masses

The Experimental Data

$$I^{exp}(s_0) \equiv \frac{12\pi^2}{s_0} \int_0^{s_0} \mathrm{d}s\,\omega\left(\frac{s}{s_0}\right) \rho_{V/A}^{exp}(s) \tag{23}$$

Inclusive Hadronic Tau Decay Ratio

- Spectral function $\rho^{(1+0)}(s)$ is a measurable from the inclusive hadronic tau decay ratio

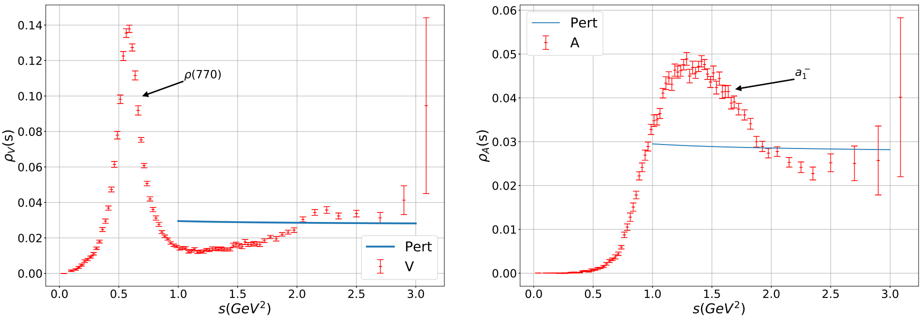
$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \tag{24}$$

- Inclusive Hadronic Tau Decay Ratio is given by ($s \equiv -q^2$)

$$R_\tau = 12\pi \, |V_{ud}|^2 \, S_{EW} \int_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2} \left(1 + 2\frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right] \tag{25}$$

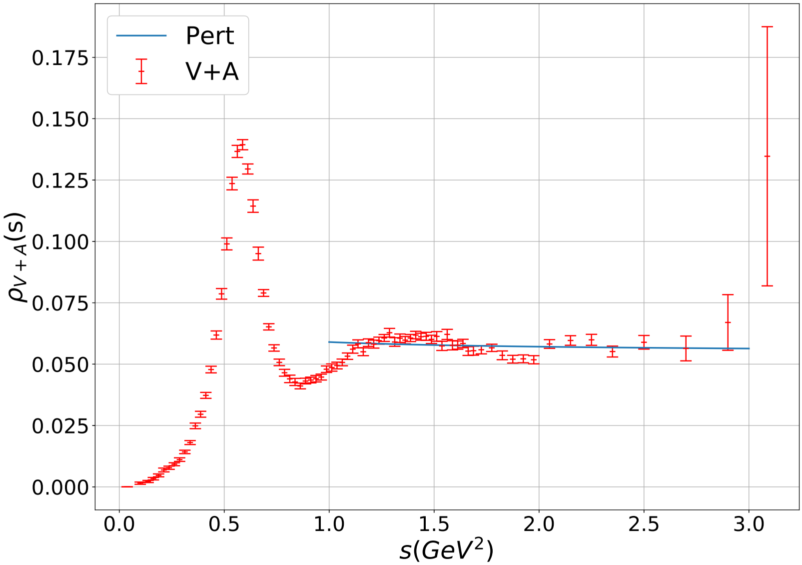
- A central value is the inclusive hadronic tau decay ratio (i.e. all decays containing hadrons)
- The ratio can be calculated by using the optical theorem
- V_{ud} is the Cabbibo matrix element, S_{EW} the electroweak correction
- We have to integrate the two-point function from $0 \to m_\tau^2$
- The two-point function has poles on the positive real axis, on the remaining s plane the two-point function is analytic
- $\Pi^{(0)}$ will be neglected? There is no $J = 0$ vector contribution. The $J = 0$ axial-vector contribution is the pion pole. Which is missing in the experimental data.

