The QCD Strong Coupling from Hadronic Tau Decays

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The Running of the Strong Coupling

- The strong coupling depends on energy

$$\alpha_s(m_\tau^2) \approx 0.33 \qquad \alpha_s(m_Z^2) \approx 0.12 \tag{1}$$

$$m_\tau = 1776.86(12)\,\text{MeV}^1 \qquad m_Z = 91.1876(21)\,\text{GeV}^1 \tag{2}$$

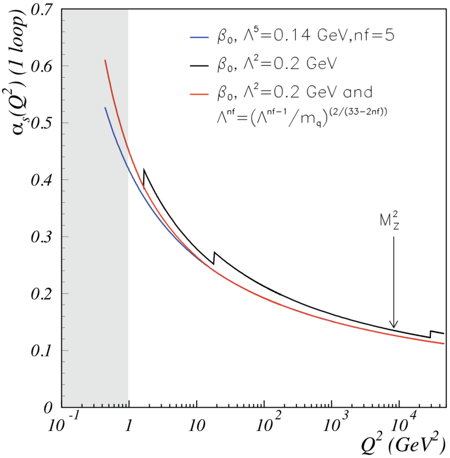


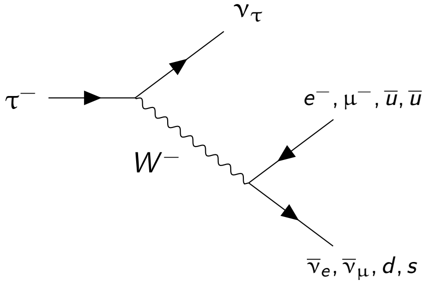
Figure: Taken from Deur, Brodsky, and Teramond, "The QCD Running Coupling", 2016

[1] Tanabashi et al., "Review of Particle Physics", 2018

- Depends on energy
- Referred to as "running of the strong coupling"
- E.g. $\alpha_s(m_\tau^2) \approx 0.33$
- Compare at m_τ^2 scale
- Plot which shows the running of α_s
- α_s decreases with increasing energy
- Asymptotic freedom: at high energies quarks and gluons interact weakly and can be treated perturbatively
- Confinement: at low energies quarks are bound. An isolated quark has never been measured. They appear in hadrons, two or three quarks
- Marked the perturbative critical region with a grey background
- for $\alpha_s > 0.5$ PT breaks down
- Hadronic tau decays good for measuring α_s
  - α small enough for PT
  - α large enough to be sensitive

Tau decays

- Feynman diagram of the tau decay



- Mesons produced by tau decays

Symbol	Quark content	Rest mass
π^-	$\overline{u}d$	$139.570\,61(24)$ MeV
π^0	$(u\overline{u} - d\overline{d})/\sqrt{2}$	$134.9770(5)$ MeV
K^-	$\overline{u}s$	$493.677(16)$ MeV
K^0	$d\overline{s}$	$497.611(13)$ MeV
η	$(u\overline{u} + d\overline{d} - 2s\overline{s})/\sqrt{6}$	$547.862(17)$ MeV

- Strong coupling constant from tau decays
- Described by Feynman Diagram
  - Tau decay into W boson and ν_τ
  - W decays into e^-, μ^- and their corresponding neutrinos or u, d or s quarks
  - only lepton decaying into quarks
- Confinement: Don't measure quarks but hadrons
- Hadrons: Composite particles that consist of quarks
- Table shows produced mesons
- Use duality ansatz: theoretically quark-gluon picture, experimentally measure hadrons
- Duality is not always valid (Duality violations)

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Two-Point Function

Two-Point Function:

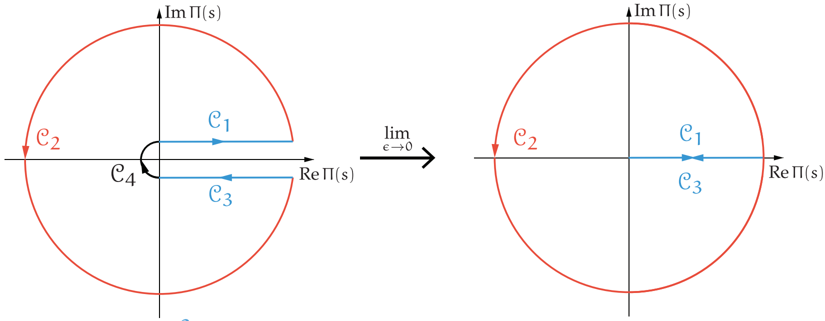
$$\begin{aligned}\Pi^{\mu\nu}_{V/A}(q^2) &\equiv i\int d^4x\, e^{iqx}\langle 0|T\left\{J^{\mu}_{V/A}(x)J^{\nu}_{V/A}(0)\right\}|0\rangle \\ &= (q^{\mu}q^{\nu}-q^2g^{\mu\nu})\Pi^{(1)}_{V/A}(q^2)+q^{\mu}q^{\nu}\Pi^{(0)}_{V/A}(q^2)\end{aligned} \quad (3)$$

where the current is given by

$$J^{\mu}_{V}=\bar{u}\gamma^{\mu}d \quad \text{and} \quad J^{\mu}_{A}=\bar{u}\gamma^{\mu}\gamma_5 d$$

