Game Theory

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1 Introduction

Game theory is the systematic study of the strategic interdependence in games. In contrast, decision theory studies the decision-making of individuals when choices of others do not affect outcomes.

Rational Choice Theory

- Suppose a decision-maker is given a set of available actions A such that each action leads to a different outcome.
- We assume the decision-maker has well-defined preferences, i.e. given any two actions $a, b \in A$, we either have $a \succeq b$ or $b \succeq a$. Furthermore, this relationship is transitive, i.e. if $a \succeq b$ and $b \succeq c$, then $a \succeq c$. (Note that the relation $\succeq$ is defined by the individual decision-maker's payoff function which provides an ordinal ranking of the actions in A.)
- With these assumptions, we say a rational decision-maker will choose the action that is at least as good, according to her preferences, as every other available action.

2 Nash Equilibrium

Definition (Strategic Game). A game in strategic form consists of (a) a set $N = \{1, \ldots, n\}$ of n players; (b) for each player $i \in N$, a (possibly infinite) set of actions A_i; and (c) for each player $i \in N$, preferences over the set of action profiles given by a utility function $u_i : A_1 \times \cdots \times A_n \to \mathbb{R}$.

An **action profile** is a tuple $a \in A_1 \times \cdots \times A_n$. The set of all such action profiles is denoted by A. We use A_{-i} to denote the set of all action profiles without player i.

In strategic games we assume all players act simultaneously, i.e. without the knowledge of other players' choices; and that once all players fix their choice, the outcome of the game is realized. We also assume players are rational, i.e. that they are aware of other players' action sets, they have preferences over A, and they will choose the best possible action according to their preferences.

Definition (Dominant Strategy). An action $a_i \in A_i$ is dominant if $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$, $\forall a_i' \in A_i$, $a_{-i} \in A_{-i}$.

We say a_i is a **best response** to a particular strategy a_{-i} if $\forall a_i' \in A_i$, $u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$.

Definition. An action a_i is **strictly dominated** if $\exists a_i' \in A_i$ such that $u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$, $\forall a_{-i} \in A_{-i}$.

The iterated elimination of strictly dominant actions (IESDA) algorithm, identifies and deletes strictly dominated actions, and repeats until no more eliminations are possible.

We can use the strategy of IEDSA because a rational player will never select a strictly dominated action. However, it requires greater assumptions about the knowledge of players.

IESDA requires "common knowledge", i.e for the first elimination we must assume players are rational. For the second, we must assume players know other players are rational and their payoffs. For the third, we must assume players know that they know each others payoffs and rationality, and so on, eventually forming an infinite hierarchy of knowledge.

Definition (Dominance Solvable). If a unique profile of actions survives IESDA then the game is called dominance solvable.

Proposition 2.1. Given a finite strategic game all iterated eliminations of strictly dominated strategies yield the same outcome.

Definition. An action a_i is **weakly dominated** if $\exists a_i' \in A_i$ such that $u_i(a_i', a_{-i}) \geq u_i(a_i, a_{-i})$, $\forall a_{-i} \in A_{-i}$, with at least one strict inequality.

Note that order of elimination can affect the outcome of IEWDA, even in finite games.

Definition (Nash Equilibrium). We say $a = (a_1, \ldots, a_n) \in A$ is a Nash equilibrium if $\forall i \in N, \forall a_i' \in A_i$, $u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$. In other words, every action in a is a best response to a_{-i}.

Theorem 2.2

- A strictly dominated strategy can never be part of a Nash equilibrium.
- An action eliminated by IESDA can never be part of a Nash equilibrium.
- If a profile is a unique survivor of IESDA, then it is a Nash equilibrium.
- Weakly dominated actions can be part of a Nash equilibrium.

Definition (Best Response Function). For all i and all $a_{-i} \in A_{-i}$, define

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i\}.$$

A profile $a^* \in A$ is a Nash equilibrium if $a_i^* \in B_i(a_{-i}^*) \forall i$.

