Le Modèle de Trenor-Black

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In this short note, we summarize the mathematical elements of the classical portfolio theory of Trenor-Black (Treynor & Black, 1973).

Assets excess return is modeled by a single factor model:

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where α_i is the idiosyncratic excess return of asset i, and $e_i \sim N(0, \sigma_i^2)$ is the specific risk.

Recall two results from Markowitz and Sharpe:

- The expression for the tangency portfolio:

$$w^* = \frac{\Sigma^{-1}\tilde{R}}{\mathbf{1}^T\Sigma^{-1}\tilde{R}} \tag{1}$$

- Given two assets, A and M, the allocation that maximizes the Sharpe ratio is given by:

$$w_A = \frac{R_A\sigma_M^2 - R_M\sigma_A\sigma_M\rho_{AM}}{R_A\sigma_M^2 + R_M\sigma_A^2 - (R_A + R_M)\sigma_A\sigma_M\rho_{AM}} \tag{2}$$

Calculation of the active portfolio

The active portfolio is determined by the idiosyncratic excess return and the specific risk of each asset.

The specific risks are assumed to be independent:

$$\Sigma_A = \begin{bmatrix} \sigma^2(e_1) & & \\ & \ddots & \\ & & \sigma^2(e_n) \end{bmatrix}$$

Using equation (1), we get:

$$w_{Ai} = \frac{\alpha_i/\sigma_i^2}{\sum \alpha_i/\sigma_i^2}$$

So that the active portfolio has an excess return and variance given by:

$$\begin{aligned} R_A &= \alpha_A + \beta_A R_M \\ \sigma_A^2 &= \beta_A^2 \sigma_M^2 + \sigma^2(e_A) \end{aligned}$$

with

$$\begin{aligned} \alpha_A &= \sum w_{Ai}\alpha_i \\ \beta_A &= \sum w_{Ai}\beta_i \\ \sigma^2(e_A) &= \sum w_{Ai}^2 \sigma^2(e_i) \end{aligned}$$

Allocation of wealth between the active portfolio and the market portfolio

A fraction w_A of wealth is allocated to the active portfolio, and the balance $(1 - w_A)$ to the market portfolio so as to maximize the Sharpe ratio of the global portfolio $xA + (1 - x)M$.

Using equation (2) we get after some algebra:

$$w_A = \frac{\alpha_A \sigma_M^2}{\alpha_A \sigma_M^2 (1 - \beta_A) + R_M \sigma^2(e_A)}$$

Separability of the Sharpe ratio in the active portfolio

The first order condition for the optimal active portfolio is:

$$w_A = \lambda_A \Sigma^{-1} \alpha \tag{3}$$

Substitute in the expression

$$\alpha_A = w_A^T \alpha$$

to get:

$$\frac{\alpha_A}{\lambda_A} = \alpha^T \Sigma^{-1} \alpha \tag{4}$$

We next get an expression for λ_A in terms of known quantities:

$$\begin{aligned} \sigma^2(e_A) &= w_A^T \Sigma w_A \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \Sigma \Sigma^{-1} \alpha \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \alpha \end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{\sigma^2(e_A)}{\lambda_A^2} &= \alpha^T \Sigma^{-1} \alpha \\
&= \frac{\alpha_A}{\lambda_A}
\end{aligned}
$$

Which yields:

$$
\lambda_A = \frac{\sigma^2(e_A)}{\alpha_A}
$$

Use this result in equation (4) to get:

$$
\begin{aligned}
\frac{\alpha_A^2}{\sigma^2(e_A)} &= \alpha^T \Sigma^{-1} \alpha \\
&= \sum_i \frac{\alpha_i^2}{\sigma^2(e_i)}
\end{aligned}
$$

which shows that the square of the Sharpe ratio of the active portfolio is the sum of the squares of the Sharpe ratios of the assets forming that portfolio.

