Chapter 1

Further Results in Stochastic Analysis

1.1 The Martingale Representation Theorem for Brownian Motion

Let $W_t, t \geq 0$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_t$ be the natural filtration: $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$.

Theorem 1 *Let $T > 0$ and suppose that $X \in L_2(\Omega, \mathcal{F}_T, P)$. Then there exists an adapted process g_t such that $E\int_0^T g^2(s)ds < \infty$ and*

$$X = EX + \int_0^T g(s)dW_s. \tag{1.1}$$

The proof follows from the Lemmas below. First, recall that a subset $\mathcal{D}$ of $L_2(\Omega, \mathcal{F}_T, P)$ is *dense* if for every $X \in L_2(\Omega, \mathcal{F}_T, P)$ we have $\mathcal{D} \cap B \neq \emptyset$ for every neighbourhood B of X. In particular, there exists a sequence $X_n \in \mathcal{D}$ such that $X_n \to X$.

Lemma 1 *Theorem 1 holds if the representation (1.1) holds for every X in some dense subset $\mathcal{D}$ of $L_2(\Omega, \mathcal{F}_T, P)$.*

Proof: Let $X \in L_2(\Omega, \mathcal{F}_T, P)$ and take $X_n \in \mathcal{D}, X_n \to X$ as described above. Then $EX_n \to EX$ and there exist integrands g_n such that

$$X_n = EX_n + \int_0^T g_n(s)dW_s. \tag{1.2}$$

Taking $\tilde{X}_n = X_n - EX_n$ we have the Ito isometry

$$E(\tilde{X}_n - \tilde{X}_m)^2 = E\int_0^T (g_n(s) - g_m(s))^2 ds \tag{1.3}$$

Since $\tilde{X}_n$ is convergent, it is a Cauchy sequence, and hence from (1.3) the sequence g_n is convergent in $L_2(\Omega \times [0,T], dP \times dt)$. Thus there exists g such that

$$E\int_0^T (g_n(s) - g(s))^2 ds \to 0 \quad \text{as } n \to \infty$$

and (1.1) holds with this integrand g. $\diamondsuit$

Let $\mathcal{D}_T$ be the subset of $L_2(\Omega, \mathcal{F}_T, P)$ consisting of random variables X of the form $X = h(W_{t_1}, W_{t_2}, \ldots W_{t_n})$, where n is an integer, h is a bounded continuous function from R^n to R, and $0 \leq t_1 < \ldots < T_n \leq T$. The proof of the following result is an elegant application of the martingale convergence theorem. See Øksendal[1], Lemma 4.3.1.

Lemma 2 *$\mathcal{D}_T$ is dense in $L_2(\Omega, \mathcal{F}_T, P)$.*

To prove the Theorem, it remains to show that any $X \in \mathcal{D}_T$ has the representation property, and this we can show by a direct argument. In the following, we take $n = 2$; the extension to $n > 2$ is obvious. First, a fact about conditional expectation.

Lemma 3 *Let X, Y be random variables taking values in R^n, R^m respectively, on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and suppose that X is independent of $\mathcal{G}$ while Y is $\mathcal{G}$-measurable. Then for any measurable function $f : R^{n+m} \to R$ such that $E|f(X,Y)| < \infty$, we have*

$$E[f(X,Y)|\mathcal{G}] = b(Y),$$

where

$$b(y) = \int_{R^n} f(x,y)\mu_X(dx).$$

Here μ_X is the distribution of X, the measure on the Borel sets $\mathcal{B}^n$ of R^n defined by $\mu_X(B) = P(X \in B)$ for $B \in \mathcal{B}^n$.

PROOF: We have to show that for all bounded real-valued $\mathcal{G}$-measurable random variables Z we have

$$E[Zf(X,Y)] = E[Zb(Y)].$$

Let $\mu_{X,Y,Z}$ be the distribution of the R^{n+m+1}-valued r.v. (X,Y,Z). Since X is independent of $\mathcal{G}$, the random variables X and (Y,Z) are independent, so that $\mu_{X,Y,Z}(dx,dy,dz) = \mu_X(dx)\mu_{Y,Z}(dy,dz)$. Hence

$$\begin{aligned}
E[Zf(X,Y)] &= \int zf(x,y)\mu_{X,Y,Z}(dx,dy,dz) \\
&= \int z \int f(x,y)\mu_X(dx)\mu_{Y,Z}(dy,dz) \\
&= \int zb(y)\mu_{Y,Z}(dy,dz) \\
&= E[Zb(Y)].
\end{aligned}$$

