* Worksheet 1 :

1) (a) $A = \{1, 2, a, d, 3\}$ (b) A^3 (c) $= 5^3 = 125$

2) 2^{500}

3) (a) $|A \cup B| \leq |A| + |B| = 7 \Rightarrow \boxed{\max |A \cup B| = 7}$

(b) $|A \cup B| \geq \max(|A|, |B|) = 4 \Rightarrow \boxed{\min |A \cup B| = 4}$

(c) $|A \cap B| \leq \min(|A|, |B|) = 3 \Rightarrow \boxed{\max |A \cap B| = 3}$

$|A \cap B| \geq 0 \Rightarrow \boxed{\min |A \cap B| = 0}$

4) $4! = 24$

5) 26^5

6) $\binom{10}{3} = \frac{10!}{3!\,7!} = 120$

7) $10 \times 9 \times 8 \times 7 \times 6$ OR $\binom{10}{5} \times 5! = \frac{10!}{5!} = 10 \times 9 \times 8 \times 7 \times 6$

($\binom{10}{5}$: pick 5 from 10; $5!$: order them)

*Worksheet 2 :

1) (a) $\Omega = \{A, B\}$ (b) $\Omega = \{H, T\}$ $\Omega = \{Jan, Feb., \ldots, dec\} \times \{1, 2, \ldots, 31\}$
(d) $\Omega = \{$student 1, student 2, $\ldots$, student 10$\}$ (e) $\Omega = \{$red, black, silver, green$\} \times \{$black, beige$\}$

2) (a) $\Omega = \{H, T\}^{200}$ (b) $\Omega = \{$all non-negative integers$\}$
(c) $\Omega = \{$all English words in Hamlet$\}$

3) (a) $A \cap B \cap C$ (b) $A \cup B \cup C$ (c) $A \cap B \cap C^c = (A \cap B) - C$

4) (a) $Pr(C) = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ (b) $\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 7$ (7 non-empty events)
(c) $Pr(a) = 1/2$ $Pr(b) = 1/3$ $Pr(c) = 1/6$ $Pr(a,b) = 5/6$ $Pr(a,c) = 2/3$ $Pr(b,c) = 1/2$
$Pr(a,b,c) = 1$

5) (a) event of first toss outcome to be head, $Pr\{a\} = 1/2$
(b) event of same outcome in all three tosses, $Pr\{b\} = \frac{1}{2^3} + \frac{1}{2^3} = \frac{1}{4}$
(c) event of exactly one tail in three tosses, $Pr(c) = \binom{3}{1} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$

6) $Pr(A) = 1 - Pr(A^c) = \frac{2}{3} \Rightarrow Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = \frac{2}{3} + \frac{1}{2} - \frac{1}{4} = \frac{11}{12}$

7) $\binom{6}{1} \times 1/6 \times 1/6 = 1/6$

8) $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^3 = \frac{1}{2^9}$

9) $Pr(\text{die} = n) = \frac{n}{6+5+4+3+2+1} = \frac{n}{21} \Rightarrow Pr(\text{even}) = \frac{2+4+6}{21} = \frac{12}{21} = \frac{4}{7}$

10) $Pr(\text{win}) = 1 - Pr(\text{none of three balls are from 3 chosen}) = 1 - \frac{97}{100} \times \frac{97}{100} \times \frac{97}{100} = 0.087$

* Worksheet 2 Cont'd :

11) $\frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{120}$

12) $\frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 5 \times 5 \times 5 \times 5} = \frac{5!}{5^5}$

13) $\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} \times \frac{4}{48} = 3.07 \times 10^{-7}$

14) $\frac{\binom{99}{10}}{\binom{100}{10}} = 0.33048$ OR $\frac{90}{100} \times \frac{89}{99} \times \cdots \times \frac{81}{91} = 0.33048$

15) $\frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{24}$

16) $\binom{6}{3} \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^3 = \frac{20}{64} = \frac{5}{16}$

17) (a) $\frac{1 \times \binom{6}{2}}{\binom{7}{3}} = \frac{3}{7}$

(b) $\frac{1 \times 1 \times \binom{5}{1}}{\binom{7}{3}} = \frac{1}{7}$

(c) $\frac{\binom{5}{3}}{\binom{7}{3}} = \frac{2}{7}$

* Worksheet 3 :

