Graph Theory

Spring 2019

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1 Graph Terminology

An **graph** G is an ordered triple $G = (V, E, f)$ where V (non-empty) and E are sets. We call the elements of V the **vertices** of G and the elements of E the **edges** of G. We say G is **undirected** if $f : E \to V_2$ is a relation that maps edges to $V_2 = \{S \subset V : |S| = 1 \text{ or } 2\}$. A **directed graph** is a graph where f is a relation from E to $V \times V$.

For simplicity we often denote the edge $\{v_i, v_j\}$ (or (v_i, v_j) in the case of a directed graph) by v_iv_j. A **loop** is an edge of the form v_iv_i. We say a graph is a **multigraph** if there exist two vertices, which are connected by more than one edge. A graph that is not a multigraph and has no loops is called **simple**.

A graph $H = (V(H), E(H))$ is a **subgraph** of a graph $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

In a graph G, we called $|V|$ the **order** of G and $|E|$ the **size** of G. We say two vertices $v_1, v_2 \in V(G)$ are **adjacent** if there exists an edge $e = v_1v_2 \in E$. In this case, we say v_1 and v_2 are **neighbors**. We say two edges $e_1, e_2 \in E(G)$ are **adjacent** if there exists a vertex $u \in V(G)$ such that $e_1 = vu$ and $e_2 = uw$ for some vertices v and w. Given an edge $e = v_1v_2$ we say e is **incident** with the vertices v_1 and v_2 (or in the case of a directed graph e would only be incident to v_2).

In a graph $G = (V, E)$, a **walk of length** k or simply k**-walk** is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$, which begins and ends with a vertex. A **trail** is a walk in which all edges are distinct. A **path** is a trail in which all vertices (except possibly the first and last) are distinct.

We say a graph is **connected** if there exists a path between any two distinct vertices. A graph that is not connected is **disconnected**.

A walk is **closed** if it begins and ends at the same vertex. Otherwise, we say the walk is **open**. A **circuit** is a closed trail with at least one edge. A **cycle** is a closed path.

A **clique** in a graph is a set of pairwise adjacent vertices. An **independent set** in a graph is a set of pairwise non-adjacent vertices.

1.1 Families of Graphs

A **cycle graph** is a graph consisting of a single cycle. The cycle graph of order n is denoted C_n. The **path graph** is denoted P_n. A simple graph is a **complete graph** if every vertex is adjacent to every other vertex. The complete graph of order n is denoted K_n. An **empty graph** is a graph with $|E| = 0$.

A graph is **bipartite** if its vertex set can be partitioned into two independent sets called the **partite sets**. A bipartite graph with bipartite sets V and W is a **complete bipartite graph** if v is adjacent to w for all $v \in V$ and $w \in W$. It is denoted by $K_{m,n}$ where $|V| = m$ and $|W| = n$.

An **(open) neighborhood** of a vertex $v \in V(G)$ is $N_G(v) = \{u : uv \in E(G)\}$. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The **degree** of a vertex v is $\deg_G(v) = |N_G(v)|$.

A graph G is r**-regular** if $\deg_G(v) = r$ for all $v \in V(G)$. For example, C_n is 2-regular and K_n is $(n-1)$-regular. Graphs which are 3-regular are called cubic. Define $\delta(G) = \min_{v\in V(G)}\{\deg_G(v)\}$ and $\Delta(G) = \max_{v\in V(G)}\{\deg_G(v)\}$.

The **complement** of a simple graph G of order n is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_n)\backslash E(G)$. A graph G is **self-complementary** if $G \cong \overline{G}$.

Proposition 1.1. A self-complementary graph of order n must have $\frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$ edges. Thus, $n \equiv 0$ or $1 \pmod 4$. Moreover, this is a necessary and sufficient condition for the existence of a self-complementary graph.

Proof. Given a self-complementary graph G of order n we can construct a self-complementary graph of order $n+4$ as follows: add $P_4 = [v_1, v_2, v_3, v_4]$ and join v_2 and v_3 to every vertex in G. Since there exist self-complementary graphs of order 4 and 5, we're done. ∎

1.2 Matrices in Graph Theory

If G is a graph of order n, then the **adjacency matrix** $A(G)$ is an $n \times n$ matrix where a_{ij} is the number of edges from vertex i to vertex j. In the case of an undirected graph, the adjacency matrix is symmetric.

The **unoriented incidence matrix** (or simply incidence matrix) of an undirected simple graph G is a $n \times m$ matrix $M(G)$, where $n = |V|$ and $m = |E|$, such that $m_{ij} = 1$ if vertex v_i and edge e_j are incident and 0 otherwise.

Given an adjacency matrix A, product $a_{ik}a_{kj}$ is 1 if and only if there is path from i to j through k. Thus the total number $N_{ij}^{(2)}$ of 2-walks from i to j is

$$N_{ij}^{(2)} = \sum_{k=1}^{n} a_{ik}a_{kj} = [A^2]_{ij}.$$

In general we have

$$N_{ij}^{(r)} = [A^r]_{ij}.$$

2 Miscellanea

Proposition 2.1. For any graph $G = (V(G), E(G))$, we have

$$\sum_{v \in V(G)} \deg(v) = 2|E|.$$

Proof. For any $e \in E(G)$, when we sum the degrees of the vertices, edge e get counted twice (once at each vertex with which it is adjacent). ■

Corollary. In any graph, there are an even number of odd degree vertices.

Two simple graphs G and H are **isomorphic** if there exists a bijection $\varphi : V(G) \to V(H)$ such that $\varphi(v)$ is adjacent to $\varphi(u)$ in H if and only if v is adjacent to u in G. We write $G \cong H$.

Necessary conditions for graph isomorphism:

- Must have same order and size.
- Same number of vertices of every degree.
- Connectedness
- Number of connected components
- Number of loops

See Pólya's enumeration theorem for a method of counting the number of graphs of order n up to isomorphism.

Proposition 2.2. Let G be a finite graph in which $\deg(v) \geq 2$ for all $v \in V(G)$. Then G contains a cycle.

Proof. If G is no simple then it must contain a cycle. Suppose G is simple and let $P = v_1, \ldots, v_k$ be the longest path in G. Then there exists a vertex $v \in V(G)$ such that v_k is adjacent to v. If $v \notin P$, then $P' = v_1, \ldots, v_k, v$ is a longer path in G. Thus $v \in P$. Say $v = v_i$, then $v_i, \ldots, v_k$ is a cycle in G. ■

Proposition 2.3. Let $n \geq 4$ be even. Then there exists a connected 3-regular graph of order n.

Proof. Note K_4 is 3-regular. Let $k \geq 4$ be even and suppose G is a connected, 3-regular graph of order k. Then we can construct a connected 3-regular graph of order $k+2$ as follows. Let $v \in V(G)$ with neighbors u and v. We'll two new vertices x and y by first deleting edges uv and vw, then adding edges ux, vx, xy, vy, and yw. ■

A **vertex-induced** subgraph of a graph G is a graph $G[V'] = (V', E')$ where V' is a subset $V' \subset V(G)$ and $E' = \{e = uv \in E(G) : u, v \in V'\}$. A **spanning** subgraph of a graph G is is the graph $H = (V(G), E')$ where $E' \subset E(G)$.

The **distance** between two vertices u and v in a graph G, is the length of the shortest path from u to v. We denote it $d_G(u, v)$. Note if there does not exist a path from u to v, then $d_G(u, v) = \infty$.

Proposition 2.4. If $x, y \in V(G)$ are distinct vertices of G, then every xy-walk in G contains an xy-path.

Theorem 2.5

A graph G is bipartite if and only if it contains no odd cycle.

Proposition 2.6. If G is a simple, n-vertex connected graph with $\delta(G) \geq \frac{n-1}{2}$, then G is connected. (Pf. Let $u, v \in V(G)$, non-adjacent. They must have a common neighbor.)

Proposition 2.7. Every loopless graph G has a bipartite subgraph with at least $e(G)/2$ edges.

Def$^{\mathbf{n}}$. A graph G is called H-free if no induced subgraph of G is isomorphic to H.

Proposition 2.8. The maximum number of edges in an n-vertex, triangle-free, simple graph G is $\lfloor \frac{n^2}{4} \rfloor$. (Pf. Let $x \in V(G)$ with $\deg(x) = k = \Delta(G)$. Then $N(x)$ is indep. as G is triangle-free. So summing over the vertices not in $N(x)$ counts every edge at least once, $e(G) \leq k(n-k)$. Note $k(n-k) \leq \lfloor \frac{n^2}{4} \rfloor$.)

3 Shortest Path

An **edge-weighted graph** is a graph G together with a weight function $\alpha : E(G) \to \mathbb{N}$. Given a path P in G, we denote the distance of the path by $\alpha(P) = \sum_{e \in P} \alpha(e)$. The shortest path problem is, given a connected graph G with weight function α, find for given $u, v \in V(G)$,

$$d_G^{\alpha}(u, v) = \min\{\alpha(P) | P \text{ is a path from } u \to v\}.$$

3.1 Dijkstra's Algorithm

Suppose we are given start vertex v_s and end vertex v_f. Let d be an array of size $|V(G)|$.

