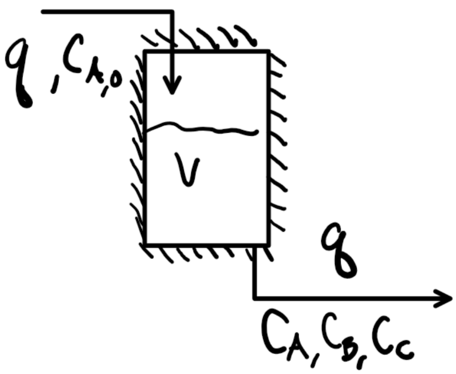
# 1)

**System:** Tank with contents and valves




- A $\xrightarrow{k_1}$ B, A $\xrightarrow{k_2}$ C
- $r_A = -(k_1 + k_2)C_A$, $r_B = k_1 C_A$, $r_C = k_2 C_A$
- $\rho, V, q =$ const.
- Perfect mixing, Adiabatic, Isothermal
- $C_{B,0} = C_{C,0} = 0$

**Variables:**

| Inputs | Units |
|---|---|
| $C_{A,0}$ | kmol/m$^3$ |
| **Outputs** | **Units** |
| $C_A, C_B, C_C$ | mol/m$^3$ |

**Parameters (Constant):**

| Var. | Units |
|---|---|
| $q$ | m$^3$/min |
| $V$ | m$^3$ |
| $k_1, k_2$ | min$^{-1}$ |

## Component Balances

Since $V$ is constant,

$$V\frac{dC_A}{dt} = q\left(C_{A,0} - C_A\right) + r_A V,$$

Using $r_A = -(k_1 + k_2)C_A$ and collecting the $C_A$ terms,

$$V\frac{dC_A}{dt} + \left[q + (k_1 + k_2)V\right]C_A = qC_{A_0}.$$

Isolating for $C_A$,

$$\left[\frac{V}{q + (k_1 + k_2)V}\right]\frac{dC_A}{dt} + C_A = \left[\frac{q}{q + (k_1 + k_2)V}\right]C_{A_0}.$$

Let,

$$\boxed{\tau_A = \frac{V}{q + (k_1 + k_2)V}, \quad K_A = \frac{q}{q + (k_1 + k_2)V}},$$

then,

$$\boxed{\tau_A\frac{dC_A}{dt} + C_A = K_A C_{A_0}}.$$

The B balance follows similarly, but with $C_{B,0} = 0$,

$$V\frac{dC_B}{dt} = -qC_B + k_1 C_A V,$$

$$\implies \left[\frac{V}{q}\right]\frac{dC_B}{dt} + C_B = \left[\frac{k_1 V}{q}\right] C_A.$$

Let,

$$\boxed{\tau_B = \frac{V}{q}, \quad K_B = \frac{k_1 V}{q}},$$

then,

$$\boxed{\tau_B \frac{dC_B}{dt} + C_B = K_B C_A}.$$

By analogy, C is nearly the same as B,

$$V\frac{dC_C}{dt} = -qC_C + k_2 C_A V,$$

$$\implies \left[\frac{V}{q}\right]\frac{dC_C}{dt} + C_C = \left[\frac{k_2 V}{q}\right] C_A.$$

Let,

$$\boxed{\tau_C = \frac{V}{q}, \quad K_C = \frac{k_2 V}{q}}.$$

then,

$$\boxed{\tau_C \frac{dC_C}{dt} + C_C = K_C C_A}.$$

## Transfer Functions

Define steady state and deviation variables. The steady state variables are found by setting the time derivatives to 0,

$$\overline{C}_A = K_A \overline{C}_{A,0}, \quad \overline{C}_B = K_B \overline{C}_A, \quad \overline{C}_C = K_C \overline{C}_A.$$

Therefore, subtracting the steady state variables from the balance equations gives the deviation form,

$$\tau_A \frac{dC'_A}{dt} + C'_A = K_A C'_{A,0}, \quad \tau_B \frac{dC'_B}{dt} + C'_B = K_B C'_A, \quad \tau_C \frac{dC'_C}{dt} + C'_C = K_C C'_A,$$

where,

$$C'_i = C_i - \overline{C}_i.$$

Taking the Laplace transform of each balance equation,

$$C'_A(s)\left[\tau_A s + 1\right] = K_A C'_{A,0}(s),$$

$$C'_B(s)\left[\tau_B s + 1\right] = K_B C'_A(s),$$

$$C'_C(s)\left[\tau_C s + 1\right] = K_C C'_A(s).$$

Rearranging,

$$C'_A(s) = \left[\frac{K_A}{\tau_A s + 1}\right] C'_{A,0}(s),$$

$$C'_B(s) = \left[\frac{K_B}{\tau_B s + 1}\right] C'_A(s) = \left[\frac{K_B}{\tau_B s + 1}\right]\left[\frac{K_A}{\tau_A s + 1}\right] C'_{A,0}(s),$$

$$C'_C(s) = \left[\frac{K_C}{\tau_C s + 1}\right] C'_A(s) = \left[\frac{K_C}{\tau_C s + 1}\right]\left[\frac{K_A}{\tau_A s + 1}\right] C'_{A,0}(s).$$