Lemma 4 *Let $h : R^2 \to R$ be a bounded continuous function and let t_1, t_2, t satisfy $0 \leq t_1 \leq t \leq t_2$. Then*

$$E[h(W_{t_1}, W_{t_2})|\mathcal{F}_t] = v_1(t, W_{t_1}, W_t),$$

where

$$v_1(t,x,y) = \int h(x,z)\frac{1}{\sqrt{2\pi(t_2-t)}}e^{(z-y)^2/2(t_2-t)}dz. \tag{1.4}$$

[1] B. Øksendal, Stochastic Differential Equations, 6th ed., Springer-Verlag 2003

PROOF: Writing $h(W_{t_1}, W_{t_2}) = h(W_{t_1}, (W_{t_2} - W_t) + W_t)$, this follows immediately from Lemma 4, on taking $X = W_{t_2} - W_{t_1}, Y = (W_{t_1}, W_t) \in R^2$ and $f(x, y) = h(y_1, x + y_2)$, and recalling that $X \sim N(0, t_2 - t)$. ♢

Lemma 5 *The random variable $X = h(W_{t_1}, W_{t_2})$, as defined in Lemma 4, has the representation property.*

PROOF: It can be checked directly from (1.4) that the function v_1 satisfies

$$\frac{\partial v_1}{\partial t}(t, x, y) + \frac{1}{2}\frac{\partial^2 v_1}{\partial y^2}(t, x, y) = 0$$

and $v_1(T, x, y) = h(x, y)$. Hence by the Ito formula

$$h(W_{t_1}, W_{t_2}) = v_1(T, W_{t_1}, W_{t_2}) = v_1(t_1, W_{t_1}, W_{t_1}) + \int_{t_1}^{t_2} \frac{\partial v_1}{\partial y}(s, W_{t_1}, W_s) dW_s, \tag{1.5}$$

and we know from Lemma 4 that $v_1(t_1, W_{t_1}, W_{t_1}) = E[h(W_{t_1}, W_{t_2})|\mathcal{F}_{t_1}]$. Now define $v_0(t_1, x) = v_1(t_1, x, x)$, and, for $t < t_1$

$$v_0(t, x) = \int v_0(t, z) \frac{1}{\sqrt{2\pi(t_1 - t)}} e^{(z-x)^2/2(t_1-t)} dz. \tag{1.6}$$

As above, we have

$$\frac{\partial v_0}{\partial t}(t, x) + \frac{1}{2}\frac{\partial^2 v_1}{\partial x^2}(t, x) = 0,$$

and the Ito formula gives

$$v_0(t_1, W_{t_1}) = v_1(t_1, W_{t_1}, W_{t_1}) = v_0(0, 0) + \int_0^{t_1} \frac{\partial v_0}{\partial y}(s, W_s) dW_s. \tag{1.7}$$

From (1.5),(1.7) we now see that

$$h(W_{t_1}, W_{t_2}) = v_0(0, 0) + \int_0^{t_2} g(s) dW_s,$$

where

$$g(s) = \begin{cases} (\partial v_0/\partial y)(s, W_s), & s < t_1 \\ (\partial v_1/\partial y)(s, W_{t_1}, W_s), & t_1 \le s < t_2 \end{cases},$$

and that

$$v_0(0, 0) = E[h(W_{t_1}, W_{t_2})].$$

1.2 Changes of Measure

1.2.1 Normal distributions

A random variable X is normally distributed, written $X \sim N(\mu, \sigma^2)$, if its characteristic function ψ takes the form

$$\psi_\mu(u) = Ee^{iuX} = \exp\left(iu\mu - \frac{1}{2}u^2\sigma^2\right). \tag{1.8}$$

This corresponds to the density function ϕ given by

$$\phi_\mu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

μ and σ are the mean and standard deviation respectively. (σ is fixed in the following and so is not included in the notation.)

