INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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**N° 4989**

Novembre 2003

THÈME 2

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4989--FR+ENG

# Segmentation of Discrete Curves into Fuzzy Segments

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Thème 2 — Génie logiciel
et calcul symbolique
Projet Adage

Rapport de recherche n° 4989 — Novembre 2003 — 17 pages

**Abstract:** A new concept, fuzzy segments, is introduced which allows for flexible segmentation of discrete curves, so taking into account some noise in them. Relying on an arithmetic approach of discrete lines, it generalizes them, admitting that some points are missing. Thus, a larger class of objects is considered. A very efficient detection algorithm for fuzzy segments and its application to curve segmentation are presented.

**Key-words:** Segmentation, Fuzzy Segment, Discrete Line, Noisy Curve

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# Segmentation de courbes discrètes en segments flous

**Résumé :** Nous introduisons dans ce rapport un nouveau concept, les segments flous, qui permet une segmentation des courbes discrètes plus flexible, tenant compte du bruit dans les courbes. Lié à l'approche arithmétique des droites discrètes, il la généralise, en admettant que certains points soient absents. On peut ainsi considérer une classe d'objets plus vaste. Un algorithme très efficace de reconnaissance de segments flous et son application à la segmentation de courbes discrètes sont présentés.

**Mots-clés :** Segmentation, segment flou, droite discrète, courbe bruitée

# 1 Introduction

Numerous techniques for the segmentation of planar discrete curves have been proposed for the last thirty years (such as [2, 7, 9]). Some of them are based on the principle of the polygonal approximation where the curve is split into a sequence of straight line segments. Several authors [5, 3] proposed linear algorithms for the segmentation into exact discrete straight lines, based on precise mathematical definitions of discrete straight lines.
However, in order to fulfil the needs for an approached segmentation of discrete curves, taking into account noises due to data processing operations, such as skeletisation in the case of image data, we present in this paper a new notion, the fuzzy segments, which relies on an arithmetical definition of discrete straight lines [6] where thickness may be parameterized. A fuzzy segment is an 8-connected sequence of points which belong to a discrete straight line with a given thickness. A parameter, the order of a fuzzy segment, permits to control the amplitude of the authorized noise by fixing the thickness of the straight line bounding the fuzzy segment. Adding a point to a fuzzy segment is translated into the calculation of the slope and thickness of a new bounding straight line. It corresponds to very easy calculations. It leads to an incremental and very efficient algorithm for the splitting of a discrete curve into fuzzy segments with fixed order.
In Section 2, after recalling definitions and basic properties of discrete straight lines, we define the related notion of fuzzy segment and bounding straight line. Then, in Section 3, a fundamental theorem on the growing of a fuzzy segment is proved. It leads to the incremental and algorithm for the recognition of a fuzzy segment with a fixed order detailed in Section 4. At last, the algorithm for the segmentation of a curve into fuzzy segments with fixed order is presented and illustrated by a few examples.

# 2 Definitions and first properties

## 2.1 Discrete lines

In this section, we briefly recall some results of [6] and [3] that we shall need.

**Definition 1** *A* **discrete line** *[6], named $\mathcal{D}(a, b, \mu, \omega)$, is the set of integer points $(x, y)$ verifying the inequalities $\mu \leq ax - by < \mu + \omega$ where $a, b, \mu, \omega$ are integers. $\frac{a}{b}$ with $b \neq 0$ and gcd(a,b)= 1 is the slope of the discrete line, $\mu$ is named lower bound and $\omega$ arithmetical thickness.*

Among the discrete lines we shall distinguish, according to their topology [6]:

- the *naive lines* which are 8-connected and for which the thickness $\omega$ verifies $\omega = max(|a|, |b|)$,
- the $*$-*connected lines* for which the thickness $\omega$ verifies $max(|a|, |b|) < \omega < |a| + |b|$,
- the discrete lines said *standard* where $\omega = |a| + |b|$, this thickness is the smallest one for which the discrete line is 4-connected,

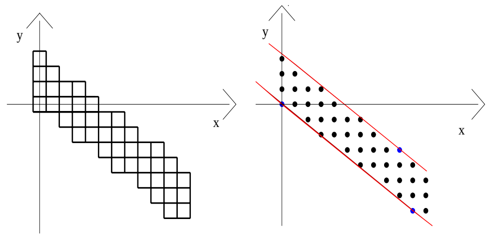


Figure 1: On the left hand side a representation by pixels (each integer point is represented by a square centered at the point) of a segment of the thick line $\mathcal{D}(7,-10,0,34)$ whose equation is $0 \leq 7x+10y < 34$, for $x \in [0,10]$, on the right hand side the points of this line are represented by disks to get a better visualisation of the leaning lines.

- the *thick lines* where $\omega > |a|+|b|$, they are 4-connected

**Definition 2** *Real straight lines $ax-by=\mu$ et $ax-by=\mu+\omega-1$ are named the* **leaning lines** *of the discrete line $\mathcal{D}(a,b,\mu,\omega)$. An integer point of these lines is named* **a leaning point***.*

The leaning line located above (resp. under) $\mathcal{D}$ in the first quadrant ($0 \leq a$ and $0 \leq b$) respects the following equation $ax-by=\mu$ (resp. $ax-by=\mu+\omega-1$), it is named **upper leaning line** (resp. **lower leaning line**) of $\mathcal{D}$, noted $d_U$ (resp. $d_L$).

