Problem 2. Let $s_i \in [n]$ be the pair's location at time $i \in [m]$. Let $d_i \in \mathbb{Z}_{\geq 0}$ be the odometer reading at time $i \in [m]$.

For $i, j \in [n]$, define LP_{ij} with variable x and $n \times n$ matrix of variables A:

$\max a_{ij}$, subject to

- $a_{kk} = 0$ for all $k \in [n]$
- $a_{rs} \leq a_{rk} + a_{ks}$ for all $r, s, k \in [n]$
- $d_k \leq x + \sum_{r=1}^{k-1} a_{s_r, s_{r+1}} \leq d_k + 1$ for all $k \in [m]$
- $x \geq 0$ and $a_{rs} \geq 0$ for all $r, s \in [n]$.

The output is $\hat{A}$ where $\hat{A}_{ij} = \text{value}(LP_{ij})$.

Intuitively, x represents the initial (real-valued) reading of the odometer and a_{rs} represents the shortest distance from city r to city s. The 2nd set of constraints encodes the requirement that the shortest path from r to s through k consists of the shortest path from r to k and the shortest path from k to s. The 3rd set of constraints encodes the requirement that the distances are consistent with the travel log readings.

Let D_{ij} be the optimal upper bound on the shortest-path distance from city i to city j which can be inferred from the travel log. For $0 < \epsilon < 1$, let LP_{ij}^{ϵ} denote LP_{ij}, but with the 3rd set of constraints replaced by

$$d_k \leq x + \sum_{r=1}^{k-1} a_{s_r, s_{r+1}} \leq d_k + 1 - \epsilon$$

If $\epsilon' < \epsilon < 1$, then $\text{value}(LP_{ij}^{\epsilon'}) \geq \text{value}(LP_{ij}^{\epsilon})$ since we're relaxing a constraint. In particular, $\text{value}(LP_{ij}) \geq \text{value}(LP_{ij}^{\epsilon})$. Moreover, for any $0 < \epsilon < 1$,

$$\text{value}(LP_{ij}^{\epsilon}) \leq D_{ij}$$

since this implies there exists a shortest-path matrix which is consistent with the odometer readings where the distance from i to j equals $\text{value}(LP_{ij}^{\epsilon})$. ($D_{ij} = \lim_{\epsilon \to 0^+} \text{value}(LP_{ij}^{\epsilon})$)

Since $\text{value}(LP_{ij}^{\epsilon})$ is monotone increasing as $\epsilon \to 0^+$, it follows that $\text{value}(LP_{ij}^{\epsilon})$ converges to a finite number or $\text{value}(LP_{ij}^{\epsilon}) \to \infty$ as $\epsilon \to 0^+$. In the latter case, it follows that there is no path from i to j through a sequence of consecutive cities visited in the travel log (if there were, we could find a finite bound on a_{ij}), so no finite upper bound can be inferred. So assume $\text{value}(LP_{ij}^{\epsilon})$ converges to a finite value, say L.

Ideally, we'd want to show $\text{value}(LP_{ij}) = L$, which would follow if we could argue that $\text{value}(LP_{ij}) - \text{value}(LP_{ij}^{\epsilon}) \leq k\epsilon$ for some fixed constant k, but I don't see a way to do this.