3 Mixed Strategy Nash Equilibria

A mixed strategy is a strategy where players use more than one available action with positive probability. For each player i, we denote the lottery over all available actions A_i by σ_i, where σ_i is a probability vector. Specifically, $\sigma_i(a_i)$ denotes the probability that player i plays a_i.

In general, $\sigma = (\sigma_1, \ldots, \sigma_n)$ denotes a profiles of of mixed strategies. Thus, given an action profile $a = (a_1, \ldots, a_n)$,

$$\mathbb{P}(a) = \prod_{i=1}^{n} \sigma_i(a_i).$$

We rank outcomes in a mixed strategy game using expected utility. We define the expected utility function U_i for each player i by

$$U_i(\sigma) = U_i(\sigma_i, \sigma_{-i}) = \sum_{a \in A} \mathbb{P}(a) u_i(a).$$

Remark. We assume u_i are cardinal payoff functions to calculate expected utility. However, two cardinal payoff functions u, v are equivalent if $u = cv + d$ for $c, d \in \mathbb{R}$.

Definition (Strategic Game with vNM Preferences).

- Set of players $N = \{1, \ldots, n\}$
- Set of actions A_i for all $i \in N$
- For each player i, a cardinal utility function $u_i : A \to \mathbb{R}$
- We assume player randomize amongst their available actions and use expected utility to compute payoffs.

Definition (Nash Equilibrium). A mixed strategy profile σ^* is a Nash equilibrium if for all players i,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*),$$

for all σ_i. Equivalently, if

$$\sigma_i^* \in BR_i(\sigma_{-i}^*)$$

for all players i.

Theorem 3.1

All finite games in strategic form have at least one Nash equilibrium in pure or mixed strategies.

Theorem 3.2

A profile σ is a Nash equilibrium if and only if for all players i:

- If $\sigma_i(a_i) > 0$ and $\sigma_i(a_i') > 0$, then

$$U_i(a_i, \sigma_{-i}) = U_i(a_i', \sigma_{-i}) = U_i(\sigma_i, \sigma_{-i}).$$

- For all $a_i'' \in A_i$ with $\sigma_i(a_i'') = 0$,

$$U_i(a_i'', \sigma_{-i}) \leq U_i(\sigma_i, \sigma_{-i}).$$

Theorem 3.3

A profile σ^* is a Nash equilibrium if for all players i,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(a_i, \sigma_{-i}^*),$$

for all pure strategies $a_i \in A_i$.

Furthermore, note that a strictly dominated strategy must have zero probability in a mixed Nash equilibrium and that almost always, each player mixes with the same number of strategies.

Definition. A strategy σ_i is strictly dominated by σ_i' if

$$U_i(\sigma_i, a_{-i}) < U_i(\sigma_i', a_{-i}),$$

for all $a_{-i} \in A_{-i}$.

Interpretations of Nash Equilibria

- (Harsanyi) Mixed strategy Nash equilibria are the limit of pure strategy equilibria in games of incomplete information as uncertainty tends to zero.
- (Nash) The probabilities of mixed strategy equilibria are the frequency of the pure strategies of different people in the population.
- (Aumann) Player 2's mixed strategy represents player 1's belief about player 1's behavior. Both players play a pure strategy. Overtime the observed frequency of various pure strategy choices must support these equilibrium beliefs.

Only strategies that survive IESDA can be part of a Nash equilibrium. These strategies are called **rationalizable**. In other words, there exists a reasonable conjecture of play by the other players that renders these strategies best responses, even if not part of NE.

4 Extensive Form Games

Definition (Extensive Form Game with Perfect Information).

- A set of players N;
- A set of sequences O, of terminal histories, with the property that no sequence is a proper subhistory of any other sequence.
- A player function P, that assigns a player to every sequence that is a proper subhistory of a terminal history.
- For each $i \in N$, a utility function $u_i : O \to \mathbb{R}$.

