Combinatorics

Fall 2018

Contents

1 Basic Counting

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Theorem 1.1 (Cauchy-Schwartz). Let $(a_i)_i$ and $(b_i)_i$ be sequences of nonnegative real numbers. Then

$$\sum_i a_i b_i \le \Big(\sum_i a_i^2\Big)^{\frac{1}{2}} \Big(\sum_i b_i^2\Big)^{\frac{1}{2}}.$$

Proof. Let $A = \left(\sum_i a_i^2\right)^{\frac{1}{2}}$ and $B = \left(\sum_i b_i^2\right)^{\frac{1}{2}}$. Recall that $ab \le \frac{a^2+b^2}{2}$. Then

$$\sum_i \frac{a_i}{A}\frac{b_i}{B} \le \sum_i \frac{1}{2}\big(\frac{a_i^2}{a^2} + \frac{b_i^2}{B^2}\big) = 1.$$

∎

Example. Consider an $n \times n$ matrix with entries in $\mathbb{Z}_2$, with no rectangle of entries which are ones (that is $a_{ij}a_{ij'} = 1$ for at most one value of i). Find an upper bound on the number of entries which can be ones.

$$\begin{aligned}\sum_{i,j=1}^{n} a_{ij} &= \sum_{i=1}^{n}\big(\sum_{j=1}^{n} a_{ij}\big)\cdot 1 \\ &\le n^{\frac{1}{2}}\big(\sum_i \big(\sum_j a_{ij}\big)^2\big)^{\frac{1}{2}} \\ &= n^{\frac{1}{2}}\big(\sum_i\sum_j a_{ij}^2 + \sum_i \sum_{j\ne j'} a_{ij}a_{ij'}\big)^{\frac{1}{2}} \\ &\le n^{\frac{1}{2}}(n^2 + n(n-1)) \\ &\le \sqrt{2}\, n^{\frac{3}{2}}.\end{aligned} \tag{1.1}$$

Define an *incidence* $\mathcal{I} = (p, l)$, where p is a point on line l. Given a set of points P and a set of lines L, the incidence matrix I of P and L, where $I(i,j) = 1$ if and only if (p_i, l_j) is an incidence, corresponds to the matrix restriction in the previous example.

Problem 1.2 (Erdos Distance Problem)**.** Given $P \subset \mathbb{R}^2$ with n points, define $\Delta(P) = \{|x-y| : x, y \in P\}$, where $|x-y|$ is the standard Euclidean distance. How many distances are possible.

If the n points are chosen randomly, then we'd expect $\binom{n}{2}$ distinct distances. However, if all n points are equally spaced along a line we can achieve as little at n distances. If we choose n points in the integer lattice $\mathbb{Z}^2 \cap [0,\sqrt{n}]^2$, then for any point $p = (x_1, x_2)$, $0 \le x_1^2 + x_2^2 \le 2n$. Hence there can be up to $2n+1$ distances.

Theorem 1.2 (Moser). $|\Delta(P)| \gtrsim n^{\frac{2}{3}}$.

Proof. Choose $X, Y \in P$ so that $|X-Y| = \min\{|p-p'| : p, p' \in P\}$. Let O be the midpoint of XY. Assume WLOG that half the points are above the line connecting X and Y, call this set of points

P'. Draw annuli centered at O with thickness $|X - Y|$ until every point in P is in some annuli. Alternatively color annuli red, green, blue. Let P'' be the set of points in the green annuli.

Note that the distances from X and Y to points in one of the green annuli cannot be the same as the distances which occur in another green annuli. Let $\mathcal{A}_j$ be the set of points of P'' in annulus j. Let $n_j = |\mathcal{A}_j|$. Let $\{d_1, \ldots, d_{k_j}\}$ be the distinct distances determined by points in $\mathcal{A}_j$ to X or Y.

