Le Modèle de Markowitz

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In this short note, we summarize the mathematical elements of the classical portfolio theory of Markowitz. Today, the benefits of diversification seem unquestionable. But it has not always been that way: in 1934, the famous economist and investor J. M. Keynes wrote this to the chairman of an insurance company:

> "As time goes on I get more and more convinced that the right method in investment is to put fairly large sums into enterprises which one thinks one knows something about and in the management of which one thoroughly believes."

We too will keep this advice in mind as we review Markowitz's contribution.

Arithmetic vs. Geometric mean

Let r_A and r_G be, respectively, the arithmetic and geometric means of a series of returns:

$$
\begin{aligned}
r_A &= \frac{1}{n}\sum_{k=1}^{n} r_k \\
r_G &= \prod_{k=1}^{n}(1+r_k)^{1/n} - 1
\end{aligned}
$$

and let V be the variance of r_k. We show that the geometric mean, which correctly represents the increase in wealth from an investment, is lower than the arithmetic mean.

The MacLaurin series for $(1+x)^{1/n}$ is:

$$
(1+x)^{\frac{1}{n}} = 1 + \frac{1}{n}x + \frac{1-n}{n^2}\frac{x^2}{2} + o(x^2)
$$

$$
r_G \approx \prod_{k=1}^{n}\left(1 + \frac{1}{n}r_k + \frac{1-n}{n^2}\frac{r_k^2}{2}\right) - 1
$$

Developping the product and keeping terms of order 2,

$$r_G \approx \frac{1}{n}\sum_k r_k + \frac{1}{n^2}\sum_{k\neq l} r_k r_l + \frac{1-n}{2n^2} r_k^2$$

$$\begin{aligned}
r_G &\approx r_A - \frac{1}{2}\left[\frac{1}{n}\sum_k r_k^2 - \frac{1}{n^2}\left(\sum_k r_k^2 + 2\sum_{k\neq l} r_k r_l\right)\right] && (1)\\
&\approx r_A - \frac{1}{2}\left[\frac{1}{n}\sum_k r_k^2 - \left(\frac{1}{n}\sum_k r_k\right)^2\right] && (2)\\
&\approx r_A - \frac{1}{2}V,\ \ V >= 0 && (3)
\end{aligned}$$

Quadratic Programming

QP with equality constraints

$$\begin{aligned}
&\min\ \frac{1}{2} w^T \Sigma w\\
&\text{s.t.}\\
&\quad A^T w = b
\end{aligned}$$

Lagrangian:

$$L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda^T \left(A^T w - b\right)$$

First order conditions:

$$\begin{cases} \Sigma w - A\lambda = 0 \\ A^T w = b \end{cases}$$

or,

$$\begin{bmatrix} \Sigma & -A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

Special case of Minimum Variance problem

$$A = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad b = 1$$

Solution:

$$w = \lambda \Sigma^{-1} A$$

Normalize so that weights sum to 1:

$$w = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T \Sigma^{-1}\mathbf{1}}$$

Mean-Variance model (Markowitz, 1952)

$$\begin{aligned} &\min \ \frac{1}{2} w^T \Sigma w \\ &\text{s.t.} \\ &\mathbf{1}^T w = 1 \\ &R^T w = R_P \end{aligned}$$

Lagrangian:

$$L(w, \lambda_1, \lambda_2) = \frac{1}{2} w^T \Sigma w - \lambda_1 (\mathbf{1}^T w - 1) - \lambda_2 (R^T w - R_P)$$

Solution of first order conditions:

$$\begin{cases} \Sigma w - \lambda_1 \mathbf{1} - \lambda_2 R = 0 \\ \mathbf{1}^T w = 1 \\ R^T w = R_P \end{cases} \tag{4}$$

Determination of λ_1 and λ_2:

$$w = \Sigma^{-1}(\lambda_1 \mathbf{1} + \lambda_2 R)$$

Define:

$$\begin{aligned} a &= \mathbf{1}^T \Sigma^{-1} \mathbf{1} \\ b &= \mathbf{1}^T \Sigma^{-1} R \\ c &= R^T \Sigma^{-1} R \end{aligned}$$

Substitute in (4):

$$\begin{cases} \lambda_1 a + \lambda_2 b = 1 \\ \lambda_1 b + \lambda_2 c = R_P \end{cases}$$

Solution:

$$\lambda_1 = \frac{c - bR_P}{\Delta}$$
$$\lambda_2 = \frac{aR_P - b}{\Delta}$$
$$\Delta = ac - b^2$$

