Markowitz & Treynor-Black

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In this short note, we summarize the mathematical elements of the classical portfolio theory of Markowitz and Trenor-Black.

Arithmetic vs. Geometric mean

Let r_A and r_G be, respectively, the arithmetic and geometric means of a series of returns:

$$
\begin{aligned}
r_A &= \frac{1}{n}\sum_{k=1}^{n} r_k \\
r_G &= \prod_{k=1}^{n}(1+r_k)^{1/n} - 1
\end{aligned}
$$

and let V be the variance of r_k. We show that the geometric mean, which correctly represents the increase in wealth from an investment, is lower than the arithmetic mean.

The MacLaurin series for $(1+x)^{1/n}$ is:

$$
(1+x)^{\frac{1}{n}} = 1 + \frac{1}{n}x + \frac{1-n}{n^2}\frac{x^2}{2} + o(x^2)
$$

$$
r_G \approx \prod_{k=1}^{n}\left(1 + \frac{1}{n}r_k + \frac{1-n}{n^2}\frac{r_k^2}{2}\right) - 1
$$

Developping the product and keeping terms of order 2,

$$
r_G \approx \frac{1}{n}\sum_{k} r_k + \frac{1}{n^2}\sum_{k\neq l} r_k r_l + \frac{1-n}{2n^2} r_k^2
$$

$$
\begin{aligned}
r_G &\approx r_A - \frac{1}{2}\left[\frac{1}{n}\sum_k r_k^2 - \frac{1}{n^2}\left(\sum_k r_k^2 + 2\sum_{k\neq l} r_k r_l\right)\right] && (1)\\
&\approx r_A - \frac{1}{2}\left[\frac{1}{n}\sum_k r_k^2 - \left(\frac{1}{n}\sum_k r_k\right)^2\right] && (2)\\
&\approx r_A - \frac{1}{2}V, \;\; V >= 0 && (3)
\end{aligned}
$$

Quadratic Programming

QP with equality constraints

$$
\begin{aligned}
\min \;\; & \frac{1}{2} w^T \Sigma w \\
\text{s.t.} & \\
& A^T w = b
\end{aligned}
$$

Lagrangian:

$$
L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda^T \left(A^T w - b\right)
$$

First order conditions:

$$
\begin{cases}
\Sigma w - A\lambda = 0 \\
A^T w = b
\end{cases}
$$

or,

$$
\begin{bmatrix} \Sigma & -A \\ A^T & 0 \end{bmatrix}
\begin{bmatrix} w \\ \lambda \end{bmatrix}
=
\begin{bmatrix} 0 \\ b \end{bmatrix}
$$

Special case of Minimum Variance problem

$$
A = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad b = \mu^*
$$

Solution:

$$
w = \lambda \Sigma^{-1} A
$$

Normalize so that weights sum to 1:

$$w = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

Mean-Variance model (Markowitz, 1952)

$$\begin{aligned}
&\min\ \frac{1}{2}w^T\Sigma w \\
&\quad \text{s.t.} \\
&\mathbf{1}^T w = 1 \\
&R^T w = R_P
\end{aligned}$$

Lagrangian:

$$L(w, \lambda_1, \lambda_2) = \frac{1}{2}w^T\Sigma w - \lambda_1(\mathbf{1}^T w - 1) - \lambda_2(R^T w - R_P)$$

Solution of first order conditions:

$$\begin{cases} \Sigma w - \lambda_1\mathbf{1} - \lambda_2 R = 0 \\ \mathbf{1}^T w = 1 \\ R^T w = R_P \end{cases} \tag{4}$$

Determination of λ_1 and λ_2:

$$w = \Sigma^{-1}(\lambda_1\mathbf{1} + \lambda_2 R)$$

Define:

$$\begin{aligned}
a &= \mathbf{1}^T\Sigma^{-1}\mathbf{1} \\
b &= \mathbf{1}^T\Sigma^{-1}R \\
c &= R^T\Sigma^{-1}R
\end{aligned}$$