We can deduce the set of actions available to a player who moves after h, $A(h) = \{a : (h, a) \text{ is a history}\}$. If the longest terminal history is finite, then the game has a **finite horizon**. If a game has a finite horizon and O is finite, then we say the game is **finite**.

A **strategy** of player i in an EFGWPI is a function that assigns to each history h, with $P(h) = i$, an action in $A(h)$. That is, a strategy of a player specifies her action after *every* history, after which it is her turn to move. Thus, if player i has n decision nodes, she must specify n actions for any strategy, one for each decision node.

Given a strategy profile s, we call the terminal history constructed from s, the **outcome** of s, denoted $O(s)$.

Definition (Nash Equilibrium). A strategy profile s^* in an EFGWPI is a Nash equilibrium if, for each player i,

$$u_i(s^*) \geq u_i(r_i, s^*_{-i})$$

for every strategy r_i of player i.

We can construct a strategic form game from an extensive form game G:

- The set of players is N
- For all i, A_i is the set of strategies of player i in G
- Each players payoff to each action profile a is $u_i(s_i)$ for

Definition (Subgame). Let Γ be a EFGWPI, with player function P. For any nonterminal history h of Γ, the subgame $\Gamma(h)$ following the history of h is the following extensive game

- Players: Those of Γ
- Terminal histories: $\{h' : (h, h') \text{ is a terminal history of } \Gamma\}$
- Player function: $P(h, h')$ is assigned to each proper subhistory h' of a terminal history.
- Preferences: Each player prefers h' to h'' iff she prefers $(h.h')$ to (h, h'') in Γ.

Definition. Strategy profile s^* in an EFGWPI is a subgame perfect equilibrium if for all players i and all histories h such that $P(h) = i$

$$u_i(s^*) \geq u_i(r_i, s^*_{-i})$$

for every strategy r_i of player i.

All *finite* games of perfect information have at least one SPNE in pure strategies.

Backwards induction. Whenever a player has to move, she deduces, for each of her possible actions, the actions that the players (including herself) will subsequently rationally take, and chooses the action that yields the terminal history she most prefers.

***Finite* games of perfect information such that no player is indifferent between any two terminal histories have a unique SPNE.**

All pure strategy SPNE of *finite* games of perfect information can be recovered by applying backwards induction.

Exercise 1. (Rotten Kid Theorem). Consider a game with two players: a parent and a child. First, the child declares an action a, resulting in her income $c(a)$ and her parent's income $p(a)$. The parent can then decide to transfer $t > 0$ income to the child. Suppose

$$u_C(a,t) = c(a) + t$$

and

$$u_P(a,t) = \min\{c(a)+t, p(a)-t\}$$

(The child is selfish, the parent spoils the child, up to a point).

The parent will choose t so as to maximize u_P, i.e. so that $c(a)+t = p(a)-t$. Thus $t = \frac{p(a)-c(a)}{2}$. Then the child selects a to maximize u_C, that is to maximize $c(a) + \frac{p(a)-c(a)}{2} = \frac{p(a)+c(a)}{2}$. Which is to say, the spoiled child will act so as to maximize the sum of her *and* her parent's income.

Perfect Information. At every decision node, the player whose turn it is has observed all prior action choices by all players and players move sequentially.

4.1 Sequential to Simultaneous

Given a two-person extensive form game, we can construct an equivalent two-person simultaneous move game where the player's respectively action sets are their strategies in the corresponding extensive form game. The payoffs of the simultaneous move game are the payoffs of the outcome resulting from the chosen strategies in the extensive form game. **A strategy profile in the simultaneous move game is a Nash equilibrium if and only if it corresponds to a Nash equilibrium of the extensive form game.**

4.2 Zero-Sum Games

A **zero-sum game** is a game in which the sum of the players' payoffs is constant for all outcomes.

Proposition 4.1. Let G two-person zero-sum game with Nash equilibrium (σ_1, σ_2). Then each player acts so as to minimize the other player's payoff.

