LINEAR ALGEBRA

FALL 2017

$$\begin{bmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

Contents

1 Vector Spaces

1.1 Vector Space

Definition (Vector Space). A vector space V over a field $\mathbb{F}$ is a set with two binary operations, $(+)$ and $(\cdot)$, satisfying

1) $x + y \in V$ for all $x, y \in V$
2) $cx \in V$ for all $c \in \mathbb{F}$ and $x \in V$

VS1) $x + y = y + x$ for all $x, y \in V$

VS2) $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$

VS3) $\exists \bar{0} \in V$ such that $x + \bar{0} = x = \bar{0} + x$ for all $x \in V$

VS4) $\forall x \in V, \exists y \in V$ such that $x + y = \theta$

VS5) $\exists e \in \mathbb{F}$ such that $ex = x$ for all $x \in V$

VS6) $c(dx) = (cd)x$ for all $c, d \in \mathbb{F}$ and $x \in V$

VS7) $c(x + y) = cx + cy$ for all $x, y \in V$ and $c \in \mathbb{F}$

VS8) $(c + d)x = cx + dx$ for all $x \in V$ and $c, d \in \mathbb{F}$

Theorem 1.1: Cancellation law

Let V be a vector space and $x, y, z \in V$ such that $x + z = y + z$, then $x = y$.

Corollary. The vector $\bar{0}$ in (VS3) is unique.

Corollary. The vector y in (VS5) is unique.

Theorem 1.2

Let V be a vector space. Then,

1) $0 \cdot x = \bar{0}$ for all $x \in V$
2) $(-a) \cdot x = -(a \cdot x) = a \cdot (-x)$ for all $a \in \mathbb{F}$ and $x \in V$
3) $a \cdot \bar{0} = \bar{0}$ for all $a \in \mathbb{F}$.

1.2 Subspaces

Definition (Linear Subspace). Let $(V, +, \cdot)$ be a vector space over $\mathbb{F}$ and let $W \subset V$. W is a subspace of V if $(W, +, \cdot)$ is itself a vector space over $\mathbb{F}$.

Theorem 1.3

Let V be a vector space and $W \subset V$. Then W is a subspace of V if and only if the following hold:

1) $0_V \in W$
2) $x + y \in W$ for all $x, y \in W$
3) $ax \in W$ for all $a \in \mathbb{F}$ and $x \in W$.

Example. Let $V = \mathcal{M}_{n \times n}(\mathbb{F})$ and $W \subset V$ be the set of symmetric $n \times n$ matrices. Then, W is a subspace of V.

Theorem 1.4

Let V be a vector space. Then any intersection of subspaces of V is also a subspace.

Warning. In general, the union of two subspaces is not a subspace.

1.3 Linear Combinations

Definition (Linear Combination). Let V be a vector space and $S \subset V$. A vector $v \in V$ is called a linear combination of vectors in S if we can write

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n,$$

with $a_i \in \mathbb{F}$ and $u_i \in S$. The scalars a_i are called coefficients.

Definition (Span). Let S be a nonempty subset of V. The span of S, denoted $\operatorname{span}(S)$, is the set of all linear combinations of the vectors in S. By convention, $\operatorname{span}(\varnothing) = \{\bar{0}\}$.

Theorem 1.5

Let V be a vector space and $S \subset V$. Then, $\operatorname{span}(S)$ is a subspace of V. Moreover, any subspace of V that contains S must also contain $\operatorname{span}(S)$.

Definition (Generates). Let S be a subset of a vector space V. We say S generates V if $\operatorname{span}(S) = V$.

1.4 Linear Independence

Definition (Linear Dependence). A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors, $u_1, u_2, \ldots, u_n$, in S and scalars $a_1, a_2, \ldots, a_n$ not all zero such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0.$$

Remark 1.1. If $a_1 = a_2 = \cdots = a_n = 0$, then we always have $a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0$. This is called the trivial representation of 0 as a linear combination of $u_1, u_2, \ldots, u_n$.