1) Set $U \leftarrow V(G) \backslash \{v_s\}$, $c \leftarrow v_s$, $v_m \leftarrow v_f$ and $m \leftarrow \infty$.
2) Set $d[v] = \infty$ for all $v \in V(G)$ and set $d[v_s] = 0$.
3) For each $v \in N_G(c) \cap U$, do
   - $d[v] = \min\{d[v], d[c] + \alpha(cv)\}$
   - if $d[v] \leq d[v_m]$, then $m \leftarrow d[v]$ and $v_m \leftarrow v$
4) If $v_s \notin U$ or $m = \infty$, then halt.
5) Otherwise, set $c \leftarrow v_m$ and $U \leftarrow U \backslash \{v_m\}$, then goto step 3.

Then, the length of the shortest path from v_s to v_f is precisely $d[v_f]$.

4 Eulerian Trails & Tours

An **Euler trail** is a trail that visits every edge once. An **Euler tour** is a closed Euler trail. If a graph has an Euler tour, it if called **Eulerian**.

Theorem 4.1

A connected graph G is Eulerian if and only if every vertex of G is even.

Proof. Let G be a connected Eulerian graph and let W from u to u be an Euler tour. Then, for any $v \neq u$, if v occurs k times in W, then $\deg(v) = 2k$. Moreover, since W is closed $\deg(u)$ must be even.

Suppose G is a non-trivial connected graph whose vertices are all even. Let T be the longest trail in G. Say $T = v_0, e_1, v_1, \ldots, e_l, v_l$. Then $v_l = v_0$ since all edges incident of v_l are used in the walk and v_l has even degree. Thus T is closed. Now suppose T is not an Euler tour. Since G is connected, there exists an edge $f = v_i u$ where $f \notin T$. However, this implies $f, e_{i+1}, \ldots, e_l, e_1, \ldots, e_i$ is a longer trail in G, a contradiction. ∎

Theorem 4.2

A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.

Proof. If G has an Euler trail from u to v, then as in the proof of the previous theorem, all $w \notin \{u, v\}$ are even. Thus, there are either 0 or 2, vertices of odd degree.

Now suppose G is a graph with at most 2 vertices of odd degree. If there are no vertices of odd degree, then G has an Euler tour. Thus G has an Euler trail. So suppose G has two vertices of odd degree, say u and v. Construct the graph H by adding vertex w and edges uw and vw to G. Then H all the vertices of H have even degree, so H has an Euler tour. Take T to be an Euler tour starting and ends at w, then the portion of the tour with w removed is an Euler trail from v to u in G. ∎

We say a graph is **even** if all of its vertices have even degree. Similarly, a graph is odd if all of its vertices have odd degree.

5 Degree Sequences

A sequence $d_1, \ldots, d_n$ of non-negative integers is called **degree sequence** of a graph G, if the vertices of G can be labeled $v_1, \ldots, v_n$ such that $\deg(v_i) = d_i$ for all $i = 1, \ldots, n$. If a sequence s is a degree sequence of some graph, then s is called **graphical**.

Theorem 5.1: Havel/Hakimi

Let $n \geq 2$. A non-increasing sequence $s = (d_1, \ldots, d_n)$ of non-negative integers with $d_1 \geq 1$ is graphical if and only if the sequence $s_1 = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphical.

Proof. Suppose s_1 is graphical. That is, there exists a graph G_1 of order $n-1$ such with degree sequence s_1. Hence we can label $V(G_1)$ with $v_1, \ldots, v_n$ such that

$$\deg(v_i) = \begin{cases} d_i - 1 & \text{for } i = 2, 3, \ldots, d_1 + 1 \\ d_i & \text{for } i = d_1 + 2, \ldots, n \end{cases}$$

We construct a new graph G as follows: start with G_1, add vertex v_1 and d_1 new edges; $v_1 v_i$ for $i = 2, \ldots, d_i + 1$. Thus, in G we have s is graphical since $\deg(v_i) = d_i$ for $i = 1, \ldots, n$. Now suppose s is graphical. So there exists graphs of order n with degree sequence s. Among all such graphs let G be the one such that $V(G) = \{v_1, \ldots, v_n\}$ with $\deg(v_i) = d_i$ for all $i = 1, \ldots, n$ and $\sum_{v \in N_G(v_1)} \deg(v)$ is maximized. I claim that v_1 is adjacent to vertices having degrees $d_2, \ldots, d_{d_1+1}$.

Suppose the claim is not true. Then there exists vertices v_r and v_s with $d_r > d_s$ such that v_1 is adjacent to v_s, but not adjacent to v_r. Since $d_r > d_s$, there exists a vertex v_t such that v_t is a neighbor of v_r but v_t is not a neighbor of v_s. Construct a new graph G' by removing edges $v_1 v_s$ and $v_r v_t$ from G, then adding $v_1 v_r$ and $v_s v_t$. Then G' has the same degree sequence as G, but $\sum_{v \in N_{G'}(v_1)} \deg(v) > \sum_{v \in N_G(v_1)} \deg(v)$, a contradiction. Thus v_1 is adjacent to vertices with degrees $d_2, \ldots, d_{d_1+1}$. Thus, the graph $G \backslash v_1$ has degree sequence $d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$. Thus s_1 is graphical. ∎

6 Directed Graphs

In a digraph, the edges are ordered pairs of vertices. In the edge $e = (v_i, v_j)$, we call v_i the **tail** of e and v_j the **head** of e. A simple digraph may have loops but no multiple edges. The i, j entry of the adjacency matrix $A(G)$ of a digraph G is the number of edges from v_i to v_j. For the incidence matrix $M(G)$ we put $m_{ij} = 1$ if v_i is the tail of e_j and $m_{ij} = -1$ if v_i is the head of e_j.

Def$^{\mathbf{n}}$. A digraph is **weakly connected** if its underlying graph is connected. It is **strongly connected** if for each order pair of vertices u, v there is a u, v-path. The strong components are the maximal strongly connected subgraphs.

The **outdegree** $d^+(v)$ is the number of edges with tail v. **Indegree** $d^-(v)$ is the number of edges with head v. **In** and **Out-neighborhoods** are defined analogously. $\Delta^+(G)$ is the maximum outdegree, $\Delta^-(G)$ is the maximum indegree. $\delta^+(G), \delta^-(G)$ are their analogous counterparts.

In a digraph $\sum_v d^+(v) = e(G) = \sum_v d^-(G)$.

Lemma. If a digraph G has $\delta^+(G) \geq 1$ (or $\delta^-(G) \geq 1$) then G contains a cycle.

Proposition 6.1. A digraph is Eulerian if and only if $\delta^+(G) = \delta^-(G)$ and the underlying graph has at most 1 non-trivial component.

Def$^{\mathbf{n}}$. An **orientation** of a graph G is a digraph obtained by orienting the edges of G. An **oriented graph** is an orientation of a simple graph. A **tournament** is an orientation of K_n.

Def$^{\mathbf{n}}$. A **king** in a digraph is a vertex from which every other vertex is reachable by a path of length at most 2.

Theorem 6.2

Every tournament has a king.

Proof. Let x be a vertex in a tournament T. If x is not a king some vertex y is not reachable from x via a path of length at most 2. So no successor of x is a predecessor of y. Since T is an orientation of a clique, every successor of x must be a successor of y and also $x \rightsquigarrow y$. Thus $d^+(y) > d^+(x)$. If y is not a king, we can repeat to find a vertex of larger degree. T is finite so this process must terminate. It can only terminate once a king is found. ∎

7 Graph Decomposition

A **decomposition** of a graph G is a family $\mathcal{F}$ of edge-disjoint subgraphs of G such that

$$\bigcup_{F \in \mathcal{F}} E(F) = E(G).$$

If every graph in $\mathcal{F}$ is a cycle, then the decompositions is called a cycle decomposition. A path decomposition is defined similarly.

Theorem 7.1

A graph G admits a cycle decomposition if and only if it is even.

Proof. Suppose there is a cycle decomposition of G. Take any $v \in V(G)$. Then, either v is isolated, in which case $\deg(v) = 0$, or v belongs to k cycles in the decomposition, in which case $\deg(v) = 2k$. Regardless, G is even.

Now suppose G is even. Note that if $|E(G)| = 0$, then G has a cycle decomposition, namely $\mathcal{F} = \varnothing$. So suppose all even graphs with less than m edges admit a cycle decomposition. Take G to be an even graph with m edges. Let $X = \{v \in V(G) : \deg(v) > 0\}$ and take $F = G[X]$. Note that F is an even graph all of whose vertices have degree at least 2. Thus, F contains a cycle, say C. Take $G' = G \backslash E(C)$. Then G' is an even graph with fewer than m edges, so G' has a cycle decomposition, say $\mathcal{C}'$. Then, $\mathcal{C}' \cup \{C\}$ is a cycle decomposition of G. ∎

8 Hamiltonian Paths

A **Hamiltonian path** is a path that covers every vertex once. A **Hamiltonian cycle** is a cycle that covers every vertex once. A graph is called **Hamiltonian** if it contains a Hamilton cycle.

We say two vertices $u, v \in V(G)$ are **similar** if there exists an automorphism α such that $\alpha(u) = v$. A **vertex transitive graph** is a graph in which all vertices are similar. Note that all vertex transitive graphs are regular.

