Physical System

Consider a pendulum on a cart attached to a spring. We assume that the cart can only move horizontally and that it has an equilibrium position which is the origin of the coordinate system. The pendulum is free to move so long that the distance from the center of mass of the cart is constant. We also assume the angle of the pendulum is measured relative to the vertical axis with the pendulum in the upright position.

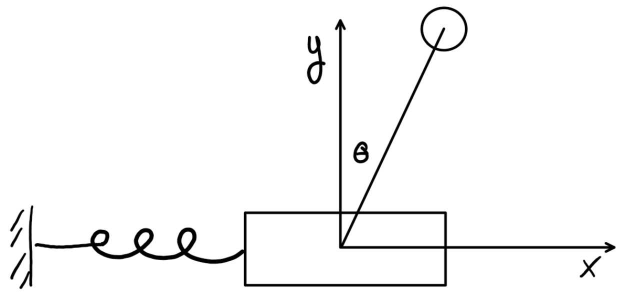

Figure 1: Pendulum on a cart

Lagrangian

For a mechanical system with a potential that does not depend on the generalized velocity, we may write the Lagrangian, $\mathcal{L}$, as the difference between the kinetic energy, T, and potential energy, V. The kinetic energy of the cart is simply given by,

$$T_c = \frac{1}{2}M\dot{x}^2,$$

where M is the mass of the cart and the overdot denotes the time derivative. The kinetic energy of the pendulum is given by,

$$T_p = \frac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right),$$

where m is the mass of the pendulum and the subscript p denotes a pendulum property. By trigonometry,

$$x_p = x + \ell\cos(\theta) \implies \dot{x}_p = \dot{x} - \ell\sin(\theta)\dot{\theta},$$
$$y_p = \ell\sin(\theta) \implies \dot{y}_p = \ell\cos(\theta)\dot{\theta}.$$

where ℓ is the length of pendulum string. Substituting the velocity expressions in the T_p equation,

$$\begin{aligned} T_p &= \frac{1}{2}m\dot{x}^2 + m\ell\dot{x}\cos(\theta)\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2\left[\cos^2(\theta) + \sin^2(\theta)\right] \\ &= \frac{1}{2}m\dot{x}^2 + m\ell\dot{x}\cos(\theta)\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2, \end{aligned}$$

where last term is simplified by the Pythagorean identity. The potential energy is given by the gravitational potential of the pendulum and elastic energy of spring,

$$V = mg\ell\cos(\theta) + \frac{1}{2}kx^2,$$

where g is the gravitational acceleration and k is the spring constant. Now that the kinetic and potential energies are known, we may write the Lagrangian,

$$\begin{aligned}\mathcal{L} &= T_c + T_p - V \\ &= \frac{1}{2}(m+M)\dot{x}^2 + m\ell\dot{x}\cos(\theta)\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos(\theta) - \frac{1}{2}kx^2.\end{aligned}$$

Hamiltonian

The Hamiltonian, $\mathcal{H}$, is formally defined as,

$$\mathcal{H} \coloneqq \mathbf{p}\cdot\dot{\mathbf{q}} - \mathcal{L}.$$

$\dot{\mathbf{q}}$ is the vector of generalized velocities and $\mathbf{p}$ is the vector of conjugate momenta, defined as,

$$p_i \coloneqq \frac{\partial \mathcal{L}}{\partial \dot{q}^i},$$

for all generalized velocities, $\dot{q}^i$. Since we are working with a mechanical system with $V = V(\mathbf{q})$, the Hamiltonian is more simply obtained by the total energy of the system,

$$\begin{aligned}\mathcal{H} &= T + V \\ &= \frac{1}{2}(m+M)\dot{x}^2 + m\ell\dot{x}\cos(\theta)\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta) + \frac{1}{2}kx^2.\end{aligned}$$

We must reexpress the above equation in terms the momenta, p_i, instead of the velocities, $\dot{q}^i$. To this end, we calculate the momenta by their definition,

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m+M)\dot{x} + m\ell\cos(\theta)\dot{\theta},$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell\dot{x}\cos(\theta) + m\ell^2\dot{\theta}.$$

The above relations may be conveniently be written in matrix form,

$$\underbrace{\begin{pmatrix} p_x \\ p_\theta \end{pmatrix}}_{\mathbf{p}} = \underbrace{\begin{pmatrix} m+M & m\ell\cos(\theta) \\ m\ell\cos(\theta) & m\ell^2 \end{pmatrix}}_{\mathbf{M}} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}}_{\dot{\mathbf{q}}},$$

where $\mathbf{M}$ is the (symmetric) mass matrix. Thus, symbolically,

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \mathbf{M}^{-1}\mathbf{p}.$$

Using this, the kinetic energy may be written as,

$$\begin{aligned} T &= \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}} \\ &= \frac{1}{2}\left(\mathbf{M}^{-1}\mathbf{p}\right)^{\mathrm{T}}\mathbf{M}\left(\mathbf{M}^{-1}\mathbf{p}\right) \\ &= \frac{1}{2}\mathbf{p}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{p}, \end{aligned}$$

where we have used the fact that $\mathbf{M}$ is symmetric and $\left(\mathbf{M}^{-1}\mathbf{p}\right)^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{M}^{-1}$. Thus, writing the Hamiltonian becomes a matter of inverting $\mathbf{M}$. Since we have a 2×2 system, this is readily done using standard formulas,

$$\begin{aligned} \mathbf{M}^{-1} &= \frac{1}{(m+M)m\ell^2 - m^2\ell^2\cos^2(\theta)}\begin{pmatrix} m\ell^2 & -m\ell\cos(\theta) \\ -m\ell\cos(\theta) & m+M \end{pmatrix} \\ &= \frac{1}{mM\ell^2 + m^2\ell^2\sin^2(\theta)}\begin{pmatrix} m\ell^2 & -m\ell\cos(\theta) \\ -m\ell\cos(\theta) & m+M \end{pmatrix}, \end{aligned}$$

where the denominator term simplifies by the Pythagorean identity. We now proceed by applying Hamilton's equations,

