Laminar Boundary Layer Over a Flat Plate

Scaling Analysis

The 2D incompressible Navier-Stokes and continuity equations give,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g_x - \frac{1}{\rho}\frac{\partial P}{\partial x},$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + g_y - \frac{1}{\rho}\frac{\partial P}{\partial y}.$$

To simplify analysis, we will only consider steady flow and neglect gravity, meaning the time derivative will vanish and $\rho\mathbf{g} = \mathbf{0}$. To further simplify the equations, an order of magnitude estimate may be performed in terms of "characteristic" variables. Let capital letters denote the characteristic variables, then the above equations approximately become,

$$\frac{U}{X} + \frac{V}{Y} \sim 0,$$

$$\rho\left(U\frac{U}{X} + V\frac{U}{Y}\right) \sim \mu\left(\frac{U}{X^2} + \frac{U}{Y^2}\right) - \frac{P}{X},$$
$$\rho\left(U\frac{V}{X} + V\frac{V}{Y}\right) \sim \mu\left(\frac{V}{X^2} + \frac{V}{Y^2}\right) - \frac{P}{Y}.$$

If we consider the x direction as the main flow direction, we expect that the X variable will be much larger than the Y variable. If this was not the case, then there would be a significant flow in the y direction since the characteristic continuity equation implies that V may be written as,

$$V \sim \frac{UY}{X}.$$

To have a large X and a small V, we must have that Y cannot be large; mathematically, we impose that $Y << X$. Using this information in the characteristic x momentum equation,

$$\rho\left(U\frac{U}{X} + \frac{UY}{X}\frac{U}{Y}\right) \sim \mu\left(\frac{U}{X^2} + \frac{U}{Y^2}\right) - \frac{P}{X},$$

we see that the U/X^2 term drops out since X is large. We also see that both terms of the convective part of the equation are the same order of magnitude, U^2/X. That is,

$$u\frac{\partial u}{\partial x} \sim v\frac{\partial u}{\partial y}.$$

By combining these arguments, we have,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial P}{\partial x}.$$

Additionally, since,

$$V \sim \frac{UY}{X},$$

we can replace V the characteristic y momentum equation to find,

$$\frac{Y}{X}\left[\rho\left(U\frac{U}{X} + V\frac{U}{Y}\right)\right] \sim \frac{Y}{X}\left[\mu\left(\frac{U}{X^2} + \frac{U}{Y^2}\right)\right] - \frac{P}{Y}.$$

The square bracketed terms are found in the x momentum equation and are of the order of magnitude were are interested in. However, these terms are also scaled by a factor of Y/X. Since we assumed $Y << X$, the bracketed terms must vanish as X becomes very large. Therefore, in the limit, we have,

$$0 \sim -\frac{P}{Y},$$

which implies (approximately),

$$\frac{\partial P}{\partial y} = 0,$$

or $P = P(x)$. In total, we have,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial P}{\partial x},$$

$$\frac{\partial P}{\partial y} = 0.$$

We may further simplify these equations by assuming that the pressure gradient is small since Y is small and X is large. We can therefore assume that the pressure gradient is approximately given by the invicid solution to the Navier-Stokes equations (Bernoulli equation),

$$\frac{1}{\rho}\frac{dP}{dx} + U\frac{dU}{dx} + \frac{d(gy)}{dx} = 0,$$

or,

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx}.$$

Thus, we have,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + U\frac{dU}{dx},$$

For a flat plate, U is constant, thus,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}.$$

The boundary conditions are no slip at the plate surface ($y = 0$) and as y grows, the u velocity should approach the invicid solution, U. Mathematically,

$$u\Big|_{y=0} = 0, \qquad v\Big|_{y=0} = 0, \qquad u\Big|_{y\to\infty} = U.$$

Blasius Solution

The flat plate problem was solved by P.R.H. Blasius in 1908 by the method of similarity variables [1]. The Blasius solution starts with a common step in 2D incompressible flows by introducing the stream function, Ψ, defined such that,