The Lovász conjecture (1970) states that every finite connected vertex transitive graph has a Hamiltonian path. To date, there are only 5 known vertex transitive non-Hamiltonian graphs.

Lemma. If G is Hamiltonian, then $c(G \backslash S) \leq |S|$ for every non-empty subset $S \subseteq V(G)$.

9 Trees

An edge $e \in E(G)$ is a **bridge** or **cut-edge** of G if $G \backslash e$ has more connected components than G.

Theorem 9.1

An edge $e \in E(G)$ is a bridge if and only if e is not in any cycle of G.

Proof. Suppose $e = uv$ is on a cycle C. Then in $G \backslash e$, u and v still lie on the same path and are thus in the same connected component. So e was not a bridge.

Now suppose e was not a bridge. Say $e = uv$, then in $G \backslash e$ there is a path from u to v. Thus, in G there was a cycle from u to u containing e. ∎

Lemma. Let e be a bridge in a connected graph G. Then $c(G \backslash e) = 2$. (Note $c(H)$ denotes the number of connected components in H.)

Proof. Let $e = uv$ be a bridge. So u and v are in different connected components of $G \backslash e$. Thus $c(G \backslash e) \geq 2$. Let $w \in V(G)$, since G is connected there exists a path P from w to v in G. If P does not use edge e, then P is also a path in $G \backslash e$. So suppose P uses edge e. Then P is path from w to v that goes through u. Thus P has a sub-path from w to u. Therefore, any $w \in V(G)$ is either connected to u or v in $G \backslash e$, so $C(G \backslash e) = 2$. ∎

A graph with no cycles is **acyclic**. A **tree** is a connected acyclic graph. A **forest** is an acyclic graph. If T is a tree, and $v \in V(T)$ with $\deg(v) = 1$, then v is a **leaf**. A **spanning tree** of a graph G is a spanning subgraph of G that is a tree.

Corollary. A connected graph is a tree if and only if all of its edges are bridges.

Proposition 9.2. Every tree T with at least 2 vertices has at least 2 leaves and deleting a vertex from an n vertex tree produces an $n-1$ vertex tree.

Theorem 9.3

The following are equivalent:

(a). G is a tree.

(b). G is connected and has $n-1$ edges.

(c). G is acyclic and has $n-1$ edges.

(d). G is loopless and for any $u, v \in V(G)$, there is a unique u, v-path.

Proof.
- (a $\Rightarrow$ b + c). Induction on n.
- (b $\Rightarrow$ a + c). Let G be connected with $n-1$ edges. If G has a cycle with $n-1$ edges, we can delete any edge on the cycle and G will remain connected. We repeat until we have a tree on n vertices and $< n-1$ edges. This contradicts (a $\Rightarrow$ b + c).
- (c $\Rightarrow$ a + b). Let G be acyclic with $n-1$ edges and $k \geq 1$ components. Let n_j be the number of vertices in component j. Then $\sum n_j = n$. Each component is a tree, hence G has $\sum(n_j - 1) = n - k$ edges. So $k = 1$.
- (a $\Rightarrow$ d). If G is connected and acyclic, G has no loops. G is connected so there is at least one path between any pair of vertices. If some pair is connected by more than one, choose a pair P, Q of distinct paths with the same endpoints having shortest total length. By this extremal choice, P and Q can have no common internal vertices. So $P \cup Q$ is a cycle, a contradiction.
- (d $\Rightarrow$ a). G is clearly connected. If G has a cycle (it cannot be a loop) then G has two u, v-paths for some u, v. Hence G must be acyclic.

∎

Corollary. A forest of order n with k connected components has $m = n - k$ edges.

Corollary. Every connected graph G has a spanning tree.

Corollary. Every connected graph has $|E(G)| = m \geq n - 1 = |V(G)|$.

> **Examples.**
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> - P_n: the path graph of order n is a tree.
> - $K_{1,n}$: the 'star' graph is a tree.
> - Other types of tree include: double star tree and the caterpillar tree.

Cayley's Formula. There are n^{n-2} trees on a vertex set V of n elements.

Given a tree T, whose vertex set if labeled with $\{1, \ldots, n\}$, we can convert T to a Prüfer sequence $S = (a_1, \ldots, a_{n-2})$ as follows (initialize $i \leftarrow 1$):

- Find the leaf v of T with the smallest label.
  - Set a_i to be the neighbor of v and delete v from T. Then $i \leftarrow i + 1$.
- Repeat until only the K_2 graph remains.

Conversely, given Prüfer sequence $S = (a_1, \ldots, a_{n-2})$ with $a_i \in L = \{1, \ldots, n\}$, we can construct a tree T as follows (initialize $i \leftarrow 1$):

- Find the smallest $x \in L$ not in S, join x to a_i in T. Then remove x from L and $i \leftarrow i + 1$.
- Repeat until L has 2 elements, then join these two elements in T.

This construct forms a one-to-one correspondence between labeled trees on n vertices and Prüfer sequences of length $n - 2$. Since there are n^{n-2} Prüfer sequences, this proves Cayley's formula.

Proposition 9.4. Let $\tau(G)$ be the number of distinct spanning trees of G. Let $G \cdot e$ denote the graph G with edge e contracted. Then $\tau(G) = \tau(G - e) + \tau(G \cdot e)$. ($e$ is not a loop)

Theorem 9.5: Matrix Tree Theorem

Let G be a loopless graph with $V(G) = \{v_1, \ldots, v_n\}$. Let $A(G)$ be the adjacency matrix of G, $D(G)$ be a diagonal matrix with $d_{ii} = \deg(v_i)$, and $Q = D - A$. If $Q^*_{s,t}$ is the matrix obtained by deleting row s and column t from Q, then

$$\tau(G) = (-1)^{s+t}\det(Q^*_{s,t}).$$

Proposition 9.6. A sequence $d_1, \ldots, d_n$ of n positive integers where $n \geq 2$ is a degree sequence of a tree of order n if and only if $\sum_{i=1}^{k} d_i = 2(n-1)$.

Proof. If $d_1, \ldots, d_n$ is a degree sequence of a tree, then $\sum_{i=1}^{k} d_i = 2|E(T)| = 2(n-1)$. Now suppose $\sum_{i=1}^{k} d_i = 2(n-1)$. For $n = 2$, we have $d_1 + d_2 = 2$, so $d_1 = d_2 = 1$, and $T = P_2$. Suppose the result holds for $n = k$ where $k \geq 2$. Let $s = (d_1, \ldots, d_{k+1})$ be a sequence of $k+1$ positive integers such that $\sum_{i=1}^{k+1} d_i = 2k$. WLOG suppose we have s is non-increasing. Note that we must have $d_1 \geq 2$ and $d_k = d_{k+1} = 1$. Now consider $s' = (d_1 - 2, d_2, \ldots, d_k)$. This is a sequence of k positive integers which sum to $2(k-1)$. Thus there exists a tree T' of order k with $V(T') = \{v_1, \ldots, v_k\}$ such that $\deg(v_i) = d_i$ for $i = 2, \ldots, k$ and $\deg(v_1) = d_1 - 1$. Let T be the tree obtained from T' by adding vertex v_{k+1} and edge $v_1 v_{k+1}$. Then $d_1, \ldots, d_{k+1}$ is the degree sequence of T. ∎

Proposition 9.7. Let T be a non-trivial tree with $\Delta(T) = k$ and let n_i be the number of vertices of degree i in T for $i = 1, \ldots, k$. Then $n_1 = n_3 + 2n_4 + 3n_5 + \ldots + (k-2)n_k + 2$.

Proof. Suppose T is a non-trivial tree and $|E(G)| = m$ and $|V(T)| = n$. Then $m = n - 1$. Also $\sum_{i=1}^{k} i n_i = 2m = 2(n-1) = 2\sum_{i=1}^{k} n_i - 2$, from which the result follows. ∎

Proposition 9.8. Let T and T' be distinct spanning trees of a graph G.

(a). If $e \in E(T) - E(T')$, then there exists $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.

(b). If $e \in E(T) - E(T')$, then there exists $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G.

Theorem 9.9

Let T be a tree of order k and G a graph with $\delta(G) \geq k$. Then G contains a subgraph isomorphic to T.

Proof. Induction on k. For $k = 0$ this is obvious. Assume the claim holds for $k - 1$. Let T be a tree with k edges and G a simple graph with $\delta(G) \geq k$. Delete a leaf from T to create a tree T'. By the IH, T' is a subgraph of G. Let $v \in V(T')$. The number of edges in T' is $k - 1$, so v has $\leq k - 1$ neighbors in T'. Since $\delta(G) \geq k$, there exists a neighbor x of v in G such that x is not in T'. Therefore, it a vertex and edge is added to any vertex in T', the result is still a subgraph of G. Since T is constructed from T' in this way, T is a subgraph of G. ∎

Suppose G and H are graphs with $V(G) = \{u_1, \ldots, u_m\}$ and $V(H) = \{v_1, \ldots, v_n\}$. Then $G \times H$ is the graph with $V(G \times H) = V(G) \times V(H)$ and e is an edge of $G \times H$ if and only if $e = (u_i, v_j)(u_k, v_l)$ where either (1) $i = k$ and $v_j v_l \in E(H)$, or $j = l$ and $u_i u_k \in E(G)$.

