Chapter 3

Multi-Asset Options

This chapter covers pricing of options where the exercise value depends on more than one risky asset. Section 3.1 describes a very useful formula for pricing exchange options, while section 3.2 gives a model for the FX market, where the option could be directly an FX option or an option on an asset denominated in a foreign currency.

3.1 The Margrabe Formula

This is an expression, originally derived by Margrabe [6], for the value

$$C = E[e^{-rT}\max(S_1(T) - S_2(T), 0)] \tag{3.1}$$

of the option to exchange asset 2 for asset 1 at time T. It is assumed that under the risk-neutral measure P, $S_1(t)$ and $S_2(t)$ satisfy

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dw_1, \quad S_1(0) = s_1 \tag{3.2}$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dw_2, \quad S_2(0) = s_2, \tag{3.3}$$

where w_1, w_2 are Brownian motions with $E[dw_1 dw_2] = \rho dt$. The riskless rate is r. The Margrabe formula is

$$C(s_1, s_2) = s_1 N(d_1) - s_2 N(d_2) \tag{3.4}$$

where $N(\cdot)$ is the normal distribution function,

$$\begin{aligned} d_1 &= \frac{\ln(s_1/s_2) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \\ \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \end{aligned} \tag{3.5}$$

3.1.1 The Probabilistic Method

First C, defined by (3.1), does not depend on the riskless rate r. Indeed, $S_i(t) = e^{rt}\tilde{S}_i(t), i = 1, 2,$ where $\tilde{S}_i(t)$ is the solution to (3.2),(3.3) with $r = 0$, and hence

$$\begin{aligned} C &= E[\max(\tilde{S}_1(T) - \tilde{S}_2(T), 0)] \\ &= E[\tilde{S}_2(T)\max(\tilde{S}_1(T)/\tilde{S}_2(T) - 1, 0)] \end{aligned} \tag{3.6}$$

Henceforth, take $r = 0$ so that $\tilde{S}_i(t) = S_i(t)$. By the Ito formula, $Y(t) = S_1(t)/S_2(t)$ satisfies

$$dY = Y(\sigma_2^2 - \sigma_1\sigma_2\rho)dt + Y(\sigma_1 dw_1 - \sigma_2 dw_2) \tag{3.7}$$

and

$$\frac{1}{s_2}S_2(T) = \exp\left(\sigma_2 w_2(T) - \frac{1}{2}\sigma_2^2 T\right)$$

is a Girsanov exponential defining a measure change

$$\frac{d\tilde{P}}{dP} = \frac{1}{s_2}S_2(T). \tag{3.8}$$

Thus from (3.6)

$$C = s_2\tilde{E}[\max(Y(T) - 1, 0)]. \tag{3.9}$$

By the Girsanov theorem, under measure $\tilde{P}$ the process

$$d\tilde{w}_2 = dw_2 - \sigma_2 dt$$

is a Brownian motion. We can write w_1 as $w_1(t) = \rho w_2(t) + \sqrt{1-\rho^2}w'(t)$ where $w'(t)$ is a Brownian motion independent of $w_2(t)$ (under measure P). You can check that with $\tilde{P}$ defined by (3.8), *w' remains a Brownian motion under $\tilde{P}$, independent of $\tilde{w}_2$*. Hence $d\tilde{w}_1$ defined by

$$\begin{aligned} d\tilde{w}_1 &= \rho d\tilde{w}_2(t) + \sqrt{1-\rho^2}dw'(t) \\ &= dw_1(t) - \rho\sigma_2 dt \end{aligned}$$

is a $\tilde{P}$-Brownian motion. The equation for Y under $\tilde{P}$ turns out – miraculously – to be

$$dY = Y(\sigma_1 d\tilde{w}_1 - \sigma_2 d\tilde{w}_2)$$

which we can write

$$dY = Y\sigma dw, \tag{3.10}$$

where w is a standard Brownian motion and σ is given by (3.5). In view of (3.9), (3.10) the exchange option is equivalent to a call option on asset Y with volatility σ, strike 1 and riskless rate 0. By the Black-Scholes formula, this is (3.4).

