Problem 3.1.6. Let $[m] = \{0, \dots, m\}$. Suppose $f : [m] \to [m]$. Fix $X_0 \in [m]$ and define the sequence $\langle X_n \rangle_{n \geq 0}$ by $X_n = X_{n-1}$ for $n \geq 1$.

By Pigeonhole, there exist an integer $\mu + p$ such that $X_0, \dots, X_{\mu+p-1}$ are distinct but $X_{n+p} = X_n$ for $n \geq \mu$. That is, the sequence is p-periodic starting at μ.

Claim. For $a > b$, $X_a = X_b$ iff $a \geq \mu$ and p divides $b - a$.

Proof. The reverse direction is obvious. For the forward direction, suppose $a < \mu$. By definition of μ, we require $b > \mu + p$ (otherwise, $X_a = X_\mu$). But then $X_{b-kp} = X_a$ for some k such that $\mu \leq b - kp < \mu + p$. A contradiction as $X_0, \dots, X_{\mu+p-1}$ are distinct.

So WLOG say $\mu \leq a < \mu + p$. Suppose $b - a$ is not a multiple of p. Thus $b - a = pk + r$ for some $0 < r < p$. Then

$$X_b = X_{a+pk+r} = X_{a+r} = X_a.$$

If $a + r \geq \mu + p$, then $X_{a+r-p} = X_a$ where $a + r - p < \mu + p$, so in any case this is a contradiction. $\square$

Thus $X_{2n} = X_n$ iff n is a multiple of p and $n \geq \mu$. So taking n to be the smallest multiple of p bigger than μ, it follows $X_n = X_{2n}$. Furthermore, if there is an r such that $X_r = X_{2r}$, then either $n = r$ or $r \geq \mu$ and $n \equiv r \bmod p$; hence $X_n = X_r$.

Problem 3.1.7. Let $\ell(n)$ be the largest power of 2 less than or equal to n. That is, if $\ell(n) = 2^q$, we can write $n = \ell(n) + r$ where $0 \leq r < 2^q$.

Now, if $X_n = X_{\ell(n)-1}$, then by the claim above, $\ell(n) \geq \mu + 1$ and $n - \ell(n) + 1 \leq \ell(n)$ is a positive multiple of p. Hence $\ell(n) \geq p$. Therefore,

$$q \geq \lg(\max\{\mu + 1, p\}).$$

Thus, the smallest n such that $X_n = X_{\ell(n)-1}$ is $n = 2^{\lceil \lg \max\{\mu+1,p\} \rceil} + p - 1$.