For example, the **hypercube graph** $Q_n = K_2^n$.

The **eccentricity** of a vertex $v \in V(G)$ is $\epsilon(v) = \max_{u \in V(G)} d(u, v)$. The **diameter** of a graph is $\operatorname{diam}(G) = \max_{v \in V(G)} \epsilon(v)$ and the **radius** of a graph is $\operatorname{rad}(G) = \min_{v \in V(G)} \epsilon(v)$.

If $\epsilon(v) = \operatorname{diam}(G)$, then v is a **peripheral vertex**. The set of all peripheral vertices is the **periphery** of a graph. If $\epsilon(v) = \operatorname{rad}(G)$, then v is a **central vertex** and the set of all central vertices is the **center** of a graph.

Proposition 9.10. $\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2 \operatorname{rad}(G)$.

Proposition 9.11. If G is a disconnected graph, then $\overline{G}$ is connected and $\operatorname{diam}(G) \leq 2$.

Proof. Let G be a disconnected graph and let G_1 be one if the connected components of G. Let $V_1 = V(G_1)$ and $V_2 = V(G) \backslash V(G_1)$. Let $u \in V_1$, then for every $v \in V_2$, $uv \notin E(G)$, so $uv \in E(\overline{G})$. Thus $d_{\overline{G}}(u, v) = 1$ and for every $v_1, v_2 \in V_2$, $d_{\overline{G}}(v_1, v_2) \leq 2$ and for every $u_1, u_2, d_{\overline{G}}(u_1, u_2) \leq 2$. ∎

Proposition 9.12. If G is simple and $\operatorname{diam}(G) \geq 3$ then $\operatorname{diam}(\overline{G}) \leq 3$.

Proposition 9.13. The center of a tree is an isolated vertex or an edge with its endpoints.

Proposition 9.14. Every graph G is the center of some connected graph.

Proof. Let G be a graph. We construct a connected graph H as follows: add two vertices u_1, u_2 and add all edges $u_1 x$ and $u_2 x$ where $x \in V(G)$. Then add vertices v_1, v_2 together with edges $u_1 v_1, u_2 v_2$. In the new graph, we have $e_H(v_1) = 4 = e_H(v_2)$, $e_H(u_1) = 3 = e_H(u_2)$ and $e_H(x) = 2$ for all $x \in V(G)$. Thus $\operatorname{rad}(H) = 2$ and $\operatorname{Cen}(H) = V(G)$. ∎

A vertex $v \in V(G)$ is called a **cut-vertex** if $G \backslash v$ has more connected components than G.

Proposition 9.15. A vertex $v \in V(G)$ is a cut-vertex of G if and only if there exists $u, w \in V(G)$ with $u, w \neq v$ such that v is on every uw-path in G.

Proposition 9.16. Every non-trivial connected graph contains at least 2 vertices that are not cut-vertices. In fact, no peripheral vertex can be a cut-vertex.

A non-trivial connected graph is called **nonseparable** or **2-connected** if it has no cut-vertices. A **block** is a maximal nonseparable subgraph.

Proposition 9.17. If G is a graph with at least one-cut vertex, then at least 2 of the blocks of G contain exactly one cut-vertex. (These blocks are called the end-blocks of G.)

Proposition 9.18. The center of every connected graph is in a single block.

Proof. Suppose not, then there exists a connected graph G where $\operatorname{Cen}(G)$ does not lie in a single block of G. Thus G has a cut-vertex $v \in V(G)$ such that $G \backslash v$ contains connected components G_1, G_2 which each contain vertices in $\operatorname{Cen}(G)$. Let $u \in V(G)$ such that $d_G(u, v) = e(v)$ and let P be a shortest uv-path. At least one of G_1, G_2 has no vertices in P, say G_2. Now let $w \in V(G_2) \cap \operatorname{Cen}(G)$

and let Q be a shortest uw-path. Then $P \cup Q$ is a uw-path which necessarily has length $d(u, w)$. So $e(w) > e(v)$, a contradiction. ∎

Theorem 9.19

Every graph G with $\Delta(G) = r$ is an induced subgraph of an r-regular graph.

Proof. If G is r-regular, we're done. Suppose G is not r-regular, i.e. $\delta(G) < r$. Let G' be a copy of G and join the corresponding vertices of G and G' with an edge if they had degree less than r. Call The resulting graph G_1. Repeat this process until G_k is constructed, where $k = r - \delta(G)$. G_k will be an r-regular graph, with induced subgraph G. ∎

10 Matchings

Def$^{\mathbf{n}}$. A **matching** in a graph G is a set of non-loop edges in G that don't share a common vertex. If a vertex is incident to an edge in a matching M, we say the vertex is **saturated** by M. Otherwise, the vertex is **unsaturated**. If all vertices in G are saturated by M we say M is a **perfect** matching.

For a matching M in a graph G, an M**-alternating path**, is a path in G, where every other edge is in M. An M**-augmenting path** is an M-alternating path whose endpoints are unsaturated. Let $X \triangle Y = (X - Y) \cup (Y - X)$ denote the symmetric difference of the sets X and Y.

Proposition 10.1. If G has no M-alternating paths if and only if M is a maximum matching.

Proof. If G has an M-augmenting path P, then $M \triangle P$ is a larger matching in G. Conversely, if M' is a larger matching than M. Since M, M' are matchings, each vertex in G is incident to at most 2 edges in $M \triangle M'$. Hence each component of $M \triangle M'$ is a path or cycle. The edges of a cycle must alternate between $M' - M$ and $M - M'$, hence any cycle in the $M \triangle M'$ is even. Cycles consist of the same number of edges from M as M'. Since M' is larger than M, $M \triangle M'$ must have a path starting and ending with an edge in M'. This is an M-augmenting path. ∎

Proposition 10.2. (Hall's Theorem). A bipartite graph G with partite sets X, Y has a matching saturating X if and only if $|N(S)| \geq |S|$ for each $S \subseteq X$.

Proof. If a matching saturates X, then for any $S \subseteq X$, the $|S|$ vertices of S must be matched with vertices in $N(S)$. So $|N(S)| \geq |S|$. Conversely, suppose no matching saturates X and let M be a maximum matching. Say $x_0 \in X$ is unsaturated. If $N(x_0)$ is empty, we're done. Otherwise, each neighbor of x_0 is saturated by M.

Let S be the subset of X reachable via M-alternating paths starting at x_0. Let $\tilde{y} \in N(S)$ and $\tilde{x} \in S$ be a neighbor of $\tilde{y}$. Now, either $\tilde{x}\tilde{y} \in M$ or we can add the edge $\tilde{x}\tilde{y}$ to the $x_0 \to \tilde{x}$ M-alternating path, to get an $x_0 \to \tilde{y}$ M-alternating path. Since $\tilde{y}$ must be saturated, so there's a $\tilde{z} \in X$ such that $\tilde{y}\tilde{z} \in M$. But $\tilde{z} \in S$. Hence every $\tilde{y} \in N(S)$ is matched to some vertex in S. Because x_0 is unsaturated, $|S| > |N(S)|$. ∎

Corollary. Every k-regular bipartite graph has a perfect matching.

Proof. Recall a k-regular bipartite graph has $|X| = |Y|$, so any matching saturating X also saturates Y (hence perfect). Let $S \subseteq X$. Let $E(S)$ be the edges incident to vertices in S. Then $|E(S)| = |S|k$. Every edge in $E(S)$ is incident to a vertex in $N(S)$. Thus $|E(N(S))| \geq |E(S)|$ as $E(S) \subseteq E(N(S))$. By k-regularity, the result follows. ∎

10.1 Vertex Covers

Def$^{\mathbf{n}}$. A **vertex cover** in a graph G is a set $Q \subseteq V$ such that every edge in G is incident to at least one vertex in Q.

Observe that for any matching M and vertex cover Q, if $e \in M$ it is incident to at least one vertex in Q and any vertex in Q is incident to at most one edge in M. Hence $|M| \leq |Q|$. In particular, if $|M| = |Q|$, then M is a maximum matching and Q is a minimum vertex cover.

Proposition 10.3. (Konig-Egervary). In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

Proof. Let G be bipartite with partite sets X, Y. Let Q be a minimum vertex cover of G. Partition Q via $P_X = Q \cap X$ and $P_Y = Q \cap Y$. Let H and H' be the subgraphs of G induced by vertex sets $P_X \cup (Y - P_Y)$ and $P_Y \cup (X - P_X)$, respectively.

Since $P_X \cup P_Y$ is a vertex cover, there is no edge of G between $X - P_X$ and $Y - P_Y$. For each $S \subseteq P_X$ consider $N_H(S) \subseteq Y - P_Y$, by definition of H. If $|N_H(S)| < |S|$, we could substitute $N_H(S)$ for S in Q to obtain a smaller vertex cover, since $N_H(S)$ covers all edges incident to S, not covered by P_Y. So by minimality of Q, Hall's condition applies to H. So H has a matching saturating P_X. Similarly, H' has a matching saturating P_Y. As H, H' are disjoint, these matchings together are size $|Q|$. ∎

10.1.1 Augmenting Path Algorithm

```
# Compute maximum matching & minimum vertex cover.
def augmentingPath(G: X,Y-bigraph, M: matching in G):
    let E(X) = {X → Y directed, unmatched edges in G}
    let E(Y) = {Y → X directed, matched edges in G}
    let H be digraph with edges E(X) and E(Y)
    add vertices s and t to H
    with edges from s to unsaturated X-vertices
    and from unsaturated Y-vertices to t
    return shortest augmenting path from s to t in H #BFS
```

$\mathcal{M}$ is a maximum when augmentingPath returns empty path. The minimum vertex cover $Q = T \cup (X - S)$ where S is the set of X-vertices reachable from unsaturated X-vertices. T is the set of Y-vertices reachable from unsaturated X-vertices.