- Two-point function is the vacuum expectation value of the time-ordered product of two currents
- Non-strange V or A currents, distinguished by a γ^{μ} or $\gamma^{\mu}\gamma_5$
- Lorentz decompose to obtain a scalar functions Π of different spin (0) and (1)
- Two-point function has poles on the positive real axis, but elsewhere analytic

Cauchy's Theorem



$$\int_0^{m_\tau^2} \mathrm{Im}\,\Pi(s)\,ds = -\frac{1}{2\pi i}\oint_{|s|=m_\tau^2} \Pi(s)\,ds$$

- Circumvent the positive real axis by Cauchy's theorem
- Closed contour integral over an analytic function is zero
- Construct closed contour integral
- Red is the outer circle, which will be calculated theoretically
- The blue line integral is experimentally accessible
- If we take the limit of $\epsilon \to 0$ the red circle is equal the blue line
- ϵ is the radius of the inner circle
- The contributions of the correlator close to positive real axis will be suppressed by weights

Finite Energy Sum Rules

- Spectral Function:

$$\rho(s) = \frac{1}{\pi} \operatorname{Im} \Pi(s) \tag{4}$$

Integral Moment

$$I^{(\omega)}_{V/A}(s_0) \equiv \frac{12\pi^2}{s_0} \int_0^{s_0} ds \omega\left(\frac{s}{s_0}\right) \rho^{exp}_{V/A}(s) = \frac{6\pi i}{s_0} \oint_{|s|=s_0} ds\, \omega\left(\frac{s}{s_0}\right) \Pi^{th}_{V/A}(s) \tag{5}$$

- The lhs is given by experiment, the rhs is theoretically calculated.

- Experimental data given in form of spectral function
- Connect the experiment with theory via integral moment
- Define the experimental integral moment, introducing a weight ω
- Apply Cauchy's theorem to get theoretical integral moment
- Note: Moments depend on ω and s_0, we only take part of the data into account
- Will construct chi-squared from moments

The Theoretical Computation

$$I^{th}(s_0) \equiv -\frac{1}{2\pi i s_0}\oint_{|s|=s_0} \mathrm{d}s\,\omega\left(\frac{s}{s_0}\right)\Pi_{V/A}(s) \tag{6}$$

Operator Product Expansion

- The correlator is approximated by the operator product expansion

$$\Pi \to \Pi^{OPE}(s) = \sum_D \frac{1}{(-s)^{D/2}} \sum_{\dim \mathcal{O}=D} C_D(-s,\mu)\langle \mathcal{O}(\mu)\rangle \equiv \sum_{k=0}^{\infty} \frac{C_{2k}(s)}{(-s)^k} \tag{7}$$

- C_D are the Wilson coefficients, which can be calculated perturbatively
- $\mathcal{O}$ are higher dimensional operators, e.g. $D=4$
  - Quark condensate: $m\langle \overline{q}q\rangle$
  - Gluon condensate: $G_a^{\mu\nu} G^a_{\mu\nu}$
- The term with $D=0$ corresponds to the perturbative contribution

- QCD vacuum contains NPT contributions
- Approximate correlator with OPE
- The OPE separates short distances (high energies/ PT) from long distances (NPT)
- Short distances ⇒ Wilson coefficients calculated by Feynman diagrams
- Long distances ⇒ vacuum expectation value of higher dimensional operators
- E.g. $D=4$ are the quark condensate and gluon condensate
- Have to be obtained by NPT methods like lattice QCD or from our fits
- Will fit up to dimension 12
- The term $D=0$ corresponds to PT

Quark-Hadron Duality

- The equality of the quark-gluon picture and the hadronic picture is called quark-hadron duality
- Differences between the physical spectral function and its OPE approximation are referred to as duality violations
- DV are connected to the behaviour of the correlator close to the positive real axis

- DV can be modelled with the following ansatz:

$$\rho^{DV}_{V/A}(s) = e^{-(\delta_{V/A}+\gamma_{V/A}s)} \sin(\alpha_{V/A} + \beta_{V/A}s) \tag{8}$$

Boito et al., "A new determination of α_s from hadronic τ decays", 2011

- The Model is theoretically well motivated, but cannot be derived from first principles

- Theoretically work in quark-gluon picture, experimentally observe hadrons $\Rightarrow$ quark-hadron duality
- This is an ansatz, but cannot be derived from first principles
- The physical spectral function differs from its OPE approximation $\Rightarrow$ Duality Violations
- DV can be parametrised via a model
- Four parameters V + four parameters A
- Too many parameters: e.g. α_s, ρ_6, ρ_8 three parameters vs eight!
- We investigate contribution of DV, if sufficient suppressed