If $X \sim N(\mu, \sigma^2)$ then for any bounded function f,

$$E[f(X)] = \int f(x)\phi_\mu(x)dx,$$

For any ν we can trivially write this as

$$E[f(X)] = \int f(x)\frac{\phi_\mu(x)}{\phi_\nu(x)}\phi_\nu(x)dx, \tag{1.9}$$

and we find that

$$\frac{\phi_\mu(x)}{\phi_\nu(x)} = \exp\left(\frac{1}{\sigma^2}(\mu-\nu)x - \frac{1}{2\sigma^2}(\mu^2-\nu^2)\right). \tag{1.10}$$

Let us denote by Λ the random variable $\Lambda = \phi_\mu(X)/\phi_\nu(X)$. We find that

- $\Lambda > 0,\ \ E_\nu[\Lambda] = 1$
- $E_\mu[f(X)] = E_\nu[f(X)\Lambda]$, where E_μ denotes integration wrt $N(\mu, \sigma^2)$

To see the first of these, take $f(x) \equiv 1$ in (1.9), or use (1.10) and the fact that if $X \sim N(\nu, \sigma^2)$ then

$$Ee^X = e^{\nu + \frac{1}{2}\sigma^2}.$$

We can thus flip between E_μ and E_ν by introducing Λ, the *likelihood ratio* or *Radon-Nikodym derivative*. In most applications, $\nu = 0$.

1.2.2 A General Setting

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and Λ be a r.v. such that $\Lambda \geq 0$ a.s. and $E\Lambda = 1$. Then we can define a measure Q on $(\Omega, \mathcal{F})$ by

$$QF = \int_F \Lambda dP,\ \ F \in \mathcal{F}. \tag{1.11}$$

Λ is often written dQ/dP and is the Radon-Nikodym derivative of Q wrt P. Note that $PF = 0 \Rightarrow QF = 0$; we say that Q is *absolutely continuous wrt* P, written $Q \ll P$. The *Radon-Nikodym theorem* states that any Q that is absolutely continuous wrt P can be written as (1.11) for some Λ. If $\Lambda > 0$ a.s. then P is absolutely continuous wrt Q, with RN derivative $dP/dQ = 1/\Lambda$. In this case P and Q are said to be *equivalent*, written $P \sim Q$. Measures P and Q are equivalent if and only if they have *the same null sets*: $PF = 0 \Leftrightarrow QF = 0$.

Conditional Expectations

Let X be an integrable r.v. and $\mathcal{G}$ a sub-sigma-field of $\mathcal{F}$. Recall that the conditional expectation of X given $\mathcal{G}$ is the unique $\mathcal{G}$-measurable r.v., denoted $E[X|\mathcal{G}]$ such that

$$\int_G X dP = \int_G E[X|\mathcal{G}] dP.$$

Key properties:

1. $E[X|\mathcal{G}] = X$ if X is $\mathcal{G}$-measurable
2. $E[X|\mathcal{G}] = EX$ if X is independent of $\mathcal{G}$
3. $E[YX|\mathcal{G}] = YE[X|\mathcal{G}]$ if Y is $\mathcal{G}$-measurable
4. For $\mathcal{H} \subset \mathcal{G}$, $E[X|\mathcal{H}] = E[E[X|\mathcal{G}]|\mathcal{H}]$. In particular, $EX = E(E[X|\mathcal{G}])$ for any sub-σ-field $\mathcal{G}$.

Existence of $E[X|\mathcal{G}]$ follows from the Radon-Nikodym theorem. Indeed, the formula $Q(A) = \int_A X dP$ defines a measure on $(\Omega, \mathcal{G})$ that is absolutely continuous wrt P', the restriction of P to $\mathcal{G}$. Hence there exists a $\mathcal{G}$-measurable function Λ such that $Q(A) = \int_A \Lambda dP'$.

The following result will be needed in Section 1.2.3 below.

Lemma 6 *Suppose $X, X_1, X_2, \ldots$ is a sequence of integrable random variables such that $X_n \to X$ in L_1. Then for any σ-field $\mathcal{G}$, $E[X_n|\mathcal{G}] \to E[X|\mathcal{G}]$ in L_1.*

