# Analysis Qual Solutions

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# Contraction Mapping Theorem

## 1 Theorems and Definitions

**Theorem 1** (Banach Fixed Point Theorem)**.** If $f : X \to X$ is a mapping on a complete metric space $X$ such that there exists $\alpha \in [0, 1)$ such that

$$|f(x) - f(y)| \leq \alpha|x - y|$$

for all $x, y, \in X$, then $f$ has a unique fixed point. Moreover, for any $x_0 \in X$, the sequence of functional iterates $x_0, f(x_0), f(f(x_0)), \ldots$ converges to the fixed point in $X$.

## 2 Problems

### 2.1 Fall 2022 #2

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Consider a Banach space $V$, a contraction map $T : V \to V$, and the equation

$$v = T(v) + y$$

The operator $T$ is not necessarily linear.

(a) Show that for any $y \in V$, the solution exists and is unique.

(b) Based on (a), call $u(y)$ this solution. Show that $u(y)$ is a continuous function of $y$.

(c) Show that if $T : K \to K \subset V$, with $T(0) = 0$, and $K = \{v \in V : ||v|| \leq r\}$, with a fixed $r > 0$, then the unique solution $u(y)$ lies in $K$, assuming $||y||$ is sufficiently small.

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Solution (a):

Fix some $y \in V$. Define $S : V \to V$ by

$$S(v) = T(v) + y$$

Since $T$ is a contraction map, there exists some constant $0 \leq \alpha < 1$ for which

$$||Tv - Tu|| \leq \alpha||u - v||$$

for all $u, v \in V$. Observe that $S$ is a contraction map on $V$ with contraction constant $\alpha$ because,

$$||Sv - Su|| = ||Tv + y - (Tu + y)|| = ||Tv - Tu|| \leq \alpha||u - v||.$$

By the Banach fixed point theorem, given that $V$ is a Banach space and thus, is a complete normed vector space with $S$ a contraction map on $V$, we may conclude that there exists a unique fixed point such that $Sv = v$. Because we fixed $y \in V$, the solution exists and is unique for all such $y \in V$.

Solution (b):

For $y \in V$ let $u(y) = T(v) + y$ be the solution. Let $\varepsilon > 0$ be given, and set $\delta = \varepsilon > 0$. Let $y_1, y_2 \in V$ be chosen such that $||y_1 - y_2|| < \delta$. Then, we compute

$$||u(y_1) - u(y_2)|| = ||Tv + y_1 - Tv - y_2|| = ||y_1 - y_2|| < \delta = \varepsilon.$$

Therefore, $u(y)$ is a continuous function of $y$.

Solution (c):

Let $\varepsilon > 0$ be given.

Observe that because $T$ is a contraction on $V$ and $K \subset V$, $T$ is a contraction on $K$ with the same contraction constant $\alpha < 1$. In particular this means that for any $v \in K$

$$||Tv|| = ||Tv - 0|| = ||Tv - T0|| \leq \alpha||v - 0|| = \alpha||v|| < \alpha \cdot r < r.$$

Hence, let $y \in V$ be such that $||y|| < \varepsilon$ and we compute,

$$||u(y)|| = ||Tv + y|| \leq ||Tv|| + ||y|| < \varepsilon + r$$

and because the above holds for all $\varepsilon > 0$, we may take $\varepsilon \to 0$ and conclude that $||u(y)|| \leq r$ for all such $y \in V$ with sufficiently small $||y||$. That is, $u(y)$ lies in $K$.

### 2.2 Fall 2020 #1

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Let
$$F(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$
for all $x > 0$, where $a > 0$ is constant. This problem is concerned with sequences $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = F(x_n)$ for $n \geq 0$, with $x_0$ chosen arbitrarily.

(a) Prove that there exists a closed interval $I$ with positive finite length such that $F$ maps $I$ into $I$.

(b) Prove that for every $x_0 \in I$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a limit that is independent of the choice of $x_0$. What is this limit?

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**Solution:** (a)

We wish to construct a closed interval $I$ of positive finite length such that $F$ maps $I$ onto $I$. Observe that $F$ has a fixed point at $x = \sqrt{a} > 0$ because
$$F(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \sqrt{a},$$
which is found by computing the solution to the equation $F'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) = 0$ and and solving for $x > 0$. Suppose $x \in \mathbb{R}$ such that $0 < x \leq \sqrt{a}$. Then by manipulating the inequality $x \leq \sqrt{a}$ we obtain
$$x = \frac{1}{2}(2x) \leq \frac{x + \sqrt{a}}{2} \leq \frac{1}{2}\left(x + \frac{a}{x}\right) = F(x). \tag{1}$$
Similarly, for any $x \in \mathbb{R}$ with $\sqrt{a} < x$ we obtain,
$$F(x) = \frac{1}{2}\left(x + \frac{a}{x}\right) < \frac{x + \sqrt{a}}{2} < \frac{1}{2}(2x) = x \tag{2}$$
That is, choose some $b \in \mathbb{R}$ such that $\sqrt{a} < b$ and by the inequalities in (1) and (2), for any $x \in \mathbb{R}$ with $\sqrt{a} \leq x < b$,
$$\sqrt{a} = F(\sqrt{a}) \leq F(x) < F(b) < b.$$
Define $I$ to be the closed interval $[\sqrt{a}, b]$ with positive finite length (because $\sqrt{a} < b$). The above statement allows us to conclude that $F(I) \subset I$, i.e., $F$ maps $I$ onto $I$.

**Solution:** (b)

Choose $b \in \mathbb{R}$ with $\sqrt{a} < b$ and consider the mapping $F : I \to I$ where $I = [\sqrt{a}, b]$. We want to show that $F$ is contraction mapping on $I$ and apply the Banach fixed point theorem which states:

If $f : X \to X$ is a mapping on a complete metric space $X$ such that there exists $\alpha \in [0, 1)$ such that
$$|f(x) - f(y)| \leq \alpha|x - y|$$
for all $x, y, \in X$, then $f$ has a unique fixed point. Moreover, for any $x_0 \in X$, the sequence of functional iterates $x_0, f(x_0), f(f(x_0)), \ldots$ converges to the fixed point in $X$.