Define

$$A_l = \{p \in \mathcal{A}_j : |p - X| = d_l\}$$

and

$$B_l = \{p \in \mathcal{A}_j : |p - Y| = d_l\}.$$

Now by construction $A_l = \cup_i (A_l \cap B_i)$ and clearly $\cup_l A_l = \cup_{l,i}(A_l \cap B_i)$. We have $|\cup_l A_l| = n_j$ and

$$|\cup_{l,i} (A_l \cap B_i)| \leq k^2 \max_{i,l}\{|A_l \cap B_i|\},$$

but A_l and B_i are on circles centered at X and Y, respectively, so $\max_{i,l}\{|A_l \cap B_i|\} \leq 1$. Thus

$$k_j \geq \sqrt{n_j}.$$

Thus

$$|\Delta(P)| \geq |\Delta(P'')| = \sum_j k_j \geq \sum_j \sqrt{n_j}.$$

However,

$$\frac{n}{6} \leq \sum_j n_j \leq \sqrt{n_{max}} \sum_j \sqrt{n_j}.$$

Thus $|\Delta(P)| \geq \frac{n}{6\sqrt{n_{max}}}$ and $\Delta(P) \gtrsim n_{max}$, from which the result follows. ∎

2 Graph Theory

Proposition 2.1. Connected, simple, planar graph $G(V, E)$. Then

$$n - e + f = 2.$$

Proof. Induction on e the number of edges. Given any graph with one edge, there are two vertices and one face. To add another edge, we can either add another vertex, or connected to an existing one. Connecting to an existing vertex encloses another face. ∎

Proposition 2.2. If G is a simple, planar, graph, then $3f \leq 2e$.

Proof. Counting argument. Each edge of the graph has 2 sides. Every face requires at least 3 sides. ∎

Corollary 2.2.1. G connected, simple, planar, implies $e \leq 3n - 6$.

Definition 2.3. The crossing number of a graph $G(V, E)$, denoted $\mathrm{cr}(G)$, is the fewest possible crossings in a plane drawing of G.

Corollary 2.3.1. $\mathrm{cr}(G) \geq e - 3n + 6$.

Proof. Remove edges that cause crossings until the graph is planar. If we removed e_{rem} edges, then $e - e_{rem} \leq 3n - 6$. Observe that e_{rem} is a lower bound for crossing number, as each removed edge eliminates at least one crossing. Thus $\mathrm{cr}(G) \geq e_{rem} \geq e - 3n + 6$. ∎

Theorem 2.4. If $e \geq 4n$, then $\mathrm{cr}(G) \gtrsim \frac{e^3}{n^2}$.

Proof. Let $p \in (0, 1)$. Construct a random subgraph G' by keeping each vertex with probability p. So $\mathbb{E}[|V'|] = pn$, $\mathbb{E}[|E'|] = p^2 e$, and $\mathbb{E}[\mathrm{cr}(G')] \leq p^4 \,\mathrm{cr}(G)$. We know

$$0 \leq 6 \leq \mathrm{cr}(G') - |E'| + 3|V'|.$$

Taking expectation, we have

$$0 \leq p^4 \,\mathrm{cr}(G) - p^2 e + 3pn,$$

hence $\mathrm{cr}(G) \geq p^{-2} e - 3p^{-3} n$. We want $p^{-2} e > 3p^{-3} n$, so take $p = \frac{4n}{e}$. Then

$$\mathrm{cr}(G) \geq \frac{e^3}{16n^2} - \frac{3e^3}{64n^2} \gtrsim \frac{e^3}{n^2}$$

∎

Theorem 2.5 (Szemeredi-Trotter). $\mathcal{I}(P, L) \lesssim (|P||L|)^{2/3} + |P| + |L|$.

Proof. Given P, L we construct a graph $G(V, E)$ by setting each vertex to be a point and the edges to be between adjacent vertices on the same line. Note $e = \sum_l I(P, l) - 1 = I(P, L) - |L|$.

If $e < 4n$, then $I(P, L) = e + |L| < |L| + 4|P|$. If $e \geq 4n$, then by (2.4) we have

$$|L|^2 \geq \mathrm{cr}(G) \geq \frac{e^3}{v^2} = \frac{(I(P, L) - |L|)^3}{|P|^2}.$$

∎

3 Finite Fields

Definition 3.1 (Fourier Transform). Given $f : \mathbb{Z}_p^d \to \mathbb{C}$ define

$$\hat{f}(m) = p^{-d} \sum_{x \in \mathbb{Z}_p^d} \chi(-x \cdot m) f(x)$$

where $\chi(t) = e^{\frac{2\pi i}{p} t}$.

