* Worksheet 4 :

1. $Pr(X=k) = Pr(\min(X_1, X_2) = k) =$

$Pr(X_1 = k, X_2 \geq k \cup X_2 = k, X_1 \geq k) = Pr(X_1 = k, X_2 \geq k) +$

$\underbrace{Pr(X_2 = k, X_1 \geq k) - Pr(X_1 = k, X_2 = k)}$ =

X_1 & X_2 are independent

$$\frac{1}{6} \times \frac{|k-7|}{6} + \frac{1}{6}\,\frac{|k-7|}{6} - \frac{1}{6} \times \frac{1}{6} = \frac{2|k-7|-1}{36}$$

OR brute force

$Pr(X=1) = Pr(X_1 = 1 \cup X_2 = 1) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$

$Pr(X=2) = Pr(X_1 = 2 \wedge X_2 \geq 2 \vee X_2 = 2 \wedge X_1 \geq 2) = 2 \times \frac{1}{6} \times \frac{5}{6} - \frac{1}{36} = \frac{9}{36}$

$Pr(X=3) = Pr((X_1 = 3 \wedge X_2 \geq 3) \vee (X_2 = 3 \wedge X_1 \geq 3)) = \frac{1}{6} \times \frac{4}{6} \times 2 - \frac{1}{36} = \frac{7}{36}$

$Pr(X=4) = \frac{1}{6} \times \frac{3}{6} \times 2 - \frac{1}{36} = \frac{5}{36}$

$Pr(X=5) = \frac{1}{6} \times \frac{2}{6} \times 2 - \frac{1}{36} = \frac{3}{36}$

$Pr(X=6) = Pr(X_1 = 6 \wedge X_2 = 6) = \frac{1}{36}$

2. X = expected number of rolls until six is seen

$$E\{X\} = \underbrace{1}_{\text{6 is seen at first roll}} + \underbrace{\frac{5}{6} E\{X\}}_{\text{6 is not seen at first roll}} \Rightarrow E\{X\} = 6$$

OR $E\{X\} = \sum_{k=1}^{\infty} k \underbrace{\left(\frac{5}{6}\right)^{k-1}}_{\text{k-1 roll and no 6}} \underbrace{\frac{1}{6}}_{\text{kth roll is 6}} = \frac{1}{(1-\frac{5}{6})^2} \times \frac{1}{6} = 6$

$\frac{1}{p}$ rule

$$\sum_{\alpha=0}^{\infty} \alpha r^{\alpha-1} = \frac{1}{(1-r)^2}$$

3. $P(S) = 0.8$ $\quad P(nd) = 0.25$ $\quad P(be) = 0.5$

$P(S \text{ and } nd \text{ and } be) = 0.8 \times 0.25 \times 0.5 = 0.1$

$1/p$ rule $\Rightarrow E\{\#\text{ days befor all 3 happens}\} = \frac{1}{0.1} = 10$

OR $E\{\#\text{ days befor all 3 happens}\} = \sum_{k=0}^{\infty} k \left(\frac{9}{10}\right)^{k-1} \frac{1}{10} = 10$

6. # students ending up in their own bed $= X = X_1 + \dots + X_n$

$$X_i = \begin{cases} 1 & \text{if student } i \text{ ends up in his/her own bed} \\ 0 & \text{otherwise} \end{cases}$$

$E\{X\} = E\{X_1\} + \dots + E\{X_n\}$

$E\{X_i\} = 1 \times \frac{1}{n} + 0 \times \frac{n-1}{n} = 1/n$

as we assume more than one student can end up in one bed, at any point the probability of student i to get to his/her own bed is $\frac{1}{n}$

$\Rightarrow E\{X\} = \frac{1}{n} + \dots + \frac{1}{n} = n \times \frac{1}{n} = 1$

7. (a) not independent

$Pr(X=1) = \frac{1}{n}$ $\quad Pr(Y=1) = \frac{1}{n}$ $\quad Pr(X=1, Y=1) = \frac{1}{n} \times \frac{1}{n-1}$

$Pr(X=1 \wedge Y=1) \neq Pr(X=1)\, Pr(Y=1)$

(c) independent $\quad Pr(X=k) = \begin{cases} \frac{48}{52} & k=0 \\ \frac{4}{52} & k=1 \end{cases}$ $\quad Pr(Y=k) = \begin{cases} \frac{39}{52} & k=0 \\ \frac{13}{52} & k=1 \end{cases}$

to be Cont'd

7.(c) $\Pr(X=k_1 \cap Y=k_2) = \begin{cases} \frac{48}{52} \times \frac{36}{48} & k_1=k_2=0 \\ \frac{48}{52} \times \frac{12}{48} & k_1=0,\ k_2=1 \\ \frac{4}{52} \times \frac{3}{4} & k_2=0,\ k_1=1 \\ \frac{4}{52} \times \frac{1}{4} & k_1=k_2=1 \end{cases}$

in all four cases $\Pr(X=k_1 \cap Y=k_2) = \Pr(X=k_1)\Pr(Y=k_2)$

8.(a) $E(Z) = \frac{1}{8}(1+2+3+4) + \frac{1}{4}(5+6) = 4$

$\mathrm{var}(Z) = E(Z^2) - E(Z)^2 = \frac{1}{8}(1+2^2+3^2+4^2) + \frac{1}{4}(5^2+6^2)$

$- 4^2 = \frac{30}{8} + \frac{61}{4} - 16$

$\mathrm{var}(Z) = 3$

(b) $X = X_1 + \ldots + X_{10}$

$E(X) = E\{X_1\} + \ldots + E\{X_{10}\} = 10 \times 4 = 40$

X_i's are independent → $\mathrm{var}(X) = \mathrm{var}\{X_1\} + \ldots + \mathrm{var}\{X_{10}\} = 10 \times 3 = 30$

(c) $A = \frac{X_1 + \ldots + X_n}{n} \Rightarrow E\{A\} = \frac{1}{n}\left[E\{X_1\} + \ldots + E\{X_n\}\right]$

$= \frac{4 \times n}{n} = 4$

$\mathrm{var}\{A\} = \frac{1}{n^2}\left[\mathrm{var}\{X_1\} + \ldots + \mathrm{var}\{X_n\}\right] =$

$= \frac{1}{n^2} \times n \times 3 = \frac{3}{n}$

12. $E\{X\} = \underbrace{1}_{\substack{\text{first toss} \\ \text{either head} \\ \text{or tail}}} + \overbrace{E\{Y\}}^{\text{expected \# of toss before the first tail/head}} = 1 + \frac{1}{1/2} = 3$

X : two heads / tails

OR $E\{X\} = \Pr(\text{first toss head})\, E\{X \mid \text{first head}\}$

$+ \Pr(\text{first toss tail})\, \underbrace{E\{X \mid \text{first tail}\}}_{1 + E\{\text{next tail}\}}$

$\Rightarrow E\{X\} = 1/2\,(1 + E\{\text{next head}\})$

$+ 1/2\,(1 + E\{\text{next tail}\}) \rightarrow 1/1/2 \leftarrow 1/\text{prob}$

$= 1/2\,(1 + 1/(1/2)) + 1/2\,(1 + 1/(1/2)) = 3$