For any distinct $x, y \in I$, we compute
$$|F(x) - F(y)| = \frac{1}{2}\left|x + \frac{a}{x} - \left(y + \frac{a}{y}\right)\right|$$

$$= \frac{|x-y|}{2}\left|1-\frac{a}{xy}\right|.$$

So, in order for $F$ to be a contraction mapping we must show that

$$\left|1-\frac{a}{xy}\right| < 1 \text{ for all } x, y \in I. \tag{3}$$

Suppose $x = y = \sqrt{a}$, then $0 = \left|1-\frac{a}{a}\right| < 1$. Conversely, if $x = y = b$, then $\sqrt{a} < b$ implies that $\frac{a}{b^2} < 1$ and so, $0 = \left|1-\frac{a}{b^2}\right| < 1$. Lastly, let $x, y \in I$ be distinct and such that $\sqrt{a} < x, y < b$. Then,

$$\frac{1}{a} > \frac{1}{xy} \quad \text{and} \quad \frac{1}{b^2} < \frac{1}{xy} \tag{4}$$

imply that

$$0 \leq \left|1-\frac{a}{a}\right| < \left|1-\frac{a}{xy}\right| \leq \left|1-\frac{a}{b^2}\right| < 1$$

for all $x, y \in I$. Using inequality (3) for any distinct $x, y \in I$

$$|F(x)-F(y)| = \frac{|x-y|}{2}\left|1-\frac{a}{xy}\right| < \frac{1}{2}|x-y|.$$

Since $I \subseteq \mathbb{R}$ is a closed and bounded set in $\mathbb{R}$, $I$ is complete and because $F$ is a contraction mapping (from above and by part (a)) with contraction constant $\alpha = 12$, the Banach fixed point theorem states that: (1) $F$ has a unique fixed point, and (2) for any $x_0 \in I$ the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point and is independent of the choice of $x_0$.

**Notes: Fall 2020 #1**

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**Notes:** The above question is regarding the Banach fixed point theorem and the consequence that the sequence of functional iterates converges to the fixed point no matter what point in the complete space you choose. However, if the space is not complete and you find a complete space within the set for which $f$ is defined in, eventually the sequence will be inside of the complete space. Of course, you need to show that $f$ is a contraction mapping. But, only finitely many terms will be outside of the complete set. Then, there will be a point where the Banach fixed point theorem kicks in and we can guarantee that the sequence of functional iterates converges to to the unique fixed point of $f$.

Another thing with this problem is that it utilizes working with inequalities and boils down to manipulating the inequality $b > \sqrt{a}$ in many different ways for some fixed $a > 0$.

You learned a lot in this problem.

**Revision: Fall 2020 #1**

**Part (a) Revision:** (Notes in magenta)

There are some issues with the above old solution that need revision:

~~Define $I = [\sqrt{a}, b]$, where $a > 0$ is given and choose $b \in \mathbb{R}$ such that $b > \sqrt{a} > 0$.~~

Yes, but move to end of proof.

~~To show that $F(I) \subseteq I$, we first compute the solution to $F'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) = 0$ to find that $F$ has a minimum at $x = \sqrt{a}$ (since $x > 0$). Also, observe $F$ has a fixed point at $x = \sqrt{a}$ because~~

$$F(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \sqrt{a}.$$

Change to just include that that $F$ has a fixed point at $x = \sqrt{a}$ because

$$F(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \sqrt{a}.$$

Then take any $x \in \mathbb{R}$ with $0 < x \leq \sqrt{a}$ and manipulate the inequalities as

$$\begin{aligned} x &\leq \sqrt{a} \\ \sqrt{a}x &\leq a \\ \sqrt{a} &\leq {}^{a}\!/\!_{x} \\ x + \sqrt{a} &\leq x + {}^{a}\!/\!_{x} \end{aligned} \qquad\qquad \begin{aligned} x &\leq \sqrt{a} \\ 2x &\leq \sqrt{a} + x \end{aligned}$$

and conclude with

$$x \leq \frac{x + \sqrt{a}}{2} \leq F(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

~~Thus, $F$ is bounded below by $x = \sqrt{a}$.~~

Yes, but you don't need this and it makes things more confusing.

~~Since $0 < \sqrt{a} < b$, we use the two inequalities $\frac{a}{b} < \sqrt{a}$ and $\sqrt{a} + b < 2b$ to conclude that~~

$$F(b) = \frac{1}{2}\left(b + \frac{a}{b}\right) < \frac{1}{2}(\sqrt{a} + b) < \frac{1}{2}(2b) = b.$$

~~That is, for any $b \in \mathbb{R}$ with $\sqrt{a} < b$, $F$ is bounded above by $b$. Therefore, for all $x \in I$, $\sqrt{a} \leq F(x) < b$, and so, $F(I) \subseteq I$.~~

Replace $b$ with $x$ to show that for any $x \in \mathbb{R}$ with $\sqrt{a} < x$, then $F(x) < x$. Then, choose the bounds as you did in the original write up. Say the same conclusion.

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**Part (b) Revise Flow:**

To prove the statement, we will show that $F$ is a contraction mapping on $I$ with constant $\alpha = \frac{1}{2} \in [0, 1)$ and apply the Banach fixed point theorem which states:

If $f : X \to X$ is a mapping on a complete metric space $X$ such that there exists $\alpha \in [0, 1)$ such that

$$|f(x) - f(y)| \leq \alpha|x - y|$$

for all $x, y, \in X$, then $f$ has a unique fixed point. Moreover, for any $x_0 \in X$, the sequence of functional iterates $x_0, f(x_0), f(f(x_0)), \ldots$ converges to the fixed point in $X$.