3.1.2 The PDE Method

This follows the original Black-Scholes "perfect hedging" argument. This time we work under the "objective" probability measure, under which S_1 and S_2 have drifts μ_1, μ_2 (rather than the riskless drift r)in (3.2),(3.3). Form a portfolio

$$X_t = C - \alpha_1 S_1 - \alpha_2 S_2 - \alpha_3 P, \tag{3.11}$$

where $P(t) = \exp(-r(T-t))$ is the zero-coupon bond and the hedging component is self-financing, i.e. satisfies

$$d(\alpha_1 S_1 + \alpha_2 S_2 + \alpha_3 P) = \alpha_1 dS_1 + \alpha_2 dS_2 + \alpha_3 dP.$$

Assuming $C(t, s_1, s_2)$ is a smooth function and writing $C_1 = \partial C/\partial S_1$ etc we have by the Ito formula,

$$
\begin{aligned}
dX_t &= C_t + C_1 dS_1 + \frac{1}{2}C_{11}\sigma_1^2 S^2 dt + C_2 dS_2 + \frac{1}{2}C_{22}\sigma_2^2 S_2^2 dt \\
&\quad + C_{12}\rho\sigma_1\sigma_2 S_1 S_2 dt - \alpha_1 dS_1 - \alpha_2 dS_2 - \alpha_3 rP\, dt.
\end{aligned}
$$

If we choose

$$\alpha_1 = C_1, \quad \alpha_2 = C_2$$

this reduces to

$$dX_t = (C_t + \frac{1}{2}C_{11}\sigma_1^2 S_1^2 + \frac{1}{2}C_{22}\sigma_2^2 S_2^2 + C_{12}\rho\sigma_1\sigma_2 S_1 S_2 - \alpha_3 rP)dt \tag{3.12}$$

and by standard "no-arbitrage" arguments X_t must grow at the riskless rate, i.e. satisfy

$$dX_t = rX_t dt. \tag{3.13}$$

We see that (3.11),(3.12),(3.13), are satisfied if C satisfies the PDE

$$C_t + \frac{1}{2}C_{11}\sigma_1^2 s_1^2 + \frac{1}{2}C_{22}\sigma_2^2 s_2^2 + C_{12}\rho\sigma_1\sigma_2 s_1 s_2 = rC - rC_1 s_1 - rC_2 s_2 \tag{3.14}$$

(The terms involving α_3 cancel.) Now from the definition (3.1) of C and the price processes (3.2),(3.3) it is clear that C satisfies

$$C(t, \lambda s_1, \lambda s_2) = \lambda C(t, s_1, s_2), \tag{3.15}$$

for any $\lambda > 0$. Differentiating with respect to λ and setting $\lambda = 1$ we see that any function satisfying (3.15) satisfies

$$C(t, s_1, s_2) = s_1 C_1(t, s_1, s_2) + s_2 C_2(t, s_1, s_2).$$

In particular, this shows that the right-hand side of (3.14) is equal to zero, so that the riskless rate r drops out of the picture, as it must.

Taking $\lambda = 1/s_2$ in (3.15) we have

$$C(t, s_1, s_2) = s_2 f(t, s_1/s_2) \tag{3.16}$$

where $f(t, y) = C(t, y, 1)$. We can now calculate derivatives of C in terms of those of f; for example

$$C_1 = \frac{\partial f}{\partial y}, \quad C_2 = f - \frac{s_1}{s_2}\frac{\partial f}{\partial y}.$$

Substituting these expressions into (3.14) we find that (3.14) is equivalent to the following PDE for f:

$$\frac{\partial f}{\partial t} + \frac{1}{2}y^2\sigma^2\frac{\partial^2 f}{\partial y^2} = 0, \tag{3.17}$$

where σ is as defined above. The boundary condition is

$$f(T, y) = C(T, y, 1) = \max(y - 1, 0). \tag{3.18}$$

But (3.17) (3.18) is just the Black-Scholes PDE whose solution is (3.4).