Lemma 3.2. $\sum_{x \in \mathbb{Z}_p^d} \chi(a \cdot x) = \begin{cases} 0 \text{ if } a \neq 0 \\ p^d \text{ if } a = 0 \end{cases}$.

Proof. It's a geometric series if $d = 1$. The analogous result in $\mathbb{Z}_p^d$, follows easily. ∎

Lemma 3.3.
$$f(x) = \sum_{m \in \mathbb{Z}_p^d} \chi(x \cdot m)\hat{f}(m).$$

Proof.
$$\begin{aligned}
\sum_{m \in \mathbb{Z}_p^d} \chi(x \cdot m)\hat{f}(m) &= \sum_{m \in \mathbb{Z}_p^d} \chi(x \cdot m) p^{-d} \sum_{y \in \mathbb{Z}_p^d} \chi(-y \cdot m) f(y) \\
&= \sum_{y} f(y) p^{-d} \sum_{m} \chi((x-y) \cdot m) \\
&= f(x).
\end{aligned} \tag{3.1}$$

Note the 3rd equality comes from lemma 2, since $\chi((x-y)\cdot m) = 1$, if and only if $x = y$. Otherwise, it is 0. So when $y = x$, for all $m \in \mathbb{Z}_p^d$, $\chi((x-y)\cdot m) = 1$, and the sum cancels out the p^{-d}. ∎

Lemma 3.4. If $f : \mathbb{Z}_p^d \to \mathbb{C}$, then $\sum_{m \in \mathbb{Z}_p^d} |\hat{f}(m)|^2 = p^{-2} \sum_{x \in \mathbb{Z}_p^d} |f(x)|^2$.

Proof. Use conjugation. Note $\overline{\chi(-x \cdot m)} = \chi(x \cdot m)$.

$$\sum_{m \in \mathbb{Z}_p^d} |\hat{f}(m)|^2 = \sum_m \left(\left(p^{-d} \sum_{x \in \mathbb{Z}_p^d} \chi(-x \cdot m) f(x) \right) \left(p^{-d} \sum_{y \in \mathbb{Z}_p^d} \chi(y \cdot m) \overline{f}(y) \right) \right) \tag{3.2}$$

Reorder summation.

$$= p^{-d} \sum_{x,y} f(x)\overline{f}(y) p^{-d} \sum_m \chi(-(x-y)\cdot m) \tag{3.3}$$

Hence
$$= p^{-d} \sum_{x \in \mathbb{Z}_p^d} |f(x)|^2. \tag{3.4}$$

∎

3.1 Applications to Erdős

Let $E \subset \mathbb{Z}_p^2$. Define
$$\nu(t) = \sum_{x,y} 1_E(x) 1_E(y) 1_{S_t}(x-y),$$
for all $t \in \mathbb{Z}_p$, where 1_A is the characteristic function and $S_t = \{x \in \mathbb{Z}_p^2 : x_1^2 + x_2^2 = t\}$. Then

$$\begin{aligned}
\nu(t) &= \sum_{x,y} 1_E(x) 1_E(y) \sum_{m \in \mathbb{Z}_p^2} \chi((x-y)\cdot m) \hat{1}_{S_t}(m) \\
&= p^4 \sum_m \hat{1}_{S_t}(m) \left(\sum_y p^{-2} \chi(-y \cdot m) 1_E(y) \right) \left(\sum_x p^{-2} \chi(x \cdot m) 1_E(x) \right) \\
&= p^4 \sum_m |1_E(m)|^2 \hat{1}_{S_t}(m).
\end{aligned} \tag{3.5}$$

We want to show $\nu(t) > 0$ for all t when $|E|$ is sufficiently large.

Theorem 3.5. Suppose that $|E| > 4p^{\frac{3}{2}}$, then $\Delta(E) = \mathbb{Z}_p$.