Substitute in (4):

$$\begin{cases} \lambda_1 a + \lambda_2 b = 1 \\ \lambda_1 b + \lambda_2 c = R_P \end{cases}$$

Solution:

$$
\lambda_1 = \frac{c - bR_P}{\Delta}
$$
$$
\lambda_2 = \frac{aR_P - b}{\Delta}
$$
$$
\Delta = ac - b^2
$$

Note that:

$$
\begin{aligned}
\sigma_P^2 &= w^{*T}\Sigma w^* \\
&= w^{*T}\Sigma\left(\lambda_1\Sigma^{-1}\mathbf{1} + \lambda_2\Sigma^{-1}R\right) \\
&= \lambda_1 + \lambda_2 R_P
\end{aligned}
$$

Two remarkable solutions:

- Minimum variance portfolio

$$
\begin{aligned}
\frac{\partial\sigma_P^2}{\partial R_P} &= 0 \implies \\
\frac{2aR_P - 2b}{\Delta} &= 0 \implies \\
R_P &= \frac{b}{a} \\
\sigma_P^2 &= \frac{1}{a} \\
\lambda_1 &= \frac{1}{a} \\
\lambda_2 &= 0
\end{aligned}
$$

The weights of the minimum variance portfolio:

$$
\begin{aligned}
w_g &= \lambda_1\Sigma^{-1}\mathbf{1} \\
&= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}
\end{aligned}
$$

- $\lambda_1 = 0$

This second solution gives $\lambda_2 = \frac{1}{b}$ and the optimal weights:

$$
\begin{aligned}
w_d &= \lambda_2\Sigma^{-1}R \\
&= \frac{\Sigma^{-1}R}{\mathbf{1}^T\Sigma^{-1}R}
\end{aligned}
$$

Theorem 1. *Any MV optimal portfolio w_P^* with expected excess return R_P can be decomposed into two MV portfolios.*

$$w_P^* = Aw_g + (1-A)w_d$$

Proof. Since w_P is MV optimal,

$$\begin{aligned} w_P &= \lambda_1 \Sigma^{-1}\mathbf{1} + \lambda_2 \Sigma^{-1} R \\ &= \lambda_1 a w_g + \lambda_2 b w_d \end{aligned}$$

One can verify that

$$\lambda_1 a + \lambda_2 b = 1$$

□

MV model with riskless asset

The tangency portfolio corresponds to the point on the efficient frontier where the slope of the tangent $\frac{R_M - r_f}{\sigma_M}$ is maximized, where:

$$\frac{R_M - r_f}{\sigma_M} = \frac{w^T(R - R_f)}{\sqrt{w^T \Sigma w}}$$

Noting that the slope is unchanged when the weights w are multiplied by a constant, the tangency portfolio is found by solving the following QP problem for an arbitrary $R^* > R_f$:

$$\begin{gathered} \min \ \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \\ \tilde{R}^T w = R^* \end{gathered}$$

with $\tilde{R} = R - R_f$.

Lagrangian:

$$L(w, \lambda) = \frac{1}{2} w^T \Sigma w - \lambda \left(\tilde{R}^T w - R^* \right)$$

Which yields:

$$w^* = \lambda^* \Sigma^{-1} \tilde{R} \tag{5}$$

Normalize so that the weights sum to 1:

$$w^* = \frac{\Sigma^{-1}\tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}} \tag{6}$$

The corresponding expected excess return is given by:

$$E(R_P^*) = \frac{\tilde{R}^T \Sigma^{-1} \tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}}$$

Maximum Sharpe ratio for two risky assets

Given two assets, A and M, the allocation that maximizes the Sharpe ratio is given by:

$$w_A = \frac{R_A \sigma_M^2 - R_M \sigma_A \sigma_M \rho_{AM}}{R_A \sigma_M^2 + R_M \sigma_A^2 - (R_A + R_M) \sigma_A \sigma_M \rho_{AM}} \tag{7}$$

