Physics: Mechanics

Kinematics

Suppose that the acceleration of a rigid body is given by, $a(t) = \text{constant}$ for $t \geq t_0$. Then,

$$v(t) = v_0 + at \tag{1}$$

where v_0 is the body's initial velocity at t_0. Furthermore,

$$\begin{aligned} x(t) &= \int_{t_0}^{t} v(s)\ ds \\ &= c + v_0 t + \frac{1}{2}at^2. \end{aligned}$$

Since $x(0) = x_0$ we have

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2. \tag{2}$$

Finally, substituting (1) into (2) we have

$$v^2 = v_0^2 + 2a(x - x_0). \tag{3}$$

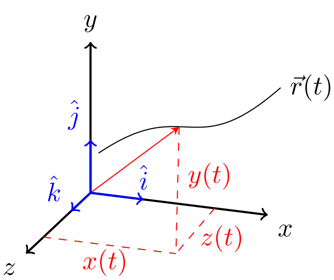

Figure 1: 3-D Coordinate System.

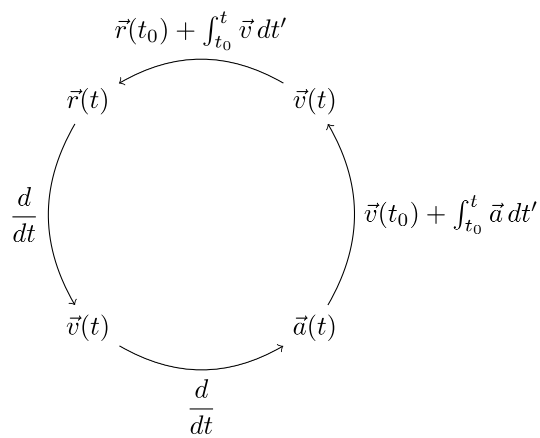

Figure 2: Relationship between position, velocity, and acceleration.

Work-Energy Theorem

The work done by a net force on a particle is equal to the change in the particle's kinetic energy.

$$W = \Delta K$$

Proof. Let $\vec{F}$ be the parallel component of the net force acting on a particle in the direction $\vec{s}$. Then, $W = Fs = (ma)s$, by Newton's 2nd Law. Substituting $as = \frac{v^2 - v_0^2}{2}$, we have

$$\begin{aligned} W &= \frac{1}{2}m(v^2 - v_0^2) \\ &= K_f - K_i. \end{aligned}$$

□

Conservation of Energy

Consider a system of two or more objects where a force acts between a rigid object and the rest of the system. Let W_1 be the transfer of kinetic energy to another type of energy in the system. Suppose to reverse the change we do W_2 work. When $W_1 = -W_2$, the other type of energy was potential energy and the force was a conservative force. Examples of conservative forces are the gravitational and spring forces.

A non-conservative force, such as kinetic friction or drag, is a process that is irreversible without adding additional energy.

Potential Energy. In general, the change in potential energy, ΔU, is defined to be

$$\Delta U = -W = \int_{x_i}^{x_f} F(x)\ dx.$$

Therefore, if $\Delta U(x)$ is known, we can find

$$F(x) = \frac{dU(x)}{dx}.$$

Finally, assume we have a system wherein only conservative forces are acting and there are no external forces from outside the system. Then,

$$\Delta K = W = -\Delta U. \tag{4}$$

$$K_2 - K_1 = U_1 - U_2. \tag{5}$$

$$\boxed{K_1 + U_1 = K_2 + U_2} \tag{6}$$

Equation (3) is the law of the conservation of mechanical energy.

Gravity

Newton's Law of Gravity. The force of gravity, F_g between two masses m_1 and m_2, separated by a distance r, is given by

$$\boxed{F_g = \frac{Gm_1m_2}{r^2}},$$

where $G = 6.67 \cdot 10^{-11}\text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$ is the gravitational constant.

