Monte-Carlo

Pricing by Simulation

P. Hénaff

Version: 10 Mar 2025

Use of Monte-Carlo Simulations

Simulations are used for:

- Risk measurement:
  - Scenario analysis, VaR
  - Stress testing (Basel II, Solvency II)
- Derivatives pricing
  - Easy to implement (just code the payoff formula)
  - Flexible: it dissociates the dynamics of the risk factors from the evaluation of the option.

Monte-Carlo Simulations: Historical vs. Risk-Neutral

- Simulations for scenario analysis and VaR are based on actual probabilities (historical VaR)
- Simulations for pricing are based on risk-neutral probabilities.
- Economic scenarii generation for Solvency II calculations use hybrid methods.
  - Market consistent
  - Use historical data when implied volatility from option market is not available

Simulating a Log-Normal Process

Given a stock with expected continuous compounded return μ:

$$\ln(S_t) - \ln(S_0) \approx \mathcal{N}\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right)$$

or

$$S_t = S_0 e^{x_t}$$

with $x_t \approx \mathcal{N}\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right)$.

$$x_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\epsilon$$

with $\epsilon \approx \mathcal{N}(0, 1)$.

How to simulate a log-normal process

To simulate a time series of stock prices following a log-normal process, observed at intervals Δt:

1. Start with S_0 at $t = 0$
2. for $t = \Delta t, 2\Delta t, \ldots, T$:
   2.1 Simulate $\epsilon \approx \mathcal{N}(0, 1) : \epsilon_t$
   2.2 Compute
   $$S_t = S_{t-\Delta t} e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_t}$$

Simulation of a Lognormal Process: Example

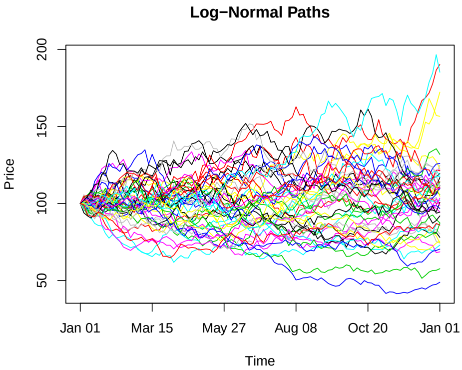
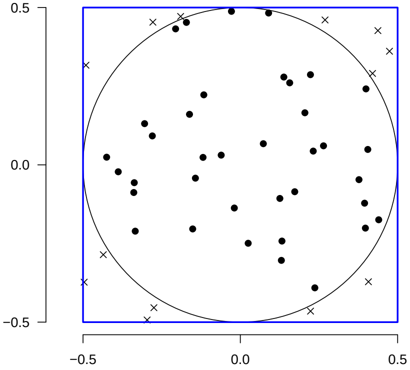


Figure: Risk-neutral log normal process,
$S_0 = 100$, $T = 1, r = 10\%, \sigma = 30\%$

Monte Carlo Simulation: Introductory Example

Estimating π by randomly throwing darts at the square.



Area of square: 1, area of circle: π

Computing circle area by simulation

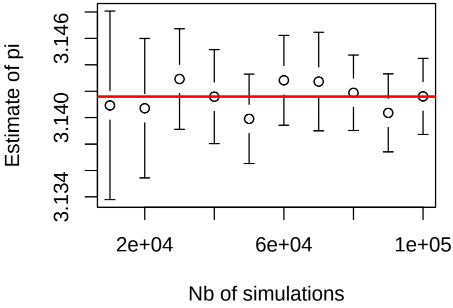
1. Simulate N throws of dart
2. Count the number of darts that land in the circle, let M be that number
3. Estimate for π is:

$$\pi = 4\frac{M}{N}$$

Estimate of π vs number of simulations

	mean	sd
10000	3.1424	0.0153
20000	3.1419	0.0122
30000	3.1389	0.0093
40000	3.1419	0.0080
50000	3.1427	0.0078
60000	3.1435	0.0068
70000	3.1415	0.0071
80000	3.1425	0.0053
90000	3.1413	0.0062
1e+05	3.1411	0.0049

Estimate of π vs. number of simulations



Estimation error

Accuracy, measured by standard deviation, improves at the rate $\sqrt{N}$.
To reduce error by a factor of 10, one needs the increase the number of simulations by a factor of 100.
This slow convergence rate has motivated research on how to improve the accuracy of the simulation by other means than increasing the number of simulations.

