Chapter 2

The Black Scholes World

2.1 The Model

To start with we consider a world with just one risky asset with price process S_t and a risk-free savings account paying constant interest rate r with continuous compounding. Everything takes place in a finite time interval $[0, T]$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, (w_t)_{t\in[0,T]})$ be Wiener space, i.e. w_t is Brownian motion, $\mathcal{F}_t$ is the natural filtration of w_t and $\mathcal{F} = \mathcal{F}_t$. The price process S_t is supposed to be *geometric Brownian motion*: S_t satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t dw_t \tag{2.1}$$

for given *drift* μ and *volatility* σ. (2.1) has a unique solution: if S_t satifies (2.1) then by the Ito formula

$$d \log S_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dw_t,$$

so that S_t satisfies (2.1) if and only if

$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma w_t). \tag{2.2}$$

Note that this makes S_t very easy to simulate: for any increasing sequence of times $0 = t_0 < t_1 \ldots$,

$$S_{t_i} = S_{t_{i-1}} \exp\left((\mu - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}} X_i\right),$$

where $X_1, X_2, \ldots$ is a sequence of independent $N[0, 1]$ random variables. This representation is *exact*. Another thing that follows from (2.2) is that $ES_t = S_0 e^{\mu t}$.

In the interests of symmetry we want the savings account also to be expressed as a traded asset, i.e. we should invest in it by buying a certain number of units of something. A convenient 'something' is a *zero-coupon bond*

$$B_t = \exp(-r(T - t)).$$

This grows, as required, at rate r:

$$dB_t = rB_t dt \tag{2.3}$$

Note that (2.3) does not depend on the final maturity T (the same growth rate is obtained from any ZC bond) and the choice of T is a matter of convenience as we will see below.

2.2 Portfolios and Trading Strategies

If we hold ϕ and ψ units of S and B respectively at time t then we have a portfolio whose time-t value is $\phi S_t + \psi B_t$. The assumptions of Black-Scholes are that we have a *frictionless market*, meaning that S and B can be traded in arbitrary amounts with no transaction costs, and short positions are allowed. In particular this means we can invest in, or borrow from, the riskless account at the same rate r of interest. A *trading strategy* is then a triple (ϕ_t, ψ_t, x_0), where x_0 is the initial endowment and (ϕ_t, ψ_t) is a pair of adapted processes satisfying

$$\int_0^T \phi_t^2 dt < \infty \quad \text{a.s.}, \qquad \int_0^T |\psi_t| dt < \infty.$$

The *gain from trade* in $[s, t]$ is then

$$\int_s^t \phi_u dS_u + \int_s^t \psi_u dB_u.$$

(Note how this matches up with the definition of the Ito integral!) A portfolio is *self-financing* if

$$\phi_t S_t + \psi_t B_t - \phi_s S_s - \psi_s B_s = \int_s^t \phi_u dS_u + \int_s^t \psi_u dB_u.$$

The increase in portfolio value is entirely due to gains from trade.

2.3 Arbitrage and Valuation

Denote by V_t the portfolio value at time t, i.e. $V_t = \phi_t S_t + \psi_t B_t$. An *arbitrage opportunity* is the existence of a self-financing trading strategy and a time t such that $V_0 = 0$, $V_t \geq 0$ a.s. and $P[V_t > 0] > 0$ (or, equivalently, $EV_t > 0$.) It is axiomatic that arbitrage cannot exist in the market, so no mathematical model should permit arbitrage opportunities.

2.3.1 Forwards

Consider a forward contract in which we fix a price K now to be paid at time T for delivery of 1 unit of S_T. *The unique no-arbitrage value of* K *is* $F = e^{rT} S_0$. Indeed, suppose someone offers us a forward contract at $K < F$. We sell one share and invest the proceeds S_0 in the bank. At time T we get the share back for a payment of K but the value of our bank account is $F > K$. We make a riskless profit of $F - K$. If we are able to offer a forward at $K > F$ then we should borrow S_0 and buy the share. Again, there is a riskless profit at time T. (This argument is independent of the pricing model for S_t.)

2.3.2 Put-Call Parity

A call option with strike K and exercise time T has exercise value $[S_T - K]^+$, and a put has exercise value $[K - S_T]^+$. Clearly

$$[S_T - K]^+ - [K - S_T]^+ = S_T - K,$$

so that buying a call and selling a put at time zero is equivalent to buying a forward and agreeing to pay K at time T. Thus whatever the prices C_0 and P_0 at time 0, they must satisfy

$$C_0 - P_0 = (F - K)B_0.$$

Note again that this is completely model-independent.