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x},$$

which implies,

$$\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y \partial x} = 0,$$

meaning that Ψ automatically satisfies the continuity equation. For convenience, we hereby denote partial differentiation by subscripts,

$$u = \Psi_y, \quad v = -\Psi_x.$$

Plugging this into the momentum equation yields,

$$\Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} = \nu \Psi_{yyy}.$$

We seek to reduce this non-linear PDE into an ODE. A common method for this is the method of similarity variables in which we guess a solution of the form,

$$\lambda^a \Psi(\lambda^b x, \lambda^c y) = \Psi(x, y),$$

for some positive constant, λ. The PDE is said to be 'dilation invariant' under this scaling transformation of the variables [7]. The motivation for this is to notice that if we impose the constraints,

$$\lambda = x, \quad b = -1,$$

then the scaling transformation becomes,

$$x^a \Psi(1, x^c y) = \Psi(x, y).$$

the LHS implies that under this scaling, Ψ only depends on the variable $x^c y$. If we choose a and c carefully, we may be able to eliminate x and y explicitly from the PDE and work only with the product variable, $x^c y$. The motivation for the constraints of $\lambda = x$ and $b = -1$ is that y derivatives appear more frequently in the simplified momentum equation. If we set $\lambda = y$, then there will be a y^a term instead of a x^a term multiplying the transformed Ψ. This would make the y derivatives harder to evaluate and makes in difficult to find the constraints on a and c, thus we avoid $\lambda = y$. The $b = -1$ constraint is to simply remove the explicit x dependence of Ψ, which is the primary goal of similarity transformations. For transformation to be physically meaningful, we must also choose a constant, C, such that,

$$Cx^c y,$$

is dimensionless. For convenience, let,

$$\eta = Cx^c y,$$

and,

$$\Psi(1, Cx^c y) = Af(\eta),$$

where f is dimensionless and A is a constant with the units of Ψ. Furthermore, let,

$$\frac{df}{d\eta} = f'.$$

We can thus evaluate the relevant derivatives (by liberal use of the chain rule). For η,

$$\frac{\partial \eta}{\partial x} = cCx^{c-1}y = \frac{c\eta}{x},$$

$$\frac{\partial \eta}{\partial y} = Cx^c = \frac{\eta}{y},$$

Recall $\Psi(x, y)$ is given by,

$$\Psi(x, y) = Ax^a f(\eta).$$

Taking the x derivative,

$$\begin{aligned}
\Psi_x &= \frac{\partial}{\partial x}\left(Ax^a f\right) \\
&= A\left(ax^{a-1}f + x^a f' \frac{c\eta}{x}\right) \\
&= Ax^{a-1}\left(af + c\eta f'\right).
\end{aligned}$$

and the y derivative,

$$\begin{aligned}
\Psi_{xy} &= \frac{\partial}{\partial y}\left(Ax^{a-1}\left(af + c\eta f'\right)\right) \\
&= Ax^{a-1}\frac{\eta}{y}\left(af' + cf' + c\eta f''\right).
\end{aligned}$$

3rd order derivatives in y are required and given by,

$$\begin{aligned}
\Psi_y &= \frac{\partial}{\partial y}\left(Ax^a f\right) \\
&= Ax^a \frac{\eta}{y} f'.
\end{aligned}$$

$$\Psi_{yy} = \frac{\partial}{\partial y}\left(Ax^a \frac{\eta}{y} f'\right)$$
$$= Ax^a \left(\frac{\eta}{y}\right)^2 f''.$$

$$\Psi_{yyy} = \frac{\partial}{\partial y}\left(Ax^a \left(\frac{\eta}{y}\right)^2 f''\right)$$
$$= Ax^a \left(\frac{\eta}{y}\right)^3 f'''$$

Plugging these into the PDE,

$$A^2 x^{2a-1}\left(\frac{\eta}{y}\right)^2 f'\left(af' + cf' + c\eta f''\right) - A^2 x^{2a-1}\left(\frac{\eta}{y}\right)^2 f''\left(af + c\eta f'\right) = \nu A x^a \left(\frac{\eta}{y}\right)^3 f''',$$

$$\implies A^2 x^{2a-1}\left(\frac{\eta}{y}\right)^2 \left[af'f' + cf'f' + c\eta f''f' - af''f - c\eta f''f'\right] = \nu A x^a \left(\frac{\eta}{y}\right)^3 f''',$$

$$\implies \frac{Ax^{a-1}}{\nu}\left(\frac{y}{\eta}\right)\left[af'f' + cf'f' - af''f\right] = f'''.$$