Since the left-hand side of (3.14) is equal to zero, we see from (3.12) that $X_t \equiv 0$ if and only if $X_0 = C(0, S_0)$ and $\alpha_3 \equiv 0$. The hedging strategy places no funds in the riskless asset.

3.1.3 Exercise Probability

As in Section 3.1.1, define $Y(t) = S_1(t)/S_2(t)$. We see from (3.6) that exercise takes place when $Y(T) > 1$, and under the risk-neutral measure $Y(t)$ satisfies (3.7). Hence the forward is $F = (s_1/s_0)\exp((\sigma_2^2 - \sigma_1\sigma_2\rho)T)$, and by standard calculations

$$\text{Risk-neutral probability of exercise} = P[Y(T) > 1] = N(\hat{d}_2),$$

where

$$\begin{aligned} \hat{d}_2 &= \frac{\ln(F) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ &= \frac{\ln(s_1/s_2) + \frac{1}{2}(\sigma_2^2 - \sigma_1^2)T}{\sigma\sqrt{T}}. \end{aligned}$$

Note that the exercise probability is *not* $N(d_2)$, which is the exercise probability under the transformed measure $\tilde{P}$.

3.2 Cross-Currency Options

This section concerns valuation of an option on an asset S_t denominated in currency F (for “foreign”) which pays off in currency D (for “domestic”). We denote by f_t the exchange rate at time t, interpreted as the domestic currency price of one unit of foreign currency. Thus the currency D value of the asset S_t is $f_t S_t$ at time t. In this note we ignore interest-rate volatility and take the foreign and domestic interest rates as constants r_F, r_D respectively, so that the corresponding zero-coupon bonds have values

$$P_F(t,T) = e^{-r_F(T-t)}$$

$$P_D(t,T) = e^{-r_D(T-t)}.$$

3.2.1 Forward FX rates

To deliver one unit of currency F at time T, we can borrow $f_0 P_F(0,T)$ units of domestic currency at time 0 and buy a foreign zero-coupon bond maturing at time T. At that time the value of our short position in domestic currency is $-f_0 P_F(0,T)/P_D(0,T)$. By standard arguments, an agreement to exchange K units of domestic currency for one unit of currency F at time T is arbitrage-free if and only if $K = f_0 P_F(0,T)/P_D(0,T)$. In summary:

$$\text{Forward price} = f_0 e^{(r_D - r_F)T}.$$

This coincides with the formula for the forward price of a domestic asset with dividend yield r_F.

3.2.2 The domestic risk-neutral measure

The traded assets in the domestic economy are the domestic zero-coupon bond, value $Z_t = P_D(t,T)$, the foreign zero-coupon bond, value $Y_t = f_t P_F(t,T)$, and the foreign asset, value $X_t = f_t S_t$. An analogous set of assets is traded in the foreign economy. It is important to realize that there are *two* risk-neutral measures, depending on which economy we regard as “home”.

We will assume that in the domestic risk-neutral (DRN) measure the asset S_t is log-normal, i.e. satisfies

$$dS_t = S_t \mu dt + S_t \sigma_S dw^S(t) \tag{3.19}$$

for some drift μ and volatility σ_S. The asset is assumed to have a dividend yield q. Similarly the FX rate f_t is log-normal:

$$df_t = f_t \gamma dt + f_t \sigma_f dw^f(t), \tag{3.20}$$

with drift γ and volatility σ_f. w^S and w^f are Brownian motions with $E dw^S dw^f = \rho\, dt$.

