Chapter 4

Fixed Income

4.1 Bonds: the basics

4.1.1 The price/yield relationship

A bondholder receives interest payments or *coupons* on fixed dates at regular intervals (say, every 6 months) and at the final maturity date receives the final coupon plus the *par value*, which we will normalize as 1.

The coupon payments are specified by a *rate* (5.5%,..), a *frequency* (1,2,4: the number of payments per year) and a *basis* stating how the *accrual* or *day count* is calculated. Typical bases are actual/actual, actual/365, 30/360.. For example if we have a rate of 5.5% paid semi-annually (frequency = 2) on an actual/365 basis then the payment dates are 6 months apart and the coupon payment on a particular payment date is $(d/365)*0.055$, where d is the number of days since the last coupon date. The accrual factor $(d/365)$ is very nearly, but not exactly, equal to 1/2. The first coupon is paid 6 months after the bond is issued.

For simplicity, consider a bond with frequency 1, coupon c and basis actual/actual (or 30/360), so that the accrual factor is 1, and maturity n years. If the price at issue is p, the *yield* is the number y satisfying $p = B(y)$ where

$$B(y) = \sum_{i=1}^{n} \frac{c}{(1+y)^i} + \frac{1}{(1+y)^n}$$

(Interpretation: all the coupon payments could be financed by depositing at time 0 the amount p in an account paying annual interest y.) Note the inverse relationship: high yield $\Leftrightarrow$ low price. The *(modified) duration* of the bond is

$$D(y) = -\frac{1}{B(y)}\frac{dB(y)}{dy}.$$

Note that this has units of *years*. For a *zero-coupon* bond $(c = 0)$ the duration is $n/(1+y) \approx n$, whereas a coupon bond has shorter duration: maybe 7 years for a 10-year bond issued at par. The *convexity* of the bond is

$$C(y) = \frac{d^2B(y)}{dy^2}.$$

To illustrate the effect of convexity, suppose that the yield is a random variable Y with expected value y_0. We define the *yield volatility* as $\sigma = \sqrt{E(Y-y_0)^2}/y_0$. Then

$$B(Y) = B(y_0) + \frac{dB(y_0)}{dy}(Y-y_0) + \frac{1}{2}\frac{d^2B(y_0)}{dy^2}(Y-y_0)^2 + \cdots$$

so that to second order

$$EB(Y) = B(y_0) + \frac{1}{2}C(y_0)y_0^2\sigma^2.$$

Yield volatility increases the expected bond value due to the convexity effect.

4.1.2 Floating rate notes

Suppose an annual interest rate L_i is set at the beginning of year $i, i = 1, 2, \ldots$, so that \$1 at the beginning of year i becomes $\$(1 + L_i)$ at the end of the year. The corresponding discount factor is $1/(1 + L_i)$. A floating rate note is an n-year bond whose coupon c_i paid at the end of year i is equal to L_i. A key fact is the following: *at coupon dates, a floating rate note is always at par.* You can think of this as a consequence of the identity

$$\sum_{i=1}^{n} \frac{L_i}{\prod_{j=1}^{i}(1+L_j)} + \frac{1}{\prod_{j=1}^{n}(1+L_j)} = 1.$$

In general the rate L_i is a random variable whose value is not known until the beginning of year i. We need extra information to be able to value a *fixed-coupon* bond.

