Modèle Moyenne-Variance et Extensions

P. Hénaff

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Optimisation de Portefeuille: Notation (1)

- Rendement

$$R_i \sim N(\mu_i, \sigma_i^2)$$

$$\mathrm{cov}(R_i, R_j) = \sigma_{i,j}$$

- Allocation exprimée en fraction de la richesse initiale

$$\sum_i w_i = 1$$

- Espérance de rendement du portefeuille

$$w^T \mu = \mu_{p,w}$$

- Variance du portefeuille

$$w^T \Sigma w = \sigma^2_{p,w}$$

Optimisation de Portefeuille: Notation (2)

- Distribution du rendement du portefeuille

$$R_{p,w} \sim N(\mu_{p,w}, \sigma^2_{p,w})$$

- Covariance entre deux portefeuilles

$$\text{cov}(R_{p,w_1}, R_{p,w_2}) = w_1^T \Sigma w_2$$

Données

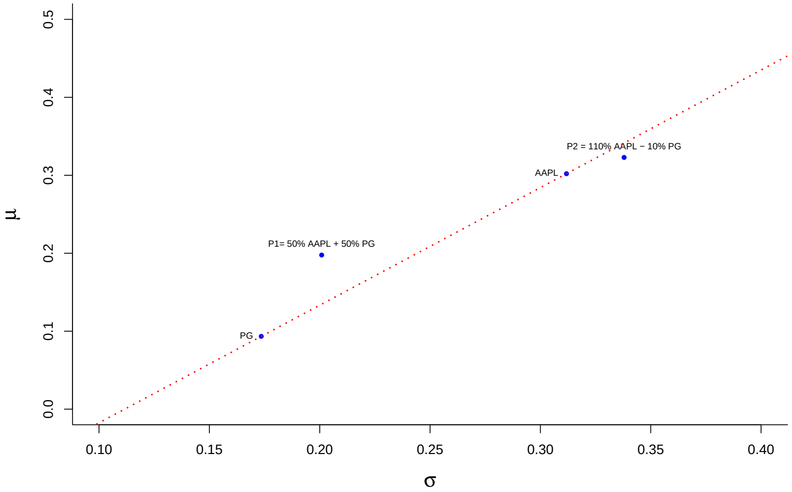
	AAPL	AMZN	MSFT	F	SPY	QQQ	XOM	MMM	HD	PG	KO
Observations	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000	3308.0000
NAs	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Minimum	-0.1792	-0.1278	-0.1171	-0.2500	-0.0984	-0.0896	-0.1395	-0.1295	-0.0822	-0.0790	-0.0867
Quartile 1	-0.0077	-0.0094	-0.0073	-0.0103	-0.0038	-0.0047	-0.0068	-0.0055	-0.0067	-0.0046	-0.0047
Median	0.0010	0.0008	0.0005	0.0000	0.0006	0.0010	0.0001	0.0008	0.0006	0.0004	0.0007
Arithmetic Mean	0.0012	0.0015	0.0008	0.0005	0.0004	0.0006	0.0001	0.0004	0.0008	0.0004	0.0005
Geometric Mean	0.0010	0.0012	0.0006	0.0001	0.0003	0.0005	0.0000	0.0003	0.0006	0.0003	0.0004
Quartile 3	0.0112	0.0123	0.0088	0.0106	0.0056	0.0070	0.0073	0.0070	0.0082	0.0055	0.0059
Maximum	0.1390	0.2695	0.1860	0.2952	0.1452	0.1216	0.1719	0.0988	0.1407	0.1021	0.1388
SE Mean	0.0003	0.0004	0.0003	0.0005	0.0002	0.0002	0.0003	0.0002	0.0003	0.0002	0.0002
LCL Mean (0.95)	0.0005	0.0006	0.0002	-0.0005	0.0000	0.0002	-0.0004	-0.0001	0.0002	0.0000	0.0001
UCL Mean (0.95)	0.0019	0.0023	0.0013	0.0014	0.0008	0.0011	0.0006	0.0009	0.0013	0.0007	0.0009
Variance	0.0004	0.0006	0.0003	0.0007	0.0001	0.0002	0.0002	0.0002	0.0003	0.0001	0.0001
Stdev	0.0196	0.0243	0.0170	0.0266	0.0121	0.0130	0.0150	0.0140	0.0162	0.0109	0.0113
Skewness	-0.2151	1.4889	0.4319	0.7627	0.1379	-0.0084	0.4199	-0.3815	0.5114	0.0555	0.5004
Kurtosis	6.2706	16.8872	10.2176	20.9458	15.2824	7.3976	15.4203	7.3856	6.4641	8.1017	14.3236