We can calculate the net gravitational force on an object by n other particles

$$F_{0,\text{net}} = \sum_{i=1}^{n} F_{0,i}.$$

A uniform spherical shell of matter attracts a particle outside the shell as if all the shell's mass were concentrated at its center. However, no net gravitational force is exerted when a particle is located inside.

Force of Gravity Inside a Vertical Tunnel. Suppose a mass m is located a distance r away from the center of a plant of radius R and mass M. Then, the force exerted by gravity is given by

$$F = \frac{GMm}{R^3} r.$$

Gravitational Potential Energy. Defined to be the negative of the work done by gravity if the separation between two particles were changed from ∞ to r.

$$\boxed{U = -\frac{GMm}{r}}.$$

Shell Theorem.

- A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its center.
- If the body were a spherically symmetric shell no net gravitational force is exerted by the shell on any object inside, regardless of its position within the shell.

Kepler's Laws.

1. All planets move in elliptical orbits, with the sun at one focus.
2. A line connecting a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal time intervals.
3. The square of the period of any planet's orbit, T, is proportional to the cube of the semimajor axis of its orbit.

   $$\boxed{T^2 = \left(\frac{4\pi^2}{GM} r^3\right)}$$

   Where M is the mass of the system. In our solar system, $M_{\text{sun}} \gg M_p$ for any planet p, thus $M = M_{\text{sun}}$ provides a sufficiently close approximation.

Linear Momentum

Momentum of a particle with mass m and velocity v is defined to be $\vec{p} = m\vec{v}$. Note, that using this definition we can redefine Newton's 2nd law

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}.$$

We define impulse to be the change in momentum, i.e.

$$\vec{J} = \Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}(t)\ dt.$$

Conservation of Momentum. If the component of the net external force acting on a closed system of zero along some axis, then the component of linear momentum along that axis cannot change.

Elastic Collision. Collision in which total kinetic energy is conserved. In an **inelastic collision**, kinetic energy is not conserved

Rotational Motion

Moment of Inertia.
For a discrete system, the moment of inertia is defined to be,

$$I = \sum m_i r_i^2.$$

For a continuous system it is,

$$\boxed{I = \int r^2 \ dm}.$$

Shape	Axis of Rotation	Moment of Inertia
Solid Sphere	Any	$\frac{2}{5}mr^2$
Cylinder	Vertical (center)	$\frac{1}{2}mr^2$
Rod	Center	$\frac{1}{12}mL^2$
Rod	End	$\frac{1}{3}mL^2$
Disc	Vertical (center)	$\frac{1}{2}mr^2$
Circular Hoop	Vertical (center)	mr^2

Table 1: Selected formulae for moment of inertia.

Rotational Kinetic Energy. For an object with moment of inertia I and angular speed ω, its kinetic energy of rotation is given by

$$\boxed{K = \frac{1}{2}I\omega^2}.$$

Parallel Axis Theorem. Let h be the perpendicular distance between the given axis and the axis through the center of mass (assumed to be parallel). Then,

$$I = \boxed{I_{\text{com}} + Mh^2}.$$

Proof. Let O be the center of mass of an arbitrary body, placed at the origin. Consider an axis through O, perpendicular to the plane and another axis through P, parallel to the first.

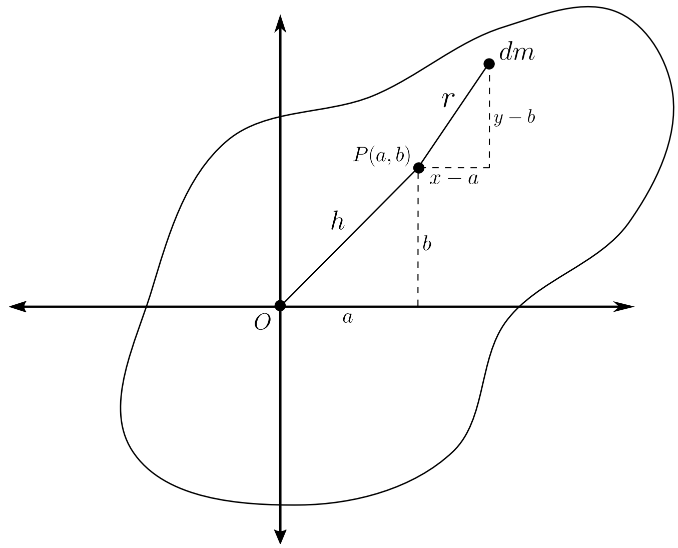

Figure 3: Arbitrary body.

