1 Definitions

a) What is a wavefunction and what does it describe?

A wavefunction is a representation of quantized state and it decribes the probability of how much a system is allowed on that state.

b) What is an expectation value? How would you calculate an expectation value?

Expectation value is the result of a measurement of a quantum mechanics ensemble.

$\langle \Omega \rangle = \langle \psi | \Omega | \psi \rangle$

c) What is the Hamiltonian?

Hermitian operator H is a Hamiltonian.

d) What is an operator?

An operator Ω is an instruction for transforming a given vector $|V\rangle$ into another, $|V'\rangle$.

e) What are eigenvalues and eigenfunctions/vectors?

In Schrodinger's Equation, $H|\psi\rangle = E|\psi\rangle$

H operator is acting on $|\psi\rangle$ and get back with the same $|\psi\rangle$ with a multiplicate constant E, which is an eigen value.

$|\psi\rangle$ is an eigenvector

f) What is degereracy in an eigen value?

When finding eigen values on wavefunction, sometimes, we get same eigenvalues. These repected eigenvalues are called degeneracy

2. Definitions

a) What does it mean for a matrix to be Hermitian?

Let $\Omega = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and its adjoint is $\Omega^{+} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$,

then the operator Ω is Hermitian.

b) What is a Unitary Matrix?

A unitary matrix is defined as I, where its diagonals are 1s and the rest are 0s. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a Unitary Matrix

c) How is the commutator defined?

The commutator is defined as $[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega$.

d) Why do we use the Dirac bracket notation-why not always write out the Schroedinger equation in x?

We use the Dirac bracket notation because the energy operator is likely to depend on two positions, which is a "non-local" operator and we want to choose to enforce "locality" by requiring that this operator depends on only 1 position.

3. $|1\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $|2\rangle = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

(a) $|1\rangle \cdot |2\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \end{bmatrix} = 1 + i^2 = 1 - 1 = 0$ #

They are orthogonal.

(b) Check normalized.

For $|1\rangle$, $\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + i^2 = 1 - 1 = 0$ #

For $|2\rangle$, $\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = 1 + i^2 = 1 - 1 = 0$ #

They are normalized

(c) $|\psi\rangle = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \alpha|1\rangle + \beta|2\rangle = \alpha \begin{bmatrix} 1 \\ i \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\langle 1|\psi\rangle = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 + 3i$

$\langle 2|\psi\rangle = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 - 3i$

$\alpha = 2 + 3i$, $\beta = 2 - 3i$

4. $$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{6}}|2\rangle + \frac{1}{\sqrt{2}}|4\rangle$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

(a) $$\int_{-\infty}^{+\infty} \psi^* \psi = \left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2$$

$$= \frac{1}{3} + \frac{1}{6} + \frac{1}{2}$$

$$= \frac{2+1+3}{6}$$

$$= \frac{6}{6} = 1 \ \#$$

The wavefunction is normalized

(b) Probability of finding the particle in ground state $(n=0)$ is $\left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}$.

(c) Probability of finding the particle in the $n=4$ state is $\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$.

(d) $$\langle E \rangle = \frac{1}{3}\left(0 + \frac{1}{2}\right)\hbar\omega + \frac{1}{6}\left(2+\frac{1}{2}\right)\hbar\omega + \frac{1}{2}\left(4 + \frac{1}{2}\right)\hbar\omega$$

$$= \frac{1}{6}\hbar\omega + \frac{5}{12}\hbar\omega + \frac{9}{4}\hbar\omega$$

$$= \frac{2+5+3}{12}\hbar\omega = \frac{\cancel{10}^{\,5}}{\cancel{12}_{\,6}}\hbar\omega = \frac{5}{6}\hbar\omega$$

5. $|a\rangle = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \qquad |b\rangle = \begin{bmatrix} -i \\ 0 \\ +i \end{bmatrix} \qquad \Omega = \begin{bmatrix} 2 & -2 & -i \\ -2 & 1 & 1 \\ i & 1 & 0 \end{bmatrix}$

(a) $|a\rangle = \dfrac{|a\rangle}{||a\rangle|} = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$

$|b\rangle = \dfrac{|b\rangle}{||b\rangle|} = \dfrac{1}{\sqrt{2}\,i}\begin{bmatrix} -i \\ 0 \\ +i \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -i \\ 0 \\ +i \end{bmatrix} = -i + 0 + i = 0$

They are orthogonal.

(c) $\Omega|a\rangle = \begin{bmatrix} 2 & -2 & -i \\ -2 & 1 & 1 \\ i & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 + 2i - i \\ -2 - i + 1 \\ i - i \end{bmatrix} = \begin{bmatrix} 2+i \\ -1-i \\ 0 \end{bmatrix}$

(d) compute $\langle b|\Omega|a\rangle$.