Proof: First we show that if Y is any integrable r.v. then

$$|E[Y|\mathcal{G}]| \leq E[|Y|\,|\mathcal{G}] \text{ a.s.} \tag{1.12}$$

Indeed, denoting as usual $Y^+ = \max(Y, 0)$ and $Y^- = Y^+ - Y$, we have

$$E[Y|\mathcal{G}]^+ = E[Y^+ - Y^-|\mathcal{G}]^+ \leq E[Y^+|\mathcal{G}]^+ = E[Y^+|\mathcal{G}]$$

and

$$E[Y|\mathcal{G}]^- = E[-Y|\mathcal{G}]^+ \leq E[(-Y)^+|\mathcal{G}] = E[Y^-|\mathcal{G}],$$

from which (1.12) follows. Now if $X_n \to X$ in L_1 then using (1.12)

$$\begin{aligned} E\,|E[X_n|\mathcal{G}] - E[X|\mathcal{G}]| &= E\,|E[X_n - X|\mathcal{G}]| \\ &\leq E\,(E[|X_n - X|\,|\mathcal{G}]) \\ &= E|X_n - X| \ \to 0. \end{aligned}$$

Conditional expectation under change of measure

If P, Q are measures on $(\Omega, \mathcal{F})$ such that $Q \ll P$ with RN derivative $\Lambda = dQ/dP$, and $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ then

$$E_Q[X|\mathcal{G}] = \frac{E[X\Lambda|\mathcal{G}]}{E[\Lambda|\mathcal{G}]} \quad \text{a.s. } Q \tag{1.13}$$

To see this, calculate $E[X\Lambda|\mathcal{G}]$ by taking a set $G \in \mathcal{G}$ and using the above properties of conditional expectation. We get

$$\begin{aligned}\int_G E[X\Lambda|\mathcal{G}]dP &= \int_G X\Lambda dP \\ &= \int_G X dQ \\ &= \int_G E_Q[X|\mathcal{G}]dQ \\ &= \int_G E_Q[X|\mathcal{G}]\Lambda dP \\ &= \int_G E_Q[X|\mathcal{G}]E[\Lambda|\mathcal{G}]dP\end{aligned}$$

Thus $\int_G Z dP = 0$ for all $G \in \mathcal{G}$, where $Z = E[X\Lambda|\mathcal{G}] - E_Q[X|\mathcal{G}]E[\Lambda|\mathcal{G}]$ is a $\mathcal{G}$-measurable random variable. Hence $Z = 0$ a.s. This gives (1.13) on noting that, by definition, the set $\{\omega : E[\Lambda|\mathcal{G}] = 0\}$ has Q-measure 0.

Changes of measure and martingales

Take a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t, t \in [0,T])$. Assume for convenience that $\mathcal{F} = \mathcal{F}_T$, and suppose there is another measure Q, defined by $dQ/dP = \Lambda$, where Λ is a non-negative r.v. with $E\Lambda = 1$. An adapted process (X_t) is a *martingale* (under measure P) if it is integrable and for $s \leq t$

$$X_s = E[X_t|\mathcal{F}_s] \quad \text{a.s.}$$

The main result we need is this: *a process Y_t is a Q-martingale if and only if the process $X_t = Y_t\Lambda_t$ is a P-martingale, where $\Lambda_t = E[\Lambda|\mathcal{F}_t]$.* This follows from (1.13). Indeed, for $s < t$ we have

$$\begin{aligned}E_Q[Y_t|\mathcal{F}_s] &= \frac{E[Y_t\Lambda|\mathcal{F}_s]}{E[\Lambda|\mathcal{F}_s]} \\ &= \frac{E[Y_t\Lambda_t|\mathcal{F}_s]}{\Lambda_s}\end{aligned}$$

If Y_t is a Q-martingale the left-hand side is equal to Y_s, so that $Y_t\Lambda_t$ is a martingale, while if $Y_t\Lambda_t$ is a martingale then the right-hand side is equal to Y_s, showing that Y_t is a Q-martingale.

A process X_t is a *local martingale* if there exists a sequence of stopping times τ_n such that $\tau_n \to \infty$ a.s. and for each n the process $X_t^n = X_{t\wedge\tau_n}$ is a martingale. It is also true that *a process Y_t is a Q-local martingale if and only if the process $X_t = Y_t\Lambda_t$ is a P-local martingale.* Exercise: show this.

