Chapter 8
Introduction to Graphs

Discrete Structures for Computing on 09 May 2014

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Contents

Motivations

The need of the graph

- Representation/Storing
- Searching/sorting
- Optimization

Its applications

- Electric circuit/board
- Chemical structure
- Networking
- Map, geometry
- ...

Graph

Definition

A graph (*đồ thị*) G is a pair of (V, E), which are:

- V – nonempty set of vertices (nodes) (*đỉnh*)
- E – set of edges (*cạnh*)

A graph captures abstract relationships between vertices.

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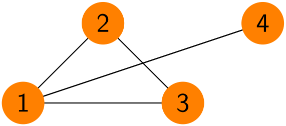

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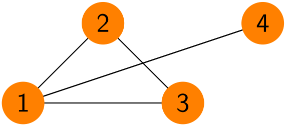
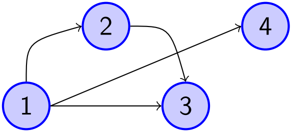


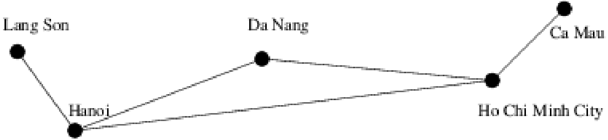


Undirected Graph (Đồ thị vô hướng)

Definition (Simple graph (đơn đồ thị))

- Each edge connects two different vertices, and
- No two edges connect the same pair of vertices

An edge between two vertices u and v is denoted as $\{u, v\}$



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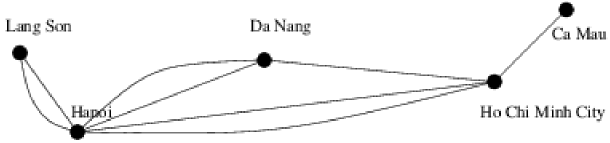
Undirected Graph

Definition (Multigraph (đa đồ thị))

Graphs that may have multiple edges connecting the same vertices.

An unordered pair of vertices $\{u, v\}$ are called multiplicity m (*bội* m) if it has m different edges between.

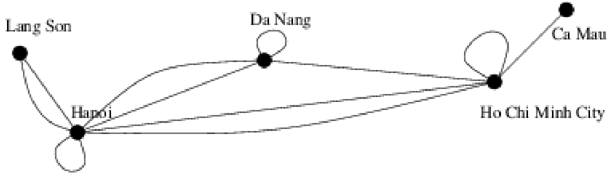


Undirected Graph

Definition (Pseudograph (giả đồ thị))

Are multigraphs that have

- loops (*khuyên*)– edges that connect a vertex to itself



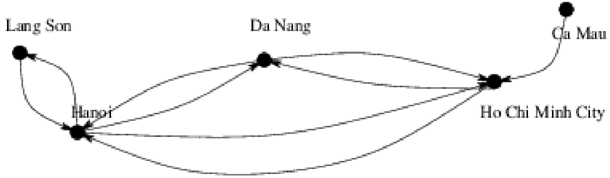
Directed Graph

Definition (Directed Graph (đồ thị có hướng))

A directed graph G is a pair of (V, E), in which:

- V – nonempty set of vertices
- E – set of directed edges (*cạnh có hướng*)

A directed edge start at u and end at v is denoted as (u, v).



Terminologies For Undirected Graph

Neighborhood

In an undirected graph $G = (V, E)$,

- two vertices u and $v \in V$ are called **adjacent** (*liền kề*) if they are **end-points** (*điểm đầu mút*) of edge $e \in E$, and
- e is **incident with** (*cạnh liên thuộc*) u and v
- e is said to **connect** (*cạnh nối*) u and v;

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The degree of a vertex

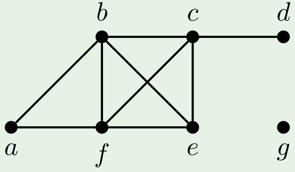
The **degree of a vertex** (*bậc của một đỉnh*), denoted by $\deg(v)$ is the number of edges incident with it, except that a loop contributes twice to the degree of that vertex.

- isolated vertex (*đỉnh cô lập*): vertex of degree 0
- pendant vertex (*đỉnh treo*): vertex of degree 1

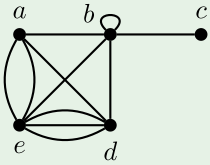
Example

Example

What are the degrees and neighborhoods of the vertices in these graphs?