**Definition 3** *Let $M(x_M, y_M)$ be an integer point, the* **remainder** *at the point $M$ as a function of $\mathcal{D}(a,b,\mu,\omega)$, noted $r(M)$, is defined by:*

$$r(M) = ax_M - by_M$$

To simplify the writing, we shall suppose hereafter that *the slope coefficients verify* $0 \leq a \leq b$ which corresponds to the first octant.

**Proposition 4** *Let $\mathcal{D}(a,b,\mu,\omega)$ be a discrete straight line. For each relative integer $k$, it exactly exists one point $P_k$ whose coordinates $(x_{P_k}, y_{P_k})$ satisfy both conditions:*

$$\begin{cases} r(P_k) = k \\ 0 \leq x_{P_k} \leq b-1 \end{cases}$$

**Proof (Existence)** As $a$ and $b$ are relatively prime between them, according to Bezout's theorem, it exists $x$, $y$ such that $ax-by=1$. Therefore, the integers $kx$ (noted $x'$) and $ky$ (noted $y'$) satisfy: $ax'-by'=k$. Let $x''$ be the remainder of the Euclidian division of $x'$ by $b$. By definition, $x''$ satisfies the inequalities $0 \leq x'' \leq b-1$ and there is an integer $q$ such that $x'-x''$ is equal to $qb$. Let $y''=y'-aq$. The point $P$, with coordinates $(x'', y'')$, satisfies the required conditions.

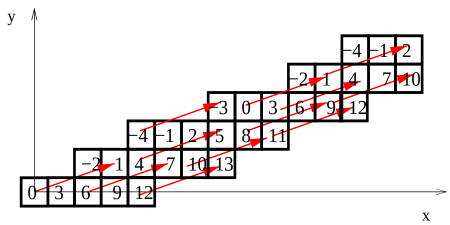

Figure 2: Representations of the positive shift vector of the line $\mathcal{D}(3, 8, -4, 18)$, the remainder value is indicated on each pixel of the segment.

**(Uniqueness)** Let $P' = (x', y')$ and $P'' = (x'', y'')$ be two points satisfying the required conditions. Let us prove that $P' = P''$ or, equivalently, that $x' = x''$. Let $x = x' - x''$ and $y = y' - y''$. By hypothesis, $x$ satisfies $-b < x < b$. Moreover, by substraction, $ax - by = 0$. So $b$ divides $ax$ and, as $a$ and $b$ are relatively prime between them, $b$ divides $x$ and therefore, necessarily, $x = 0$. $\square$

**Definition 5** *We call* **positive shift vector** *(resp.* **negative shift vector***) of* $\mathcal{D}(a, b, \mu, \omega)$*, noted* $V_{+1}$ *(resp.* $V_{-1}$*), a vector between two points of* $\mathcal{D}$ *for which remainder values are different from +1 (resp. -1) and such that* $0 \leq x_{V_{+1}} \leq b$ *(resp.* $0 \leq x_{V_{-1}} \leq b$*)*

**Definition 6** *An integer point* $M$ *is* **k-exterior** *to* $\mathcal{D}$ *if* $r(M) = \mu - k$ *or* $r(M) = \mu + \omega + k - 1$ *with* $k$ *being a strictly positive integer. If* $k > 1$*, this point is named* strongly exterior *to* $\mathcal{D}$ *and if* $k = 1$*, it is named* **weakly exterior** *to* $\mathcal{D}$*.*

**Proposition 7** *Let us suppose that* $a > 0$*, the vertical distance (ordinate difference) between the leaning lines of a line* $\mathcal{D}(a, b, \mu, \omega)$ *is equal to* $\frac{\omega-1}{b}$ *and the horizontal distance (abscissa difference) between the leaning lines is equal to* $\frac{\omega-1}{a}$*.*

**Proof** Let us consider a point $M(x, y_M)$ on the upper leaning line and the point $N(x, y_N)$ on the lower leaning line. By definition $ax - by_M = \mu$ and $ax - by_N = \mu + \omega - 1$, and therefore $y_M - y_N = \frac{\omega-1}{b}$. The reasoning is similar for the horizontal distance. $\square$

**Proposition 8** *[6] The vertical (resp. horizontal) steps of* $\mathcal{D}(a, b, \mu, \omega)$ *are the segments obtained by intersecting* $\mathcal{D}$ *and the vertical line whose equation is* $x = k$ *(resp. horizontal one whose equation is* $y = k$*), where* $k$ *is integer.*
*The lengths of the vertical steps are the consecutive integer values* $\left[\frac{\omega}{b}\right]$ *and* $\left[\frac{\omega}{b}\right] + 1$ *if* $b$ *does not divide* $\omega$ *and the integer* $\frac{\omega}{b}$ *if* $b$ *divides* $\omega$*.*
*As well, the lengths of the horizontal steps are the consecutive integer values* $\left[\frac{\omega}{a}\right]$ *and* $\left[\frac{\omega}{a}\right] + 1$ *if* $a$ *does not divide* $\omega$ *and the integer* $\frac{\omega}{a}$ *if* $a$ *divides* $\omega$*.*

The value $\frac{\omega}{b}$ will allow us to define the notion of fuzzy segment introduced in the next paragraph.

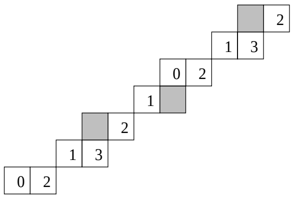


Figure 3: In white, a fuzzy segment with order $\frac{4}{3}$, in gray the pixels which do not belong to the segment but which are contained in the bounding line $\mathcal{D}(2,3,0,4)$, the value $2x - 3y$ is indicated on the pixels of the fuzzy segment.