2.3.3 Replication

Suppose there is a contingent claim with exercise value $h(S_T)$ at time T (for example a put or call option) and there exists a self-financing trading strategy (ϕ, ψ, x_0) such that $V_T = h(S_T)$ a.s. Then x_0 is the unique no-arbitrage price of the contingent claim. The arbitrage, if available, is realized by selling the contingent claim and going long the replicating portfolio, or *vice versa.* This argument is sometimes known as the *law of one price*: if two assets have identical cash flows in the future then they must have the same value now.

2.4 Black-Scholes: the Original Proof

Black's and Scholes' original proof of the famous formula [1] was a very direct argument showing that a replicating portfolio exists for the European call option. Here is a version of that argument.

The idea is to assume a whole lot of things and then show they are all true. The first assumption is that there is a smooth function $C(t, S)$ such that the call option has a value $C(t, S_t)$ at time $t < T$, with $\lim_{t\uparrow T} C(t, S) = [S - K]^+$. Suppose we form a portfolio in which we are long one unit of the call option and short a self-financing portfolio $(\phi, \psi, C(0, S_0))$. The value of this portfolio at time t is then

$$X_t = C(t, S_t) - \phi_t S_t - \psi_t B_t,$$

with, in particular, $X_0 = 0$. By the Ito formula and the self-financing property,

$$dX_t = \frac{\partial C}{\partial S} dS + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt - \phi_t dS_t - \psi_t dB_t.$$

If we choose $\phi_t = \partial C/\partial S$ and use the fact that $dB = rBdt$ we see that

$$dX_t = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} dt - \psi_t r B_t \right) dt.$$

Let us now choose

$$\psi_t = \frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}}{rB_t},$$

heroically assuming that in doing so we have not destroyed that self-financing property. Then $X_t \equiv 0$, so that

$$C = \phi S + \psi B = S\frac{\partial C}{\partial S} + \frac{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}}{r},$$

showing that C must satisfy the Black-Scholes PDE

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{2.4}$$

with boundary condition

$$C(T, S) = [S - K]^+. \tag{2.5}$$

Equations (2.4),(2.5) are enough to determine the function C, as we will show below. Is (ϕ, ψ, x_0) in fact self-financing? By definition $\phi S + \psi B = C$ (since $X_t \equiv 0$) and

$$\begin{aligned}
\int_0^t \phi dS + \int_0^t \psi dB &= \int_0^t \frac{\partial C}{\partial S} dS + \int_0^t \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) du \\
&= \int_0^t dC \\
&= C(t, S_t) - C(0, S_0).
\end{aligned}$$

This confirms the self-financing property. We have now shown that the call option can be replicated by a self-financing portfolio with initial endowment $C(0, S_0)$, so by the argument in section 2.3.3 this is the unique arbitrage-free price.

In the above argument, no special role is played by the call option exercise function $[S_T - K]^+$. It simply provides the boundary condition for the Black-Scholes PDE (2.4). If we used another boundary condition $C(T, S) = h(S)$ then the corresponding solution of (2.4) would give us the no-arbitrage value and hedging strategy for a contingent claim with exercise value $h(S_T)$.

Example: the Forward Price. It is easy to check that

- $C(t, S) = S$ satisfies (2.4) with boundary condition $C(T, S) = S$.
- For a constant K, $C(t, S) = e^{-r(T-t)}K$ satisfies (2.4) with boundary condition $C(T, S) = K$.

Since (2.4) is a linear equation, the value at time 0 of receiving $S_T - K$ at time T is therefore $S_0 - Ke^{-rT}$, which is equal to zero when $K = e^{rT}S_0$, the forward price. You can check (please do!) that the hedging strategy $\phi = \partial C/\partial S$ implied by Black-Scholes coincides with the strategy given in section 2.3.1.

2.4.1 Probabilistic solution of the Black-Scholes PDE

Suppose we have an SDE

$$dx_t = m(x_t)dt + g(x_t)dw_t$$

where m, g are Lipschitz continuous functions so that a solution exists. The differential generator of x_t is the operator $\mathcal{A}$ defined by

$$\mathcal{A}f(x) = m(x)\frac{\partial f}{\partial x} + \frac{1}{2}g^2(x)\frac{\partial^2 f}{\partial x^2}$$

so the Ito formula can be written

$$df(x_t) = \mathcal{A}f(x_t)dt + \frac{\partial f}{\partial x}g(x_t)dw_t.$$

Now consider the following PDE for a function $v(t, x)$

$$\begin{aligned}
\frac{\partial v}{\partial t} + \mathcal{A}v(t, x) - rv(t, x) &= 0, \quad t < T, && (2.6) \\
v(T, x) &= \Xi(x), && (2.7)
\end{aligned}$$

where r is a given constant and Ξ a given function. If v satisfies this then applying the Ito formula we find that

$$\begin{aligned} d(e^{-rt}v(t,x_t)) &= -re^{-rt}v(t,x_t)dt + e^{-rt}\left(\frac{\partial v}{\partial t} + \mathcal{A}v(t,x_t)dt + \frac{\partial v}{\partial x}g(t,x_t)dw_t\right) \\ &= e^{-rt}\frac{\partial v}{\partial x}g(t,x_t)dw_t. \end{aligned} \tag{2.8}$$