Observe that in order for $F$ to be a contraction mapping we must show that

$$\left|1 - \frac{a}{xy}\right| < 1 \text{ for all } x, y \in I. \tag{5}$$

Suppose $x = y = \sqrt{a}$, then $0 = \left|1 - \frac{a}{a}\right| < 1$. Conversely, if $x = y = b$, then $b > \sqrt{a}$ implies that $\frac{a}{b^2} < 1$ and so, $0 = \left|1 - \frac{a}{b^2}\right| < 1$. Lastly, let $x, y \in I$ be distinct and such that $\sqrt{a} < x, y < b$s. Then,

$$\frac{1}{a} > \frac{1}{xy} \quad \text{and} \quad \frac{1}{b^2} < \frac{1}{xy}$$

imply that

$$0 \leq \left|1 - \frac{a}{a}\right| < \left|1 - \frac{a}{xy}\right| \leq \left|1 - \frac{a}{b^2}\right| < 1$$

for all $x, y \in I$. Using inequality (??) for any $x, y \in I$

$$\begin{aligned} |F(x) - F(y)| &= \frac{1}{2}\left|x + \frac{a}{x} - \left(y + \frac{a}{y}\right)\right| \\ &= \frac{|x - y|}{2}\left|1 - \frac{a}{xy}\right| \\ &< \frac{1}{2}|x - y|. \end{aligned}$$

Since $I \subseteq \mathbb{R}$ is a closed and bounded set in $\mathbb{R}$, $I$ is complete and because $F$ is a contraction mapping (from above and by part (a)) with contraction constant $\alpha = 12$, the Banach fixed point theorem states that: (1) $F$ has a unique fixed point, and (2) for any $x_0 \in I$ the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point and is independent of the choice of $x_0$.

The main issue is that the flow should start with the calculation of why we care about the inequality. Also, should state that $I$ is taken from part (a) to be fixed. And the biggest thing is that STICK WITH THE INEQUALITY OF LESS THAN AND WRITE THINGS FROM SMALLEST TO BIGGEST. It will pay off.

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### 2.3 Jordan Week 1 #3

**Problem Jordan Qual Prep Week 1 (& 7), # 3.** Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipshitz continuous function. That is,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L \quad \text{for some constant } L \text{ and for all } x \neq y.$$

(a) Prove that $f$ is uniformly continuous.

(b) Prove that if $L < 1$, then $f$ has a fixed point.

(c) Let $x_{n+1} = \frac{1}{2(1+x_n)}$ with $x_0 = 0$. Prove that the sequence $\{x_n\}$ is convergent.

**Solution:** (a)

Let $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{L} > 0$ and let $x \in \mathbb{R}$ be arbitrary. Then, for any $y \in \mathbb{R}$ such that $|x - y| < \delta$,

$$|f(x) - f(y)| \leq L|x - y| < \varepsilon.$$

Therefore, $f$ is uniformly continuous because $\delta > 0$ is independent of $x \in \mathbb{R}$.

**Solution:** (b)

Since $\mathbb{R}$ is complete, $L < 1$, and $f(\mathbb{R}) \subseteq \mathbb{R}$, we conclude that $f$ is a contraction mapping on $\mathbb{R}$. By the Banach fixed point theorem (see Theorem 1), $f$ has a unique fixed point.

**Solution:** (c)

Let $t \in \mathbb{R}$. Observe that for any $t \in \mathbb{R}$, $[t, \infty) \subset \mathbb{R}$ is a closed subspace of $\mathbb{R}$ because the complement of $X_t$ is open. Since any closed subset of a complete space, is also complete, we conclude that $X_t$ is complete.

Define $f(x) = \frac{1}{2(1+x)}$. We will show that for any $t \in \mathbb{R}$ such that $0 \leq t \leq \frac{\sqrt{3}-1}{2}$, $f$ is a contraction mapping on a complete space defined by $X_t = [t, \infty)$, then conclude that $f$ has a unique fixed point and the sequence of functional iterates for any $x_0 \in X_t$ converges to the fixed point.

First, $x = \frac{\sqrt{3}-1}{2}$ is a fixed point of $f$ by solving $f(x) = x$ and using the quadratic formula on $x^2 + x - \frac{1}{2} = 0$. Now suppose that $t > \frac{\sqrt{3}-1}{2}$. Then,

$$t > \frac{\sqrt{3}-1}{2} = f(\frac{\sqrt{3}-1}{2}) = \frac{1}{2(1 + \frac{\sqrt{3}-1}{2})} > \frac{1}{2(1+t)} = f(t),$$

and $f(t) < t$ implies that $f(t) \notin X_t$ for any $t$ larger than the fixed point. Next, $f$ is bounded below by 0 because $f$ is defined on $(-1, \infty)$ and for any $x > -1$, $f(x) > 0$. To see that $t = 0$ is also true, we compute $f(0) = \frac{1}{2} \in X_0 = [0, \infty)$. Thus, for any $0 \leq t \leq \frac{\sqrt{3}-1}{2}$, $f(X_t) \subseteq X_t$.

Next, we will show that $f$ is a contraction mapping. Fix $t$, where $0 \leq t \leq \frac{\sqrt{3}-1}{2}$. Then for any distinct $x, y \in X_t$, assume $0 \leq t \leq x < y$. Then, we have $1 + y > 1 + t > 1$ and $1 + x \leq 1 + t \leq 1$ implies that

$$\frac{1}{(1+x)(1+y)} < \frac{1}{(1+t)^2} \leq 1.$$

Because $x$ and $y$ are distinct, $\frac{1}{(1+x)(1+y)} < 1$ for all $x, y \in X_t$. So, observe that

$$|f(x) - f(y)| = \frac{1}{2}\left|\frac{1}{(1+x)} - \left(\frac{1}{(1+y)}\right)\right| \tag{6}$$

$$= \frac{|x-y|}{2}\left|\frac{1}{(1+x)(1+y)}\right| \tag{7}$$

$$< \frac{1}{2}|x-y| \tag{8}$$

which shows that $f$ is a contraction mapping on a complete space with constant $\frac{1}{2} \in [0, 1)$. Therefore, by Banach fixed point theorem by choosing $t = 0$, we can conclude that the sequence defined by $f(x_n) = x_{n+1} = \frac{1}{2(1+x_n)}$ with $x_0 = 0 \in X_t$ converges.