### 2.2 The fuzzy segments

**Definition 9** *A set $\mathcal{S}f$ of consecutive points ($|\mathcal{S}f| \geq 2$) of a 8-connected curve is a* **fuzzy segment with order d** *if and only if there is a discrete line $\mathcal{D}(a,b,\mu,\omega)$ such that all points of $\mathcal{S}f$ belong to $\mathcal{D}$ and $\frac{\omega}{max(|a|,|b|)} \leq d$. The line $\mathcal{D}$ is said* **bounding** *for $\mathcal{S}f$*

The order of a fuzzy segment allows to limit the thickness of the discrete line framing the 8-connected sequence of points of the fuzzy segment and, so doing, to control the length of vertical steps of the bounding line . In order to be reasonably close to the points of the fuzzy segment, we introduce more restrictive conditions to the discrete line with the notion of strictly bounding line as defined hereafter.

**Definition 10** *Let $\mathcal{S}f$ be a fuzzy segment whose order is $d$, and whose the abscissa interval is $[0, l-1]$, $\mathcal{D}(a,b,\mu,\omega)$ a bounding line of $\mathcal{S}f$, $\mathcal{D}$ is named* **strictly bounding** *for $\mathcal{S}f$ if:*

- *$\mathcal{D}$ possesses at least three leaning points in the interval $[0, l-1]$,*
- *$\mathcal{S}f$ contains at least one lower leaning point and one upper leaning point of $\mathcal{D}$.*

## 3 Growth of a fuzzy segment

We present in this section a result which allows to control the growth of a fuzzy segment. The problem is as follows: let $\mathcal{S}f$ be a fuzzy segment whose order is $d$, whose interval on the x axis is $[0, l-1]$, and let $\mathcal{D}(a,b,\mu,\omega)$ be a strictly bounding line for $\mathcal{S}f$. The point $M$, connected to $\mathcal{S}f$, whose abscissa $x_M$ is equal to $l$ or $l-1$, is added to $\mathcal{S}f$. Is the line $\mathcal{D}(a,b,\mu,\omega)$ strictly bounding for $\mathcal{S}f \cup M$ and if not, how can we determine a strictly bounding line? A solution is given by the theorem hereafter whose principle is the following. Two cases are possible:

- $M \in \mathcal{D}$, in that case $\mathcal{D}$ is strictly bounding for $\mathcal{S}f \cup M$,
- $M \notin \mathcal{D}$, let us suppose that $r(M) \geq \mu + \omega$ (symmetrical case if $r(M) < \mu$), $M$ is then located under the lower leaning line of $\mathcal{D}$, the idea consists in thickening if necessary

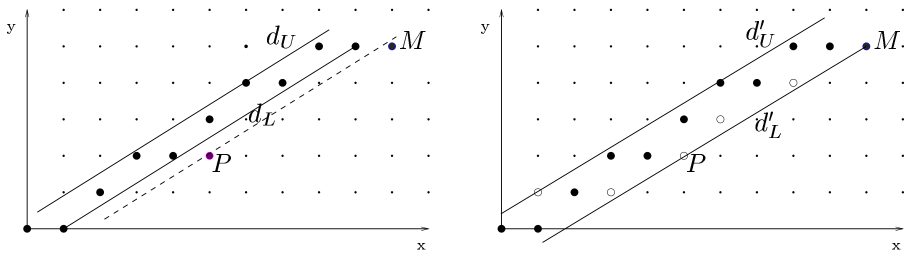


Figure 4: On the left hand side, a fuzzy segment $\mathcal{S}f$ whose order is 1, the line $\mathcal{D}(5,8,-2,8)$ is strictly bounding and we add the point $M(10,5)$ to it, $r_{\mathcal{D}}(M) = 10$. $P$ is the point from the interval $[0,7]$ such that $r_{\mathcal{D}}(P) = 9$. On the right hand side, the line $\mathcal{D}'(3,5,-2,7)$ strictly bounding for $\mathcal{S}f \cup \{M\}$ whose slope is calculated from the vector $PM$, in black the points of $\mathcal{S}f \cup \{M\}$ and in white the points of $\mathcal{D}'$ which do not belong to $\mathcal{S}f \cup \{M\}$.

the line $\mathcal{D}$ so that the lower leaning line is the line whose equation is $ax-by = r(M)-1$, then to take as new slope the one obtained from the vector $\overrightarrow{PM}$ with $P$ the point of this line whose abscissa verifies $x_P \in [0, b-1]$, the thickness of this new line will be calculated from the last upper leaning point of $\mathcal{D}$ present in $\mathcal{S}f$ (see illustrated Figure 4). The vector $\overrightarrow{PM}$ is calculated from the shift vectors of $\mathcal{D}$ (see details in Remark 12).

The principle used here is different from the one of naive lines recognition [3], indeed, at each step, the slope of the bounding line may change but its thickness too. Moreover, all the points of the line in the considered interval do not necessarily belong to the fuzzy segment. When the added point $M$ is 1-exterior to $\mathcal{D}$, the thickening step does not exist, the point of the first period to be considered for the calculation of the new slope is on the lower leaning line of $\mathcal{D}$.