10.2 Weighted Matchings & the Hungarian Algorithm

In the **weighted matching problem**, we seek a perfect matching of a (non-negative) weighted graph $K_{n,n}$ such that matching has maximum total weight. A **weighted cover** is a choice of labels

u_i for X and v_j for Y such that $u_i + v_j \geq w_{i,j} = w(e_{i,j})$. The **cost** of a cover is $\sum u_i + \sum v_i$. The **minimum weighted cover** seeks to find a vertex cover of minimum cost.

Given a weight matrix $W \in \mathbb{R}^{n \times n}_{\geq 0}$ and a vertex cover (u, v), with **cost matrix** $C_{i,j} = u_i + v_j$, we define the **excess matrix**, $\epsilon := W - C \geq 0$.

```
1  # Maximum weighted matching on K_{n,n} (partite sets X,Y)
2  def weightedMatching(W: n × n weight matrix):
3      initialize weighted cover (u,v): u[i] = max_j W_{i,j}, v[i] = 0
4      find maximum matching M in the excess graph G_{u,v}
5      let S,T be as in augmentingPath
6      let ϵ_min be the minimum excess between S and Y − T
7      u[i] -= ϵ_min for i in S, v[i] += ϵ_min for i in T
8      if M is perfect then return else goto 4
```

11 Cuts & Connectivity

Def$^{\mathbf{n}}$. A **vertex cut** or **separating set** in a graph G is a set of vertices $S \subseteq V(G)$ such that $G - S$ has more than one component. The **connectivity** of G, denoted $\kappa(G)$ is the minimum cardinality of a set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. A **k-connected** graph has $\kappa(G) \geq k$.

Examples.

- $\kappa(K_n) = n - 1$ and $\kappa(K_{m,n}) = \min\{m, n\}$.
- For any graph G, deleting the neighborhood of a vertex either disconnects G or leaves one vertex, so $\kappa(G) \leq \min_v |N(v)|$. Consequently, $\kappa(G) \leq \delta(G)$.
- If G is simple with $n(G) = n$ and $G \ncong K_n$, then $\kappa(G) \leq n - 2$.

Proposition 11.1. Q_k, $k \geq 1$ has $\kappa(Q_k) = k$.

Proof. Since Q_k is k-regular, it suffices to show for any separating set S that $|S| \geq k$. Let S be a separating set. Let Q, Q' be the two copies of Q_{k-1} used to form Q_k.

- Case 1. S contains at least one endpoint from every edge between Q, Q'. Then $|S| \geq 2^{k-1} \geq k$.
- Case 2. Deleting S leaves at least one Q, Q'-edge. Thus, deleting S must disconnect one of Q or Q'. WLOG assume $Q - S$ is disconnected. Proceed by induction. Assume $\kappa(Q_j) = j$ for $j < k$. Then we have $|S| \geq k - 1$. If S contains no vertex in Q', then $Q_k - S$ is still connected. Therefore, $|S| \geq k$.

∎

Proposition 11.2. If $\kappa(G) = k$ then $\delta(G) \geq k$, so $e(G) \geq \lceil \frac{nk}{2} \rceil$.

This bound is sharp. If n is a power of 2, then Q_k provides an example. More generally, the **Harary graph** $H_{k,n}$, $2 \leq k \leq n$ has equality in the bound above. If n is even, we construct $H_{k,n}$ as follows:

Start with n vertices in a circle and make every vertex adjacent to its $k/2$ nearest neighbors in *both* directions.

Def$^{\mathbf{n}}$. A **disconnecting set** of a graph G is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. The **edge connectivity** of G, denoted $\kappa'(G)$, is the minimum size of a disconnecting set. For $S, T \subseteq V(G)$, let $[S, T]$ denote the set of edges with one endpoint in S and the other in T. An **edge cut** is a set of edges of the form $[S, \overline{S}]$ where $S \neq \varnothing$ is a proper subset of $V(G)$ and $\overline{S}$ is the complement of S.

Note every edge cut is a disconnecting set and every minimal disconnecting set is an edge cut.

Proposition 11.3. If G is simple, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Proof. We can delete all edges incident to any vertex and disconnect G, so $\kappa'(G) \leq \delta(G)$. It remains to show $\kappa(G) \leq \kappa'(G)$. Let $F = [S, \overline{S}]$ be an edge cut of minimum size in G.

- Case 1. All vertices in S are adjacent to all vertices in $\overline{S}$. Then
$$\kappa'(G) = |F| = |S||\overline{S}| \geq |S| + |\overline{S}| - 1 = n - 1 \geq \kappa(G).$$
- For some $x \in S$ and $y \in \overline{S}$, $xy \notin E(G)$. Let $e \in F$. If e is incident to x, delete its endpoint in $\overline{S}$. Otherwise, delete its endpoint in S. This deletes no more than $|F|$ vertices, but removes every edge in F and hence disconnects G.

∎

Proposition 11.4. If G is 3-regular, then $\kappa(G) = \kappa'(G)$.

Proof. Let S be a minimum size vertex cut, so $\kappa(G) = |S|$. Since $\kappa \leq \kappa'$, it suffices to show there is a disconnecting set of size $|S|$. Let H_1, H_2 be two components of $G - S$. Each vertex in S has a neighbor in both H_1, H_2, by minimality. We'll construct a disconnecting set as follows: For $v \in S$, recall $\deg(v) = 3$.

- If v has 1 neighbor in H_1 and 2 in H_2, delete its neighbor in H_1. If v has 1 neighbor in H_2 and 2 in H_1, delete its neighbor in H_2.
- Otherwise, v has 1 neighbor in both H_1 and H_2, and 1 neighbor in S, say w. w must also have one neighbor in each of H_1, H_2, S. For both v, w delete their neighbors in H_1.

∎

Proposition 11.5. Let $S \subseteq V(G)$, then $\left|[S, \overline{S}]\right| = \sum_{v \in S} \deg(v) - 2e(G[S])$, where $G[S]$ is the subgraph of G induced by S.

Proof. $\sum_{v \in S} \deg(v)$ counts each edge in $[S, \overline{S}]$ once and each edge in $e(G[S])$ twice. ∎

Corollary. Let G be simple. If $\left|[S, \overline{S}]\right| < \delta(G)$, for some nonempty, proper subset $S \subseteq V(G)$, then $|S| < \delta(G)$.

Proof. Note $|S| > 1$ since if S contains just 1 vertex, then necessarily, $\left|[S, \overline{S}]\right| \geq \delta(G)$. Recall for simple graphs, $e(G[S]) \leq \binom{|S|}{2}$. Hence

$$2e(G[S]) \leq |S|(|S| - 1). \tag{11.1}$$

By the last proposition, with $\left|[S, \overline{S}]\right| < \delta(G)$, we have $\delta|S| \leq \sum_{v \in S} \deg(v) = \left|[S, \overline{S}]\right| + 2e(G[S]) < \delta + 2e(G[S])$. Thus

$$\delta\left(|S| - 1\right) < 2e(G[S]). \tag{11.2}$$

Combining (11.1) and (11.2) and recalling $|S| > 1$, we're done. ∎

Def$^{\mathbf{n}}$. A **bond** is a minimal edge cut.

Proposition 11.6. If G is connected, an edge cut F is a bond iff $G-F$ has exactly two components.

Proof. Let $F = [S, \overline{S}]$ be an edge cut. Suppose $G - F$ has exactly two components. Let F' be a proper subset of F. The graph $G - F'$ has the two components of $G - F$ with one edge between them, so it has one component. Thus F is minimal. Conversely, suppose $G - F$ has more than two components. Since $G - F$ is the disjoint union of $G[S]$ and $G[\overline{S}]$, one of these has two components. WLOG it is $G[S]$. So $S = A \cup B$ where no edges join A and B, so edge cut $[A, \overline{A}]$ is a proper subset of F, i.e. F isn't minimal. ∎

Def$^{\mathbf{n}}$. Two u, v-paths are **internally disjoint** if they share no vertices other that u, v.

Proposition 11.7. Let $n(G) \geq 3$. Then G is 2-connected iff for every pair $u, v \in V(G)$, there exist 2 internally disjoint u, v-paths in G.

Proof. Let S be a vertex cut in G, then there exist $u, v \in V(G)$ such that there is no u, v-path in $G - S$. So S must contains one internal vertex from each internally disjoint u, v-path, so $|S| \geq 2$. Conversely, suppose G is 2-connected and proceed by induction on $d_G(u, v)$. If $d(u, v) = 1$, then $uv \in E(G)$ and since $\kappa'(G) \geq \kappa(G) = 2$, $G - uv$ is connected, so there is a u, v-path P in $G - uv$. Now let $k = d(u, v) \geq 2$. Assume the result holds for all pairs of vertices at a distance $\leq k - 1$. Let w be the vertex preceding v on a shortest u, v-path. By the inductive hypothesis, there exist P, Q, two internally disjoint u, w-paths.

