Quantitative Finance

Benders Decomposition

Patrick Hénaff

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library(foreach)
library(doParallel)
library(kableExtra)
library(linprog)
library(gtools)
library(pracma)
library(kernlab)
library(optiSolve)
library(piqp)
library(latex2exp)
```

The algorithm due to Benders provides a practical strategy for solving two-stage stochastic optimization problems. The principle of the method is to observe that in an optimization problem, there are sometimes "complicating variables", in the sense that, if these variables were known, the remaining problem would be simple to solve. In a mixed-integer programming problem, the "complicating variables" are the integer variables. In a two-stage problem with recourse, they are the variables pertaining to the first stage decision.

We first present the method with a simple model, then explore how it may be used to solve stochastic optimization problems with recourse.

The Benders decomposition algorithm

Consider the problem

$$
\begin{aligned}
\min \quad & c^T x + f^T y \\
\text{s.t.} \quad & \\
& Ax = b \\
& Bx + Dy = d \\
& x \geq 0; y \geq 0
\end{aligned}
$$

The problem is reformulated as follows:

$$\begin{aligned}
\min\ & c^T x + z(x) \\
\text{s.t.}\ & \\
& Ax = b \\
& x \geq 0
\end{aligned}$$

with

$$\begin{aligned}
P2 : z(x) = \min\ & f^T y \\
\text{s.t.}\ & \\
& Dy = d - Bx \\
& y \geq 0
\end{aligned}$$

Let $D2$ be the dual of $P2$:

$$\begin{aligned}
D2 : z(x) = \max_p\ & p^T (d - Bx) \\
\text{s.t.}\ & \\
& Dp \leq f
\end{aligned}$$

The domain $\{p | D^T p \leq f\}$ is defined by a set of extreme points $p^i, i = 1, \ldots, I$ and of extreme rays $r^j, j = 1, \ldots, J$.

If $D2$ is unbounded, there exists an extreme ray r^j such that $(r^j)^T (d - Bx) > 0$.

If the solution $\bar{x}$ of $D2$ is finite, that solution identifies an extreme point p^i of the domain such that $z(\bar{x}) = (p^i)^T (d - Bx)$.

We can, in theory, restate $D2$ in terms of the extreme points and extreme rays of the feasible domain; this is called the full master problem:

$$\begin{aligned}
FMP : z(x) = \min\ & c^T x + z \\
\text{s.t.}\ & \\
& Ax = b \\
& x \geq 0 \\
& (p^i)^T (d - Bx) \leq z, i = 1, \ldots, I \\
& (r^j)^T (d - Bx) \leq 0, j = 1, \ldots, J
\end{aligned}$$

The set of extreme points and extreme rays may be very large, and we expect that only a few constraints will be active at the optimum. The idea behind Benders' partitioning is therefore to progressively build the set of constraints, until the optimum is reached.

The above formulation, with only a subset of extreme rays and extreme points constraints is called the restricted master problem (RMP).

Assume that we have just solved the RMP with l constraints:

$$
\begin{aligned}
FMP: z(x) = \min\ & c^T x + z \\
\text{s.t.} & \\
& Ax = b \\
& x \geq 0 \\
& (p^i)^T (d - Bx) \leq z, i = 1, \ldots, k - l \\
& (r^j)^T (d - Bx) \leq 0, j = 1, \ldots, l
\end{aligned}
$$

and obtained the solution $(\bar{x}, \bar{z})$. Does this solution violate any constraint not included thus far? To answer that question, we solve

$$
\begin{aligned}
Q(\bar{x}): \max\ & p^T (d - B\bar{x}) \\
\text{s.t.} & \\
& Dp < f
\end{aligned}
$$

If $Q(\bar{x})$ is unbounded, the linear program reveals a new extreme ray r^j such that $(r^j)^T (d - B\bar{x}) > 0$. Therefore, r^j violates the constraint $(r^j)^T (d - B\bar{x}) \leq 0$, and that constraint must be added to the RMP.

If $Q(\bar{x})$ is finite, the linear program reveals a new extreme point p^i. If $(p^i)^T (d - B\bar{x}) > \bar{z}$, the extreme point violates the constraints of the RMP, and that constraint must added to the RMP. Otherwise, we conclude that the RMP provides a solution $(\bar{x}, \bar{z})$ that is feasible and optimal for the FMP.