Thus, the transfer functions are,

$$\boxed{\begin{aligned} C'_A(s) \leftrightarrow C'_{A,0}(s): &\quad G_A(s) = \frac{K_A}{\tau_A s + 1} \\ C'_B(s) \leftrightarrow C'_{A,0}(s): &\quad G_B(s) = \left[\frac{K_B}{\tau_B s + 1}\right]\left[\frac{K_A}{\tau_A s + 1}\right] \\ C'_C(s) \leftrightarrow C'_{A,0}(s): &\quad G_C(s) = \left[\frac{K_C}{\tau_C s + 1}\right]\left[\frac{K_A}{\tau_A s + 1}\right] \end{aligned}}.$$

The numerical values of the time constants and gains are given in the following table.

| $i$ | $\tau_i$ (min) | $K_i$ $(\cdot)$ |
|---|---|---|
| A | 4 | 0.4 |
| B | 10 | 1 |
| C | 10 | 0.5 |

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# 2)

## a.

$$V\frac{dP}{dt} = C_0 P_0 - C_1 P_1 (P - P_1)$$

Let,

$$f(P_0, P_1, P) = C_0 P_0 - C_1 P_1 (P - P_1).$$

By a Taylor series expansion about the steady state,

$$\begin{aligned} f' &= \left(\frac{\partial f}{\partial P_0}\right)_{SS} P'_0 + \left(\frac{\partial f}{\partial P_1}\right)_{SS} P'_1 + \left(\frac{\partial f}{\partial P}\right)_{SS} P' \\ &= [C_0]\, P'_0 + \left[-C_1\overline{P} + 2C_1\overline{P}_1\right] P'_1 - \left[C_1\overline{P}_1\right] P'. \end{aligned}$$

Plugging this back into the original model,

$$V\frac{dP'}{dt} + \left[C_1\overline{P}_1\right] P' = [C_0]\, P'_0 + \left[-C_1\overline{P} + 2C_1\overline{P}_1\right] P'_1.$$

By rearrangement,

$$\left[\frac{V}{C_1\overline{P}_1}\right]\frac{dP'}{dt}+P'=\left[\frac{C_0}{C_1\overline{P}_1}\right]P_0'+\left[\frac{-C_1\overline{P}+2C_1\overline{P}_1}{C_1\overline{P}_1}\right]P_1'.$$

Define,

$$\boxed{\tau=\frac{V}{C_1\overline{P}_1},\quad K_0=\frac{C_0}{C_1\overline{P}_1},\quad K_1=\frac{-\overline{P}+2\overline{P}_1}{\overline{P}_1}}.$$

Therefore,

$$\tau\frac{dP'}{dt}+P'=K_0P_0'+K_1P_1'.$$

Taking the Laplace transform,

$$P'(s)\left[\tau s+1\right]=K_0P_0'(s)+K_1P_1'(s),$$

or by rearrangement,

$$P'(s)=\left[\frac{K_0}{\tau s+1}\right]P_0'(s)+\left[\frac{K_1}{\tau s+1}\right]P_1'(s).$$

Thus, the transfer functions are,

$$\boxed{\begin{aligned}P_0'(s)\leftrightarrow P'(s):&\quad G_0(s)=\frac{K_0}{\tau s+1}\\ P_1'(s)\leftrightarrow P'(s):&\quad G_1(s)=\frac{K_1}{\tau s+1}\end{aligned}}.$$

The numerical values of the time constants and gains are given in the following table.

| $i$ | $\tau_i$ (min) | $K_i$ $(\cdot)$ |
|---|---|---|
| 0 | 2.083 | 0.1667 |
| 1 | 2.083 | 0.6667 |

## b.

By the linear model from part a, the time derivative is set to 0,

$$P'=K_0\overline{P_0'}+K_1\overline{P_1'}.$$

By the definition of deviation variables,

$$\begin{aligned}P&=\overline{P}+K_0\left(P_0-\overline{P}_0\right)+K_1\left(P_1-\overline{P}_1\right)\\&=40\text{ psia}+(0.1667)\,(60-60)\text{ psia}+(0.6667)\,(20-30)\text{ psia}\\&=33.3\text{ psia}.\end{aligned}$$

Therefore, the new steady state is $\boxed{P = 33.3 \text{ psia}}$. For the nonlinear model, take the original equation and set the time derivative to 0,

$$0 = C_0\overline{P}_0 - C_1\overline{P}_1\left(\overline{P} - \overline{P}_1\right),$$

solving for $\overline{P}$,

$$\overline{P} = \overline{P}_1 + \frac{C_0\overline{P}_0}{C_1\overline{P}_1}.$$

Numerically,

$$\boxed{\overline{P} = 20 \text{ psia} + \frac{\left(4 \ \frac{\text{ft}^3}{\text{min}}\right)(60 \text{ psia})}{\left(0.8 \ \frac{\text{ft}^3}{\text{min}\cdot\text{psia}}\right)(20 \text{ psia})} = 35 \text{ psia}}.$$

