EUCLIDEAN GEOMETRY

SUMMER 2017

Contents

1 Euclid's Postulates

Postulate 1. Every pair of distinct points can be joined by a straight line.

More precisely, every such pair of points can by connected by one and only one straight line.

Postulate 2. It is possible to produce a finite straight line from a straight line.

This postulate implies that the plane and space extend infinitely in all directions.

Postulate 3. Given a point A and a line segment AB, there exists a circle centered at A with radius AB.

We denote such a circle as $(A; AB)$ or $(A; r)$ where $r = AB$.

Postulate 4. All right angles are equal.

Postulate 5. If a straight line falling on two straight lines make interior angles on the same side less than two right angles [in sum], then the two straight lines, if produced indefinitely, meet on that side on which there are angles less than two right angles.

This postulate implies the existence of parallel lines, that is, straight lines which never intersect, and whose adjacent interior angles sum to two right angles.

1.1 Supplementary Axioms

Postulate 6. The straight line, the triangle, and the circle separate the plane into two portions.

In the case of a line these are called sides, in the circle/triangle case, they're referred to as interior and exterior.

Postulate 7. Given $\triangle ABC$ and $\triangle A'B'C'$, it is possible to superimpose ABC to $A'B'C'$ such that A falls on A', side AB falls on $A'B'$ and C falls on the same side of $A'B'$ as C'.

2 Propositions

Proposition 2.1: Equilateral Triangle

Given a line segment AB, $\triangle ABC$ can be constructed such that $AB = BC = AC$.

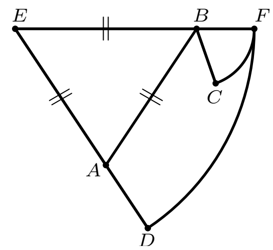
CONSTRUCTION: Draw circles $(A; AB)$ and $(B; AB)$ [PT 3]. Let C be one of their intersections [PT 6]. Then $\triangle ABC$ has the desired properties.

PROOF: By definition, we have $AB = AC$ and $BA = BC$. Thus, $AC = BC$. ■

Proposition 2.2

Given a point A and a line segment BC, it is possible to construct line segment AD such that $AD = BC$.

CONSTRUCTION: Let $\triangle ABE$ be equilateral [PN 2.1], let $F = EB \cap (B; BC)$, and let $D = EA \cap (E; EF)$ [PT 3]. Then, $AD = BC$.



PROOF: $ED = EF$ [radii of the same circle] and $EA = EB$ [sides of equilateral triangle]. Thus, $AD = EF$. However $BC = BF$, hence $AD = BC$. ■

Proposition 2.3

Given line segments $AB > CD$, construct a point E on AB such that $AE = CD$.

CONSTRUCTION: Let F be a point such that $AF = CD$ [PN 2.2]. Then, $E = AB \cap (A; AF)$.

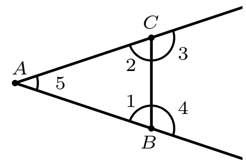
Proposition 2.4: SAS Congruence

If triangles ABC and DEF have $AB = DE$, $AC = DF$, $\angle BAC = \angle EDF$, then $BC = EF$, $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$, and $[ABC] = [DEF]$.

PROOF: Apply ABC to DEF such that A falls on D, AB falls alongside DE and C falls on the same side of DE as F [PT 7]. Then, B falls on E [given], AC falls on DF [given], C falls on F [given], BC falls on EF [PT 1], hence $BC = EF$, $[ABC] = [DEF]$, and corresponding angles are equal. We denote $\triangle ABC \cong \triangle DEF$. ■

Proposition 2.5

Given isosceles triangle ABC, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.



PROOF: By SAS, $\triangle ABC \cong \triangle ACB$, hence $\angle 1 = \angle 2$. Since $\angle 1 + \angle 4 = \angle 2 + \angle 3$ we have $\angle 3 = \angle 4$.

Proposition 2.6

The converse of PN 2.5: Given $\triangle ABC$ with $\angle 1 = \angle 2$, $AC = AB$.

PROOF: Suppose AB and AC are unequal and WLOG $AB > AC$. Let D be on the interior of AB such that $BD = AC$. Then, $\triangle BCD \cong \triangle CBA$ by SAS. Thus, $[CBA] = [BCD]$, a contradiction since $\triangle DCB$ is only part of $\triangle ACB$. ∎

Proposition 2.7

Given a line segment AB with points C, D on the same side of AB such that $CA = DA$ and $CB = DB$, the points C and D coincide.

