

Modèle Moyenne-Variance et Extensions

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Optimisation de Portefeuille: Notation (1)

- Rendement

$$R_i \sim N(\mu_i, \sigma_i^2)$$

$$\text{cov}(R_i, R_j) = \sigma_{i,j}$$

- Allocation exprimée en fraction de la richesse initiale

$$\sum_i w_i = 1$$

- Espérance de rendement du portefeuille

$$w^T \mu = \mu_{p,w}$$

- Variance du portefeuille

$$w^T \Sigma w = \sigma_{p,w}^2$$

Optimisation de Portefeuille: Notation (2)

- Distribution du rendement du portefeuille

$$R_{p,w} \sim N(\mu_{p,w}, \sigma_{p,w}^2)$$

- Covariance entre deux portefeuilles

$$\text{cov}(R_{p,w_1}, R_{p,w_2}) = w_1^T \Sigma w_2$$

Données

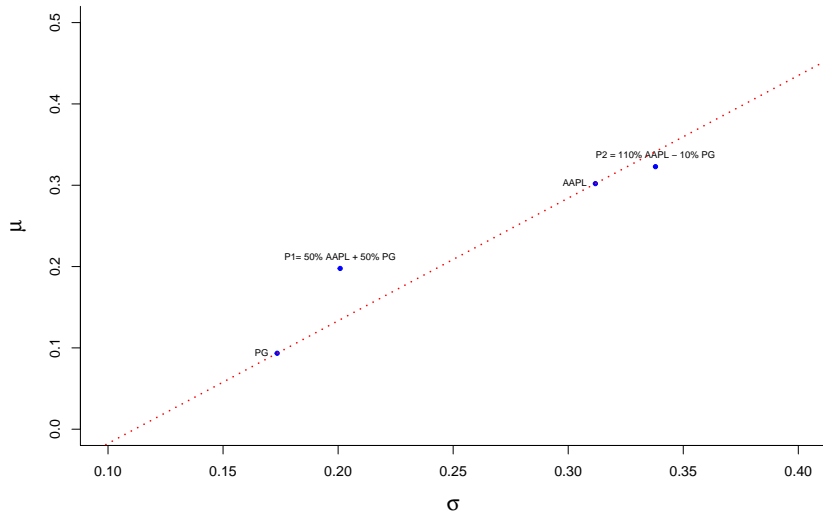
| | AAPL | AMZN | MSFT | F | SPY | QQQ | XOM | MMM | HD | PG | KO |
|-----------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Observations | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 | 3308.0000 |
| NAs | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Minimum | -0.1792 | -0.1278 | -0.1171 | -0.2500 | -0.0984 | -0.0896 | -0.1395 | -0.1295 | -0.0822 | -0.0790 | -0.0867 |
| Quartile 1 | -0.0077 | -0.0094 | -0.0073 | -0.0103 | -0.0038 | -0.0047 | -0.0068 | -0.0055 | -0.0067 | -0.0046 | -0.0047 |
| Median | 0.0010 | 0.0008 | 0.0005 | 0.0000 | 0.0006 | 0.0010 | 0.0001 | 0.0008 | 0.0006 | 0.0004 | 0.0007 |
| Arithmetic Mean | 0.0012 | 0.0015 | 0.0008 | 0.0005 | 0.0004 | 0.0006 | 0.0001 | 0.0004 | 0.0008 | 0.0004 | 0.0005 |
| Geometric Mean | 0.0010 | 0.0012 | 0.0006 | 0.0001 | 0.0003 | 0.0005 | 0.0000 | 0.0003 | 0.0006 | 0.0003 | 0.0004 |
| Quartile 3 | 0.0112 | 0.0123 | 0.0088 | 0.0106 | 0.0056 | 0.0070 | 0.0073 | 0.0070 | 0.0082 | 0.0055 | 0.0059 |
| Maximum | 0.1390 | 0.2695 | 0.1860 | 0.2952 | 0.1452 | 0.1216 | 0.1719 | 0.0988 | 0.1407 | 0.1021 | 0.1388 |
| SE Mean | 0.0003 | 0.0004 | 0.0003 | 0.0005 | 0.0002 | 0.0002 | 0.0003 | 0.0002 | 0.0003 | 0.0002 | 0.0002 |
| LCL Mean (0.95) | 0.0005 | 0.0006 | 0.0002 | -0.0005 | 0.0000 | 0.0002 | -0.0004 | -0.0001 | 0.0002 | 0.0000 | 0.0001 |
| UCL Mean (0.95) | 0.0019 | 0.0023 | 0.0013 | 0.0014 | 0.0008 | 0.0011 | 0.0006 | 0.0009 | 0.0013 | 0.0007 | 0.0009 |
| Variance | 0.0004 | 0.0006 | 0.0003 | 0.0007 | 0.0001 | 0.0002 | 0.0002 | 0.0002 | 0.0003 | 0.0001 | 0.0001 |
| Stdev | 0.0196 | 0.0243 | 0.0170 | 0.0266 | 0.0121 | 0.0130 | 0.0150 | 0.0140 | 0.0162 | 0.0109 | 0.0113 |
| Skewness | -0.2151 | 1.4889 | 0.4319 | 0.7627 | 0.1379 | -0.0084 | 0.4199 | -0.3815 | 0.5114 | 0.0555 | 0.5004 |
| Kurtosis | 6.2706 | 16.8872 | 10.2176 | 20.9458 | 15.2824 | 7.3976 | 15.4203 | 7.3856 | 6.4641 | 8.1017 | 14.3236 |