Proof. In (3.5), let $M(t) = p^4|\hat{1}_E(\vec{0})|^2\hat{1}_{S_t}(\vec{0})$ and $R(t) = p^4\sum_{m\neq\vec{0}}|\hat{1}_E(m)|^2\hat{1}_{S_t}(m)$. By lemma 3.6, we have

$$M(t) = p^4(p^{-2}|E|)^2(p^{-2}|S_t|) = |E|^2\frac{p\pm 1}{p^2}.$$

Now it suffices to prove that $|R(t)| < M(t)$.

$$\begin{aligned}
\hat{1}_{S_t}(m) &= p^{-2}\sum_{x\in\mathbb{Z}_p^2}\chi(-x\cdot m)1_{S_t}(m) \\
&= p^{-3}\sum_x\sum_{s\in\mathbb{Z}_p}\chi(-x\cdot m)\chi(s(x_1^2+x_2^2-t)) \\
&= p^{-3}\sum_{s\neq 0}\chi(-st)\sum_x\chi(-x\cdot m)\chi(s(x_1^2+x_2^2-t)) \\
&= p^{-3}\sum_{s\neq 0}\chi(-st)\sum_x\chi(s(x_1-\tfrac{m_1}{2s})^2+s(x_2-\tfrac{m_2}{2s})^2-(\tfrac{m_1^2+m_2^2}{4s})) \\
&= p^{-3}\sum_{s\neq 0}\chi(-st)\chi(-\tfrac{||m||}{4s})\sum_x\chi(s||x||).
\end{aligned}\tag{3.6}$$

Note that we completed the square and used the translation invariance of $||\cdot||$. By lemma 3.7, we have that $\sum_{x\in\mathbb{Z}_p^2}\chi(s||x||) = p\alpha^2(s)$, where $s\neq 0$ and $\alpha(s)$ is a function of modulus 1. Hence

$$\hat{1}_{S_t}(m) = p^{-2}\sum_{s\neq 0}\chi(-st-\tfrac{||m||}{s})\alpha^2(s)\tag{3.7}$$

The summation is (3.7) is a Kloosterman sum and is bounded above by $2\sqrt{p}$ (Weil, 1948). Now it follows that $|R(t)| \leq p^4(2p^{-\frac{3}{2}})(p^{-2}|E|) = 2p^{\frac{1}{2}}|E|$, and so $|R(t)| < M(t)$ when $|E| > 4p^{\frac{3}{2}}$. ∎

Lemma 3.6. If $t\neq 0$, then $|S_t| = p+1$ if $p\equiv 3 \bmod 4$ and $|S_t| = p-1$ if $p\equiv 1 \bmod 4$.

Proof. Homework #2. ∎

The following two results suggest an extension of theorem 3.5 to higher dimensions.

Lemma 3.7. Let $\gamma(s) = \sum_{t\in\mathbb{Z}_p}\chi(st^2)$. If $s\neq 0$, then $|\gamma(s)| = \sqrt{p}$.

Proof.

$$|\gamma(s)|^2 = \sum_{u,v\in\mathbb{Z}_p}\chi(s(u^2-v^2))\tag{3.8}$$

Let $\alpha = u-v$ and $\beta = u+v$, so that $u = \frac{\alpha+\beta}{2}$ and $v = \frac{\beta-\alpha}{2}$. Then

$$\begin{aligned}
|\gamma(s)|^2 &= \sum_{\alpha,\beta\in\mathbb{Z}_p}\chi(s\alpha\beta) \\
&= \sum_{w\in\mathbb{Z}_p}n(w)\chi(sw)
\end{aligned}\tag{3.9}$$

where $n(w)$ is the number of pairs $(\alpha, \beta) \in \mathbb{Z}_p^2$ such that $\alpha\beta = w$. If $w = 0$, then $n(w) = 2p - 1$, since (t, it) and $(t, -it)$ for $t \in \mathbb{Z}_p$ all have magnitude 0. If $w \neq 0$, then for each choice of α, there is one and only one choice of β so that $\alpha\beta = w$. Hence $n(w) = p - 1$ in this case. Thus

$$\begin{aligned} |\gamma(s)|^2 &= (2p-1) + (p-1)\sum_{w\neq 0} \chi(sw) \\ &= p + \sum_{w} \chi(sw) = p. \end{aligned} \tag{3.10}$$

■

Theorem 3.8. Let $S_t = \{x \in \mathbb{Z}_p^d : \|x\| = t\}$. Then $|S_t| \leq p^{d-1} + p^{d/2}$ when $d \geq 3$.

