Multivariable Calculus

1.0.1 Vectors & 3-D Coordinate Systems

- Cartesian coordinate system: 3 perpendicular lines, i.e. the x, y, z-axes, intersect at the origin and form 8 octants and three coordinate planes: xy, xz, yz. First octant is $\{P(a,b,c)|a,b,c > 0\}$. Yields a 1-1 correspondence between $\mathbb{R}^3$ and 3-space.
- Surfaces: Generally determined by a single equation relating x, y, z.
  - For example, $\{P(x,y,z)|x^2 + y^2 = 1\} \to$ cylinder centered at z-axis with $r = 1$.
  - However, $\{P(x,y,z)|x^2 + y^2 = -1\} = \emptyset$ isn't a surface.
- Distance $= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$
- Vector: A directed arrow with a initial and terminal point. Two vectors are equivalent iff they have the same magnitude and direction.

1.0.2 Dot & Cross Products

Given $\vec{a}, \vec{b} \in \mathbb{R}^k$, then $\vec{a} \cdot \vec{b} = \sum_{n=1}^{k} a_n b_n$. Also known as inner or scalar product.

Properties of the dot product:

1. $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}| \cos\theta$
2. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
3. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
4. If $\vec{a}$ is orthogonal to $\vec{b}$, then $\vec{a} \cdot \vec{b} = 0$.

Scalar Projection: $\text{comp}_{\vec{x}}\vec{y} = |\vec{y}| \cos\theta$.

Vector Projection: $\text{proj}_{\vec{x}}\vec{y} = \left(\frac{\vec{x} \cdot \vec{y}}{|\vec{x}|^2}\right)\vec{x}$

Orthogonal Projection: $\text{orth}_{\vec{a}}\vec{b} = \vec{b} - \text{proj}_{\vec{a}}\vec{b}$

The cross product results in a vector orthogonal to both product vectors, whose direction is given by the right hand rule.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\hat{\imath} - (a_1 b_3 - a_3 b_1)\hat{\jmath} + (a_1 b_2 - a_2 b_1)\hat{k}$$

The area of a parallelogram determined by two vectors is equal to $|\vec{a} \times \vec{b}|$. Likewise, the volume of a parallelopiped determined by three vectors is given by the scalar triple product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of the cross product:

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
5. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
6. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

1.0.3 Lines & Planes

Any point on a line in 3D can be written as $\vec{r} = \vec{r_0} + t\vec{v}$, where $t \in \mathbb{R}$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ (arbitrary point), and $\vec{v} = \langle a, b, c \rangle$ (direction).

This yields the parametric equations: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$. Solving these equations for t and equating yields the symmetric equation for a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

A line segment can be generated by limiting the domain of t in the parameterized equations.

Two lines are **skew** if they are neither parallel nor intersecting.

A plane in 3-D is determined by 2 lines/vectors, 3 points, or 1 point and a vector orthogonal to the plane (normal vector).

Given the normal vector, $\vec{n} = \langle a, b, c \rangle$, and an arbitrary point $r_0 = (x_0, y_0, z_0)$, any arbitrary point, $r = (x, y, z)$, on the plane will form a vector, $\overrightarrow{r_0 r}$, orthogonal to $\vec{n}$. Hence, $\vec{n} \cdot \overrightarrow{r_0 r} = 0$, which yields an equation for the plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Given two planes with normal vectors, $\overrightarrow{n_1}, \overrightarrow{n_2}$, the angle between the planes is given by: $\theta = \arccos\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$.

Distance from a point (x, y, z) to a plane $(ax + by + cz + d = 0)$: $\dfrac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Distance from a point to a line: $\dfrac{|\vec{b} \times \vec{v}|}{|\vec{v}|}$, $\vec{v}$ is line's direction vector, and $\vec{b}$ is from point to the line.

1.0.4 Vector-Valued Functions & Space Curves

Limits: $\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$. Limit exists only if limits of component functions all exist.

A function $\vec{r}$ is continuous at $t = a$, if $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$. The derivative, $\vec{r}'(t)$ is the vector comprised of the derivatives of the component functions of $\vec{r}(t)$, assuming the component functions are differentiable. Likewise, definite and indefinite integrals apply componentwise.