4.2 A general valuation model

All processes are assumed to be $\mathcal{F}_t$-adapted on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ where P will be the unique risk-neutral measure. There are traded assets with price processes $S_i(t)$. The holder of asset i receives cumulative dividends $D_i(t)$, i.e. the dividend received in the time interval $]t, t+dt]$ is $dD_i(t) = D_i(t+dt) - D_i(t)$. There is a savings account paying interest at continuously-compounde rate $r(t)$, again an adapted random process. A self-financing portfolio with trading strategy ϕ (i.e. $\phi_i(t)$ is the number of units of asset i in the portfolio at time t) thus has wealth process X_t satisfying

$$dX_t = \sum_i \phi_i(dS_i + dD_i) + \left(X_t - \sum_j \phi_j S_j\right) r(t)\,dt. \tag{4.1}$$

Define

$$B(t) = \exp\left(\int_0^t r(s)ds\right).$$

Then

$$\begin{aligned} d\left(B^{-1}(t)X_t\right) &= \sum_i B^{-1}\phi_i(dS_i - rS_i dt + dD_i) \\ &= \sum_i \phi_i d(B^{-1}S_i) + \sum_i \phi_i B^{-1}dD_i). \end{aligned} \tag{4.2}$$

In particular, taking $\phi_i(t) \equiv 1$ for $i = j$ and $\phi_i(t) \equiv 0$ otherwise, we obtain

$$B^{-1}(t)X_t = B^{-1}(t)S_j(t) + \int_0^t B^{-1}(s)dD_j(s) =: M_j(t). \tag{4.3}$$

If P is the risk-netral measure corresponding to the savings account numéraire $B(t)$ then $M_j(t)$ *is a martingale for each* j. We then see that (4.2) can be written

$$d\left(B^{-1}(t)X_t\right) = \sum_i \phi_i(t)dM_i(t). \tag{4.4}$$

Thus if Z is an $\mathcal{F}_t$-measurable random variable such that $Z = X_t$ a.s. for some self-financing strategy ϕ then – assuming the stochastic integrals in (4.4) are true martingales – the value of Z at time 0 is

$$E\left[e^{-\int_0^t r(u)du} Z\right],$$

and more generally the value at some intermediate time $s \in [0, t]$ is

$$E\left[e^{-\int_s^t r(u)du} Z \middle| \mathcal{F}_s\right]. \tag{4.5}$$

The market is complete if, for each $t \in [0, T]$, *every* contingent claim Z (in some class about which we will not be too precise) with exercise time t can be replicated, i.e. $Z = X_t$ for some trading strategy ϕ.

A special but exceptionally important case is the "contingent claim" $Z \equiv 1$. Then (4.5) gives us the time-s value $p(s, t)$ of a zero-coupon (ZC) bond, paying \$1 at time t, as

$$p(s,t) = E\left[e^{-\int_s^t r(u)du} \middle| \mathcal{F}_s\right] = B(s)E[B^{-1}(t)|\mathcal{F}_s] \tag{4.6}$$

4.3 Interest rate contracts

4.3.1 Libor rates

A zero-coupon (ZC) bond value $p(s, t)$ is equivalent to a simple interest payment for the period $[s, t]$ of L satisfying

$$p(s,t) = \frac{1}{1 + \theta_{st}L},$$

or equivalently

$$L = \frac{1}{\theta_{st}}\left(\frac{1}{p(s,t)} - 1\right), \tag{4.7}$$

where θ_{st} is the accrual factor (in the appropriate basis) for the interval $[s, t]$. L defined by (4.7) is the *Libor rate.* Note that

- L is set at time s (i.e. it is $\mathcal{F}_s$ measurable) but paid at time t.
- The value at time s of the Libor payment at time t is

$$\begin{aligned} E\left[e^{-\int_s^t r(u)du}\theta_{st}L \middle| \mathcal{F}_s\right] &= p(s,t)\theta_{st}L \\ &= 1 - p(s,t) \end{aligned}$$

Because of the latter fact, the accrual factor plays no role in the theory. It is just a conventional way of specifying what the Libor *rate* is, while the actual *payment* depends only on the ZC bond values.