Proof.

$$\begin{aligned} |S_t| &= \frac{1}{p} \sum_{x\in\mathbb{Z}_p^d} \sum_{s\in\mathbb{Z}_p} \chi(s(x_1^2 + \ldots + x_d^2 - t)) \\ &= p^{d-1} + \frac{1}{p}\sum_{s\neq 0}\chi(-st)\sum_{x}\chi(s(x_1^2+\ldots+x_d^2)) \end{aligned} \tag{3.11}$$

Note that $\sum_x \chi(s(x_1^2 + \ldots + x_d^2)) = \gamma^d(s)$. So by lemma 3.7, we're basically done. ■

4 Final Review

Theorem 4.1. Let $E, F \subset \mathbb{Z}_p^d$. If $|E||F| > 4p^{d+1}$, then $\Delta(E, F) = \mathbb{Z}_p$.

Proof.

$$\begin{aligned} \nu(t) &= \sum_{x,y\in\mathbb{Z}_p^d} 1_E(x)1_F(y)1_{S_t}(x-y) \\ &= \sum_{m\in\mathbb{Z}_p^d}\sum_{x,y} 1_E(x)1_F(y)\chi(m(x-y))\hat{1}_{S_t}(m) \\ &= p^{2d}\sum_m \overline{\hat{1}_E(m)}\hat{1}_F(m)\hat{1}_{S_t}(m) \\ &= p^{-d}|E||F||S_t| + p^{2d}\sum_{m\neq 0}(\ldots) \\ &= M + R. \end{aligned} \tag{4.1}$$

The second equality follows by the inverse Fourier transform. The third follows by applying the Fourier transform twice. The last by pulling out the $m = 0$ case. Now

$$\begin{aligned} \hat{1}_{S_t}(m) &= p^{-d}\sum_{x\in\mathbb{Z}_p^d}\chi(-x\cdot m)1_{S_t}(m) \\ &= p^{-d-1}\sum_{x\in\mathbb{Z}_p^d, s\in\mathbb{Z}_p}\chi(-x\cdot m)\chi(s(||x||-t)) \\ &= p^{-d-1}\sum_{s\neq 0}\left(-st - \frac{||m||}{4s}\right)\sum_{y\in\mathbb{Z}_p^d}\chi(s||y||) \end{aligned} \tag{4.2}$$

The first equality follows by the Fourier transform, the second by the fact that $\sum_s \chi(s(||x||-t))$ is 0 unless x has norm t, in which case the sum over s, yields p. Finally, we can complete the square on $sx_j^2 - x_j m_j = sy_j^2 - \frac{m_j^2}{4s}$, where $y_j = x_j - \frac{m_j}{2s}$. Now we can apply the Gauss sum bound to the inner summation. Hence

$$\hat{1}_{S_t}(m) = p^{-\frac{d}{2}-1} \sum_{s \neq 0} \chi\left(-st - \frac{||m||}{4s}\right) \phi^d(s) \tag{4.3}$$

where $||\phi^d(s)|| = 1$. Using the bound on Kloosterman sums (which says the sum is $\leq 2\sqrt{p}$), we have that $\hat{1}_{S_t} \leq 2p^{\frac{1}{2}(-d-1)}$. This says

$$R \leq 2p^{2d} p^{\frac{1}{2}(-d-1)} \sum_{m \neq 0} \overline{\hat{1}_E(m)} \hat{1}_E(m) \leq 2p^{\frac{d-1}{2}} \sqrt{|E||F|}$$

by applying Cauchy-Schwarz and Plancharel. Finally, we have that $\nu(t) > 0$ whenever $M > R$. Since $|S_t| \sim p^{d-1}$ (by Gauss sums), we get our desired result. ∎

Theorem 4.2. Suppose $E \subset \mathbb{Z}_p^2$ and $|E| > cp^{\frac{4}{3}}$, and for any line through the origin $|E \cap l| \leq c' \sqrt{|E|}$. Then, $|\Pi(E)| \geq Cp$.