Let dm be an element of mass with coordinates (x, y). Then,

$$\begin{aligned}
I &= \int r^2\ dm = \int [(x-a)^2 + (y-b)^2]\ dm \\
&= \int (x^2+y^2)\ dm - 2a\int x\ dm - 2b\int y\ dm + \int (a^2+b^2)\ dm \\
&= I_{\text{com}} + 0 + h^2 \int dm \\
&= I_{\text{com}} + Mh^2.
\end{aligned}$$

Torque.

Defined to be

$$\vec{\tau} = \vec{r} \times \vec{F} = |\vec{r}||\vec{F}|\sin\theta.$$

Newton's second law, also applied to rotation: $\boxed{\tau_{\text{net}} = I\alpha}$.

Angular Momentum.
If the net torque on a system is zero, then the angular momentum of that system remains constant regardless of changes within the system. Defined to be

$$\boxed{|\vec{l}| = |\vec{r} \times \vec{p}| = I\omega}.$$

And in an analogous manner to Newton's 2nd law,

$$\vec{\tau}_{\text{net}} = \frac{d\vec{l}}{dt}.$$

Rolling. A rolling object has two types of kinetic energy, one due to rotation about its center of mass and the other is translational about its center of mass. Thus

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

The velocity of the center of mass of a ball can be calculated given the angular speed and the ball's radius,

$$v_{\text{com}} = \omega R.$$

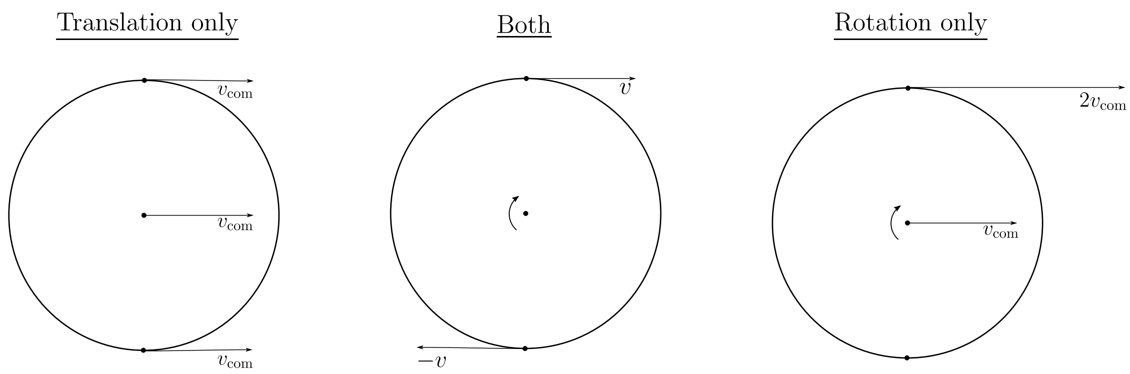

Figure 4: Types of motion.

Work and Power.
The change in rotational kinetic energy is the work done on an object. If a torque, τ, is applied to an object over an interval of angular displacement $[\theta_i, \theta_f]$, then the work done is given by

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta.$$

Finally, power is defined to be the derivative of work with respect to time.