**Theorem 11** *Let us consider a fuzzy segment $\mathcal{S}f$ whose interval on the $x$ axis is $[0, l-1]$, $\mathcal{D}(a, b, \mu, \omega)$ a strictly bounding line. In this case the order of $\mathcal{S}f$ is $\frac{\omega}{b}$. Let $M(x_M, y_M)$ be an integer point connected to $\mathcal{S}f$ whose abscissa is equal to $l$ or $l-1$.*

$(i)$ *If $\mu \le r(M) < \mu + \omega$, then $M \in \mathcal{D}$.*
*$\mathcal{S}f \cup M$ is a fuzzy segment whose order is $\frac{\omega}{b}$ with $\mathcal{D}$ as strictly bounding line.*

$(ii)$ *If $r(M) \le \mu - 1$, then $M$ is exterior to $\mathcal{D}$.*
*$\mathcal{S}f \cup M$ is a fuzzy segment whose order is $\frac{\omega'}{b'}$ and the line $\mathcal{D}'(a', b', \mu', \omega')$ is strictly bounding, with*

- *$b'$ and $a'$ coordinates of the vector $\overrightarrow{P_{r(M)+1}M}$, $P_{r(M)+1}$ being the point whose remainder is $r(M)+1$ with regard to $\mathcal{D}$ and $x_{P_{r(M)+1}} \in [0, b-1]$,*
- *$\mu' = a'x_M - b'y_M$*

- $\omega' = a'x_{L_L} - b'y_{L_L} - \mu' + 1$, *with* $L_L(x_{L_L}, y_{L_L})$ *last lower leaning point of the line* $\mathcal{D}$ *present in* $\mathcal{S}f$.

*Moreover, the shift vectors of* $\mathcal{D}'$ *are:*
$\overrightarrow{V'_{-1}} = (b'-b, a'-a) + [\frac{b}{b'}](b', a')$ *and* $\overrightarrow{V'_{+1}} = (b, a) - [\frac{b}{b'}](b', a')$

$(iii)$ *If* $r(M) \geq \mu + \omega$, *then* $M$ *is exterior to* $\mathcal{D}$.
$\mathcal{S}f \cup \{M\}$ *is a fuzzy segment whose order is* $\frac{\omega'}{b'}$ *and the line* $\mathcal{D}'(a', b', \mu', \omega')$ *is strictly bounding with*

- $b'$ *and* $a'$ *coordinates of the vector* $\overrightarrow{P_{r(M)-1}M}$, $P_{r(M)-1}$ *being the point whose remainder is* $r(M)-1$ *with regard to* $\mathcal{D}$ *and* $x_{P_{r(M)-1}} \in [0, b-1]$,
- $\mu' = a'x_{U_L} - b'y_{U_L}$ *with* $U_L(x_{U_L}, y_{U_L})$ *last upper leaning point of the line* $\mathcal{D}$ *present in* $\mathcal{S}f$,
- $\omega' = a'x_M - b'y_M - \mu' + 1$.

*Moreover, the shift vectors of* $\mathcal{D}'$ *are:*
$\overrightarrow{V'_{-1}} = (b, a) - [\frac{b}{b'}](b', a')$ *and* $\overrightarrow{V'_{+1}} = (b'-b, a'-a) + [\frac{b}{b'}](b', a')$

**Remark 12**

*1. In the cases (ii) and (iii) of the theorem, the new slope is calculated from vectors* $\overrightarrow{P_{r(M)+1}M}$ *and* $\overrightarrow{P_{r(M)-1}M}$. *Thanks to the shift vectors of* $\mathcal{D}$, *they can be very easily calculated as follow:*

$$\overrightarrow{P_{r(M)+1}M} = [\frac{x_M - x_{\overrightarrow{V_{-1}}}}{b}](b, a) + \overrightarrow{V_{-1}} \text{ and } \overrightarrow{P_{r(M)-1}M} = [\frac{x_M - x_{\overrightarrow{V_{+1}}}}{b}](b, a) + \overrightarrow{V_{+1}}.$$

*2. When* $r(M) = \mu - 1$, *the point* $M$ *is weakly exterior to* $\mathcal{D}$ *and the point* $P_{r(M)+1}$ *is the first upper leaning point of the line* $\mathcal{D}$ *in the interval* $[0, l-1]$. *As well, when* $r(M) = \mu + \omega$, *the point* $M$ *is weakly exterior to* $\mathcal{D}$ *and the point* $P_{r(M)-1}$ *is the first lower leaning point of the line* $\mathcal{D}$ *in the interval* $[0, l-1]$.

*3. Adding a point* $M$ *to a fuzzy segment* $\mathcal{S}f$ *thanks to the theorem 11 permits to obtain a strictly bounding line but this one is not always the closest to the points of* $\mathcal{S}f \cup \{M\}$.

**Proof of the theorem**
We shall only demonstrate point $(iii)$ as point $(i)$ is obvious and point $(ii)$ symmetrical of case $(iii)$.
The demonstration requires the result described hereafter (pp. 27 of [4]).

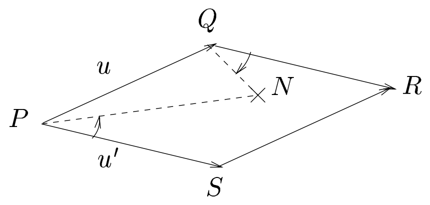


Figure 5:

**Lemma 13** *Let $P$ be a point with integer coordinates, $\overrightarrow{u}(b,a)$ and $\overrightarrow{u'}(b',a')$ two vectors with integer coordinates. If $ab' - a'b = 1$, the interior of the parallelogram $PQRS$ where $\overrightarrow{PQ} = \overrightarrow{u}$ and $\overrightarrow{PS} = \overrightarrow{u'}$ contains no point with integer coordinates.*