Definition (Linear Independence). A subset S of a vector space V is linearly independent if it is not linearly dependent.

Facts.

- By convention, $\varnothing$ is linearly independent.
- $S = \{v\}$ with $v \neq 0$ is linearly independent.
- S is linearly independent if and only if the only representation of 0 as a linear combination of vectors in S are trivial representations.

Example. Let $V = P_n(\mathbb{R})$. Let $p_0(x) = -1$ and for $k = 1, \ldots, n$ let $p_k(x) = x^k - 1$. Then $S = \{p_0, p_1, \ldots, p_n\}$ is linearly independent.

Theorem 1.6

Let V be a vector space and let $S_1 \subset S_2 \subset V$. If S_1 is linearly dependent, then so is S_2.

Corollary. If S_2 is linearly independent, then so it S_1.

Theorem 1.7

Let S be a linearly independent subset of a vector space V. Let $v \in V$ and $v \notin S$. Then, $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$.

Remark 1.2. If $0 \in S$, then S is linearly dependent.

1.5 Bases & Dimension

Definition (Basis). A basis β for a vector space V is a linearly independent subset of V that generates V.

Theorem 1.8

Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of V. Then,

β is a basis for V if and only if every vector in V can be uniquely expressed as a linear combination of the vectors in β.

Theorem 1.9

If V is a vector space and V has a finite generating set S, then some subset of S is a basis for V. Hence, V has a finite basis.

Theorem 1.10: Replacement Theorem

Let V be a vector space generated by G with $|G| = n$. Let L be a linearly independent subset of V with $|L| = m$. Then,

1) $m \leqslant n$
2) there exists a subset H of G with $|H| = n - m$ such that $L \cup H$ generates V.

Corollary. Let V be a vector space with a finite basis. Then, every basis for V is finite and contains the same number of vectors.

Definition (Dimension). A vector space V is called finite-dimensional if it has a finite basis. The unique number of vectors in each basis for V is called the dimension of V, denoted $\dim V$. A vector space that is not finite-dimensional is called infinite-dimensional.

Corollary. Let V be a vector space with $\dim V = n$. Then,

1) Any finite generating set for V contains at least n vectors and any generating set with exactly n vectors is a basis for V.
2) Any linearly independent subset of V containing exactly n vectors is a basis for V.
3) Every linearly independent subset of V can be extended to form a basis for V.

Theorem 1.11

Let W be a subspace of a vector space V. If V is finite dimensional then so is W and $\dim W \leq \dim V$. Moreover, if $\dim W = \dim V$, then $W = V$.

Corollary. If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to form a basis for V.

2 Linear Transformations & Matrices

Definition (Linear Transformation). Let V, W be vector spaces over $\mathbb{F}$. A function $T : V \to W$ is called a linear transformation from V to W if for all $x, y \in W$ and $c \in \mathbb{F}$ we have

1) $T(x + y) = T(x) + T(y)$
2) $T(cx) = cT(x)$

Facts.

1) If T is linear, then $T(0) = 0$.
2) T is linear if and only if $T(X + cy) = T(x) + cT(y)$.
3) If T is linear, then $T(x - y) = T(x) - T(y)$.
4) T is linear if and only if for all $a_1, a_2, \ldots, a_n \in \mathbb{F}$ and $v_1, v_2, \ldots, v_n \in V$ we have $T(\sum a_i v_i) = \sum a_i T(v_i)$.

Definition (Null Space and Range). Let V, W be vector space and $T : V \to W$ be linear. The null space (kernel) of T is

$$N(T) = \{x \in V : T(x) = 0\}.$$

The range (image) of T is

$$R(T) = \{T(x) : x \in V\}.$$

Theorem 2.1

Let V, W be vector spaces and $T : V \to W$ be linear. Then, $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Theorem 2.2

Let V, W be vector spaces and $T : V \to W$ be linear. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then, $R(T) = \operatorname{span}\{T(v_1), \ldots, T(v_n)\}$.

Definition (Rank & Nullity). Let $T : V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then $\dim(N(T))$ is called the nullity of T and $\dim(R(T))$ is called the rank of T.