Proof.

$$\begin{aligned}
\nu^2(t) &= \left(\sum_{x \cdot y = t} 1_E(x) 1_E(y)\right)^2 \\
&\leq |E| \left(\sum_x \sum_{x \cdot y = t, x \cdot y' = t} 1_E(x) 1_E(y) 1_E(y')\right) \\
&= |E| \sum_{x \cdot y = x \cdot y'} (\ldots) \\
&= p^{-1} |E| \sum_{x,y,y'} \sum_s 1_E(x,y,y') \chi(sx \cdot (y-y')) \\
&= p^{-1} |E|^4 + p^{-1} |E| \sum_{x,y,y',s \neq 0} (\ldots)
\end{aligned} \tag{4.4}$$

Apply CS first, Then add in the character. Then pull out the $s = 0$ case. Now we work on the remainder. First we apply the Fourier transform twice,

$$\begin{aligned}
R &= |E| p^3 \sum_x \sum_{s \neq 0} |\hat{1}_E(sx)|^2 1_E(x) \\
&= |E| p^3 \sum_x \sum_{s \neq 0} |\hat{1}_E(x)|^2 1_E(sx) \\
&\leq |E|^2 p |E \cap l_x|
\end{aligned} \tag{4.5}$$

The second step is just a change of variables. The first follows by noticing, sx as s runs through $\mathbb{Z}_p$ is a line through the origin, so $\sum_{s \neq 0} 1_E(sx) \leq |E \cap l_x|$. In addition, we applied Plancharel. To relate this back to the dot product set, we note $|E|^4 = \left(\sum_t \nu(t)\right)^2 \leq |\Pi(E)| \cdot \sum_t \nu^2(t)$. ∎

Theorem 4.3. Let $E, F \subset \mathbb{Z}_p^d$. If $|E||F| > p^{d+1}$, then $|\Pi(E, F)| = p$.

Proof.

$$\begin{aligned}\nu(t) &= \sum_{x\cdot y=t} 1_E(x)1_F(y)\\ &= p^{-1}\sum_{x,y,s} 1_E(x)1_F(y)\chi(s(x\cdot y-t))\\ &= p^{-1}|E||F| + p^{-1}\sum_{x,y,s\neq 0}(\dots).\end{aligned} \tag{4.6}$$

Now we work on the remainder. First apply CS,

$$\begin{aligned}R^2(t) &\le \left(\sum_x p^{-2}1_E^2(x)\right)\left(\sum_x \sum_{y,y',s\neq 0,s'\neq 0} 1_E(x)1_F(y,y')\chi(x\cdot(sy-s'y'))\chi(t(s'-s))\right)\\ &= |E|p^{d-2}\sum_{sy=s'y',s\neq 0,s'\neq 0} 1_F(y,y')\chi(t(s'-s))\\ &= p^{d-2}|E|\left(|F|(p-1)+\sum_{sy=s'y',s,s\neq 0,y\neq y'}(\dots)\right)\end{aligned} \tag{4.7}$$

the left sum is obvious and we sum in x on the right sum. Finally, we split off the case where $y=y'$. Now, to analyze the remaining sum, we apply the bijective transformation $(s,\frac{s}{s'})\mapsto(a,b)$, rearrange, and note the sum is clearly negative. Hence $R^2(t)\le|E||F|p^{d-1}$. The result is immediate. ■

Theorem 4.4. Plancharel. $\sum_{m\in\mathbb{Z}_p^d}|\hat f(m)|^2 = p^{-d}\sum_{x\in\mathbb{Z}_p}|f(x)|^2$.

Theorem 4.5. Szemeredi-Trotter. $I(P,L)\lesssim O(m+n+(mn)^{\frac{2}{3}})$.

Theorem 4.6. $k^2\cdot\mathrm{cr}(H)=cr(k\cdot H)$.

Proof. Taking any drawing of H with $cr(H)$ crossings. For each edge of H, draw k parallel edges sufficiently close together. Then we have $cr(k\cdot H)\le k^2\cdot cr(H)$. Conversely, taking any drawing of $k\cdot H$, we can pick a representative of each set of parallel edges in $k^{|E(H)|}$ ways to obtain a drawing of H, with at least $cr(H)$ crossings. But in this way any pair of crossing edges in $k\cdot H$ is counted $k^{|E(H)|-2}$ ways (since once we picked two representative edges that cross, there are that many different ways to pick representatives for the remaining edges.). ■

Theorem 4.7. Crossing number of multigraph of maximum multiplicity m. Assume $e\ge 5vm$. Then $cr(G)\lesssim\frac{e^3}{mv^2}$.

