MTH 627 (Prof. M. Peszynska) Problem set 3. **Madison Phelps**
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Include proper citations including online resources as in [**?**, Chap.I, Theorem 1.1].
For other results, state these.
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Problem 1. Problem #1 was chosen.

Solution: Let $\varepsilon > 0$ be chosen such that $0 < \varepsilon \ll 1$. Choose $c = \varepsilon^2(1-\varepsilon) > 0$. We will show that the given c works. Take any $u \in H^1(0,1)$ and by the hint on the problem we see that

$$\int_0^x (u')^2 = u^2(x) - u^2(0)$$

by the fundamental theorem of calculus on $(0,1)$ and implies,

$$u^2 \le u^2(0) + \int_0^1 2uu'$$

for all $x \in (0,1)$ and given that $(u')^2 = 2uu'$. Then, since $\varepsilon \neq 0$, we may multiply by one and use Young's Inequality to obtain the following estimates

$$\begin{aligned} u^2 &\le u^2(0) + \int_0^1 \left(2\varepsilon u \cdot \frac{1}{\varepsilon} u'\right) \\ &\le u^2(0) + \int 0^1 \left(\varepsilon^2 u^2 + \frac{1}{\varepsilon^2} u'\right) \\ &= u^2(0) + \varepsilon^2 \int_0^1 u^2 + \frac{1}{\varepsilon^2} \int_0^1 u' \\ &= u^2(0) + \varepsilon^2 ||u||^2_{L^2(0,1)} + \frac{1}{\varepsilon^2} ||u'||^2_{L^2(0,1)}. \end{aligned}$$

Manipulating the above and integrating on $(0,1)$ gives

$$||u||^2_{L^2(0,1)} - \varepsilon^2 ||u||^2_{L^2(0,1)} \le u^2(0) + \frac{1}{\varepsilon^2} ||u'||^2_{L^2(0,1)}$$

and

$$\varepsilon^2(1-\varepsilon^2)||u||^2_{L^2(0,1)} \le \varepsilon^2 u^2(0) + ||u'||^2_{L^2(0,1)} \le u^2(0) + ||u'||^2_{L^2(0,1)}$$

because $\varepsilon \ll 1$. Therefore, there exists some $c = \varepsilon^2(1-\varepsilon) > 0$ such that

$$c \int_0^1 u^2 = \varepsilon^2(1-\varepsilon^2)||u||^2_{L^2(0,1)} \le u^2(0) + ||u'||^2_{L^2(0,1)}$$

for any $u \in H^1(0,1)$.

Problem 2. Problem #5 chosen.

Solution:

Assume that $c : \Omega \to \mathbb{R}$ is a function that is bounded on Ω such that

$$\beta_c \leq c(x) \leq \beta_b \text{ for all } x \in \Omega. \tag{1}$$

Through discussions, we found that the most general constraint to place on

$$K = \begin{bmatrix} k_{11}(x) & k_{12}(x) \\ k_{21}(x) & k_{22}(x) \end{bmatrix} \qquad \text{for } k_{ij} : \Omega \to \mathbb{R} \tag{2}$$

is that there needs to exist some constant k_c and k_b that are real, nonzero, and positive such that

$$k_c||x||_V = k_c x^T x \leq x^T K x \leq k_b x^T x = k_b ||x||_V \tag{3}$$

holds for all nonzero $x \in \Omega$ and, which in doing so implies that K is symmetric positive definite.

Coercive: Using the left hand side of the inequalities in Equations (**??**) and (**??**), we compute

$$\begin{aligned} |a(u,u)| &= \left| \int_\Omega (K\nabla u) \cdot \nabla u + \int_\Omega c(x) u^2 \right| \\ &\geq \left| k_c \int_\Omega (\nabla u \cdot \nabla u) + \beta_c \int_\Omega u^2 \right| \\ &\geq k_c ||\partial u||^2_{L^2(0,1)} + \beta_c ||u||^2_{L^2(0,1)} \end{aligned}$$

where we have obtained the last inequality because we have assumed that $(K\nabla u) \cdot \nabla u = \nabla u^T K \nabla u$ and, $\nabla u \in H^1(0,1)$ and $\nabla u \in H^1_0(0,1)$. If we take the constant of coercivity to be $c = \min\{k_c, \beta_c\} > 0$, then $|a(u,u)| \geq c||u||^2_{H^1(0,1)}$ which shows that a is coercive. Otherwise, if $u \in H^1_0(0,1)$ and $\beta_c > k_c > 0$ then,

$$|a(u,u)| \geq k_c ||\partial u||^2_{L^2(0,1)} + \beta_c ||u||^2_{L^2(0,1)} \geq k_c ||\partial u||^2_{L^2(0,1)} = k_c ||u||^2_{H^1_0(0,1)}$$

where we take the constant of coercivity to be $c = k_c$ on $H^1_0(0,1)$. In either case, we have shown that a is coercive, as desired. There are a few cases that I thought up of in class for which are weaker assumptions that shows $a(\cdot,\cdot)$ is coersive on $H^1(0,1)$ and $H^1_0(0,1)$. The proofs will be at the end of the problem.