For $t < T_1 < T_2$ the *forward bond* $p^f(t; T_1, T_2)$ and *forward Libor rate* $L^f(t; T_1, T_2)$ at t for the period $[T_1, T_2]$ are

$$p^f(t; T_1, T_2) = \frac{p(t, T_2)}{p(t, T_1)}$$

and

$$\begin{aligned} L^f(t; T_1, T_2) &= \frac{1}{\theta_{T_1 T_2}} \left(\frac{1}{p^f(t; T_1, T_2)} - 1 \right), \\ &= \frac{1}{\theta_{T_1 T_2}} \left(\frac{p(t, T_1)}{p(t, T_2)} - 1 \right) \end{aligned}$$

Suppose that, at time t, I agree to make a Libor payment at time T_2 (with rate set at time T_1) in exchange for a payment at a rate K fixed now, at time t; this is a *forward rate agreement (FRA)*. Fact: *the unique arbitrage-free value of the fixed side in a FRA is $K = L^f$, the forward Libor rate.* Indeed, the corresponding hedging strategy is as follows: at time t,

- borrow a number $\theta_{T_1 T_2} L^f$ of T_2-ZC bonds, value $p(t, T_2)(p(t, T_1)/p(t, T_2) - 1) = p(t, T_1) - p(t, T_2)$; the fixed payment θL^f at time T_2 exactly redeems these bonds.
- buy one T_1-ZC bond and sell one T_2-ZC bond.
- at time T_1, these bonds have value $1 - p(T_1, T_2)$, enough to buy a number $(1 - p(T_1, T_2))/p(T_1, T_2)$ of T_2-ZC bonds. At time T_2 these have value $(1/p(T_1, T_2) - 1) = \theta_{T_1 T_2} L$.

4.3.2 Swap rates

An interest rate swap is specified by maturity, frequency, basis, notional amount N and fixed side rate K. On each coupon date t_i one party (the 'fixed side') pays $N\theta_i K$ while the other (the 'floating side') pays $N\theta L_i$ where L_i is the Libor rate set at t_{i-1}. Here $\theta_i = \theta_{t_1 t_2}$. We will take $N = 1$ henceforth.

Fictitiously adjoin to the swap equal and opposite payments of 1 at the maturity date. Then the floating side is equivalent to a floating rate note, with value 1 at time 0, while the fixed side is equivalent to a coupon bond, with value

$$\sum_{i=1}^{n} K\theta_i p(0, t_i) + p(0, t_n).$$

The *swap rate* is the value of K such that the swap has value 0 at time 0. Clearly this value is

$$K_0 = \frac{1 - p(0, t_n)}{\sum_{i=1}^{n} \theta_i p(0, t_i)} \tag{4.8}$$

At later times this swap does not generally have value zero because the same fixed-side rate K_0 is maintained throughout. For example the value at t_j, to the party paying fixed, is

$$1 - \sum_{i=j+1}^{n} K_0 \theta_i p(t_j, t_i) - p(t_j, t_n), \tag{4.9}$$

since the floating side always has value 1. The swap rate K_j at t_j is, in our model, an $\mathcal{F}_{t_j}$-measurable random variable. The *forward swap rate* at t_j is, by analogy with (4.8)

$$\begin{aligned} K_j^f &= \frac{1 - p^f(0; t_j, t_n)}{\sum_{i=j+1}^n \theta_i p^f(0; t_j, t_i)} \\ &= \frac{p(0, t_j) - p(0, t_n)}{\sum_{i=j+1}^n \theta_i p(0, t_i)} \end{aligned}$$

EXERCISE: Show that $K = K_j^f$ is the unique no-arbitrage value of an agreement, made at time 0, to enter a swap at time t_j at fixed rate K.

