1

These ideas are inspired by L. E. J. Brouwer and others.

Let us imagine both f and g as inexhaustible streams of bits. They are ever-unfinished sequences, which is to say functions from $\mathbb{N}$ to $\mathbb{B}$.

If $f = 0101...$ and $g = 0101...$, then we can't yet differentiate f from g.

Here we assume that no rule governs either f or g. This means that $f(n) = g(n)$ for $n < N$ is never proof that $f = g$ for any fixed value of N.

Perhaps f and g are "different" streams in the sense of having different sources in the non-mathematical "real" world. Perhaps two different humans are picking bits whimsically. Or perhaps two different machines are generating random or pseudo-random bits by sampling electromagnetic noise in their different environments.

This suggests that (maybe) it never makes sense to claim that $f = g$. Perhaps we should hope only for equivalence.

Let's look at two rules that generate the same stream. Let $f = \overline{01}$ and let $g = \overline{0101}$. Now it's clear that $f \equiv g$ in the sense that $\forall n \; f(n) = g(n)$.

This is functional equality in mainstream mathematics. Usually we'd say that $f = g$ and leave it at that.

Yet f and g were defined differently. So such functional equality is itself "mere" equivalence. Basically the equivalence class of all rules that result in the same output.

But this equivalence class is not made explicit, because the "real" function is just its platonic (completed) stream of values.

I think it's fair to say that mainstream math made a good choice in practical terms. It's messy to deal with the incomplete object, and it's just as messy to plug that incomplete sequence into the world.

2

Let's switch to "free choice" Cauchy sequences of rational numbers. We require that $f : \mathbb{N} \to \mathbb{Q}$ satisfy $|f(n) - f(m)| < \epsilon$ for $n, m > N$ in the usual way.

But we don't demand a rule. We don't demand a rate of convergence. Such a stream can take arbitrarily long to settle down. This concept is so open that its "limit" (if it has one in a non-mainstream context) can't be guessed at from any finite prefix.

This concept is so open that we'd never be able to tell whether a stream was a Cauchy sequence in the first place. Unless of course we assume or declare that it is.

If we just happened upon a stream of rational numbers in the wild which we took to be unfinished and unfinishable and governed by no rule, we'd never be able to tell whether it was a Cauchy sequence.

That helps us a little. If we say that f is one of these sequences, then those are the rules we are playing by.

We might say $f \approx g$ if $|f(n) - g(n)| \to 0$. But even assuming both are Cauchy, we could never be sure they were equivalent. Not without access to a rule for each stream.

This is probably why Brouwer added a constraint.

3

One intuitive approach that doesn't involve an arbitrary convergence rate uses intervals.

Let $f(0)$ be the closed interval $[a_0, b_b] \subset \mathbb{Q}$. Then require that $f(n+1) \subset f(n)$ and $|f(n)| \to 0$. Where $|f(n)|$ is just the length of the interval.

Now f can take as long as it wants to "converge," but we can reason about it, compute on its "prefixes."

What we did with streams of bits above could be interpreted in this way. For instance, $f = 01...$ would indicate $[0, 1/2], [1/4, 1/2], ...$ We might say that

$$f = 0\overline{1111} \approx g = 1\overline{0000}$$

and so on. We've just made the binary streams into streams of intervals or Cauchy sequence of rationals. Which requires a slightly more complex notion of equivalence.

4

We could also use "zig-zags" of rational numbers. We start with $f : \mathbb{N} \to \mathbb{Q}$ and require $f(n)$ to be *between* $f(n-1)$ and $f(n-2)$. We can leave $f(0)$ and $f(1)$ totally unconstrained. Or we can force $f(0) < f(1)$. Then $f(0) < f(2) < f(1) \implies f(2) < f(3) < f(1)$. So $f(0)$ is a floor and $f(1)$ is a ceiling. This gives us $f(2k) < f(2k+1)$. The odd elements dominate the even elements. The odds descend, the evens ascend.

5

Such sequences would an analogue of real numbers. But they are unfinished. If we are committed to unfinished streams of rationals governed only by a zigzag rule, then we need functions that behave properly as they operate on "prefixes" — on partial developments of their inputs, to generate partial developments of their outputs.

We demand that a function $f : \mathbb{R} \to \mathbb{R}$ ignore everything but the prefix of the stream it transforms. So if $x = .2, .22, .222, ...$ and $y = .2, .22, .222, ...$, then $f(.2, .22, .222, ...)$ has a definite value, something like $.4, .44, .444, ...$ for instance.

Let's assume that $f(x_n) = 2x_n$. Then f will always be a function on prefixes. If $x_n = y_n$ for $n < N$, then $f(x_n) = f(y_n)$ for $n < N$. This actually guarantees that f is continuous. So we have a version of the real number system where all functions are continuous. We don't *allow* functions that aren't continuous.

This basically follows from our concept of real numbers as unfinished. We don't want functions to have to see into a future that doesn't exist yet.

This requirement is at least necessary for total functions that output real numbers. But that's assumed in $f : \mathbb{R} \to \mathbb{R}$.

Could we integrate such functions ? We'd probably want a narrow conception of the integral which would determine a particular stream. The definite integral would be a definite stream.

Let $F(n) = \int_a^b f(x)dx$. Let $h = \frac{b-a}{n}$. Then $F(n) = \sum_{i=1}^{n} f_n(a + ih)h$