- Case 1. One of P or Q contains v. Then $P \cup Q$ is a cycle containing u, v, so we're clearly done.
- Case 2. v is not on $P \cup Q$. Since G is 2-connected, $G - w$ is connected, so there is a u, v-path R in $G - u$. If R contains no vertex on $P \cup Q$, we're done. Otherwise, let z be the last vertex shared by R and $P \cup Q$ (moving from u to v). WLOG $z \in P$. Let A be the path along P from u to z together with the path along R from z to v and $B = Q \cup \{wv\}$. Then A and B are internally disjoint u, v-paths.

∎

Theorem 11.8: Characterization of 2-Connectivity

For $n \geq 3$, TFAE:

a) G is connected and has no cut-vertex, i.e. G is 2-connected.

b) $\forall x, y \in V(G)$, there are 2 internally disjoint u, v-paths.

c) $\forall x, y \in V(G)$, there is a cycle containing x, y.

d) $\delta(G) \geq 1$ and every pair of edges lies on a common cycle.

Def$^{\mathbf{n}}$. Let $x, y \in V(G)$ and $S \subset V(G) - \{x, y\}$. We call S an x, y**-cut** if $G - S$ contains no x, y-path. Let $\kappa(x, y)$ be the minimum size of any x, y-cut (**local connectivity**) and $\lambda(x, y)$ be the maximum size of any collection of pairwise internally disjoint x, y-paths in G.

Note that if there are j internally disjoint x, y-paths, then $\kappa(x, y) \geq j$. Hence in general, $\kappa(x, y) \geq \lambda(x, y)$.

Theorem 11.9: Menger

If $x, y \in V(G)$ are non-adjacent, then $\kappa(x, y) = \lambda(x, y)$.

Proof. We'll show $k = \kappa(x, y) \leq \lambda(x, y)$ by constructing k pairwise internally disjoint (PID) x, y-paths. Proceed by induction of $n = n(G)$. If $n = 2$, G has only two vertices which are non-adjacent, so $\kappa(x, y) = 0 = \lambda(x, y)$. Now suppose $n \geq 3$ and assume the result for graph with less than n vertices.

- Case 1. G has a min. size x, y-cut S other that $N(x)$ or $N(y)$. Note: S cannot contain either $N(x)$ or $N(y)$, so $N(x) - S$ and $N(y) - S$ are nonempty. Let $V_1 = \{x, S\text{-paths}\}$ and $V_2 = \{y, S\text{-paths}\}$ (paths to a vertex in S with no internal vertices in S). Claim: $V_1 \cap V_2 = S$. $S \subseteq V_1 \cap V_2$ since every element of S is on an x, y-path as S is a min. x, y-cut. Conversely, if $v \in S_1 \cap S_2$ but $v \notin S$, then v lies on some x, S-path and some y, S-path, but then $x \to v \to y$ is an x, y-path in $G - S$, a contradiction. Now, construct H_1 as follows: Take $G[V_1]$ and add y' adjacent to each vertex in S. Then S is an x, y'-cut in H_1 of min. size and $n(H_1) < n$ as H_1 doesn't contain y and some vertex in $N(y)$ which are in G, but H_1 only contains y', which isn't in G. Similarly, we construct H_2 from $G[V_2]$ and adding x' connected to S. By the IH, we have k PID x, y'-paths and k PID y, x'-path. Removing x', y', we can match up these path at their S-vertices.
- Case 2. Every min. size x, y-cut is $N(x)$ or $N(y)$.
  a) There is some $v \notin N(x) \cup N(y) \cup \{x, y\}$. Then v is not in any min. size x, y-cut, so if v is in an x, y-cut, it is size $\geq k + 1$. Hence $\kappa_{G-v}(x, y) = \kappa_G(x, y) = k$. By induction, we're done.
  b) Some $w \in N(x) \cap N(y)$. Then $G - w$ has $n - 1$ vertices and $\kappa_{G-w}(x, y) = k - 1$. By induction we get $k - 1$ PID x, y-paths, adding x, w, y, we're done.
  c) $V(G) = N(x) \cup N(y) \cup \{x, y\}$ are $N(x) \cap N(y) = \varnothing$. Thus $N(x)$ and $N(y)$ partition $G - \{x, y\}$. Let H be the bipartite subgraph of $G - \{x, y\}$ keeping only edges in $[N(x), N(y)]$. An x, y-cut in G must remove at least on endpoint of each edge in H, i.e. x, y-cuts are vertex covers of H. Let S be a min. size x, y-cut. Then S is a min. size vertex cover in H, so by Konig-Egervary, there is matching of size k in H, from which we construct our k PID x, y-paths.

∎

Define $\lambda'(x, y)$ to be the maximum number of edge disjoint x, y-paths in G and $\kappa'(x, y)$ to be the minimum size of an x, y-disconnecting set.

Given a graph G, the **line graph** of G, denoted $L(G)$, is the simple graph whose vertices are the edges of G and $ef \in E(L(G))$ iff e and f share a common endpoint in G.

Theorem 11.10

$\lambda'(x,y) = \kappa'(x,y)$ for all $x, y \in V(G)$

Given G with $x, y \in V(G)$, add a vertex s adjacent to x and add t adjacent to y. Call this new graph G'. Then $L(G')$ is $L(G)$ with edges from the vertex sx to each edge incident to x in G and edges from yt to each edge incident to y in G. In general, we have $\lambda'_G(x,y) = \lambda_{L(G')}(sx, yt)$ and $\kappa'_G(x,y) = \kappa_{L(G')}(sx, yt)$. Applying Menger's theorem to $L(G')$ gives the result.

Corollary. $\kappa(G) = \min \lambda(x,y)$ and $\kappa'(G) = \min \lambda'(x,y)$.

12 Network Flows

Def$^{\mathbf{n}}$. A **network** is a digraph G with a capacity function $c : E(G) \to \mathbb{R}_{\geq 0}$ together with two designated nodes, the **source** node s and the **sink** node t. A **flow** f in G assigns a value $f(e)$ to each edge in G. For each $v \in V(G)$, let $f^+(v)$ be the total flow leaving v and $f^-(v)$ be the total flow entering v. A flow is **feasible** if $0 \leq f(e) \leq c(e)$ for every edge $e \in E(G)$ and $f^+(v) = f^-(v)$ for all $v \in V(G) - \{s,t\}$. The **value** of a flow, $\texttt{val}(f) := f^-(t) - f^+(t)$. A **maximum flow** is a feasible flow of maximum value.

Given a feasible flow f, an f-augmenting path is a source-to-sink path P in the underlying graph such that for each $e \in E(P)$, (a) $f(e) < c(e)$ if e is a forward edge in P (b) $f(e) > 0$ if e is a backwards edge in P. We say the **tolerance** of an f-augmenting path P, is $\texttt{tol}(P) := \min_{e \in E(P)} \epsilon(e)$ where

$$\epsilon(e) = \begin{cases} c(e) - f(e) & \text{if } e \text{ is forward} \\ f(e) & \text{if } e \text{ is backwards.} \end{cases}$$

Lemma. If P is an f-augmenting path with $\texttt{tol}(P) = z$, then changing the flow along P by $+z$ on forward edges and $-z$ on backwards edges results in a feasible flow f' with $\texttt{val}(f') = \texttt{val}(f) + z$.

Def$^{\mathbf{n}}$. A **source-sink cut** is an edge cut $[S,T]$ such that $s \in S$ and $t \in T$ and S, T partition $V(G)$. The **capacity** of a source-sink cut, is $\texttt{cap}(S,T) = \sum_{e \in [S,T]} c(e)$.

Lemma. Let $U \subseteq V(G)$ in a network. The net flow out of U, denoted $f^+(U) - f^-(U)$ is equal to $\sum_{v \in U}(f^+(v) - f^-(v))$. In particular, if f is feasible, and $[S,T]$ is a source-sink cut, then the net flow out of S is the net flow into T.

Proof. Edge $e = xy$ where $x \in U$ and $y \in \overline{U}$ contributes $+f(e)$ to $\sum_{v \in U}(f^+(v) - f^-(v))$. Edge $e = yx$ where $x \in U$ and $y \in \overline{U}$ contributes $-f(e)$ to $\sum_{v \in U}(f^+(v) - f^-(v))$. Edge $e = xy$ where $x, y \in U$ contributes 0 to $\sum_{v \in U}(f^+(v) - f^-(v))$. Thus we can partition the sum as desired. Now, $f^+(S) - f^-(S) = f^+(s) - f^-(s) = -\texttt{val}(f) = f^+(t) - f^-(t) = f^+(T) - f^-(T)$. ∎

Corollary. If f is a feasible flow and $[S,T]$ is a source-sink cut then $\texttt{val}(f) \leq \texttt{cap}(S,T)$.

```
Floyd-Fulkerson(N: network, f: feasible flow):
    R ← {s}, S ← ∅
    pick v ∈ R − S:
        for forward edge vw with f(vw) < c(vw) and w ∉ R
            put w ∈ R, parent(w) = v
        for backward edge uv with f(uv) > 0 and u ∉ R
            put u ∈ R, parent(u) = v
        if t was reached: return the f-augmenting path
    if R − S is non-empty: goto 3
    else: return the cut [S, S̄].
```

R is the set of vertices reached along paths from s with positive tolerance. S is the set of searched vertices.

