Options: Valorisation et Couverture dans le cadre Black-Scholes

Patrick Hénaff

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Objectives

At the end of this lecture, you should understand:

- The assumptions and key concepts of the Black-Scholes model
- How to interpret de "delta" of an option
- How to set up a dynamic strategy to hedge the market risk of an option
- The risks associated with dynamic hedging, and the weaknesses of the Black-Scholes model,
- Why and how the Black-Scholes model is still used, despite its limitations

Historical Perspective

- Bachelier (1900) models stock prices as a random walk, and derives the first option pricing formula.
- James Boness (1964) derives almost the same option valuation formula as Black-Scholes, but uses the expected stock return as opposed to the risk-free rate
- Paul Samuelson (1965) also derives and option pricing formula similar to the Black-Scholes model, but it involves both the expected return of the stock and the expected return of the option.
- Ed Thorp (1967) reports to have hedged an option portfolio using Boness' formula, but replacing the expected return by the risk-free rate.

The Black Scholes Model

A formal justification of the Black-Scholes model requires many assumptions:

- There is no arbitrage opportunity;
- It is possible to borrow and lend cash at a known constant risk-free interest rate r;
- It is possible to buy and sell any amount of stock (this includes short selling);
- Transactions do not incur any costs;
- The stock price follows a log-normal distribution with constant and known drift μ and volatility σ.

The Black-Scholes Key Concept

"It is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock."

Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities.". In: *Journal of Political Economy* 81 (1973), pp. 637–654.

The Black-Scholes Formula

- S_0 Price of underlying asset
- T Option expiry
- K Strike
- r Interest rate
- σ Volatility

Call price:

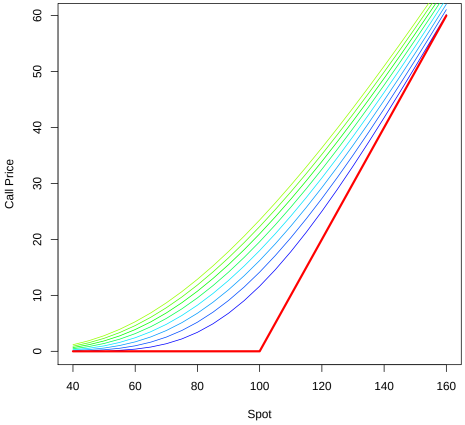
$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

with:

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

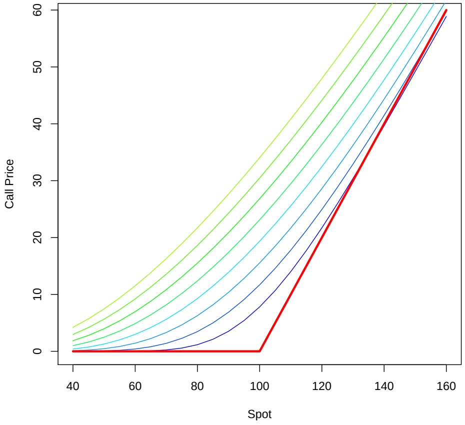
Call Price as a function of Spot and T

Strike = 100, Time to maturity 100 to 900 days. $\sigma = .3$.



Call Price as a function of Spot and Volatility

Strike = 100, Time to maturity = 1 Yr, $\sigma = .1, \ldots, .9$.



The Replicating Portfolio

The Black-Scholes formula can be directly interpreted as the description of the replicating portfolio:

$$C = S_0 N(d_1) - e^{-rT} K N(d_2)$$

The replicating portfolio has:

- $N(d_1)$ stock
- $KN(d_2)$ € of nominal of a zero-coupon bond expiring at T.

Review from Binomial Model

An option in a binomial model is equivalent to a portfolio:

- long Δ units of stock
- funded by borrowing B € at the risk-less rate

Such that, for a call worth C:

$$C = S_0 \Delta - B$$

The same principle applies with the Black-Scholes model:

$$C = S_0 N(d_1) - e^{-rT} K N(d_2)$$

Black-Scholes Delta

The delta is the change in option price for a change of one e in the price of the underlying asset.

$$\text{Delta} = \frac{\text{Change in Option Value}}{\text{Change in Underlying Value}}$$

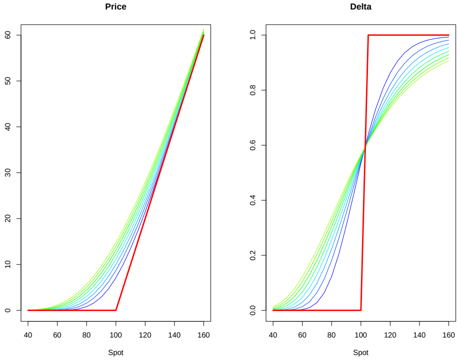
For a European call:

$$\Delta_C = N(d_1)$$

For a European put:

$$\Delta_p = N(d_1) - 1$$

Price and Delta of a Call as a function of Maturity



Construction of a hedge portfolio

Notation:

- C_t Value of derivative
- V_t Value of hedge portfolio
- B_t Amount borrowed/lent at the risk-free rate
- Δ_t Delta of derivative

At $t = 0$, the derivative is sold at price C_0 and the proceeds are used to purchase a hedge portfolio.The initial hedge is $\Delta_0 S_0 - B_0$, where B_0 is computed from the accounting identity:

$$C_0 = \Delta_0 S_0 - B_0$$

Example

A financial institution writes (sells) an at-the-money option on a stock worth €100. The option expires in two months, the hedge will be rebalanced every week (for illustration). Interest rate is 2% and volatility 30%.

