

Conditional Probability and Volatility Clustering

Patrick Hénaff

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The purpose of this note is to reproduce and comment some results on the conditional distribution of returns, presented by K. Chen *et al* in (Chen et al., 2008). We provide the code and data to reproduce the calculations presented in the paper, and are able to reproduce its figures. We however show that the advocated model does not fully account for the “volatility clustering”, an important stylized fact of financial time series.

1 Characteristics of Financial Time Series

The characteristic features of financial time series have been extensively documented (Cont, 2001). As mentioned, a distribution of returns with "fat tails", and volatility clustering are the key aspects of these series. (Chen et al., 2008) shows that for many financial time series, the return at any time scale is, beyond some threshold, distributed according to a power law. The parameters of this power law depend upon the magnitude of return in the previous period (this is the volatility clustering feature). Moreover, when return is scaled conditionally to the magnitude of the previous period return, the distribution is found to follow a universal form, asymmetrical and with fat tails.

Let $\{r\}$ be a series of returns, computed at intervals Δt . Following Chen's notation, the values of r are binned according to r_p , the magnitude of return in the previous period. The standard deviation of each bin is $w(r_p)$. the distribution of return, conditional upon the magnitude of the previous return, can be expressed as:

$$P(r|r_p) = \frac{1}{w(r_p)} f\left(\frac{r}{w(r_p)}\right) \quad (1)$$

with:

r return over a time interval Δt

r_p absolute value of return in the previous period

$w(r_p)$ standard deviation of r in the bin defined by r_p

$f()$ distribution of return, conditionally normalized

It follows that the joint distribution of r and r_p given by (1) captures both the features described by $f()$ (fat tails, asymmetry) and the conditional variance (volatility clustering).

Numerous studies (Gopikrishnan et al., 1998) have documented that the cumulative distribution of returns, beyond some threshold, can be described by a power law, that is, in the notation of (1):

$$f(x) = \frac{\alpha - 1}{x_{\min}} \left(\frac{x}{x_{\min}} \right)^{-\alpha}, \quad x > x_{\min} \quad (2)$$

Chen et al. (2008, p. 2) notes that the distribution of scaled conditional return, $P(r|r_p)$ "collapses to a universal curve", and this can indeed be observed on some time series. Following Chen, we consider a long series of the Dow Jones Industrial Average, from 1900 to 2004, provided by Williamson (2013). The series is sampled every $T = 2$ days, and we use 8 bins of equal size to partition r_p .

```
# number of bins
nb <- 8
ticker <- 'djia'
r.djia <- get.ts(ticker, returns=TRUE)
ticker <- 'qqq'
r.QQQ <- get.ts(ticker, returns=TRUE)
```

The next function allocate observations r_t into nb bins, conditional upon the value of $|r_{t-1}|$.

```
bins.djia <- make.bins(r.djia, NBIN=nb, model='abs', params)
bins.QQQ <- make.bins(r.QQQ, NBIN=nb, model='abs', params)
```

Following the article in reference, we plot the conditional density of return for each bin. The width of the density is positively correlated to the magnitude of $|r_{t-1}|$, although the pattern is more visible for QQQ (as reported in the paper) as for, for example, for DJIA. This is illustrated in the following two figures.

```
par(mar=c(4,4,0.1,0.1))
plot.conditional.r(r=r.QQQ, bins=bins.QQQ, ticker='QQQ', NBIN=nb)
plot.conditional.r(r=r.djia, bins=bins.djia, ticker='DJIA', NBIN=nb)
```