**Proof of the lemma**
A point $N(x,y)$ inside the parallelogram is located between both lines $(PS)$ and $(QR)$ (see Figure 5), it implies that the angles $(\overrightarrow{PS}, \overrightarrow{PN})$ and $(\overrightarrow{QR}, \overrightarrow{QN})$, and therefore their sinus, are with opposite signs. It is equivalent to

$$\begin{vmatrix} b' & x - x_P \\ a' & y - y_P \end{vmatrix} \begin{vmatrix} b' & x - x_P - b \\ a' & y - y_P - a \end{vmatrix} < 0 \ ,$$

thus $(b'(y - y_P) - a'(x - x_P))(b'(y - y_P) - a'(x - x_P) - (ab' - a'b)) < 0$.
If $ab' - a'b = 1$, we obtain, by calling $\Delta$ the expression $b'(y - y_P) - a'(x - x_P)$, $\Delta(\Delta - 1) < 0$, which implies $\Delta > 0$ and $\Delta - 1 < 0$. But it is impossible to have $0 < b'(y - y_P) - a'(x - x_P) < 1$ with x and y being integer as $a'$, $b'$, $x_P$, $y_P$ are integer by hypothesis. □

(a) Firstly let us prove that all points of $\mathcal{S}f \cup \{M\}$ belong to $\mathcal{D}'$. For this, we only have to prove that all points of $\mathcal{S}f \cup \{M\}$ are located between both leaning lines of $\mathcal{D}'$, i.e. between the line through $M$ whose main vector is $\overrightarrow{P_{r(M)-1}M} = (b', a')$ that we'll call $d'_L$ and the line through $U_L$ with the same main vector named $d'_U$.
By looking at Figure 6, we can see that two triangles must be studied more precisely, the one coming from the point $U_L$ and the other coming from the point $Q$, intersection of $d'_L$ with the lower leaning line of $\mathcal{D}$, named $d_L$.
Let us consider the triangle $U_L S S'$, $S$ being the point whose abscissa is $x_M$ on the upper leaning line of $\mathcal{D}$ (named $d_U$) and $S'$ the point whose abscissa is $x_M$ on the line $d'_U$ which is the upper leaning line of $\mathcal{D}'$. We must verify that there is no point of $\mathcal{S}f$ contained inside the triangle $U_L S S'$ other than those located on the line $d'_U$. There is no point of $\mathcal{S}f$ on the line $d_U$ other than $U_L$ because $U_L$ is, by hypothesis, the last upper leaning point of $\mathcal{D}$.
Let us show that $ab' - a'b = 1$ (*):

$$\begin{aligned} ab' - a'b &= a(x_M - x_{P_{r(M)-1}}) - (y_M - y_{P_{r(M)-1}})b \\ &= (ax_M - by_M) - (ax_{P_{r(M)-1}} - by_{P_{r(M)-1}}) \\ &= r(M) - (r(M) - 1) = 1. \end{aligned}$$

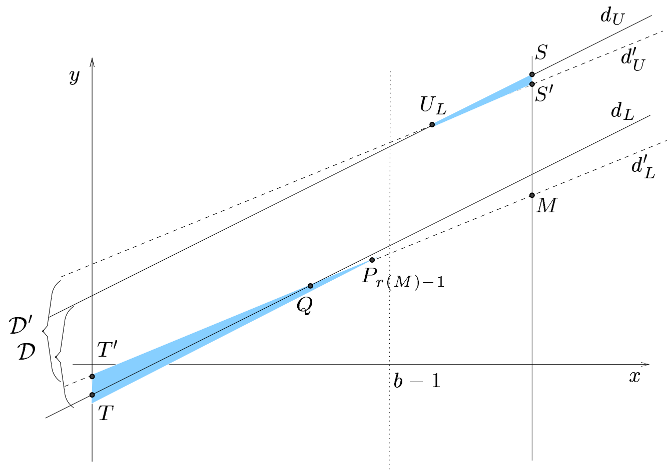


Figure 6:

According to lemma, there is no point whose coordinates are integer inside the parallelogram whose origin is $U_L$ generated by both vectors $u$ and $u'$. The abscissa of the vertex of this parallelogram which does not belong to $d_U$ and $d'_U$ is $x_{U_L} + b + b'$. However we have, by definition of $b'$, $b' = x_M - x_{P_{r(M)-1}} > x_M - b$, and therefore $x_{U_L} + b + b' > x_M$. This vertex does not belong to the segment $\mathcal{S}f$. It is a fortiori true for the vertices, which do not belong to the lines $d_U$ and $d'_U$, of the parallelograms deduced from the previous one by successive translations by vector $u$ or $u'$, possibly necessary to cover the triangle $U_L SS'$.
For the points of $\mathcal{S}f$ which are close to the origin, it is clear that in the case where the abscissa of $Q$ is negative, the points of $\mathcal{S}f$ are above $d_L$ as they belong to $\mathcal{D}$, and therefore above $d'_L$. In the other case, we consider the triangle $QTT'$ where $T$ and $T'$ are the points intersection of the axis $Oy$ with $d_L$ and $d'_L$. This triangle is contained inside the triangle delimited by the line $d'_L$, the axis $Oy$ and the parallel to $d_L$ through $P_{r(M)-1}$ (in gray on Figure 6). As we did for the triangle $U_L SS'$, we can cover this triangle by the parallelogram whose origin is $P_{r(M)-1}$ generated by both vectors $-u$ and $-u'$ and one or several parallelograms obtained by successive translations by vector $-u'$. The points of $\mathcal{S}f$ are necessarily some vertices of these parallelograms. Those ones which belong to $d'_L$ also belong to $\mathcal{D}'$. The other ones have a negative abscissa because $x_{P_{r(M)-1}}$ is less than $b$.