Question: Compute the option price and the hedge portfolio.

Initial Hedge Portfolio

- Call price (Black-Scholes): $C_0 = 5.04$
- Delta: $\Delta_0 = 0.5352$
- Amount borrowed:

$$
\begin{aligned}
B_0 &= C_0 - \Delta_0 S_0 \\
&= -48.48
\end{aligned}
$$

Initial hedge portfolio

	Hedge Portfolio		
Call Price C_0	Stock $\Delta_0 \times S_0$	Bond B_0	Total
5.04	53.52	-48.48	5.04

Rebalancing of hedge portfolio

The hedge must be adjusted periodically. At each step i, the decision rule is as follows:

1. Compute the value of the hedge portfolio formed at the previous time step:

$$V_i = -B_{i-1}e^{r\Delta t} + \Delta_{i-1}S_i$$

2. Compute the amount of stock to hold:

$$\Delta_i = \frac{\partial C_i}{\partial S_i}$$

3. The new hedge portfolio is $\Delta_i S_i + B_i$, with borrowing B_i determined by:

$$-B_i = V_i - \Delta_i S_i$$

At expiry of the derivative, the residual wealth is:

$$-C_T + \Delta_{T-1}S_T - B_{T-1}e^{r\Delta t}$$

Hedge Effectiveness

The quality of a model is ultimately measured by the residual error:

$$
\begin{aligned}
E_T &= -B_{T-1}e^{r\Delta t} + \Delta_{T-1}S_T - C_T \\
&= V_T - C_T
\end{aligned}
$$

Simulation 1: Call expiring out of the money

Week	stock price	Δ	call	bond	hedge port.
1	100.00	0.54	5.05	-48.9	5.05
2	98.16	0.47	3.79	-42.0	4.05
3	90.05	0.18	0.90	-16.0	0.23
4	88.01	0.11	0.42	-9.8	-0.15
5	90.28	0.13	0.50	-11.6	0.09
6	94.67	0.25	1.02	-23.0	0.66
7	94.17	0.17	0.53	-15.5	0.53
8	95.65	0.16	0.34	-14.5	0.78
9	94.67	0.00	0.00	+.6	0.62

Hedging discrepancy: 0.62 €.

Example 2: Call expiring in the money

Week	stock price	Δ	call	bond	hedge port.
1	100.00	0.54	5.05	-48.9	5.05
2	95.38	0.37	2.63	-32.7	2.55
3	93.58	0.29	1.73	-25.26	1.87
4	102.46	0.63	5.39	-60.1	4.45
5	101.23	0.58	4.22	-55.0	3.66
6	103.78	0.71	5.39	-68.6	5.12
7	103.34	0.72	4.56	-69.6	4.77
8	109.01	0.98	9.09	-98.0	8.82
9	103.94	1.00	3.94	-100.	4.06

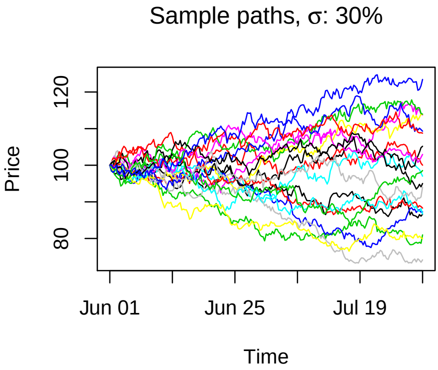
Hedging discrepancy: 4.06 - 3.94 = .12 €.

Large Scale Dynamic Hedging Simulation

To test the effectiveness of delta-hedging with the Black-Scholes model:

1. Simulate price scenarios
2. Simulate the dynamic rebalancing of the hedge portfolio
3. For each path, observe the hedging error at expiry

Simulated paths - log-normal process $\sigma = 30\%$, $T = 2$ months



First Simulation

Simulations in a "perfect Black-Scholes world"

- The volatility is known and constant
- the interest rate is constant
- No transaction costs

Delta-hedging simulation, maturity: 2 months,
$\sigma = .3$, $r = .02$, $K = 100$, $S_0 = 100$. Option price: 5.05.
200 time steps, 1000 simulations.

Distribution of Hedging Error - ATM European Call

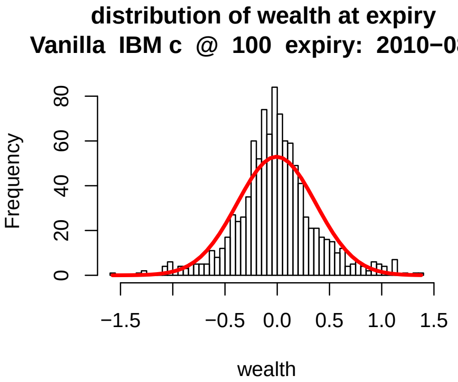


Figure: Distribution of hedging error $\sigma = 0.4$

Delta Hedging Error

Sources of hedging error:

1. Hedging frequency (Black-Scholes assumes continuous rebalancing);
2. Along one path, the volatility that is experienced may not be the theoretical volatility σ.