Proof. For $0\le i\le\lg m$, let G_i denote the subgraph of G where we only keep pairs of edges whose multiplicity lies between 2^i and 2^{i+1}. Put

$$A=\{i\in[0,\lg m]:|E(G-i)|\le 2^{i+1}v\}$$

$$B=[0,\lg m]\backslash A.$$

$$\sum_{i\in B}|E(G_i)|\ge|E(G)|-\sum_{i\in A}|E(G_i)|\ge|E(G)|-4mv\ge|E(G)|/5$$

since $\sum_{i\in A}|E(G_i)| \le 4mv$ (as geometric series can be bounded by their largest term). Now let G_i^* denote the simple graph gotten from identifying parallel edges in G_i. Then

$$\begin{aligned} cr(G) &\ge \sum_{i=0}^{\lg m} cr(G_i) \ge \sum_{i\in B} 2^{2i} cr(G_i^*) \\ &\ge \sum_{i\in B} c\frac{|E(G_i^*)|^3}{v^2} 2^{2i}. \end{aligned} \tag{4.8}$$

The first inequality follows since taking subgraphs can only reduce the number of crossings. The second follows by theorem 9. The third using the crossing number lemma for simple graphs. Now, use the fact that $2^{i+1}|E(G_i^*)| \ge |E(G_i)|$ (let c absorb an extra factor of 2).

$$cr(G) \ge \sum_{i\in B} c\frac{|E(G_i)|^3}{v^2\cdot 2^{3i}}2^{2i} = \frac{c}{n^2}\sum_{i\in B}\left(\frac{|E(G_i)|}{2^{i/3}}\right)^3. \tag{4.9}$$

Now apply Holder's inequality (first inequality), with $p=3, q=\frac{3}{2}$, we have

$$cr(G) \ge \frac{c}{n^2}\left(\sum_{i\in B}\frac{|E(G_i)|}{2^{i/3}}2^{i/3}\right)^3 \Bigg/ \left(\sum_{i\in B}(2^{i/3})^{3/2}\right)^2 \ge \frac{c}{v^2 m}\left(\sum_{i\in B}|E(G_i)|\right)^3 \ge \frac{ce^3}{mv^2}.$$

■

Theorem 4.8. Let P be a set of n points in the plane. Let m_k denote the number of lines with at least k points. Applying ST,

$$m_k \lesssim \frac{n}{k} + \frac{n^2}{k^3}.$$

Theorem 4.9. (Beck). Let P be a set of n points in the plane. Either there is a line containing a positive proportion of the points in P, or there is cn^2 ($c>0$ constant) different lines each containing at least two points of P.

Proof. Let $L_{u,v}$ denote the number of lines incident to at least u but no more than v points of P. By theorem 11, as there are at most $\binom{v}{2}$ pairs of points in P which determine each line with at most v points, we have $L_{u,v} \lesssim v^2(\frac{n}{u}+\frac{n^2}{u^3})$. Fix a constant C. Then

$$\begin{aligned} L_{c,n/c} &\le \sum_{i=0}^{\lfloor \lg(n/c^2)\rfloor} L_{C2^i,C2^{i+1}} \\ &= \sum O(\frac{4n^2}{C2^i} + 4Cn2^i) \\ &= O\left(\frac{n}{C^2}\right) \end{aligned} \tag{4.10}$$

where we bound the geometric sums by their largest terms. Hence for some $C_0 > 0$, $L_{C,n/C} \le C_0(n^2/C)$. For an appropriate choice of C, at least have of the pairs of points will determine a line with between C and n/C points. Either, a quarter of the pairs will go through fewer than C points (lots of lines with few points), or there is a line with n/C points. ■

A corollary to this theorem is that there exists c_0 such that $c_o n$ points of P lie on $c_0 n$ lines of L.

Theorem 4.10. (Application of theorem 6). If $A \subset \mathbb{Z}_p$ such that $|A| > p^{\frac{3}{4}}$, then $A \cdot A + A \cdot A$ contains $\mathbb{Z}_p^*$.