Un portefeuille avec 2 actifs

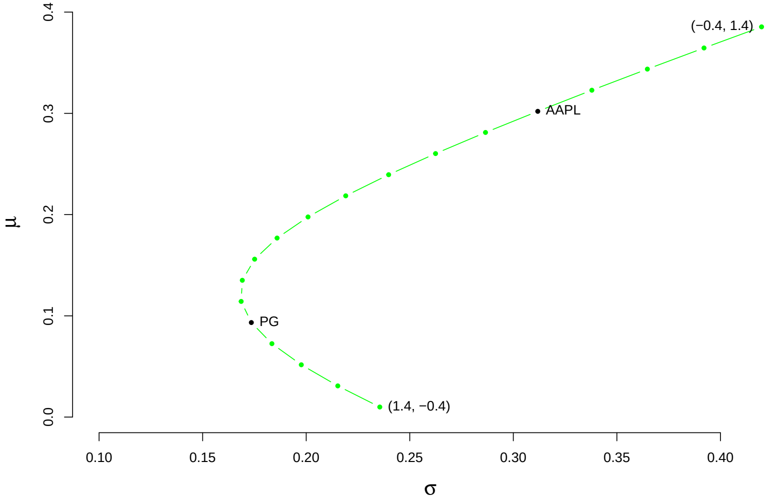
On choisit PG (faible vol) et AAPL (forte vol)

	PG	AAPL
mu	0.0933800	0.3019870
sigma	0.1735107	0.3118083
rho	0.3150221	NA

Rendement et Risque de deux actifs



Frontière de deux actifs



Calcul du Portefeuille de Variance Minimale (1)

$$
\begin{aligned}
\min \quad & w^T \Sigma w \\
\text{s.t.} \quad & \\
& \mathbf{1}^T w = 1
\end{aligned}
$$

Solution Analytique

```r
n <- 2
A.mat <- matrix(rep(1,n), ncol=1)
w.min <- solve(2*sig2, A.mat)
w.min <- w.min / sum(w.min)
names(w.min) <- assets
```

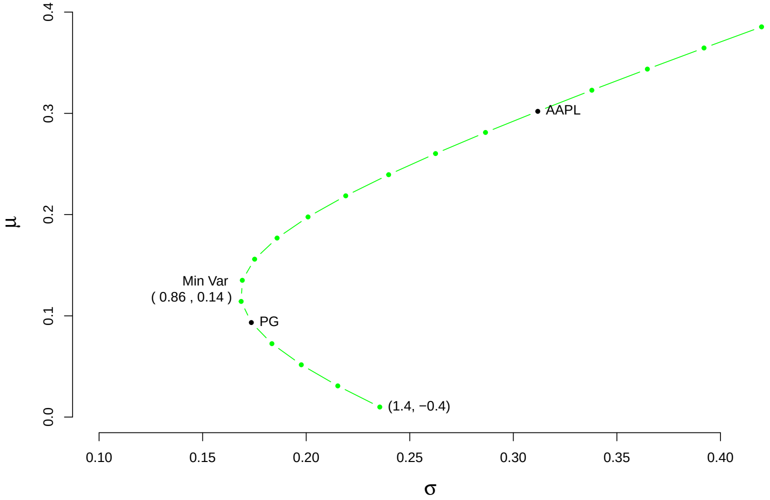
PG	0.8599089
AAPL	0.1400911

Utilisation de solve.QP - Solution Numérique

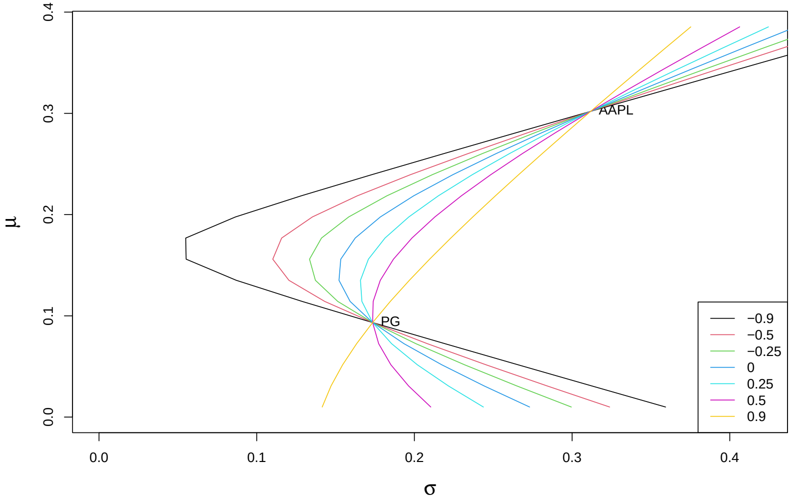
```
A.mat <- matrix(1, nrow=2, ncol=1)
b <- 1
sol <- solve.QP(2*sig2, rep(0,2),
                A.mat, b, meq=1)
w.min <- sol$solution
names(w.min) <- assets
```