G



H

Solution

In G, deg$(a) = 2$, deg$(b) =$ deg$(c) =$ deg$(f) = 4$, deg$(d) = 1$, …

Neiborhoods of these vertices are

$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, \ldots$

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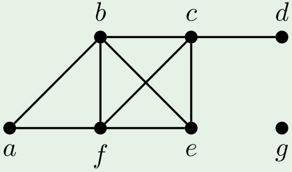

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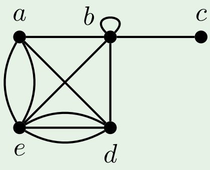

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Neiborhoods of these vertices are
$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, \ldots$
In H, deg$(a) = 4$, deg$(b) =$ deg$(e) = 6$, deg$(c) = 1$, ...
Neiborhoods of these vertices are
$N(a) = \{b, d, e\}, N(b) = \{a, b, c, d, e\}, \ldots$

Basic Theorems

Theorem (The Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Basic Theorems

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$$2m = \sum_{v \in V} deg(v)$$

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Theorem

An undirected graph has an even number of vertices of odd degree.

Terminologies for Directed Graph

Neighborhood

In an directed graph $G = (V, E)$,

- u is said to be **adjacent to** (*nôi tới*) v and v is said to be adjacent from (*được nối từ*) u if (u, v) is an edge of G, and
- u is called **initial vertex** (*đỉnh đầu*) of (u, v)
- v is called **terminal** (*đỉnh cuối*) or **end vertex** of (u, v)
- the initial vertex and terminal vertex of a loop are the same.

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The degree of a vertex

In a graph G with directed edges:

- **in-degree** (*bậc vào*) of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex.
- **out-degree** (*bậc ra*) of a vertex v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex.

Note: a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

Basic Theorem

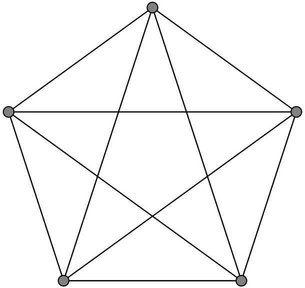
Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

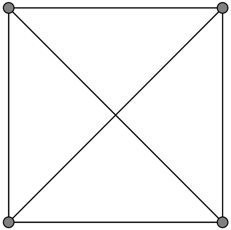
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Complete Graphs

A complete graph (*đồ thị đầy đủ*) on n vertices, K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.



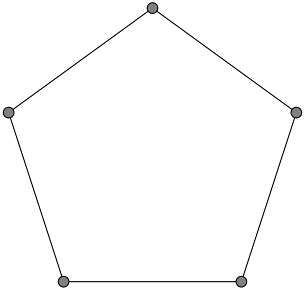
K_5



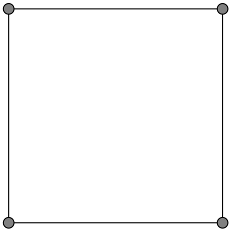
K_4

Cycles

A cycle (*đồ thị vòng*) C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



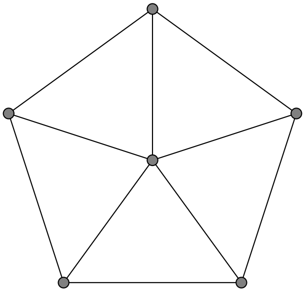
C_5



C_4

Wheels

We obtain a wheel (*đồ thị hình bánh xe*) W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n.



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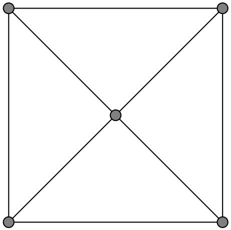
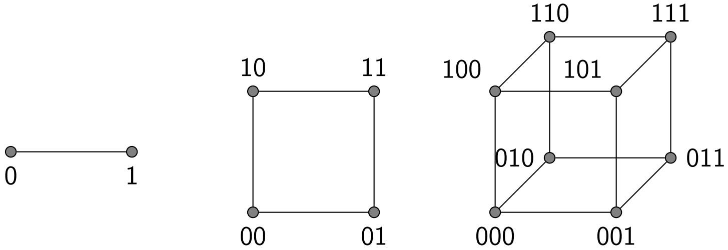


W_4

n-cube

An n-dimensional hypercube (*khối* n *chiều*), Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent iff the bit strings that they represent differ in exactly one bit position.



Q_1 Q_2 Q_3

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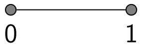
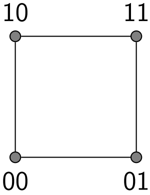
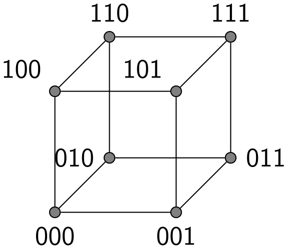


Q_1



Q_2



Q_3

What's about Q_4?

Applications of Special Graphs

- Local networks topologies
  - Star, ring, hybrid
- Parallel processing
  - Linear array
  - Mesh network

Exercise

Exercise (1)

Is there any simple graph including four vertices that their degrees are respectively 1, 1, 2, 2 ?