Using the definition of η, the y/η term may be put in terms of x,

$$\frac{Ax^{a-1}}{\nu}\left(\frac{y}{\eta}\right) = \frac{A}{C\nu}x^{a-c-1}.$$

We wish for x to disappear from the PDE, thus, we must choose a and c such that,

$$a - c - 1 = 0.$$

The second condition on a and c comes from the boundary condition

$$\Psi_y\Big|_{y\to\infty} = U,$$

which implies,

$$ACx^{a+c} f'\Big|_{\eta\to\infty} = U,$$

from which we desire,

$$a + c = 0.$$

Using these conditions, we find that,

$$a = 1/2, \quad c = -1/2.$$

Thus, the PDE becomes an ODE,

$$f''' = -\frac{A}{2C\nu} f'' f.$$

Since f''' and $f''f$ are dimensionless, we must have that,

$$\frac{A}{2C\nu},$$

is dimensionless. Additionally, from the boundary conditions, we have,

$$\frac{U}{AC},$$

is dimensionless. Taking the ratios of these quantities,

$$\left[\frac{1}{2}A^2\frac{1}{U\nu}\right] = 1,$$

where $[\,\cdot\,]$ denotes "the units of". The simplest way to make the LHS unitless is to choose,

$$A = \sqrt{U\nu}.$$

Meaning for C,

$$\left[\frac{AC}{U}\right] = 1,$$

$$\implies [C] = \left[\frac{U}{A}\right],$$

which is satisfied by choosing,

$$C = \sqrt{\frac{U}{\nu}}.$$

Meaning we have,

$$\frac{A}{2C\nu} = \frac{1}{2}.$$

At this point, we must find the remaining boundary conditions for f to solve the ODE. The u no slip condition gives,

$$u\Big|_{y=0} = \Psi_y\Big|_{y=0} = ACx^{a+c}f'\Big|_{y=0} = 0,$$

which implies,

$$f'\Big|_{\eta=0} = 0.$$

The v no slip condition gives,

$$v\Big|_{y=0} = -\Psi_x\Big|_{y=0} = -Ax^{a-1}\left(af + c\eta f'\right)\Big|_{y=0} = 0,$$

which implies,

$$f\Big|_{\eta=0} = 0.$$

In total,

$$f''' + \frac{1}{2}f''f = 0,$$

$$f(0)=0, \quad f'(0)=0, \quad f'(\infty)=1.$$

Where the main physical quantities are given by,

$$u = Uf',$$

$$v = \frac{1}{2}\sqrt{\frac{U\nu}{x}}\left(\eta f' - f\right),$$

$$\eta = y\sqrt{\frac{U}{\nu x}},$$

$$\tau_{yx} = \mu\frac{\partial u}{\partial y} = \mu U\sqrt{\frac{U}{\nu x}}f''.$$

Numerical Solution

The derived equation for f is 3rd order non-linear ODE which cannot be solved in closed form. Blasius originally used a power series solution along with asymptotic methods to give an approximate form of f [3]. However, with the advent of computers, ODE's of this form can be readily integrated with methods such as with a Runge-Kutta scheme. Often, these integrators take in a system of first order equations. This ODE can easily be transformed to a first order system by defining,

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix},$$

which implies,

$$\frac{d}{d\eta}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \\ -0.5 f_1 f_3 \end{pmatrix},$$

which is a first order system for (f_f, f_2, f_3).

Most integrators can only solve initial values problems and not boundary value problems. We have two conditions on f and f' at $\eta = 0$, but the remaining boundary condition at $\eta \to \infty$ poses a problem for the integrator. The solution is simply to guess an initial condition on f'' and see if it satisfies the boundary condition at $\eta \to \infty$. Practically, we also just choose a large value for η to approximate the asymptotic condition. Explicitly, we let the initial condition vector be,

$$\begin{pmatrix} f_1(0) \\ f_2(0) \\ f_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix},$$

where a is to be guessed. This amounts to a root finding problem; we define the objective function as,

$$\mathcal{L}(a) = f_2(\eta = N, a) - 1,$$

where N is a large number. We wish to solve for the a such that $\mathcal{L}(a) = 0$, this can be done iteratively using the secant method,

$$a_{n+1} = a_n - \mathcal{L}(a_n)\frac{a_n - a_{n-1}}{\mathcal{L}(a_n) - \mathcal{L}(a_{n-1})},$$

which evaluates to,

$$a_{n+1} = a_n - [f_2(a_n) - 1]\,\frac{a_n - a_{n-1}}{f_2(a_n) - f_2(a_{n-1})}.$$

This can be simply iterated until convergence is met, say,

$$|a_{n+1} - a_n| < \varepsilon.$$

where ε is as small as we would like. Here, we implicitly assume that the solutions, f_2, do not significantly vary significantly between iterations and that the initial guess is "sufficiently close" to the real solution.

