Singular Value Decomposition

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1 Introduction:

Singular Value Decomposition (SVD) is a factorization of matrix base on in decomposition of eigenvectors and eigenvalues. However, SVD is used in several fields such as signal processing and statistics to analyse the variability of the data. In the last decade, SVD was applied in remote sensing applications like feature extraction (dimensionality reduction) [1] or spatio-temporal analysis using multiple remote sensing data sets [2]. In this project the SVD is used to analyse the spatio-temporal relations between temperature-based phenological indices and land surface phenological metrics.

SVD is a generalization of the eigenvalue-vector decomposition, which is well-known by the Principal Components Analysis (PCA) [3]. A sort review of eigenvalue-vector decomposition and SVD mathematical definitions are describe below.

2 Definitions:

This section is based on the work [4].

2.1 Eigenvalue-vector decomposition

Let $\mathbf{A}$ be a square $(N \times N)$ matrix, it is possible to compute its spectral decomposition into eigenvalues and eigenvectors ($\mathbf{A} = \mathbf{U}^{-1}\mathbf{\Lambda}\mathbf{U}$) if $\mathbf{A}$ satisfies the linear equation:

$$\mathbf{A}\mathbf{U} = \mathbf{\Lambda}\mathbf{U} \tag{1}$$

where $\mathbf{U}$ is the matrix whose columns are the eigenvectors ($\mathbf{u}_i$) of $\mathbf{A}$ and $\mathbf{\Lambda}$ is a diagonal matrix whose principal diagonal is formed by the eigenvalues (λ_i) of $\mathbf{A}$. The equation (1) is known as the *standard eigenvalue problem*. If $\mathbf{U}$ is an orthonormal matrix ($\mathbf{U}^\top\mathbf{U} = 1$) then $\mathbf{A}$ is orthogonally diagonalizable: $\mathbf{A} = \mathbf{U}^\top\mathbf{\Lambda}\mathbf{U}$. There exist many ways of solving the equation (1) depending on the matrix size and the matrix rank.

2.2 Singular value decomposition

Let $\mathbf{A}$ be a matrix ($N \times M$), the *Singular Value Decomposition* (SVD) is a factorization of the matrix in singular values and singular vectors that consists on: $\mathbf{A} = \mathbf{U\Sigma V}^\top$ where $\mathbf{U} \in \mathbb{R}^{N\times N}$ is made up of unit eigenvectors associated with non-zero eigenvalues of $\mathbf{AA}^\top$ (and it is also known *right eigenvectors*), $\mathbf{V} \in \mathbb{R}^{M\times M}$ is made up of unit eigenvectors associated with non-zero eigenvalues of $\mathbf{A}^\top\mathbf{A}$ (and it is also known *left eigenvectors*), and $\mathbf{\Sigma}$ is a diagonal matrix that contains the Singular Values of $\mathbf{A}$ sorted in descending order. The Singular Values are the square root eigenvalues of $\mathbf{AA}^\top$. We can express also the SVD in vectorial from: $\mathbf{A} = \sum_{i=1}^{d_f} \lambda_i \mathbf{u}_i \mathbf{v}_i^\top$.

So, to calculated the PCA of the data matrix ($\mathbf{X}$) is possible by two ways:

a. Eigenvalue decomposition of the covariance matrix $\mathbf{C} = X^\top X$ ($\mathbf{C} = \mathbf{V\Lambda V}^\top$). Then the principal components are the projection matrix $\mathbf{V}$ and follows eq.(1).

b. Singular value decomposition of $\mathbf{X}$ ($\mathbf{X} = \mathbf{U\Sigma V}^\top$). In this case,

$$\mathbf{C} = X^\top X = (\mathbf{U\Sigma V}^\top)^\top(\mathbf{U\Sigma V}^\top) = \mathbf{V}\frac{\mathbf{\Sigma}^2}{n-2}\mathbf{V}^\top. \tag{2}$$

Then, the principal components are the projection matrix $\mathbf{V}$ and follows:

$$\mathbf{XV} = \mathbf{U\Sigma}, \tag{3}$$

2.2.1 centred of matrices:

The principal components are sensitive to the scaling of the data. In general the data have to be scaling, although it is possible to find some examples about uncentered PCA [5-6]. Different ways to scale the data are possible to implement (fixing or not the the origin of the data). In this case, three implementations will be done:

a. Without centrer the data: $\mathbf{X}$

b. Centring the original data (the mean of the data per dimension is equal to zero): $\mathbf{X} - mean(\mathbf{X})$

c. *Decorrelating* the original data, i.e. their covariance are the identity matrix: $(\mathbf{X} - mean(\mathbf{X}))/std(\mathbf{X})$

3 Case study: Spatio-temporal phenological analysis

As mentioned before, we would like to analysis the spatio-temporal relations between temperature-based phenological indices and land surface phenological metrics. In this case, we have the temperature-based phenological index (matrix $\mathbf{A} \in \mathbb{R}^{M_1 \times N}$) and surface phenological metric (matrix $\mathbf{B} \in \mathbb{R}^{M_2 \times N}$), where M_1 and M_2 are the spatial dimension, and they are equal if and only if the spatial resolution in both data sets are equal. And N is the temporal dimension. To analyse the relations between them, the singular value decomposition of $\mathbf{AB}^\top$ is calculated.

In this case, $\mathbf{AB}^\top\mathbf{V} = \mathbf{\Sigma U}$ (eq.(3)), where the projection matrix $\mathbf{U} \in \mathbb{R}^{M_1 \times M_1}$ (right eigenvectors) are the *spatial principal components* and $\mathbf{V} \in \mathbb{R}^{N \times N}$ (left eigenvectors) are the *temporal principal components.*

References

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