1) (a) $Pr(2H \mid \text{first } H) = \binom{2}{1} \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

(b) $Pr(2H \mid \text{first } T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(c) $Pr(2H \mid \text{first two } H) = \frac{1}{2}$

(d) $Pr(2H \mid \text{first two } T) = 0$

(e) $Pr(2H \mid \text{first } H, \text{third } T) = Pr(\text{second } H) = \frac{1}{2}$

2) $Pr(\text{art}) = \frac{5}{8}$ $Pr(\text{French}) = \frac{5}{8}$ $Pr(\text{art} \wedge \text{French}) = \frac{1}{4}$

(a) $Pr(\text{math}) = Pr(\text{math} \wedge \text{art}) + Pr(\text{math} \wedge \text{French}) = 1 - Pr(\text{art} \wedge \text{French}) = 1 - \frac{1}{4} = \frac{3}{4}$

(b) $Pr(\text{art} \vee \text{French}) = Pr(\text{art}) + Pr(\text{French}) - Pr(\text{art} \wedge \text{French}) = \frac{5}{8} + \frac{5}{8} - \frac{1}{4} = \frac{8}{8} = 1$

3) $Pr(H) = \frac{6}{10}$ $Pr(S) = \frac{8}{10}$ $Pr(H \vee S) = \frac{9}{10}$

$$Pr(H \wedge S) = Pr(H) + Pr(S) - Pr(H \vee S) = \frac{6}{10} + \frac{8}{10} - \frac{9}{10} = \frac{5}{10} = \frac{1}{2}$$

4) $\underbrace{Pr(\text{eye})}_{A} \geq \frac{7}{10}$ $\underbrace{Pr(\text{ear})}_{B} \geq \frac{75}{100}$ $\underbrace{Pr(\text{hand})}_{C} \geq \frac{8}{10}$ $\underbrace{Pr(\text{leg})}_{D} \geq \frac{85}{100}$

$$Pr(A \wedge B \wedge C \wedge D) = 1 - Pr(A^c \vee B^c \vee C^c \vee D^c) \geq 1 - (Pr(A^c) + Pr(B^c) + Pr(C^c) + Pr(D^c))$$

$$= 1 - (1 - Pr(A) + 1 - Pr(B) + 1 - Pr(C) + 1 - Pr(D)) = Pr(A) + Pr(B) + Pr(C) + Pr(D) - 3$$

$$\geq \frac{70 + 75 + 80 + 85}{100} - 3 = 0.1 \Rightarrow \boxed{Pr(\text{eye} \wedge \text{ear} \wedge \text{hand} \wedge \text{leg}) \geq 0.1}$$

* Worksheet 3 Cont'd :

5) (a) $Pr(h|r) = \frac{Pr(h)}{Pr(r)} Pr(r|h) = \frac{1/4}{1/2} \times 1 = 1/2$

(b) $Pr(\text{card} > 10 | h) = Pr(J, Q, K, A | h) = 4/13$

(c) $Pr(J | \text{card} > 10) = Pr(J | J, Q, K, A) = \frac{Pr(J)}{Pr(J,Q,K,A)} \underbrace{Pr(J,Q,K,A|J)}_{1}$

$= \frac{4/52}{\frac{16}{52}} \times 1 = 4/16 = 1/4$

6) $Pr(B') = 1/4 \Rightarrow Pr(B) = \frac{3}{4} \Rightarrow Pr(A|B) = \frac{Pr(A \wedge B)}{Pr(B)} = 1/2$

$\Rightarrow Pr(A \wedge B) = 1/2 \times \frac{3}{4} = \frac{3}{8}$

7) (a) $Pr(S > 7 | \text{first } 4) = Pr(\text{Second} \in \{3,4,5,6\}) = \frac{4}{6} = \frac{2}{3}$

(b) $Pr(S > 7 | \text{first } 1) = 0$

(c) $Pr(S > 7 | \text{first} > 3) = \frac{Pr(S > 7 \wedge \text{first} > 3)}{Pr(\text{first} > 3)} =$

$$\frac{Pr(\{(4,4),(4,5),(4,6),(5,3),(5,4),(5,5),(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\})}{1/2}$$