Theorem 12.1: Max-Flow, Min-Cut

In every network, the maximum value of a feasible flow equals the minimum values of a source-sink cut.

Floyd-Fulkerson requires rational capacities to guarantee convergence. Edmonds-Karp (finding shortest f-augmenting path) works for all real capacities.

13 Planar Graphs

A **planar graph** is a graph that can be drawn in the plane without any edge-crossings, i.e. it has a **planar-embedding**. A **plane graph** is a particular planar-embedding of a planar graph. The **dual** G^* of a plane graph G has one vertex v_i for each face f_i of G. Vertices v_i, v_j are adjacent if their corresponding faces in G share an edge on their boundary (add a loop on v_i for cut-edges).

- If G is a plane graph, so is G^*.
- G^* is always connected and $(G^*)^* \cong G$ iff G is connected. Note different planar-embeddings of G can yield different duals.
- $e(G^*) = e(G)$ and $n(G^*) = f(G)$.
- If G is connected, the boundary of a face is a closed walk.

For a face f in a plane graph G, the **length of** f, denoted $\ell(f)$ is the number of edges bounding f (double count cut-edges).

Proposition 13.1. If f_i are the faces of a plane graph, $2e(G) = \sum_i \ell(f_i)$.

Theorem 13.2: Euler

If G is a a connected plane graph with n vertices, e edges, and f regions, then $n - e + f = 2$.

Proof. For $e = 0$, G is K_1. Suppose the theorem holds for all connected graphs with less than e edges, for $e \geq 1$. Now consider a connected plane graph G on e edges, n vertices, and f regions. Either

- G is a tree, in which case $e = n - 1$ and $f = 1$.
- G is not a tree. Hence G has a cycle C. Let e be an edge of C. Remove e (since e is on a cycle, e is not a bridge so $G \backslash e$ is connected). Then $G \backslash e$ is planar and has n vertices, $e - 1$ edges, and $f - 1$ regions. Thus $n - (e - 1) + (f - 1) = 2$.

In any case, we have $n - e + f = 2$. ∎

Corollary. If G is a plane graph with $c(G)$ connected components, then $n - m + r = 1 + c(G)$.

Proposition 13.3. If G is a simple planar graph with n vertices ($n \geq 3$) and e edges, then $e \leq 3n - 6$.

Proof. Take a plane drawing of G on f regions. The boundary of every region is a triangle, thus $2e = \sum_{\text{regions } R} (\# \text{ of edges on } \partial R) = 3f$. Applying Euler's formula, we obtain our desired result. ∎

Proposition 13.4. If G is a planar graph with $n \geq 3$ vertices, e edges, and no triangles, then $e \leq 2n - 4$.

Using these theorems, we can show K_5 and $K_{3,3}$ aren't planar. An **elementary subdivision** of a nonempty graph G is a graph obtained from G by removing an edge $e = uv$ and adding a new vertex w together with edges uw and wv. A **subdivision** of a graph G is a graph obtained from G by a sequence of zero or more elementary subdivisions.

Theorem 13.5: Kuratowski

A graph G is planar if and only if G contains no subgraph which is a subdivision of K_5 or $K_{3,3}$.

Let G be a graph and $e = uv \in E(G)$. Suppose $w \notin V(G)$. Contracting an edge in G results in the graph G/e obtained from G by: removing edge e, replacing u and v with w, and making w adjacent to the neighbors of u and v. A graph H is called a **minor** of G if it can be produced from G by successive application of these reductions:

- deleting an edge
- contracting an edge
- deleting an isolated vertex.

Theorem 13.6: Wagner

A graph G is planar if and only if it has no K_5 or $K_{3,3}$ minor.

14 Colorings

Let G be a graph. A **(proper) vertex coloring** of G is a label of the vertex set

$$f : V(G) \to \{1, \dots, k\}$$

where the labels are called colors and no two adjacent vertices have the same color. A k**-coloring** of a graph G is a coloring of G using k colors. If G has a proper k-coloring, it is k**-colorable**. The **chromatic number** of a graph G, denoted $\chi(G)$, is the smallest k such that G is k-colorable. If $\chi(G) = k$, we say G is k**-chromatic**.

Facts.

- If G has order n, then $\chi(G) \leq n$.
- $\chi(G) = 1$ if and only if G is empty
- $\chi(C_{2n}) = 2$ and $\chi(C_{2n+1}) = 3$
- $\chi(K_n) = n$
- If G is the Petersen graph, then $\chi(G) = 3$.
- If $H \leq G$, then $\chi(H) \leq \chi(G)$
- Every tree with at least 2 vertices is 2-chromatic.
- $\chi(G) = k$ if and only if G is k-partite.

A k-coloring of G partitions the vertex set in k independent sets, called the **color classes**.

A graph G is k**-critical** if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph $H \subset G$.

- The only 1-critical graph is K_1. The only 2-critical graph is K_2. The only 3-critical graphs are odd cycles.

Recall the following graph parameters:

- $\alpha(G)$ = max size of an independent set in G.
- $\alpha'(G)$ = max size of a matching in G.
- $\beta(G)$ = min size of a vertex cover in G. Note $\alpha'(G) \leq \beta(G)$ for all graphs, with equality if G is bipartite.
- $\beta'(G)$ = min size of edge cover in G. Note $\alpha(G) \leq \beta'(G)$ for all graphs with no isolated vertices, with equality if G is bipartite (and no isolated vertices).

Define $\omega(G)$ to be the maximum size of a clique in G.

Proposition 14.1. For all graphs G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

Proof. If G has a k-clique, we require at least k colors to properly color G. To show the second inequality, we'll establish $n(G) \leq \alpha(G)\chi(G)$. Suppose $\chi(G) = k$. Then in a proper k-coloring in G if n_j is the size of the j-th color class, then $n_j \leq \alpha(G)$ (as each color class is independent). There are k color classes, so the claim follows. ∎

Proposition 14.2. For every graph, $\chi(G) \leq \Delta(G) + 1$.

Proof. We'll use greedy coloring to show G is $(\Delta(G) + 1)$-colorable. Take an arbitrary ordering of the vertices, $v_1, \ldots, v_n$. Color v_1 with 1. At each step, color the next vertex in the list with the least number available (so the adjacency condition is satisfied). At the j-th step in this process, there must be some color available in $\{1, \ldots, \Delta(G) + 1\}$ to color v_j since v_j has at most $\Delta(G)$ neighbors already colored. ∎

Def$^{\mathbf{n}}$. An **interval graph** is a graph G that admits an **interval representation**, i.e. a function f mapping $V(G)$ to closed intervals of $\mathbb{R}$ such that $f(u) \cap f(v)$ is non-empty iff u, v are adjacent.

Proposition 14.3. If G is an interval graph, then $\chi(G) = \omega(G)$.

Proof. Consider an interval representation of G. Order the vertices by their interval's left endpoints (breaking ties arbitrarily). Suppose $\chi(G) = k$ and we color G greedily as above. Suppose vertex x is assigned color k. By greedy choice, x must be adjacent to $k - 1$ other vertices each of which must be pairwise adjacent. So G has a k-clique. Thus $k \leq \omega(G) \leq \chi(G) = k$. ∎

Proposition 14.4. (Brooke). If G is connected and not complete or an odd cycle, then $\chi(G) \leq \Delta(G)$.

If $\omega(G) = \chi(G)$, we say G is a **perfect graph**. For example, every interval graph is perfect.

Def$^{\mathbf{n}}$. Given a simple graph G, **Mycielski's construction** produces a simple graph $M(G)$ containing G as follows:

- If $V(G) = \{v_1, \ldots, v_n\}$, add new vertices $U = \{u_1, \ldots, u_n\}$ such that u_i is adjacent to $N_G(v_i)$.

- Add w adjacent to U.

Proposition 14.5. If G is $\triangle$-free and k-chromatic, then $M(G)$ is $\triangle$-free and $(k+1)$-chromatic.

Proof. Since G is $\triangle$-free and no two u_i's can be adjacent, the only possible triangles in G have to v vertices and one u vertex, but this would imply the v vertices have a common neighbor, a contradiction. So $M(G)$ is $\triangle$-free. Assume $\chi(G) = k$ and f is a proper k-coloring of G. By coloring u_i and v_i the same and w a new color, we have a $(k+1)$-coloring of $M(G)$. Furthermore, suppose $\chi(G') = j$ and WLOG w is colored j. Then U is colored with $1, \ldots, j-1$. For each v_i with $f(v_i) = j$, reassign so $f(v_i) = f(u_i)$. This gives a proper $j-1$ coloring of G (not necessarily $M(G)$), as neighbors of v_i in G are also neighbors of u_i. ∎

Proposition 14.6. Every k-chromatic graph with n vertices has $\geq \binom{k}{2}$ edges with equality for K_k with $n-k$ isolated vertices.