**Notes: Jordan Week 1 #3**

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**Notes:** Well, this is the problem for you if you want to conquer inequalities. The goal is to think smarter and not harder. When you catch yourself doing the thing that is not “obvious” you should revert back to the original idea you are trying to prove. Also, keep track at each step that your inequalities are all facing the correct way at each step in your scratch work. It is very easy to make < a > and then sit there for hours thinking about how stupid inequalities are. Also, computing the fixed point doesn’t seem as obvious that it is a fixed point because

$$f((\sqrt{3}-1))/2) = 1/(\sqrt{3}+1)$$

meaning that for someone that is not good at numbers (which is me) this doesn’t make sense. However, in cases like these you should think to multiple by the conjugate. Then, remember the difference of two squares formula of

$$(a-b)(a+b) = a^2 - b^2$$

that really comes up a lot on this test.

In a nutshell, make sure that you have all of the inequalities worked out before you write the problem. Make sure you know how you want to write it up before you write it. There is no time to write and rewrite. Also, if you know what you are doing, the writing is easy and will not come across as someone that is confused.

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**Problem #5, Spring 2021.** Let $(a_n)$, $(\beta_n)$, $(\gamma_n) \in \ell_\infty$ be fixed sequences. For a sequence $(b_n)$ define

$$T(b_n) = a_n + \beta_n b_{n+1} + \gamma_n b_{n+2}, \quad n \in \mathbb{N}.$$

(a) Prove that $T$ is a well-defined continuous mapping from $\ell_\infty$ to $\ell_\infty$.

(b) Prove that if $\| |\beta_n| + |\gamma_n| \|_\infty < 1$, then $T$ has a fixed point. (Here $(|\beta_n|)$, $(|\gamma_n|)$ are the sequences of the absolute values.)

(c) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $|\beta| + |\gamma| < |\alpha|$. Prove that for any $(a_n) \in \ell_\infty$ there exists $(b_n) \in \ell_\infty$ such that for all $n \in \mathbb{N}$
$$a_n = \alpha b_n + \beta b_{n+1} + \gamma b_{n+2}.$$
Also show that the result fails if the assumption $|\beta| + |\gamma| < |\alpha|$ is removed.

**Solution:** (a)

Let $\varepsilon > 0$ be given. Since $(\beta_n)$, $(\gamma_n) \in \ell_\infty$ are fixed sequences, set $\delta = \varepsilon / \| |\beta_n| + |\gamma_n| \|_\infty > 0$ and consider any $(x_n)$ and $(y_n)$ in $\ell_\infty$ with $\|x_n - y_n\| < \delta$. Observe that for any $n \in \mathbb{N}$, $|x_n - y_n| < \|x_n - y_n\| < \delta$ also holds. We compute,

$$\begin{aligned}
|T(x_n) - T(y_n)| &= |a_n + \beta_n x_{n+1} + \gamma_n x_{n+2} - (a_n + \beta_n y_{n+1} + \gamma_n y_{n+2})| \\
&= |\beta_n (x_{n+1} - y_{n+1}) + \gamma_n (x_{n+2} - y_{n+2})| \\
&\leq |\beta_n||x_{n+1} - y_{n+1}| + |\gamma_n||x_{n+2} - y_{n+2}| \\
&\leq |\beta_n| \, \| |x_n - y_n| \|_\infty + |\gamma_n| \, \| |x_n - y_n| \|_\infty \\
&= (|\beta_n| + |\gamma_n|) \, \| |x_n - y_n| \|_\infty
\end{aligned}$$

By taking the supremum over all $n \in \mathbb{N}$ of the inequality $|T(x_n) - T(y_n)| \leq (|\beta_n| + |\gamma_n|) \, \| |x_n - y_n| \|_\infty$ we obtain

$$\|T(x_n) - T(y_n)\|_\infty \leq \| |\beta_n| + |\gamma_n| \|_\infty \, \| |x_n - y_n| \|_\infty < \| |\beta_n| + |\gamma_n| \|_\infty \, \varepsilon / \| |\beta_n| + |\gamma_n| \| = \varepsilon$$

Thus, for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all sequences in $\ell_\infty$ with $\|x_n - y_n\|_\infty < \delta$, $\|T(x_n) - T(y_n)\|_\infty < \varepsilon$, and so, $T$ is continuous.

Since each sequence in the definition of $T$ is in $\ell_\infty$, the sup-norm of each sequence is finite which means that the sup-norm, $\|T(b_n)\|_\infty < \infty$ for any sequence $(b_n) \in \ell_\infty$.

Therefore, $T$ is a well-defined continuous mapping on $\ell_\infty$.

**Solution:** (b)

Since $\ell_\infty$ is complete with respect to the sup-norm and $\| |\beta_n| + |\gamma_n| \|_\infty < 1$ from part (a) we see that

$$\|T(x_n) - T(y_n)\|_\infty \leq \| |\beta_n| + |\gamma_n| \|_\infty \, \| |x_n - y_n| \|_\infty$$

implies that $T$ is a (strict) contraction mapping on $\ell_\infty$. By the Banach fixed point theorem, $T$ has a (unique) fixed point.