1.2.3 The Lévy characterization of Brownian Motion

Quadratic variation of Brownian motion

Let W_t be a Brownian motion process and let T be a fixed time. For $n = 1, 2, \ldots$ let $\{t_i^n, i = 0..k_n\}$ be an increasing sequences of times with $t_0^n = 0, t_{k_n}^n = T$. Denote $\Delta W_i = W_{t_{i+1}^n} - W_{t_i^n}$,$\Delta t_i =$

$t^n_{i+1}-t^n_i$ and $S_n = \sum_i \Delta W_i^2$. Note that the r.v. ΔW_i are independent with $E\Delta W_i = 0, E\Delta W_i^2 = \Delta t_i$. Hence that $ES_n = T$ and

$$ES_n^2 = 2\sum_i \Delta t_i^2 + T^2. \tag{1.14}$$

The latter follows from a short calculation using the fact that if $X \sim N(0,\sigma^2)$ then $EX^4 = 3\sigma^4$. From (1.14),

$$\begin{aligned} \operatorname{var}(S_n) &= E(S_n - T)^2 \\ &= 2\sum_i \Delta t_i^2 \\ &\leq 2\max_i\{\Delta t_i\}\sum_i \Delta T_i \\ &= 2T\max_i\{\Delta t_i\}. \end{aligned} \tag{1.15}$$

Hence $S_n \to T$ in L_2 as $n \to \infty$ as long as $\max_i\{\Delta t_i\} \to 0$.

Let us now specialize to the case $t_i^n = i/2^n$. From (1.15) and the Chebyshev inequality, for any $\epsilon > 0$

$$P[|S_n - T)| > \epsilon] \leq \frac{2T2^{-n}}{\epsilon^2}.$$

Taking $\epsilon = 1/n$ we find that

$$\sum_n P\left[|S_n - T)| > \frac{1}{n}\right] \leq \sum_n 2Tn^2 2^{-n} < \infty$$

Hence by the Borel-Cantelli lemma we have

$$P\left[|S_n - T| > \frac{1}{n} \quad \text{infinitely often}\right] = 0,$$

showing that $S_n \to T$ almost surely. Thus for each $T > 0$ the quadratic variation $\mathrm{QV}(T)$ is equal to the deterministic function $\mathrm{QV}(T) = T$.

Suppose now that X_t is a continuous process with sample paths of bounded variation, i.e.

$$\sup_n \sum_i \left|X_{t^n_{i+1}} - X_{t^n_i}\right| < \infty \quad a.s.$$

For example, any process of the form $X_t = \int_0^t \phi(s)ds$ with integrable ϕ satisfies this. Let us compute the quadratic variation of $Y_t = W_t + X_t$. We have

$$\begin{aligned} \sum_i (Y_{t^n_{i+1}} - Y_{t^n_i})^2 &= \sum_i (W_{t^n_{i+1}} - W_{t^n_i} + X_{t^n_{i+1}} - X_{t^n_i})^2 \\ &= \sum_i \Delta W_i^2 + \sum_i \Delta X_i^2 + 2\sum_i \Delta W_i \Delta X_i \end{aligned}$$

where $\Delta W_i = W_{t^n_{i+1}} - W_{t^n_i}$ etc. The first term converges to T and the second and third converge to 0: for the third term,

$$\sum_i (W_{t^n_{i+1}} - W_{t^n_i})(X_{t^n_{i+1}} - X_{t^n_i}) \leq \max_i |W_{t^n_{i+1}} - W_{t^n_i}| \sum_i |X_{t^n_{i+1}} - X_{t^n_i}|.$$

The sum on the right is bounded and the "max" converges to zero because W_t is a continuous function. A similar argument applies to the second term.

We have shown that the quadratic variation of W and Y are the same: the quadratic variation of W is not altered by adding a bounded variation perturbation to the sample path.

Quadratic variation of continuous martingales

We can't treat this subject in complete detail here; see [7] pages 52-55 or [2]. Let M_t be a martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Because of the martingale property,

$$E[(M_t - M_s)^2|\mathcal{F}_s] = E[M_t^2 + M_s^2 - 2M_tM_s|\mathcal{F}_s] = E[M_t^2 - M_s^2|\mathcal{F}_s]. \tag{1.16}$$

and hence with the notation above

$$E\left[\sum_i (M_{t_{i+1}^n} - M_{t_i^n})^2\right] = E\left[\sum_i E\left((M_{t_{i+1}^n} - M_{t_i^n})^2 \Big| \mathcal{F}_{t_i^n}\right)\right] = EM_T^2, \tag{1.17}$$

using (1.16). This suggests that the left-hand side has a limit as $n \to \infty$, the *quadratic variation* of (M_t).