Delta Hedging Error vs. Hedging Frequency

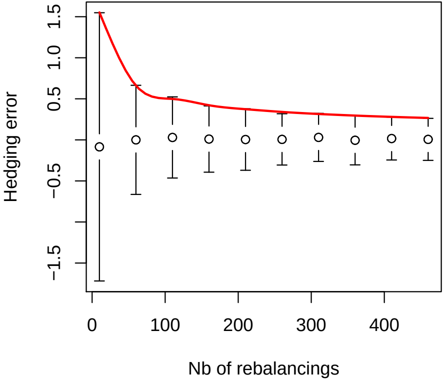


Figure: The variance of the hedging error is inversely related to the hedging frequency

Black-Scholes in Real Life

In real life, every assumption of the Black-Scholes model is invalidated:

- Interest rate is not constant
- Future volatility is unknown
- The distribution of S_T is not log-normal
- There are transaction costs and restrictions to shorting stocks

Let's measure the impact on hedging error.

Delta Hedging Error vs. Transaction Cost

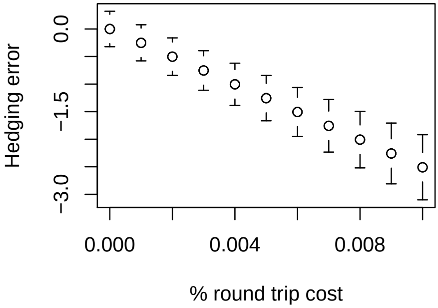


Figure: The hedging error is directly related to the magnitude of the transaction costs

Hedging Error and Unknown Volatility

Consider:

- The pricing volatility, used to determine the initial value of the derivative
- The actual volatility, the one that is experienced during the live of the option.

You sell an option and dynamically hedge your risk by trading the delta-hedging portfolio.

- If the actual volatility is lower than the pricing (expected) volatility, you will on average make money.
- If the actual volatility is higher than the pricing (expected) volatility, you will on average loose money.

More Greeks...

Hedging error is strongly dependent upon the curvature of the option price curve: the Gamma.

$$\begin{aligned}\Gamma &= \frac{\partial^2 C}{\partial S^2} \\ &\approx \frac{C(S_0+h) - 2C(S_0) + C(S_0-h)}{h^2}\end{aligned}$$

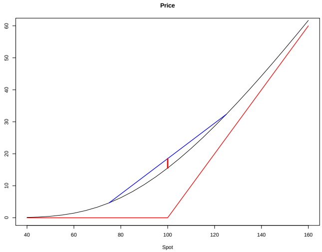


Figure: Gamma of a Call

More Greeks...

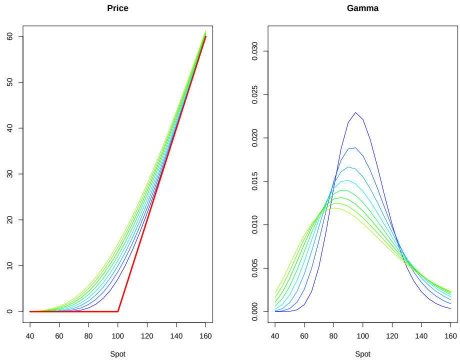


Figure: Price and Gamma of a Call as a function of maturity

Hedging Error and Unknown Volatility

In the presence of unknown volatility, it can be shown that hedging error is related to:

1. The difference between the pricing (i.e. assumed) volatility and the effective volatility
2. The gamma of the option:
   - Hedging error is small when the gamma is small and keeps the same sign
   - Hedging error is large when the gamma is large and changes sign

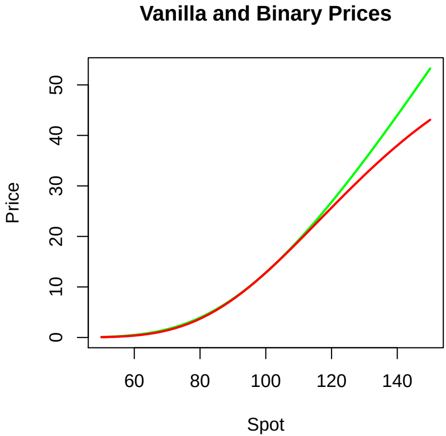
Influence of Gamma on Hedging Error

We compare the hedging error of two simple derivatives expiring in 1 year:

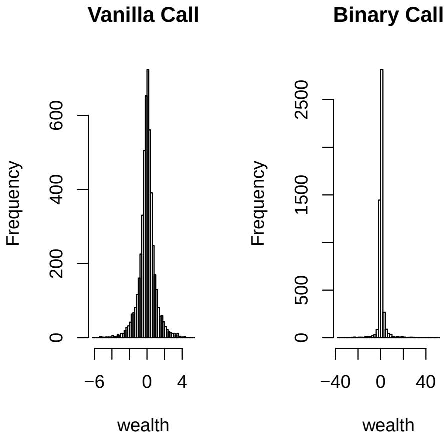
- a European call, $K = 100$,
- a binary call struck at 126.65 e, that will pay 62.09 e if the option expires in the money.

The binary call is designed to have the same premium and same initial delta as the European option.