Theorem 2.3: Rank-Nullity Theorem

Let $T : V \to W$ be linear. If V is finite-dimensional then

$$\dim(V) = \text{nullity}(T) + \text{rank}(T).$$

Theorem 2.4

Let $T : V \to W$ be linear. Then, T is injective if and only if $N(T) = \{0\}$.

Theorem 2.5

Let V, W be finite-dimensional vector space with $\dim V = \dim W$. Let $T : V \to W$ be linear. Then, the following are equivalent

1) T is injective
2) T is surjective
3) $\text{rank}(T) = \dim V$.

Theorem 2.6

Let V, W be vector spaces over $\mathbb{F}$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $w_1, w_2, \ldots, w_n$ be (not necessarily distinct) vectors in W. Then, there is exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, \ldots, n$.

2.1 Matrices of Linear Transformations

Definition (Ordered Basis). Let V be a vector space. An ordered basis is a basis for V with a specified order.

Definition (Coordinate Vector). Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for a finite-dimensional vector space V. For $x \in V$, let $a_1, a_2, \ldots, a_n$ be the unique scalars in $\mathbb{F}$ such that $x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$. The coordinate vector of x relative to β is

$$[x]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Definition (Matrix of a Linear Transformation). Let V, W be finite-dimensional vector spaces with $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_n\}$ ordered bases for V, W, respectively. Let $T : V \to W$ be linear. Then, the matrix representation of T with respect to β and γ is the $m \times n$ matrix

$$A = [T]_\beta^\gamma \text{ whose } j\text{th column is } [T(v_j)]_\gamma.$$

Theorem 2.7

Let V, W be vector spaces over $\mathbb{F}$. Let $T, U : V \to W$ be linear. Define $T + U : V \to W$ by $(T + U)(x) = T(x) + U(x)$ for all $x \in V$ and for $c \in \mathbb{F}$ define $cT : V \to W$ by $(cT)(x) = c \cdot T(x)$ for all $x \in V$. Then,

(a) $cT + U$ is linear

(b) the set of all linear transformations from $V \to W$ with addition and scalar multiplication defined as above is a vector space over $\mathbb{F}$.

The vector space describe in Theorem 2.7(b) is denoted $\mathcal{L}(V, W)$.

Theorem 2.8

Let V, W be finite-dimensional vector spaces with ordered basis β, γ, respectively. Let $T, U : V \to W$ be linear and $c \in \mathbb{F}$. Then,

(a) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

(b) $[cT]_\beta^\gamma = c[T]_\beta^\gamma$.

2.2 Composition

Theorem 2.9

Let V, W, Z be vector spaces over $\mathbb{F}$. Let $T : V \to W$ and $U : W \to Z$ be linear. Then $UT : V \to Z$ is linear.

Theorem 2.10

Let $T, U_1, U_2 \in \mathcal{L}(V, W)$. Then,

1) $T(U_1 + U_2) = T(U_1) + T(U_2)$

2) $T(U_1U_2) = (TU_1)U_2$

3) $TI_V = T = I_VT$

4) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$

Theorem 2.11

Let V, W, Z be finite-dimensional vector spaces with bases α, β, γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear. Then, $[UT]_\alpha^\gamma = [U]_\beta^\gamma[T]_\alpha^\beta$.

Definition (Kronecker Delta).

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorems 2.12 and 2.13 intentionally are omitted.

Theorem 2.14

Let V, W be finite-dimensional vector spaces with ordered basis β, γ, respectively. Let $T : V \to W$ be linear. Then, for each $v \in V$ we have

$$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta.$$

Definition. Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ and define $L_A : \mathbb{F}^n \to \mathbb{F}^m$ by $L_A(x) = Ax$ where x is a column vector in $\mathbb{F}^n$. Then L_A is called the left multiplication transformation by A.