When (M_t) is Brownian motion we have from (1.16) for $t > s$

$$\begin{aligned} E[M_t^2|\mathcal{F}_s] &= E[M_t^2 - M_s^2|\mathcal{F}_s] + M_s^2 \\ &= E[(M_t - M_s)^2|\mathcal{F}_s] + M_s^2 \\ &= t - s + M_s^2. \end{aligned}$$

Hence the process $M_t^2 - t$ is a martingale. The general situation is as follows.

Theorem 2 *Let M_t be a continuous local martingale. Then there is a unique continuous increasing process, denoted $[M]_t$, such that $M_t^2 - [M]_t$ is a local martingale. $[M]_t$ is the quadratic variation of M_t: it is the almost sure limit of approximating sums as in (1.17) taken along suitable sequences (t_i^n).*

The existence of $[M]_t$ gives us an Ito formula for continuous local martingales, analogous to the usual Ito formula for Brownian motion.

Theorem 3 *Let M_t be a continuous local martingale and f a $C^{1,2}$ function. Then*

$$df(t, M_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial M}dM_t + \frac{1}{2}\frac{\partial^2 f}{\partial M^2}d[M]_t \tag{1.18}$$

The Lévy characterization

Theorem 4 *Let M_t be a continuous local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, and suppose that $[M]_t = t,\ t \geq 0$. Then M_t is an $\mathcal{F}_t$-Brownian motion.*

PROOF: Suppose M_t is a continuous local martingale with $[M]_t = t$ and take $f(t, x) = \exp(iux + u^2t/2)$. By applying (1.18) to the real and imaginary parts of f you can check that (1.18) is also valid for complex functions. We obtain

$$df(t, M_t) = \frac{1}{2}u^2 f(t, M_t)dt + iuf(t, M_t)dM_t - \frac{1}{2}u^2 f(t, M_t)d[M]_t,$$

so that $f(t, M_t)$ is a local martingale if $[M]_t = t$. Thus for $t > s$ we have

$$E\left[e^{iuM_{t\wedge\tau_n}+\frac{1}{2}u^2 t\wedge\tau_n} \middle| \mathcal{F}_s \right] = e^{iuM_{s\wedge\tau_n}+\frac{1}{2}u^2 s\wedge\tau_n}, \tag{1.19}$$

where τ_n is a sequence of localizing times. Now the sequence $\exp(iuM_{s\wedge\tau_n} + \frac{1}{2}u^2(s \wedge \tau_n))$ is bounded and converges almost surely (and hence in L_1) to $\exp(iuM_s + \frac{1}{2}u^2 s)$. By Lemma 6, the conditional expectation in (1.19) converges in L_1 to the conditional expectation of the limit, and we conclude that

$$E\left[e^{iuM_t+\frac{1}{2}u^2 t} \middle| \mathcal{F}_s \right] = e^{iuM_s+\frac{1}{2}u^2 s},$$

or, equivalently,

$$E\left[e^{iu(M_t-M_s)} \middle| \mathcal{F}_s \right] = e^{-\frac{1}{2}u^2(t-s)}. \tag{1.20}$$

Now let Y be any $\mathcal{F}_s$-measurable random variable, and ψ_Y be the characteristic function of Y. Then by Property (3) of conditional expectation (see Section 1.2.2 above) the joint characteristic function of Y and $M_t - M_s$ is

$$\begin{aligned}
\psi_{Y,M_t-M_s}(v,u) &= E\left[e^{i(vY+u(M_t-M_s))} \right] \\
&= E\left[e^{ivY} e^{iu(M_t-M_s)} \right] \\
&= E\left[e^{ivY} E\left[e^{iu(M_t-M_s)} \middle| \mathcal{F}_s \right] \right] \\
&= E\left[e^{ivY} \right] e^{-\frac{1}{2}u^2(t-s)} \\
&= \psi_Y(v)\psi_{M_t-M_s}(u).
\end{aligned}$$

Thus Y and $(M_t - M_s)$ are independent, implying – since Y is arbitrary – that $(M_t - M_s)$ is independent of $\mathcal{F}_s$. From (1.20), $(M_t - M_s)$ is normally distributed with mean 0 and variance $t - s$. Hence (M_t) is an $(\mathcal{F}_t)$ Brownian motion. ♢

1.2.4 The Girsanov Theorem

The Girsanov theorem states that, for Brownian motion, absolutely continuous change of measure is equivalent to change of drift.

