Pricing and Hedging in the Black-Scholes Framework

Smile and Local Volatility

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The Implied Volatility Problem

- In the BS formula, we use σ as an input.
- In reality, exchanges quote options in price.
- The BS formula is used to convert an option price into the corresponding volatility.

Given the observed price C^* of a call, compute the volatility σ such that:

$$\begin{aligned} C^* &= C(S, K, T, r, \sigma) \\ &= f(\sigma) \end{aligned}$$

Option data: Settlement prices of options on the WTI Feb09 futures contract

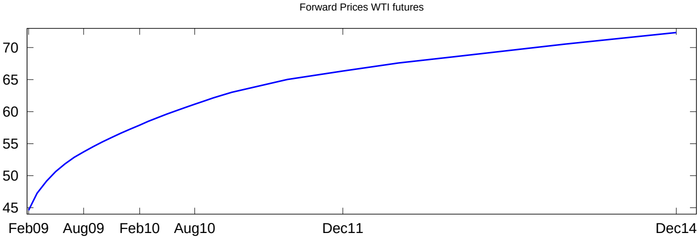
NEW YORK MERCANTILE EXCHANGE
NYMEX OPTIONS CONTRACT LISTING FOR 12/29/2008

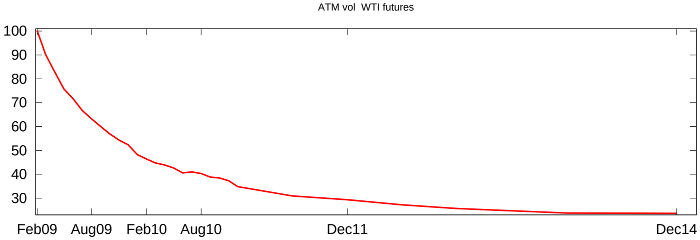
--------CONTRACT--------				TODAY'S SETTLE	PREVIOUS SETTLE	ESTIMATED VOLUME
LC	02 09	P	30.00	.53	.85	0
LC	02 09	P	35.00	1.58	2.28	0
LC	02 09	P	37.50	2.44	3.45	0
LC	02 09	C	40.00	3.65	2.61	10
LC	02 09	P	40.00	3.63	4.90	0
LC	02 09	P	42.00	4.78	6.23	0
LC	02 09	C	42.50	2.61	1.80	0
LC	02 09	C	43.00	2.43	1.66	0
LC	02 09	P	43.00	5.41	6.95	100

Option data: Options on the S&P 500 Index (Source:CBOE)

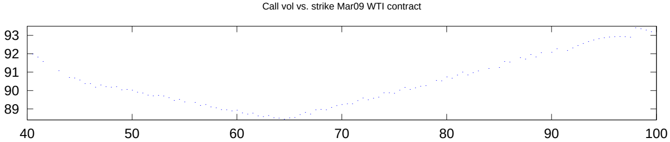
SPX (SP 500 INDEX)	1290.59	+7.24					
Jan 24 2011 @ 14:03 ET							
Calls	Last Sale	Net	Bid	Ask	Vol	Open Int	...
11 Jan 1075.00 (SPXW1128A1075-E)	0.0	0.0	215.30	217.00	0	0	...
11 Jan 1100.00 (SPXW1128A1100-E)	0.0	0.0	190.60	191.80	0	0	...

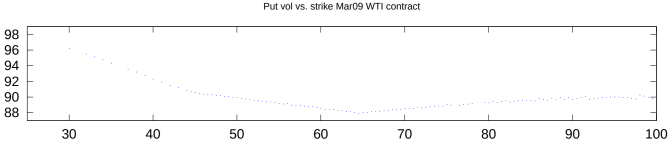
ATM Volatility



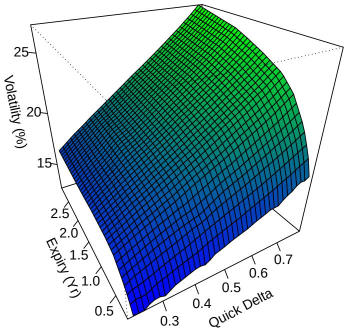


WTI Option Implied Vol





Implied Volatility of S&P 500 Index Options (24-jan-2011)



The Implied Volatility

Option traders study the implied volatility rather than the option prices.
Why?