PV of Vanilla and Binary Calls with same current price and delta



Delta Hedging Errors: Vanilla vs. Binary Calls



In identical experimental conditions, the hedging error of the binary call is about 10 times larger than the one of the European

Robustness of Black-Scholes

$$\epsilon_T = \frac{1}{2}\int_0^T \left[\Sigma^2 - \sigma_t^2\right] \frac{\partial^2 C}{\partial S^2} S^2 dt$$

N El-Karoui, M Jeanblanc-Picquè, and Steven E. Shreve. "Robustness of the Black and Scholes Formula". In: *Mathematical Finance* 8.2 (1998), pp. 93–126. ISSN: 1467-9965. DOI: 10.1111/1467-9965.00047

Hedging Error: Conclusion

In a Black-Scholes world, hedging error is determined by four factors:

1. the hedging frequency
2. the transaction costs
3. the magnitude of the option gamma
4. the difference between Σ (the BS volatility used for pricing and hedging) and the volatility that is actually experienced, σ.

Is Black-Scholes Still in Use?

- Black-Scholes in its pure form (delta hedging argument) has probably never been used in practice
- Complex derivatives have been managed with much more complex models (stochastic volatility, jumps, local volatility) since the mid 1990's.

Is Black-Scholes Still in Use?

The Black-Scholes formula is widely used in vanilla option markets, but not in the way consistent with the theory:

- The Black-Scholes formula is used to operate a change of unit of measure for vanilla options: to convert prices into volatilities (it is easier to interpret volatility differences than price differences).
- Traders recognize that the main risk is unknown volatility / non normality of returns, and therefore hedge options with other options.

Espen Gaarder Haug and Nassim Nicholas Taleb. “Option Traders Use (very) Sophisticated Heuristics, Never the Black-Scholes-Merton Formula.”. In: *Economic Behavior and Organization* 77 (2011).

Is Black-Scholes Still in Use?

The Black-Scholes assumptions (normality of returns) are unfortunately still sometimes used by regulators to define risk measures.

1. Value at Risk, Pillar I of Basel II/III;
2. Solvency II risk models.

The Implied Volatility Problem

- In the BS formula, we use σ as an input.
- In reality, exchanges quote options in price.
- The BS formula is used to convert an option price into the corresponding volatility.

Given the observed price C^* of a call, compute the volatility σ such that:

$$\begin{aligned} C^* &= C(S, K, T, r, \sigma) \\ &= f(\sigma) \end{aligned}$$

Option data: Settlement prices of options on the WTI Feb09 futures contract

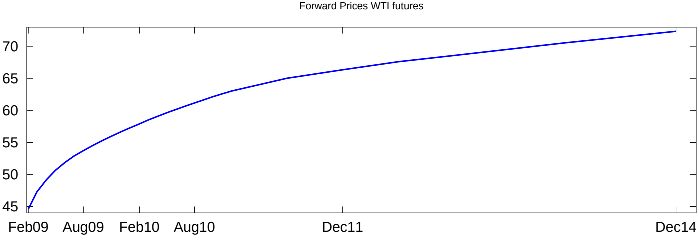
NEW YORK MERCANTILE EXCHANGE
NYMEX OPTIONS CONTRACT LISTING FOR 12/29/2008

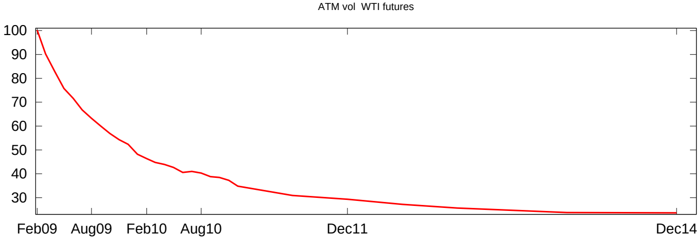
--------CONTRACT--------				TODAY'S SETTLE	PREVIOUS SETTLE	ESTIMATED VOLUME
LC	02 09	P	30.00	.53	.85	0
LC	02 09	P	35.00	1.58	2.28	0
LC	02 09	P	37.50	2.44	3.45	0
LC	02 09	C	40.00	3.65	2.61	10
LC	02 09	P	40.00	3.63	4.90	0
LC	02 09	P	42.00	4.78	6.23	0
LC	02 09	C	42.50	2.61	1.80	0
LC	02 09	C	43.00	2.43	1.66	0
LC	02 09	P	43.00	5.41	6.95	100

Option data: Options on the S&P 500 Index (Source:CBOE)

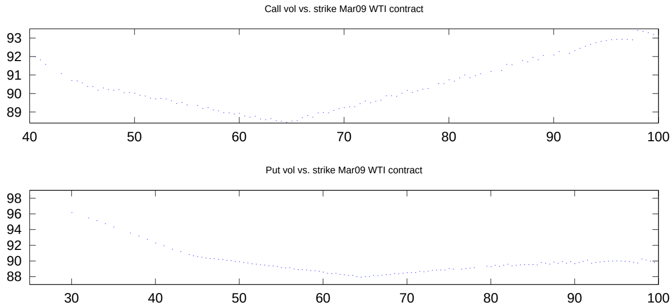
SPX (SP 500 INDEX)	1290.59	+7.24					
Jan 24 2011 @ 14:03 ET							
Calls	Last Sale	Net	Bid	Ask	Vol	Open Int	...
11 Jan 1075.00 (SPXW1128A1075-E)	0.0	0.0	215.30	217.00	0	0	...
11 Jan 1100.00 (SPXW1128A1100-E)	0.0	0.0	190.60	191.80	0	0	...