Perturbative Contribution

- In the chiral limit the vector and axial-vector contributions are equal
- The renormalisation-scale-invariant Adler function:

$$D_{OPE}^{D=0}(s) \equiv -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s) = \frac{1}{4\pi^2}\sum_{n=0}^{\infty} a^n(\mu^2)\sum_{k=1}^{n+1} k\, c_{n,k} \log\left(\frac{-s}{\mu^2}\right)^{k-1} \tag{9}$$

where

$$a(\mu^2) \equiv \frac{\alpha(\mu^2)}{\pi} \tag{10}$$

- The Adler function only depends on the coefficients $c_{n,1}$. All other $c_{n,k}$ can be expressed in terms of the $c_{n,1}$ through the RGE.

$$c_{0,1} = c_{1,1} = 1, \quad c_{2,1} = 1.63982, \quad c_{3,1} = 6.37101, \quad c_{4,1} = 49.07570, \tag{11}$$

- It is common to rewrite the two-point function in terms of the Adler function.
- In case of vector correlator the derivative (Adler Function) is a physical quantity.
- Physical quantities are renormalisation scale invariant.
- The Adler function has different defintions for the $\Pi^{(1+0)}$ and $\Pi^{(0)}$.
- Our final expression for the inclusive hadronic tau decay ratio then is given in equation 12.

Perturbative Contribution

- Perturbative Integral Moment:

$$
\begin{aligned}
I^{th,PT} &\equiv \frac{6\pi i}{s_0}\oint_{|s|=s_0} ds\, \omega\left(\frac{s}{s_0}\right)\Pi^{D=0}_{OPE}(s) \\
&= \frac{6\pi i}{s_0}\oint_{|s|=s_0} \frac{ds}{s}\omega_D\left(\frac{s}{s_0}\right) D^{D=0}_{OPE}(s) \\
&= \frac{3i}{2\pi s_0}\oint_{|s|=s_0} \frac{ds}{s}\omega_D\left(\frac{s}{s_0}\right)\sum_{n=0}^{\infty} a^n(\mu^2)\sum_{k=1}^{n+1} k\, c_{n,k}\log\left(\frac{-s}{\mu^2}\right)^{k-1}
\end{aligned}
\tag{12}
$$

where

$$
\omega_D \equiv \int_0^{s_0} \omega(s')\, ds \tag{13}
$$

FOPT vs CIPT

- Perturbative Moment $(x \equiv s/s_0)$

$$I^{th,PT} = \frac{3i}{2\pi s_0} \oint_{|x|=1} \frac{\mathrm{d}x}{x} \omega_D(x) \sum_{n=0}^{\infty} a^n(\mu^2) \sum_{k=1}^{n+1} k\, c_{n,k} \log\left(\frac{-xs_0}{\mu^2}\right)^{k-1} \tag{14}$$

Fixed-Order Perturbation Theory (FOPT)

$\mu^2 \equiv s_0$

- Constant $a(s_0)$

Contour-Improved Perturbation Theory (CIPT)

$\mu^2 \equiv -xs_0$

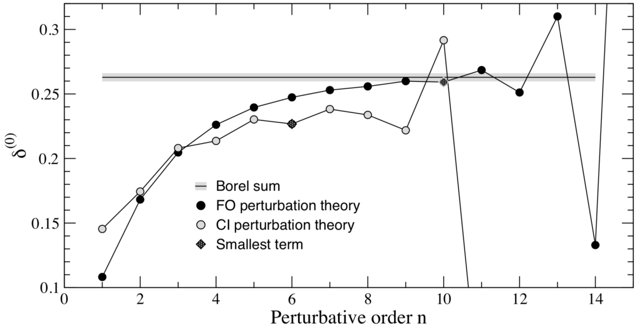
- Resums the logarithms
- Variable $a(-xs_0)$

- The general perturbative contribution δ_{pt} is defined in equation 22, where we plugged in the expanded Adler function in to the tau decay ratio and factorised $12\pi^2$
- Having the freedom to fix μ leads to two different treatments of the PT contributions
- FOPT where we fix $\mu \equiv m_\tau^2$
- This leads to a constant a_μ, so we do not have to run the strong coupling. We are left with the integration of the logarithms $\log(-x)$
- On the other hand CIPT fixed $\mu \equiv -m_\tau^2 x$, which sums up the logarithms, but leaves us with a running coupling
- Both approaches lead to different results

FOPT vs CIPT

Perturbative FOPT and CIPT contributions ($\alpha(m_\tau^2) = 0.34$):

$$\begin{aligned} & \quad\quad\;\; \alpha_s^2 \quad\;\; \alpha_s^2 \quad\;\; \alpha_s^3 \quad\;\; \alpha_s^4 \quad\;\;\;\; \alpha_s^5 \\ \delta^{(0)}_{FOPT} &= 0.1082 + 0.0609 + 0.0334 + 0.0174 (+0.0088) = 0.2200(0.2288) \quad (15) \\ \delta^{(0)}_{CIPT} &= 0.1479 + 0.0297 + 0.0122 + 0.0086 (+0.0038) = 0.1984(0.2021) \quad (16) \end{aligned}$$



Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008

- E.g. here we display the FOPT and CIPT contribution up to fifth order.
- From the table we can conclude that CIPT converges faster, but has a smaller contribution as FOPT, which leads to larger values of α_s
- The graph below has been taken from a paper of Beneke and Jamin who invested the topic
- here we see as the black dots the FOPT contribution, as the gray dots the CIPT contribution and as a straight line the Borel sum to which we will come in a minute to which we will come in a minute to which is used to sum asymptotic series like in this case
- Note that FOPT converges in line with the Borel sum, but CIPT does not
- We will make the same observation while performing our fits

Borel Summation

- Borel summation is a summation method for divergent asymptotic series, e.g. Adler function
- Beneke and Jamin introduced a physical model of the Adler function[2]:

$$B[\widehat{D}](u) = B[\widehat{D}_1^{UV}](u) + B[\widehat{D}_2^{IR}](u) + B[\widehat{D}_3^{IR}](u) + d_0^{PO} + d_1^{PO} u, \quad (17)$$

$$\widehat{D}(\alpha) \equiv \int_0^\infty \mathrm{d}t e^{-t/\alpha} B[\widehat{D}](t) \quad (18)$$

[2] Beneke and Jamin, "α_s and the τ hadronic width: fixed-order, contour-improved and higher-order perturbation theory", 2008.

- Summation method for divergent asymptotic series
- Best possible sum for Adler function
- Method consists of the Borel transform and Borel integral
- Beneke and Jamin (2008) modelled the Adler function
- Follow method of Beneke and Jamin to use BS in fits to test validity of FOPT

Non-Perturbative Contributions

- Neglect dimension two contributions
- Dimension four vacuum condensate contributions:

$$D_4 = \frac{1}{12}\left[1 - \frac{11}{18}a_s\right]\langle a_s GG\rangle + \left[1 + \frac{\pm 36 - 23}{27}a_s\right]\langle (m_u + m_d)\overline{q}q\rangle \quad (19)$$

- Gluon condensate can be quoted as

$$\langle a_s GG\rangle \approx 0.021\,\mathrm{GeV}^4 \quad (20)$$

- Higher dimensional contributions are approximated by simplest possible approach:

$$D_6 = 3\frac{\rho^{(6)}_{V/A}}{s^3}, \quad D_8 = 4\frac{\rho^{(8)}_{V/A}}{s^4}, \quad D_{10} = 5\frac{\rho^{(10)}_{V/A}}{s^5}, \quad D_{12} = 6\frac{\rho^{(12)}_{V/A}}{s^6} \quad (21)$$

- Next to the PT contribution we have to implement the NPT contributions from the OPE
- We can see that the OPE series is suppressed by powers of s thus we can approximate the series by a cutoff
- The lowest dimensional operators are given in equation 37
- In our analysis we will neglect the dimension two contributions as we work in the chiral limit and their contributions are proportional to the quark masses

The Experimental Data

$$I^{exp}(s_0) \equiv \frac{12\pi^2}{s_0} \int_0^{s_0} \mathrm{d}s\, \omega\left(\frac{s}{s_0}\right) \rho^{exp}_{V/A}(s) \tag{22}$$

Inclusive Hadronic Tau Decay Ratio

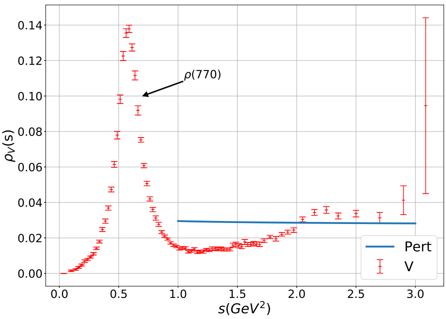
- Spectral function $\rho^{(1+0)}(s)$ is a measurable from the inclusive hadronic tau decay ratio

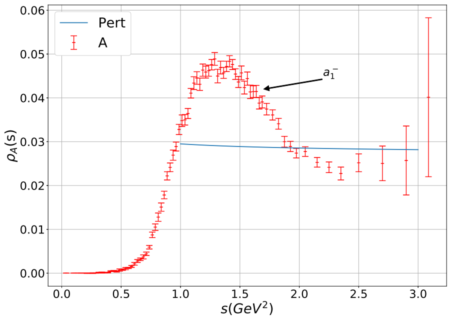
$$R_\tau = \frac{\Gamma[\tau^- \to \nu_\tau + \text{hadrons}]}{\Gamma[\tau^- \to \nu_\tau e^- \overline{\nu}_e]} \tag{23}$$

- Inclusive Hadronic Tau Decay Ratio is given by ($s \equiv -q^2$)

$$R_\tau = 12\pi \,|V_{ud}|^2\, S_{EW} \int_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2} \left(1 + 2\frac{s}{m_\tau^2}\right) \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right] \tag{24}$$