- A measure of value, irrespective of strike and maturity
- The BS model assumes a constant volatility: deviations from that assumption are useful to study

The Implied Volatility Problem

Given the observed price C^* of a call, compute the volatility σ such that:

$$\begin{aligned} C^* &= C(S, K, T, r, \sigma) \\ &= C(\sigma) \end{aligned}$$

The Implied Volatility Problem

There is a change in convexity in $C(\sigma)$:

$$\frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{T}$$

$$\frac{\partial^2 C}{\partial \sigma^2} = S\sqrt{T}n(d_1)\frac{1}{\sigma}\left[\frac{1}{\sigma^2 T}\ln(\frac{F}{K})^2 - \frac{1}{4}\sigma^2 T\right]$$

with $F = Se^{rT}$.
Thus, $C(\sigma)$ is convex on the interval $(0, \sqrt{\frac{2|\ln(F/K)|}{T}}]$, and concave otherwise.

Convergence of Newton's Method

To ensure convergence of Newton's method, one must carefully choose the initial point.

Theorem

Let f be defined on the interval $[a, b]$ and assume that:

1. $f(x^*) = 0$ *for some* $x^* \in [a, b]$
2. $f'(x) > 0$
3. $f''(x) \geq 0$

Then Newton's method converges monotonically from $x_0 = b$. *If*

1. $f(x^*) = 0$ *for some* $x^* \in [a, b]$
2. $f'(x) > 0$
3. $f''(x) \leq 0$

Then Newton's method converges monotonically from $x_0 = a$.

Convergence of Newton's Method

Consider solving $f(\sigma) = C(\sigma) - C^* = 0$. Start Newton's method at

$$\sigma_0 = \sqrt{\frac{2|\ln(F/K)|}{T}}$$

- If $f(\sigma_0) > 0$, we are in case I of theorem.
- If $f(\sigma_0) < 0$ we are in case II.

Implied Volatility by Newton's Method

The following algorithm generates a monotonic series (σ_n):

1. Set $\sigma_0 = \sqrt{\frac{2|\ln(F/K)|}{T}}$
2. While $|C(\sigma_n) - C^*| > \epsilon$:
   2.1 Let
   $$\sigma_{n+1} = \sigma_n + \frac{C^* - C(\sigma_n)}{\frac{\partial C}{\partial \sigma}}$$
   2.2 $n \leftarrow n + 1$

The Breeden-Litzenberger Formula

Probability Distribution Implied by Option prices

The Breeden-Litzenberger formula

Risk-neutral density of the underlying asset at maturity T as a function of derivative prices:

$$p_T(K) = e^{rT} \frac{\partial^2 C(S, K, T)}{\partial K^2} \tag{1}$$

where $C(S, K, T)$ is the price of a call of strike K, maturity T, when the current spot is S.

The Breeden-Litzenberger formula

By definition of the risk-neutral probability,

$$C(S,K,T) = e^{-rT}\int_K^{\infty}(S_T - K)p(S_T)dS_T \tag{2}$$

Applying Leibniz's Rule to get:

$$\frac{\partial C(S,K,T)}{\partial K} = -e^{-rT}\int_K^{\infty} p(S_T)dS_T$$

Let $F(K)$ the cumulative density function of S_T,

$$\begin{aligned} e^{rT}\frac{\partial C(S,K,T)}{\partial K} &= -\int_K^{\infty} p(S_T)dS_T \\ &= F(K) - 1 \end{aligned}$$

The Breeden-Litzenberger formula

Differentiate again with respect to K to get:

$$\frac{\partial^2 C}{\partial K^2} e^{rT} = p(K) \tag{3}$$

Local Vol: Interpretation

Consider a butterfly spread centered at K, and scaled to yield a maximum payoff of 1. Let $\phi(S_T)$ be the payoff function. The value of the butterfly is:

$$\begin{aligned} V &= \frac{1}{\Delta K}\left[C(K+\Delta K) - 2C(K) + C(K-\Delta K)\right] \\ &= e^{-rT}\int_0^\infty \phi(S)p(S)dS \end{aligned}$$

Local Vol: Interpretation

In the interval $[K - \Delta K, K + \Delta K]$, approximate $p(S)$ by the constant $p(K)$ to get:

$$\begin{aligned} V &= e^{-rT} p(K) \int_0^\infty \phi(S) dS \\ &= e^{-rT} p(K) \Delta K \end{aligned}$$

Finally, use the definition of the derivative:

$$\lim_{\Delta K \to 0} \frac{1}{\Delta K^2} \left[C(K + \Delta K) - 2C(K) + C(K - \Delta K) \right] = \frac{\partial^2 C(K)}{\partial K^2} \tag{4}$$

to get:

$$p_T(K) = e^{rT} \frac{\partial^2 C(K)}{\partial K^2} \tag{5}$$

Analytical expression for the density of S_T

$$\begin{aligned} p(K) = & n(d_2)\Big\{ \\ & \frac{1}{K\sigma\sqrt{T}} + \frac{\partial\sigma}{\partial K}\frac{2d_1}{\sigma} + \\ & \left(\frac{\partial\sigma}{\partial K}\right)^2 \frac{\sqrt{T}Kd_1d_2}{\sigma} + \\ & \frac{\partial^2\sigma}{\partial K^2}K\sqrt{T}\Big\} \end{aligned} \tag{6}$$