$= \frac{\frac{12}{36}}{1/2} = \frac{2}{3}$

(d) $Pr(S > 7 | \text{first} < 5) = \frac{Pr(\{(4,4),(4,5),(4,6),(3,5),(3,6),(2,6)\})}{4/6} = \frac{\frac{6}{36}}{\frac{4}{6}}$

$= \frac{1}{4}$

Worksheet 3 Cont'd:

8)(a) $1 \times \frac{3}{51} = \frac{3}{51}$

(b) $P_r(\underset{SR}{\text{Second rank}} > \underset{FR}{\text{first rank}}) = P_r(SR > FR \mid \text{first}=2)\, P_r(\text{first}=2) + P_r(SR > FR \mid \text{first}=3)\, P_r(\text{first}=3) + \dots P_r(SR > FR \mid \text{first}=4)\, P_r(\text{first}=4)$

$$= \frac{48}{51}\times\frac{4}{52} + \frac{44}{51}\times\frac{4}{52} + \frac{40}{52}\times\frac{4}{52} + \cdots + \underbrace{\frac{4}{51}\times\frac{4}{52}}_{\text{first king}} + \underbrace{0\times\frac{4}{52}}_{\text{first ace}}$$

$$= \frac{4}{52}\left(0 + \frac{4+8+12+\cdots+40+44+48}{51}\right) = \frac{4}{52}\left(\frac{4\times\frac{12(12+1)}{2}}{51}\right) = \frac{24}{51} = \boxed{\frac{8}{17}}$$

9) (a) $P_r(\text{def}) = P_r(\text{def}\mid F_1)P_r(F_1) + P_r(\text{def}\mid F_2)P_r(F_2) + P_r(\text{def}\mid F_3)P_r(F_3)$

$$= \frac{5}{100}\times\frac{25}{100} + \frac{4}{100}\times\frac{35}{100} + \frac{2}{100}\times\frac{40}{100} = \frac{205}{10000} = 0.0205$$

(b) $P_r(F_1 \mid \text{def}) = \frac{P_r(F_1)}{P_r(\text{def})} P_r(\text{def}\mid F_1) = \frac{\frac{25}{100}}{\frac{205}{10000}} \times \frac{5}{100} = \frac{125}{205} = 0.609$

10) $P_r(\underset{m}{\text{male}} \mid \underset{CB}{\text{colorblind}}) = \frac{P_r(m)}{P_r(CB)} P_r(CB\mid m) = \frac{1/2}{3/100} \times \frac{5}{100} = \boxed{\frac{5}{6}}$

$P_r(CB) = P_r(CB\mid m)P_r(m) + P_r(CB\mid F)P_r(F) = \frac{1}{2}\left(\frac{5}{100} + \frac{1}{100}\right) = \frac{3}{100}$

11)(a) $P_r(\text{Positive}) = P_r(P\mid d_1)P_r(d_1) + P_r(P\mid d_2)P_r(d_2) + P_r(P\mid d_3)P_r(d_3)$

$= \frac{1}{3}(0.8+0.6+0.4) = \boxed{0.6}$

(b) $P_r(d_1\mid P) = \frac{P_r(d_1)}{P_r(P)} P_r(P\mid d_1) = \frac{1/3}{6/10}\,\frac{8}{10} = \frac{4}{9}$

*Worksheet 3 Cont'd ?

11) (b) $Pr(d_2|P) = \frac{Pr(d_2)}{Pr(P)} Pr(P|d_2) = \frac{1/3}{6/10} \times \frac{6}{10} = 1/3$

(c) $Pr(d_3|P) = \frac{Pr(d_3)}{Pr(P)} Pr(P|d_3) = \frac{1/3}{6/10} \times \frac{4}{10} = \frac{2}{9}$

12) $Pr(\text{2headed}|\text{6H in a row}) = \frac{Pr(\text{2headed})}{Pr(\text{6H})} Pr(\text{6H}|\text{2headed})$

$= \frac{1/65}{2/65} \times 1 = 1/2$

$Pr(\text{6H}) = 1/65 \times 1 + \frac{64}{65}(1/2)^6 = \frac{2}{65}$

13) $Pr(\text{tiger}) = 1/3$ $Pr(\text{mammoth}) = 2/3$

$Pr(\text{Positive}|\text{tiger}) = 5/6$ $Pr(\text{Positive}|\text{mammoth}) = 1/3$