Properties of the Derivative of Vector-valued Functions:

- $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
- $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
- $\frac{d}{dt}\vec{u}(f(t)) = f'(t)\vec{u}'(f(t))$
- If $|\vec{u}'(t)| = c$, where $c \in \mathbb{R}$, then $\vec{u}(t)$ is $\perp$ to $\vec{u}'(t) \forall t$.

Arc Length, $L = \int_a^b |\vec{r}'(t)| dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2} dt$.
We define the arc length function s, for function $\vec{r}(t)$, $a \le t \le b$,

$$s(t) = \int_a^t |\vec{r}'(u)| du \tag{1}$$

Now we can reparameterize the curve with respect to arc length, $t(s)$.

A curve is **smooth** if on an interval if its derivative is continuous and non-zero. Define the curvature of a curve to be the measure of how quickly a curve is changing direction at a point, or formally, the magnitude of the rate of change in the unit tangent vector, $\vec{T}$, with respect to arc length, s,

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \tag{2}$$

Define the **principal unit normal vector**: $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$, which indicates the direction in which the curve will turn. Define the **binormal vector**: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$. Note: $\vec{T}, \vec{N}, \vec{B}$ are all orthogonal to each other. The plane formed by $\vec{T}, \vec{N}$ is called the **osculating plane**, and the plane formed by $\vec{B}, \vec{N}$ is called the **normal plane**.

1.0.5 Motion

Position: $\vec{r}(t) = \langle v_0 \cos \alpha t, v_0 \sin \alpha t - gt^2 \rangle$, given initial velocity v_0, shot at angle α.

Acceleration: $\vec{a}(t) = \vec{v}'(t) = \dfrac{d}{dt}[v(t)\vec{T}(t)] = v(t)\vec{T}'(t) + \kappa(t)v(t)^2\vec{N}(t)$

1.1 Multivariable Derivatives

1.1.1 Partial Derivatives

Given a function $f(x_1, x_2, \ldots, x_n)$, the *partial derivative*, denoted f_{x_i} or $\frac{\partial f}{\partial x_i}$, for $1 \leq i \leq n$, is computed by regarding all other variables x_j, with $j \neq i$, constant, and differentiating wrt x_i.

$$f_{x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h} \tag{3}$$

Suppose f is defined on disc D that contains (a, b). If f_{xy}, f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$. This symmetry extends to higher dimensions.

1.1.2 Tangent Planes & Linear Approximations

If f has continuous partial derivatives, then the tangent plane to $z = f(x, y)$ at (a, b) is given by:

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{4}$$

The *linearization* of $z = f(x, y)$ near (a, b) is given by:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \tag{5}$$

Let $z = f(x, y)$ and consider dx and dy to be independent. Then, the *differential*, $dz = f_x(x, y)dx + f_y(x, y)dy$.

If $z = f(x, y)$, then f *differentiable* at (a, b) if Δz can be expressed as $\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$, where $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Or more simply, if the partial derivatives exist near and are continuous at (a, b).

Note all of the above have analogous definitions in higher dimensions.

1.1.3 Chain Rule

Suppose $z = f(x, y)$ is differentiable wrt x and y, where $x = g(t)$ and $y = h(t)$ are also differentiable. Then,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{6}$$

In general, for a function u of n variables, $x_1, x_2, \ldots, x_n$, and each x_j is a differentiable function of m variables, $t_1, t_2, \ldots, t_m$. Then,

$$\frac{\partial u}{\partial t_i} = \sum_{j=1}^{m} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i} \tag{7}$$

Theorem 1.1 (Implicit Function Theorem)**.** *Given $F(x, y)$, Assuming $F(x, y) = 0$ defines y implicitly as a function of x. By, the chain rule, we can derive*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \tag{8}$$

assuming F is differentiable with continuous partial derivatives.

Likewise, an analogous theorem holds for high dimensional functions.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \tag{9}$$

1.2 Directional Derivatives & Gradient

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of unit vector $\vec{u} = \langle a, b, c\rangle$ is

$$f_{\vec{u}}(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h} \tag{10}$$

The **gradient vector**, denoted ∇F, of a function $F(x_1, x_2, \ldots, x_n)$ is the vector containing its partial derivatives.