(b) Secondly, let us prove that $\mathcal{D}'$ is strictly bounding for $\mathcal{S}f \cup \{M\}$. By construction $\mathcal{D}'$ has in $\mathcal{S}f \cup \{M\}$ a lower leaning point, the point $M$, and an upper leaning point, the point $U_L$. The third leaning point is the point $P_{r(M)-1}$ which does not necessarily belong to $\mathcal{S}f \cup \{M\}$.

(c) Finally, let us prove that $\overrightarrow{V'_{+1}} = (b'-b, a'-a) + [\frac{b}{b'}](b', a')$. Let $Q$ and $S$ be two points of $\mathcal{D}'$ such that $\overrightarrow{QS} = (b'-b, a'-a) + [\frac{b}{b'}](b', a')$. Then,

$$\begin{aligned} r'(S) - r'(Q) &= a'x_S - b'y_S - (a'x_Q - b'y_Q) \\ &= a'(b' - b + [\tfrac{b}{b'}]b') - b'(a' - a + [\tfrac{b}{b'}]a') \\ &= -a'b + ab' = 1, \text{ thanks to (*).} \end{aligned}$$

Moreover, $x_{\overrightarrow{QS}} = b' - b + [\frac{b}{b'}]b'$. As b and b' are greater than 0 and $[\frac{b}{b'}] \leq \frac{b}{b'}$ , $0 \leq x_{\overrightarrow{QS}} \leq b'$.
Consequently, $\overrightarrow{QS} = \overrightarrow{V'_{+1}}$.
The proof of formula for $\overrightarrow{V'_{-1}}$ is similar. $\square$

This theorem will allow us to determine an incremental and algorithm for the recognition of a given fuzzy segment and to deduce a algorithm for the segmentation of 8-connected curves into fuzzy segments. These algorithms are presented in the next section.

# 4 A segmentation algorithm of 8-connected curves into fuzzy segments

## 4.1 Incremental recognition of a fuzzy segment with order d in the first octant

Let $d$ be a real number, the algorithm hereafter analyses an 8-connected sequence $E$ of pixels located in the first octant and determines if $E$ is a fuzzy segment with order $d$. Moreover, in that case, the characteristics $a$, $b$, $\mu$ and $\omega$ of a strictly bounding line are calculated.
Each point $M$ of $E$ is analyzed and added to the current segment and the characteristics $a$, $b$, $\mu$ and $\omega$ of a strictly bounding line of this segment can possibly change according to the theorem of the previous section.
At each step, the value $\frac{\omega}{b}$ is evaluated and if it is greater than $d$, the recognition stops. $E$ is not a fuzzy segment whose we may calculate the strictly bounding line according to the theorem.

**Algorithm 1** *Fuzzy Segment Recognition Algorithm*

*Input:* $E$ *an 8-connected sequence of points and* $d$ *the order authorized for the fuzzy segment*
*Output: a boolean value* $isSegment$,
– *false if* $E$ *is not a fuzzy segment with order* $d$ *according to the theorem*
– *true else, in this case* $a$, $b$, $\mu$ *and* $\omega$ *are the characteristics of the fuzzy segment*
*Initialisation:* $a = 0$, $b = 1$, $\omega = b$, $\mu = 0$, $isSegment = true$, $M$ = *the first point of* $E$, $L_L = U_L = M = (0,0)$, $\overrightarrow{V_{-1}} = (0,1)$, $\overrightarrow{V_{+1}} = (0,-1)$

**while** $E$ *is not entirely scanned and* $isSegment$ **do**

$M$ = *next point of* $E$ ;
$r = ax_M - by_M$ ;
**if** $r = \mu$ **then** $U_L = M$ ; **endif**
**if** $r = \mu + \omega - 1$ **then** $L_L = M$ ; **endif**
**if** $r \leq \mu - 1$ **then**
$a_{last} = a$ ; $b_{last} = b$ ; $\mu_{last} = \mu$ ; $\omega_{last} = \omega$ ;
$a = [\frac{x_M - x_{\overrightarrow{V_{-1}}}}{b_{last}}]a_{last} + y_{\overrightarrow{V_{-1}}}$ ;
$b = [\frac{x_M - x_{\overrightarrow{V_{-1}}}}{b_{last}}]b_{last} + x_{\overrightarrow{V_{-1}}}$ ;
$\mu = ax_M - by_M$ ;
$\omega = ax_{L_L} - by_{L_L} - \mu + 1$ ;
$U_L = M$ ; *possible update of* $L_L$ ;
$\overrightarrow{V_{-1}} = (b - b_{last} + [\frac{b_{last}}{b}]b, a - a_{last} + [\frac{b_{last}}{b}]a)$ ;
$\overrightarrow{V_{+1}} = (b_{last} - [\frac{b_{last}}{b}]b, a_{last} - [\frac{b_{last}}{b}]a)$ ;
**endif**
**if** $r \geq \mu + \omega$ **then**
$a_{last} = a$ ; $b_{last} = b$ ; $\mu_{last} = \mu$ ; $\omega_{last} = \omega$ ;
$a = [\frac{x_M - x_{\overrightarrow{V_{+1}}}}{b_{last}}]a_{last} + y_{\overrightarrow{V_{+1}}}$ ;
$b = [\frac{x_M - x_{\overrightarrow{V_{+1}}}}{b_{last}}]b_{last} + x_{\overrightarrow{V_{+1}}}$ ;
$\mu = ax_{U_L} - by_{U_L}$ ;
$\omega = ax_M - by_M - \mu + 1$ ;
$L_L = M$ ; *possible update of* $U_L$ ;
$\overrightarrow{V_{-1}} = (b_{last} - [\frac{b_{last}}{b}]b, a_{last} - [\frac{b_{last}}{b}]a)$ ;
$\overrightarrow{V_{+1}} = (b - b_{last} + [\frac{b_{last}}{b}]b, a - a_{last} + [\frac{b_{last}}{b}]a)$ ;
**endif**
$isSegment = \frac{\omega}{b} \leq d$ ;
**endwhile**

