problem 3 : $\dim(u_1) = p \times 1 = \dim(u_2)$

a) $U = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} \Rightarrow \dim(U) = p \times 2$

$U^T = \begin{pmatrix} — u_1 — \\ — u_2 — \end{pmatrix} \Rightarrow \dim(U^T) = 2 \times p$

$UU^T = \underbrace{\begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix}}_{p\times 2} \underbrace{\begin{pmatrix} — u_1 — \\ — u_2 — \end{pmatrix}}_{2\times p} \Rightarrow \dim(UU^T) = p \times p$

• $u_1 u_1^T = \underbrace{\begin{pmatrix} | \\ u_1 \\ | \end{pmatrix}}_{p\times 1} \underbrace{\begin{pmatrix} — u_1 — \end{pmatrix}}_{1\times p} \Rightarrow \dim(u_1 u_1^T) = p \times p$

b) ① $x \longmapsto (\underbrace{u_1 . x, u_2 . x}_{\text{one single point}}) \Rightarrow$ projecting x onto one single point in 2-dimention (u_1, u_2)

② $x \longmapsto \underbrace{(u_1 . x) u_1}_{\text{projecting onto } u_1 \text{ in direction of } u_1} + (u_2 . x) u_2 \Rightarrow$ projecting x onto a vector in p-dimention in 2-d subspace of p

③ $x \longmapsto U^T x = \begin{pmatrix} — u_1 — \\ — u_2 — \end{pmatrix} \begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} u_1 . x \\ u_2 . x \end{pmatrix} = (u_1 . x, u_2 . x) \Rightarrow$ Same as ①

④ $x \longmapsto UU^T x = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} \begin{pmatrix} u_1 . x \\ u_2 . x \end{pmatrix} = (u_1 . x) u_1 + (u_2 . x) u_2$

$\Rightarrow$ Same as ②