$$\boxed{P = \frac{dW}{dt} = \tau\omega}.$$

Fluids

Pressure is defined to be

$$p = \frac{F}{A},$$

where F is the magnitude of the normal force on an area A. The SI units is N m^{-2} and is called a pascal. We can relate pressure to depth within a fluid in the following way. Imagine a right circular cylinder submerged in a tank of water, such that the top is a distance y_1 below the surface, and the bottom is a distance y_2. A force F_1 acts downwards on the top of the cylinder and is due to the water above, F_2 acts upwards from the bottom and is due to the water just below. Finally, the gravitational force, mg, where m is the mass of the water in the cylinder, acts downwards. If the fluid is in static equilibrium, then

$$F_2 = F_1 + mg$$

and since $F_1 = p_1 A$ and $F_2 = p_2 A$ we can derive

$$\boxed{p_2 = p_1 + \rho g(y_1 - y_2)},$$

using ρ as the density of the water. Note: Standard air pressure is 101 kPa.

The pressure at a point in a fluid in static equilibrium depends on the depth of that point but not on any horizontal dimension of the fluid or its container.

Pascal's Principle. A change in pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the fluid and to the walls of its container.

Archimedes' Principle. When a body is fully or partially submerged in a fluid, a buoyant force, F_b, from the surrounding fluid acts on the body. The force is directed upward and has a magnitude equal to the weight $m_f g$ the fluid that has been displaced by the body.

Bernoulli's Equation. Applying the principle of conservation of mechanical energy to the flow of an ideal fluid leads to Bernoullis equation along any tube of flow:

$$\boxed{p + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}}$$

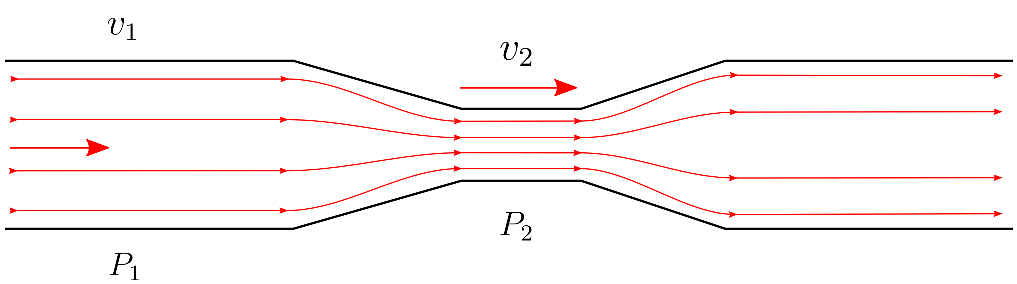

Figure 5: Bernoulli's Equation.

An ideal fluid is incompressible and lacks viscosity, and its flow is steady and confrontational. A streamline is the path followed by an individual fluid particle. A tube of flow is a bundle of streamlines. The flow within any tube of flow obeys the equation of continuity:

$$\boxed{R_V = Av = \text{constant}},$$

in which R_V is the volume flow rate, A is the cross-sectional area of the tube of flow at any point, and v is the speed of the fluid at that point. The mass flow rate R_m is $R_m = \rho R_V = \rho A v = \text{constant}$.

Simple Harmonic Motion

Motion which can be modeled via a sinusoidal function: $\boxed{x(t) = x_m \cos(\omega t + \phi)}$. The amplitude of the wave, x_m gives the maximum displacement from equilibrium. The argument of the cosine function is the phase of the motion and the constant ϕ is the phase angle. Finally, ω denotes the angular frequency of the motion. For SHM with period T and frequency $f = 1/T$, we have

$$\omega = \frac{2\pi}{T} = 2\pi f.$$

Differentiating $x(t)$ we get the velocity to be

$$v(t) = -\omega x_m \sin(\omega t + \phi).$$

We derive acceleration analogously

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi) = -\omega^2 x(t).$$

Thus, in SHM, Newton's law is equivalent to $F = -(m\omega^2)x$. Recall that Hooke's law states that $F = -kx$, thus we can relate ω to k,

$$\boxed{\omega = \sqrt{\frac{k}{m}}}.$$

Energy. Using the kinematics for SHM from above we can derive formulas for the potential and kinetic energy.

$$U(t) = \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$
$$K(t) = \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

Thus, total energy simplifies nicely to $E = \frac{1}{2}kx_m^2$.