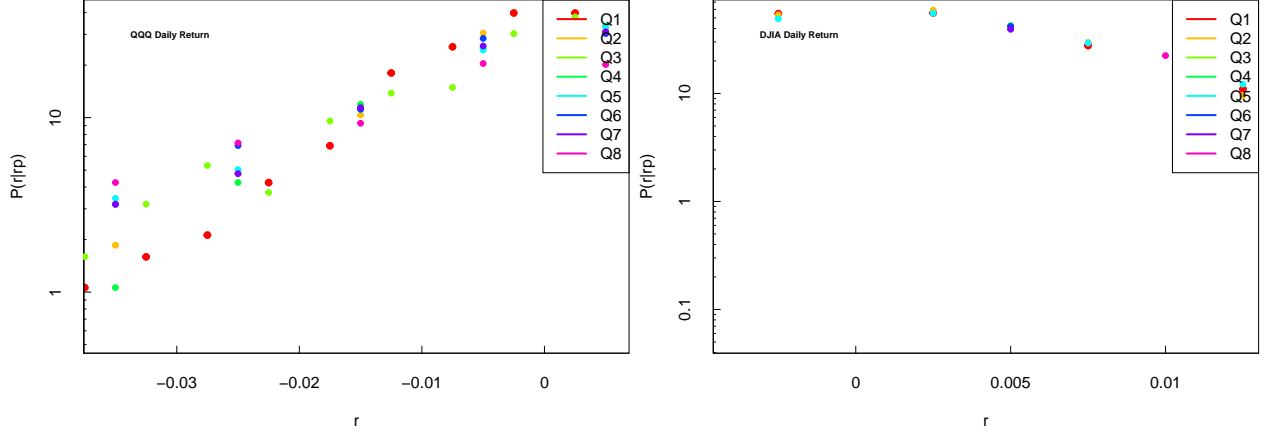


Figure 1: Conditional density of QQQ and DJIA return. r is partitioned in 8 bins of equal size, according to the absolute value of return in the previous period.

Using the estimation method introduced by Clauset et al. (2009), and considering each bin separately, we fit the tail distribution to a power law. The parameters α_{\min} and α specific to each bin are reported in Figure 2. The results are consistent with the ones reported by Chen *et al.*

```
params = list(lambda=0.94, nb.init=250)
plot.cum.cond.r(r=r.djia, bins=bins.djia, ticker='djia', sgn='all',
NBIN=nb, model='abs', params=params)
```

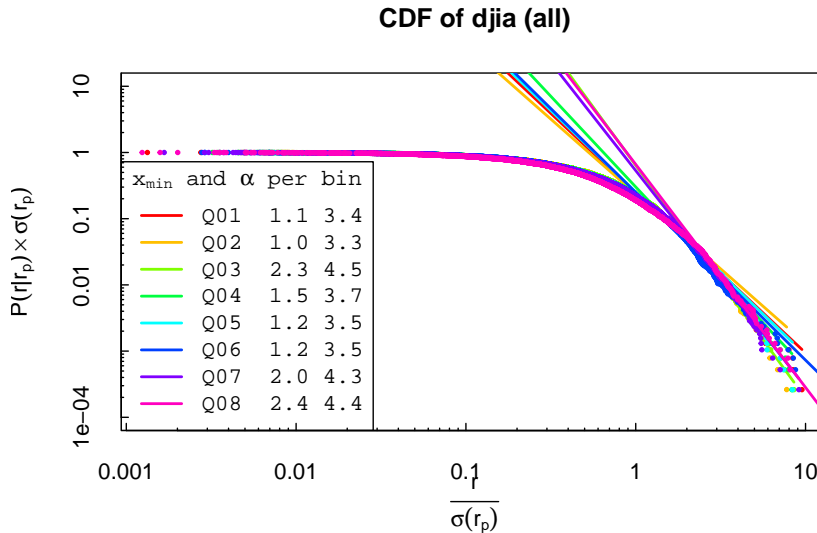


Figure 2: Cumulative density of DJIA return. r is partitioned in 8 bins of equal size, according to the absolute value of return in the previous period. For each bin, the inserted table reports α_{\min} and α .

However, the claim should perhaps not be taken literally: it is not hard to identify time series that

do not exhibit a "collapse to a universal curve". Figure 3, among many examples, displays the results of the same calculation, applied to the BNP (BNP.PA) stock, using daily close prices from 2000 to 2013. The conditional scaled tail return for the first bin (the bin corresponding to returns of small magnitude in the previous time interval) shows a marked difference from the distributions of the other bins.

```
ticker <- 'bnp'
r.bnp <- get.ts(ticker, returns=TRUE)
bins <- make.bins(r.bnp, NBIN=nb, model='abs', params)

plot.cum.cond.r(r=r.bnp, bins=bins, ticker=ticker, sgn='all',
NBIN=nb, model='abs', params=params)
```

