Deriving the Gaussian Distribution

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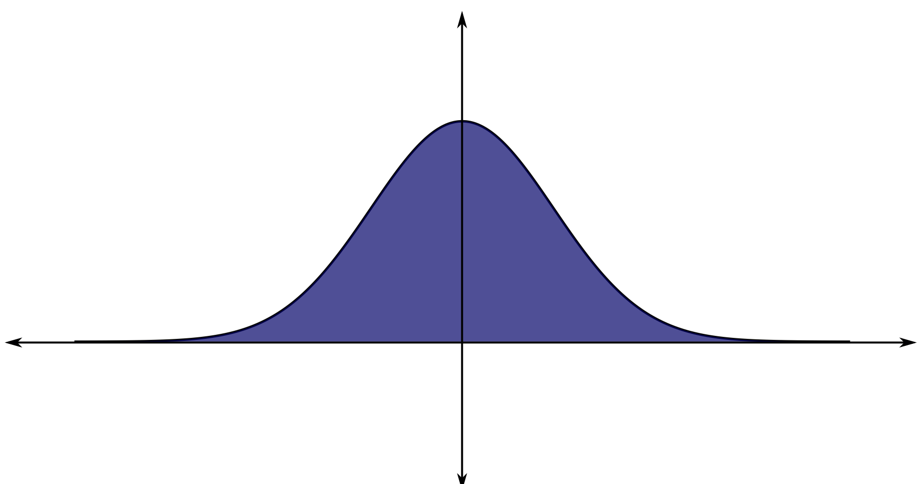

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-x^2)\ \mathrm{d}x = 1.$$

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1 The Setup

1.1 A Thought Experiment

Consider an infinite dart board on the xy-plane, as in Figure 1, with the bullseye at the origin. As we might expect the majority of the darts will cluster around the origin. Furthermore, as we move radially outwards from the origin, we would expect darts to land with decreasing frequency. Now we wish to derive

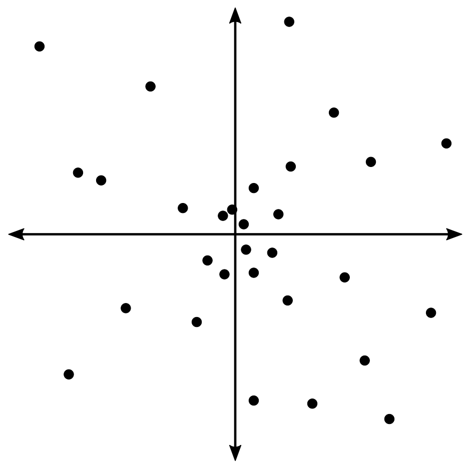

Figure 1: Infinite dart board.

a pdf φ that describes the relative probability of finding a dart at a particular location on the xy-plane. That is, given a point (x_0, y_0) on the plane, we want to describe φ, such that the probability that a dart lands within a box with area $\mathrm{d}A$, centered at (x_0, y_0), is $\varphi(x_0, y_0)\mathrm{d}A$.

However, since φ is only dependent on the radial distance from the origin, we have $\varphi(x_0, y_0) = \varphi(r)$, where $r = \sqrt{x_0^2 + y_0^2}$. Next, we'll assume that the x and y coordinates are statistically independent. Therefore, for some particular pdf, f, we have $\varphi(r)\mathrm{d}A = f(x)f(y)\mathrm{d}A$. Hence,

$$\varphi(\sqrt{x^2 + y^2}) = f(x)f(y). \tag{1.1}$$

Now, consider when $y = 0$. Then, we have $\varphi(x) = f(0)f(x)$, that is for $y = 0$, φ is a scalar multiple of f. Let $\lambda = f(0)$, and we'll rewrite Equation 1.1 as

$$\lambda f(\sqrt{x^2 + y^2}) = f(x)f(y). \tag{1.2}$$

Or equivalently,

$$\frac{f(\sqrt{x^2 + y^2})}{\lambda} = \frac{f(x)}{\lambda}\frac{f(y)}{\lambda}. \tag{1.3}$$

Now we much solve this functional equation.

2 The General Solution

2.1 A Similar Problem

Currently, solving Equation 1.3 seems rather difficult. So first we'll consider a much simpler problem to gain insight on a possible solution.

$$h(x)h(y) = h(x+y) \tag{2.1}$$

Astute readers will immediately recognize that this the functional equation for the exponential function. Hence, $h(x) = e^{Cx}$ for some arbitrary constant C.

2.2 Solving Equation 1.3

Using the substitution, $g(x) = \frac{f(x)}{\lambda}$, Equation 1.3 becomes

$$g(\sqrt{x^2+y^2}) = g(x)g(y) \tag{2.2}$$

Notice that if we first, square the input of g, then 2.2 simplifies to

$$g(x^2+y^2) = g(x^2)g(y^2)$$

which is just the exponential function. Hence g is the composition of the exponential function and the function $x \mapsto x^2$. Explicitly,

$$g(x) = e^{Cx^2}. \tag{2.3}$$

Therefore,

$$f(x) = \lambda e^{Cx^2}. \tag{2.4}$$

2.3 Simplifying the Constants

First, since f is a probability density function we must have

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1. \tag{2.5}$$

Therefore, $\lim_{x\to\pm\infty} f(x)$ must equal 0, i.e. we require that $C < 0$. Suppose $C = -h^2$, then $f(x) = \lambda e^{-(hx)^2}$. Using Equation 2.5, we can relate λ to h via $h^2 = \pi\lambda^2$. So our normalized function is

$$f(x) = \lambda e^{-\pi\lambda^2 x^2} \tag{2.6}$$