4.3.3 Interest rate options

The standard interest-rate options are *caps, floors* and *swaptions.* A *cap* pays a cash amount $\theta_i[L_i - K]^+$ at each coupon date $i, i = 1 \dots n$. In view of (4.7) we have

$$\theta_i[L_i - K]^+ = \left[\frac{1}{p(t_{i-1}, t_i)} - (1 + \theta_i K)\right]^+$$

and the value of this payment at time t_{i-1} is

$$p(t_{i-1}, t_i)\left[\frac{1}{p(t_{i-1}, t_i)} - (1 + \theta_i K)\right]^+ = (1 + \theta_i K)[\kappa_i - p(t_{i-1}, t_i)]^+,$$

where $\kappa_i = 1/(1 + \theta_i K)$. Thus a cap is equivalent to a series of *caplets*, each caplet being equivalent to a put option on the ZC bond. In our model the caplet value is

$$\frac{1}{\kappa_i} E\left(e^{-\int_0^{t_{i-1}} r(s)ds}[\kappa_i - p(t_{i-1}, t_i)]^+\right)$$

A *floor* pays $\theta_i[K - L_i]^+$.

A *swaption* is the right to enter a swap at a fixed-side rate K, starting at a time t_j in the future. It is a 'payer's swaption' if the holder will enter the swap paying the fixed side, and a 'receiver's swaption' otherwise. From (4.9) the value of a payer's swaption with strike K is

$$E\left(e^{-\int_0^{t_j} r(s)ds}[1 - \sum_{i=j+1}^n K\theta_i p(t_j, t_i) - p(t_j, t_n)]^+\right).$$

It is equivalent to a put option on a coupon bond, with coupon K, with strike 1.

4.3.4 Futures

Very briefly, a futures contract maturing at time T on an asset S_i is a traded asset with 'price' F_t such that

- The futures contract can be entered at zero cost at any time;
- A holder of the contract receives a payment $F_{t+dt} - F_t$ in the interval $[t, t + dt]$.

- At maturity T, $F_T = S_i(T)$.

From this description it is clear that the futures 'price' is not a price at all. It is a dividend. The future is an asset S_j with price $S_j(t) \equiv 0$ and dividend process $D_j(t) = F_t$. In view of (4.3) we see that

$$M_j(t) = \int_0^t B^{-1}(s)dF_s$$

is a martingale, so that

$$F_t = \int_0^t B(s)dM_j(s)$$

is a martingale. Since $F_T = S_i(T)$, this shows that

$$F_t = E[S_i(T)|\mathcal{F}_t], \qquad t \leq T.$$

Recall that the *forward price* G_t is the no-arbitrage exchange price for $S_i(T)$ fixed at time t, i.e. G_t satisfies

$$E\left[e^{-\int_t^T r(s)ds}(G_t - S_i(T)) \middle| \mathcal{F}_t \right] = 0.$$

Hence

$$G_t = \frac{1}{p(t,T)} E\left[e^{-\int_t^T r(s)ds} S_i(T) \middle| \mathcal{F}_t \right].$$

The difference between forward and futures prices is therefore

$$\begin{aligned} F_t - G_t &= E[S_i(T)|\mathcal{F}_t] - \frac{1}{p(t,T)} E\left[e^{-\int_t^T r(s)ds} S_i(T) \middle| \mathcal{F}_t \right] \\ &= \frac{1}{p(t,T)} E\left[S_i(T)(p(t,T) - e^{-\int_t^T r(s)ds}) \middle| \mathcal{F}_t \right] \\ &= \frac{-1}{p(t,T)} \mathrm{cov}_{\mathcal{F}_t}\left(S_i(T), e^{-\int_t^T r(s)ds} \right), \end{aligned} \tag{4.10}$$

where $\mathrm{cov}_{\mathcal{F}_t}(X, Y)$ denotes the conditional covariance of X and Y. In particular, *forward and future are the same if there is no interest-rate volatility.*

Exchange-traded futures include the *Eurodollar futures contract*, whose settlement value at time T is $100(1-L)$, where L is the 3-month Libor rate set at T. (The reason for this convention is to maintain the 'high rate $\Leftrightarrow$ low price' relationship, as for bonds.) It is important to note that a futures price of, say, 94.5 does not mean that forward Libor is 5.5%: this figure has to be adjusted by the 'convexity correction' (4.10). Note that when S_i in (4.10) is a Libor rate, it is generally positively correlated with $r(s)$ and therefore negatively correlated with $e^{-\int_t^T r(s)ds}$. Thus the right-hand side of (4.10) is positive, so the futures price is *bigger* than the forward price.