ATM Volatility

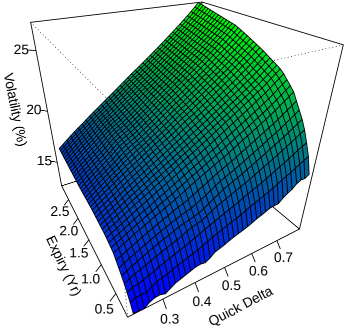




WTI Option Implied Vol



Implied Volatility of S&P 500 Index Options (24-jan-2011)



The Implied Volatility

Option traders study the implied volatility rather than the option prices.
Why?

- A measure of value, irrespective of strike and maturity
- The BS model assumes a constant volatility: deviations from that assumption are useful to study

The Implied Volatility Problem

Given the observed price C^* of a call, compute the volatility σ such that:

$$\begin{aligned} C^* &= C(S, K, T, r, \sigma) \\ &= f(\sigma) \end{aligned}$$

The Implied Volatility Problem

There is a change in convexity in $C = f(\sigma)$:

$$\frac{\partial C}{\partial \sigma} = Sn(d_1)\sqrt{T}$$

$$\frac{\partial^2 C}{\partial \sigma^2} = S\sqrt{T}n(d_1)\frac{1}{\sigma}\left[\frac{1}{\sigma^2 T}\ln(\frac{F}{K})^2 - \frac{1}{4}\sigma^2 T\right]$$

with $F = Se^{rT}$.
Thus, $C(\sigma)$ is convex on the interval $(0, \sqrt{\frac{2|\ln(F/K)|}{T}}]$, and concave otherwise.

Convergence of Newton's Method

To ensure convergence of Newton's method, one must carefully choose the initial point.

Theorem

Let f be defined on the interval $[a, b]$ *and assume that:*

1. $f(x^*) = 0$ *for some* $x^* \in [a, b]$
2. $f'(x) > 0$
3. $f''(x) \geq 0$

Then Newton's method converges monotonically from $x_0 = b$. *If*

1. $f(x^*) = 0$ *for some* $x^* \in [a, b]$
2. $f'(x) > 0$
3. $f''(x) \leq 0$

Then Newton's method converges monotonically from $x_0 = a$.

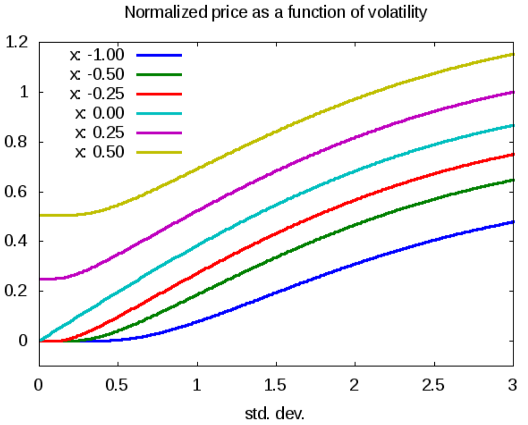
Convergence of Newton's Method

Consider now Newton's method started at

$$\sigma_0 = \sqrt{\frac{2|\ln(F/K)|}{T}}$$

- If $f(\sigma_0) > 0$, we are in case I of theorem.
- If $f(\sigma_0) < 0$ we are in case II.

Normalized Call price as a function of σ



Implied Volatility by Newton's Method

The following algorithm generates a monotonic series (σ_n):

1. Set $\sigma_0 = \sqrt{\frac{2|\ln(F/K)|}{T}}$
2. While $|C(\sigma_n) - C^*| > \epsilon$:
   2.1 Let
   $$\sigma_{n+1} = \sigma_n + \frac{C^* - C(\sigma_n)}{\frac{\partial C}{\partial \sigma}}$$
   2.2 $n \leftarrow n + 1$

Jackel's Method

Given an option price p, we must solve for σ

$$p = \delta\theta \left[F\Phi\left(\theta\left[\frac{\ln(F/K)}{\sigma} + \frac{\sigma}{2}\right]\right) - K\Phi\left(\theta\left[\frac{\ln(F/K)}{\sigma} - \frac{\sigma}{2}\right]\right)\right]$$

where:

- δ discount factor
- θ 1 for call, -1 for put
- F Forward price: $Se^{(r-d)T}$
- σ $\sigma\sqrt{T}$

Jackel's Method

Set:

$$\begin{aligned} x &= \ln(F/K) \\ b &= \frac{p}{\delta\sqrt{FK}} \end{aligned}$$

The Black-Scholes equation becomes:

$$b = \theta\left[e^{x/2}\Phi\left(\theta\left[\frac{x}{\sigma}+\frac{\sigma}{2}\right]\right) - e^{-x/2}\Phi\left(\theta\left[\frac{x}{\sigma}-\frac{\sigma}{2}\right]\right)\right]$$

The normalized price function changes convexity at $\sigma_c = \sqrt{|x|}$

Jaeckel's Method

Further defines

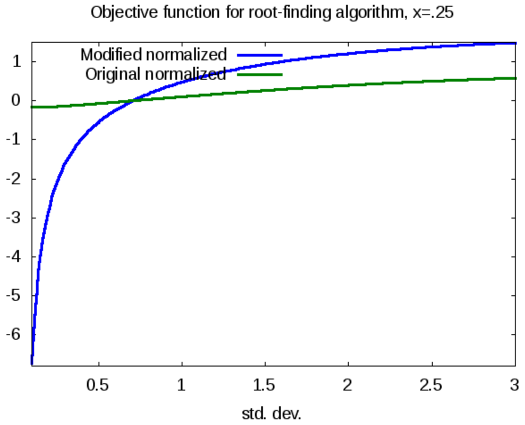
$$f(\sigma) = \begin{cases} \ln\left(\frac{b-i}{\bar{b}-i}\right) & \text{if} \bar{b} < b_c \\ b - \bar{b} & \text{otherwise} \end{cases}$$

with:

- b_c $b(x, \sigma_c, \theta)$
- i normalized intrinsic value: $1_{\theta x > 0} \theta \left(e^{x/2} - e^{-x/2} \right)$
- $\bar{b}$ target normalized price

The function $f(\sigma)$ is now monotonously concave.