Proof. Player 2's strategy maximizes her payoff subject to player 1's strategy... ∎

Proposition 4.2. In every finite *two-person* zero-sum game, the *same* expected payoff is achieved for both players in *all* Nash equilibria.

Proposition 4.3. (Zermelo) In any two-person zero-sum game (not affected by chance), either player 1 wins, player 2 wins, or the players force a draw in *all* Nash equilibria of the game.

4.3 Ultimatum Game & Bargaining

Discrete Ultimatum Game. Say we split \$1 in increments of a penny. Then, player 2 accepts any positive offer and is indifferent for an offer of \$0. Thus, we have two SPNE

1) Player 1 offers \$0, and player 2 accepts any offer.
2) Player 1 offers \$0.01, and players 2 accepts any positive offer, but rejects \$0.

Continuous Ultimatum Game. Again player 2 accepts any positive offer and is indifferent for an offer of \$0. If player 2 rejects \$0, then there is no optimal action for player 1, so the only SPNE is when player 1 offers \$0, and players 2 accepts all offers.

Finite Bargaining. Players have discount factor $0 < \delta_i < 1$, $i = 1, 2$ (an offer of $\delta_i^t x$ now, is worth x, t periods from now).

- $T = 2$. As before, if round 2, player 2 offers $y^2 = (0, 1)$. Thus, player 2 will accept no less than δ_2 in round 1. Thus the only SPNE is for player 1 to offer $y^1 = (1 - \delta_2, \delta_2)$ in round 1 and accept any offer in round 2, and player 2 to accept any offer $y_2^1 \geq \delta_2$ in round 1 and offer $(0, 1)$ in round 2.
- $T = 3$. Similar analysis. Player 1 offers $y^1 = (1 - \delta_2(1 - \delta_1), \delta_2(1 - \delta_1))$ and $y^3 = (1, 0)$ and accepts any offer $y_1^2 \geq \delta_1$. Player 2 offers $y^2 = (\delta_1, 1 - \delta_1)$ and accepts offers $y_2^1 \geq \delta_2(1 - \delta_1)$ and $y_2^3 \geq 0$.
- $T = \infty$. (Subgames are indistinguishable) Player 1 always proposes $(y_1, 1 - y_1)$ and player 2 always proposes $(1 - y_2, y_2)$. Player i accepts offers which allocate at least $\delta_i y_i$. Thus we have

$$1 - y_1 = \delta_2 y_2 \tag{4.1}$$
$$1 - y_2 = \delta_1 y_1 \tag{4.2}$$

  So $y_2 = \frac{1-\delta_1}{1-\delta_1\delta_2}$ and $y_1 = \frac{1-\delta_2}{1-\delta_1\delta_2}$. Thus player 1 always proposes $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, 1 - \frac{1-\delta_2}{1-\delta_1\delta_2}\right)$ and player 2 accepts if and only if she gets at least $1 - \frac{1-\delta_2}{1-\delta_1\delta_2}$. Player 2 always offers $\left(1 - \frac{1-\delta_1}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right)$ and player 1 accepts if and only if she gets at least $1 - \frac{1-\delta_1}{1-\delta_1\delta_2}$.

We see that there is no delay in SPNE, the first offer is always accepted. It pays to be patient (δ_i close to 1). It is advantageous to be the proposer.

4.4 Binary Sequential Voting & Agenda Manipulation

In any binary sequential vote, the majority preferred *outcome* prevails in all subgame perfect Nash equilibria. This is not necessarily the case for general Nash equilibria.

Agenda Manipulation. e.g. Three alternatives x, y, and z, committee may vote between x and y first, then vote between z and the winner of the first vote. In this case, it is possible that voters may vote against their preference in early votes, even in SPNE.

Interestingly, if committee preferences are intransitive (majorities prefer x to y to z to x, which is possible even when personal preferences are transitive), the voting outcome is contingent on order of sequential binary agenda.