Pendulum. A pendulum of length L has period

$$\boxed{T_p = 2\pi\sqrt{\frac{L}{g}} \qquad T_s = 2\pi\sqrt{\frac{m}{k}}}.$$

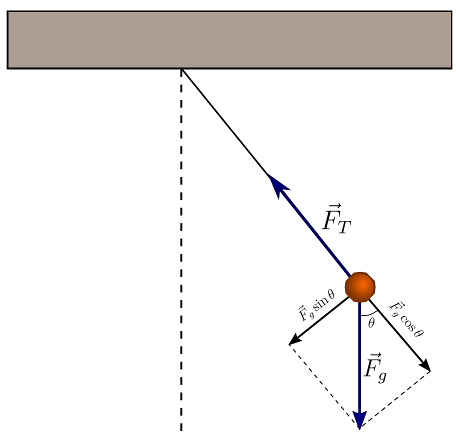

Figure 6: Pendulum force diagram.

Waves

Sinusoidal wave function: $y(x,t) = y_m \sin(kx - \omega t)$.

- Amplitude: y_m
- Phase: $kx - \omega t$
- Wavelength: λ
- Angular Wave Number: $\boxed{k = \frac{2\pi}{\lambda}}$
- Angular Frequency: $\omega = \frac{2\pi}{T}$
- Frequency: $f = \frac{1}{T} = \frac{\omega}{2\pi}$
- Wave Velocity: $\boxed{\nu = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f}$

Sound & Vibration

The velocity of a wave traveling across a string with tension τ and linear density μ is

$$\boxed{\nu = \sqrt{\frac{\tau}{\mu}}}.$$

The average rate at which energy is transmitted across a string with speed ν, linear density μ, amplitude y_m, and angular frequency ω is

$$P_{\text{avg}} = \tfrac{1}{2}\mu\nu\omega^2 y_m^2.$$

Wave Equation. The general differential equation which governs the travel of all wave types.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\nu^2}\frac{\partial^2 y}{\partial t^2} \tag{7}$$

Interference. The superposition of two waves, shifted by a phase constant ϕ, given by

$$y_1(x,t) = y_m \sin(kx - \omega t)$$
$$y_2(x,t) = y_m \sin(kx - \omega t + \phi)$$

is given by

$$y'(x,y) = [2y_m \cos(\tfrac{1}{2}\phi)]\sin(kx - \omega t + \tfrac{1}{2}\phi).$$

The superposition of two waves which are identical except that they travel in opposite direction produces a standing wave given by

$$y'(x,t) = 2y_m \sin(kx)\cos(\omega t).$$

Resonance Frequencies.

$$\boxed{f_n = \frac{\nu}{\lambda} = n\frac{\nu}{2L}}$$

Sound

The speed of sound is given by

$$\boxed{\nu = \sqrt{\frac{B}{\rho}}},$$

where B is the bulk modulus and ρ is the traveling medium's density. The bulk modulus is defined as

$$B = -\frac{\Delta p}{\Delta V/V},$$

where $\Delta V/V$ is the fractional change in volume produced by a change in pressure Δp.

Decibel Scale. The loudness of sound of intensity I, is measured in decibels (dB) on a logarithmic scale.

$$\boxed{\beta = (10\text{ dB})\log_{10}\left(\frac{I}{I_0}\right)}.$$

Here $I_0 = 10^{-12}$ W/m^2 is the threshold of human hearing.

Sound Waves. Individual particles move according to a sinusoidal displacement pattern: $s(x,t) = s_m\cos(kx - \omega t)$. The change in pressure from equilibrium is therefore, $\Delta p = \Delta p_m \sin(kx - \omega t)$, where $\Delta p_m = (Bk)\cdot s_m$.

Path Length Differences. Clearly, $\frac{\phi}{2\pi} = \frac{\Delta L}{\lambda}$, thus $\phi = \frac{2\pi \Delta L}{\lambda}$. Fully constructive interference occurs when $\Delta L/\lambda$ is an integer, and fully destructive occurs at half integer multiples.

