CHOICE

1

The following speculations are inspired by Brouwer's choice sequences.

Let f be a "living" (unfinishable) sequence of bits. For instance, $f = 01110101...$

The dots represent where our knowledge ends and our ignorance begins.

In this intuitionistic world, the future does not exist yet.

Consider $g = 0000000000000000000000...$

We might *hypothesize* that $g(k) = 0$ for all k. But let us recall Hume's problem of induction. And that these are *empirical* numbers, "signals from outer space. "

"Empirical" means work-in-progress and wait and see. Wait how long ? Forever. If we want to fully possess such a transcendent number.

Let $f = 1000...$ and $g = 1000...$

Are f and g "equal" ?

First answer: It's too early to tell.

Second answer: It will *always* be too early tell.

Third surprising answer: *No!*

This third answer is based on a rejection of the idea that $\forall k(f(k) = g(k)) \implies f = k$.

Here we consider f and g to be like two different signals from outer space. Or f is a sequence freely chosen by Bob, while g is a sequence freely chosen by Alice.

We *can* embrace the idea of equivalence.

$\forall k(f(k) = g(k)) \implies f \equiv k$.

2

But we still can't establish or verify equivalence.

Now we wax philosophical, and embrace the deflationary conception of truth. In other words, in this brave new world we only have belief. We never have truth. For "truth" is just a mystified way of talking about belief.

We can *hypothesize* that $f = \overline{10} = 1010101010...$, where the dots now express the continuation of a pattern.

Now we can say that $f = \overline{10} \wedge g = \overline{10} \implies f \equiv g$. If we assume the same computable patterns for both f and g, we can establish their equivalence.

In mainstream math, two functions are really the same function if they agree like this. And this makes sense if one takes computable functions as the core of math. Computable functions feed into the intuition that the future is already present.

Surely a Turing machine either halts or not, even if there's no upper bound on how long we might have to wait to find out ?

The realist platonic position answers yes. The antirealist position answers no.

The realist position makes our uncertainty accidental and subjective. In one sense it insists on the transcendence of the object. But in another sense it cancels this transcendence.

The object is what it is, independent of my uncertainty. But the anti-realist retort is that such uncertainty *is* the (genuine) transcendence of the object.

The computable numbers all have finite expressions. They are countable. They have a measure of 0. The computable numbers are analogous to the rational numbers. The rational numbers are easier to work with, but both of these countable numbers have arbitrarily long but always finite expressions. Both have measure 0. But the measure of $[0, 1]$ is of course 1. So the bulk of the real line is constituted by uncomputable numbers. Each contains an infinity of information.

I mention real numbers here, even though so far I've only discussed sequences of bits concretely. The bit sequence approach is offered as a simplification that highlights the time issue.

3

There are at least 3 paths available.

(1) We can stick with the mainstream, which is practically justified even if it is not pure in a philosophical-aesthetic sense. It is often very beautiful.

(2) We can follow Bishop into constructivism. We can insist that all real numbers be computable. This involves a serious modification of mainstream analysis.

(3) We can follow Brouwer into the strange territory of choice sequences, of something like the unfinishable bit sequences presented above.

Fortunately, we can explore all 3 paths. But here we explore the third path.

4

Given a "bitstream" $f = 0011....$, we call 0011 a **prefix** of f. While these sequences are relatively lawless, they don't get to change their mind after delivering a bit. So any prefix of f remains a prefix forever.

Our metaphor is that f is a signal from outer space. The future of f is uncertain and in fact inexistent for now. Its past is present as history.

Let ϕ be a function on such bitstreams. We define ϕ as an inverter of bits. So $\phi(f) = 1100...$

The key point here is that $\phi(f)$ is another unfinishable bitstream. Because f is never fully present, ϕ can only transform prefixes into prefixes.

As we receive new bits of f, we can compute new bits of $\phi(f)$.

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Let's look at a more complicated function ψ. We define $\psi(k) = \bigoplus_{i=0}^{i=k} f(i)$. So we are recording the parity of various prefixes of f. If the number of 1s in $\{f(1), ...f(k)\}$ is odd, then $\psi(k) = 1$, else $\psi(k) = 0$.

So $f = 0011... \implies \psi(f) = 0010...$

The point of this example is to emphasize that functions have access to all prefixes so far. Any intuitively meaningful function involving the "present history" of the argument is allowed.

Let $g = 1....$ Then $\phi(g) = 1...$

The point here is to emphasize that we can use the same function on different bitstreams.

We can even do something like a redundancy function ρ that doubles every bit in its argument.

So $\rho(f) = 00001111..$ and $\rho(g) = 11..$

Output prefixes can be longer or shorter than input prefixes. What matters is that the same input prefixes are mapped to the same output prefixes.

6

We could include rational numbers in such bitstreams and develop real numbers in such a language, but it's more intuitive to use intervals.

Why not Cauchy sequences ? We don't want to choose a particular convergence rate, and without such a rate, Cauchy sequences are just too wild. Because no finite prefix gives us any information about the intuitive location of the number on the real line. And of course we only ever have finite prefixes.

But let $f(1) = (3/7, 4/7)$. Assume $f(n+1) \subset f(n)$ and that $|f(k)| \to 0$, where $|f(k)|$ is just the distance between the rational endpoints of the interval. Note that we do not require a convergence rate. But at least the indeterminateness of f is at least itself determined.

Now our metaphor might be more and more precise measurements of an empirical quantity. Since no prefix can be revised (a sequence can't change its mind), this is an idealization in which no measurement is thrown out. Measurements can only get better, and we have a guarantee that they will get arbitrarily good, though we don't know when.

Interval arithmetic is well developed. So we have fg and $f+g$ already well defined. We also have polynomials. Do we have e^x ? Not yet, since we've insisted that our real numbers are sequences of rationals. We could however build an e^x that would work.

Let our exponential function be

$$E(f(k)) = (\sum_{i=0}^{i=k} \frac{a_k^i}{i!}, \sum_{i=0}^{i=k} \frac{a_k^i}{i!} + \frac{1}{k})$$

.

Here a_k is the lower bound of $f(k)$. Note that $E(f(k))$ converges (intuitively at least) to e^f. This is one of many ways to accomplish the same goal.

But is the notion of convergence changed in a context where

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In 2 dimensions, we have a sequence of circles. Each circle must fit within all previous circles. The radius of next circle gets arbitrarily small eventually.

We can project this idea back on the 1 dimensional case by expressing the intervals above in terms of $f(k) = [c_k - r_k, c_k + r_k]$, with $r_k \geq 0$. This makes c_k a fundamental sequence of rational numbers. We can also use $f(k) = (c_k - r_k, c_k + r_k)$, with $r_k > 0$. This forces the "circles" to shrink at every click forward in time.

We can construct/hypothesize rational real numbers like $f(k) = (q - \frac{1}{k}, q + \frac{1}{k})$.

We can also define a weaker notion of equivalence. This weaker notion is the normal notion, basically associating intervals or fundamental sequences when their distance goes to zero.