PROOF: Suppose C and D are distinct and draw CD. Since $\triangle ACD$ is isosceles it follows that $\angle ACD = \angle ADC$. Thus, $\angle BCD < \angle ACD = \angle ADC < \angle BDC$. However, since $\triangle BDC$ is isosceles $\angle BCD = \angle BDC$, a contradiction. ∎

Proposition 2.8: SSS Congruence

Given $\triangle ABC$, $\triangle DEF$ with $AB = DE$, $BC = EF$, $AC = DF$, $\triangle ABC \cong \triangle DEF$.

PROOF: Apply $\triangle ABC$ to $\triangle DEF$ such that B, C fall on E, F, respectively and so that vertex A falls in position G on side EF that contains point D [PT 7]. Then, $GE = AB = DE$ and $GF = AC = DF$. It follows that G and D coincide [PN 2.7]. Consequently, $\angle ABC = \angle GEF = \angle DEF$ and so $\triangle ABC \cong \triangle DEF$ [SAS]. ∎

Proposition 2.9: Angle Bisector

Given $\angle BAC$ construct a line AF such that $\angle BAF = \angle FAC$.

CONSTRUCTION: Let D be any point on AB and let E be a point on

AC such that $AE = AD$ [PN 2.3]. Let DEF be an equilateral triangle [PN 2.1], then $\angle BAF = \angle FAC$.

PROOF: $\triangle DAF \cong \triangle EAF$ [SSS]. ∎

Proposition 2.10: Median

Given line segment AB construct a point D on AB such that $AD = DB$.

CONSTRUCTION: Construct an equilateral $\triangle ABC$ [PN 2.1] and let D be the intersection of the bisector of $\angle ACB$ with AB [PN 2.9], then $AD = DB$.

PROOF: $\triangle DAC \cong \triangle DBC$ [SAS]. ∎

The point D is the median of AB and the segment CD is a median, denoted m_c or m_{AB}, of $\triangle ABC$.

Proposition 2.11: Perpendicullar Line

Given a point C on line AB, construct $CF \perp AB$.

CONSTRUCTION: Let D be a point on AC and E a point on BC such that $CD = CE$ [PN 2.3]. Construct equilateral $\triangle DEF$, then $CF \perp AB$.

PROOF: $\triangle DCF \cong \triangle ECF$ [SSS], so $\angle DCF = \angle ECF$. Thus, $CF \perp AB$. ∎

Proposition 2.12

Every point on the perpendicular bisector of a line is equidistant from the segments endpoints.

PROOF: Given segment AB with midpoint M, let C be an arbitrary point on the perpendicular bisector. Then, $\triangle AMC \cong \triangle BMC$ [SAS], hence $AC = BC$. ∎

Proposition 2.13

Every point that is equidistant from the endpoints of a line segment is on its perpendicular bisector.

PROOF: Given segment AB, let D be a point such that $AD = BD$. Let M be the intersection of AB and the bisector of $\angle ADB$ [PN 2.9]. Then, $\triangle AMD \cong \triangle BMD$ [SAS]. Therefore, M is the median of AB and $\angle AMD = \angle BMD$, hence $DM \perp AB$. ∎

Proposition 2.14: Altitude

Given $\overleftrightarrow{AB}$ and point C not on $\overleftrightarrow{AB}$. Construct a straight line CH such that $CH \perp \overleftrightarrow{AB}$.

CONSTRUCTION: Let D be any point on the side of $\overleftrightarrow{AB}$ that does not contain C. Let the circle $(C;CD)$ [PT 3] intersect $\overleftrightarrow{AB}$ at the points E and G [PT 7]. Let H be the midpoint of EG [PN 2.10]. Then, $CH \perp \overleftrightarrow{AB}$.

PROOF: Draw CG and CE. Then $\triangle CGH \cong \triangle CEH$ [SSS]. Therefore, $\angle GHC = \angle EHC$, thus $CH \perp \overleftrightarrow{AB}$. ■

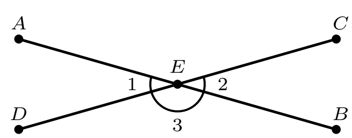
The straight line that joins a vertex of a triangle to a point on the opposite side and is perpendicular to that side is called an altitude of the triangle. The altitude of sides A, B, C are denoted h_a, h_b, h_c, respectively.