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i}.$$

Since V does not depend on the momenta, the first set of equations for $\dot{q}^i$ are simply given by,

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \mathbf{M}^{-1}\mathbf{p}.$$

Performing this calculation yields,

$$\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} = \frac{1}{D}\begin{pmatrix} m\ell^2 p_x - m\ell\cos(\theta)p_\theta \\ -m\ell\cos(\theta)p_x + (m+M)p_\theta \end{pmatrix},$$

where D is a determinant given by,

$$D = mM\ell^2 + m^2\ell^2\sin^2(\theta).$$

The equation for p_x is simple since x only appears in the elastic energy term, giving,

$$\frac{dp_x}{dt} = -\frac{\partial \mathcal{H}}{\partial x} = -kx.$$

The most involved calculation is the equation for p_θ since many terms depend on θ. Symbolically,

$$\frac{dp_\theta}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{1}{2}\frac{\partial}{\partial \theta}\left(\mathbf{p}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{p}\right) - \frac{\partial}{\partial \theta}\left(mg\ell\cos(\theta)\right).$$

We treat $\mathbf{p}$ as independent of θ, meaning the kinetic energy derivative is given by,

$$\frac{\partial T}{\partial \theta} = \frac{1}{2}\mathbf{p}^{\mathrm{T}} \left(\frac{\partial \mathbf{M}^{-1}}{\partial \theta} \right) \mathbf{p},$$

thus, we must calculate a matrix derivative. By the chain rule,

$$\begin{aligned}
\left(\frac{\partial \mathbf{M}^{-1}}{\partial \theta} \right) &= \frac{-m^2\ell^2 \sin(\theta)\cos(\theta)}{D^2} \begin{pmatrix} m\ell^2 & -m\ell\cos(\theta) \\ -m\ell\cos(\theta) & m+M \end{pmatrix} \\
&\quad + \frac{1}{D} \begin{pmatrix} 0 & m\ell\sin(\theta) \\ m\ell\sin(\theta) & 0 \end{pmatrix} \\
&= \frac{1}{D^2} \begin{pmatrix} -m^3\ell^4 \sin(\theta)\cos(\theta) & m^3\ell^3\sin(\theta)\cos^2(\theta) + Dm\ell\sin(\theta) \\ m^3\ell^3\sin(\theta)\cos^2(\theta) + Dm\ell\sin(\theta) & -(m+M)m^2\ell^2\sin(\theta)\cos(\theta) \end{pmatrix},
\end{aligned}$$

We may simplify the off diagonal terms by expanding the determinant,

$$\begin{aligned}
m^3\ell^3\sin(\theta)\cos^2(\theta) + Dm\ell\sin(\theta) &= m^3\ell^3\sin(\theta)\cos^2(\theta) + m^2M\ell^3\sin(\theta) + m^3\ell^3\sin^3(\theta) \\
&= \sin(\theta)\left(m^2M\ell^3 + m^3\ell^3\right) \\
&= (m+M)m^2\ell^3\sin(\theta).
\end{aligned}$$

Plugging in these simplifications,

$$\begin{aligned}
\left(\frac{\partial \mathbf{M}^{-1}}{\partial \theta} \right) &= \frac{1}{D^2} \begin{pmatrix} -m^3\ell^4\sin(\theta)\cos(\theta) & (m+M)m^2\ell^3\sin(\theta) \\ (m+M)m^2\ell^3\sin(\theta) & -(m+M)m^2\ell^2\sin(\theta)\cos(\theta) \end{pmatrix} \\
&= \frac{m^2\ell^2\sin(\theta)}{D^2} \begin{pmatrix} -m\ell^2\cos(\theta) & (m+M)\ell \\ (m+M)\ell & -(m+M)\cos(\theta) \end{pmatrix}.
\end{aligned}$$

With this matrix, the p_θ equation may now be readily calculated,

$$\frac{dp_\theta}{dt} = \frac{m^2\ell^2\sin(\theta)}{D^2}\left[m\ell^2\cos(\theta)p_x^2 - 2(m+M)\ell p_x p_\theta + (m+M)\cos(\theta)p_\theta^2\right] + mg\ell\sin(\theta).$$

In summary, Hamilton's equations for this system are given by,

$$\frac{dx}{dt} = \frac{m\ell^2 p_x - m\ell\cos(\theta)p_\theta}{D},$$

$$\frac{dp_x}{dt} = -kx,$$

$$\frac{d\theta}{dt} = \frac{-m\ell\cos(\theta)p_x + (m+M)p_\theta}{D},$$

$$\frac{dp_\theta}{dt} = \frac{m^2\ell^2\sin(\theta)}{D^2}\left[m\ell^2\cos(\theta)p_x^2 - 2(m+M)\ell p_x p_\theta + (m+M)\cos(\theta)p_\theta^2\right] + mg\ell\sin(\theta).$$

This is a dynamical system of the form,

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}),$$

which may be readily integrated numerically given intitial conditions.