$\langle b|\Omega|a\rangle = [-i \quad 0 \quad +i]\begin{bmatrix} 2+i \\ -1-i \\ 0 \end{bmatrix}$

$= (-2i - \cancel{i^2}) + 0 + 0$

$= -2i$

6

$$L_x = \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix} \quad L_y = \begin{bmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{bmatrix} \quad L_z = \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}.$$

(a) Verify the commutation relation

$$[L_x, L_y] = i\hbar L_z$$

$$[\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega$$

$$[L_x, L_y] = L_x L_y - L_y L_x$$

$$= \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ -\frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{i\hbar^2}{2} & 0+0+0 & -\frac{i\hbar^2}{2} \\ 0 & -\frac{i\hbar^2}{2} + 0 + \frac{i\hbar^2}{2} & 0 \\ -\frac{i\hbar^2}{2} & 0 & -\frac{i\hbar^2}{2} \end{bmatrix} - \begin{bmatrix} -\frac{i\hbar^2}{2} & 0 & -\frac{i\hbar^2}{2} \\ 0 & \frac{i\hbar^2}{2} + \frac{i\hbar^2}{2} & 0 \\ \frac{i\hbar^2}{2} & 0 & \frac{i\hbar^2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} i\hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i\hbar^2 \end{bmatrix}$$

$$i\hbar L_z = i\hbar \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix} = \begin{bmatrix} i\hbar^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i\hbar^2 \end{bmatrix}$$

(b)

$$L_x = \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\det L_x = \det \begin{bmatrix} -\lambda & \frac{\hbar}{2} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{bmatrix} = 0$$

$$-\lambda\left((-\lambda)(-\lambda) - \left(\frac{\hbar}{\sqrt{2}} \cdot \frac{\hbar}{\sqrt{2}} \right)\right) + 0 + 0 = 0$$

$$-\lambda\left(\lambda^2 - \frac{\hbar^2}{2} \right) = 0$$

$$\lambda = 0 \quad , \quad \lambda^2 = \frac{\hbar^2}{2} \Rightarrow \lambda = \hbar \, , -\hbar$$

(c) For $\lambda = 0$,

$$\begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\hbar}{\sqrt{2}} b = 0 \Rightarrow b = 0$$

$$\frac{\hbar}{\sqrt{2}} a + \frac{\hbar}{\sqrt{2}} c = 0$$

$$a = -c$$

pick $c = 1$, $a = -1$

$$\lambda = 0 \Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = \hbar$, $\begin{bmatrix} -\hbar & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\hbar & \hbar \\ 0 & \frac{\hbar}{\sqrt{2}} & -\hbar \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-\hbar a + \frac{\hbar b}{\sqrt{2}} = 0 \Rightarrow a = \frac{1}{\sqrt{2}} b$

$\frac{\hbar}{\sqrt{2}} b - \hbar c = 0 \Rightarrow c = \frac{1}{\sqrt{2}} b$

If $b = \sqrt{2}$, $a = 1$, $c = 1$

$\lambda = \hbar \Rightarrow \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$, Normalize $\Rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}$

For $\lambda = -\hbar$, $\begin{bmatrix} \hbar & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & \hbar & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & \hbar \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\hbar a + \frac{\hbar}{\sqrt{2}} b = 0 \Rightarrow a = -\frac{1}{\sqrt{2}} b$

$\frac{\hbar}{\sqrt{2}} b + \hbar c = 0 \Rightarrow c = -\frac{1}{\sqrt{2}} b$

pick $b = -\sqrt{2}$, $a = 1$, $c = 1$

$\lambda = -\hbar \Rightarrow \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$ Normalize $\Rightarrow \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix}$

(d) $|\psi\rangle = \frac{1}{\sqrt{2}}|-1\rangle - i\frac{1}{2}|+1\rangle$

$$\langle\psi|L_x|\psi\rangle = \frac{1}{\sqrt{2}}\langle -1| - i\frac{1}{2}\langle +1| \cdot L_x \cdot \frac{1}{\sqrt{2}}|-1\rangle - i\frac{1}{2}|+1\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} - \frac{i}{2}\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \cdot L_x \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} - \frac{i}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} - \begin{bmatrix} \frac{i}{4} & \frac{i}{2\sqrt{2}} & \frac{i}{4} \end{bmatrix} \cdot L_x \cdot \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} - \begin{bmatrix} \frac{i}{4} \\ \frac{i}{2\sqrt{2}} \\ \frac{i}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\hbar}{\sqrt{2}}\left(-\frac{1}{2} - \frac{i}{2\sqrt{2}}\right) & \sqrt{2}\,\hbar\left(\frac{1}{2\sqrt{2}} - \frac{i}{4}\right) & \frac{\hbar}{\sqrt{2}}\left(-\frac{1}{2} - \frac{i}{2\sqrt{2}}\right) \end{bmatrix}$$

7. $\lambda = 0$ $\qquad$ $\lambda = -\hbar$ $\qquad$ $\lambda = \hbar$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

Take $U = \begin{bmatrix} -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/2 & \frac{1}{\sqrt{2}} & 1/2 \end{bmatrix}$ $\qquad U^{T} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$UU^{\dagger} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