The discounted domestic value of the foreign zero-coupon bond is

$$e^{-r_D t} f_t P_F(t,T) = e^{-r_F T} f_t e^{-(r_D - r_F)t}.$$

This is a martingale in the DRN measure, which is true if and only if

$$\gamma = r_D - r_F. \tag{3.21}$$

Now consider a self-financing portfolio of foreign assets in which we hold ϕ_t units of asset S_t and keep the remaining value in foreign zero-coupon bonds. The portfolio value process V_t then satisfies

$$\begin{aligned} dV_t &= \phi_t dS_t + q\phi_t S_t dt + (V_t - \phi_t S_t) r_F dt. \\ &= V_t r_F dt + \phi_t S_t(\mu + q - r_F)dt + \phi_t S_t \sigma_S dw_t^S. \end{aligned}$$

Using (3.20),(3.21) and the Ito formula we find that the domestic value $U_t = f_t V_t$ of this portfolio satisfies

$$dU_t = r_D U_t dt + \sigma_f U_t dw_t^f + \phi_t f_t S_t \sigma_S dw_t^S + \phi_t f_t S_t(\mu + q - r_F + \rho\sigma_S\sigma_f)dt.$$

Again, the discounted value $e^{-r_D t}U_t$ is a martingale in the DRN measure, and this holds if and only if

$$\mu = r_F - q - \rho\sigma_S\sigma_f. \tag{3.22}$$

In summary, under the DRN measure the FX rate and asset value satisfy the following equations

$$df_t = f_t(r_D - r_F)dt + f_t\sigma_f dw^f(t) \tag{3.23}$$

$$dS_t = S_t(r_F - q - \rho\sigma_S\sigma_f)dt + S_t\sigma_S dw^S(t) \tag{3.24}$$

By applying the Ito formula to (3.23),(3.24) we find that $X_t := S_t f_t$, the asset price expressed in domestic currency, satisfies

$$dX_t = X_t(r_D - q)dt + X_t(\sigma_S dw^S(t) + \sigma_f dw^f(t)). \tag{3.25}$$

By computing variances we find that

$$\sigma_S w^S(t) + \sigma_f w^f(t) = \tilde{\sigma} w(t), \tag{3.26}$$

where $w(t)$ is a standard Brownian motion and

$$\tilde{\sigma} = \sqrt{\sigma_S^2 + \sigma_f^2 + 2\rho\sigma_S\sigma_f} \tag{3.27}$$

$$E[dw dw^f] = \frac{1}{\tilde{\sigma}}(\sigma_f + \rho\sigma_S). \tag{3.28}$$

Thus (3.25) becomes

$$dX_t = X_t(r_D - q)dt + X_t\tilde{\sigma} dw(t). \tag{3.29}$$

3.2.3 Option Valuation

Options on Foreign Assets

This refers to, for example, a call option with value at maturity time T

$$\max[X_T - K, 0],$$

i.e. the foreign asset value is converted to domestic currency at the spot FX rate f_T and compared to a domestically-quoted strike K. Since X_t satisfies (3.29) we see that the option value is just the Black-Scholes value for a domestic asset with volatility $\tilde{\sigma}$ given by (3.28).[1]

Currency-Protected (Quanto) Options

Here the option value at maturity is $A_0 \max[S_T - K, 0]$ units of domestic currency, where A_0 is an arbitrary exchange factor, for example the time-zero exchange rate. The option value at time zero is

$$A_0 e^{-r_D T} E(\max[S_T - K, 0]).$$

The expectation is taken under the DRN measure, in which S_t satisfies (3.24). Note that the volatility is σ_S and the drift is $r_F - q - \rho\sigma_S\sigma_f = r_D - (q + r_D - r_F + \rho\sigma_S\sigma_f)$. We can therefore calculate the option value in two equivalent ways:

(i) Use the "forward" form of the BS formula with forward $F_T = S_0 \exp((r_F - q - \rho\sigma_S\sigma_f)T)$ and discount factor $\exp(-r_D T)$.

(ii) Use the "stock" form of BS with riskless rate r_D and dividend yield $q + r_D - r_F + \rho\sigma_S\sigma_f$.

