$\cancel{k = k_0 k_1 k_2 k_3}$

$k = k_{n-1} k_{n-2} \text{---} k_2 k_1 k_0 \quad \forall \ell \in \mathbb{N}$

$QFT_n |k\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \exp\left(2i\pi \frac{kj}{2^n}\right) |j\rangle$

$= \frac{1}{\sqrt{2^n}} \bigotimes_{\ell=1}^{n} \left(|0\rangle + \exp\left(2i\pi \frac{k}{2^\ell}\right) |1\rangle \right)$

$\boxed{n=1}$

$|0\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$

$|1\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$

$|k\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle+(-1)^k|1\rangle)$

$\boxed{n=2}$ Produit $= \cancel{\frac{1}{2}(|0\rangle +}$

$\ell=1 \quad \exp\left(2i\pi \frac{k}{2}\right) = (-1)^k$

$\ell=2 \quad \exp\left(2i\pi \frac{k}{2^2}\right) = \exp\left(\frac{i\pi k}{2}\right) = i^k$

Produit $= \frac{1}{2}(|0\rangle+(-1)^k|1\rangle)(|0\rangle+ i^k|1\rangle)$

$= \frac{1}{2}(|00\rangle + i^k|01\rangle + (-1)^k|10\rangle + (-i)^k|11\rangle)$

$= \frac{1}{2}(|00\rangle \times i^{0\times k} + |01\rangle . i^{1\times k} + |10\rangle . i^{2\times k} + |11\rangle i^{3\times k})$

$2^2 \times 3^2 = 6^2 = 36$

$4 \times 9 = 36$

Cas général

Si $k = k_{n-1} k_{n-2} \text{---} k_1 k_0 = \sum_{j=0}^{n-1} 2^j k_j$

$\frac{k}{2^\ell} = \underbrace{k_{n-1} 2^{n-1-\ell} + \dots + k_\ell}_{\text{entier}} + \underbrace{\frac{k_{\ell-1}}{2} + \text{---} + \frac{k_0}{2^\ell}}_{0,k_{\ell-1}\text{---}k_0}$

$= p + 0,k_{\ell-1}k_{\ell-2}\text{---}k_1k_0$

$e^{2i\pi k/\ell} = \underbrace{e^{2i\pi p}}_{1} e^{2i\pi \frac{k_{\ell-1}}{2}} . e^{2i\pi \frac{k_{\ell-2}}{4}} . \text{---} . e^{2i\pi \frac{k_0}{2^\ell}}$

$= \exp(2i\pi \times 0,k_{\ell-1} \text{---} k_1k_0)$

Donc le produit $=$

$\frac{1}{\sqrt{2^n}} (|0\rangle + \exp(2i\pi \times 0,k_0)|1\rangle)$

$\otimes (|0\rangle + \exp(2i\pi \times 0,k_1k_0)|1\rangle)$

$\otimes (|0\rangle + \exp(2i\pi \times 0,k_2k_1k_0)|1\rangle)$

$\otimes$

Pourquoi le produit

$$F(k) = \bigotimes_{l=1}^{n} \left(|0\rangle + \exp\left(2i\pi \frac{k}{2^l}\right)|1\rangle\right) \overset{?}{=} \sum_{l=0}^{2^n-1} \exp\left(2i\pi \frac{k.l}{2^n}\right)|l\rangle.$$

Si j_l le bit de rang l, il vaut 0 ou 1 pour un entier j ($\neq k$)

S'il vaut $|0\rangle$ on va prendre un terme $|0\rangle$ qui va contribuer pour $1 = e^{2i\pi \times \frac{k}{2^l} \times 0} \otimes |0\rangle$

Si 1 on va contribuer pour $e^{2i\pi k/2^l}|1\rangle$

De manière synthétique le bit $\{j_l\}$ de j

j_{n-l} va contribuer pour $\exp\left(2i\pi \frac{k . j_{n-l}}{2^l}\right)$

$$|j\rangle = |j_{n-1} j_{n-2} \ldots j_0 j_1\rangle = \bigotimes_{l=1}^{n} |j_{n-l}\rangle$$

il va contribuer pour

$$\bigotimes_{l=1}^{n} \exp\left(2i\pi \frac{k\, j_{n-l}}{2^l}\right)|j_{n-l}\rangle$$

$$= \left(\prod_{l=1}^{n} \exp\left(2i\pi \frac{k . j_{n-l}}{2^l}\right)\right) \bigotimes_{l=1}^{n} |j_{n-l}\rangle.$$

$\Downarrow$

$$\exp\left(2i\pi k \sum_{l=1}^{n} \frac{j_{n-l}}{2^l}\right)$$

$$\exp\left(2i\pi \frac{k}{2^n} \underbrace{\sum_{l=0}^{n-1} j_l 2^l}_{j}\right)$$

$$\exp\left(2i\pi \frac{k}{2^n} \cdot j\right)$$

Circuit $\quad R_k = \begin{pmatrix} 1 & 0 \\ 0 & \exp(2i\pi/2^k) \end{pmatrix}$

Si $n=0 \to I$

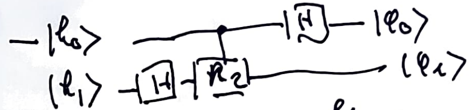
$n=1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to Z$

$n=2 \quad e^{2i\pi/4} = i \qquad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = S$

$n=3 \quad e^{2i\pi/8} = e^{i\pi/4}. \quad R_3 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = T.$

1 qubit $\quad |k_0\rangle$ —[H]— $|\varphi_0\rangle \qquad |\varphi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{k}|1\rangle\right)$

2 qubits



Si $k_0 = 0 \quad |\varphi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{k_1}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{2i\pi}{2^1} \cdot k_1}|1\rangle\right)$

Si $k_0 = 1 \quad$ H et R_2

$|\varphi_1\rangle = R_2 H |k_1\rangle = \frac{1}{\sqrt{2}} R_2 \left(|0\rangle + e^{2i\pi \frac{k_1}{2}}|1\rangle\right)$

$\sqrt{2}\, \varphi_1 \quad R_2|0\rangle + \exp\left(2i\pi \frac{k_1}{2}\right) R_2|1\rangle$

$= |0\rangle + \exp\left(2i\pi \frac{k_1}{2}\right) \underset{\exp\left(2i\pi \frac{k_0}{4}\right)}{\underset{\Downarrow}{\exp\left(2i\pi \frac{1}{4}\right)}} |1\rangle$

synthétiquement

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \exp\left(2i\pi \frac{k_1}{2}\right) \exp\left(2i\pi \frac{k_0}{4}\right)|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \exp(2i\pi \times 0{,}k_1 k_0)\,|1\rangle\right)$$