4.4 Pricing interest-rate options

The standard market convention for pricing plain-vanilla interest-rate options is to use the Black 'forward' formula

$$p(0,T)[FN(d_1) - KN(d_2)], \tag{4.11}$$

where d_1, d_2 are the usual volatility-related factors. This can be applied to caplets, with F as the forward Libor rate, or to swaps with F as the forward swap rate. There is some apparent inconsistency with this approach: the whole point is that interest rates in the future are random, but we treat the discount factor $p(0,T)$ in (4.11) as deterministic. In this section we show that something close to this approach is in fact consistent if we re-interpret things in terms of 'forward measures'. A good reference for this material is Hunt and Kennedy [4].

4.4.1 The forward measure

In the framework of Section 4.2, the *forward price* $F_i(t,T)$ of a traded asset S_i is the price agreed at time t for exchange at time T, i.e. the value of κ such that

$$E\left[e^{-\int_t^T r(u)du}(\kappa - S_i(T))\middle|\mathcal{F}_t\right] = 0,$$

or equivalently

$$\kappa p(t,T) = E\left[e^{-\int_t^T r(u)du}S_i(T)\middle|\mathcal{F}_t\right]. \tag{4.12}$$

Since $M_i(t)$ given by (4.3) is a martingale, we see that

$$F_i(t,T) = \frac{1}{p(t,T)}S_i(t) - \frac{1}{p(t,T)}E\left[\int_t^T e^{-\int_t^s r(u)du}dD_i(s)\middle|\mathcal{F}_t\right]. \tag{4.13}$$

The value $C(t)$ at time t of an option on S_i, maturing at T with exercise value $h(S_i(T))$ is, as usual,

$$C(t) = E\left[e^{-\int_t^T r(u)du}h(S_i(T))\middle|\mathcal{F}_t\right] \tag{4.14}$$

Definition: The T*-forward measure* on $(\Omega, \mathcal{F}_T)$ is the measure P^T defined by the Radon-Nikodým derivative

$$\frac{dP^T}{dP} = \frac{e^{-\int_0^T r(u)du}}{p(0,T)} = \frac{\frac{1}{p(0,T)}}{B(T)}. \tag{4.15}$$

Note that P^T is well-defined in that the right hand side of (4.15) is strictly positive and has expectation 1. We see from (4.15) and the general change-of-numéraire formula that P^T is the risk-neutral measure corresponding to a numéraire $N(t)$ where $N(T) = 1/p(0,T)$. Since $N(t)/B(t)$ is a P-martingale, this implies that $N(t) = p(t,T)/p(0,T)$. Thus *moving to the T-forward measure is equivalent to changing the numéraire from the savings account $B(t)$ to the zero-coupon bond $p(t,T)/p(0,T)$.*

By the standard formula for conditional expectation under change of measure,

$$\begin{aligned} E^T[h(S_i(T))|\mathcal{F}_t] &= \frac{E\left[e^{-\int_0^T r(u)du}h(S_i(T))\middle|\mathcal{F}_t\right]}{E\left[e^{-\int_0^T r(u)du}\middle|\mathcal{F}_t\right]} \\ &= \frac{E\left[e^{-\int_t^T r(u)du}h(S_i(T))\middle|\mathcal{F}_t\right]}{p(t,T)}, \end{aligned}$$

so that

$$C(t) = p(t,T)E^T[h(S_i(T))|\mathcal{F}_t].$$

The key fact about the forward measure is this:

Proposition 3 *The forward price is a martingale under the T-forward measure.*

PROOF: Indeed, for $s < t$, we have from (4.12)

$$\begin{aligned}
E^T[F_i(t,T)|\mathcal{F}_s] &= \frac{E\left[e^{-\int_0^T r(u)du} \frac{E\left[e^{-\int_t^T r(u)du} S_i(T) \Big| \mathcal{F}_t \right]}{p(t,T)} \Bigg| \mathcal{F}_s \right]}{E\left[e^{-\int_0^T r(u)du} \Big| \mathcal{F}_s \right]} \\
&= \frac{E\left[e^{-\int_s^t r(u)du} E\left[e^{-\int_t^T r(u)du} S_i(T) \Big| \mathcal{F}_t \right] \Big| \mathcal{F}_s \right]}{p(s,T)} \\
&= \frac{E\left[e^{-\int_s^T r(u)du} S_i(T) \Big| \mathcal{F}_s \right]}{p(s,T)} = F_i(s,T).
\end{aligned}$$

This is the martingale property. $\diamondsuit$

Proposition 3 implies in particular that

$$\begin{aligned}
E^T[S_i(T)] &= E^T[F(T,T)] \\
&= F(0,T), \qquad (4.16)
\end{aligned}$$

where $F(0,T)$ is given by (4.13) with $t = 0$. This gives us our first pricing formula.

Proposition 4 *Suppose $S_i(t)$ is log-normally distributed in the T-forward measure, with volatility σ. Then the no-arbitrage price at time 0 of a call option with exercise value $[S_i(T) - K]^+$ is given by the Black formula*

$$C(0) = p(0,T)[F(0,T)N(d_1) - KN(d_2)], \qquad (4.17)$$

where

$$d_1 = \frac{\log(F(0,T)/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

PROOF: In view of (4.16) the price $S_i(T)$ is given by

$$S_i(T) = F(0,T)\exp\left(-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}X \right)$$

where $X \sim N(0,1)$. The result follows by standard calculations. $\diamondsuit$

4.4.2 Forwards and futures

In section 4.3.4 we showed that the futures price is a martingale in the risk-neutral measure, whereas Proposition 3 shows that the forward is a martingale in the T-forward measure. Thus the convexity correction (4.10) is equal to the difference in expected value under the two measures, $E[S_i(T)] - E^T[S_i(T)]$.

4.4.3 Caplets

Consider a caplet where the Libor rate is set at T_1 and paid at T_2. Let θ be the accrual factor. Then the forward Libor rate at $t \leq T_1$ is

$$L_t^f = \frac{1}{\theta}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right).$$

In this case *the forward Libor rate is a martingale in the T_2-forward measure.* Indeed for $s < t$

$$\begin{aligned} E^{T_2}[L_t^f|\mathcal{F}_s] &= \frac{E\left[e^{-\int_0^{T_2} r(u)du}\frac{1}{\theta}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right)|\mathcal{F}_s\right]}{E\left[e^{-\int_0^{T_2} r(u)du}|\mathcal{F}_s\right]} \\ &= \frac{E\left[e^{-\int_s^{t} r(u)du}p(t,T_2)\frac{1}{\theta}\left(\frac{p(t,T_1)}{p(t,T_2)} - 1\right)|\mathcal{F}_s\right]}{p(s,T_2)} \\ &= \frac{1}{\theta}\frac{p(s,T_1) - p(s,T_2)}{p(s,T_2)} = L_s^f. \end{aligned}$$

Thus, as in Proposition 4, if the Libor rate is assumed to be log-normally distributed in the T_2-forward measure we can use the Black formula (4.17) to price the caplet, slightly modified because of the different setting and paying times. Specifically, the price is

$$p(0,T_2)[L_0^f N(d_1) - KN(d_2)]$$

with $T := T_1$ in d_1, d_2.