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Is there any simple graph including six vertices that their degree are respectively 2, 3, 3, 3, 3, 3 ?

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What is the largest number of edges a simple graph with 10 vertices can have ?

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Exercise (3)

What is the largest number of edges a simple graph with 10 vertices can have ?

Exercise (4)

An undirected simple graph G has 15 edges, 3 vertices of degree 4 and other vertices having degree 3. What is the number of vertices of the graph G?

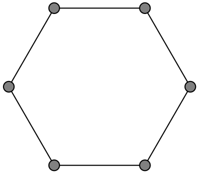
Bipartite Graphs

Definition

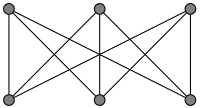
A simple graph G is called bipartite (*đồ thị phân đôi*) if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2)

Example

C_6 is bipartite



C_6

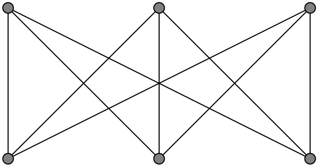


Complete Bipartite Graphs

Definition

A complete bipartite $K_{m,n}$ is a graph that

- has its vertex set partitioned into two subsets of m and n vertices, respectively,
- with an edge between two vertices iff one vertex is in the first subset and the other is in the second one

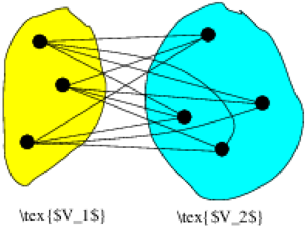


$K_{3,3}$

Bipartite graphs

Example (Bipartite graphs?)

- C_6
- C_n
- K_3
- K_n
- the following graph



New Graph From Old

Definition

A **subgraph** (*đồ thị con*) of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.

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Definition

The **union** (*hợp*) of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

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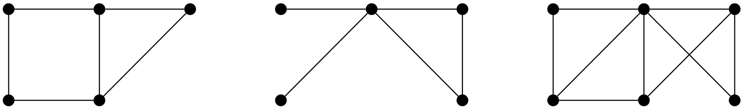
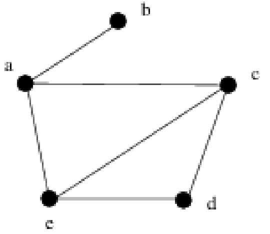


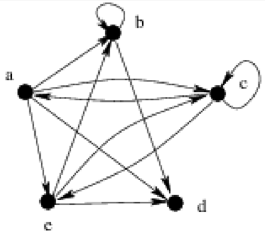
G_1 G_2 $G_1 \cup G_2$

Adjacency Lists (Danh sách kề)

Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Initial vertex	Terminal vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	c, e
e	b, c, d



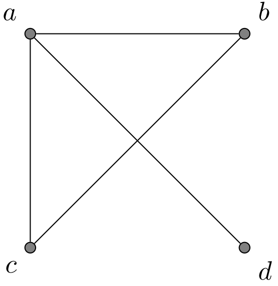


Adjacency Matrices

Definition

Adjacency matrix (*Ma trận kề*) A_G of $G = (V, E)$

- Dimension $|V| \times |V|$
- Matrix elements
  $$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Examples

Example

Give the graph defined by the following adjacency matrix

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrices

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1 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

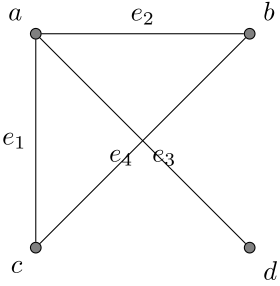
Incidence Matrices

Definition

Incidence matrix (*ma trận liên thuộc*) M_G of $G = (V, E)$

- Dimension $|V| \times |E|$
- Matrix elements

$$m_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

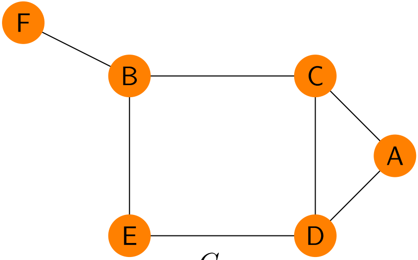


$$\begin{array}{cc} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Examples

Example

Give the incidence matrix according to the following graph

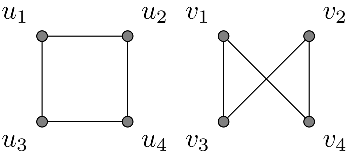


G

Graph Isomorphism

Definition

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** (*đẳng cấu*) if there is a **one-to-one function** f from V_1 to V_2 with the property that a and b are adjacent in G_1 iif $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an **isomorphism** (*một đẳng cấu*).
(i.e. there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.)