Proof. Let $X_1, \ldots, X_k$ be the partite sets (color classes) of G. Each pair of sets X_i, X_j must contain some pair of adjacent vertices, else we could combine them to obtain a smaller proper k-coloring. ∎

$K_{n_1,\ldots,n_k}$ denotes the **complete multipartite graph**, a simple k-partite graph with partite sets of size $n_1, \ldots, n_k$ and all pairs of vertices in different partite sets are adjacent. The **Turán graph**, $T_{n,r}$ is the r-partite, n-vertex multipartite complete graph the size of whose partite sets differ by at most 1.

Proposition 14.7. Among simple r-partite graphs with n vertices, $T_{n,r}$ is the unique graph with the most edges.

Proof. If there are two non-adjacent vertices in different color classes, we can add an edge. Hence the max edge k-chromatic graph is a complete multipartite graph. Suppose two partite sets X, Y of the graph differ in size by more than 1, i.e. $|X| = t > t' + 1$ where $t' = |Y|$. Let $v \in X$. Then moving v to Y deletes t' edges and adds $t-1$, but $t-1 > t'$. ∎

Proposition 14.8. (Turán's Theorem). Among n-vertex simple graphs with no $r+1$-clique, T_{n+r} has the most edges.

14.1 Chromatic Polynomial

Def$^{\mathbf{n}}$. For a graph G and $k \in \mathbb{N}$, let $\chi(G;k)$ be the number of proper k-colorings of G. As a function of k, we call $\chi(G;k)$ the **chromatic polynomial** of G.

For example, $\chi(K_n;k) = \binom{k}{n} n! = k(k-1)\ldots(k-n+1)$ and $\chi(\overline{K_n};k) = k^n$.

Proposition 14.9. If T is a tree with n vertices, $\chi(T;k) = k(k-1)^{n-1}$.

Proof. Color some vertex (k choices). Repeat until colored, choose a neighbor of a colored vertex and color it. There are always $k-1$ choices after the first step because a vertex gets colored only when it is adjacent to one other colored vertex; if it were adjacent to two, there'd be a cycle. ∎

Let $P_r(G)$ be the number of partitions of $V(G)$ into r nonempty independent sets and let $x_{(r)} = x(x-1)\dots(x-r+1)$ be the number of ways to color r independent sets with distinct colors from color set $[k]$.

Proposition 14.10. For any simply graph G,

$$\chi(G;k) = \sum_{r=1}^{n} P_r(G)k_{(r)}.$$

Proposition 14.11. If G is simple and $e \in E(G)$ then $\chi(G;k) = \chi(G-e;g) - \chi(G \cdot e;k)$.

Theorem 14.12: Whitney

Let G be simple. Then $\chi(G;k)$ is a polynomial in $\mathbb{Z}[k]$ of degree $n(G)$, whose coefficients alternate in sign and has leading terms $k^n - e(G)k^{n-1} + \dots$.

Proof. If $e(G) = 0$, then $G = \overline{K_n}$, so $\chi(G;k) = k^n$. Let $k = e(G) \geq 1$ and assume the result for graphs with less than k edges. Then $e(G-e) = e(G) - 1$ and $e(G \cdot e) < e(G)$, so by the IH,

$$\chi(G;k) = (k^n - (e(G)-1)k^{n-1} + \dots) - (k^{n-1} - \dots) = k^n - e(G)k^{n-1} + \dots$$

From which it is clear $\chi(G;k)$ has integer coefficient which alternate sign. ∎

Four Color Theorem. All planar graphs a 4-colorable.

15 Graph Algorithms

15.1 Minimum Weight Spanning Tree

```
prims(G: weighted graph, s: start vertex)
        create min-priority queue Q with all the vertices of G
        for each v ∈ Q
                v.priority ← ∞
        s.priority ← 0
        s.parent ← NULL
        while Q is not empty
                u ← Q.pop()
        for each vertex v adjacent to u
                if v ∈ Q and weight(u,v) < v.priority
                        v.parent ← u
                        v.priority ← weight(u,v)
```

```
kruskal(G: weighted graph)
        A ← ∅
        for each v ∈ V(G):
                MAKE-SET(v)
        for each (u,v) in E(G) ordered by increasing weight(u,v):
                if FIND-SET(u) ≠ FIND-SET(v):
                        A = A ∪ {(u,v)}
                        UNION(u, v)
        return A
```

Proof. Suppose Kruskal's algorithm outputs spanning tree T. Let $e_1, \ldots, e_{n-1}$ be the edges of T in non-decreasing order. Let T' be any other spanning tree of G. Say, listed in non-decreasing order the vertices of T' agree with T up to the k-th step:

$$T' : e_1, \ldots, e_k, e'_{k+1}, \ldots, e'_{n-1}.$$

Note the graph with edges $e_1, \ldots, e_k$ is acyclic as is the graph with $e_1, \ldots, e_k, e'_{k+1}$.

At the k-th step of Kruskal's, both e_{k+1} and e'_{k+1} were available so $w(e_{k+1}) \le w(e'_{k+1})$. Adding e_{k+1} to T' creates exactly one cycle. There is an edge $e' \in E(T') - E(T)$ such that $T^* = T' + e_{k+1} - e'$ is a spanning tree that agrees with T up to the $k+1$-st step and $w(e_{k+1}) \le w(e'_{k+1}) \le w(e')$ (note: $w(e')$ is larger since e'_{k+1} is the least weight edge of T' not in T). We can repeat this process until $T^* = T$. Then $w(T') - w(T) = \sum_{i=k+1}^{n-1} (w(e'_i) - w(e_i)) \ge 0$ by above. Hence $w(T) \le w(T')$. ∎

15.2 Shortest Path Problem

```
floyds(M_G: adj. matrix of weighted graph G, n: |V(G)|)
        for i ← 0 to n − 1
                M_G[i,i] ← 0
        for k ← 0 to n − 1
                for i ← 0 to n − 1
                        for j ← 0 to n − 1
```

```
                M_G[i, j] ← min(M_G[i, j], M_G[i, k] + M_G[k, j])
    return M_G
```

Apprendix

Linear Algebra

A complex inner product space is a complex vector space E equipped with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ satisfying: (a) $x \neq 0 \Rightarrow \langle x, x \rangle > 0$ (b) $\overline{\langle x, y \rangle} = \langle y, x \rangle$ (c) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (d) $\langle cx, z \rangle = c\langle x, z \rangle$. A finite-dimensional complex-IPS is called a unitary space.

We have

- $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$
- $\langle x, \theta \rangle = \langle \theta, x \rangle$
- $\|\theta\| = 0$; $\|x\| > 0$ when $x \neq \theta$
- $\|cx\| = |c| \, \|x\|$
- $\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 + \|x - y\|^2 + i \, \|x + iy\|^2 - i \, \|x - iy\|^2)$
- $\|x + y\|^2 + \|x - y\|^2 = 2 \, \|x\|^2 + 2 \, \|y\|^2$
- $|\langle x, y \rangle| \leq \|x\| \, \|y\|$
- $\|x + y\| \leq \|x\| + \|y\|$.
- $x \perp y \Rightarrow \|x\|^2 + \|y\|^2 = \|x + y\|^2$

We have $J_E : E \to E'$ a bijection where $J_E y = y'$, where $y'(x) = \langle x, y \rangle$.

For any subspace M of E, we have $E = M \oplus M^\perp$. Thus, $\dim M^\perp = \dim E - \dim M$. Every nontrivial unitary space has an orthonormal basis.

Two complex-IPS are isomorphic is there exists a linear bijection $T : E \to F$ such that $\langle Tx, Ty \rangle = \langle x, y \rangle$. The isomorphism T is called a unitary linear map. Given $T : E \to F$, define the adjoint of T to be $T^* = J_E^{-1} T' J_F$, where $T' : F' \to E'$ is the transpose of T, given by $T'g = gT$.

We have

- T^* is linear
- $\langle Tx, y \rangle = \langle x, T^* y \rangle$
- $(S + T)^* = S^* + T^*$, $(cT)^* = \overline{c}T^*$, $(T^*)^* = T$
- $(ST)^* = T^* S^*$.
- T unitary $\Leftrightarrow T^*T = I$.

Theorem 15.1

Let E and F be unitary spaces with dimension n and m, respectively. Let $T : E \to F$ be linear and $\{e_i\}$ and $\{f_j\}$ bases for E and F, respectively, and $A = [T]$ with respect to these basis. Then the matrix of T^* relative to these bases is the conjugate transpose A^* of A.

We say $A \in \mathrm{Mat}_n(\mathbb{C})$ is unitary if $A^*A = I$, normal if $A^*A = AA^*$, Hermitian if $A^* = A$ and skew-Hermitian if $A^* = -A$. Correspondingly, T is normal if $T^*T = TT^*$, self-adjoint if $T^* = T$ and skew-adjoint if $T^* = -T$.

Theorem 15.2

If E is a unitary space and $T \in \mathcal{L}(E)$ is normal, then E has an orthonormal basis consisting of eigenvectors of T.

Moreover, if A is normal, then there exists a unitary matrix U such that U^*AU is diagonal. Furthermore, if $U^*AU = \mathrm{diag}(c_1, \ldots, c_n)$, then $A^* = A$ if and only if c_i are real.