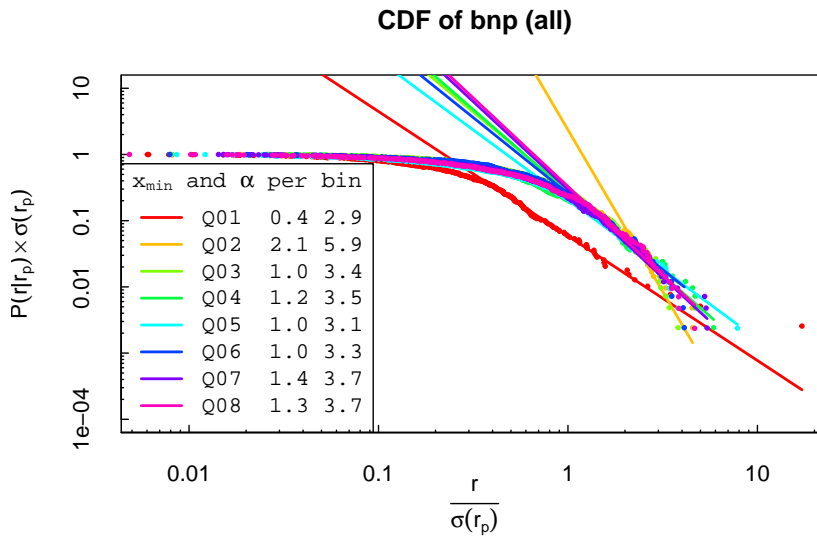


Figure 3: CDF of BNP (BNP.PA) return. The conditional scaled tail return for the first bin (the bin corresponding to returns of small magnitude in the previous time interval) shows a marked difference from the distributions of the other bins.

1.1 Relationship between the scaling factor and $|r_p|$

For r_p sufficiently large, there is an apparent linear relationship between r_p and $\sigma(r_p)$. This observation is consistent with volatility clustering, and is illustrated by the following calculation, which reproduces Figure 2 of Chen's paper.

```
ticker <- 'qqq'
p <- get.ts(ticker, returns=FALSE)
nb <- 8
```

```

# sample at T=2
indx <- seq(1, length(p), by=2)
r.2 <- returns(p[indx,])
# sample at T=20
indx <- seq(1, length(p), by=20)
r.20 <- returns(p[indx,])

plot.scaling.factor(plot.type='plot', r=r.2, NBIN=nb, type='b',
                    pch=16, forcexlim=c(0,.01),
                    forceylim=c(.0, .01),
                    main=TeX('$\\sigma(r_p)$ as a function of $r_p$'))

plot.scaling.factor(plot.type='lines', r=r.20, NBIN=nb,
                    type='b', pch=1)
labels <- c('T=2', 'T=20')
legend('topleft', labels, lty=c(1,1), pch=c(16,1))

```

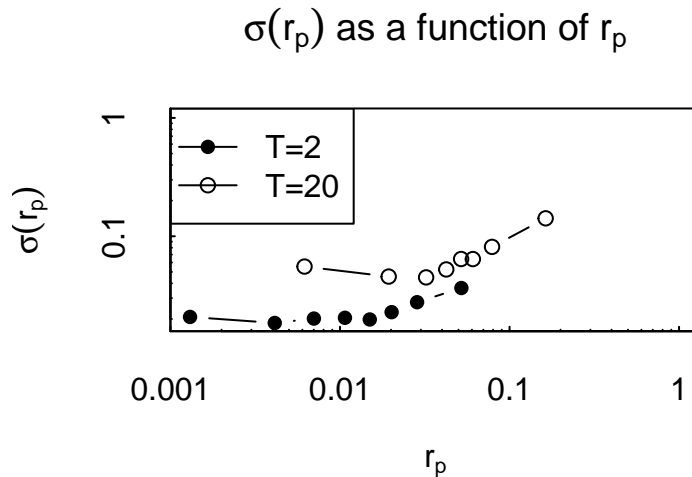


Figure 4: The scaling factor $\sigma(r_p)$ as a function of r_p . The approximate linear dependency is confirmed by the calculation.