- $n^{\frac{4}{5}}$ argument. Let P be a set of n points in the plane determining t distances. Draw circles around each point of P, going through other points of P. (So at most nt circles, with exactly $n(n-1)$ incidences). Delete circles with at most 2 points on them (eliminates at most $2nt$ incidences & since we assume $t \ll n$, this is fine). Form a graph G whose vertices are points of P and whose edges are points adjacent on some circle.
- Now our graph could have multiple edges. However, if two points have multiple edges, then the centers of the circles corresponding to those edges must lie on the perpendicular bisector of those two points. We want to eliminate high multiplicity edges.
- Lemma. Number of pairs (l, e), where l (a perpendicular bisector) has at least k points and e is an edge corresponding to an arc in G is $O(t(\frac{n^2}{k^2} + n \log n))$.
- Proof. The number of lines with at least $m_k = 2^i \leq \sqrt{n}$ points is at most $n^2/2^{3i}$. ????
- Using the lemma, with $k = c\sqrt{t}$ we can delete all edges with multiplicity $\geq k$. By the lemma, we still have $\Omega(n^2)$ edges. Applying multigraph crossing number lemma, $\frac{e^3}{kn^2} \leq cr(G) \leq n^2 t^2$, hence as $k \sim \sqrt{t}$ and $e \sim n^2$, we have our $\frac{4}{5}$ths result.

- $n^{\frac{6}{7}}$.
- P set of n points in the plane, L set of lines determined by P.
- Beck's Lemma.

  (Many points, one line $\iff$ Few points, many lines)

  So for some constant $c_0 > 0$, we can find $c_0 n$ points of P which lie on $c_0 n$ distinct lines of L (there must be a middle-ground). Let B be the set of such points (so $|B| = \Omega(n)$). Let $a \in B$. Draw lines through a going through other points of P. ($\Omega(n)$ lines).
- On each of these lines, choose a point $\neq a$ and draw a circle centered at a through that point (remove circles with ≤ 3 points, if many circles has ≤ 3 we'd have many distances).
- For each circle, (after deleting at most 2 points from the circle), partition the remaining points into disjoint consecutive triples. We still have the number of such triples is $\Omega(n)$.
- Call a triple *bad* if all three bisectors formed by its points, go through at least k points. We call *a bad* is half its triples are bad. (Note the smaller k is the more bad points, we want k to be big, $k = \frac{cn^2}{t^2}$).
- We want to show, $n^2/t^{\frac{2}{3}} \leq I(L_k, P) \leq t^4/n^2$, where L_k is the k-rich lines (with k incidences) and P is now all the bad points.
- For the upper bound, use the $m_k \leq \frac{n^2}{k^3}$ bound and S-T, to get $I(L_k, P) \leq (n\frac{n^2}{k^3})^{2/3} \sim \frac{t^4}{n^2}$
- Lemma. Arithmetic triples. T is a set of N triples (a_i, b_i, c_i) s.t. $a_i < b_i < c_i$ for all i and $c_i < a_{i+1}$ for all but $t-1$ of the i. If $W = \{\frac{a_i+b_i}{2}, \frac{a_i+c_i}{2}, \frac{b_i+c_i}{2}\}$, then $|W| \geq \frac{N}{t^{2/3}}$. (Here we interpret the a_i, b_i, c_i as angles of points on the circles and their averages are the angles of the bisectors).

- The number of k-rich lines incident to a is $n/t^{2/3}$, so since $|B| = \Omega(n)$, we have $I(L_K, P) \geq n^2/t^{2/3}$ and taking $k = cn^2/t^2$, in our inequality gives the result.

- Fefferman's trick gives a $n^{4/3}$ lower bound in $\mathbb{F}_p^2$ for distance sets (an improvement over $2p^{3/2}$).
- Triangle congruence's $|E| > 10p^{8/5}$?
- The proof of Szemeredi-Trotter only uses the fact that a line is determined by a finite number of points. A family of r-pseudo-lines is a collection of subsets of the plane so that any such subset is determined by r points. I f L is a family of r-psuedo-lines and P is a set of points, then $I(P, L) \leq (|P||L|)^{2/3}$.

  `www.ms.uky.edu/~pkoester/research/sztrotter.pdf`