The Treynor-Black model in the notation of the 1973 paper and separability of the Sharpe ratio between the active and market portfolios

The investment universe is composed of n assets with asset-specific excess return:

$$
\begin{aligned}
r_i &= \alpha_i + \beta_i r_M + e_i \;\; i = 1, \ldots, n && (5) \\
E(r_i) &= \alpha_i + \beta_i E(r_M) = \mu_i && (6)
\end{aligned}
$$

and of the market asset itself. Let $w_i, i = 1, \ldots, n$ be the investment in the assets with asset-specific excess returns, and w_M the investment in the market asset.

Treynor and Black restate this portfolio as an investment in $n+1$ assets, asset 1 to n being only exposed to the specific risk, and the $n+1$ asset being only exposed to the market risk:

$$
w_{n+1} = w_M + \sum_{i=1}^{n} \beta_i w_i
$$

Note that these $n+1$ assets are independent. The mean and variance of the portfolio are:

$$E(r_P) = \sum_{i=1}^{n+1} w_i E(r_i) = \mu_P \tag{7}$$

$$\sigma_P^2 = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 \tag{8}$$

As usual, maximize the Sharpe ratio by solving:

$$\begin{gathered} \min \ \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \\ \mu^T w = \mu_P \end{gathered}$$

Keeping in mind that the assets are independent, the Lagrangian is:

$$L(w, \lambda) = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 - 2\lambda \left(\sum_{i=1}^{n+1} w_i \mu_i - \mu_P \right)$$

First order conditions for optimality yield:

$$2 w_i \sigma_i^2 - 2\lambda \mu_i = 0 \quad i = 1, \ldots, n+1$$

or,

$$w_i = \lambda \frac{\mu_i}{\sigma_i^2} \tag{9}$$

Substitute in (6) to get:

$$\mu_P = \lambda \sum_{i=1}^{n+1} \mu_i^2 / \sigma_i^2 \tag{10}$$

$$\sigma_P^2 = \lambda^2 \sum_{i=1}^{n+1} \mu_i^2 \sigma_i^2 \tag{11}$$

so that,

$$\lambda = \frac{\sigma_P^2}{\mu_P}$$

To summarize, the weights of the assets in the active portfolio are:

$$w_i = \frac{\mu_i}{\mu_P} \frac{\sigma_P^2}{\sigma_i^2} \quad i = 1, \ldots, n$$

To determine the investment in the market asset, w_M, recall that,

$$\mu_{n+1} = E(r_M) = \mu_M \tag{12}$$
$$\sigma^2_{n+1} = \sigma^2_M \tag{13}$$

Thus,

$$w_{n+1} = \sum_{i=1}^{n} w_i \beta_i + w_M \tag{14}$$
$$= \lambda \frac{\mu_M}{\sigma^2_M} \tag{15}$$

From equation (9, we have:

$$\sum_{i=1}^{n} w_i \beta_i = \lambda \sum_{i=1}^{n} \frac{\beta_i \mu_i}{\sigma^2_i}$$

So that the investment in the market asset can be written as

$$w_M = \lambda \left[\frac{\mu_M}{\sigma^2_M} - \sum_{i=1}^{n} \frac{\beta_i \mu_i}{\sigma^2_i} \right]$$

To establish the separability of the Sharpe ratio between the active and the market portfolios, combine equations (10) and (11) to get:

$$\frac{\mu^2_P}{\sigma^2_P} = \sum_{i=1}^{n+1} \frac{\mu^2_i}{\sigma^2_i}$$

Denoting S_A, S_M, S_P the Sharpe ratios of, respectively, the active, market and overall portfolios, we can restate the previous equation as:

$$S^2_P = \sum_{i=1}^{n} \frac{\mu^2_i}{\sigma^2_i} + S^2_M \tag{16}$$
$$= \frac{\alpha^2_A}{\sigma^2_A + S^2_M} \tag{17}$$
$$S^2_A + S^2_M \tag{18}$$

Treynor and Black call $\alpha_A = \sum_{i=1}^{n} w_i \alpha_i$ the "appraisal premium" and $\sigma^2_A = \sum_{i=1}^{n} w^2_i \sigma^2_i$ the "appraisal risk".

Bibliography

Treynor, J. L., & Black, F. (1973). How to Use Security Analysis to Improve Portfolio Selection. *The Journal of Business*, *46*(1), 66–86. http://www.jstor.org/stable/2351280