Finally, it should be noted that the scaling factor $\sigma(r_p)$ does not transform returns into independent random variables: Figure 5 displays the autocorrelation of the scaled return and of its absolute value. This observation should not come as a surprise, as volatility is commonly modeled by an autoregressive process, and not by a function of the previous observation alone.

```

bins <- make.bins(r.QQQ, NBIN=nb, model='abs', params)
par(mar=c(4,4,0.1,0.1))
acf(bins$r.sc)
acf(abs(bins$r.sc))

```

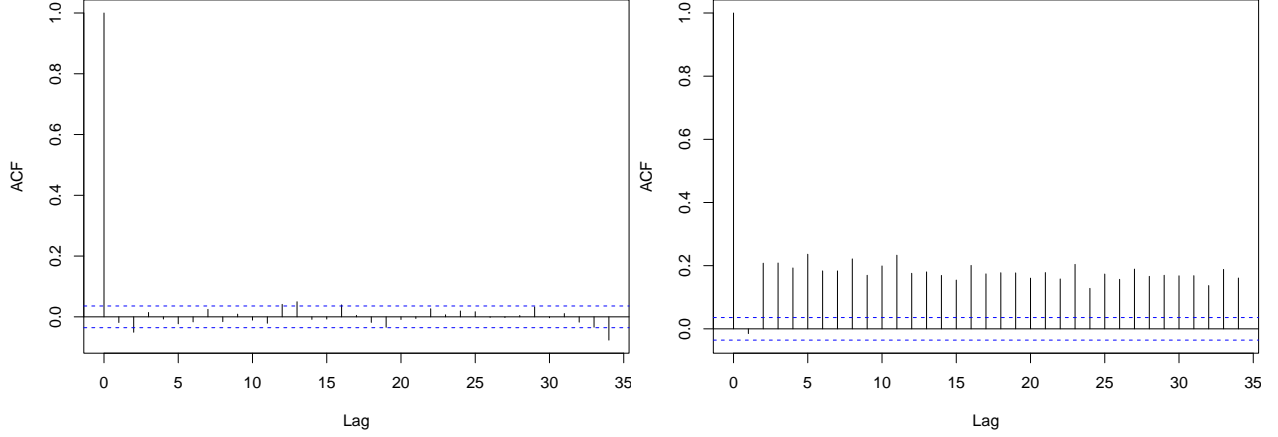


Figure 5: Autocorrelation of $r_t/\sigma(r_p)$ and $|r_t|/\sigma(r_p)$. QQQ return serie, sampling $T = 2$

The effect of scaling can be measured by comparing Figure 5 to the same calculation, performed on the original return series. Scaling by $\sigma(r_p)$ reduces the autocorrelation of absolute return, but does not completely cancel it.

```
par(mar=c(4,4,0.1,0.1))
acf(r.QQQ)
acf(abs(r.QQQ))
```

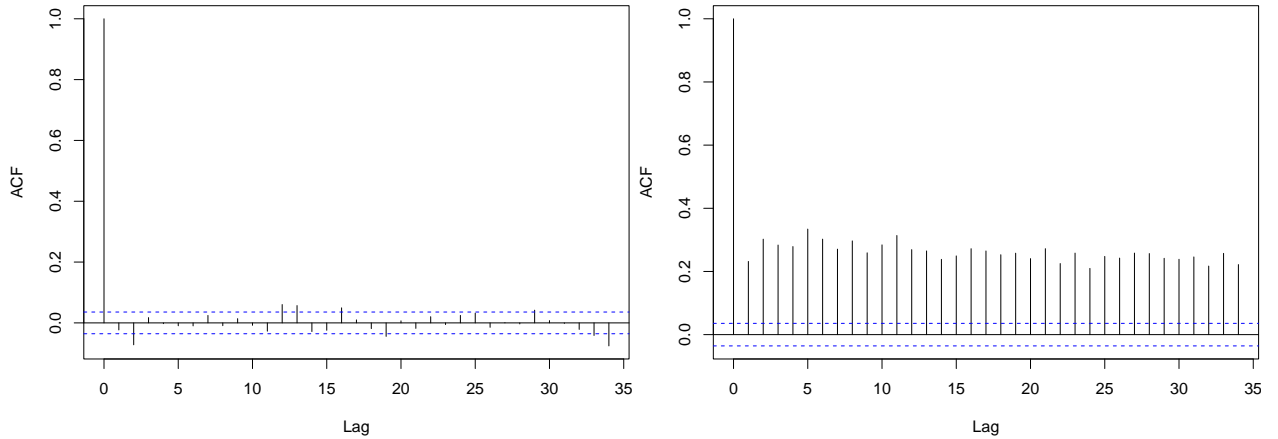


Figure 6: Autocorrelation of r_t and $|r_t|$. QQQ return serie, sampling $T = 2$

2 Conclusion

Scaling return series by a factor which is a function of the absolute value of return in the previous period provides a model for “fat tails”, but only partially captures the volatility clustering feature of financial time series.

Bibliography

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