Un portefeuille avec 2 actifs

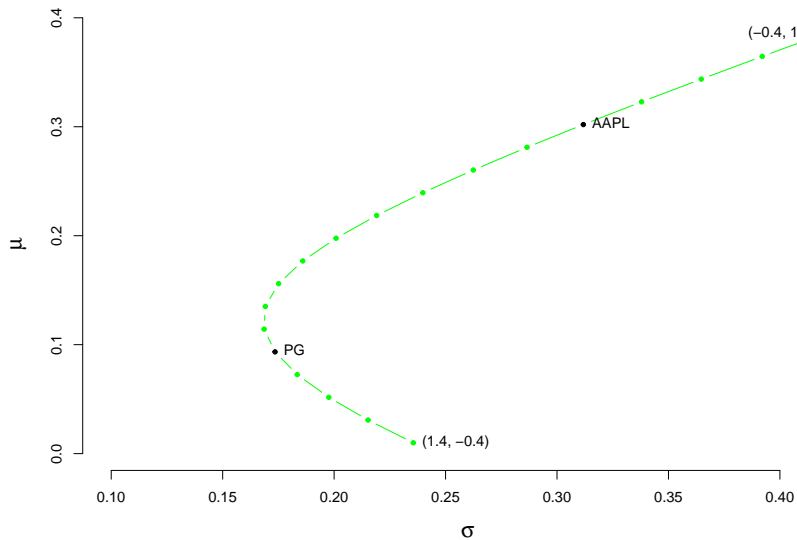
On choisit PG (faible vol) et AAPL (forte vol)

| | PG | AAPL |
|-------|-----------|-----------|
| mu | 0.0933800 | 0.3019870 |
| sigma | 0.1735107 | 0.3118083 |
| rho | 0.3150221 | NA |

Rendement et Risque de deux actifs



Frontière de deux actifs



Calcul du Portefeuille de Variance Minimale (1)

$$\min w^T \Sigma w$$

s.t.

$$\mathbf{1}^T w = 1$$

Solution Analytique

```
n <- 2
A.mat <- matrix(rep(1,n), ncol=1)
w.min <- solve(2*sig2, A.mat)
w.min <- w.min / sum(w.min)
names(w.min) <- assets
```