Proof. Use equation (6) with

$$\Sigma = \begin{bmatrix} \sigma_A^2 & \rho \sigma_A \sigma_M \\ \rho \sigma_A \sigma_M & \sigma_M^2 \end{bmatrix} \tag{8}$$

$$\Sigma^{-1} = \frac{1}{(1-\rho^2)\sigma_A^2 \sigma_M^2} \begin{bmatrix} \sigma_M^2 & -\rho \sigma_A \sigma_M \\ -\rho \sigma_A \sigma_M & \sigma_A^2 \end{bmatrix}$$

□

Treynor-Black Model (Treynor & Black, 1973)

Assets excess return is modeled by a single factor model:

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where α_i is the idiosyncratic excess return of asset i, and $e_i \sim N(0, \sigma_i^2)$ is the specific risk.

Calculation of the active portfolio

The active portfolio is determined by the idiosyncratic excess return and the specific risk of each asset.

The specific risks are assumed to be independent:

$$\Sigma_A = \begin{bmatrix} \sigma^2(e_1) & & \\ & \ddots & \\ & & \sigma^2(e_n) \end{bmatrix}$$

Using equation (6), we get:

$$w_{Ai} = \frac{\alpha_i / \sigma_i^2}{\sum \alpha_i / \sigma_i^2}$$

So that the active portfolio has an excess return and variance given by:

$$\begin{aligned} R_A &= \alpha_A + \beta_A R_M \\ \sigma_A^2 &= \beta_A^2 \sigma_M^2 + \sigma^2(e_A) \end{aligned}$$

with

$$\begin{aligned} \alpha_A &= \sum w_{Ai}\alpha_i \\ \beta_A &= \sum w_{Ai}\beta_i \\ \sigma^2(e_A) &= \sum w_{Ai}^2 \sigma^2(e_i) \end{aligned}$$

Allocation of wealth between the active portfolio and the market portfolio

A fraction w_A of wealth is allocated to the active portfolio, and the balance $(1 - w_A)$ to the market portfolio so as to maximize the Sharpe ratio of the global portfolio $xA + (1 - x)M$.

Using equation (7) we get after some algebra:

$$w_A = \frac{\alpha_A \sigma_M^2}{\alpha_A \sigma_M^2 (1 - \beta_A) + R_M \sigma^2(e_A)}$$

Separability of the Sharpe ratio in the active portfolio

The first order condition for the optimal active portfolio is:

$$w_A = \lambda_A \Sigma^{-1} \alpha \tag{9}$$

Substitute in the expression

$$\alpha_A = w_A^T \alpha$$

to get:

$$\frac{\alpha_A}{\lambda_A} = \alpha^T \Sigma^{-1} \alpha \tag{10}$$

We next get an expression for λ_A in terms of known quantities:

$$\begin{aligned} \sigma^2(e_A) &= w_A^T \Sigma w_A \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \Sigma \Sigma^{-1} \alpha \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \alpha \end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{\sigma^2(e_A)}{\lambda_A^2} &= \alpha^T \Sigma^{-1} \alpha \\
&= \frac{\alpha_A}{\lambda_A}
\end{aligned}
$$

Which yields:

$$
\lambda_A = \frac{\sigma^2(e_A)}{\alpha_A}
$$

Use this result in equation (10) to get:

$$
\begin{aligned}
\frac{\alpha_A^2}{\sigma^2(e_A)} &= \alpha^T \Sigma^{-1} \alpha \\
&= \sum_i \frac{\alpha_i^2}{\sigma^2(e_i)}
\end{aligned}
$$

which shows that the square of the Sharpe ratio of the active portfolio is the sum of the squares of the Sharpe ratios of the assets forming that portfolio.