Thus $\exp(-rt)v(t,x_t)$ is a local martingale. If it is a martingale then integrating from t to T and using (2.7) we see that

$$v(t,x_t) = E_{t,x}\left[e^{-r(T-t)}\Xi(x_T)\right]$$

This is the probabilistic representation of the solution of the PDE (2.6),(2.7).

Comparing (2.4),(2.5) with (2.6),(2.7) we see that these equations match up when $m(x) = rx$ and $g(x) = \sigma x$, i.e. x_t satisfies

$$dx_t = rx_t\,dt + \sigma x_t\,dw_t. \tag{2.9}$$

Now return to the price model (2.1) and introduce a measure change

$$\frac{dQ}{dP} = \exp\left(\alpha w_T - \frac{1}{2}\alpha^2 T\right),$$

where α is a constant. By Girsanov, $d\breve{w} = dw - \alpha dt$ is a Q-Brownian motion, in terms of which (2.1) becomes

$$dS_t = \mu S_t\,dt + \sigma S_t(d\breve{w}_t + \alpha\,dt).$$

Choosing $\alpha = (r-\mu)/\sigma$ we get

$$dS_t = rS_t\,dt + \sigma S_t d\breve{w}_t, \tag{2.10}$$

the same equation as (2.9). Equation (2.10) is the price process expressed in the *risk-neutral measure* Q, and the above argument shows that the probabilistic solution of the Black-Scholes PDE (2.4),(2.5) is

$$C(t,S) = E^Q_{t,S}\left(e^{-r(T-t)}[S_T - K]^+\right).$$

This is however easily computed since S_T is given explicitly in terms of w_T by (2.2) (with r replacing μ). We get

$$C(t,S) = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[S\exp((r-\sigma^2/2)(T-t) - \sigma x\sqrt{T-t}) - K]^+ e^{-x^2/2}dx.$$

A short calculation gives the final expression

$$C(t,S) = SN(d_1) - e^{-r(T-t)}KN(d_2) \tag{2.11}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution function and

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

We can now tie up the loose ends of the argument. It can be checked directly that C defined by (2.11) does satisfy the Black-Scholes PDE (2.4),(2.5), and another short calculation (see Problems II!) shows that $\partial C/\partial S = N(d_1)$, so that in particular $0 < \partial C/\partial S < 1$. Hence the integrand in (2.8) is square-integrable and the stochastic integral is a martingale, as required.

Another version of the formula, often more useful, is this. Recall that the *forward price* at t for delivery at T is $F = Se^{r(T-t)}$. We can therefore express (2.11) as

$$C(t,S) = e^{-r(T-t)}(FN(d_1) - KN(d_2)), \tag{2.12}$$

and d_1 can be expressed as

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

2.5 Proof by Martingale Representation

Let ϕ be an adapted process with

$$\int_0^T \phi_t^2 S_t^2\, dt < \infty \text{ a.s.} \tag{2.13}$$

and let X_t be a process defined by

$$dX_t = \phi_t\, dS_t + (X_t - \phi_t S_t)r\, dt, \quad X_0 = x_0. \tag{2.14}$$

The interpretation is that X_t is the portfolio value corresponding to the trading strategy (ϕ, ψ, x_0) where

$$\psi_t = \frac{X_t - \phi_t S_t}{B(t)}, \tag{2.15}$$

i.e. ϕ_t units of the risky asset are held and the remaining value $X_t - \phi_t S_t$ is held in the savings account. This strategy is always self-financing since X_t is by definition the gains from trade process, while the value is $\phi S + \psi B = X$. Applying the Ito formula we find that, with $\tilde{X}_t = e^{-rt}X_t$,

$$\begin{aligned} d\tilde{X}_t &= \phi e^{-rt}S((\mu - r)\, dt + \sigma\, dw) && (2.16) \\ &= \phi\tilde{S}\sigma d\check{w}, && (2.17) \end{aligned}$$

where $\tilde{S}_t = e^{-rt}S_t$. (The first line (2.16) shows incidentally that (2.14) has a unique solution.) Thus $e^{-rt}X_t$ is a local martingale in the risk-neutral measure Q.