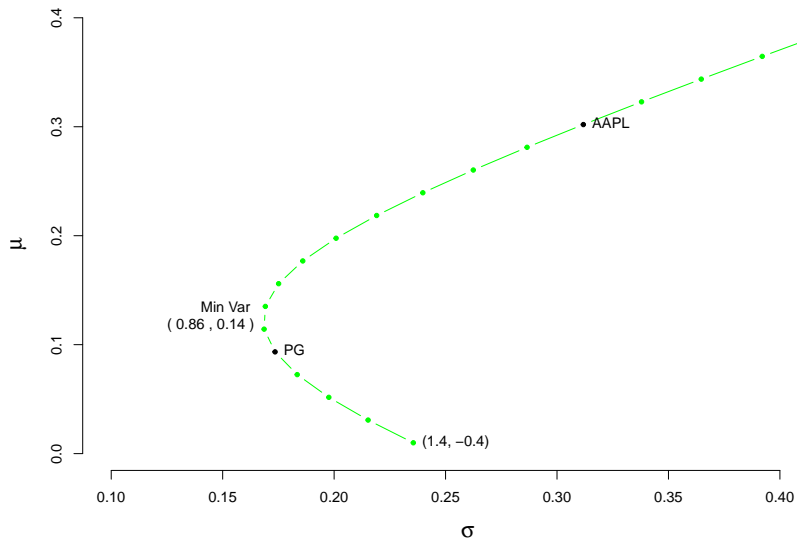
| | |
|------|-----------|
| PG | 0.8599089 |
| AAPL | 0.1400911 |

Utilisation de solve.QP - Solution Numérique

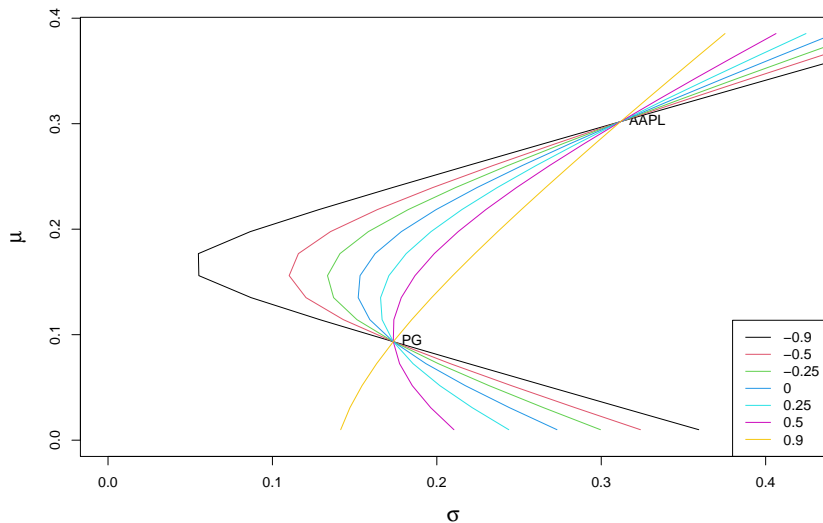
```
A.mat <- matrix(1, nrow=2, ncol=1)
b <- 1
sol <- solve.QP(2*sig2, rep(0,2),
               A.mat, b, meq=1)
w.min <- sol$solution
names(w.min) <- assets
```

| | X |
|------|-----------|
| PG | 0.8599089 |
| AAPL | 0.1400911 |

Portefeuille de Variance Minimale (2)



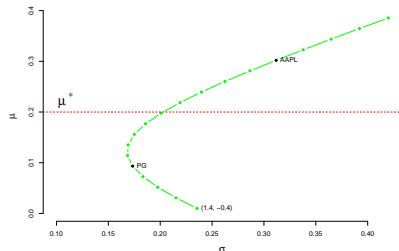
Frontière en fonction de la corrélation



Modèle de Markowitz

$$\begin{aligned}
 \min \quad & w^T \Sigma w \\
 \text{s.t.} \quad & \mu^T w = \mu^* \\
 & \mathbf{1}^T w = 1
 \end{aligned}$$

Voir note pour solution analytique et théorème de séparation des fonds.