Centipede Game. Payoffs are structured in such a way (doubling after every period) such that players take after any history, and the least fruitful outcome prevails.

5 Imperfect Information

An **extensive form game with imperfect information** is an extensive form game coupled with an **information partition** of the decision nodes. Formally, an information partition is a partition of H into $\{I_j\}_{j=1,\ldots,m}$, such that for all $h, k \in I_j$, we have $P(h) = P(k)$ and $A(h) = A(k)$. We interpret information sets as follows:

- Players that move at decision nodes the belong to the same information set cannot distinguish between these node. (Hence the requirement all nodes belonging to an information set must belong to the same player and have the same set of available actions)
- Moreover, we assume players have **perfect recall** so they do not forget past action choices of their own or any action choice by other players which they observed.

A **pure strategy** s_i in an EFGWII, is a specification of an action choice for every information set corresponding to player i. A **mixed strategy** σ_i is a lottery over player i's pure strategies.

A strategy profile (σ_i, σ_{-i}) is a **Nash equilibrium** if for all players i

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(s_i, \sigma_{-i}) \text{ for all } s_i \in S_i.$$

Extensive Form to Strategic Form. Determine the players strategy sets and for each profile of strategies, assign payoffs corresponding to the terminal history reach with these strategies.

Strategic Form to Extensive Form. Assign a player to some root. Draw a branch for each strategy available to that player. Add a decision node at the end of these branches for another player with as many branches as her strategies.

Collect all these nodes with an information set and repeat for all remaining players. Finally, assign payoffs to terminal histories.

Nash equilibria of the strategic form game correspond to Nash equilibria of the imperfect information game.

A **subgame** must be rooted at a decision node h contained in a singleton information set and all decision nodes that follow h must not be connected in an information set with nodes that do not follow h. A **subgame perfect Nash equilibrium** is a Nash equilibrium that is Nash in every game and Nash in every subgame.

- In finite games of imperfect information, existence of SPNE is not guaranteed in pure strategies, but it is guaranteed in mixed strategies.
- Backwards induction can recover SPNE in finite games of imperfect information.

5.1 Backwards Induction 2.0

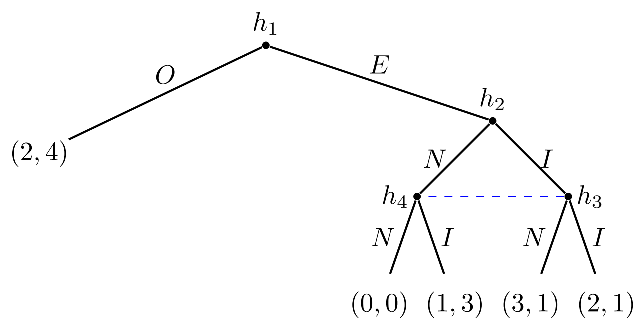
1) Identify minimal subgames.
   (a) Write the equivalent strategic form of each subgame and solve for NE.
   (b) Calculate expected payoffs and called these subgames *solved*.
2) Proceed to subgames that only contain solved subgames.
   (a) Assume payoffs of solved subgames.
   (b) Apply step (1).
3) Repeat until all subgames are solved.
4) Resulting strategy profiles are SPNE.

In games of perfect recall, Kuhn's theorem assures us that it is equivalent to report player behavior at each information set. Strategies reported this way are called behavioral strategies.

Example. Consider the following game which has two subgames rooted at decision nodes: h_1 and h_2. Define the information partition: $\mathcal{I}_1 = \{h_1\}, \mathcal{I}_2 = \{h_2\}$, and $\mathcal{I}_3 = \{h_3, h_4\}$. For the subgame rooted at h_2, we find three Nash equilibria: (I, N), (N, I) and $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}))$.