Isomorphism function $f : U \longrightarrow V$ with

$$f(u_1) = v_1 \quad f(u_2) = v_4$$
$$f(u_3) = v_3 \quad f(u_4) = v_2$$

Graph

- Does there exists the graphs with all the vertices of degree three ?
- One goat, a cabbage and a wolf is on the side of a river; a boatman wishes to transport them on the other side but, his boat being too small, he could transport that the only one of them at once. How does he have to proceed not to leave them together without surveillance: the wolf and the goat, as well as the goat and the cabbage?

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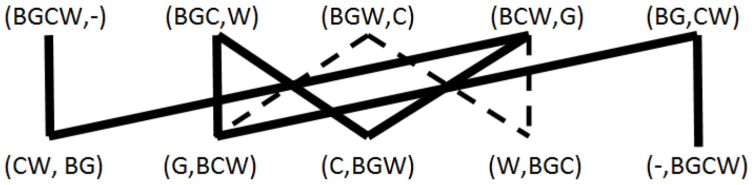
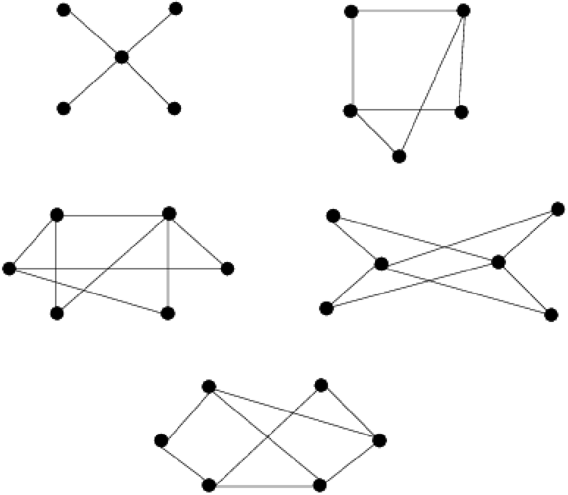
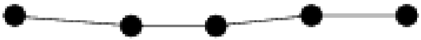


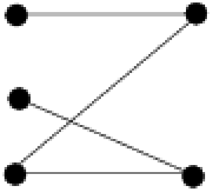
Bipartie graph

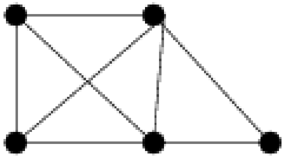


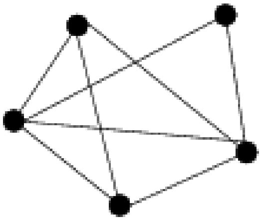
Contents

Isomorphism ?









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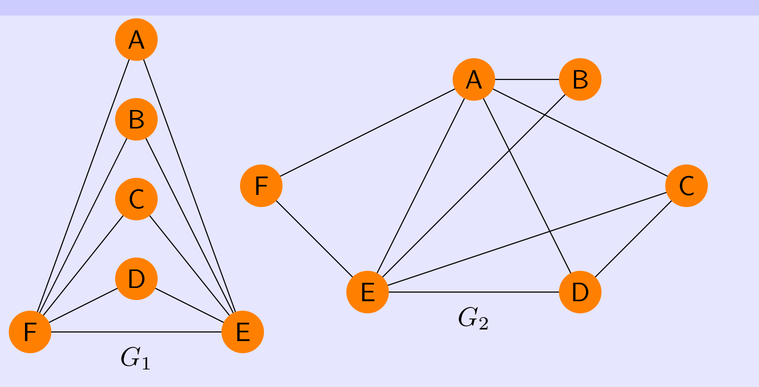
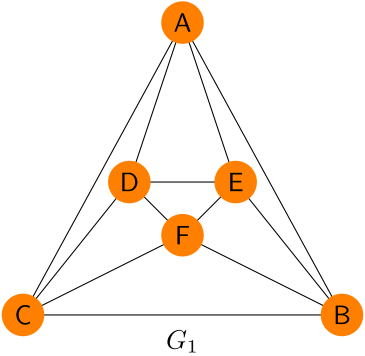
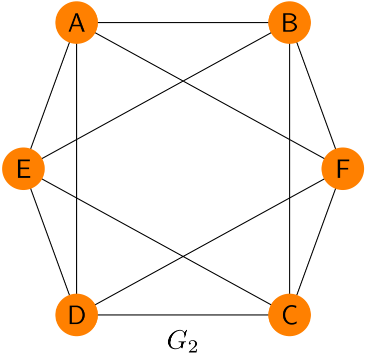
Isomorphism?



Isomorphism?





Isomorphism

Are the simple graphs with the following adjacency matrices isomorphic ?

1. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

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2. $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

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3. $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

Isomorphism

Determine whether the graphs without loops with the incidence matrices are isomorphic.

- $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

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- $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
- Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.
- Define isomorphism of directed graphs