In general, the directional derivative can be computed by

$$f_{\vec{u}} = \nabla f \cdot \vec{u} \tag{11}$$

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative is $|\nabla f(a, b, \ldots)|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(a, b, \ldots)$.

The gradient vector at a point P is perpendicular to the tangent vector to any curve on a surface that passes through P. Therefore, we can easily define the equations for the tangent plane and normal lines to a level surface. For instance, the tangent plane to a level surface is: $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$.

1.3 Minima & Maxima

Second Derivative Test: Suppose the 2nd order partial derivatives of f are continuous on a disc with center (a, b), and that $f_x(a, b) = f_y(a, b) = 0$, i.e. (a, b) is a *critical point.*

$$\text{Let } H(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2,$$

- If $H > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- If $H > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- If $H < 0$, then $f(a, b)$ is a *saddle point.*
- If $H = 0$, then the test is inconclusive.

Note: H is determined by the determinant of the Hessian matrix, i.e.

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \tag{12}$$

Theorem 1.2 (Extreme Value Theorem)**.** *If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute maximum and absolute minimum at some points in D.*

To determine the absolute minima and maxima, consider both the critical points in D, and the extreme values along the boundary of D.

1.4 Lagrangian Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$,

- Find all values of x, y, z, λ, such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = k$$

- Then, evaluate f at the points from above.

2 Multiple Integrals

The *double integral* of f over the rectangle $R = [a, b] \times [c, d]$, where $\Delta x = \frac{b-a}{m}$ and $\Delta y = \frac{d-c}{n}$ is

$$\iint_R f(x, y)\, dx\, dy = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=i}^{n} f(x_i, y_j) \Delta x \Delta y \tag{13}$$

Fubini's Theorem: If f is continuous of the rectangle R, then

$$\iint_R f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx \tag{14}$$

The general procedure for determining bounds over a general domain is:

1. Choose the order of integration based on whichever is easier given the domain. If the domain is given in the form $y = f(x)$, then it's generally better to integrate with respect to y first.
2. To determine the bounds on the outer integral, determine the range of values of the second variable over the entire domain.
3. For the inner integral, determine for a fixed value of the outer what the inner variable ranges over.

2.1 Polar Transformations

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r \cos\theta, r \sin\theta) r\, dr d\theta$$

2.2 Triple Integrals

The definition of the triple integral is analogous to that of the double integral. Furthermore, an analogous form of Fubini's theorem holds.

2.3 Coordinate Transformations

Cylindrical
$C : (x, y, z) \to (r, \theta, z)$

$x = r\cos\theta$	$r > 0$
$y = r\sin\theta$	$0 \le \theta \le 2\pi$
$z = z$	$-\infty < z < \infty$

Spherical
$S : (x, y, z) \to (\rho, \theta, \phi)$

$x = \rho\sin\phi\cos\theta$	$r > 0$
$y = \rho\sin\phi\sin\theta$	$0 \le \theta \le 2\pi$
$z = \rho\cos\phi$	$0 \le \phi \le \pi$

General Change of Variables:
Suppose T is a linear transformation that maps from S in the uv-plane to R is the xy-plane. If T is one-to-one, except perhaps on the boundary of S, then

$$\iint_R f(x,y)\ dA = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\ dv$$

where the *Jacobian* of the transformation T is given by

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right\|.$$

3 Vector Calculus

A vector field $\mathbf{F}$ is a *conservative* vector field if it is the gradient of some scalar function, this is, if there exists a function f such that $\mathbf{F} = \nabla f$. Furthermore, f is the *potential* function for $\mathbf{F}$.

3.1 Line Integral

Throughout this section, let C be a curve parameterized by

$$x = x(t) \qquad y = y(t) \qquad a \leq t \leq b.$$

If f is a function defined along a smooth curve C then the line integral of f along C is

$$\int_C f(x,y)\ ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Note that one can also define line integrals with respect to x and y, instead of respect to arc length. An analogous definition for the line integral holds in 3-dimensions.

Theorem 3.1. *If $\mathbf{F}$ is a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$. Then the line integral of $\mathbf{F}$ along C is*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_C \mathbf{F}\cdot\mathbf{T}ds.$$

3.1.1 Fundamental Theorem for Line Integrals

Provides a method to evaluate a line integral along a conservative vector field.

