MTH 627 (Prof. M. Peszynska) Problem set 1. **Madison Phelps**
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Include proper citations including online resources as in [1, Chap.I, Theorem 1.1].
For other results, state these.

Problem 1. Solve a modification of I.2.2: consider

$$p(x) = p_1(x_1) + 5p_2(x_2), q(x) = \max(10p_1(x_1), p_2(x_2)) \tag{1}$$

defined for $V = V_1 \times V_2 \ni x = (x_1, x_2)$. Are $p(x), q(x)$ seminorms on V. If yes, which is stronger? (Provide appropriate scaling constants). Under what assumptions are they norms?

Solution:

First, we show that $p(x)$ is a seminorm. Assume that p_1 and p_2 are seminorms on V. Then for any $x, y \in V$ we compute,

$$\begin{aligned} p(x+y) &= p_1(x_1 + y_1) + 5p_2(x_2 + y_2) \\ &\leq p_1(x_1) + p_1(y_1) + 5p_2(x_2) + 5p_2(y_2) \\ &= p(x) + p(y) \end{aligned}$$

and any scalar $\alpha \in \mathbb{R}$,

$$p(\alpha x) = p_1(\alpha x_1) + 5p_2(\alpha x_2) = \alpha p(x).$$

So, p has the triangle inequality and absolute homogeneity.

Next, observe that

$$10p_1(x_1) \leq \max(10p_1(x_1), p_2(x_2)) \text{ and } p_2(x_2) \leq \max(10p_1(x_1), p_2(x_2)), \tag{2}$$

and similarly with $10p_1(y_1)$ and $p_2(y_2)$. Thus, for any $x, y \in V \times V$ we calculate

$$\begin{aligned} q(x+y) &= \max(10p_1(x_1 + y_1), p_2(x_2 + y_2)) \\ &\leq \max(10p_1(x_1) + 10p_1(y_1), p_2(x_2) + p_2(y_2)) \\ &\leq \max(10p_1(x_1), p_2(x_2)) + \max(10p_1(y_1), p_2(y_2)) \\ &= q(x) + q(y) \end{aligned}$$

which is obtained by adding the inequalities above in (2) and using the triangle inequality of seminorms p_i with $i = 1, 2$. Also, for any scalar α,

$$q(\alpha x) = \max(10p_1(\alpha x_1), p_2(\alpha x_2)) = \max(|\alpha| 10p_1(x_1), |\alpha| p_2(x_2)) = |\alpha| q(x).$$

Therefore, p and q are seminorms.

The above seminorms are equivalent. Observe,

$$q(x) = \max(10p_1(x_1), p_2(x_2)) \leq 10p_1(x_1) + 50p_2(x_2) = 10p(x)$$

and

$$p(x) = p_1(x_1) + 5p_2(x_2) \leq \max(p_1(x_1), p_2(x_2)) + \max(50p_1(x_1), 5p_2(x_2)) \leq 6q(x).$$

If at least one of p_i is a norm, then q and p are both norms. However, it could also be true that if the kernel of p_1 is of the form $(0, x_2)$ and p_2 is of the form $(x_1, 0)$ then the only element sent to 0 is $(0, 0)$ for which we can conclude that p and q are norms.

Problem 2. Solve I.4.3.

Solution:

$\Longrightarrow$ Let $f \in V'$, then f is continuous and limits are preserved by continuity. By the reverse triangle inequality,

$$||\|x_n|| - ||x|||\leq ||x_n - x|| \to 0, \text{ as } n \to \infty$$

meaning that $\lim_{n\to\infty} ||x_n|| = ||x||$. Therefore, if $\lim x_n = x$ in V then $\lim ||x_n|| = ||x||$ and $\lim f(x_n) = f(x)$ for all f in the algebraic dual.

$\Longleftarrow$ We want to show that $||x_n - x|| \to 0$, as $n \to \infty$ which is equivalent to showing that $\lim ||x_n - x||^2 = 0$. Let x_n be a sequence in V such that $\lim ||x_n|| = ||x||$ and $\lim f(x_n) = f(x)$ for all f in the algebraic dual. Then, $f(\cdot) = (\cdot, x)$ is a continuous linear functional and so, $f(x_n) = (x_n, x) \to (x, x) = ||x||^2$. Using this, we calculate

$$\begin{aligned}
||x_n - x||^2 &= (x_n - x, x_n - x) \\
&= (x_n, x_n) - 2(x_n, x) + (x, x) \\
&= ||x_n||^2 - 2f(x_n) + ||x||^2 \\
&\to ||x||^2 - 2||x||^2 + ||x||^2 = 0,
\end{aligned}$$

as $n \to \infty$. Therefore, $\lim x_n = x$ in V.

References

[1] Ralph Showalter, *Hilbert Space Methods in Partial Differentia;l Equations*, Dover, (2010)
[2] CTAN archive of the LaTeX package `listings` `https://ctan.org/pkg/listings`