Theorem 5 *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a filtered probability space, where $0 < T < \infty$ and we assume for convenience that $\mathcal{F} = \mathcal{F}_T$. Let w_t be an $(\mathcal{F}_t, P)$-Brownian motion.*
(a) Let $g(t)$ be an adapted process satisfying $\int_0^T g^2(s)ds < \infty$ a.s. and define

$$\Lambda_T = \exp\left(\int_0^T g(s)dw_s - \frac{1}{2}\int_0^T g^2(s)ds \right). \tag{1.21}$$

Suppose that $E[\Lambda_T] = 1$, and define a measure Q on $(\Omega, \mathcal{F})$ by $dQ/dP = \Lambda_T$. Then under measure Q the process $\tilde{w}_t$ defined by

$$\tilde{w}_t = w_t - \int_0^t g(s)ds$$

is an $\mathcal{F}_t$ Brownian motion.
(b) Suppose $\mathcal{F}_t$ is the natural filtration of w_t and that Q is a measure such that $Q \sim P$. Then there exists a process $g(t)$ such that dQ/dP is equal to Λ_T defined by (1.21).

Proof: (a) The assumption that $E\Lambda_T = 1$ ensures that Q is a probability measure. Applying the Ito formula, we find that

$$d(\tilde{w}\Lambda) = \Lambda(\tilde{w}g + 1)dw,$$

so that $\tilde{w}\Lambda$ is a local martingale which implies, as shown in section 1.2.2, that $\tilde{w}$ is a Q-local martingale. Certainly $\tilde{w}$ has continuous sample paths, and by the argument in section 1.2.3 the quadratic variation of $\tilde{w}$ is equal to t. By the Lévy characterization, $\tilde{w}$ is a Q-Brownian motion.
(b) Let Q be an equivalent measure and define $\Lambda_T = dQ/dP$. Then $\Lambda_T > 0$ a.s. and $E\Lambda_T = 1$. For any $t \in [0, T]$ let P^t, Q^t denote the restrictions of P and Q to $\mathcal{F}_t$. Then $P^t \sim Q^t$ and the Radon-Nikodym derivative is $dQ^t/dP^t := \Lambda_t = E[\Lambda_T|\mathcal{F}_t]$. Hence $\Lambda_t > 0$ a.s. By the martingale representation theorem for Brownian motion, there exists an integrand ϕ such that $\int_0^T \phi^2(t)dt < \infty$ and

$$\Lambda_t = 1 + \int_0^t \phi(s)dw_s, \quad 0 \leq t \leq T. \tag{1.22}$$

Now apply the Ito formula to calculate

$$d\log\Lambda_t = \frac{1}{\Lambda_t}\phi(t)dw_t - \frac{1}{2}\frac{1}{\Lambda_t^2}\phi^2(t)dt.$$

Thus Λ_T is given by (1.21) with $g(t) = \phi(t)/\Lambda_t$. $\diamondsuit$

Remarks (a) Let M_t be a non-negative local martingale, i.e. for times $\tau_n \uparrow \infty$, for $t > s$

$$M_{s\wedge\tau_n} = E[M_{t\wedge\tau_n}|\mathcal{F}_s].$$

Thus, using Fatou's lemma for conditional expectation,

$$\begin{aligned} M_s &= \liminf{}_n M_{s\wedge\tau_n} \\ &= \liminf{}_n E[M_{t\wedge\tau_n}|\mathcal{F}_s] \\ &\geq E[\liminf{}_n M_{t\wedge\tau_n}|\mathcal{F}_s] \\ &= E[M_t|\mathcal{F}_s]. \end{aligned}$$

Thus any non-negative local martingale is a supermartingale, so that in particular EM_t is a decreasing function of t. Now Λ_T defined by (1.21) is a non-negative local martingale, so the assumption that $E\Lambda_T = 1$ implies that $E\Lambda_t = 1$ for all $t \in [0, T]$, since $\Lambda_0 = 1$ a.s.

(b) The best general sufficient condition implying $E\Lambda_T = 1$ is the *Novikov condition*

$$E\exp\left(\frac{1}{2}\int_0^T g^2(s)ds\right) < \infty.$$