Implementation of the numerical scheme with an RK45 integrator and initial guess of $a = 0.5$ is shown in the following figure.

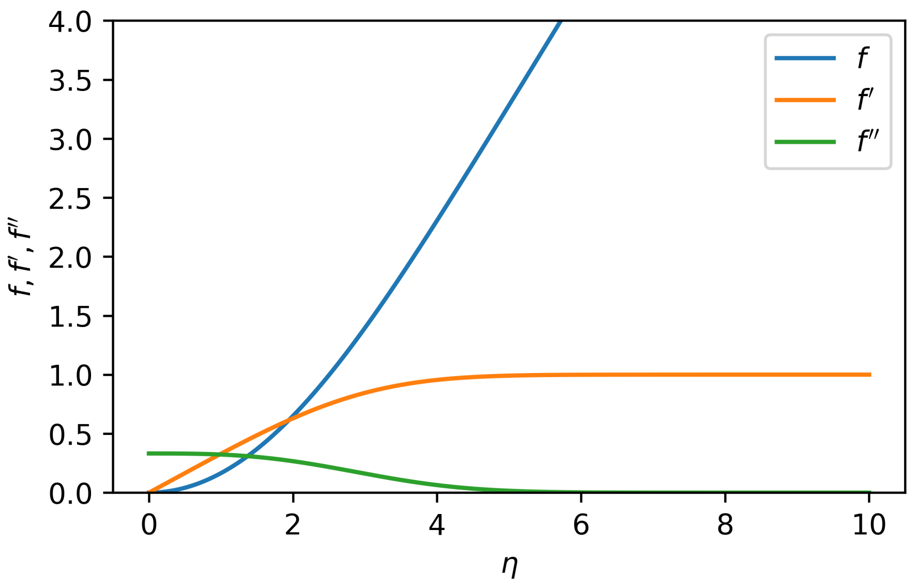

Figure 1: Blasius boundary value problem

Analysis of the Numerical Solution

From the integrated solution, we find that the u velocity profile reaches 99% of the free stream velocity at $\eta = 4.913$. However, for convenience, this value is often rounded up to 5.0, which is (arbitrarily) *defined* as where the boundary layer ends. The corresponding y value where this occurs is called the "displacement thickness" and is denoted by δ. Therefore, by the definition of η, we have,

$$5.0 = \delta\sqrt{\frac{U}{\nu x}}.$$

Rearranging,

$$\delta = 5.0\sqrt{\frac{\nu x}{U}} = \frac{5.0\,x}{\sqrt{Re_x}},$$

where we define the "local" Reynolds number as,

$$Re_x := \frac{Ux}{\nu}.$$

Another useful quantity to extract is the shear force acting on the plate, τ_w,

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U \sqrt{\frac{U}{\nu x}} f''(0).$$

From the solution, we found that $f''(0) = 0.332$. In common practice, instead of directly calculating τ_w, we define a dimensionless number, C_f, called the skin friction coefficient,

$$C_f := \frac{\tau_w}{\frac{1}{2}\rho U^2}.$$

Plugging in the previous expression for τ_w, we may calculate the local skin friction coefficient, $C_{f,x}$,

$$C_{f,x} = \mu U \sqrt{\frac{U}{\nu x}} \frac{0.332}{\frac{1}{2}\rho U^2},$$

rearranging,

$$C_{f,x} = \frac{0.664}{\sqrt{Re_x}}.$$

Thus, the local shear stress on the plate can be calculated as,

$$\tau_w = C_{f,x}\frac{1}{2}\rho U^2.$$

Even more commonly, instead of working with the local skin friction coefficient, we assume we are interested in the total shear force acting on a plate of length, L and width, W. The total force is given by integrating τ_w over the entire plate area. Since the only non-constant quantity is $C_{f,x}$, we wish to average this quantity to get rid of the integral. Assuming L spans the x direction, the average skin friction coefficient, $C_{f,L}$, may be calculated,

$$C_{f,L} = \frac{1}{L}\int_0^L \frac{0.664}{\sqrt{Re_x}}\,dx = \frac{1.328}{\sqrt{Re_L}},$$

where,

$$Re_L := \frac{LU}{\nu}.$$

The shear force, F, is then simply given by,

$$F = \tau_w LW = C_{f,L}\frac{1}{2}\rho U^2 LW.$$