Intensity. Defined as power per unit area. A sound wave, with velocity ν, angular frequency ω, and amplitude s_m, in medium with density ρ, has intensity

$$I = \tfrac{1}{2}\rho\nu\omega^2 s_m^2.$$

Furthermore, the sound intensity, with power source P_s at a distance r from the source is given by

$$\boxed{I = \frac{P_s}{4\pi r^2}}$$

Sound in Pipes.
Pipe, two open ends.

$$f = \frac{n\nu}{2L}, \qquad \text{for } n \in \mathbb{Z}^+ \tag{8}$$

Pipe, one open ends.

$$f = \frac{n\nu}{4L}, \qquad \text{for odd } n \tag{9}$$

Beat Frequency. When two waves with slightly different frequencies are detected together a beat arises, such that

$$\boxed{f_{\text{beat}} = f_1 - f_2}.$$

Doppler Effect. The emitted frequency f and the detected frequency f' are given by

$$\boxed{f' = f\left(\frac{v \pm v_D}{v \mp v_S}\right)}$$

where v is the speed of sound in air, v_D is the detectors speed with respect to the air, and v_S is the source's speed with respect to the air. Signs are chosen such that when the motion of the detector or source is toward the other, the sign on its speed must give an upward shift in frequency, and vice versa.

Mach Speeds. Mach number is defined as $M = \frac{v_o}{v_s}$, where v_o is the velocity of the object and v_s is the speed of sound. The mach cone angle is the half-angle θ such that

$$\sin\theta = \frac{v_s}{v_o} = \frac{1}{M}.$$

Table of Equations

Module 1: Unit Conversions & 1-D Motion

- $v = v_0 + at$
- $v^2 = v_0^2 + 2a\Delta x$
- $x = x_0 + v_0 t + \frac{1}{2}at^2$

Module 2: Vectors & Scalars

Module 3: Newton's Laws

- $\vec{F}_{\text{net}} = \sum m\vec{a}$
- $F_g = mg = -F_N$
- $F_f = \mu F_N$

Module 4: Circular Motion & Work

- $F_c = -\frac{mv^2}{r}$
- $a_c = \frac{v^2}{r}$
- $v = \omega r$
- $KE = \frac{1}{2}mv^2$
- $W = \int \vec{F} \cdot d\vec{x}$
- $U_g = mgh$
- $U_s = \frac{1}{2}kx^2$
- $W = \Delta KE$

Module 5: Conservation of Energy & Center of Mass

- $KE_1 + U_1 = KE_2 + U_2 + W$
- $x_{\text{com}} = \frac{1}{M}\sum m_i x_i$

Module 6: Momentum

- $\vec{p}_1 = m_1\vec{v}_1 = m_2\vec{v}_2 = \vec{p}_2$
- $\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} = \frac{dm}{dt}\vec{v} + \frac{d\vec{v}}{dt}m$
- $J = \Delta p = \int \vec{F}dt$

Module 7: Rotation of Particles

- $\omega = \frac{v_t}{r}$
- $\alpha = \frac{a_t}{r}$
- $\tau = \vec{F} \times \vec{r}$

Module 8: Inertia & Rotation of Rigid Bodies

- $I = \int r^2 dm$
- $I = I_{\text{com}} + Mh^2$
- $KE = \frac{1}{2}I\omega^2$
- $\tau = I\alpha$
- $l = I\omega$ (conserved)
- $W = \int \tau d\theta$
- $P = \tau\omega$

Module 9: Gravity

- $F_g = \frac{Gm_1m_2}{r^2}$
- $U_g = -\frac{GMm}{r}$
- $T^2 = \left(\frac{4\pi^2}{GM}r^3\right)$

Module 10: Fluids

- $p = \frac{F}{A}$
- $p_2 = p_1 + \rho g(y_1 - y_2)$
- $A_1 v_1 = A_2 v_2$
- $p + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$