Illustration: Shimko's Model

Fit a quadratic model to the implied volatility, in order to get analytical expressions for $\frac{\partial \sigma}{\partial K}$, $\frac{\partial^2 \sigma}{\partial K^2}$:

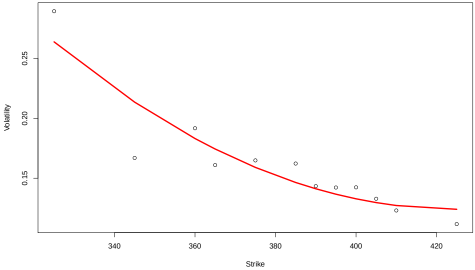


Figure: Quadratic Volatility Model

Illustration: Shimko's Model

The density implied from the quadratic volatility smile cleary exhibit "fat tails".

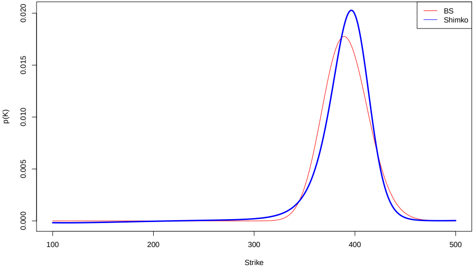


Figure: Density of S_T, with constant volatility and quadratic model for implied volatility.

Consequence for Option Pricing

To illustrate the importance of correctly accounting for the volatility smile, we now consider a digital option maturing at the same time as our European options. We want to price this option in a way that is consistent with the observed volatility smile.

A naive approach would be to look up the Black-Scholes volatility corresponding to the strike, and price the digital option accordingly. The price of a digital cash-or-nothing call is given by:

$$C = e^{-rT}\Phi(d_2) \tag{7}$$

with:

$$\begin{aligned} \Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} \\ d_2 &= \frac{\ln\frac{S}{K} + (r - d - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \end{aligned}$$

Consequence for Option Pricing

However, since we know the density of S_T, we can directly compute the expected discounted value of the digital payoff:

$$C = e^{-rT} \int_K^{\infty} p(S_T) dS_T$$

Digital Price with and without Smile

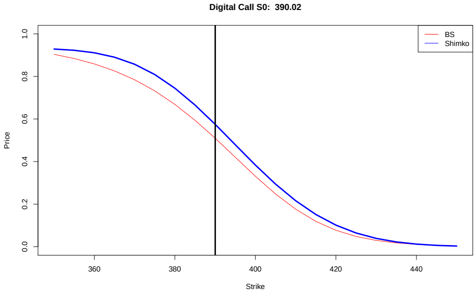


Figure: Comparison of prices of a digital option, using a log-normal density for S_T, and using the density implied by the volatility smile fitted to a quadratic function.

WTI Smile

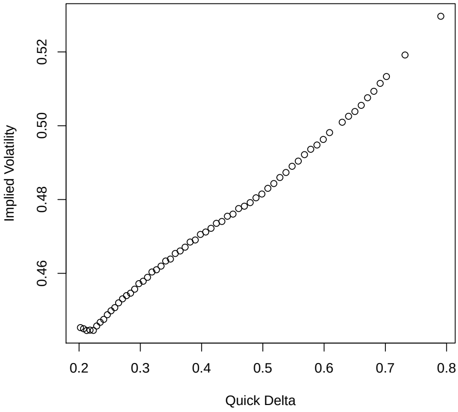


Figure: Implied volatility of WTI NYMEX options on the December 2009 Futures, observed on April 21, 2009

WTI Smile

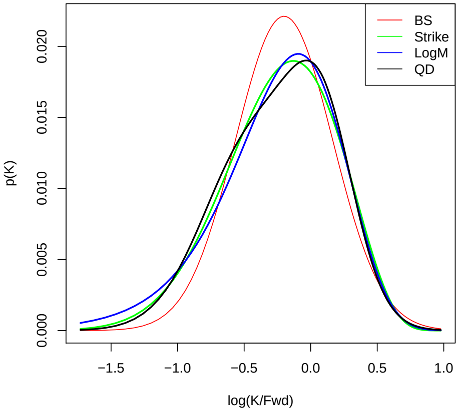


Figure: Implied density of F_T, the December 2009 WTI Nymex Futures contract. Calculation is performed by finite difference, with implied

References