	N	I
N	$0, 0$	$3, 1$
I	$1, 3$	$2, 1$

These equilibria obtain payoffs: $(1, 3), (3, 1)$, and $(\frac{3}{2}, 1)$, respectively. For each equilibrium of the h_3 subgame we apply backwards induction to solve for SPNE of the subgame rooted at h_1.



The SPNE of game are (OI, N), (EN, I), and in the mixed strategy case

- Player 1 picks O at $\mathcal{I}_1$ and at $\mathcal{I}_3 = \{h_3, h_4\}$, plays N with probability $\frac{2}{3}$ and I with probability $\frac{1}{3}$.
- Player two picks N with probability $\frac{1}{2}$ and I with probability $\frac{1}{2}$ at $\mathcal{I}_2$.

6 Repeated Games

Given a strategic form game $G = (N; A_1, \dots, A_n; u_1, \dots u_n)$, called the **stag game**, a **repeated game** is constructed by repeating G for T periods. A strategy for player i consists of an action choice from A_i for every history/information set and every period $t = 1, \dots, T$. Formally, a strategy for player i is a set of functions $s_i^t : A^{t-1} \to A_i$, one for each period $t = 1, \dots, T$. Payoffs are the sum of the payoffs from every period.

Proposition 6.1. Given an n player strategic form game that has a unique Nash equilibrium $a^* = (a_1^*, \dots, a_n^*)$ for all T, there is a unique SPNE of the T-times repeated game. It is such that player i chooses a_i^* after all histories.

Thus for any finite T, prisoner's dilemma repeated T periods has a unique SPNE, for both players to defect after every history.

6.1 Infinitely Repeated Games

Payoffs are now discounted by a factor $\delta < 1$. Typically, the payoffs are normalized, hence the sequence of outcomes $(a^1, a^2, \dots,)$ yields payoff

$$S_i = (1 - \delta) \sum_{k=1}^{\infty} u_i(a^k) \delta^{k-1} \text{ for player } i.$$

The discount factor can be interpreted as the probability play continues in the next period.

Theorem 6.2: One-stage-deviation Principle

A profile is SPNE if for all players and all subgames, no player can profitably deviate *just* in the first period of the subgame, while she and all others stick to their strategy forever after.

Corollary. If $a^* = (a_1^*, \ldots, a_n^*)$ is NE of the stage game, then the profile of strategies where player i always plays a_i^* is SPNE of the ∞-repeated game for any δ.

Using the one-stage-deviation principle we see that cooperation can be supported in PD if $\delta \geq \frac{1}{2}$.

6.2 Folk Theorem

Nash Threat. Consider any 2-player stage game with at least two NE a^* and a^{**} such that $u_1(a^*) > u_1(a^{**})$. Let a be a strategy such that $u_1(a) \geq u_1(a^{**})$. Then the equilibrium a^{**} can be used to punish player 1 for deviating from strategy a.

Proposition 6.3. There exists some even number of periods T^* such that for all $T > T^*$, profile a is the outcome for the first $T - T^*$ period in some SPNE of the repeated game.

7 Bayesian Games

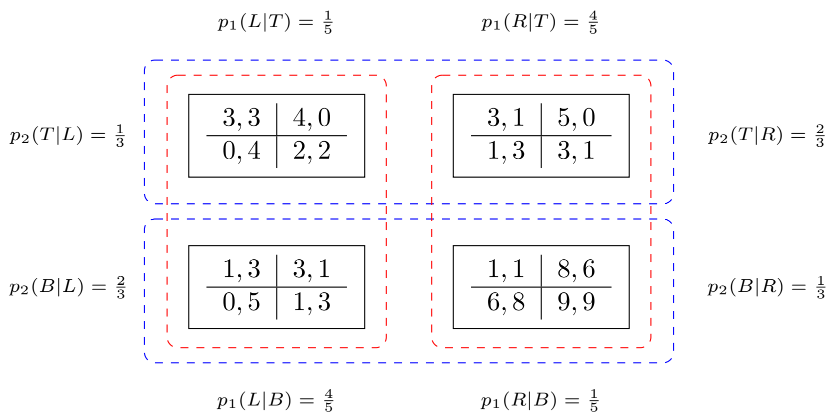
Definition. A **Bayesian Game** is a game of incomplete information, that is, a game where player only have probabilistic beliefs about the other players payoffs. A Bayesian game consists of (a) a set of players N, (b) a set of actions A_i for each player (c) a set of types T_i for each player (d) for each type t_i of player i, beliefs about the types of other players, $p_i(t_{-i}|t_i)$, and (e) a payoff function $u_i(a;t)$ for each profile of actions a and profile type t.