Utilisation dans l'industrie

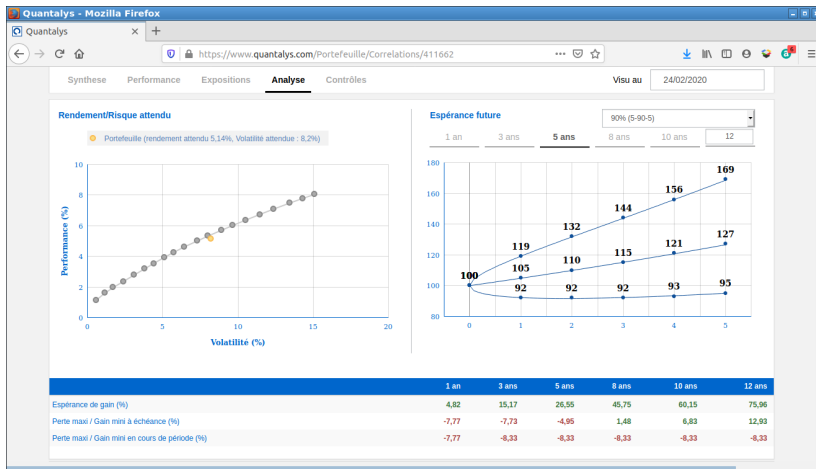
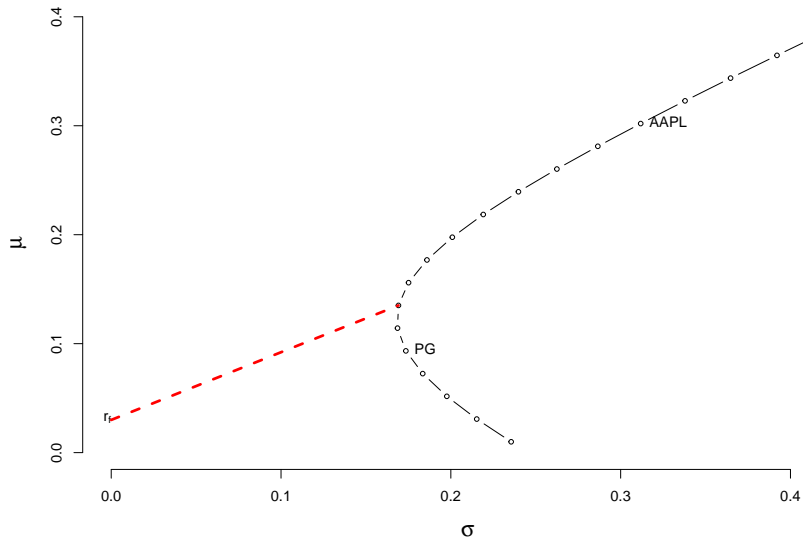


Figure 1: Diagramme Moyenne / Ecart Type (www.quantalys.com)

Ajout d'un actif sans risque (1)



Portefeuille Tangent (1)

Portefeuille sur la frontière efficiente qui maximise le ratio de Sharpe:

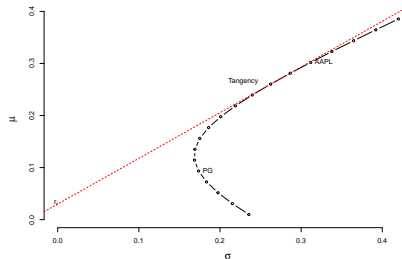
$$\max_w \frac{w^T \mu - r_f}{(w^T \Sigma w)^{\frac{1}{2}}}$$

s.t.

$$\mathbf{1}^T w = 1$$

Solution:

$$w = \frac{\Sigma^{-1}(\mu - r_f \mathbf{1})}{\mathbf{1}^T \Sigma^{-1}(\mu - r_f \mathbf{1})}$$