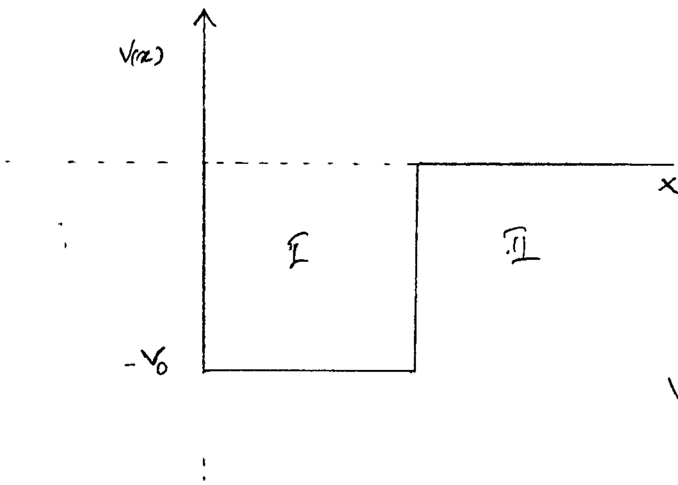
$$L_x UU^{\dagger} = \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{bmatrix}$$

$$L_z UU^{\dagger} = \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}$$

8.



$$V_{(x)} = \begin{cases} \infty & x < 0 \\ -V_0 & 0 \leq x < a \\ 0 & x \geq a \end{cases}$$

a). Region II , $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -E_0\psi$

Region I , $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = -E_0\psi$

b) Region II , $\frac{d^2\psi}{dx^2} - \frac{2mE_0\psi}{\hbar^2} = 0$

$\boxed{\psi_{II}(x) = Ae^{Kx} + Be^{-Kx}}$, $K = \sqrt{\frac{2mE_0}{\hbar^2}}$

Region I , $\frac{d^2\psi}{dx^2} + \frac{2mV_0}{\hbar^2}\psi = \frac{2mE_0}{\hbar^2}\psi$

$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(V_0 - E_0)\psi = 0$

$V_0 > E_0$ so $\frac{2m}{\hbar^2}(V_0 - E_0) > 0$

$\boxed{\psi_I(x) = Ce^{ikx} + De^{-ikx}}$, $k = \sqrt{\frac{2m(V_0-E_0)}{\hbar^2}}$

Region II , $\psi_{II}(x) = Ae^{Kx} + Be^{-Kx}$

when $x \to +\infty$, since Ae^{Kx} grows exponentially.

$\boxed{\psi_{II}(x) = Be^{-Kx}}$

Region I , $\psi_I(x) = Ce^{ikx} + De^{-ikx}$

since $\psi_I(0) = 0$, $\psi_I(x) = Ce^{ikx} - Ce^{-ikx}$ $(C = -D)$

$\boxed{\psi_I(x) = C(e^{ikx} - e^{-ikx})}$

(c) $\varphi_I(x) = C\,(e^{ikx} - e^{-ikx})$ $\quad \varphi_I'(x) = ik\,C\,(e^{ikx} + e^{-ikx})$

$\varphi_{II}(x) = B\,e^{-Kx}$ $\quad \varphi_{II}'(x) = -K\,B\,e^{-Kx}$

$K = \sqrt{\dfrac{2mE_0}{\hbar^2}}$ $\qquad k = \sqrt{\dfrac{2m(V_0 - E_0)}{\hbar^2}}$

From continuity,

$$\varphi_I(a) = \varphi_{II}(a) \;, \quad \varphi_I'(a) = \varphi_{II}'(a)$$

$$\frac{\varphi_I'(a)}{\varphi_I(a)} = \frac{\varphi_{II}'(a)}{\varphi_{II}(a)}$$

$$\frac{ik\,C\,(e^{ika} + e^{-ika})}{C\,(e^{ika} + e^{ika})} = \frac{-K\,B\,e^{-Ka}}{B\,e^{-Ka}}$$

$$ik = -K$$

$$i\sqrt{\frac{2m(V_0 - E_0)}{\hbar^2}} = -\sqrt{\frac{2mE_0}{\hbar^2}}$$

9. $$H|\psi\rangle = E|\psi\rangle$$

$$\langle x|H|\psi\rangle = E\langle x|\psi\rangle$$

$$\langle x|H|\psi\rangle = E\psi(x)$$

$$\int \langle x|H|x'\rangle\langle x'|\psi\rangle\, dx' = E\psi(x)$$

$$\int H(x,x')\,\psi(x')\,dx' = E\psi(x)$$

where $\langle x'|\psi\rangle = \psi(x')$, $\langle x|H|x'\rangle = H(x,x')$

Using Dirac Notation,

$$\langle x|H|x'\rangle = H(x,x') = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\delta(x-x')$$

$$\int \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\delta(x-x')\,dx' = E\psi(x)$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)}$$