A **Bayesian Nash equilibrium** of a Bayesian game is a profile of (behavioral) strategies σ such that

$$U_i(\sigma|t_i) \geq U_i(a_i, \sigma_{-i}|t_i)$$

for all player-type pairs (i, t_i) and all actions $a_i \in A_i$.

> **Example.** Consider the following Bayesian game G with players $N = \{1, 2\}$ and type set $T = \{T_1, T_2\}$ where $T_1 = \{T, B\}$ and $T_2 = \{L, R\}$. The action sets are $A_1 = \{I, O\}$ and $A_2 = \{F, C\}$.
>
> We begin by noticing that if player 1 is of type T, then she strictly prefers I to O, and if player 2 is of type L she strictly prefers F to C. Thus in any NE we have $\sigma_1(I|T) = 1$ and $\sigma_2(F|L) = 1$.



Suppose $\sigma_2(F|R) = 1$, then

$$U_1(I, \sigma_2|B) = \frac{4}{5}(1) + \frac{1}{5}(1) = 1$$

$$U_1(O, \sigma_2|B) = \frac{4}{5}(0) + \frac{1}{5}(6) = \frac{5}{6}.$$

So player 1 strictly prefers O to I. Now assume that $\sigma_1(O|B) = 1$, then

$$U_2(F, \sigma_1|R) = \frac{2}{3}(1) + \frac{1}{3}(8) = \frac{10}{3}$$

$$U_2(C, \sigma_1|R) = \frac{2}{3}(0) + \frac{1}{3}(9) = 3$$

Thus player 2 strictly prefers F to C. Therefore, we've found a pure strategy Nash equilibrium. Similarly, we can find another pure strategy NE, where $\sigma_1(I|B) = 1$ and $\sigma_2(C|R) = 1$. To find the mixed equilibrium,

$$U_1(I, \sigma_2|B) = \frac{1}{5}(12 - 7\sigma_2(F|R))$$

$$U_1(O, \sigma_2|B) = \frac{1}{5}(9 - 3\sigma_2(F|R))$$

Equating yields $\sigma_2(F|R) = \frac{3}{4}$.

$$U_2(F, \sigma_1|R) = \frac{1}{3}(10 - 7\sigma_1(I|B))$$

$$U_2(C, \sigma_1|R) = \frac{1}{3}(9 - 3\sigma_1(I|B))$$

Equating yields $\sigma_1(I|B) = \frac{1}{4}$. The Nash equilibria are thus

1) $\sigma_1(I|T) = \sigma_2(F|L) = \sigma_2(F|R) = \sigma_1(O|B) = 1$.
2) $\sigma_1(I|T) = \sigma_2(F|L) = \sigma_2(C|R) = \sigma_1(I|B) = 1$.
3) $\sigma_1(I|T) = \sigma_2(F|L) = 1$ and $\sigma_2(F|R) = \sigma_1(O|B) = \frac{3}{4}$.

Proposition 7.1. If there exists any higher order uncertainty (uncertainty is present at all levels in the knowledge hierarchy), then the finite game is dominance solvable, so there is a unique Nash equilibrium.

Note. More information can actually make a player worse off in strategic games.

7.1 Weakly Sequential Equilibria

There are games of imperfect information in which SPNE do not guarantee sequential rationality at non-singleton information sets. WSE is a solution concept designed to impose this restriction.