Bounded:

Similarly, assume the right hand side of the inequalities in Equations (**??**) and (**??**). For any $u \in V$ we have that

$$\begin{aligned}|a(u,u)| &= \left|\int_\Omega (K\nabla u)\cdot\nabla u + \int_\Omega c(x)u^2\right| \\ &\leq \left|k_b\int_\Omega (\nabla u\cdot\nabla u) + \beta_b\int_\Omega u^2\right| \\ &\leq k_b||\partial u||^2_{L^2(0,1)} + \beta_b||u||^2_{L^2(0,1)}\end{aligned}$$

and take $b = \max\{k_b, \beta_b\} > 0$ we see that

$$|a(u,u)| \leq b\left(||\partial u||^2_{L^2(0,1)} + ||u||^2_{L^2(0,1)}\right) = b||u||^2_{H^1(0,1)}$$

for all $u \in H^1(0,1)$ and otherwise, if $u \in H^1_0(0,1)$ then we may use the Poincare Friedrich's inequality to obtain,

$$|a(u,u)| \leq b\left(||\partial u||^2_{L^2(0,1)} + ||u||^2_{L^2(0,1)}\right) \leq b\left(||\partial u||^2_{L^2(0,1)} + C_{PF}||\partial u||^2_{L^2(0,1)}\right) \leq b'||u||^2_{H^1_0(0,1)}$$

with a different bounding cosntant $b' = \max\{k_b, \beta_b\}(1 + C_{PF})$.

Therefore with the assumptions given in (**??**) and (**??**), we have shown that a is coercive on both spaces.

Solution for weaker conditions on K that show the coercive property:

There are a few weaker conditions that we could place on K to ensure that $a(\cdot,\cdot)$ is coersive. If we further assume that $u \in C^1(\Omega)$, then we can apply the Poincare Inequality such that

$$||u||^2_{L^2} \leq C_{\text{PF}}||\partial u||^2_{L^2}.$$

If $u \in H^1_0(\Omega)$, then we have that

$$||\partial u||^2_{L^2} \leq ||\partial u||^2_{L^2} + ||u||^2_{L^2} \leq ||\partial u||^2_{L^2} + C_{\text{PF}}||\partial u||^2_{L^2} \tag{4}$$

which implies that

$$\frac{1}{1 + C_{\text{PF}}}||\partial u||^2_{L^2} \leq ||\partial u||^2_{L^2} \tag{5}$$

Now, $a(\cdot,\cdot)$ is said to be coersive if there exists some constant c such that $|a(u,u)| \geq c||u||_V$. Hence, we consider the following,

$$|a(u,u)| = \left|\int_\Omega (K\nabla u)\cdot\nabla u + \int_\Omega c(x)u^2\right| \tag{6}$$

and we have the following cases to show that $a(\cdot,\cdot)$ is coersive on $V = H^1(\Omega)$ and $V = H^1_0(\Omega)$.

Case 1: Assume that K is a diagonal matrix and $k_1(x)$ and $k_2(x)$ are functions that are bounded below such that $k = \inf_{x\in\Omega}\{k_1(x), k_2(x)\} > 0$.Assume $c : \Omega \to \mathbb{R}$ is a function that is bounded below by some $\beta \in \mathbb{R}$. Then we estimate using Equation **??** and find that

$$
\begin{aligned}
|a(u,u)| &= \left| \int_\Omega (K\nabla u)\cdot \nabla u + \int_\Omega c(x)u^2 \right| \\
&= \left| \int_\Omega \left(k_1(x)(u_x)^2 + k_2(x)(u_y)^2\right) + \int_\Omega c(x)u^2 \right| \\
&\geq \left| k \int_\Omega \left((u_x)^2 + (u_y)^2\right) + \beta \int_\Omega u^2 \right| \\
&\geq \left| k \int_\Omega (\nabla u \cdot \nabla u) + \beta \int_\Omega u^2 \right| \\
&\geq k||\partial u||_{L^2}^2 + \beta ||u||_{L^2}^2 \qquad (7)
\end{aligned}
$$

If $u \in H_0^1(\Omega)$, then we use Equation **??** and conclude that

$$|a(u,u)| \geq \left(k + \frac{\beta}{1+C_{PF}}\right)||\partial u||_{L^2}^2 = \left(k + \frac{\beta}{1+C_{PF}}\right)||u||_{H_0^1(\Omega)}^2$$

which further assumes that $k > \frac{\beta}{1+C_{PF}}$ since β could be negative. On the other side, if $u \in H^1(\Omega)$ we use line (**??**) from above and we set $c = \min\{k,\beta\}$ to conclude that

$$|a(u,u)| \geq \min\{k,\beta\}\left(||\partial u||_{L^2}^2 + ||u||_{L^2}^2\right) = c||u||_{H^1(\Omega)}^2.$$

Case 2: If K is such that $k_{11}(x)$ and $k_{22}(x)$ are functions that are bounded below such that $k = \inf_{x\in\Omega}\{k_{11}(x), k_{22}(x)\}$ and $k_{12}(x) = -k_{21}(x)$ and c is bounded below by β, then we can use the same conclusions in Case 1 because $(K\nabla u)\cdot\nabla u$ becomes

$$k_{11}(x)(u_x)^2 + (k_{12}(x)+k_{21}(x))u_x u_y + k_{22}(x)(u_y)^2 = k_{11}(x)(u_x)^2 + k_{22}(x)(u_y)^2$$

and we may conclude that a is coersive.

References

[1] Ralph Showalter, *Hilbert Space Methods in Partial Differentia;l Equations*, Dover, (2010)

[2] CTAN archive of the LaTeX package `listings` `https://ctan.org/pkg/listings`