Transformed objective function



- Vega: $\nu = \frac{\partial O}{\partial \sigma}$
- Vanna: $\frac{\partial \nu}{\partial S}$
- Volga: $\frac{\partial \nu}{\partial \sigma}$

Vanna-Volga Dynamic Hedging

- A practical method for pricing and hedging derivatives, taking into account an uncertain volatility.
- Popular for Foreign Exchange derivatives
- Relates the price of a complex derivative to the known price of simpler, liquid instruments.

Vanna-Volga Dynamic Hedging

An extended version of Ito's lemma with random volatility:

$$\begin{aligned} dO(t,K) &= \frac{\partial O}{\partial t}dt + \frac{\partial O}{\partial S}dS_t + \frac{\partial O}{\partial \sigma}d\sigma_t \\ &+ \frac{1}{2}\frac{\partial^2 O}{\partial S^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 O}{\partial \sigma^2}(d\sigma_t)^2 + \frac{\partial^2 O}{\partial S \partial \sigma}dS_t d\sigma_t \end{aligned}$$

Vanna-Volga Dynamic Hedging

$$dO(t,K) - \Delta_t dS_t - \sum_{i=1}^{3} x_i dC_i(t,K_i) =$$

$$\left[\frac{\partial O}{\partial t} - \sum_i x_i \frac{\partial C_i}{\partial t}\right] dt + \left[\frac{\partial O}{\partial S} - \Delta_t - \sum_i x_i \frac{\partial C_i}{\partial S}\right] dS_t$$

$$+ \left[\frac{\partial O}{\partial \sigma} - \sum_i x_i \frac{\partial C_i}{\partial \sigma}\right] d\sigma_t + \left[\frac{\partial^2 O}{\partial S^2} - \sum_i x_i \frac{\partial^2 C_i}{\partial S^2}\right] (dS_t)^2$$

$$+ \left[\frac{\partial^2 O}{\partial \sigma^2} - \sum_i x_i \frac{\partial^2 C_i}{\partial \sigma^2}\right] (d\sigma_t)^2 + \left[\frac{\partial^2 O}{\partial S \partial \sigma} - \sum_i x_i \frac{\partial^2 C_i}{\partial S \partial \sigma}\right] dS_t d\sigma_t$$

Vanna-Volga Hedging

$$dO(t,K) - \Delta_t dS_t - \sum_{i=1}^{3} x_i dC_i(t,K_i) =$$

$$r\left[O(t,K) - \sum_i x_i C_i(t,K_i)\right] dt$$

Vanna-Volga Dynamic Hedging

The weights x_i are obtained by solving the system of linear equations:

$$
\begin{aligned}
\frac{\partial O}{\partial \sigma} &= \sum_i x_i \frac{\partial C_i}{\partial \sigma} \\
\frac{\partial^2 O}{\partial \sigma^2} &= \sum_i x_i \frac{\partial^2 C_i}{\partial \sigma^2} \\
\frac{\partial^2 O}{\partial S \partial \sigma} &= \sum_i x_i \frac{\partial^2 C_i}{\partial S \partial \sigma}
\end{aligned}
$$

or,

$$b = Ax$$

Vanna-Volga Dynamic Hedging

Choice of hedge instruments

- An at-the-money straddle:

$$C_1 = C(S) + P(S)$$

- A "risk reversal", traditionally defined as

$$C_2 = P(K_1) - C(K_2)$$

with K_1 and K_2 chosen so that the options have a delta of .25.

- A "butterfly", defined as

$$C_3 = \beta(P(K_1) + C(K_2)) - (P(S) + C(S))$$

with β determined to set the vega of the butterfly to 0.

Vanna-Volga Dynamic Hedging: Calculation Steps

- compute the risk indicators for the option to be priced:

$$b = \begin{pmatrix} \frac{\partial O}{\partial \sigma} \\ \frac{\partial^2 O}{\partial \sigma^2} \\ \frac{\partial^2 O}{\partial \sigma \partial S} \end{pmatrix} \tag{1}$$

- compute A matrix

$$A = \begin{pmatrix} \frac{\partial C_1}{\partial \sigma} & \dots & \frac{\partial C_3}{\partial \sigma} \\ \frac{\partial^2 C_1}{\partial \sigma^2} & \dots & \frac{\partial^2 C_3}{\partial \sigma^2} \\ \frac{\partial^2 C_1}{\partial \sigma \partial S} & \dots & \frac{\partial^2 C_3}{\partial \sigma \partial S} \end{pmatrix} \tag{2}$$

Vanna-Volga Dynamic Hedging: Calculation Steps

- solve for x:

$$b = Ax$$

- the corrected price for O is:

$$O^M(t,K) = O^{BS}(t,K) + \sum_{i=2}^{3} x_i \left(C_i^M(t) - C_i^{BS}(t) \right) \tag{3}$$

where $C_i^M(t)$ is the market price and $C_i^{BS}(t)$ the Black-Scholes price (i.e. with flat volatility).