Theorem 2.15

Let $A, B \in \mathcal{M}_{m\times n}(\mathbb{F})$ and let β, γ be the standard ordered basis of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively. Then, $L_A, L_B : \mathbb{F}^n \to \mathbb{F}^m$ are linear and

1) $[L_A]_\beta^\gamma = A$
2) $L_A = L_B$ if and only if $A = B$
3) $L_{aA+B} = aL_A + L_B$ for any $a \in \mathbb{F}$
4) If $T : \mathbb{F}^n \to \mathbb{F}^m$ is linear then $T = L_C$ where $C = [T]_\beta^\gamma$.
5) If $E \in \mathcal{M}_{n\times p}$ then $L_{AE} = L_A L_E$
6) If $m = n$ then $L_{I_n} = I_{\mathbb{F}^n}$.

Theorem 2.16 says that matrix multiplication is associative.

2.3 Invertibility & Isomorphism

Definition. Let $T : V \to W$ be linear. A function $U : W \to V$ is called an inverse of T if $UT = I_V$ and $TU = I_W$. If U exists then T is called invertible.

Facts.

- If T is invertible, then its inverse is unique. We denote it by T^{-1}.
- $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ for any invertible S, T.
- $(T^{-1})^{-1} = T$

- T is invertible if and only if T is one-to-one and onto.
- If $T : V \to W$ is linear and $\dim V = \dim W$, then T is invertible if and only if $\dim V = \text{rank}T$.

Theorem 2.17

Let $T : V \to W$ be linear and invertible. Then, T^{-1} is linear.

Lemma. Let $T : V \to W$ be linear and invertible. Then

1) V is finite-dimensional if and only if W is finite-dimensional.
2) If V is finite-dimensional, then $\dim V = \dim W$.

Theorem 2.18

If V, W are finite-dimensional vector spaces with ordered basis β and γ, respectively and $T : V \to W$ is linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Moreover, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Definition (Isomorphism). Let V, W be vector spaces. We say V is isomorphic to W, denoted $V \cong W$ if there exists an invertible linear transformation $T : V \to W$. WE say T is an isomorphism.

Theorem 2.19

Let V, W be finite-dimensional vector spaces. Then $V \cong W$ if and only if $\dim V = \dim W$.

Corollary. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim V = n$. Then $V \cong \mathbb{F}^n$.

Theorem 2.20

Let V, W be finite-dimensional vector spaces with $\dim V = n$ and $\dim W = m$. Define $\Phi : \mathcal{L}(V, W) \to \mathcal{M}_{m\times n}(\mathbb{F})$ by $\Phi(T) = [T]_\beta^\gamma$ where β, γ are ordered bases for V and W, respectively. Then Φ is an isomorphism.

Corollary. $\mathcal{L}(V, W)$ is finite-dimensional and $\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$.

The function $\phi : V \to \mathbb{F}^n$ by $\phi_\beta(v) = [v]_\beta$ is an isomorphism called the standard representation of V with respect to the basis β. Then, $\phi_\gamma \circ T = L_A \circ \phi_\beta$.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\
\mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^m
\end{tikzcd}$$

2.4 Change of Basis

Theorem 2.21

Let β, β' be ordered basis for V and let $Q = [I_V]_{\beta'}^{\beta}$. Then

1) Q is invertible
2) for any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

Theorem 2.22

Let V be a finite-dimensional vector space with ordered bases β and β'. Let $T : V \to V$ be linear. Let Q be the change of basis matrix from β' to β. Then

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q.$$

3 Elementary Matrices

Definition (Elementary Row Operations). Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. The following are called elementary row (column) operations:

1) Interchanging two rows (columns) of A
2) Multiplying a row (column) of A be a non-zero scalar
3) Adding a scalar multiple of one row (column) to another row (column)

Definition (Elementary Matrices). An $n \times n$ matrix is elementary if it is obtained by performing one elementary row operation on $I_{n\times n}$.