$Pr(\text{tiger}|\text{negative}) = \frac{Pr(\text{tiger})}{Pr(\text{negative})} Pr(\text{negative}|\text{tiger})$

$= \frac{1/3}{1/2} \times \frac{1}{6} = \boxed{1/9}$

$Pr(\text{negative}) = 1 - Pr(\text{Positive}) = 1 - (1/3 \times \frac{5}{6} + 1/3 \times \frac{2}{3}) = 1/2$

$Pr(\text{negative}|\text{tiger}) = 1 - Pr(\text{Positive}|\text{tiger}) = 1 - 5/6 = 1/6$

14) $Pr(\text{bear}|\text{scratch}) = \frac{Pr(\text{bear})}{Pr(\text{scratch})} Pr(\text{scratch}|\text{bear}) =$

$= \frac{1/4}{9/40} \times \frac{3}{5} = \boxed{\frac{2}{3}}$

$Pr(\text{Scratch}) = 1/10 \times \frac{3}{4} + \frac{3}{5} \times \frac{1}{4} = \frac{9}{40}$

* worksheet 3 cont'd *

15) $Pr(A)=\frac{1}{2}$ $Pr(B)=\frac{1}{2}$ $Pr(C)=\frac{1}{2}$ $Pr(D)=\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^3=\frac{1}{4}$

$Pr(E)=\binom{3}{1}\frac{1}{2}\times\left(\frac{1}{2}\right)^2=\frac{3}{8}$

24) (1) $Pr(A\wedge B)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}=Pr(A)\,Pr(B)\Rightarrow$ independent ✓

(2) $Pr(A\wedge D)=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{2}=\frac{1}{4}=Pr(A)\,Pr(D)\Rightarrow$ independent ✓

(3) $Pr(A\wedge E)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}\neq Pr(A)\,Pr(E)\Rightarrow$ not independent ✗

(4) $Pr(D\wedge E)=0\neq Pr(D)\,Pr(E)\Rightarrow$ not independent ✗

25) (1) $Pr(A\wedge B\wedge C)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}=Pr(A)\,Pr(B)\,Pr(C)\Rightarrow$ independent ✓

(2) $Pr(A\wedge B\wedge D)=0\neq Pr(A)\,Pr(B)\,Pr(D)\Rightarrow$ not independent ✗

(3) $Pr(C\wedge D\wedge E)=0\neq Pr(C)\,Pr(D)\,Pr(E)\Rightarrow$ not independent ✗

16) (1) $Pr(A)=\frac{13}{52}$ $Pr(B)=\frac{13}{52}$ $Pr(A\wedge B)=\frac{13}{52}\times\frac{12}{51}\Rightarrow$ not independent ✗

(2) $Pr(A)=\frac{13}{52}$ $Pr(B)=\frac{1}{4}$ $Pr(A\wedge B)=\frac{1}{52}\Rightarrow$ independent ✓

(3) $Pr(A)=\frac{4}{52}$ $Pr(B)=\frac{4}{52}$ $Pr(A\wedge B)=\frac{4}{52}\times\frac{4}{51}\Rightarrow$ not independent ✗

(4) $Pr(A)=\frac{13}{52}$ $Pr(B)=\frac{4}{52}$ $Pr(A\wedge B)=P(A\wedge B\mid \text{first is }A)\,Pr(\text{first is }A)+P(A\wedge B\mid \text{first not }A)\,Pr(\text{first not }A)$

$\Rightarrow Pr(A\wedge B)=Pr(A)\,Pr(B)\Rightarrow$ independent ✓ $=\frac{4}{52}\left(\frac{1}{4}\times\frac{3}{51}\right)+\frac{48}{52}\left(\frac{1}{4}\times\frac{4}{51}\right)=\frac{1}{52}$

17) (a) $Pr(VC\mid SD\wedge VC\wedge A)=\dfrac{Pr(VC\wedge SD\wedge VC\wedge A)}{Pr(VC\wedge A)}=\dfrac{0.2}{0.5}=\dfrac{2}{5}$

(b) $Pr(VC\wedge A)\times Pr(VC\wedge SD)=0.5\times 0.3\neq Pr(\text{both})=0.2$

not independent ✗