**Solution:** (c)

Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $|\gamma| + |\beta| < |\alpha|$. Let $(a_n) \in \ell_\infty$ be arbitrary. Observe that $\alpha \neq 0$ since the left hand side of the inequality is always positive, which means we can divide by $\alpha$. Define the (fixed) sequences

$$\alpha_n := \frac{a_n}{\alpha}, \quad , \beta_n := \frac{-\beta}{\alpha}, \text{ and } \gamma_n := \frac{-\gamma}{\alpha} \qquad \text{for all } n \in \mathbb{N}.$$

Then $T$ is given by

$$T(b_n) = \alpha_n + \beta_n b_{n+1} + \gamma_n b_{n+2}$$

for each $(b_n) \in \ell_\infty$. Since $\||\beta_n|\|_\infty = \left|\frac{\beta}{\alpha}\right|$ and $\||\gamma_n|\| = \left|\frac{\gamma}{\alpha}\right|$ for which the sequence defined by $(|\gamma_n|+|\beta_n|)$ has norm equal to

$$\||\beta_n| + |\gamma_n|\|_\infty = \left|\frac{\gamma}{\alpha}\right| + \left|\frac{\beta}{\alpha}\right| < 1,$$

because each of the fixed sequences are constant sequences and by the assumptions on $\alpha, \beta, \gamma \in \mathbb{R}$, we conclude by part (b) that $T$ has (unique) fixed point. So, there exists $(b_n) \in \ell_\infty$ such that $T(b_n) = b_n$ for all $n \in \mathbb{N}$. Using the definition of $T$ with our fixed sequences, consider

$$T(b_n) = \alpha_n + \beta_n b_{n+1} + \gamma_n b_{n+2} = b_n,$$

which gives

$$\frac{a_n}{\alpha} - \frac{\beta}{\alpha} b_{n+1} - \frac{\gamma}{\alpha} b_{n+2} = b_n,$$

and multiplying by $\alpha$ and rearranging the terms we have that

$$a_n = \alpha b_n + \beta b_{n+1} + \gamma b_{n+2}$$

for all $n \in \mathbb{N}$ and any arbitrary $(a_n) \in \ell_\infty$.

Now assume that the condition $|\gamma| + |\beta| < |\alpha|$ is removed. Let $(a_n)$ be a nonzero sequence in $\ell_\infty$ and choose $\alpha = \beta = \gamma = 0$, and let $(b_n) \in \ell_\infty$ be arbitrary. Because $(a_n)$ is a nonzero sequence, there exists some $m \in \mathbb{N}$ such that the term $a_m \neq 0$ which means that when we compute

$$0 \neq a_m = 0 \cdot b_m + 0 \cdot b_{m+1} + 0 \cdot b_{m+2} = 0$$

the claim fails to hold for all $n \in \mathbb{N}$.

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**Notes:** Well fuuuuuucking yay. This one has been a mystery to me for a year. I remember spending hours on the problem with Tyler and Josh in a room in Kidder hall and we were all confused on how to actually attack the problem. But, this problem has hints riddled throughout. First, it's weird that they did not choose to use $(\alpha_n)$ as the fixed sequences, they used $(a_n)$ instead. At first, this still felt weird but I moved on to try and prove part (a). I looked at part (b) and saw that they probably wanted us to use part (a) to prove part (b) to apply the contraction mapping theorem. So, this gave me the idea to somehow get the norm of the sum of the absolute values of the gamma and beta fixed sequences to appear in my proof of continuity of $T$. The harder part was to determine when exactly I needed to take the supremum over $n \in \mathbb{N}$. This is a common trick when proving various things with inequalities such as continuity, equicontinuity, contractions, etc. However, replacing the terms $|x_{n+1} - y_{n+1}|$ with the sup-norm works to eliminate the weirdness of the indices not matching. This took me a while, I started to apply the triangle inequality to write things out... but then I felt like I was working too hard to I wasn't getting the "trick". These questions which horrendous notation normally boil down to an easy trick if you see it. Praveeni helped with the last trick of dividing by $\alpha$. Also, the fact that each sequence in the definition of $T$ were fixed and how they chose $(a_n)$ and not $(\alpha_n)$ made me think to just define sequences that work. From part (b) we get the existence of the sequence $(b_n)$ and then all is left to show is that the equation is satisfied for the fixed sequences we chose.

The last part to show that the conclusion fails if we do not have the conditions on the real numbers $\alpha, \beta$, and $\gamma$ stumped me for a while. But, why not choose them all to be zero and choose a random sequence with at least one nonzero term. Then, the sequence is in the space because were chosing it to be. I guess I could choose the sequence $(a_n) := (1, 0, 0, \ldots, )$ as my sequence because the sup-norm equals one and then we get that $1 \neq 0$. I was trying to be tricky and overcomplicate the idea but it boils down to something doesn't equal something else when the condition is removed.

Really good problem and I am proud of myself that I figured it out.

# Fundamental Theorem of Calculus Trick

## 1 Convergence and Completeness

### 1.1 Problems

**Problem #1, Fall 2021.** Let $X = \{f \in C[0,1] : f(0) = 0\}$. For each $f \in X$, let

$$\|f\| = \max\{|f'(x)| : x \in [0,1]\}.$$

Note that the norm of $f$ is defined in terms of the *derivative* of $f$.

(a) Show that if the definition of $X$ is changed by deleting the condition, $f(0) = 0$, then $\|\cdot\|$ is not a norm.

(b) Show that if a sequence $\{f_n\}_{n=1}^{\infty}$ converges in $X$ with respect to the norm $\|\cdot\|$, then the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[0,1]$.

(c) Show that the space $X$ is complete with respect to the norm defined above.

**Solution:** (a)

Choose $f(x) = 1 \in C^1[0,1]$ for all $x \in [0,1]$. We compute, and find that $\|f\| = 0$ but $f \neq 0$ on $[0,1]$. Therefore, if the condition that $f(0) = 0$ is removed, then $\|\cdot\|$ does not define a norm.

**Solution:** (b)

Let $\varepsilon > 0$ be given. By the Fundamental Theorem of Calculus, write

$$f_n(x) = \int_0^x f_n'(t)dt \qquad \text{for all } x \in [0,1] \text{ and every } n \in \mathbb{N}, \tag{9}$$

and similarly, for $f \in X$. Since the sequence $\{f_n\}_{n=1}^{\infty}$ converges in $X$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n - f\| < \varepsilon$. Let $x \in [0,1]$ be arbitrary and consider the following,

$$\begin{aligned}
|f_n(x) - f(x)| &= \left| \int_0^x f_n'(t)dt - \int_0^x f(t)dt \right| \\
&\leq \int_0^x |f_n'(t) - f'(t)|\, dt \\
&\leq \int_0^x \|f_n - f\|\, dt \\
&< \varepsilon \cdot x
\end{aligned}$$

By taking the supremum over all $x \in [0,1]$, we have that $\sup_{x\in[0,1]} |f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$. That is, $f_n$ converges uniformly to $f$ on $[0,1]$.