Suppose we have an option whose exercise value at time T is H, where H is an $\mathcal{F}_T$-measurable random variable with $EH^2 < \infty$. By the martingale representation theorem there is an integrand g such that

$$e^{-rT}H = E_Q[e^{-rT}H] + \int_0^T g_t\, d\check{w}_t.$$

Define $\phi_t = e^{rt}g_t/(\sigma S_t)$ and then ψ by (2.15) and $x_0 = E_Q[e^{-rT}H]$. Then $\phi\tilde{S}\sigma = g$ and the trading strategy (ϕ, ψ, x_0) generates a portfolio value process such that $X_T = H$ a.s., i.e. (ϕ, ψ, x_0) is a replicating portfolio for H. It follows that the option value is

$$x_0 = E_Q[e^{-rT}H].$$

Note this is a much more general result than that obtained by the previous argument, in that the option payoff can be an arbitrary, possibly 'path-dependent', random variable, whereas before we assumed it took a value of the form $H = h(S_T)$. On the other hand the above argument only asserts that a replicating portfolio exists: it does not give an explicit formula for ϕ.

Theorem 6 *Let Φ be the class of investment strategies ϕ_t such that (a) the integrability condition (2.13) is satisfied, and (b) there exists a positive constant A_ϕ such that $X_t \geq -A_\phi$ for all $t \in [0, T]$, where X_t is the process defined by (2.14). In the Black-Scholes model, no strategy $\phi \in \Phi$ is an arbitrage opportunity.*

PROOF: Suppose X_t is given by (2.14) for some strategy $\phi \in \Phi$ and $X_0 = 0$. Then, from (2.17), the discounted process $\tilde{X}_t$ is a Q-local martingale which is bounded below by the constant $-A_\phi$. Thus $\tilde{X}_t + A_\phi$ is a non-negative local martingale, and hence a supermartingale. Therefore $\tilde{X}_t$ is a supermartingale and has decreasing expectation: for any $t > 0$

$$0 = E_Q[\tilde{X}_0] \geq E_Q[\tilde{X}_t]. \tag{2.18}$$

On the other hand, if $X_t \geq 0$ a.s.(P) and $P[X_t > 0] > 0$ then, since P and Q are equivalent measures, $E_Q[\tilde{X}_t] > 0$, which is incompatible with (2.18). Hence there cannot be an arbitrage opportunity as defined in Section 2.3. $\diamondsuit$

2.6 Robustness of Black-Scholes Hedging

If we assume the Black-Scholes price model (2.1) then the price at time t of an option with exercise value $h(S_T)$ is $C_h(S_t, r, \sigma, t) = C(t, S_t)$ where $C(t, S)$ satisfies the Black-Scholes PDE (2.4) with boundary condition $C(T, S) = h(S)$.

Suppose we sell an option at implied volatility $\hat{\sigma}$, i.e. we receive at time 0 the premium $C_h(S_0, r, \hat{\sigma}, 0)$, and we hedge under the assumption that the model (2.1) is correct with $\sigma = \hat{\sigma}$. The hedging strategy is then 'delta hedging': the number of units of the risky asset held at time t is the so-called option 'delta' $\partial C / \partial S$:

$$\phi_t = \frac{\partial C}{\partial S}(t, S_t). \tag{2.19}$$

Suppose now that the model (2.1) is *not* correct, but the 'true' price model is

$$dS_t = \alpha(t, \omega) S_t dt + \beta(t, \omega) S_t dw_t, \tag{2.20}$$

where w_t is an $\mathcal{F}_t$-Brownian motion for some filtration $\mathcal{F}_t$ (not necessarily the natural filtration of w_t) and α_t, β_t are $\mathcal{F}_t$-adapted, say bounded, processes. It is no loss of generality to write the drift and diffusion in (2.20) as $\alpha S, \beta S$: since $S_t > 0$ a.s. we could always write a general diffusion coefficient γ as $\gamma_t = (\gamma_t / S_t) S_t \equiv \alpha_t S_t$. In fact the model (2.20) is saying little more than that S_t is a positive process with continuous sample paths.