Proposition 2.15

Given straight lines AB and CD with B on CD; $\angle CBA + \angle ABD = 2$ right angles. That is, the angles are supplementary. Conversely, given $\angle ABC$ and $\angle ABD$ are supplementary with C and D on opposite sides of AB, BC and BD form a single straight line. ■

Proposition 2.16: Vertical Angles

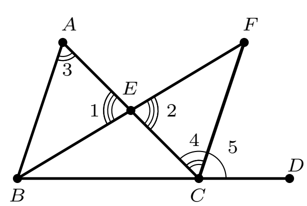
Given lines AB and CD intersecting at E, $\angle AED = \angle BEC$.



PROOF: $\angle 1 + \angle 3 = 2$ right angles $= \angle 2 + \angle 3$. Hence, $\angle 2 = \angle 3$. ■

Proposition 2.17

Given $\triangle ABC$ with BC extended to D, $\angle 5 > \angle 3$.



PROOF: Let E be the midpoint of AC [PN 2.10] and extend BE to F so that $BE = EF$ [PT 2, PN 2.3]. Then, $\triangle AEB \cong \triangle CEF$. So $\angle 3 = \angle 4$, $\angle 4 < \angle 5$, we have $\angle 3 < \angle 5$. ■

Proposition 2.18

Given $\triangle ABC$ the sum of two interior angles is less than 2 right angles.

PROOF: Extend BC to D [PT 2]. Then, $\angle ACB + \angle ACD = 2$ right angles [PN 2.15] and $\angle ACD > \angle ABC$ [PN 2.17]. ■

Proposition 2.19

Given $\triangle ABC$ with $AC > AB$, $\angle ABC > \angle ACB$.

PROOF: Let D be on the interior of AC such that $AD = AB$ [PN 2.3], and draw BD. Then, $\angle ABD < \angle ABC$, $\angle ABD = \angle ADB$ [PN 2.5], and $\angle ACB < \angle ADB$ [PN 2.17]. ■

Proposition 2.20

Given $\triangle ABC$ with $\angle ABC > \angle ACB$, $AC > AB$.

PROOF: Suppose not, then either $AC = AB$ or $AC < AB$. If $AC = AB$ then $\angle ABC = \angle ACB$ [PN 2.5], or if $AC < AB$ then $\angle ABC > \angle ACB$ [PN 2.19], a contradiction. ■

Proposition 2.21

Given straight line $\overleftrightarrow{AB}$, point C not on AB, points $P \neq Q$ on $\overleftrightarrow{AB}$, and $CP \perp \overleftrightarrow{AB}$, $CP < CQ$.

PROOF: $\angle CPQ + \angle CQP < 2$ right angles and $\angle CPQ$ is a right angle so $\angle CQP$ is less than a right angle. Thus, $CP < CQ$ [PN 2.20]. ■

Proposition 2.22

Given $\triangle ABC$, $AB + AC > BC$.

PROOF: Extend BA to D such that $AD = AC$ [PT 2, PN 2.3] and draw CD. Then $\angle BCD > \angle ACD$, and $\angle ACD = \angle ADC$ [$\triangle ABC$ is isosceles]. Therefore, $\angle ADC < \angle BCD$, $BC < BD$ [PN 2.23], and $AB + AC = BA + AD = BD > BC$. ■

Proposition 2.23

Proposition 2.24

Proposition 2.25: ASA Congruence

Given $\triangle ABC$, $\triangle DEF$, with $BC = EF$, $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, $\triangle ABC \cong \triangle DEF$.

PROOF: Suppose $AB \neq DE$, then WLOG $AB < DE$. Let G be a point in the extension of AB such that $GB = DE$. Then, $\triangle GBC \cong \triangle DEF$ [SAS]. Thus, $\angle GCB = \angle DFE$, which implies $\angle GCB = \angle ACB$, a contradiction. Thus, $AB = DE$ and $\triangle ABC \cong \triangle DEF$ [SAS]. ■

Proposition 2.26: AAS Congruence

Given $\triangle ABC$, $\triangle DEF$, with $\angle BAC \cong \angle EDF$, $\angle BCA \cong \angle EFD$, and $AB = DE$, $\triangle ABC \cong \triangle DEF$.

PROOF: Suppose WLOG $AC < DF$, let H be on the interior of AC such that $AH = DF$ and join BH. Then, $\triangle BHA \cong \triangle EFD$ and $\angle BHA = \angle EFD$. However, $\angle EFD = \angle BCA$ a contradiction of PN 2.17, since $\angle BHA$ is an exterior angle of $\triangle BHC$. Thus, $BC = EF$ and $\triangle ABC \cong \triangle DEF$ [ASA]. ■