**Solution:** (c)

Let $\varepsilon > 0$ be given and let $\{f_n\}_{n=1}^{\infty}$ be an arbitrary Cauchy sequence in $X$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|f_n'(x) - f_m'(x)| \leq \max_{x\in[0,1]} \{|f_n'(x) - f_m'(x)|\} < \varepsilon.$$

Since the above inequality holds for all $x \in [0, 1]$, we conclude that the sequence $\{f_n'\}_{n=1}^{\infty}$ is uniformly Cauchy on $[0, 1]$. A sequence is uniformly Cauchy if and only if the sequence converges uniformly, and, moreover, the sequence converges to a continuous limit. So,

$$\lim_{n\to\infty} f_n'(x) = g(x) \quad \text{for some } g \in C[0,1] \tag{10}$$

Define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in [0, 1]$. By the Fundamental Theorem of Calculus, (as defined by equation (9) we may write

$$f_n(x) = \int_0^x f_n'(t)dt \quad \text{for all } x \in [0,1] \text{ and every } n \in \mathbb{N}.$$

Now observe that,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_0^x f_n'(t)dt = \int_0^x \lim_{n\to\infty} f_n'(t)dt = \int_0^x g(t)dt$$

(note that the above equalities come from our definition of $f$, equations (10) and (9), and that if a function converges uniformly then we can interchange the integral and limits). Since $g \in C[0, 1]$ and $\int_0^x g(t)dt$ is continuous, $f \in C[0, 1]$ and it follows that $f'(x) = g(x) \in C[0, 1]$, i.e., $f \in C^1[0, 1]$. Lastly, because $f_n \in X$ for all $n \in \mathbb{N}$, $f(0) = \lim_{n\to\infty} f_n(0) = 0$.

Therefore, $f \in X$ and $X$ is complete with respect to the $\|\cdot\|$ defined above.

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**Notes:** Well, this was a beast of a problem and it combined a lot of different facts together. In addition, it has you write the convergence of sequence using the limit definition and also use the the epsilon limit definition of a convergent sequence. One important thing is that converging uniformly is equivalent to uniformly Cauchy in the sup-norm. And, even more, the limit of a sequence of functions that converges uniformly on some space is continuous. In the last part we were kind of clever by writing $f_n$ using the derivative and the fundamental theorem of calculus.

One nice thing that you did is that you wrote down your ideas and then wrote up the solution. Go you.

Next, the last part kind of hinges on the theorem that states: if you have a sequence of real valued functions each of which is continuously differentiable and that the sequence of the derivatives converges uniformly to some continuous function and if the sequence of functions converges point wise for some $x_0 \in [a, b]$, then the sequence of real valued functions converges uniformly and the sequence of derivatives converges to the derivative of the sequence of functions.

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# Arzeli-Ascoli Theorem

## 1 Problems

**Problem #6, Fall 2019.** Let $C[0,1] = \{f : [0,1] \to \mathbb{R}\}$ denote the space of all continuous functions. You can use that this is a complete metric space with respect to the sup-norm, $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$. Define

$$\mathcal{F} = \{f \in C[0,1] : |f(x) - f(y)| \leq |x-y|, \text{ for all } x, y \in [0,1]\}$$

(a) Show that $\mathcal{F}$ is closed, but not compact in $C[0,1]$.

(b) Show that

$$\mathcal{F}_1 = \{f \in \mathcal{F} : \int_0^1 f^2(x)dx = 1\}$$

is compact in $C[0,1]$.

**Solution:** (a)

Define $f_n(x) = n$ for $n \in \mathbb{N}$. Then, $f_n \in \mathcal{F}$ because $|f_n(x) - f_n(y)| = 0 \leq |x-y|$ for all $n \in \mathbb{N}$. However, as $n \to \infty$, $f_n(x) \to \infty$ implying that $\mathcal{F}$ is not bounded and therefore, is not totally bounded and is not compact as a result. [1]

Let $\varepsilon > 0$ and let $\{f_n\}_{n=1}^{\infty}$ be an arbitrary convergent sequence in $\mathcal{F}$ that converges to $f \in C[0,1]$ (since $C[0,1]$ is complete). Then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|f_n(x) - f(x)| \leq \sup_{x \in [0,1]} \{|f_n(x) - f(x)|\} < \frac{\varepsilon}{2} \quad \text{for all } x \in [0,1].$$

Observe,

$$\begin{aligned}
|f(x) - f(y)| &\leq |f_n(x) - f(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&< \frac{\varepsilon}{2} + |x-y| + \frac{\varepsilon}{2} \\
&= \varepsilon + |x-y|
\end{aligned}$$

holds for all $\varepsilon > 0$ and as $\varepsilon \to 0$, $|f(x) - f(y)| < |x-y|$ which means that $f \in \mathcal{F}$. Therefore, $f$ contains all its limit points and $\mathcal{F}$ is closed.

**Solution:** (b)

We will show that $\mathcal{F}_1$ is closed, equicontinuous, and uniformly bounded.

1. Let $\{f_n\}_{n=1}^{\infty}$ be an arbitrary convergent sequence in $\mathcal{F}_1$ that converges to $f$. Since $\mathcal{F}_1 \subseteq C[0,1]$ and $C[0,1]$ is complete under the sup-norm, convergence of this sequence implies that $f_n$ converges uniformly to $f$ on $[0,1]$. Since each $f_n \in \mathcal{F}_1$, we see that $F_1$ is closed because

$$1 = \lim_{n\to\infty} \int_0^1 f_n^2(x)dx = \int_0^1 \lim_{n\to\infty} f_n^2(x)dx = \int_0^1 f^2(x)dx,$$

[1] Also note that you can choose $f_n(x) = n + x$. Here we have created a sequence of functions in the set that is unbounded. Hence, the sequence does not have a Cauchy subsequence because any two functions of our set will differ by 1. A set is totally bounded if and only if every sequence in the space has a Cauchy subsequence. So, if we choose our definition of compact as "Complete and Totally Bounded", then $\mathcal{F}$ is not compact because $\mathcal{F}$ is not totally bounded.

that is $f \in \mathcal{F}_1$.

2. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon > 0$. Then, for any $f \in \mathcal{F}_1 \subset \mathcal{F}$,
$$|f(x) - f(y)| \leq |x - y| < \delta = \varepsilon.$$
So, $\mathcal{F}_1$ is equicontinuous.

3. Let $f \in \mathcal{F}_1 \subset \mathcal{F}$ be arbitrary, then observe
$$|f(x) - f(0)| \leq |x - 0| = |x|$$
and expanding the inequality (as $-|x| \leq f(x) - f(0) \leq |x|$), repackaging, and using the triangle inequality we have that
$$|f(x)| \leq |f(0)| + |x|$$
for all $f \in \mathcal{F}_1$. Suppose $f(0) = \pm 2$. Since $f$ is Lipschitz with constant 1 at best we can bound $f$ above by the line $g_1(x) = 2 - x$ and below by the line $g_2(x) = -2 + x$. Then, by squaring and integrating the inequality $|f(x)| \leq |f(0)| + |x|$, we have
$$1 = \int_0^1 f^2(x)dx < \int_0^1 (2-x)^2 dx = \frac{7}{3}$$
which shows that the $y$-intercept of $f$ cannot be above 2 nor below $-2$ for any $f \in \mathcal{F}_1$. Thus,
$$|f(x)| \leq |f(0)| + |x| \leq 2 + |x|$$
for all $f \in \mathcal{F}_1$. By taking the supremum over all $x \in [0,1]$, we conclude that the set $\{f(x) : x \in [0,1], f \in \mathcal{F}_1\}$ is bounded with an upper bound of $M = 3 > 0$. Therefore, $\mathcal{F}_1$ is uniformly bounded.

Therefore, by the Arzela-Ascoli theorem, given that $[0,1]$ is a compact metric space and $\mathcal{F}_1$ is closed, equicontinuous, and uniformly bounded, $\mathcal{F}_1$ is compact.

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So, this is a really good problem because it makes you think about the behavior of the functions themselves. Also, things may “look” like they work out, but then, you can find a counterexample that breaks it.

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# Sequential Compactness to the Rescue

## 1 Problems

**Problem #, Spring 2018.** INCLUDE PROBLEM STATEMENT.

**Solution:** (a)

(Proof by Contradiction).

Let $0 < \alpha < \beta < 1$ be arbitrary and suppose

$$d(K_\alpha, \partial K_\beta) = 0.$$

We first show that $K_\alpha$ and $\partial K_\beta$ are compact in $\mathbb{R}^m$. Since $\liminf_{|x|\to\infty} f(x) \geq 2$, there exists some $M > 0$ in $\mathbb{R}$ such that for any $x \in \mathbb{R}^n$ with $|x| \leq M$, $f(x) < \frac{3}{2}$ (otherwise, 2 would not be the limit inferior). Then, $K_\alpha$ and $\partial K_\beta$ are bounded by $M$ because $f(x) \leq \alpha < \frac{3}{2}$ for all $x \in K_\alpha$ and $f(x) = \beta < \frac{3}{2}$ for all $x \in \partial K_\beta$. Observe that the sets can be written as the pre-image of $f$ under a specific set, i.e.,

$$K_\alpha = f^{-1}((-\infty, \alpha])$$
$$\partial K_\beta = f^{-1}(\{\beta\}).$$

Since $(-\infty, \alpha]$ and $\{\beta\}$ are closed in $\mathbb{R}^m$ (under the Euclidean metric) and $f$ is continuous, the pre-image of closed sets under continuous functions are closed. Therefore, by the Heine-Borel theorem in $\mathbb{R}^m$, each set is compact if and only if they are closed and bounded.

Let $\{x_n\} \subset K_\alpha$ and $\{y_n\} \subset \partial K_\beta$ be sequences such that

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

(which follows by the assumption that $d(K_\alpha, \partial K_\beta) = 0$). Since $K_\alpha$ is compact there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to $x \in K_\alpha$. Similarly, by the compactness of $\partial K_\beta$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that converges to $y \in \partial K_\beta$. Also, $x \neq y$ because if so, we would have that $x \in \partial K_\beta$ which means that $f(x) = \beta$ and $f(x) \leq \alpha$ with $\alpha < \beta$ and this contradicts our assumption that $f$ is a (continuous) function. So, we can assume that each subsequence converges to distinct points in their respective compact spaces. However, we also find that

$$\lim_{k\to\infty} |x_{n_k} - y_{n_k}| = 0,$$

because every subsequence of a convergent sequence converges to the same limit point. Yet, by the properties of convergent sequences,

$$\lim_{k\to\infty} |x_{n_k} - y_{n_k}| = |x - y| \neq 0$$

which contradicts our assumption that there exists sequences in $K_\alpha$ and $\partial K_\beta$ whose difference converges to 0.

Therefore, $d(K_\alpha, \partial K_\beta) > 0$ for all $0 < \alpha < \beta < 1$.

**Solution:** (b)

FINISH ME. Define a function that is a bunch of spikes that have a constant height and the width of each spike is equal to $1/n$ for all $n \in \mathbb{N}$ or something like this.

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**Notes:** So, this was a super super hard problem because the notation is kind of aweful. Yet, they are conveying a "simple" idea. In the first problem, because the limit inferior of the function stays bounded below by 2 as $|x|$ approaches infinity, there are only "finitely" many values of $x \in \mathbb{R}^m$ that map to values below 2. The choice of the word finite is weird because the real numbers are uncountable. Yet, we can visualize the idea as a bounded subset of real numbers that map to values less than 2 as we approach infinity. Still feels weird. You're tired.

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# Continuity and Limits

## 1 Problems

### 1.1 Spring 2022 #2

**Problem #2, Spring 2022.** INCLUDE PROBLEM STATEMENT

**Solution:** (a)

Let $\varepsilon > 0$ be given and let $f \in C[0,1]$ be arbitrary.