	x
PG	0.8599089
AAPL	0.1400911

Portefeuille de Variance Minimale (2)



Frontière en fonction de la corrélation



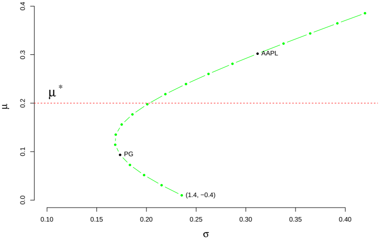
Modèle de Markowitz

$$\min \quad w^T \Sigma w$$

s.t.

$$\mu^T w = \mu^*$$

$$\mathbf{1}^T w = 1$$



Voir note pour solution analytique et théorème de séparation des fonds.

Utilisation dans l'industrie

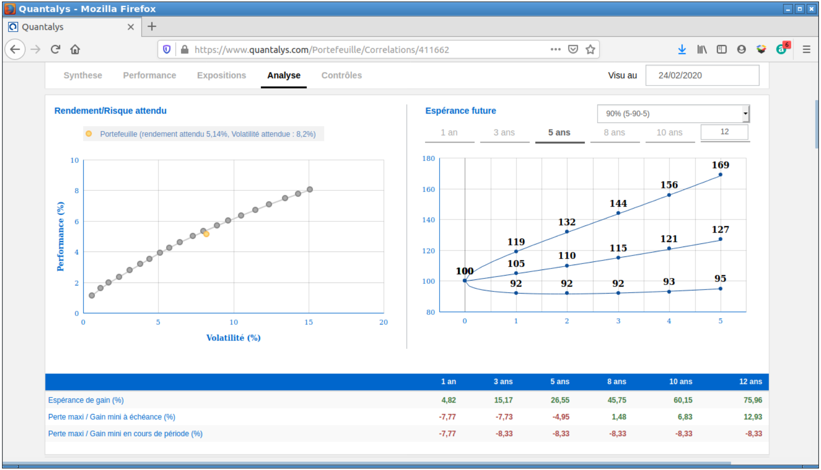
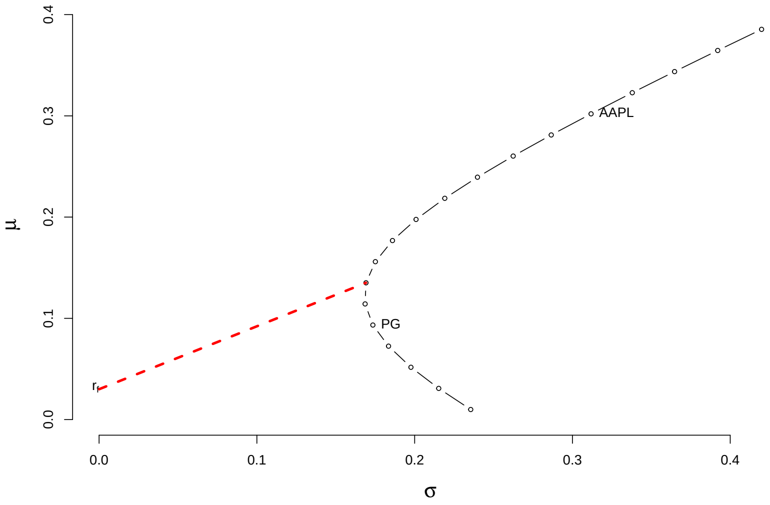


Figure 1: Diagramme Moyenne / Ecart Type (www.quantalys.com)

Ajout d'un actif sans risque (1)



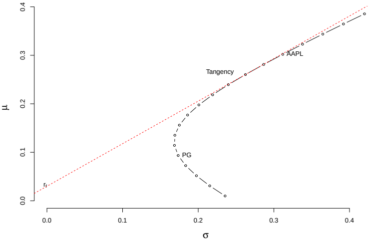
Portefeuille Tangent (1)