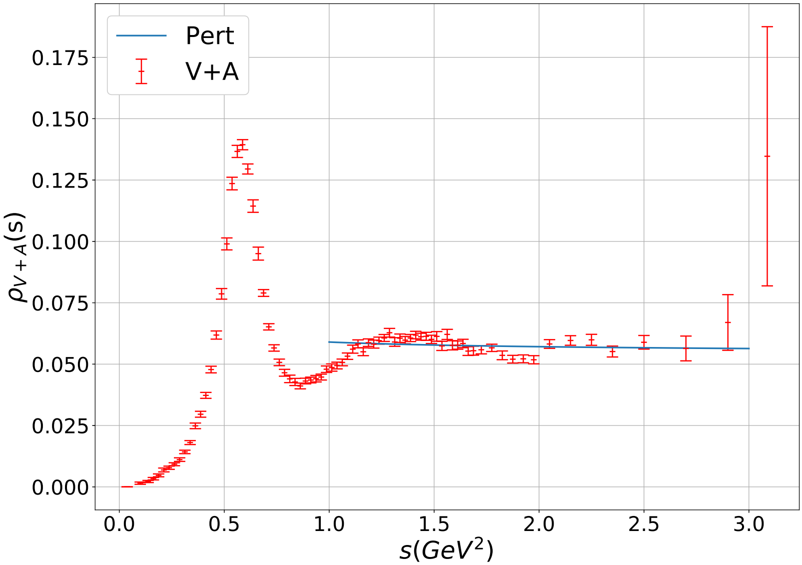
- A central value is the inclusive hadronic tau decay ratio (i.e. all decays containing hadrons)
- The ratio can be calculated by using the optical theorem
- V_{ud} is the Cabbibo matrix element, S_{EW} the electroweak correction
- We have to integrate the two-point function from $0 \to m_\tau^2$
- The two-point function has poles on the positive real axis, on the remaining s plane the two-point function is analytic
- $\Pi^{(0)}$ will be neglected? There is no $J = 0$ vector contribution. The $J = 0$ axial-vector contribution is the pion pole. Which is missing in the experimental data.





- The data we use is given by the ALEPH group
- ALEPH was a particle detector on the Large Electron-Positron collider in the nineties
- The data is given as a the normalised invariant mass squared distribution $\mathrm{d}N/N/\mathrm{d}s$ for each channel V, A and $V+A$
- In the two graphs we see the contribution of the V channel (left) and the A channel (right)
- In the vector channel we see the $\rho(770)$ resonance
- In the axial channel we see the a_1^- resonance
- We also plotted the Perturbative contribution, which cannot reproduce the experimental data, especially for lower energies

ALEPH Data



- Here we see the experimental spectral function of the $V + A$ channel
- Note that for higher energies the perturbative contribution matches the spectral function far better
- Also note that we still see a wavy behaviour of the spectral function in the data, which is connected to Duality Violations
- We assume that in the $V + A$ channel DV are sufficiently suppressed to avoid modelling their contributions

Experimental Spectral Functions

- Experimental Spectral Functions:

$$\frac{1}{N}\frac{\Delta N^{(1)}_{V/A}(s_i)}{\Delta s_i} \approx \frac{1}{N}\frac{\mathrm{d}N^{(1)}_{V/A}}{\mathrm{d}s} = B_e \frac{\mathrm{d}R^{(1)}_{\tau,V/A}}{\mathrm{d}s}(s)$$
$$= \frac{12\pi^2}{m_\tau^2} B_e S_{EW} |V_{ud}|^2 \left(1-\frac{s}{m_\tau^2}\right)^2 \left(1+\frac{2s}{m_\tau^2}\right) \rho^{(1)}_{V/A}(s) \qquad (25)$$

$$\frac{1}{N}\frac{\Delta N^{(0)}_{V/A}(s_i)}{\Delta s_i} \approx \frac{1}{N}\frac{\mathrm{d}N^{(0)}_{V/A}}{\mathrm{d}s} = B_e \frac{\mathrm{d}R^{(0)}_{\tau,V/A}}{\mathrm{d}s}(s)$$
$$= \frac{12\pi^2}{m_\tau^2} B_e S_{EW} |V_{ud}|^2 \left(1-\frac{s}{m_\tau^2}\right)^2 \rho^{(0)}_{V/A}(s) \qquad (26)$$

- $\Delta N^{(0,1)}_{V/A}(s_i)$ is the number of V/A events with $J = 0, 1$ in the bin centred at s_i.

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Chi-Squared

- The integral moments depend on the weight ω and selected energy s_0

$$I^{th}(s_0, \omega) \quad \text{and} \quad I^{exp}(s_0, \omega)$$

- For a fit we choose a weight and select multiples $s_0 s$
- The chi-squared is then given by:

$$\chi^2 = (I_i^{exp} - I_i^{th}(\vec{\alpha}))C_{ij}^{-1}(I_j^{exp} - I_j^{th}(\vec{\alpha})), \quad \text{with} \quad C_{ij} = \text{cov}(I_i^{exp}, I_j^{exp}) \tag{27}$$