Theorem 3.1

Let $A, B \in \mathcal{M}_{m\times n}(\mathbb{F})$ and suppose B is obtained from A by performing one elementary row (column) operation. Then there exists an $m \times m$ ($n \times n$) elementary matrix E such that $B = EA$ ($B = AE$). In fact, E is the elementary matrix obtained from the identity matrix by the same row (column) operation. Conversely, if E is an elementary matrix, then EA (AE) is the matrix obtained by performing the corresponding row (column) operation on A.

Theorem 3.2. Elementary matrices are invertible and their inverse is an elementary matrix of the same type.

3.1 Matrix Rank

Definition (Rank of a Matrix). Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Then rank of A is the rank of $L_A : \mathbb{F}^n \to \mathbb{F}^m$.

Theorem 3.3

Let V, W be finite-dimensional vector spaces and $T : V \to W$ be linear. Let β and γ be ordered basis for V, W, respectively. Then, $\operatorname{rank}[T]_\beta^\gamma = \operatorname{rank} T$.

Theorem 3.4

Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $P_{m\times m}$, $Q_{n\times n}$ be invertible matrices. Then,

1) $\operatorname{rank}(PAQ) = \operatorname{rank}(A)$
2) $\operatorname{rank}(PA) = \operatorname{rank}(A) = \operatorname{rank}(AQ)$.

Theorem 3.5

The rank of $A_{m\times n}$ is the number of linearly independent columns of A.

Theorem 3.6

Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ with rank r. Then

1) $r \le \min\{m, n\}$
2) A can be transformed by finitely-many EROs/ECOs into

$$D = \left[\begin{array}{c|c} I_{r\times r} & O \\ \hline O & O \end{array}\right].$$

Corollary. The matrix A from theorem 3.6 can be written as $A = BDC$ where B, C are invertible matrices of appropriate dimension.

Furthermore, (a) $\operatorname{rank}(A) = \operatorname{rank}(A^t)$ (b) $\operatorname{rank}(A)$ is the dimension of the subspace of $\mathbb{F}^n$ generated by the columns of A. (c) the column space and row space of A have the same dimension.

Lastly, every invertible matrix can be expressed as a product of elementary matrices.

Theorem 3.7

Let V, W, Z be finite-dimensional vector spaces and let $T : V \to W$ and $U : W \to Z$ be linear. Let A, B be matrices such that AB is defined. Then

1) $\operatorname{rank} UT \le \min\{\operatorname{rank} U, \operatorname{rank} T\}$
2) $\operatorname{rank} AB \le \min\{\operatorname{rank} A, \operatorname{rank} B\}$

4 Determinants

Definition (n-linear Function). Let $f : V_1 \times \cdots \times V_n \to W$, where V_i and W are vector spaces over $\mathbb{F}$, be a function such that for each i if we fix all variables but v_i, then $f(v_1, \ldots, v_n)$ is linear in v_i.

Definition (Determinant Function). A function $D : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ satisfying

1) D is n-linear in the columns of $M \in \mathcal{M}_{n\times n}(\mathbb{F})$.
2) If M has two identical columns, then $D(M) = 0$.
3) $D(I) = 1$.

is called a determinant function.

Lemma. Let D be a determinant function and let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$. Suppose B is obtained from A by interchanging two columns. Then $D(B) = -D(A)$, i.e. D is alternating.

Let A be an $n \times n$ matrix, then the ij-minor of A, denoted $\widetilde{A}_{ij}$, is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

Theorem 4.1: Existence and Uniqueness of det

Suppose $D : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is a determinant function. Fix i such that $1 \leq i \leq n$. For $A \in \mathcal{M}_{n\times n}(\mathbb{F})$ define

$$\widetilde{D}(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} D(\widetilde{A}_{ij}),$$

then $\widetilde{D}$ is a determinant function. Moreover, such a function is unique for each n and we denote $\widetilde{D}$ by det.

Theorem 4.2

For $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$, $\det(AB) = \det(A)\det(B)$.

Theorem 4.3

Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$.

1) If B is obtained from A by interchanging two rows of A, then $\det(B) = -\det(A)$.
2) If B is obtained from A by multiplying a row of A by a scalar k, then $\det(B) = k\det(A)$.