Note that:

$$\begin{aligned}
\sigma_P^2 &= w^{*T}\Sigma w^* \\
&= w^{*T}\Sigma\left(\lambda_1\Sigma^{-1}\mathbf{1} + \lambda_2\Sigma^{-1}R\right) \\
&= \lambda_1 + \lambda_2 R_P
\end{aligned}$$

Two remarkable solutions:

- Minimum variance portfolio

$$\begin{aligned}
\frac{\partial\sigma_P^2}{\partial R_P} &= 0 \implies \\
\frac{2aR_P - 2b}{\Delta} &= 0 \implies \\
R_P &= \frac{b}{a} \\
\sigma_P^2 &= \frac{1}{a} \\
\lambda_1 &= \frac{1}{a} \\
\lambda_2 &= 0
\end{aligned}$$

The weights of the minimum variance portfolio:

$$\begin{aligned}
w_g &= \lambda_1\Sigma^{-1}\mathbf{1} \\
&= \frac{\mathbf{\Sigma^{-1}1}}{\mathbf{1^T\Sigma^{-1}1}}
\end{aligned}$$

- $\lambda_1 = 0$

This second solution gives $\lambda_2 = \frac{1}{b}$ and the optimal weights:

$$
\begin{aligned}
w_d &= \lambda_2 \Sigma^{-1} R \\
&= \frac{\Sigma^{-1} R}{\mathbf{1}^T \Sigma^{-1} R}
\end{aligned}
$$

Theorem 1. *Any MV optimal portfolio w_P^* with expected excess return R_P can be decomposed into two MV portfolios.*

$$
w_P^* = A w_g + (1 - A) w_d
$$

Proof. Since w_P is MV optimal,

$$
\begin{aligned}
w_P &= \lambda_1 \Sigma^{-1} \mathbf{1} + \lambda_2 \Sigma^{-1} R \\
&= \lambda_1 a w_g + \lambda_2 b w_d
\end{aligned}
$$

One can verify that

$$
\lambda_1 a + \lambda_2 b = 1
$$

$\square$

MV model with riskless asset

The tangency portfolio corresponds to the point on the efficient frontier where the slope of the tangent $\frac{R_M - r_f}{\sigma_M}$ is maximized, where:

$$
\frac{R_M - r_f}{\sigma_M} = \frac{w^T (R - R_f)}{\sqrt{w^T \Sigma w}}
$$

Noting that the slope is unchanged when the weights w are multiplied by a constant, the tangency portfolio is found by solving the following QP problem for an arbitrary $R^* > R_f$:

$$
\begin{gathered}
\min \ \frac{1}{2} w^T \Sigma w \\
\text{s.t.} \\
\tilde{R}^T w = R^*
\end{gathered}
$$

with $\tilde{R} = R - R_f$.

Lagrangian:

$$
L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda \left(\tilde{R}^T w - R^* \right)
$$

Which yields:

$$w^* = \lambda^* \Sigma^{-1} \tilde{R} \tag{5}$$

Normalize so that the weights sum to 1:

$$w^* = \frac{\Sigma^{-1} \tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}} \tag{6}$$

The corresponding expected excess return is given by:

$$E(R_P^*) = \frac{\tilde{R}^T \Sigma^{-1} \tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}}$$

Maximum Sharpe ratio for two risky assets

Given two assets, A and M, the allocation that maximizes the Sharpe ratio is given by:

$$w_A = \frac{R_A \sigma_M^2 - R_M \sigma_A \sigma_M \rho_{AM}}{R_A \sigma_M^2 + R_M \sigma_A^2 - (R_A + R_M) \sigma_A \sigma_M \rho_{AM}} \tag{7}$$

Proof. Use equation (6) with

$$\Sigma = \begin{bmatrix} \sigma_A^2 & \rho \sigma_A \sigma_M \\ \rho \sigma_A \sigma_M & \sigma_M^2 \end{bmatrix} \tag{8}$$

$$\Sigma^{-1} = \frac{1}{(1 - \rho^2) \sigma_A^2 \sigma_M^2} \begin{bmatrix} \sigma_M^2 & -\rho \sigma_A \sigma_M \\ -\rho \sigma_A \sigma_M & \sigma_A^2 \end{bmatrix}$$

□

Bibliography

Markowitz, H. M. (1952). Portfolio Selection. *The Journal of Finance*, *7*(1), 77–91.