An **assessment** (σ, μ) in a game of imperfect information is a profile of strategies σ and beliefs μ such that for each player i, it specifies

- i's behavioral strategy $\sigma_i(a|I)$ for each of her information sets I, and
- i's beliefs at every such information set.

If I comprises of decision nodes $h_1, \dots, h_k$ then i believes h_j occurs with probability $\mu(h_j)$ and $\sum \mu(h_i) = 1$.

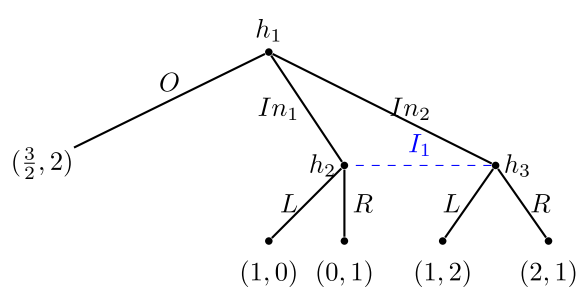
An assessment (σ, μ) constitutes a **WSE** if for all i

- (Sequential rationality). i's behavioral strategy at all I is optimal given i's beliefs at I and subsequent play according to σ.
- (Weak consistency of beliefs). i's beliefs are obtained via Bayes' rule whenever possible.

In particular, to verify sequential equilibrium, we must compute $U_i(a, \sigma, \mu|I)$. Then ensure indifference between any actions a_1, a_2 assigned positive probability and ensure that there is no profitable deviation to an unused pure action a_3 at I. Furthermore, beliefs along the path of play, must be computed using Bayes' rule, whilst beliefs off the path of play can be arbitrary as long as they're consistent with equilibrium.

Note that all WSE are Nash; however, there are SPNE that aren't WSE and vice versa.

Example. In pure strategies, the only NE (and SPNE) is (O, L). There are no mixed NE, since if R is assigned positive probability, player 1 strictly prefers In_2. So $p(In_1) = 0$. If In_2 has positive probability, player 2 strictly prefers L. In which case player 1, would be better off playing O.



Now we consider WSE (σ, μ) in pure and mixed strategies. Let $p = \mu(h_2)$. Setting $U_2(L, \sigma, \mu|I_1) \geq U_2(R, \sigma, \mu|I_1)$, player 2 strictly prefers L iff $p \leq \frac{1}{2}$.

If $\sigma_2(L|I_1)$, then $\sigma_1(O|h_1) = 1$. This is a pure strategy WSE $((O, L), p)$. If $p \geq \frac{1}{2}$, player 2 prefers R. But then player 1 strictly prefers In_2, so $p = 0$. Hence no WSE exists with $\sigma_2(R|I_1) = 1$.

To find the mixed WSE, set $s = \sigma_2(L|I_1)$ and $t_i = \sigma_1(In_i|h_1), i = 1, 2$. We must have $p = \frac{1}{2}$, since player 2 is indifferent between L and R. Thus $t_1 = t_2$.

Player 1 is not indifferent between the two choices of In unless $s = 1$. Hence $t_1 = 0$. If player 1 choses O over In_2, then $\frac{3}{2} \geq s + 2(1 - s)$ or $s \geq \frac{1}{2}$. Thus

- $\sigma_1(O|h_1) = 1$.
- $\sigma_2(L|I_1) \geq \frac{1}{2}$ and $p = \frac{1}{2}$.

A **signaling game** is a dynamic game of incomplete information with

1) an informed player, the *sender*, who observes all information chosen by nature and take action; and
2) uninformed players, the *receivers*, who don't observe the information, but observe the sender's action and update their beliefs about the sender's type accordingly.

There are three types of equilibria in signaling games: **Pooling equilibria** where all types of the sender choose the same action, **Separating equilibria** where different types of the sender choose different actions, and **Semi-separating equilibria** where mixed strategies are used by the sender.