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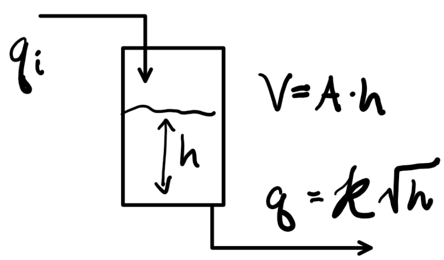
## 3)

**System:** Tank with contents and valves

- $A, \rho = \text{const.}$
- $q = k\sqrt{h}$




**Variables:**

| Inputs | Units |
|---|---|
| $q_i$ | $\text{ft}^3/\text{hr}$ |
| **Outputs** | **Units** |
| $h$ | ft |

**Parameters (Constant):**

| Var. | Units |
|---|---|
| $A$ | $\text{ft}^2$ |
| $k$ | $\text{ft}^3/\text{hr}/\text{ft}^{1/2}$ |

### a.

By a mass balance,

$$\frac{d(\rho V)}{dt} = \rho\left(q_i - q\right).$$

Using $\rho, A = \text{const.}$ and $q = k\sqrt{h}$,

$$\boxed{A\frac{dh}{dt} = q_i - k\sqrt{h}}.$$

## b.

Let $f(q_i, h) = q_i - k\sqrt{h}$, then by a Taylor series expansion about the steady state,

$$\begin{aligned} f' &= \left(\frac{\partial f}{\partial q_i}\right)_{ss} q_i' + \left(\frac{\partial f}{\partial h}\right)_{ss} h' \\ &= [1]\, q_i' - \left[\frac{k}{2\bar{h}^{1/2}}\right] h'. \end{aligned}$$

Plugging this into the model,

$$A\frac{dh'}{dt} + \left[\frac{k}{2\bar{h}^{1/2}}\right] h' = q_i'$$

isolating for $h'$,

$$\left[\frac{2A\bar{h}^{1/2}}{k}\right] \frac{dh'}{dt} + h' = \left[\frac{2\bar{h}^{1/2}}{k}\right] q_i'.$$

Let,

$$\boxed{\tau = \frac{2A\bar{h}^{1/2}}{k}, \quad K = \frac{2\bar{h}^{1/2}}{k}},$$

then,

$$\tau\frac{dh'}{dt} + h' = Kq_i'.$$

Taking the Laplace transform and rearranging,

$$h'(s) = \left[\frac{K}{\tau s + 1}\right] q_i'(s).$$

Therefore, the transfer function of the linearized model is,

$$\boxed{h'(s) \leftrightarrow q_i'(s): \quad G(s) = \frac{K}{\tau s + 1}}.$$

## c.

Calculate the cross-sectional area of the tank, $A$,

$$A = \frac{\pi D^2}{4} = \frac{\pi\,(36 \text{ ft}^2)}{4} = 28.27 \text{ ft}^2.$$

At steady state, $q = k\sqrt{h}$ must still hold, thus,

$$k = \frac{\bar{q}}{\sqrt{\bar{h}}} = \frac{16 \text{ ft}^3/\text{hr}}{(5 \text{ ft})^{1/2}} = 7.155 \text{ ft}^3/\text{hr}/\text{ft}^{1/2}.$$

By the definition of $\tau$,

$$\boxed{\tau = \frac{2A\bar{h}^{1/2}}{k} = \frac{2(28.27 \text{ ft}^2)(5 \text{ ft})^{1/2}}{7.155 \text{ ft}^3/\text{hr}/\text{ft}^{1/2}} = 17.67 \text{ hr}}.$$

By the definition of $K$,

$$\boxed{K = \frac{2\overline{h}^{1/2}}{k} = \frac{2(5\text{ ft})^{1/2}}{7.155\text{ ft}^3/\text{hr}/\text{ft}^{1/2}} = 0.625\text{ hr}/\text{ft}^2}.$$

## d.

At the new steady state, the linearized model predicts,

$$h' = Kq_i',$$

or,

$$h = \overline{h} + K\left(q_i - \overline{q}_i\right).$$

Numerically,

$$\boxed{h = 5\text{ ft} + \left(0.625\text{ hr}/\text{ft}^2\right)(24 - 16)\,\text{ft}^3/\text{hr} = 10\text{ ft}}.$$

Under the linearized model, there is no problem since the liquid level is exactly 10 ft, which is the height of the tank. However, this is a linearized model, and does not account for nonlinear effects. Using the original dynamic model, the steady state condition is given as,

$$\overline{h} = \left(\frac{\overline{q}_i}{k}\right)^2 = \left(\frac{16\text{ ft}^3/\text{hr}}{7.155\text{ ft}^3/\text{hr}/\text{ft}^{1/2}}\right)^2 = 11.25\text{ ft}.$$

This is larger than the tank height, which is problematic.