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

A curve is considered *closed* if its terminal point coincides with its initial point.

Theorem 3.2. *We say $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in a domain D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.*

A domain D is considered *open* if it contains none of its boundary points. In addition, we say that D is *connected* if any two points in D can be joined by a path that lies within D.

Theorem 3.3. *Therefore, for a vector field $\mathbf{F}$ which is continuous on an open connected region D, if $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.*

Theorem 3.4. *If $\mathbf{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ is a conservative vector field, where P and Q have first order partial derivatives on a domain D, then throughout D we have*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

We say a curve is *simple* if it does not intersect itself anywhere between its endpoints. A *simply-connected region* in the plane is a connected region D such that every simple closed curve in D encloses only point that are in D.

Theorem 3.5. *Let $\mathbf{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Then $\mathbf{F}$ is conservative.

3.2 Green's Theorem

Positive orientation of a simple closed curve C refers to a counterclockwise traversal of C. Note that reversing orientation, negates the value of the line integral calculated along the curve.

Theorem 3.6 (Green's Theorem)**.** *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then*

$$\int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Note the notation $\oint_C P\ dx + Q\ dy$ is sometimes used to denote the line integral calculated with positive orientation around a curve C.

3.3 Curl & Divergence

Define the vector differential operator $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$.

The *curl* of some vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given by

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}.$$

Theorem 3.7. *If f is a function of three variables that has continuous second order partial derivatives, then* $\text{curl}(\nabla f) = \mathbf{0}$.

Theorem 3.8. *If $\mathbf{F}$ is a vector field defined on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and* $\text{curl}(\mathbf{F}) = \mathbf{0}$*, then $\mathbf{F}$ is a conservative vector field.*

The *divergence* of a vector field is given by

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}.$$

Theorem 3.9. *If $\mathbf{F}$ is a vector field defined on $\mathbb{R}^3$ whose component functions have continuous second order partial derivatives, then*

$$\text{div curl } \mathbf{F} = 0.$$

3.4 Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

3.5 Surface Area & Parametric Surfaces

If a smooth parametric surface S is given by the equation

$$\bar{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k} \qquad (u,v) \in D$$

and S is covered once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$\iint_D |\bar{r}_u \times \bar{r}_v| \; dA.$$

In the case where S is a function, then

$$\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2} \; dA.$$

3.6 Surface Integrals

The surface integral of a function f over the surface S, on a domain D is

$$\iint_S f(x,y,z) \; dS = \iint_D f(\mathbf{r}(t))|\mathbf{r}_u \times \mathbf{r}_v| \; dA.$$

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface (flux) integral is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \; dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \; dA.$$

In the special case where S is a function

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial f}{\partial x} - Q \frac{\partial f}{\partial y} + R \right) dA.$$

3.7 Divergence Theorem

Let E be a solid region and let S be the boundary surface of E, given positive orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \; dV.$$

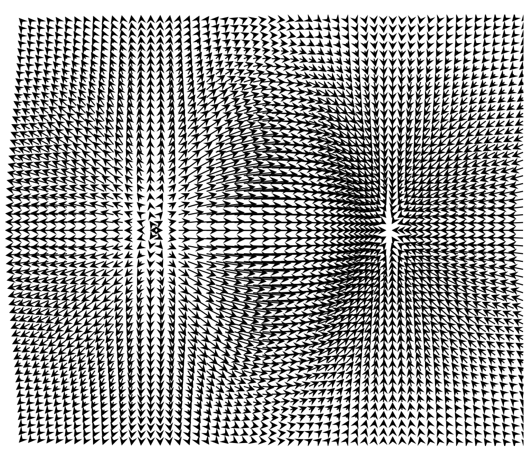

Figure 1: $\mathbf{F}(x,y) = \langle (1-2x^2)e^{-x^2-y^2}, -2xye^{-x^2-y^2} \rangle$.

"...I look at politicians as, they are doing what inherently they need to do to retain power. Their job is to consolidate power. When you go to the zoo and you see a monkey throwing poop, you go, 'That's what monkeys do, what are you gonna do?' But what I wish the media would do more frequently is say, 'Bad monkey.'"

– John Stewart