Using strategy (2.19) the value X_t of the hedging portfolio is given by $X_0 = C(0, S_0)$ and

$$dX_t = \frac{\partial C}{\partial S} dS_t + \left(X_t - \frac{\partial C}{\partial S} S_t \right) r\, dt$$

where S_t satisfies (2.20). By the Ito formula, $Y_t \equiv C(t, S_t)$ satisfies

$$dY_t = \frac{\partial C}{\partial S} dS + \left(\frac{\partial C}{\partial t} + \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$

Thus the hedging error $Z_t \equiv X_t - Y_t$ satisfies

$$\frac{d}{dt} Z_t = rX_t - rS_t \frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} - \frac{1}{2}\beta^2 S_t^2 \frac{\partial^2 C}{\partial S^2}.$$

Using (2.4) and denoting $\Gamma_t = \Gamma(t, S_t) = \partial^2 C(t, S_t)/\partial s^2$, we find that

$$\frac{d}{dt} Z_t = rZ_t + \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2).$$

Since $Z_0 = 0$, the final hedging error is

$$\boxed{Z_T = X_T - h(S_T) = \int_0^T e^{r(T-s)} \frac{1}{2} S_t^2 \Gamma_t^2 (\hat{\sigma}^2 - \beta_t^2) dt.}$$

Comments:

This is a key formula, as it shows that successful hedging is quite possible even under significant model error. It is hard to imagine that the derivatives industry could exist at all without some result of this kind. Notice that:

- Successful hedging depends entirely on the relationship between the Black-Scholes implied volatility $\hat{\sigma}$ and the true 'local volatility' β_t. For example, if we are lucky and $\hat{\sigma}^2 \geq \beta_t^2$ a.s. for all t then the hedging strategy (2.19) makes a profit *with probability one* even though the true price model is substantially different from the assumed model (2.1), as long as $\Gamma_t \geq 0$, which holds for standard puts and calls.

- The hedging error also depends on the option convexity Γ. If Γ is small then hedging error is small even if the volatility has been underestimated.

2.7 Options on Dividend-paying Assets

Holders of ordinary shares receive dividends, which are cash payments normally quoted as "x pence per share", paid on specific dates with the value x being announced some time in advance. For a stock index, where the constituent stocks are all paying different dividends at different times, it makes sense to think in terms of a *dividend yield*, the dividend per unit time expressed as a fraction of the index value. In mathematical terms, we assume that a dividend is a continuous-time payment stream, the dividend paid in a time interval dt being $qS_t dt$. Thus q is the dividend yield. In this section we analyse the case where q is a fixed constant. Equation (2.14), describing the evolution of a self-financing portfolio, must be modified to

$$\begin{aligned} dX_t &= \phi_t \, dS_t + q\phi_t S_t dt + (X_t - \phi_t S_t) r \, dt, \quad X_0 = x_0. && (2.21) \\ &= \phi_t S_t (\mu + q - r) dt + X_t r \, dt + \phi_t S_t \sigma dw_t, && (2.22) \end{aligned}$$

so that

$$d\left(e^{-rt} X_t\right) = \phi \tilde{S} (\mu + q - r) dt + \phi \tilde{S} \sigma \, dw. \quad (2.23)$$

Now change to a martingale measure Q_q such that

$$dw^q = dw + \frac{\mu + q - r}{\sigma} dt$$

is a Q_q-Brownian motion. Then (2.23) becomes simply

$$d\left(e^{-rt}X_t\right) = \phi\tilde{S}\sigma\, dw^q.$$

Thus by the argument of the previous section, the price at time 0 of a contingent claim H is

$$p = E_{Q^q}\left[e^{-rT}H\right]. \tag{2.24}$$

In particular, take $H = S_T$. Then p is the no-arbitrage price now for delivery of 1 unit of the asset at time T, or, equivalently, $e^{rT}p$ is the forward price.

In (2.21), take $X - \phi S = 0$, so that all receipts are re-invested in the risky asset S, nothing being held in the bank account. Then $\phi = X/S$, so that

$$\begin{aligned} dX &= \frac{X}{S}dS + q\frac{X}{S}S\,dt \\ &= X(\mu\,dt + \sigma\,dw) + qX\,dt \\ &= X((\mu+q)dt + \sigma\,dw). \end{aligned} \tag{2.25}$$

On the other hand, if we define $\hat{S}_t = e^{qt}S_t$ and use (2.1) and the Ito formula, we find that

$$d\hat{S}_t = \hat{S}_t((\mu+q)dt + \sigma\,dw). \tag{2.26}$$

From (2.25) and (2.26) we see that $X_t = \hat{S}_t = e^{qt}S_t$ for all $t > 0$ if $X_0 = S_0$. Now the solution of (2.25) is linear in the initial condition, so if $X_0 = e^{-qT}S_0$ then $X_T = S_T$ a.s. We have shown the following.