Vanna-Volga Dynamic Hedging

Neglecting the off diagonal terms in A, a simplified procedure is to estimate x_i by:

$$x_2 = \frac{\frac{\partial^2 O}{\partial \sigma^2}}{\frac{\partial^2 C_2}{\partial \sigma^2}}$$

$$x_3 = \frac{\frac{\partial^2 O}{\partial \sigma \partial S}}{\frac{\partial^2 C_3}{\partial \sigma \partial S}}$$

In practice, the weights x_i are scaled to better fit market prices.

Smile Parametrisation

Risk reversals and butterflies provide a simple parametrisation of the smile:

- the risk reversal measures assymetry
- the butterfly measures the convexity

Volatility Interpolation

Given the ATM volatility and at two other strikes, determine the volatility at an arbitrary strike K.

Vana-Volga Smile Interpolation

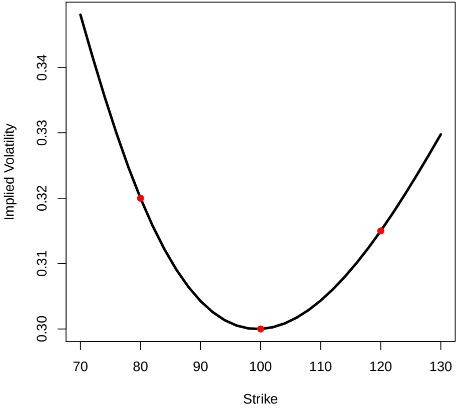


Figure: Interpolated Volatility Curve with Vanna-Volga algorithm

Vanna-Volga Pricing of a Binary Option

Consider a one-year binary call, struck at the money. Assume that the smile is quadratic.
Use the traditional benchmark instruments of the FX market:

- Straddle,
- Risk-reversal
- Butterfly

Vanna-Volga Pricing of a Binary Option

Steps:

- Compute the strikes corresponding to a 25Δ call and put,
- Compute the value of each benchmark instrument. The butterfly must be vega-neutral,
- Compute the risk indicators (vega, vanna, volga) for the binary option,
- ... and for the benchmark instruments,
- Compute the smile cost of each benchmark: the price with the smile effect less price at the ATM volatility.
- Compute the price correction for the binary option.

Pricing a Binary Option

In summary, we get:

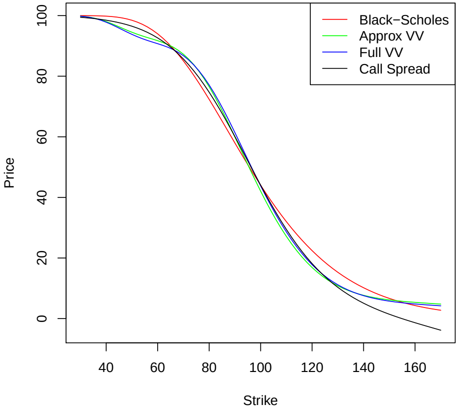
- Black-Scholes price: 32
- With approximate Vanna-Volga correction: $32 + -4.86 = 27.14$
- With acurate Vanna-Volga correction: $32 + -3.39 = 28.61$
- the approximation by a call spread is: 29.44

Pricing Comparison

comparison of binary option value for a range of strikes, computed with four methods:

1. the regular Black-Scholes method, assuming a flat volatility
2. the Black-Scholes price with approximate Volga-Vanna correction
3. same a above, but with an accurate calculation of the Vanna-Volga correction
4. the value of a call spread centered at the strike

Vana-Volga Pricing of a Binary Option



The Breeden-Litzenberger formula

Risk-neutral density of the underlying asset at maturity T as a function of derivative prices:

$$p_T(K) = e^{rT} \frac{\partial^2 C(S, K, T)}{\partial K^2} \tag{4}$$

where $C(S, K, T)$ is the price of a call of strike K, maturity T, when the current spot is S.

The Breeden-Litzenberger formula

By definition of the risk-neutral probability,

$$C(S,K,T) = e^{-rT}\int_K^{\infty}(S_T - K)p(S_T)dS_T \tag{5}$$

Applying Leibniz's Rule to get:

$$\frac{\partial C(S,K,T)}{\partial K} = -e^{-rT}\int_K^{\infty} p(S_T)dS_T$$

Let $F(K)$ the cumulative density function of S_T,

$$\begin{aligned} e^{rT}\frac{\partial C(S,K,T)}{\partial K} &= -\int_K^{\infty} p(S_T)dS_T \\ &= F(K) - 1 \end{aligned}$$

The Breeden-Litzenberger formula

Differentiate again with respect to K to get:

$$\frac{\partial^2 C}{\partial K^2} e^{rT} = p(K) \tag{6}$$

Local Vol: Interpretation

Consider a butterfly spread centered at K, and scaled to yield a maximum payoff of 1. Let $\phi(S_T)$ be the payoff function. The value of the butterfly is:

$$\begin{aligned} V &= \frac{1}{\Delta K}\left[C(K+\Delta K) - 2C(K) + C(K-\Delta K)\right] \\ &= e^{-rT}\int_0^{\infty} \phi(S)p(S)dS \end{aligned}$$

Local Vol: Interpretation

In the interval $[K - \Delta K, K + \Delta K]$, approximate $p(S)$ by the constant $p(K)$ to get:

$$\begin{aligned} V &= e^{-rT} p(K) \int_0^\infty \phi(S) dS \\ &= e^{-rT} p(K) \Delta K \end{aligned}$$