- A typical fit then looks like this

- 9 Moments
- max nine parameters

#	9 Moments		
1	I_1	s_1	ω
2	I_2	s_2	ω
$\vdots$		$\vdots$	$\vdots$
9	I_3	s_9	ω

- The chi-squared function is constructed from the theoretical and experimental moments
- The indices i and j represent the dependency of the moments on the chosen weight and s_0
- The fits are highly correlated.
- The correlation matrix is given with the data.
- A good fit is characterised by a $\chi^2/dof \approx 1$
- As we have to deal with missing correlations, we will also interpret fits with a χ^2/dof smaller than 1 as good

How to choose Weights

- Weight functions have to be analytic:

$$\omega(x) \equiv \sum_i a_i x^i \tag{28}$$

- We choose weights to two major criteria: pinching and contained monomials
- E.g. the kinematic weight

$$\begin{aligned} \omega_\tau &\equiv (1-x)^2(1+2x) \\ &= 1 - 3x^2 + 2x^3 \end{aligned} \tag{29}$$

$\Rightarrow$ double pinched, no monomial term x, D6 and D8

- The weight is an analytic function
- Thus we can define it as an arbitrary polynomial
- As an example we can take the natural appearing kinetic weight ω_τ
- It is double pinched, does not contain a monomial and as we will see has active D6 and D8 contributions

How to choose Weights

- Pinched weight suppress the correlator close to the not analytic positive real axis, which is known for Duality Violations

$$\omega(x) = (1-x)^k \tag{30}$$

- The active OPE Dimensions depend on the monomials the weight carries:

$$\oint_C x^k \, \mathrm{d}x = i \int_0^{2\pi} \left(e^{i\theta}\right)^{k+1} \mathrm{d}\theta = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{otherwise} \end{cases} . \tag{31}$$

$$R(x)\Big|_{D=0,2,4,\ldots} = \oint_{|x|=1} \mathrm{d}x \, x^{k-D/2} C^{(D)} \quad \Rightarrow \quad D = 2(k+1) \tag{32}$$

monomial:	x^0	x^1	x^2	x^3	x^5	x^6	x^7
dimension:	$D^{(2)}$	$D^{(4)}$	$D^{(6)}$	$D^{(8)}$	$D^{(10)}$	$D^{(12)}$	$D^{(14)}$

- The theoretical two-point function contains DV close to the positive real axis
- To suppress DV contributions we introduce pinched weights
- The order of the pinching is given by the exponent k in equation 50
- The higher the pinching the fewer the contributions close to the positive real axis. This can be seen by plotting the weights. Blue is single pinched and decreases linear. Higher pinched weights decrease faster.
- Thus implementing a sufficient pinching should avoid DV

Strategy

- Extract α_s
- Probe Duality Violations
- FOPT vs CIPT

- To extract α_s at the m_τ^2 scale, we perform fits with multiple s_0 moments.
- We check isolated weights for stability for different s_0 moments
- Check stability for different weights and pinchings. If we obtain similar weights DV should not be present.
- Perform additional fits with the BS. If parameters are similar to FOPT, then FOPT should be the preferred framework.

Chosen Weights

	Symbol	Term	Expansion	OPE Contributions
Pinched	ω_τ	$(1-x)^2(1+2x)$	$1-3x^2+2x^3$	$D6$, $D8$
	ω_{cube}	$(1-x)^3(1+3x)$	$1-6x^2+8x^3-3x^4$	$D6$, $D8$, $D10$
	$\omega_{quartic}$	$(1-x)^4(1+3x)$	$1-10x^2+20x^3-15x^4+4x^5$	$D6$, $D8$, $D10$, $D12$
Monomial	ω_{M2}	$1-x^2$	$1-x^2$	$D6$
	ω_{M3}	$1-x^3$	$1-x^3$	$D8$
	ω_{M4}	$1-x^4$	$1-x^4$	$D10$
Pinched $+x$	$\omega_{1,0}$	$(1-x)$	$1-x$	$D4$
	$\omega_{2,0}$	$(1-x)^2$	$1-2x+x^2$	$D4$, $D6$
	$\omega_{3,0}$	$(1-x)^3$	$1-3x+3x^2-x^3$	$D4$, $D6$, $D8$
	$\omega_{4,0}$	$(1-x)^4$	$1-4x+6x^2-4x^3+x^4$	$D4$, $D6$, $D8$, $D10$

- To apply the strategy we have to choose several weights
- We selected three categories:
  - Pinched weights without a monomial term x, these are double, triple or quadruple pinched,
  - Monomial weights, these weights are single pinched and do not contain a monomial term x
  - "Pichs optimal" weights, these weights are single up to quadruple pinched and contain a term monomial in x
- We cannot apply FOPT to weights with a monomial term $x \Rightarrow$ BS

Kinematic Weight: $\omega_\tau(x) \equiv (1-x)^2(1+2x)$

	$s_{min}[GeV^2]$	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.2	7	0.3274(42)	-0.82(21)	-1.08(41)	0.21
FOPT	2.1	8	0.3256(38)	-0.43(15)	-0.25(28)	1.30
	2.2	7	0.3308(44)	-0.72(20)	-0.85(38)	0.19
	2.3	6	0.3304(52)	-0.69(25)	-0.80(50)	0.25
	2.4	5	0.3339(70)	-0.91(39)	-1.29(83)	0.10
	2.6	4	0.3398(15)	-1.3(1.0)	-2.3(2.5)	0.01