Portefeuille sur la frontière efficiente qui maximise le ratio de Sharpe:

$$\max_w \quad \frac{w^T \mu - r_f}{(w^T \Sigma w)^{\frac{1}{2}}}$$

s.t.

$$\mathbf{1}^T w = 1$$



Solution:

$$w = \frac{\Sigma^{-1}(\mu - r_f \mathbf{1})}{\mathbf{1}^T \Sigma^{-1}(\mu - r_f \mathbf{1})}$$

Optimisation Moyenne/Variance: Procédure de calcul

- Déterminer les espérances de rendement des actifs et la matrice de covariance
- Définir des contraintes du portefeuille (ex: encours max par ligne)
- Résoudre
- Faire varier r^* pour dessiner la frontière efficiente et le portefeuille tangent
- Choisir l'allocation en fonction de la tolérance au risque de l'investisseur

Optimisation M/V - avecR (1)

```
idx <- (time(daily.ret) >= dmy("01jan2014")) &
  (time(daily.ret) <= dmy("01jan2016"))
ret <- daily.ret[idx,]
# covariance matrix
Sigma <- cov(ret) * 252
# expected return
mu <- colMeans(ret) * 252
```

Portefeuille à Variance Minimale (1)

```
A.mat <- matrix(rep(1,length(mu)), ncol=1)
b <- 1
qp <- solve.QP(2*Sigma, mu*0, A.mat, b, meq=1)
```

Portefeuille à Variance Minimale (2)

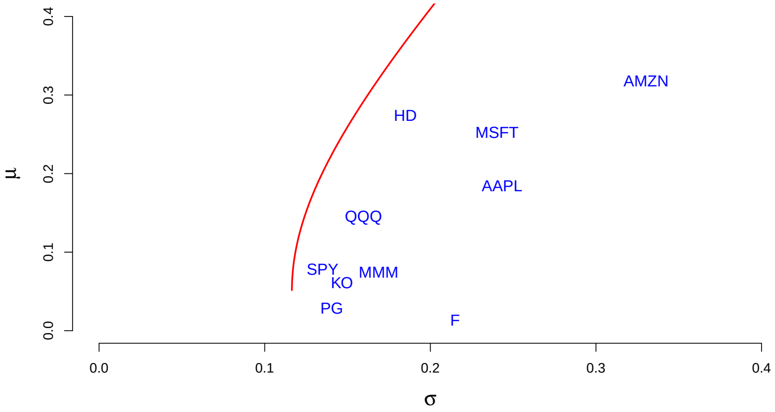
	allocation
AAPL	0.0204498
AMZN	0.0006842
MSFT	-0.0644735
F	-0.0195378
SPY	0.8323712
QQQ	-0.2902213
XOM	-0.0805829
MMM	0.0161995
HD	0.0301382
PG	0.3006411
KO	0.2543316

	Min Var Portfolio
return	0.0507792
stdev	0.1164107

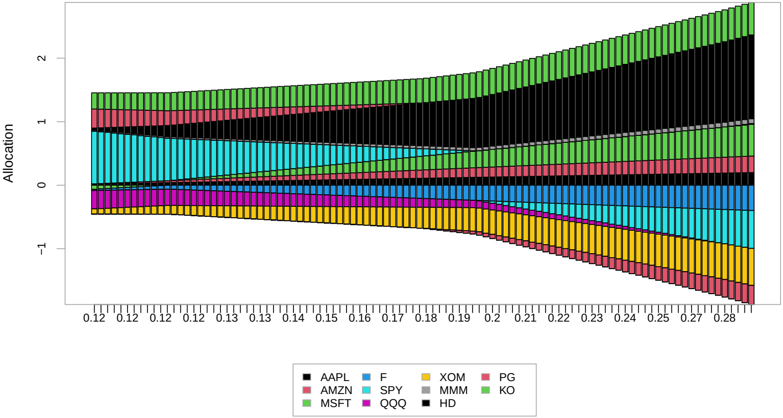
Frontière

$$
\begin{aligned}
\min \quad & w^T \Sigma w \\
\text{s.t.} \quad & \\
& \mu^T w = \mu^* \\
& \mathbf{1}^T w = 1
\end{aligned}
$$