Module 11: Simple Harmonic Motion

- $x(t) = x_m \cos(\omega t + \phi)$
- $\omega = \sqrt{\frac{k}{m}}$
- $F_s = -kx$
- $T_p = 2\pi\sqrt{\frac{L}{g}}$
- $T_s = 2\pi\sqrt{\frac{m}{k}}$

Module 12: Waves I

- $k = \frac{2\pi}{\lambda}$
- $\omega = 2\pi f = \frac{2\pi}{T}$
- $\nu = \lambda f$
- $f_n = n\frac{\nu}{2L}$
- $\nu = \sqrt{\frac{\tau}{\mu}}$

Module 13: Waves II - Sound

- $\nu = \sqrt{\frac{B}{\rho}}$
- $\beta = (10 \text{ dB}) \log_{10}\left(\frac{I}{10^{-12}}\right)$
- $I = \frac{P_s}{4\pi r^2}$
- $f_{\text{beat}} = f_1 - f_2$
- $f' = f'\left(\frac{v \pm v_D}{v \mp v_S}\right)$
- $f = \frac{n\nu}{2L}$ (open) $\qquad f = \frac{n\nu}{4L}$ (closed)

Shell Theorem Problem.
Acceleration due to gravity at a depth d. Let $r = R_E - d$.

$$\begin{aligned} F_g &= ma_g \\ \frac{GmM_{\text{eff}}}{r^2} &= ma_g \\ \frac{GM_{\text{eff}}}{r^2} &= a_g \end{aligned}$$

Note that $M_{\text{eff}} = \rho_E(\frac{4}{3}\pi r^3) = \frac{M_E}{\frac{4}{3}\pi R_E^3}(\frac{4}{3}\pi r^3) = M_E\left(\frac{r}{R_E}\right)^3$. Thus,

$$\begin{aligned} a_g &= \left(\frac{GM_E}{R_E^3}\right) r \\ a_g &= \left(\frac{GM_E}{R_E^3}\right)(R_E - d) \\ a_g &= g - d\left(\frac{GM_E}{R_E^3}\right). \end{aligned}$$

Dampened Harmonic Motion. Consider a harmonic oscillator with mass m, spring constant k, and damping force $F_d = -b\dot{x}$. Assuming no external forces, by Newton's 2nd Law we have

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

- Case I. (Underdamped oscillator) If $b^2 - 4mk < 0$, then the position function is given by

  $$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi),$$

  where $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$.

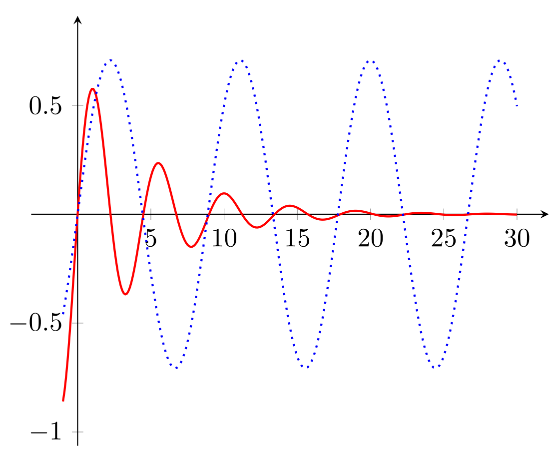

Figure 7: Underdampened (red): $5\ddot{x} + 2\dot{x} + 10x = 0$. No damping (blue): $5\ddot{x} + 10x = 0$.

- Case II. (Critically damped) If $b^2 - 4mk < 0$, then our position function is given by

  $$x(t) = e^{-bt/2m}(c_1 + c_2 t).$$

- Case III. (Overdamped)

Harmonics

Open-Open Pipe

There must be anti-nodes at the open ends for resonance to occur.