3) If B is obtained from A by adding a scalar multiple of one row to another, then $\det(B) = \det(A)$.

Theorem 4.4

If $A \in \mathcal{M}_{n\times n}(\mathbb{F})$, then A is invertible if and only if $\det(A) \neq 0$.

Theorem 4.5

For any $A\mathcal{M}_{n\times n}(\mathbb{F})$, $\det(A) = \det(A^t)$. As a corollary, we can compute $\det(A)$ by expanding along any column.

5 Diagonalization

5.1 Eigen-everything

Definition (Diagonalizable). A linear operator $T : V \to V$ is diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is diagonal. An $n \times n$ matrix A is diagonalizable iff L_A is diagonalizable.

Definition (Eigenvector & Eigenvalues). Let $T : V \to V$ be linear. An eigenvector of T is a non-zero vector $v \in V$ such that $T(v) = \lambda v$ for some scalar λ. The scalar λ is the eigenvalue of T corresponding to the eigenvector v. Eigenvector/values of a matrix are defined similarly.

Theorem 5.1

$T : V \to V$ is diagonalizable if and only if there is an ordered basis for V consisting of eigenvectors of T. Moreover, if T is diagonalizable and $\beta = \{v_1, \ldots, v_n\}$ is an ordered basis of eigenvectors. Then $[T]_\beta$ is a diagonal matrix and $([T]_\beta)_{jj} = \lambda_j$ the eigenvalue corresponding to eigenvector v_j.

Theorem 5.2

Let $A \in \mathcal{M}_{n\times n}(\mathbb{F})$. Then $\lambda \in \mathbb{F}$ is an eigenvalue if and only if $\det(A-\lambda I) = 0$.

Definition. The characteristic polynomial of A is $f(t) = \det(A - tI)$.

Theorem 5.4

Let $T : V \to V$ be linear. A vector $v \in V$ is an eigenvalue of T corresponding to the eigenvalue λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.

Example. Consider $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$. The characteristic polynomial of A is $f(t) = (t-2)(t-3)$. Thus $\lambda = 2, 3$ are eigenvalues of A. By theorem 5.4, these eigenvalues respectively correspond to the eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Since they are linearly independent they form a basis β. If we let $Q = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$, then $[L_A]_\beta = Q^{-1}AQ$ is a diagonal matrix by theorem 5.1.

Theorem 5.5

Let $T : V \to V$ be linear and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of T. Let $v_1, \ldots, v_k$ be eigenvectors of T such that v_i corresponds to λ_i for each $1 \leq i \leq k$. Then $\{v_1, \ldots, v_k\}$ is linearly independent.

Corollary. Let $T : V \to V$ be linear where $\dim V = n$. If T has n distinct eigenvalues then T is diagonalizable.

Definition. A polynomial $f(t)$ in $\mathcal{P}(\mathbb{F})$ splits over $\mathbb{F}$ if there are scalars $c, a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that $f(t) = c(t - a_1) \cdots (t - a_n)$.

Theorem 5.6

If T is diagonalizable, then the characteristic polynomial of T splits.

Definition (Eigenspace). Let $T : V \to V$ be linear and λ an eigenvalue of T. The eigenspace of T corresponding to λ is the subspace $E_\lambda = \{x \in V : T(x) = \lambda x\}$.

Theorem 5.7

Let $T : V \to V$ be linear with eigenvalue λ having multiplicity m. Then $1 \leq \dim E_\lambda \leq m$.

Lemma. Let $T : V \to V$ be linear. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues and for each $1 \leq i \leq n$ choose $v_i \in E_{\lambda_i}$. If $v_1 + \cdots + v_n = 0$, then $v_i = 0$ for all $1 \leq i \leq n$.

Theorem 5.8

let $T : V \to V$ be linear and $\lambda_1, \lambda_2, \ldots, \lambda_n$ distinct eigenvalues. For each

$1 \leq i \leq n$ let $S_i \subset E_{\lambda_i}$ be linearly independent. Then,

$$S = \bigcup_{i=1}^{n} S_i$$

is linearly independent.