Optimisation Moyenne/Variance: Procédure de calcul

- ▶ Déterminer les espérances de rendement des actifs et la matrice de covariance
- ▶ Définir des contraintes du portefeuille (ex: encours max par ligne)
- ▶ Résoudre
- ▶ Faire varier r^* pour dessiner la frontière efficiente et le portefeuille tangent
- ▶ Choisir l'allocation en fonction de la tolérance au risque de l'investisseur

Optimisation M/V - avecR (1)

```
idx <- (time(daily.ret) >= dmy("01jan2014")) &
      (time(daily.ret) <= dmy("01jan2016"))
ret <- daily.ret[idx,]
# covariance matrix
Sigma <- cov(ret) * 252
# expected return
mu <- colMeans(ret) * 252
```

Portefeuille à Variance Minimale (1)

```
A.mat <- matrix(rep(1,length(mu)), ncol=1)
b <- 1
qp <- solve.QP(2*Sigma, mu*0, A.mat, b, meq=1)
```

Portefeuille à Variance Minimale (2)

| allocation | | Min Var Portfolio | |
|------------|------------|-------------------|-----------|
| AAPL | 0.0204498 | return | 0.0507792 |
| AMZN | 0.0006842 | stdev | 0.1164107 |
| MSFT | -0.0644735 | | |
| F | -0.0195378 | | |
| SPY | 0.8323712 | | |
| QQQ | -0.2902213 | | |
| XOM | -0.0805829 | | |
| MMM | 0.0161995 | | |
| HD | 0.0301382 | | |
| PG | 0.3006411 | | |
| KO | 0.2543316 | | |