Since $f$ is continuous on a compact set, $f$ is uniformly continuous, and there exists $\delta > 0$ such that for any $x, y \in [0,1]$ with $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Choose $n \in \mathbb{N}$ by the Archimedean property such that $1/n < \delta$ and define the intervals

$$E_j = \left[\frac{j-1}{n}, \frac{j}{n}\right) \text{ for all } j \in \{1, \ldots, n-1\} \text{ and } E_n = \left[\frac{n-1}{n}, 1\right].$$

Observe that $\bigcup_{j=1}^{n} E_j = [0,1]$ and for any $k \in \{1, \ldots, n\}$ with $k \neq j$, $E_k \cap E_j = \emptyset$, i.e., the intervals $E_j$ are pairwise disjoint. Define $c_j = f\left(\frac{j-1}{n}\right) \in \mathbb{R}$ for $j \in \{1, \ldots, n\}$. Then, $\phi : [0,1] \to \mathbb{R}$ is a step function with

$$\phi(x) = \sum_{j=1}^{n} c_j I_j(x)$$

where $I_j(x)$ is the characteristic function on $E_j$.

Let $x \in [0,1]$ be arbitrary. Since there are finitely many intervals, there exists some $j \in \{1, \ldots, n\}$ such that $x \in E_j$. Also, observe that $y = \frac{j-1}{n} \in E_j$ and $|x - y| < \frac{1}{n} < \delta$. Using the (uniform) continuity of $f$ on $[0,1]$, it follows that

$$|f(x) - c_j| = \left|f(x) - f\left(\frac{j-1}{n}\right)\right| = |f(x) - f(y)| < \varepsilon.$$

Because $x \in [0,1]$ was arbitrary, the maximum over all $x \in [0,1]$ will also be less that $\varepsilon$ by construction of our step function. Therefore, for every $\varepsilon > 0$ and every $f \in C[0,1]$, there exists a step function $\phi(x)$ such that

$$\max_{0 \leq x \leq 1} |f(x) - \phi(x)| < \varepsilon.$$

**Solution:** (b)

Let $\phi(x)$ be an arbitrary step function. We compute,

$$\begin{aligned}
\left|\int_0^1 \phi(x)\cos(nx)dx\right| &= \left|\int_0^1 \sum_{j=1}^{n} c_j I_j(x)\cos(nx)dx\right| \\
&= \left|\int 0^1 c_1 I_1(x)\cos(nx)dx + \cdots + \int_0^1 c_n I_n(x)\cos(nx)dx\right|
\end{aligned}$$

$$= \left| \int_{E_1} c_1 \cos(nx)dx + \cdots + \int_{E_n} c_n \cos(nx)dx \right|$$
$$= \left| c_1 \frac{\sin(nx)}{n}\bigg|_{E_1} + \cdots + c_n \frac{\sin(nx)}{n}\bigg|_{E_n} \right|$$

where the last two lines come from the fact that the intervals $E_j$ are pairwise disjoint. Now, consider some $j \in \{1, \ldots, n\}$ and let $a_j$ and $b_j$ be arbitrary points in $E_j$, and $0 \leq a_j < b_j \leq 1$. Observe that by the triangle inequality and $|\sin(y)| \leq 1$ for all $y \in \mathbb{R}$, we have that

$$\left| c_j \left( \frac{\sin(b_j n)}{n} - \frac{\sin(a_j n)}{n} \right) \right| \leq |c_j| \left( \frac{|\sin(b_j n)|}{n} - \frac{|\sin(a_j n)|}{n} \right) \leq \frac{2|c_j|}{n}.$$

Define $C = \sum_{j=1}^{n} 2|c_j|$, then

$$0 \leq \left| \int_0^1 \phi(x) \cos(nx)dx \right| = \left| c_1 \frac{\sin(nx)}{n}\bigg|_{E_1} + \cdots + c_n \frac{\sin(nx)}{n}\bigg|_{E_n} \right| \leq \frac{C}{n}$$

and as $n \to \infty$, $\frac{C}{n} \to \infty$ and so,

$$\lim_{n \to \infty} \int_0^1 \phi(x) \cos(nx)dx = 0.$$

**Solution:** (c)

Let $\varepsilon > 0$ be given and let $f \in C[0,1]$ be arbitrary. By part (a) there exists a step function $\phi$ on $[0,1]$ such that $\max_{0 \leq x \leq 1} |f(x) - \phi(x)| < \varepsilon$. We compute,

$$\begin{aligned}
\int_0^1 f(x)\cos(nx)dx &= \int_0^1 (f(x) - \phi(x) + \phi(x))\cos(nx)dx \\
&= \int_0^1 (f(x) - \phi(x))\cos(nx)dx + \int_0^1 \phi(x)\cos(nx)dx \\
&\leq \int_0^1 \max_{0 \leq x \leq 1} |f(x) - \phi(x)| \cos(nx)dx + \int_0^1 \phi(x)\cos(nx)dx \\
&< \varepsilon \int_0^1 \cos(nx)dx + \int_0^1 \phi(x)\cos(nx)dx \\
&= \varepsilon \frac{\sin(n)}{n} + \int_0^1 \phi(x)\cos(nx)dx \\
&\leq \frac{\varepsilon}{n} + + \int_0^1 \phi(x)\cos(nx)dx
\end{aligned}$$

and observe that

$$0 \leq \left| \int_0^1 f(x)\cos(nx)dx \right| \leq \frac{\varepsilon}{n} + \left| \int_0^1 \phi(x)\cos(nx)dx \right|.$$

Then, by taking the limit as $n \to \infty$ and applying the result from part (b) (and because $1/n \to 0$ as $n \to \infty$) we have that

$$0 \leq \lim_{n \to \infty} \left| \int_0^1 f(x)\cos(nx)dx \right| \leq 0.$$

Therefore, $\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)dx = 0$ for every $f \in C[0,1]$.

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**Notes:** This one is hard.