3.2.4 Hedging Quanto Options

Deriving the Hedge

The value of the quanto option given above has the usual interpretation as the initial endowment of a perfect hedging portfolio, but the formula does not indicate how the hedging takes place. To discover this, we re-derive the formula using the traditional Black-Scholes perfect hedging argument. For this we use the "objective" probability measure - not the risk-neutral measure - under which $X_t = S_t f_t$ and f_t are log-normal processes satisfying

$$\begin{aligned} dX_t &= \lambda X_t dt + \tilde{\sigma} X_t d\tilde{w}(t) \quad (3.30) \\ df_t &= \upsilon f_t dt + \sigma_f f_t dw^f(t) \quad (3.31) \end{aligned}$$

for some drift coefficients λ, υ the value of which, it turns out, we do not need to know. The point about the hedging argument is that from the perspective of a domestic investor, S_t itself is not a traded asset: the traded assets are X_t (the domestic value of S_t) and the foreign and domestic bonds Y_t, Z_t. From (3.31), the equation satisfied by Y_t is

$$dY_t = (\upsilon + r_F)Y_t dt + \sigma_f Y_t dw^f. \quad (3.32)$$

[1]Note that the sign of ρ would be reversed if we had written the FX model in terms of $1/f_t$ rather than f_t.

We know from section 3.2.3 that the quanto call option value at time t is a function $C(t, S_t) = C(t, X_t/f_t)$ but we need to regard it as a function of X_t, f_t separately for hedging purposes. Note that if we define $g(t, x, f) := C(t, x/f)$ then with $C' = \partial C/\partial S$ we have

$$\begin{array}{lll} \frac{\partial g}{\partial t} = \frac{\partial C}{\partial t} & \frac{\partial g}{\partial x} = \frac{1}{f}C' & \frac{\partial g}{\partial f} = -\frac{x}{f^2}C' \\ \frac{\partial^2 g}{\partial x^2} = \frac{1}{f^2}C'' & \frac{\partial^2 g}{\partial f^2} = \frac{2x}{f^3}C' + \frac{x^2}{f^4}C'' & \frac{\partial^2 g}{\partial x \partial f} = -\frac{1}{f^2}C' - \frac{x}{f^3}C'' \end{array} \tag{3.33}$$

Let $C(t, S_t)$ be the call value and consider the portfolio

$$V_t = C(t, X_t/f_t) - \phi_t X_t - \psi_t Y_t - \chi_t Z_t, \tag{3.34}$$

where ϕ_t, ψ_t, χ_t are the number of units of X_t, Y_t, Z_t respectively in the putative hedging portfolio. Recall that S (and hence X) pays dividends at rate q. Applying the Ito formula using (3.33) and (3.30),(3.31) and then substituting $S = X/f$ we eventually obtain

$$\begin{aligned} dV_t &= \left(\frac{\partial C}{\partial t} - SC'(v + \rho\sigma_S\sigma_f) + \frac{1}{2}\sigma_S^2 S^2 C'' - \psi(v + r_F)Y_t - \chi r_D Z_t\right) dt \\ &\quad + (\frac{1}{f}C' - \phi)dX - \phi q X dt - (\psi Y_t + SC')\sigma_f dw^f \end{aligned}$$

Taking

$$\phi = \frac{1}{f}C' \tag{3.35}$$

$$\psi = -\frac{1}{Y}SC' \tag{3.36}$$

this becomes

$$dV_t = \left(\frac{\partial C}{\partial t} + SC'(r_F - y - \rho\sigma_S\sigma_f) + \frac{1}{2}\sigma_S^2 s^2 C'' - \chi r_D Z_t\right) dt \tag{3.37}$$

The usual no-arbitrage argument implies that V_t must grow at the domestic riskless rate, i.e.