Finally, use the definition of the derivative:

$$\lim_{\Delta K \to 0} \frac{1}{\Delta K^2} \left[C(K + \Delta K) - 2C(K) + C(K - \Delta K) \right] = \frac{\partial^2 C(K)}{\partial K^2} \tag{7}$$

to get:

$$p_T(K) = e^{rT} \frac{\partial^2 C(K)}{\partial K^2} \tag{8}$$

Analytical expression for the density of S_T

$$
\begin{aligned}
p(K) = & n(d_2)\Big\{ \\
& \frac{1}{K\sigma\sqrt{T}} + \frac{\partial\sigma}{\partial K}\frac{2d_1}{\sigma} + \\
& \left(\frac{\partial\sigma}{\partial K}\right)^2 \frac{\sqrt{T}Kd_1d_2}{\sigma} + \\
& \frac{\partial^2\sigma}{\partial K^2}K\sqrt{T}\Big\}
\end{aligned} \tag{9}
$$

Illustration: Shimko's Model

Fit a quadratic model to the implied volatility, in order to get analytical expressions for $\frac{\partial \sigma}{\partial K}$, $\frac{\partial^2 \sigma}{\partial K^2}$:

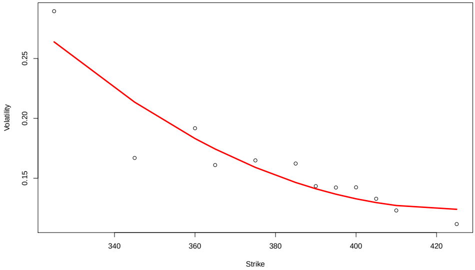


Figure: Quadratic Volatility Model

Illustration: Shimko's Model

The density implied from the quadratic volatility smile cleary exhibit "fat tails".

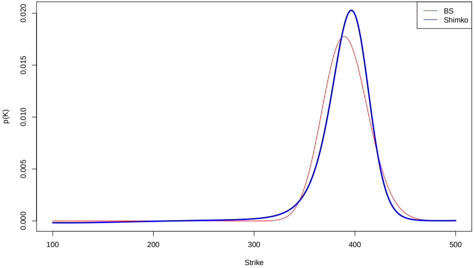


Figure: Density of S_T, with constant volatility and quadratic model for implied volatility.

Consequence for Option Pricing

To illustrate the importance of correctly accounting for the volatility smile, we now consider a digital option maturing at the same time as our European options. We want to price this option in a way that is consistent with the observed volatility smile.

A naive approach would be to look up the Black-Scholes volatility corresponding to the strike, and price the digital option accordingly. The price of a digital cash-or-nothing call is given by:

$$C = e^{-rT}\Phi(d_2) \tag{10}$$

with:

$$\begin{aligned}
\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2} \\
d_2 &= \frac{\ln\frac{S}{K} + (r - d - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}
\end{aligned}$$

Consequence for Option Pricing

However, since we know the density of S_T, we can directly compute the expected discounted value of the digital payoff:

$$C = e^{-rT} \int_K^{\infty} p(S_T) dS_T$$

Digital Price with and without Smile

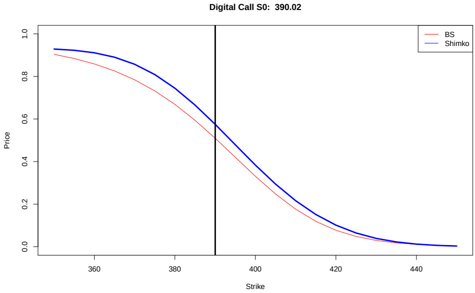


Figure: Comparison of prices of a digital option, using a log-normal density for S_T, and using the density implied by the volatility smile fitted to a quadratic function.

WTI Smile

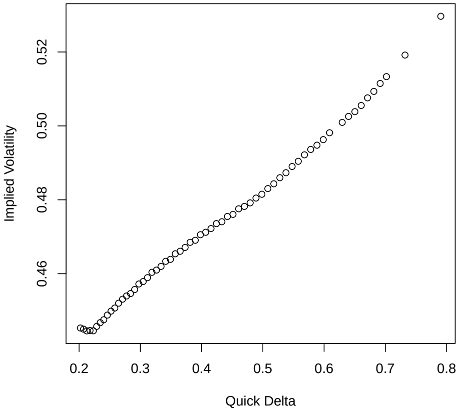


Figure: Implied volatility of WTI NYMEX options on the December 2009 Futures, observed on April 21, 2009

WTI Smile

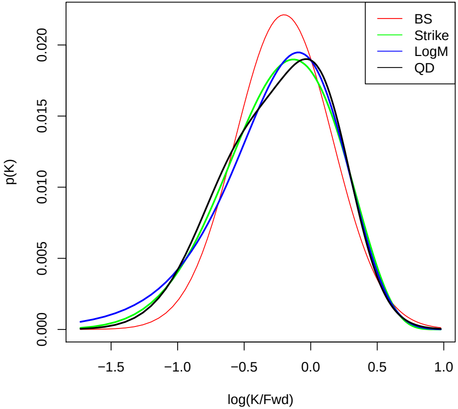


Figure: Implied density of F_T, the December 2009 WTI Nymex Futures contract. Calculation is performed by finite difference, with implied

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