The Treynor-Black model in the notation of the 1973 paper and separability of the Sharpe ratio between the active and market portfolios

The investment universe is composed of n assets with asset-specific excess return:

$$
r_i = \alpha_i + \beta_i r_M + e_i \quad i = 1, \ldots, n \tag{11}
$$

$$
E(r_i) = \alpha_i + \beta_i E(r_M) = \mu_i \tag{12}
$$

and of the market asset itself. Let $w_i, i = 1, \ldots, n$ be the investment in the assets with asset-specific excess returns, and w_M the investment in the market asset.

Treynor and Black restate this portfolio as an investment in $n+1$ assets, asset 1 to n being only exposed to the specific risk, and the $n+1$ asset being only exposed to the market risk:

$$
w_{n+1} = w_M + \sum_{i=1}^{n} \beta_i w_i
$$

Note that these $n+1$ assets are independent. The mean and variance of the portfolio are:

$$E(r_P) = \sum_{i=1}^{n+1} w_i E(r_i) = \mu_P \tag{13}$$

$$\sigma_P^2 = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 \tag{14}$$

As usual, maximize the Sharpe ratio by solving:

$$\begin{aligned} \min \ & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} & \\ \mu^T w &= \mu_P \end{aligned}$$

Keeping in mind that the assets are independent, the Lagrangian is:

$$L(w, \lambda) = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 - 2\lambda \left(\sum_{i=1}^{n+1} w_i \mu_i - \mu_P \right)$$

First order conditions for optimality yield:

$$2 w_i \sigma_i^2 - 2\lambda \mu_i = 0 \quad i = 1, \dots, n+1$$

or,

$$w_i = \lambda \frac{\mu_i}{\sigma_i^2} \tag{15}$$

Substitute in (12) to get:

$$\mu_P = \lambda \sum_{i=1}^{n+1} \mu_i^2 / \sigma_i^2 \tag{16}$$

$$\sigma_P^2 = \lambda^2 \sum_{i=1}^{n+1} \mu_i^2 \sigma_i^2 \tag{17}$$

so that,

$$\lambda = \frac{\sigma_P^2}{\mu_P}$$

To summarize, the weights of the assets in the active portfolio are:

$$w_i = \frac{\mu_i}{\mu_P} \frac{\sigma_P^2}{\sigma_i^2} \quad i = 1, \dots, n$$

To determine the investment in the market asset, w_M, recall that,

$$\mu_{n+1} = E(r_M) = \mu_M \tag{18}$$
$$\sigma_{n+1}^2 = \sigma_M^2 \tag{19}$$

Thus,

$$w_{n+1} = \sum_{i=1}^{n} w_i \beta_i + w_M \tag{20}$$
$$= \lambda \frac{\mu_M}{\sigma_M^2} \tag{21}$$

From equation (15, we have:

$$\sum_{i=1}^{n} w_i \beta_i = \lambda \sum_{i=1}^{n} \frac{\beta_i \mu_i}{\sigma_i^2}$$

So that the investment in the market asset can be written as

$$w_M = \lambda \left[\frac{\mu_M}{\sigma_M^2} - \sum_{i=1}^{n} \frac{\beta_i \mu_i}{\sigma_i^2} \right]$$

To establish the separability of the Sharpe ratio between the active and the market portfolios, combine equations (16) and (17) to get:

$$\frac{\mu_P^2}{\sigma_P^2} = \sum_{i=1}^{n+1} \frac{\mu_i^2}{\sigma_i^2}$$

Denoting S_A, S_M, S_P the Sharpe ratios of, respectively, the active, market and overall portfolios, we can restate the previous equation as:

$$S_P^2 = \sum_{i=1}^{n} \frac{\mu_i^2}{\sigma_i^2} + S_M^2 \tag{22}$$
$$= \frac{\alpha_A^2}{\sigma_A^2 + S_M^2} \tag{23}$$
$$S_A^2 + S_M^2 \tag{24}$$

Treynor and Black call $\alpha_A = \sum_{i=1}^{n} w_i \alpha_i$ the "appraisal premium" and $\sigma_A^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2$ the "appraisal risk."

Bibliography

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