Proposition 1 *(i) For an asset with a constant dividend yield q, the forward price at time T is $F_T = e^{(r-q)T}S_0$. The replicating strategy that delivers one unit of the asset at time T consists of buying e^{-qT} units of the asset at time 0 and reinvesting all dividends in the asset.*

(ii) The value of a call option on the asset with exercise time T and strike K is

$$C(S_0, K, r, q, \sigma, T) = e^{-rT}(F_T N(d_1) - KN(d_2)), \tag{2.27}$$

where

$$d_1 = \frac{\log(F_T/K) + \sigma^2 T/2}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

PROOF: Only part (ii) remains to be proved. Under measure Q_q, S_t satisfies

$$dS_t = (r-q)S_t dt + \sigma S_t dw_t^q, \tag{2.28}$$

and the option value is given by (2.24) with $H = [S_T - K]^+$. This is exactly the same calculation as standard Black-Scholes, but with $(r-q)$ replacing r in the price equation (2.28) (but not in the 'discount factor' e^{-rT} in (2.24)). Formula (2.27) follows from (2.12). ♢

2.8 Barrier Options

Let S_t be a price process and let $M_t = \max_{0 \le u \le t} S_u$ be the maximum price to date. An *up-and-out* call option has exercise value

$$[S_T - K]^+ \mathbf{1}_{M_T < B}.$$

It pays the standard call payoff if $S_t < B$ for all $t \in [0, T]$ and zero otherwise. B is the 'barrier', and to make sense, we must have $S_0 < B, K < B$. An *up-and-in* call option pays

$$[S_T - K]^+ \mathbf{1}_{M_T \ge B}.$$

The sum of these two payoffs is an ordinary call, so we only need to value one of the above. There are analogous definitions for down-and-out and down-and-in options.

Remarkably, there are analytic formulas for the values of these options in the Black-Scholes world. These formulas – but not the proofs – can be found on pages 462-464 of Hull's book [5]

The starting point is the so-called reflection principle for Brownian motion. Let x_t be standard Brownian motion starting at zero and $m_t = \max_{s \le t} x_s$. The reflection principle states that for $y > 0$ and $x \le y$,

$$P[m_t \ge y, x_t < x] = N\left(\frac{x - 2y}{\sqrt{t}}\right). \tag{2.29}$$

The idea is that those paths that do hit level y before time t 'restart' from level y with symmetric distribution (see figure 2.1), so there is equal probability that they will be below $x = y - (y - x)$ or above $y + (y - x) = 2y - x$ at time t. But

$$\begin{aligned} P[m_t \ge y, x_t \ge 2y - x] &= P[x_t \ge 2y - x] \\ &= 1 - N\left(\frac{2y - x}{\sqrt{t}}\right) \\ &= N\left(\frac{x - 2y}{\sqrt{t}}\right) \end{aligned}$$

Now $[x_t < x] = [m_t < y, x_t < x] \cup [m_t \ge y, x_t < x]$, so

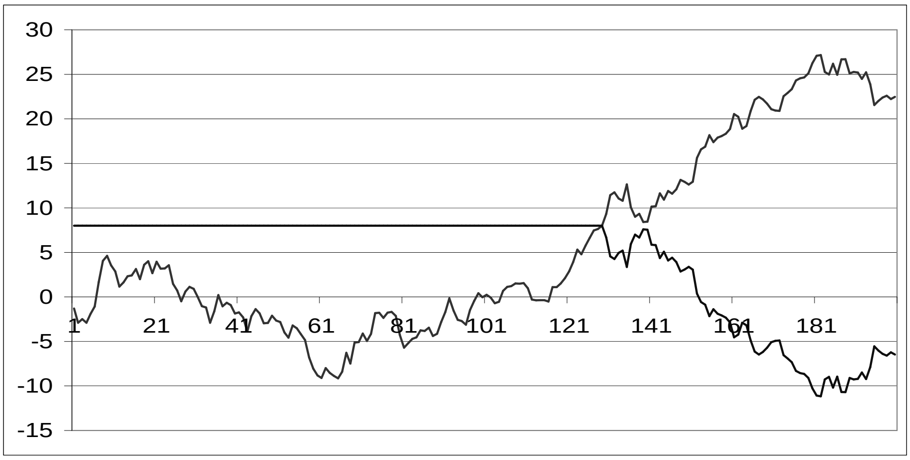

Figure 2.1: Reflection principle

$$N\left(\frac{x}{\sqrt{t}}\right) = P[x_t < x] = P[m_t < y, x_t < x] + P[m_t \geq y, x_t < x]. \tag{2.30}$$

We have shown the following.