ALEPH Data



- The data we use is given by the ALEPH group
- ALEPH was a particle detector on the Large Electron-Positron collider in the nineties
- The data is given as a the normalised invariant mass squared distribution $\mathrm{d}N/N/\mathrm{d}s$ for each channel V, A and $V + A$
- In the two graphs we see the contribution of the V channel (left) and the A channel (right)
- In the vector channel we see the $\rho(770)$ resonance
- In the axial channel we see the a_1^- resonance
- We also plotted the Perturbative contribution, which cannot reproduce the experimental data, especially for lower energies

ALEPH Data



- Here we see the experimental spectral function of the $V + A$ channel
- Note that for higher energies the perturbative contribution matches the spectral function far better
- Also note that we still see a wavy behaviour of the spectral function in the data, which is connected to Duality Violations
- We assume that in the $V + A$ channel DV are sufficiently suppressed to avoid modelling their contributions

Experimental Spectral Functions

- Experimental Spectral Functions:

$$\frac{1}{N}\frac{\Delta N^{(1)}_{V/A}(s_i)}{\Delta s_i} \approx \frac{1}{N}\frac{\mathrm{d}N^{(1)}_{V/A}}{\mathrm{d}s} = B_e \frac{\mathrm{d}R^{(1)}_{\tau,V/A}}{\mathrm{d}s}(s)$$
$$= \frac{12\pi^2}{m_\tau^2} B_e S_{EW} |V_{ud}|^2 \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \rho^{(1)}_{V/A}(s) \quad (26)$$

$$\frac{1}{N}\frac{\Delta N^{(0)}_{V/A}(s_i)}{\Delta s_i} \approx \frac{1}{N}\frac{\mathrm{d}N^{(0)}_{V/A}}{\mathrm{d}s} = B_e \frac{\mathrm{d}R^{(0)}_{\tau,V/A}}{\mathrm{d}s}(s)$$
$$= \frac{12\pi^2}{m_\tau^2} B_e S_{EW} |V_{ud}|^2 \left(1 - \frac{s}{m_\tau^2}\right)^2 \rho^{(0)}_{V/A}(s) \quad (27)$$

- $\Delta N^{(0,1)}_{V/A}(s_i)$ is the number of V/A events with $J = 0, 1$ in the bin centred at s_i.

Weights

- Weight functions have to be analytic:

$$\omega(x) \equiv \sum_i a_i x^i \tag{28}$$

- We choose weights to two major criteria: pinching and contained monomials
- E.g. the kinematic weight

$$\begin{aligned} \omega_\tau &\equiv (1-x)^2(1+2x) \\ &= 1 - 3x^2 + 2x^3 \end{aligned} \tag{29}$$

$\Rightarrow$ double pinched, no monomial term x, D6 and D8

- The weight is an analytic function
- Thus we can define it as an arbitrary polynomial
- As an example we can take the natural appearing kinetic weight ω_τ
- It is double pinched, does not contain a monomial and as we will see has active D6 and D8 contributions

Pinched Weights and the OPE

- Pinched weight suppress the correlator close to the not analytic positive real axis, which is known for Duality Violations

$$\omega(x) = (1-x)^k \tag{30}$$

- The active OPE Dimensions depend on the monomials the weight carries:

$$\oint_C x^k \,\mathrm{d}x = i\int_0^{2\pi} \left(e^{i\theta}\right)^{k+1} \mathrm{d}\theta = \begin{cases} 2\pi i & \text{if } k=-1, \\ 0 & \text{otherwise} \end{cases}. \tag{31}$$

$$R(x)\Big|_{D=0,2,4,\ldots} = \oint_{|x|=1} \mathrm{d}x\, x^{k-D/2} C^{(D)} \quad \Rightarrow \quad D = 2(k+1) \tag{32}$$

monomial:	x^0	x^1	x^2	x^3	x^5	x^6	x^7
dimension:	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

- The theoretical two-point function contains DV close to the positive real axis
- To suppress DV contributions we introduce pinched weights
- The order of the pinching is given by the exponent k in equation 50
- The higher the pinching the fewer the contributions close to the positive real axis. This can be seen by plotting the weights. Blue is single pinched and decreases linear. Higher pinched weights decrease faster.
- Thus implementing a sufficient pinching should avoid DV

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Chi-Squared

- Chi-Squared function:

$$\chi^2 = (I_i^{exp} - I_i^{th}(\vec{\alpha}))C_{ij}^{-1}(I_j^{exp} - I_j^{th}(\vec{\alpha})) \tag{33}$$