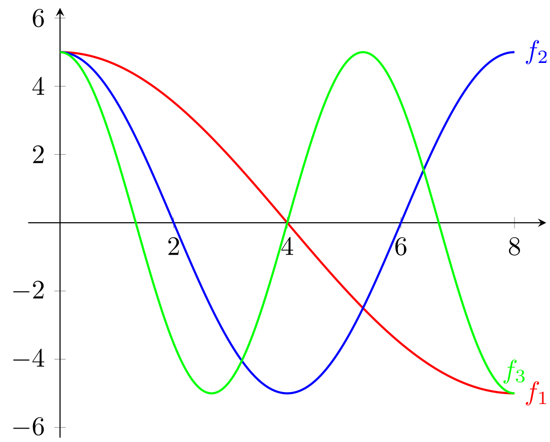

Figure 8: First three harmonic on an open-ended tube with $L = 8$.

Open-Closed Pipe

There must be a node at the pipe's closed end for resonance to occur.

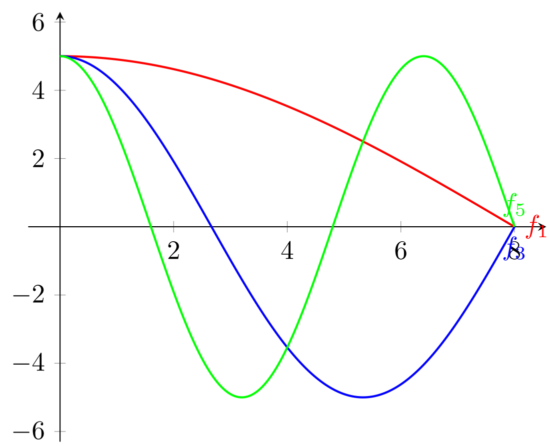

Figure 9: First three harmonics on an close-ended tube with $L = 8$.

Calculating Moments of Inertia

The goal is to use a substitution to a variable we can integrate over, e.g. radius/length, instead of mass.

- First, determine $\frac{dm}{dV} = \rho$.
- Then, determine $\frac{dV}{dr}$.
- Substituting, we obtain $dm = \rho \frac{dV}{dr} dr$.
- If ρ is constant, then
$$I = \rho \int r^2 \frac{dV}{dr} dr.$$

Cylinder about its central axis. Let R be the radius of the cylinder with height h and total mass M. Assume the mass is evenly distributed. Then,

$$\frac{dm}{dV} = \frac{M}{\pi R^2 h} = \rho.$$

Furthermore, we have

$$\frac{dV}{dr} = 2\pi r h.$$

Using the substitution $dm = \rho(2\pi r h)dr$, the rotational inertia is given by

$$\begin{aligned} I &= 2\pi\rho h \int_0^R r^3 \, dr \\ &= 2\pi\rho h \frac{R^4}{4} \\ &= \frac{1}{2} M R^2. \end{aligned}$$

Rectangular plate about its center. Let h, w be the plate's height and width, respectively. Note that

$$\frac{dm}{dA} = \rho,$$

thus

$$\begin{aligned} I &= \iint \rho r^2 \, dA \\ &= \rho \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} (w^2 + h^2) \, dw \, dh \\ &= \frac{1}{12} M (w^2 + h^2). \end{aligned}$$

Sphere. Let R be the radius of the sphere. Consider a disc, of width dz and height z above the center of the sphere, perpendicular to the axis of rotation. The radius, r, of these discs is determined by $r^2 + z^2 = R^2$. Using the moment of inertia of a cylinder we have

$$\frac{dI}{dm} = \frac{1}{2} r^2.$$

Furthermore, $\frac{dm}{dV} = \rho$ and $\frac{dV}{dz} = \pi r^2$, thus $dm = \rho\pi r^2 dz$. Substituting for dm this simplifies to

$$\begin{aligned} dI &= \frac{\pi}{2} \rho r^4 dz \\ &= \frac{\pi}{2} \rho (R^2 - z^2)^2 dz. \end{aligned}$$

To finalize, we must integrate over $[-R, R]$ to obtain

$$I = \frac{2}{5} M R^2$$