4.4.4 Swaptions

Here we have to be a little more ingenious. As discussed in section 4.3.3, the value at time t of the right to enter a swap at time $t_0 > t$ at fixed-side rate K is

$$SV_t = E\left\{e^{-\int_t^{t_0} r(u)du}\left[1 - K\sum_{i=1}^{n}\theta_i p(t_0,t_i) - p(t_0,t_n)\right]^+ \middle| \mathcal{F}_t\right\}, \tag{4.18}$$

where the swap coupon dates are $t_1, \ldots, t_n$ and θ_i are the accrual factors. The forward swap rate is

$$F_t = \frac{p(t,t_0) - p(t,t_n)}{\sum_i \theta_i p(t,t_i)} = \frac{p(t,t_0) - p(t,t_n)}{p_A(t)},$$

where $p_A(t) = \sum_i \theta_i p(t,t_i)$ known as the 'present value of a basis point' (it is the value at time t of unit payments received at $t_1, \ldots, t_n$).

The swaption value (4.18) can be written

$$\begin{aligned} SV_t &= E\left\{ e^{-\int_t^{t_0} r(u)du} \sum_{i=1}^n \theta_i p(t_0,t_i)[F_{t_0}-K]^+ \middle| \mathcal{F}_t \right\} \\ &= E\left\{ \sum_i \theta_i e^{-\int_t^{t_i} r(u)du}[F_{t_0}-K]^+ \middle| \mathcal{F}_t \right\}. \end{aligned} \tag{4.19}$$

Now define the *annuity measure* P_A as

$$\frac{dP_A}{dP} = \frac{\sum_i \theta_i e^{-\int_0^{t_i} r(u)du}}{p_A(0)}. \tag{4.20}$$

The swaption value is then expressed in terms of the annuity measure as

$$SV_t = p_A(t)E_A([F_{t_0}-K]^+|\mathcal{F}_t). \tag{4.21}$$

Expression (4.21) shows that a payer's swaption is equivalent to a call option on the swap rate.

Proposition 5 *The forward swap rate F_t is a martingale in the annuity measure, on the interval $t \in [0, t_0]$.*

PROOF: Exercise! (The calculation is very similar to the forward Libor rate case.)

This gives us the Black formula for pricing swaptions. Assume that the swap rate F_{t_0} is log-normal in the annuity measure. In view of Proposition 5, $E_A[F_{t_0}] = F_0$ and the swaption price at time 0 is

$$p_A(0)[F_0N(d1) - KN(d_2)].$$

Finally, we want to understand the change-of-numéraire aspects of the annuity measure. These are complicated by the fact that the swaption exercise value is $\mathcal{F}_{t_0}$-measurable but dP_A/dP given by (4.20) is not $\mathcal{F}_{t_0}$-measurable. The Radon-Nikodym derivative restricted to the σ-field $\mathcal{F}_{t_0}$ is just the conditional expectation

$$\begin{aligned} \left.\frac{dP_A}{dP}\right|_{\mathcal{F}_{t_0}} &= E\left[\frac{\sum \theta_i e^{-\int_0^{t_i} r(s)ds}}{p_A(0)} \middle| \mathcal{F}_{t_0} \right] \\ &= \frac{e^{-\int_0^{t_0} r(s)ds} \sum \theta_i p(t_0,t_i)}{p_A(0)} \\ &= \frac{p_A(t_0)/p_A(0)}{B(t_0)}. \end{aligned}$$

Since this process is a P-martingale, we have shown that moving to the annuity measure P_A is equivalent to a change of numéraire from $B(t)$ to the normalized annuity $p_A(t)/p_A(0)$. Thus the value at time 0 of any $\mathcal{F}_{t_0}$-measurable payment Y received at time t_0 is

$$p_A(0)E_A\left[\frac{Y}{p_A(t_0)}\right].$$

In the swaption case, $Y = p_A(t_0)[F_{t_0}-K]^+$, so the value is $Y = p_A(0)E_A([F_{t_0}-K]^+)$, as we found earlier.

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