Basic MC pricing

Pricing an European Call option by simulation:

1. Simulate N normal random variables $\epsilon_i, i = 1, \ldots N$
2. Compute $S_T^i, i = 1, \ldots, N$
3. Evaluate the payoff:

$$V_i = \max(S_T^i - K, 0)$$

4. Compute price

$$P = e^{-rT} \frac{1}{N} \sum_{i=i}^{N} V_i$$

MC estimate of price vs. number of simulations

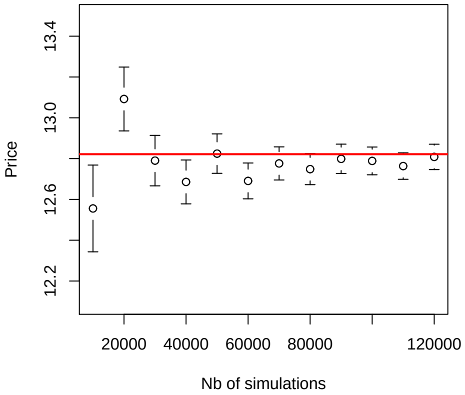
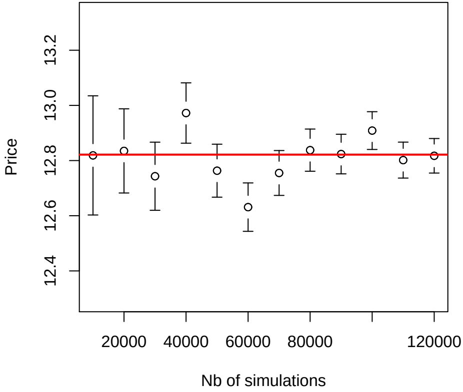


Figure: MCpricing of a call option, $S_0 = K = 100, \sigma = 30\%$

Methods for improving the convergence of Monte-Carlo simulations

- Antithetic variables: draw random numbers symetrically around the mean to ensure that the sample mean is accurate.
- Use random number generators (Sobol) that generate well-distributed samples, regardless of sample size.
- Use Control Variate methods

MC with antithetic variables. Price vs. number of simulations



Comparison of Sobol sequence and default uniform random variates

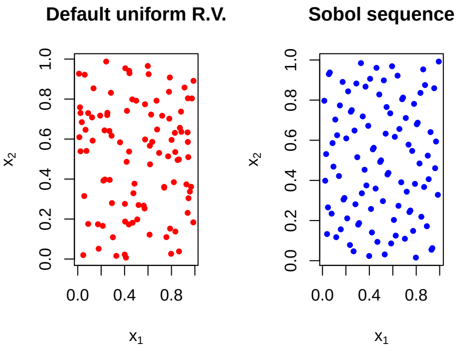
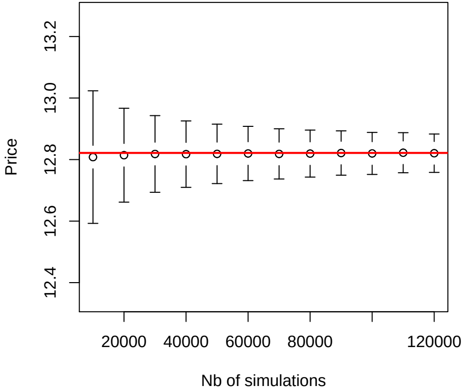


Figure: The points generated by the Sobol sequence are more evenly distributed in the rectangle

MC with Sobol sequences. Price vs. number of simulations



Summary

- Variance of MC estimator is a function of $\sqrt{\frac{1}{N}}$.
- The quality of the random number generator has a major impact on the accuracy of the MC estimation.

MC with Control Variate (Version 1)

A and B are very similar derivatives, function of the same risk factors X. We know the exact value of B: f_B.

1. Compute the values of $f_B^* = E(f_B(X))$ and $f_A^* = E(f_A(X))$ by MC simulation.
2. The quantity $e = f_B - f_B^*$ is a measure of the MC simulation bias
3. Use this measure to correct the MC price of A:

$$f_A = f_A^* + (f_B - f_B^*)$$

MC with Control Variate (Version 2)

introduce a factor α:

$$f_A = f_A^* + \alpha(f_B - f_B^*)$$

Choose α to minimize the variance of f_A.

$$V(f_A) = V(f_A^*) + \alpha^2 V(f_B^*) + 2\alpha Cov(f_A^*, f_B^*)$$

Or,

$$\alpha = \frac{Cov(f_A^*, f_B^*)}{V(f_B^*)}$$

α is the regression coefficient of f_A^* on f_B^*.