- Kinematic weight is double pinched (suppressed DV), contains no monomial term x
- OPE D=6 and D=8
- Three fitting parameters: $\alpha_s, \rho^{(6)}$ and $\rho^{(8)}$
- s_{min} smallest invariant mass squared value
- Probed weight down to 1.5 GeV
- Increasing number of s_0's the χ^2/dof increases, until point where χ^2/dof jumps (threshold/ phase transition)
- s_0 becomes to low for a good theoretical description
- Select fits with maximum number of s_0 that is still below above s_0 threshold as best fit (blue background)
- Same behaviour in all fits, also select best fit
- Parameters are within the weight very stable $\alpha_s \approx 0.33$
- We are aware that the χ^2/dof are small, caused by missing correlations
- Performed BS for best fit, also compatible
- CIPT causes higher values for $\alpha_s \Rightarrow$ FOPT more valid

Comparison

weight	PT	$\#s_0$'s	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
$(1-x)^2(1+2x)$	FO	7	0.3308(44)	2.1*	-0.72(20)	-0.85(38)	0.19
$(1-x)^2(1+2x)$	BS	7	0.3274(42)	2.1*	-0.82(21)	-1.08(41)	0.21
$(1-x)^3(1+2x)$	FO	8	0.3302(40)	2.1*	-0.52(11)	-0.58(22)	0.43
$1-x^2$	FO	7	0.3248(52)	2.1*	-0.77(22)	0*	0.38
$1-x^3$	FO	7	0.3214(49)	2.1*	0*	-1.01(39)	0.41
$1-x$	BS	7	0.3246(52)	-0.2262(59)	0*	0*	0.38
$(1-x)^2$	BS	7	0.3270(54)	-0.0254(61)	-0.77(21)	0*	0.74
$(1-x)^3$	BS	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46

- Gathered the “best” fits (from phase transition)
- Fits of different pinching
- We left out the fourth pinched weights (fits did not converge)
- Parameters with asterisks have been fixed
- Weights higher dimensional OPE omitted
- α_s compatible
- $\rho^{(6)}$, $\rho^{(8)}$ within error boundaries
- $\Rightarrow$ DV are sufficiently suppressed
- Applied BS to weights containing x
- Still stable $\Rightarrow$ FOPT valid
- BS yields $\langle aGG\rangle$ with opposite sign, has to be investigated

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- Obtained values for the strong coupling:

$$\alpha_s(m_\tau) = 0.3268(44)(25) = 0.3268(51)$$

First error taken from kinematic weight, second error $c_{5,1} \pm 100\%$

$$\alpha_s(m_Z) = 0.11886(53)(30)(5) = 0.11886(61)$$

Evolved using RunDec3, third error from using 5-loop or 4-loop evolution

$$\alpha_s^{(PDG)} = 0.1181(11)^3$$

- $\rho^{(6)} = -0.68(20)$ and $\rho^{(8)} = -0.80(38)$
- DV sufficiently suppressed (V+A channel and single pinched weights)
- FOPT more valid than CIPT

[3] Tanabashi et al., "Review of Particle Physics", 2018.

Questions

Constants

Quantity	Value
V_{ud}	0.9742 ± 0.00021
S_{EW}	1.0198 ± 0.0006
B_e	17.818 ± 0.023
m_τ	$1.776\,86(12000)\,\mathrm{MeV}$
$\langle aGG \rangle_I$	$0.012\,\mathrm{GeV}^2$
$\langle q_{u/d} q_{u/d} \rangle (m_\tau)$	$-272(15)\,\mathrm{MeV}$
$ss/\langle qq \rangle$	$0.8\ \pm 0.3$

DV-model

$$-\frac{1}{2\pi i}\oint_{|s|=s_0}\frac{\mathrm{d}s}{s_0}\omega(s/s_0)\Delta_{V/A}(s) = -\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s_0}\omega(s/s_0)\frac{1}{\pi}\,\mathrm{Im}\,\Delta_{V/A}(s) \tag{33}$$

Pion Pole

$$R^{\omega}_{\tau,A}(s_0,\pi) = 24\pi^2 |V_{ud}|^2 \, S_{EW} \frac{f_\pi^2}{s_0} \omega\left(\frac{s_\pi}{s_0}\right)\left[1 - \frac{2s_\pi}{s_\tau + 2s_\pi}\right] \tag{34}$$

Cubic Weight: $\omega_{cube}(x) \equiv (1-x)^3(1+3x)$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
2.000	9	0.3228(26)	-0.196(27)	0.075(28)	0.420(56)	1.96
2.100	8	0.3302(40)	-0.52(11)	-0.58(22)	-1.00(45)	0.43
2.200	7	0.3312(43)	-0.56(12)	-0.68(23)	-1.23(50)	0.55
2.300	6	0.336(11)	-0.78(47)	-1.17(98)	-2.38(22)	0.29
2.400	5	0.3330(96)	-0.63(47)	-0.82(10)	-1.51(26)	0.48