Theorem 5.9

Let $T : V \to V$ be linear such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of T with multiplicities $m_1, m_2, \ldots, m_n$, respectively. Then

1) T is diagonalizable if and only if $\dim E_{\lambda_i} = m_i$ for each $i = 1, \ldots, n$.

2) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} then $\beta = \cup \beta_i$ is an ordered basis for V consisting of eigenvalues.

Definition. Let T be a linear operator on V. A subspace W of V is T-invariant if $T(W) \subset W$.

The subspace $W = \operatorname{span}(\{x, T(x), T^2(x), \ldots\})$ is called the T-cyclic subspace of V generated by x.

Theorem 5.10

Let T be a linear operator on V and let W be a T-invariant subspace. Then the characteristic polynomial of T_W divides that of T.

Theorem 5.11

Let T be a linear operator on V and W a T-cyclic subspace generated by a nonzero vector $v \in V$. Let $\dim W = k$. Then

1) $\{v, T(v), \ldots, T^{k-1}(v)\}$ is a basis for W

2) If $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$ then the characteristic polynomial of T_W is $f(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$.

Theorem 5.12: Cayley-Hamilton

Let T be a linear operator on V and let $f(t)$ be its characteristic polynomial. Then $f(T) = T_0$.

6 Inner Product Spaces

Definition (Inner Product). Let V be a vector space. An inner product on V is a binary operation which given $x, y \in V$ outputs $\langle x, y\rangle \in \mathbb{F}$ such that

1) $\langle cx + z, y\rangle = c\langle x, y\rangle + \langle z, y\rangle$
2) $\langle x, y\rangle = \overline{\langle y, x\rangle}$
3) $\langle x, x\rangle \geq 0$ with equality if and only if $x = 0$.

An inner product space is a vector space equipped with an inner product.

Examples.

- Let $V = \mathbb{F}^n$ and $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$. Then $\langle x, y\rangle = \sum a_i \bar{b}_i$ is an inner product.
- Let $V = \mathcal{P}(\mathbb{C})$ then $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dt$ is an inner product.
- Let $V = \mathcal{M}_{m\times n}(\mathbb{F})$ and define $\langle A, B\rangle = \mathrm{tr}(B^*A)$, where B^* is the conjugate transpose of B, is an inner product. (Frobenius)

For any inner product space V, the norm is a function $\|\cdot\| : V \to \mathbb{F}$ defined by $\|v\| = \sqrt{\langle v, v\rangle}$.

Theorem 6.1: Properties of the Inner Product

Let V be an inner product space and $x, y, z \in V$ and $c \in \mathbb{F}$. Then

- $\langle x, cy + z\rangle = \bar{c}\langle x, y\rangle + \langle x, z\rangle$
- $\langle x, x\rangle = 0$ if and only if $x = 0$
- if $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$ then $y = z$.

Theorem 6.2: Properties of the Norm

- $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ (Cauchy-Schwarz inequality)
- $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)

Definition (Orthogonal). For an inner product space V, we say $x, y \in V$ are orthogonal if $\langle x, y\rangle = 0$. A subset S of V is orthogonal if x and y are orthogonal for any $x, y \in S$.

6.1 Gram-Schmidt Orthonormalization

Theorem 6.3

Let V be an inner product space and $S = \{v_1, v_2, \dots, v_n\}$ be orthogonal with $v_i \neq 0$. Suppose $y \in \text{span}S$, then $y = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ where

$$a_i = \frac{\langle y, v_i \rangle}{\|v_i\|^2}.$$

Corollary. Any set of nonzero orthogonal vectors is linearly independent.

Theorem 6.4: Gram-Schmidt Orthogonalization

Let V be an inner product space and let $S = \{w_1, w_2, \dots, w_n\}$ be a linearly independent set in V. Define $S' = \{v_1, v_2, \dots, v_n\}$ by $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j.$$

Then S' is orthogonal and span S' = span S.

Corollary. Any inner product space has an orthonormal basis.