Frontière

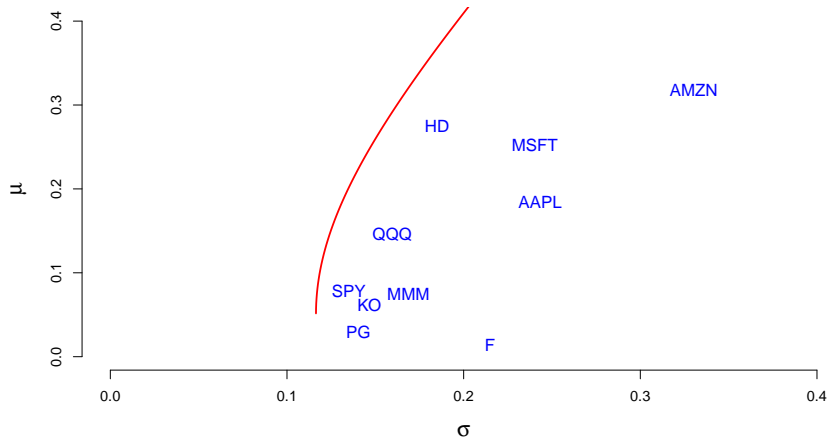
$$\min w^T \Sigma w$$

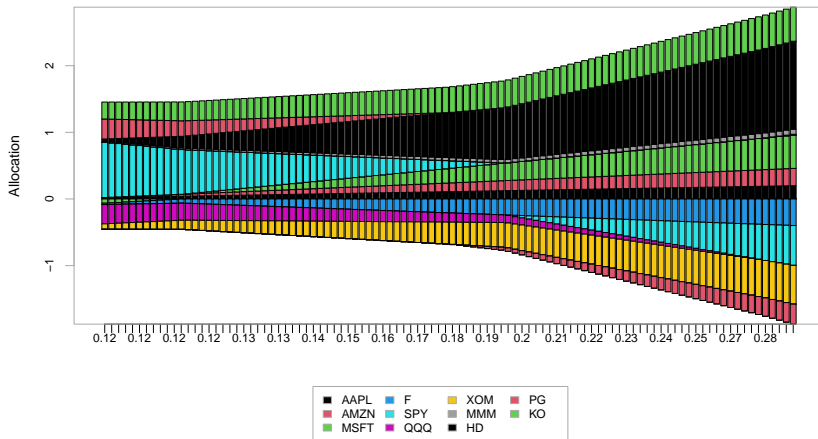
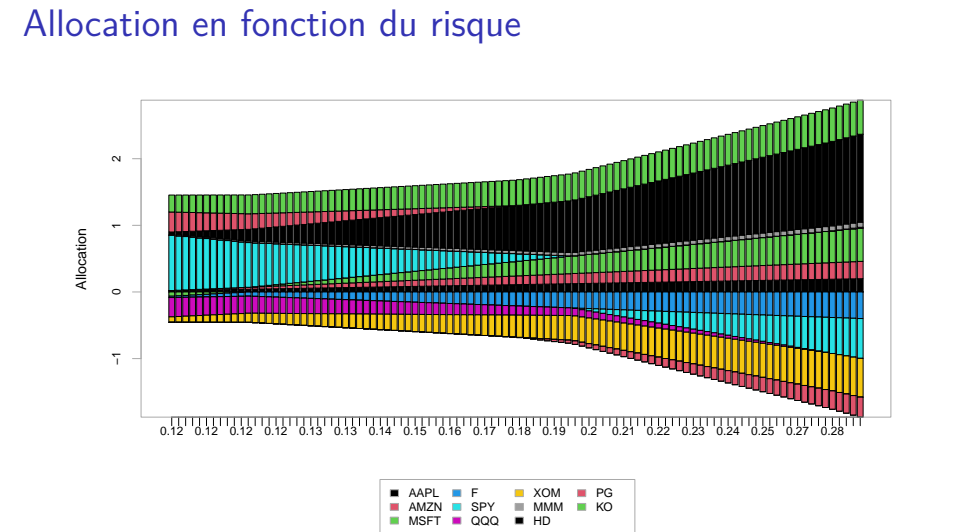
s.t.

$$\mu^T w = \mu^*$$

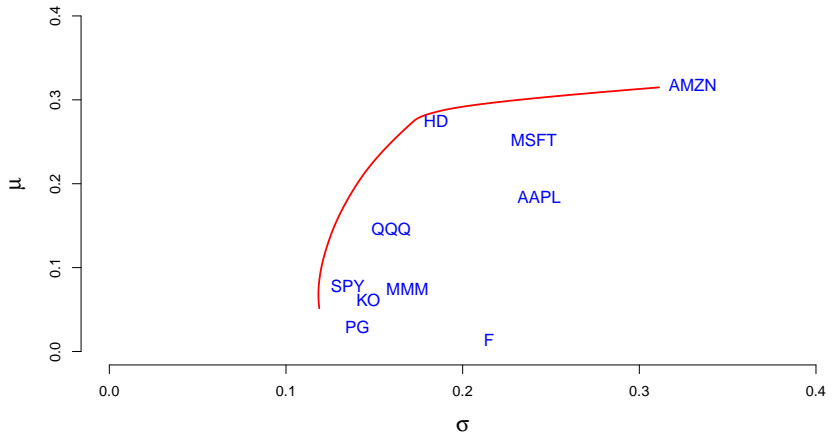
$$\mathbf{1}^T w = 1$$

Calcul de la Frontière (Long/Short)