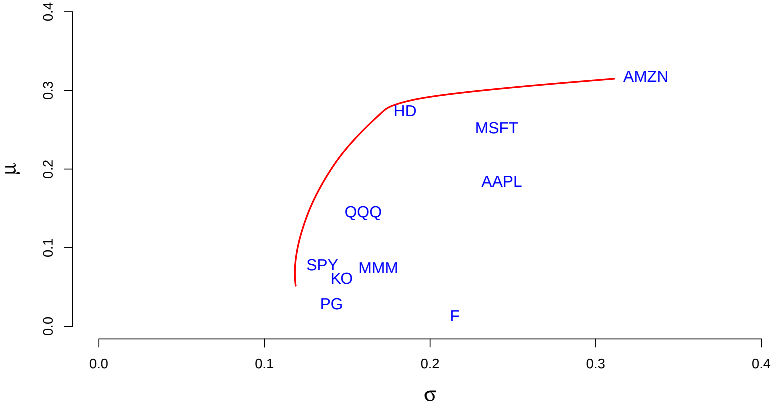
Calcul de la Frontière (Long/Short)



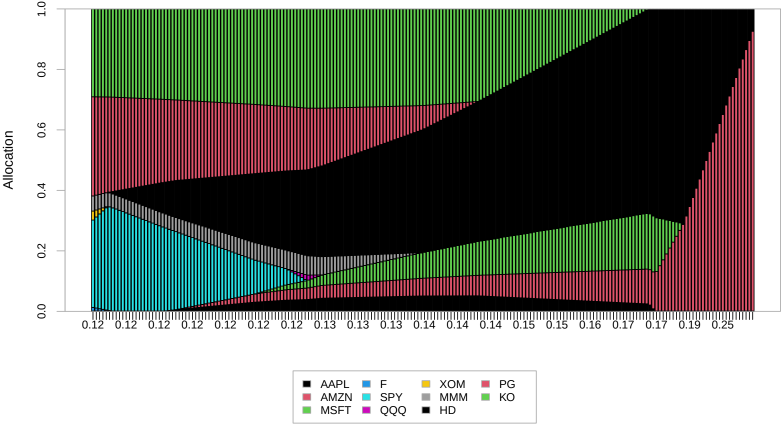
Allocation en fonction du risque



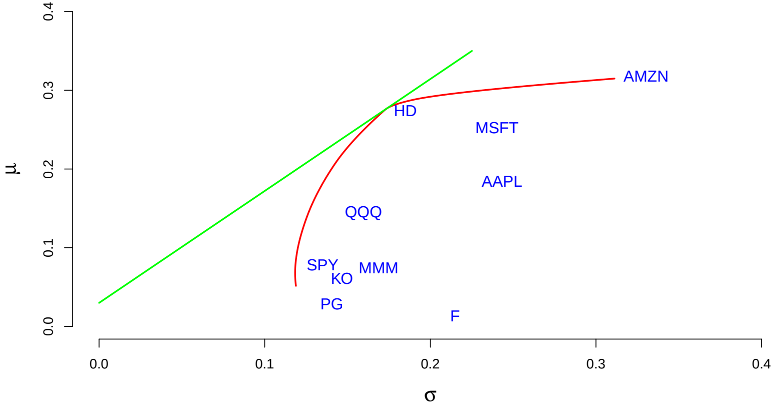
Calcul de la Frontière (Long)



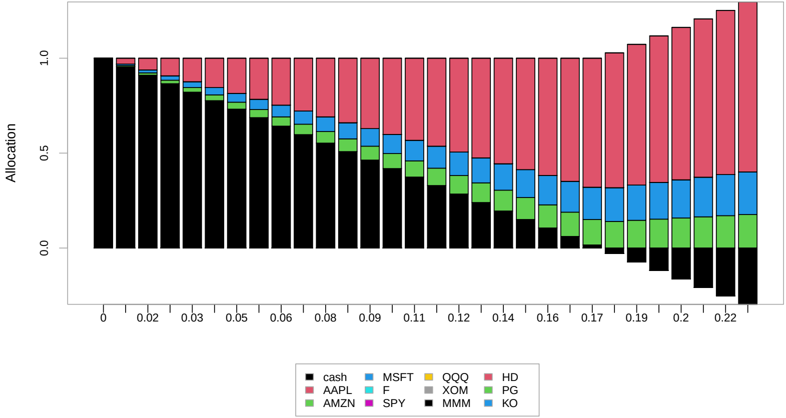
Allocation le long de la Frontière



Ajout d'un actif sans risque



Allocation le long de la droite de marché



Bibliographie

Grinhold, R.C. and Kahn, R. Active Portfolio Management, Mc Graw-Hill, 2000