MTH 627 (Prof. M. Peszynska) Problem set 2. **Madison Phelps**
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Problem 1. Consider the space of linear polynomials $V = \mathbf{P}_1(0,1)$, with the L^2 inner product.
(i) Is V a Hilbert space? (be brief).
(ii) If possible, find the kernel, its orthogonal, and Riesz representer for the functionals $F(p) = \int_0^1 xp(x)dx$, and $G(p) = \int_0^1 g(x)p(x)dx$. (Identify sufficient assumptions on $g(\cdot)$ to make the problem meaningful). You can start with G, and then conclude the case of F as a special case. Or warm-up with F, and do G next.

Solution, part i: Yes, because V is an inner product space equipped with the L^2 inner product, and V is finite which implies V is complete, and so is Hilbert.

Solution, part ii: We wish that each F and G are linear functionals. For this to happen the following calculation must be well-defined

$$G(ax+b) = \int_0^1 g(x)(ax+b)dx = a\int_0^1 xg(x)dx + b\int_0^1 g(x)dx = aG(x) + bG(1)$$

where $p(x) = ax+b$ represents an arbitrary linear polynomial in V. In particular, to make the problem meaningful we must ensure that g is Lebesgue integrable on $(0,1)$ so that IBP is well-defined for this problem. Assuming this is true, $p \in \mathrm{Ker}(G)$ if and only if

$$aG(x) + bG(1) = 0$$

which implies that $b = -a\frac{G(x)}{G(1)}$ and the kernel of G is any linear polynomial in the span of the linear polynomial given by

$$p(x) = x - \frac{G(x)}{G(1)}$$

where G is defined in terms of a given g.

Next, we find the orthogonal complement of its kernel and find that

$$\int_0^1 (ax+b)(x - \frac{G(x)}{G(1)})dx = 0$$

if and only if

$$b = \frac{a\left(-\frac{1}{3} + \frac{G(x)}{2G(1)}\right)}{\frac{1}{2} - \frac{G(x)}{G(1)}}$$

which means that

$$\mathrm{Ker}(G)^{\perp} = \mathrm{span}\left(x - \frac{\left(-\frac{1}{3} + \frac{G(x)}{2G(1)}\right)}{\frac{1}{2} - \frac{G(x)}{G(1)}}\right) = \mathrm{span}\left(f(x)\right).$$

By Reisz theorem, the Reisz representer for the functional G is the unique element in the orthogonal complement of its kernel, that is, f is the Reisz representer for G.

Now if we take $g(x) = x$, the kernel of F is given by

$$\mathrm{Ker}(F) = \mathrm{span}\left(x - \frac{F(x)}{F(1)}\right) = \mathrm{span}\left(x - \frac{2}{3}\right).$$

and its orthogonal complement

$$\mathrm{Ker}(F)^{\perp} = \mathrm{span}\left(x - \frac{\left(-\frac{1}{3} + \frac{F(x)}{2F(1)}\right)}{\frac{1}{2} - \frac{F(x)}{F(1)}}\right) = \mathrm{span}\left(x - 0\right) = \mathrm{span}(x).$$

Thus, the Reisz representer of F is $f(x) = x$.

Problem 2. Solve II.2.1.

Solution: Let $u \in H_0^1(G)$ and $\varphi \in H^1(G)$ be an arbitrary test function, then we compute

$$\begin{aligned}
(u, \phi)_{H^1(G)} &= \int_G u \cdot \varphi + \int_G \sum_{i=1}^{n} \partial_i u \cdot \partial_i \varphi \\
&= \int_G u \cdot \varphi - \int \sum_{i=1}^{n} u \cdot \partial_i^2 \varphi \\
&= \int_G u \cdot \varphi - \int u \sum_{i=1}^{n} \partial_i^2 \varphi \\
&= \int_G u \cdot \varphi - \int_G u \Delta_n \varphi
\end{aligned}$$

and have used the definition of the $H^1(G)$ inner product, the multidimensional laplacian and then used IBP with

$$u \cdot \partial_i \varphi \Big|_{\partial G} - \int_G u \cdot \partial_i^2 \varphi = -\int_G u \cdot \partial_i^2 \varphi$$

after applying the boundary conditions of u in G. Then,

$$(u, \phi)_{H^1(G)} = 0$$

if and only if $\varphi = \Delta_n \varphi$.

Applying the same thought process as above in one dimension, we seek a general solution to the ODE given by $\varphi'' + \varphi = 0$ whose solutions are in

- $H^1(0, 1)$, which gives the solutions of e^{-x} and e^x
- $H^1(0, \infty)$ only has the solutions of e^{-x} because e^x is not integrable on $(0, \infty)$, and
- $H^1(\mathbb{R})$ has only the zero function because both functions become unbounded as we approach negative and positive infinity, respectively.

Thus, the basis for each of the spaces are precisely the span of the functions mentioned above.

Problem 3. Let $v(x) = \log\log\frac{1}{r}$ for $r = \| x \|_2$, $x \in \mathbb{R}^2$. Show [with calculations and estimates] that $v \in H^1(B(0;\beta))$ for any $\beta < 1$, but that v is unbounded as $x \to 0$.

(**Note:** This shows that it is not true that $H^1(\Omega) \subset C(\overline{\Omega})$ for $\Omega \subset \mathbb{R}^d$ when $d > 1$, even if this results holds in $d = 1$ Compare to [II.Pbm.2.2]).

Solution: I am not sure where to go from here.

Problem 4: Extra. (Synthesis from reading Chapters I, II; can be done by a group of any size. Please provide the names of those in the group and submit only one solution for all group members.)

Consider the vector spaces

$$C^\infty(\mathbb{R}), C_0^\infty(\mathbb{R}), C^1(0,1), C^1[0,1], C_0(0,1), C[0,1], H_0^1(\mathbb{R}), H^1(\mathbb{R}), L^2(0,1), H^1(0,1), H_0^1(1).$$

and consider the norms that are relevant such as "*sup*", "L^2", and more.

For these (possibly normed) spaces, write at least 10 statements involving the keywords "complete", "not complete", "completion","dense", "compact" (the last two referring to the inclusions≡embeddings between spaces). If writing more, include these on at most 1 page total.

Annotate the results with references to the results in the book; no proofs are needed. (Briefness will be appreciated: e.g., one line per each of the eleven spaces suffices to state whether this space embeds in any of the others (and how): start with the largest of these spaces!).

Solution:

References

[1] Ralph Showalter, *Hilbert Space Methods in Partial Differentia;l Equations*, Dover, (2010)
[2] CTAN archive of the LaTeX package `listings` `https://ctan.org/pkg/listings`