**Remark 14** *An update of* $L_L$*, in the case* $r \leq \mu - 1$*, is necessary if a point of the new lower leaning line whose abscissa is* $x_{L_L} + kb, 1 \leq k \leq [\frac{x_r - X_{L_L}}{b}]$*, belongs to the segment, because* $L_L$ *must always be the last lower leaning point of the segment. It is similar for* $U_L$ *in the case* $r \geq \mu + \omega$*.*

An example of the algorithm processing is presented at Figure 7. The recognition of the fuzzy segment whose order is 1.5 and whose strictly bounding line is $\mathcal{D}(5, 11, -2, 14)$ required 3 steps during which the characteristics of a strictly bounding line have been calculated. On the figure, the pixels are weighted by the remainder value as a function of the strictly bounding line of the current segment.
At the beginning of the recognition, the segment is an exact segment of a naive line which is a stricly bounding line of this segment. The introduction of a strongly exterior point

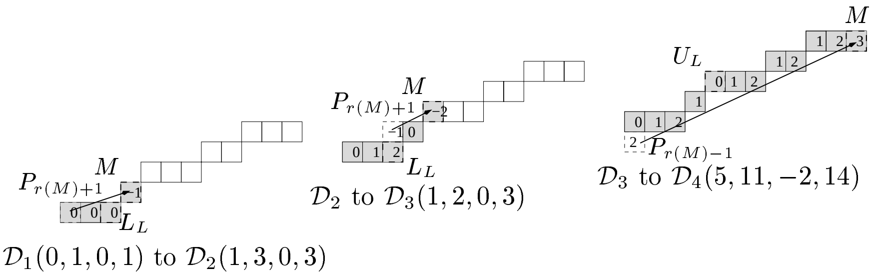


Figure 7:

changes that state, the segment is no longer exact, it does no longer contain all points of the discrete bounding line on the considered interval.

### 4.2 A algorithm for splitting an 8-connected curve into fuzzy segments with order d

The theorem of Section 3 and the algorithm of the previous paragraph are used to split a curve $\mathcal{C}$ into 8-connected fuzzy segments with a fixed order $d$. The curve $\mathcal{C}$ is incrementally scanned, each point is watched. Let $\mathcal{S}f$ be the current fuzzy segment, the point $M$ of $\mathcal{C}$ is added to $\mathcal{S}f$ by a procedure addPointSf which possibly changes the characteristics of the strictly bounding line of $\mathcal{S}f \cup M$ (according to the theorem of the previous section). According to the obtained ratio $\frac{\omega}{max(|a|,|b|)}$, the current segment does or not include the point $M$.

- If $\frac{\omega}{max(|a|,|b|)} > d$, $M$ is not included in the current fuzzy segment $\mathcal{S}f$, this one ends at the predecessor $M_p$ of $M$ in the curve $\mathcal{C}$, the strictly bounding line of $\mathcal{S}f$ has the same characteristics as the ones obtained before the point $M$ was added. A new fuzzy segment then starts, consisting of points $M_p$ and $M$.
- If $\frac{\omega}{max(|a|,|b|)} \leq d$, $M$ is included into $\mathcal{S}f$, $\mathcal{S}f$ becomes $\mathcal{S}f \cup \{M\}$ and the characteristics of its strictly bounding line are the last calculated ones.

Any scanning of the curve must take into account the possible changings of octants, it is therefore mandatory to include in the algorithm the detection and the management of octant changings. Several solutions are possible, we chose to do all calculations in the first octant by considering that the first point of each segment is $(0,0)$. For each added point $M$, we work with its *transformed point* $M_c$ in the first octant after having checked that $M$ belongs to the octant of the current segment. In the algorithm given below, the testOctant procedure:

- tests the validity of the point $M$ according to the octant of the current segment, and sets $isSameOctant$ to the according value,

– possibly updates the number of the octant of the current segment (variable $octantNumber$ of the algorithm).

The way the boolean value $isSameOctant$ is updated in the procedure testOctant depends on the *accepted directions* of a segment in a given octant.
When the current segment has entirely been scanned, we compute the transformed characteristics of those obtained in the first octant, according to the current octant, by using compositions of symmetries of discrete lines [6].

**Algorithm 2** *Splitting a curve into fuzzy segments*

*Input:* $\mathcal{C}$ *an 8-connected sequence of points and* $d$ *the order authorized for the fuzzy segments*
*Output: a list* $\mathcal{L}$ *of fuzzy segments, each of them being defined by its number of points* $nbPoint$ *and the characteristics* $a, b, \mu, \omega$ *of a strictly bounding line*
*Initialisation:* $\mathcal{L} = \emptyset$, $a = 0$, $b = 1$, $\mu = 0$, $\omega = b$, $nbPoint = 1$, $end = false$, $octantNumber = 0$, $isSegment = true$, $isSameOctant = true$, $M =$ *the first point of* $\mathcal{C}$, $L_L = U_L = M_c = (0,0)$, $\overrightarrow{V_{-1}} = (0,1)$, $\overrightarrow{V_{+1}} = (0,-1)$