$$\begin{aligned} dV_t &= V_t r_D dt \\ &= (C - \frac{C'}{f}X + \frac{SC'}{Y}Y - \chi Z)r_D dt \\ &= (C - \chi Z)r_D dt \end{aligned} \tag{3.38}$$

and, from (3.37) and (3.38), this equality is satisfied if C satisfies

$$\frac{\partial C}{\partial t} + SC'(r_F - q - \rho\sigma_S\sigma_f) + \frac{1}{2}\sigma_S^2 S^2 C'' - r_D C = 0. \tag{3.39}$$

The boundary condition is

$$C(T, s) = A_0[s - K]^+ \tag{3.40}$$

If we write

$$r_F - q - \rho\sigma_S\sigma_f = r_D - (q + r_D - r_F + \rho\sigma_S\sigma_f),$$

we can see that (3.39),(3.40) is just A_0 times the Black-Scholes PDE with volatility σ_S, riskless rate r_D and dividend yield $(q + r_D - r_F + \rho\sigma_S\sigma_f)$. This agrees with the valuation obtained in Section 3.2.3.

We still have to check the two key properties of the hedging portfolio, namely perfect replication and self-financing. The former is obtained by suitably defining χ_t; from (3.38), $V_t \equiv 0$ if

$$\chi_t = \frac{C(t, S_t)}{Z_t}. \tag{3.41}$$

To check the latter, note that the hedging portfolio value is $W = \phi X + \psi Y + \chi Z$ and we now know that this is equal to the option value $C(t, S_t)$. Hence

$$\begin{aligned} dW &= dC \\ &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_S^2 S^2 C''\right) dt + C' dS \\ &= r_D C dt - SC'(r_F - q - \rho\sigma_S\sigma_f)dt + C' dS. \end{aligned} \tag{3.42}$$

(The second line is just an application of the Ito formula and the third uses the Black-Scholes PDE (3.39).) Now $S = X/f$, so using (3.30), (3.31) we get from the Ito formula

$$dS = \frac{1}{f}dX - S\frac{df}{f} - \rho\sigma_f\sigma_S S dt.$$

Thus

$$\begin{aligned} dW &= r_D C dt - SC'(r_F - q - \rho\sigma_S\sigma_f)dt + \frac{C'}{f}dX - \frac{C'S}{f}df - \rho\sigma_S\sigma_f C' S dt \\ &= r_D C dt - SC'\left(r_F dt + \frac{df}{f}\right) + qSC'dt + \frac{C'}{f}dX \end{aligned} \tag{3.43}$$

Using the definitions of ϕ, ψ and χ at (3.35),(3.36),(3.41) we see that the first, third and fourth terms of (3.43) are equal to $\chi dZ, \phi qX dt$ and ϕdX respectively. Now $Y_t = e^{-r_F(T-t)} f_t$, so

$$\begin{aligned} dY_t &= r_F Y_t dt + e^{-r_D(T-t)} df_t \\ &= r_F Y_t dt + Y_t \frac{df}{f}, \end{aligned}$$

showing that the second term in (3.43) is equal to ψdY. Thus (3.43) is equivalent to

$$dW = \phi dX + q\phi X dt + \psi dY + \chi dZ,$$

which is the self-financing property.

Interpretation of the Hedging Strategy

Recall that the hedging portfolio is

$$\phi X + \psi Y + \chi Z$$

where

$$\begin{aligned} \phi &= \frac{1}{f}C' \\ \psi &= -\frac{SC'}{Y} \\ \chi &= \frac{C}{Z} \end{aligned}$$

The net value of the first two terms is zero, and this is what eliminates the FX exposure: ϕ represents a conventional delta-hedge in Currency F, financed by Currency F borrowing (this is ψ). All increments in the hedge value are immediately "repatriated" and deposited in the home currency riskless bond Z. The value in this domestic account is $\chi Z = C$ so that, in particular, the value at the exercise time T is (for a call option) $A_0[S_T - K]^+$, the exercise value of the option.