- The cubic weight is triple pinched
- Has three active OPE contributions, $D6$, $D8$, and $D10$
- Consequently we fitted four paremters
- Shows very similar behaviour to the kinematice weight (threshold, low χ^2/dof)
- Has also very stable values for α_s

Quartic Weight: $\omega_{quartic}(x) \equiv (1-x)^4(1+4x)$

$$\begin{aligned} \alpha_s(m_\tau^2) = 0.3290(11), \quad \rho^{(6)} = -0.3030(46), \quad \rho^{(8)} = -0.1874(28), \\ \rho^{(10)} = 0.3678(45) \quad \text{and} \quad \rho_{(12)} = -0.4071(77). \end{aligned} \tag{35}$$

- Too many parameters. Only one fit converged

$\omega_{M2}(x) \equiv 1 - x^2$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(6)}$	χ^2/dof
2.100	8	0.3179(47)	-0.42(17)	1.62
2.200	7	0.3248(52)	-0.77(22)	0.38
2.300	6	0.3260(60)	-0.85(28)	0.43

$\omega_{M3}(x) \equiv 1 - x^3$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(8)}$	χ^2/dof
2.100	8	0.3147(44)	-0.27(29)	1.71
2.200	7	0.3214(49)	-1.01(39)	0.41
2.300	6	0.3227(57)	-1.18(54)	0.46
2.400	5	0.3257(67)	-1.58(74)	0.39
2.600	4	0.325(10)	-1.54(1.53)	0.58
2.800	3	0.326(21)	-1.69(4.03)	1.17

Fourth Power Monomial: $\omega_{M4}(x) \equiv 1 - x^4$

s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\rho^{(10)}$	χ^2/dof
2.100	8	0.3136(43)	-0.07(54)	1.75
2.200	7	0.3203(48)	-1.64(77)	0.42
2.300	6	0.3216(56)	-2.01(1.13)	0.47
2.400	5	0.3247(66)	-2.98(1.62)	0.39
2.600	4	0.324(10)	-2.86(3.69)	0.58
2.800	3	0.325(20)	-3.43(10.74)	1.17

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG \rangle_I$	χ^2/dof
BS	2.100	8	0.3176(47)	-0.0134(48)	1.62
BS	2.200	7	0.3246(52)	-0.2262(59)	0.38
BS	2.300	6	0.3260(60)	-0.2453(73)	0.43
FOPT	2.100	8	0.357(12)	-0.072(23)	0.95
FOPT	2.200	7	0.3593(97)	-0.079(19)	0.2
FOPT	2.300	6	0.3589(99)	-0.078(20)	0.24

$\omega_{2,0} \equiv (1-x)^2$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	χ^2/dof
BS	2.100	8	0.3207(48)	-0.0170(50)	-0.45(17)	1.90
	2.200	7	0.3270(54)	-0.0254(61)	-0.77(21)	0.74
	2.300	6	0.3253(63)	-0.0232(75)	-0.69(27)	0.9
FOPT	2.100	8	0.3331(54)	-0.0108(45)	0.361(76)	1.9
	2.200	7	0.3401(57)	-0.0185(52)	0.220(88)	0.73
	2.300	6	0.3383(68)	-0.0165(67)	0.26(12)	0.89

$\omega_{3,0} \equiv (1-x)^3$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$\langle aGG\rangle_I$	$\rho^{(6)}$	$\rho^{(8)}$	χ^2/dof
BS	2.000	9	0.3169(20)	-0.0123(34)	-0.29(12)	-0.05(24)	2.0
	2.100	8	0.3239(40)	-0.0212(42)	-0.63(15)	-0.74(29)	0.46
	2.200	7	0.3251(17)	-0.02283(56)	-0.689(12)	-0.879(33)	0.56
FOPT	2.000	9	0.33985(81)	-0.01124(43)	0.002(10)	-0.242(26)	1.59
	2.100	8	0.3480(47)	-0.0201(36)	-0.264(89)	-1.03(28)	0.31
	2.200	7	0.3483(23)	-0.0204(41)	-0.27(15)	-1.05(40)	0.41

$\omega_{4,0} \equiv (1-x)^4$

	s_{min}	$\#s_0$s	$\alpha_s(m_\tau^2)$	$aGGInv$	$\rho^{(6)}$	$\rho^{(8)}$	$\rho^{(10)}$	χ^2/dof
BS	1.950	10	0.31711(67)	-0.012432(24)	-0.30013(73)	-0.06785(16)	0.26104(50)	1.09
	2.000	9	0.3206(24)	-0.0167(14)	-0.455(38)	-0.373(67)	-0.36(14)	0.83
	2.100	8	0.3248(21)	-0.02230(47)	-0.6724(63)	-0.834(14)	-1.352(28)	0.23
FOPT	1.950	10	0.3416(14)	-0.01306(83)	-0.050(22)	-0.390(59)	-0.50(19)	1.71
	2.100	8	0.3480(25)	-0.0201(27)	-0.264(91)	-1.02(23)	-339.00(20)	0.41

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