Proposition 2 *The joint distribution of x_t, the Brownian motion at time t, and its maximum-to-date m_t is given by*

$$F_0(y, x) = P[m_t < y, x_t < x] = N\left(\frac{x}{\sqrt{t}}\right) - N\left(\frac{x - 2y}{\sqrt{t}}\right) \tag{2.31}$$

This argument depends on symmetry and doesn't work if x_t has drift. We can use the Girsanov theorem to get the answer in this case. If P_ν denotes the probability measure of BM with drift ν (i.e. $x_t = \nu t + w_t$ where w_t is ordinary BM) then we know that on the interval $[0, T]$

$$\frac{dP_\nu}{dP_0} = \exp(\nu x_T - \frac{1}{2}\nu^2 T).$$

Thus if f is any integrable function then, using (2.31),

$$\begin{aligned} E_\nu[\mathbf{1}_{m_T<y} f(x_T)] &= E_0\left[\mathbf{1}_{m_T<y} f(x_T)\frac{dP_\nu}{dP_0}\right] \\ &= E_0\left[\mathbf{1}_{m_T<y} f(x_T)\exp\left(\nu x_T - \frac{1}{2}\nu^2 T\right)\right] \\ &= \int_{-\infty}^{y} f(x) e^{(\nu x - \nu^2 T/2)} \frac{1}{\sqrt{T}}\left(\phi(x/\sqrt{T}) - \phi((x-2y)/\sqrt{T})\right) dx, \end{aligned}$$

where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ is the standard normal density function. Now clearly

$$\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{1}{2T}x^2 + \nu x - \frac{1}{2}\nu^2 T\right) = \frac{1}{\sqrt{T}}\phi\left(\frac{x - \nu T}{\sqrt{T}}\right)$$

while after some calculation we find that

$$\frac{1}{\sqrt{2\pi T}}\exp\left(-\frac{1}{2T}(x-2y)^2 + \nu x - \frac{1}{2}\nu^2 T\right) = \frac{e^{2y\nu}}{\sqrt{T}}\phi\left(\frac{x - 2y - \nu T}{\sqrt{T}}\right).$$

This gives us the final result: the joint distribution function with drift ν is

$$F_\nu(y, x) = P_\nu[m_T < y, x_T < x] = \left(N\left(\frac{x - \nu T}{\sqrt{T}}\right) - e^{2y\nu} N\left(\frac{x - 2y - \nu T}{\sqrt{T}}\right)\right). \tag{2.32}$$

This does coincide with F_0 when $\nu = 0$. A good reference for the above argument is Harrison [3].

Let us now return to barrier option pricing. The price process in the risk-neutral measure is

$$S_T = S_0 \exp((r - \sigma^2/2)T + \sigma w_T)$$

which we can write as

$$S_T = S_0 e^{\sigma x_T}$$

where $x_T = w_T + \nu T$ with

$$\nu = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right).$$

The price S_T is in the money but below the barrier level when $x_T \in (a_1, a_2)$ where

$$a_1 = \frac{1}{\sigma} \log\left(\frac{K}{S_0}\right), \qquad a_2 = \frac{1}{\sigma} \log\left(\frac{B}{S_0}\right).$$

Denoting $g(y,x) = \partial F_\nu(y,x)/\partial x$, the option value can now be expressed as

$$E_\nu\left[e^{-rT}[S_T - K]^+ \mathbf{1}_{M_T<B}\right] = e^{-rT} \int_{a_1}^{a_2} (S_0 e^{\sigma x} - K) g(y,x) dx.$$

Doing the calculations we obtain the option value given in [5] as a sum of four terms of the form $c_1 N(c_2)$, as in the Black-Scholes formula. The up-and-out option price is

$$S_0 \left(N(d_1) - N(x_1) + \left(\frac{B}{S_0}\right)^{2\lambda} (N(-y) - N(-y_1)) \right)$$

$$+Ke^{-rT} \left(-N(d_2) + N(x_1 - \sigma\sqrt{T}) - \left(\frac{B}{S_0}\right)^{2\lambda-2} (N(-y+\sigma\sqrt{T}) - N(-y_1+\sigma\sqrt{T}) \right)$$

where d_1, d_2 are the usual coefficients and

$$\begin{aligned}
x_1 &= \frac{\log(S_0/B)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
y_1 &= \frac{\log(B/S_0)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
y &= \frac{\log(B^2/(S_0 K))}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T} \\
\lambda &= \frac{r+\sigma^2/2}{\sigma^2}
\end{aligned}$$

Figures 2.2,2.3,2.4 show the value, delta and gamma of an up-and-out call option with strike $K = 100$, barrier level $B = 120$ and volatility 25%. The option matures at time $T = 1$. One can clearly see the "black hole" of barrier options: the region where the time-to-go is short and the priced is close to the barrier. In this region there is high negative delta, and there comes a point where hedging is essentially impossible because of the large gamma (i.e. unrealistically frequent rehedging is called for by the theory.)