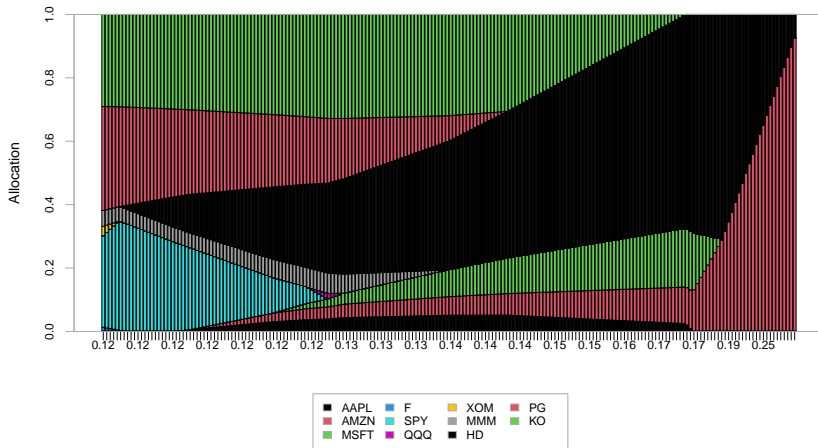




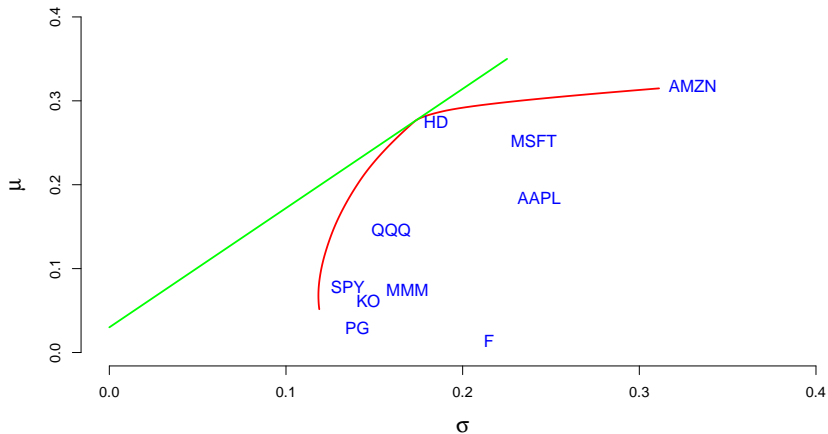
Calcul de la Frontière (Long)

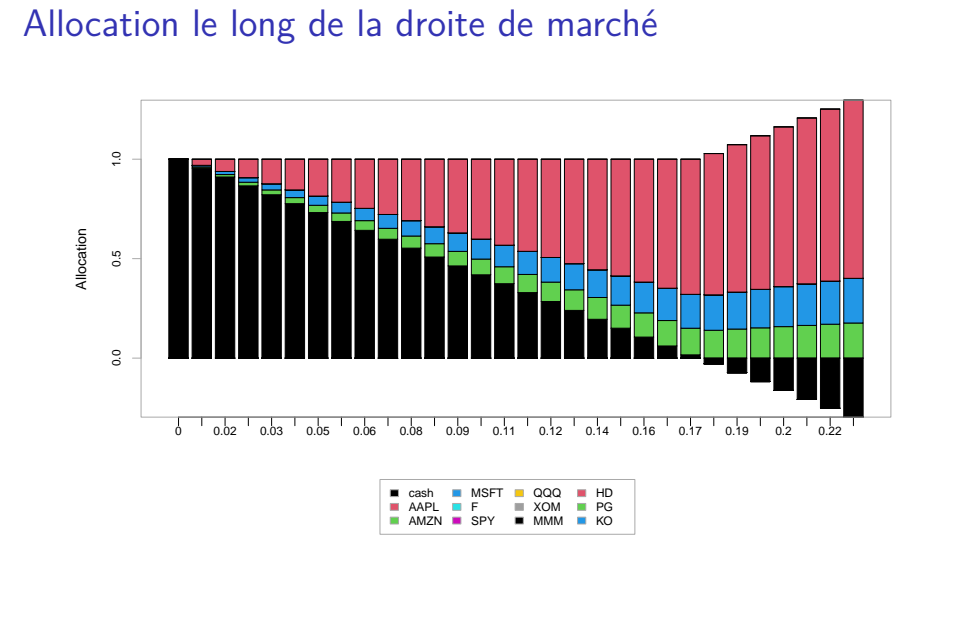


Allocation le long de la Frontière



Ajout d'un actif sans risque





Bibliographie

Grinhold, R.C. and Kahn, R. Active Portfolio Management, McGraw-Hill, 2000