- Covariance Matrix:

$$C_{ij} = \mathrm{cov}(I_i^{exp}, I_j^{exp}) \tag{34}$$

- Chi-Squared per Degrees of Freedom:

$$\frac{\chi^2}{dof} \approx 1 \tag{35}$$

- The chi-squared function is constructed from the theoretical and experimental moments
- The indices i and j represent the dependency of the moments on the chosen weight and s_0
- The fits are highly correlated.
- The correlation matrix is given with the data.
- A good fit is characterised by a $\chi^2/dof \approx 1$
- As we have to deal with missing correlations, we will also interpret fits with a χ^2/dof smaller than 1 as good

Parameters and Momenta

- 3 Moments
- max three parameters
- e.g. $\alpha_s, \rho^{(6)}_{V/A}, \rho^{(8)}_{V/A}$ (fully determined)

#	3 Moments	
1	s_1	ω
2	s_2	ω
3	s_3	ω

#	9 Moments	
1	s_1	ω
2	s_2	ω
$\vdots$	$\vdots$	$\vdots$
9	s_9	ω

- 9 Moments
- max nine parameters

- Our "data points" are the integral momenta. We will construct multiple integral momenta for every fit by varying the number of s_0.
- In principle we could also vary the weights, but have not done this
- E.g. if we want to fit three parameters as

Strategy

- Extract α_s
- Probe Duality Violations
- FOPT vs CIPT

- To extract α_s at the m_τ^2 scale, we perform fits with multiple s_0 moments.
- We check isolated weights for stability for different s_0 moments
- Check stability for different weights and pinchings. If we obtain similar weights DV should not be present.
- Perform additional fits with the BS. If parameters are similar to FOPT, then FOPT should be the preferred framework.

Chosen Weights

	Symbol	Term	Expansion	OPE Contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	$D6$, $D8$
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	$D6$, $D8$, $D10$
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	$D6$, $D8$, $D10$, $D12$
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	$D6$
	ω_{M3}	$1-x^3$	$1-x^3$	$D8$
	ω_{M4}	$1-x^4$	$1-x^4$	$D10$
Pinched $+x$	$\omega_{1,0}$	$(1-x)$	$1-x$	$D4$
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	$D4$, $D6$
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	$D4$, $D6$, $D8$
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	$D4$, $D6$, $D8$, $D10$

- To apply the strategy we have to choose several weights
- We selected three categories:
  - Pinched weights without a monomial term x, these are double, triple or quadruple pinched,
  - Monomial weights, these weights are single pinched and do not contain a monomial term x
  - "Pichs optimal" weights, these weights are single up to quadruple pinched and contain a term monomial in x
- We cannot apply FOPT to weights with a monomial term $x \Rightarrow$ BS

Kinematic Weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.200	7	0.3274(42)	-0.82(21)	-1.08(41)	0.21
FOPT	2.100	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.200	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.300	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.400	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.600	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

- Starting with the kinematic weight
- appears naturally in the inclusive hadronic tau decay ratio
- is double pinched $\Rightarrow$ should suppress DV sufficiently
- Has two active OPE dimensions, namely dimension six and eight
- Leaves us with three fitting parameters: $\alpha_s, \rho^{(6)}$ and $\rho^{(8)}$
- s_{min} is the smallest invariant mass squared value that is included in the fit
- One has to imagine that the data is binned and that we construct our moments starting from the highest available energy
- We then perform fits with an increasing number of s_0s, including more and more bins and thus include lower and lower energies
- beginning from $2.2\,\mathrm{GeV}^2$ the fits get problematic due to the appearing resonances
- Lets regard the two first lines of the FOPT table, we also applied the BS for the best fit
- Regarding the χ^2/dof we se a jump in its value, which we noted for every weight. If we go to too low energies the fits become unreliable, which is also notable from the deviating values for the parameters.
- We decided to take the fits above, but closest to this threshold to be the best fit
- For the fits above the threshold we note a great stability between the values obtained for α_s