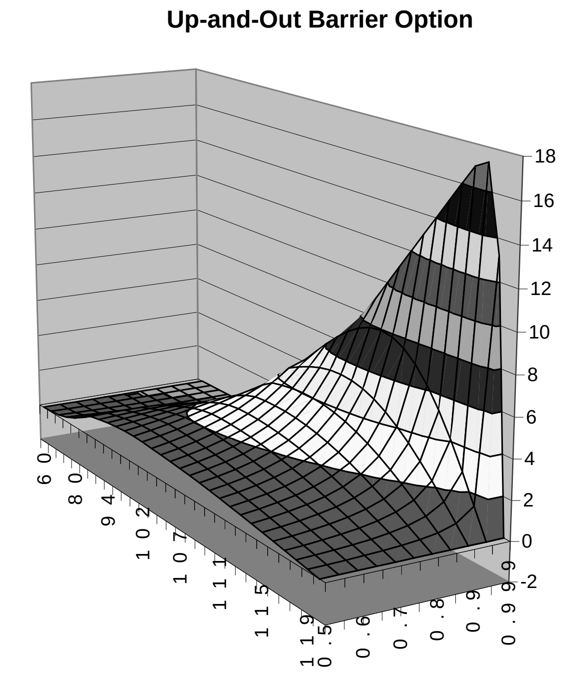

Figure 2.2: Barrier option value

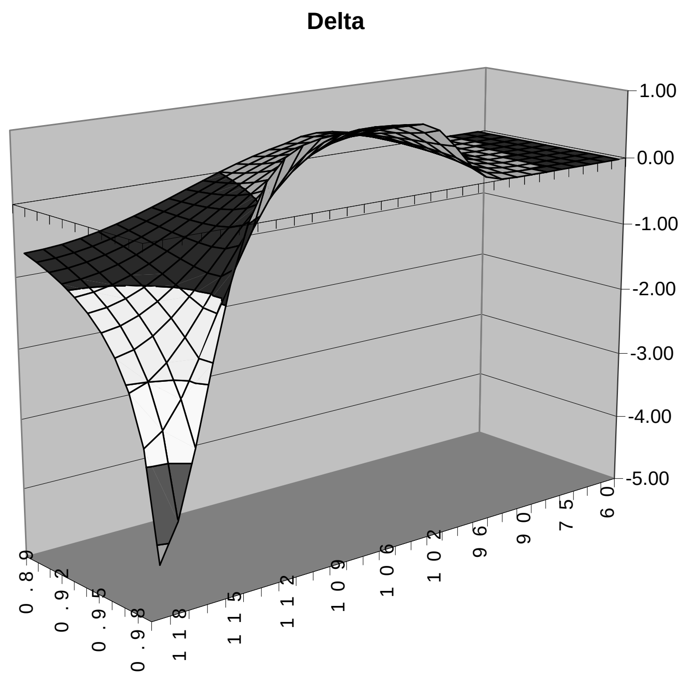

Figure 2.3: Barrier option delta

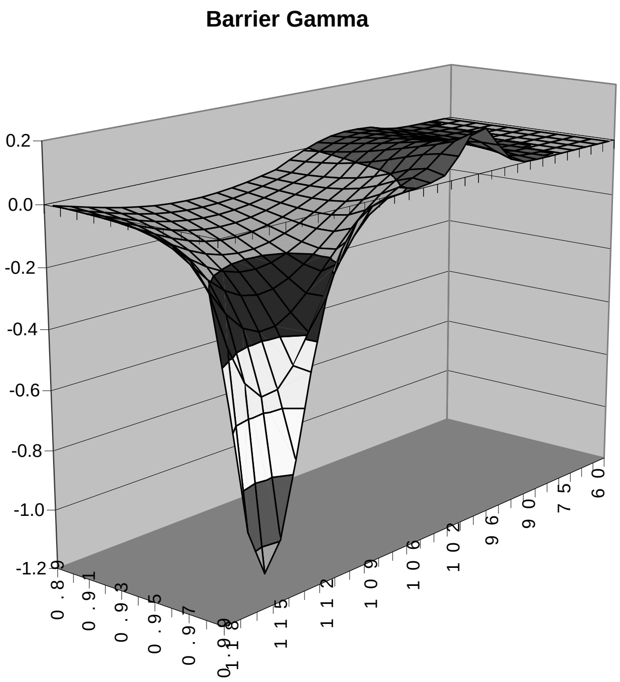

Figure 2.4: Barrier option gamma