**while** *!end* **do**
  **while** $isSegment$ *and* $isSameOctant$ *and !end* **do**
      // *Loop of determination of a fuzzy segment*
    $M_{last} = M$ ;
    $M$ = *next point of* $\mathcal{C}$ ; $M_c$ = *image of* $M$ *in the first octant* ;
    *testOctant(M)* ;
    $a_{last} = a$ ; $b_{last} = b$ ; $\mu_{last} = \mu$ ; $\omega_{last} = \omega$ ;
    **if** $isSameOctant$ **then**
      *addPointSf*($M_c$) ;       //*See 4.1*
      $isSegment = \frac{\omega}{b} \leq d$ ;
      **if** $isSegment$ **then** $nbPoint = nbPoint + 1$ ; **endif**
    **endif**
    $end = \mathcal{C}$ *is entirely scanned* ;
  **endwhile**
  **if** $isSegment$ *and* $isSameOctant$ **then**
      // *The last segment integrates* $M$
    *Add to* $\mathcal{L}$ *the fuzzy segment characterized by* $nbPoint$ *and, according to the current octant, the transformed characteristics of* $a$, $b$, $\mu$, $\omega$ ;
  **else**
      // *We add a segment which does not integrate* $M$
    *Add to* $\mathcal{L}$ *the fuzzy segment characterized by* $nbPoint$ *and, according to the current octant, the transformed characteristics of* $a_{last}$, $b_{last}$, $\mu_{last}$,

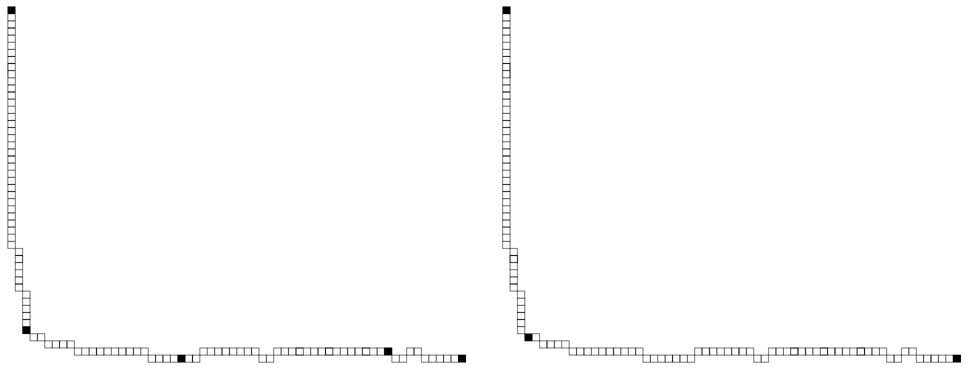

Figure 8:

$\omega_{last}$ ;
**if** *end* **then**
// *The last point of the curve is isolated*
*Add to* $\mathcal{L}$ *the fuzzy segment characterized by* $nbPoint = 2$ *and the transformed characteristics of* $a = 0$, $b = 1$, $\mu = 0$, $\omega = 1$ ;
**endif**
**endif**
// *Initialisations for the next segment*
$a = 0$ ; $b = 1$ ; $\mu = 0$ ; $\omega = b$ ; $nbPoint = 1$ ; $octantNumber = 0$ ;
$isSegment = true$ ; $isSameOctant = true$ ; $M = M_{last}$ ;
$L_L = U_L = M_c = (0, 0)$ ; $\overrightarrow{V_{-1}} = (0, 1)$ ; $\overrightarrow{V_{+1}} = (0, -1)$ ;
**endwhile**

**Remark 15** *The first point of a new fuzzy segment is the last point of the previous fuzzy segment.*

**Example 16** *Let us consider the curve given in Figure 8, on the left hand side, we can see the segmentation of the curve with an order equal to 2, the obtained fuzzy segments have the following characteristics:*

- *First fuzzy segment located in the octant 6 with a length of 46 and with* $\mathcal{D}(-33, 1, -26, 60)$ *as strictly bounding line*
- *Second fuzzy segment located in the octant 7 with a length of 22 and with* $\mathcal{D}(4, -19, -29, 38)$ *as strictly bounding line*
- *Third fuzzy segment located in the octant 0 with a length of 29 and with* $\mathcal{D}(1, 13, -10, 26)$ *as strictly bounding line*

- *Fourth fuzzy segment located in the octant 7 with a length of 11 and with* $\mathcal{D}(1,-5,-4,10)$ *as strictly bounding line.*

*On the right hand side of Figure 8, the curve segmentation is done with the order* 4, *there are only two fuzzy segments.*
*By using a variant of the algorithm, in some cases, the points whose remainders are* $r(M)-1$ *or* $r(M)+1$ *can be seeked beyond the first period, it allows to obtain a strictly bounding straight line which is closer to the points of the segment. In particular, in the above case, an horizontal segment can be detected and we obtain, from the order* 2, *three segments for the above curve, the first two ones are identical and the third fuzzy segment is located in the octant 0 with a length of 39 and with* $\mathcal{D}(0,1,-1,2)$ *as strictly bounding straight line.*

# 5 Conclusion and perspectives

We have presented in this paper a new notion of discrete segment, named fuzzy segment, which enables the splitting of discrete curves in a less strict way than with the techniques proposed in [3], by taking into account possible noises. An efficient and segmentation algorithm was presented.
Moreover, this notion opens new perspectives ; it might be used to define fuzzy tangent by extending the definition of discrete tangents given by A. Vialard [8]. The notion of discrete fuzzy arc might as well be deduced from the notion of fuzzy segment and from the work undertaken on the discrete arcs in [1].

Acknowledgments to Xavier Hilaire (QGAR project, LORIA) who asked us to segment a set of noisy curves by using a digital geometry approach.

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ISSN 0249-6399