Comparison

PT	weight	$\#s_0$'s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
FOPT	$(1-x)^2(1+2x)$	7	0.3308(44)	2.1*	-0.72(20)	-0.85(38)	0.19
	$(1-x)^3(1+2x)$	8	0.3302(40)	2.1*	-0.52(11)	-0.58(22)	0.43
	$1-x^2$	7	0.3248(52)	2.1*	-0.77(22)	0*	0.38
	$1-x^3$	7	0.3214(49)	2.1*	0*	-1.01(39)	0.41
BS	$(1-x)^2(1+2x)$	7	0.3274(42)	2.1*	-0.82(21)	-1.08(41)	0.21
	$1-x$	7	0.3246(52)	-0.2262(59)	0*	0*	0.38
	$(1-x)^2$	7	0.3270(54)	-0.0254(61)	-0.77(21)	0*	0.74
	$(1-x)^3$	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46

- Here we gathered the "best" fits, which are fits with the highest $\#s_0$'s, but being above the threshold of unstable fits
- We left also out the problematic fourth pinched weights, which include too high dimensions of the OPE
- We can clearly see that all the values obtained for α_s are very similar
- values obtained for $\rho^{(6)}$ and $\rho^{(8)}$ are within error boundaries
- Even though we used different pinchings, aka different amounts of suppression for DV
- Note that even a single pinched weights like in the second row of the BS we achieve comparable results
- Comparing the parameters obtained from FOPT, we also see that they are very similar to parameters obtained from the BS

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- We measured $\alpha_s(m_\tau^2) = 0.3261 \pm 0.0050$, which after running yields a value of $\alpha_s(m_Z^2) = 0.11940 \pm 0.00060$ and is comparable to the world average of $\alpha_s^{(PDG)}(m_Z^2) = 0.1181 \pm 0.0011$[3].
- $\rho^{(6)} = -0.68 \pm 0.2$
- $\rho^{(8)} = -0.80 \pm 0.38$
- DV not present if using single pinched weights in the V+A channel
- FOPT more valid than CIPT

[3] Tanabashi et al., "Review of Particle Physics", 2018.

Questions

Constants

Quantity	Value
V_{ud}	0.9742 ± 0.00021
S_{EW}	1.0198 ± 0.0006
B_e	17.818 ± 0.023
m_τ	$1.776\,86(12000)$ MeV
$\langle aGG \rangle_I$	$0.012\,\mathrm{GeV}^2$
$\langle q_{u/d} q_{u/d} \rangle(m_\tau)$	$-272(15)$ MeV
$ss/\langle qq \rangle$	$0.8\ \pm 0.3$

DV-model

$$-\frac{1}{2\pi i}\oint_{|s|=s_0} \frac{\mathrm{d}s}{s_0}\,\omega(s/s_0)\Delta_{V/A}(s) = -\int_{s_0}^{\infty} \frac{\mathrm{d}s}{s_0}\,\omega(s/s_0)\frac{1}{\pi}\,\mathrm{Im}\,\Delta_{V/A}(s) \qquad (36)$$

Pion Pole

$$R^{\omega}_{\tau,A}(s_0,\pi) = 24\pi^2|V_{ud}|^2\, S_{EW}\,\frac{f_\pi^2}{s_0}\,\omega\left(\frac{s_\pi}{s_0}\right)\left[1-\frac{2s_\pi}{s_\tau+2s_\pi}\right] \qquad (37)$$

Cubic Weight: $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

- The cubic weight is triple pinched
- Has three active OPE contributions, $D6$, $D8$, and $D10$
- Consequently we fitted four paremters
- Shows very similar behaviour to the kinematice weight (threshold, low χ^2/dof)
- Has also very stable values for α_s

Quartic Weight: $\omega_{quartic}(x) \equiv (1-x)^4(1+4x)$

$$\alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28),$$
$$\rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77). \tag{38}$$

- Too many parameters. Only one fit converged

$\omega_{M2}(x) \equiv 1 - x^2$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	χ^2/dof
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

$\omega_{M3}(x) \equiv 1 - x^3$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	χ^2/dof
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	χ^2/dof
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

$\omega_{1,0} \equiv (1-x)$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	χ^2/dof
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
	2.200	7	0.3246(52)	-0.2262(59)	0.38
	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
	2.200	7	0.3593(97)	-0.079(19)	0.2
	2.300	6	0.3589(99)	-0.078(20)	0.24

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	$\rho^{(6)}$	χ^2/dof
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

$\omega_{3,0} \equiv (1-x)^3$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

$\omega_{4,0} \equiv (1-x)^4$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$aGGInv$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

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