Real Analysis

Fall 2018

Contents

1 Real Numbers

Definition 1.1. Let S be a set. An **order** on S is a relation, denoted $<$, such that

1) $\forall x, y \in S$, either $x < y$ or $x = y$ or $y > x$.
2) if $x < y$ and $y < z$, then $x < z$.

Definition 1.2. A subset E of a set S is **bounded above** if there exists a number $b \in S$ such that $a \leq b$ for all $a \in E$. The number b is called an **upper bound** for E.

Definition 1.3. We say $b \in S$ is the **least upper bound** for a set $E \subset S$ if:

- b is an upper bound for E;
- if $a < b$ then a is not an upper bound for E.

The least upper bound for E is also called the **supremum** of A. The **infimum** or **greatest lower bound** of E is defined similarly. We denote the infimum and supremum of A by $\sup(A)$ and $\inf(A)$, respectively.

Definition 1.4. An ordered set S has the **least-upper-bound (LUB) property** if for any nonempty subset E that is bounded above, $\sup(E)$ exists in S.

Theorem 1.5. Let S be an ordered set with the LUB property, then S has the greatest lower bound property.

Proof. Let $B \subset S$ be a nonempty set bounded below. Let L be the set of all lower bounds for B. By assumption, L is a nonempty subset of S. Since L is the set of LBs for B, it follows that any element of B is an upper bound for L. So L is bounded above, and by the LUB property, L has a LUB, call it α. If $x < \alpha$, then x is not an upper bound for L, so $\exists y \in L$ such that $x < y$. Thus, $x \notin B$. It follows that $\alpha \leq z$ for all $z \in B$, so $\alpha \in L$. If $\alpha < x$ then $x \notin L$ since α is an upper bound for L. We have shown α is a lower bound for B but x is not if $x > \alpha$. That is $\alpha = \inf(B)$. ∎

Definition 1.6. An **ordered field** is a field F with a relation $<$ such that (i) $x + y < x + z$ whenever $y < z$ and (ii) $xy > 0$ if $x, y > 0$.

Proposition 1.7. $(\mathbb{R}, +, \cdot, <)$ is an ordered field with the LUB property that contains $\mathbb{Q}$ as a subfield.

Theorem 1.8 (Archimedian Property). If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an integer $n \geq 1$ such that $nx > y$.

Proof. Suppose not, i.e. there exists x_0, y_0, such that $nx_0 \leq y_0$ for all $n \geq 1$. Then $A = \{mx_0\}$ is bounded above by y_0. So A has a supremum, say α. Note $w - x_0$ is not an upper bound for A, so there exists $t \in A$ such that $w - x_0 < t = kx_0$, i.e. $w < (k+1)x_0$. ∎

Theorem 1.9. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proposition 1.10. $x^n = y$ is uniquely solvable for $y > 0$ and $n > 0$.

2 Basic Topology

Definition 2.1. A set X, whose elements we shall call *points*, it a **metric space** if for any points $p, q \in X$, there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(a) $d(p, q) > 0$, if $p \neq q$; $d(p, p) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these properties is called a *distance function* or *metric*.

If $a_i < b_i$ for $i = 1, \ldots, k$, the set of points $\mathbf{x} = (x_1, \ldots, x_k)$ in $\mathbb{R}^k$ such that $a_i \leq x_i \leq b_i$ is called a *k-cell*. So a 1-cell is an interval, a 2-cell is a rectangle, and so on. If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the *open* (*closed*) *ball* B with center at $\mathbf{x}$ and radius r is the set of call $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

A set $E \subset \mathbb{R}^k$ is **convex** if

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

for all $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$. For example, open and closed balls are convex, as are k-cells.

Definition 2.2. Let X be a metric space. All points or subsets reference below belong to X.

(a) A **neighborhood** of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$ for some $r > 0$.

(b) A point p is a **limit point** of the set E if every neighborhood of p contains a point $q \neq p$ with $q \in E$.

(c) If $p \in E$ but p is not a limit point of E then p is a *isolated point*.

(d) E is **closed** if every limit point of E is a point of E.

(e) p is an **interior** point of E if some neighborhood N of p is contained in E.

(f) E is **open** if every point of E is an interior point.

(g) The complement of E, denoted E^c is the set of all $p \in X$ such that $x \notin E$.

(h) E is **perfect** if E is closed and if every point of E is a limit point of E (converse).

(i) E is **bounded** if $\exists M \in \mathbb{R}$ and $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

(j) E is **dense in** X is every point of X is a limit point of E, or a point of E.

Proposition 2.3. Every neighborhood is an open set.

Proposition 2.4. If p is a limit point of a set E then every neighborhood of p contains infinitely many points of E.

Corollary 2.4.1. A finite point set has no limit points.

Proposition 2.5. If $\{E_\alpha\}$ is a collection of sets, then

$$(\cap_\alpha E_\alpha)^c = \cup_\alpha E_\alpha.$$

Theorem 2.6.

(a) A set E is open if and only if E^c is closed.

(b) Given a collection of open sets $\{G_\alpha\}$, $\cup_\alpha G_\alpha$ is open.

(c) Given a collection of closed sets $\{F_\alpha\}$, $\cap_\alpha F_\alpha$ is closed.

(d) For any finite collection $G_1, \ldots, F_n$ of open sets, $\cap_{i=1}^n G_i$ is open.

(e) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.

Definition 2.7. In a metric space X, if $E \subset X$ and E' denotes the set of limit points of E, then the **closure** of E is $\bar{E} = E \cup E'$.

Theorem 2.8. If X is a metric space and $E \subset X$, then

(a) $\bar{E}$ is closed;

(b) $E = \bar{E}$ if and only if E is closed;

(c) For any closed set $F \subset X$ with $E \subset F$, we have $\bar{E} \subset F$.

Proposition 2.9. Let $\varnothing \neq E \subset \mathbb{R}$ be bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Let $E \subset Y \subset X$, where X is a metric space. We say E is *open relative* to Y if to each point $p \in E$ there is associated a real number $r > 0$, such that $q \in E$ when $d(p, q) < r$, $q \in Y$.

Theorem 2.10. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.

2.1 Compact Sets

Definition 2.11. An **open cover** of a set in E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$. We say E is **compact** if every open cover of E contains a finite subcover.

Theorem 2.12. Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.

Theorem 2.13. Compact subsets of metric spaces are closed. Moreover, closed subsets of compacts sets are compact.

Proposition 2.14.

1) If $\{K_\alpha\}$ is a collection of compact subsets of X, such that every finite intersection of $\{K_\alpha\}$ is nonempty. Then $\cap K_\alpha$ is nonempty.
2) If E is an infinite subset of a compact set K, then E has a limit point in K.
3) If $\{I_n\}$ is a sequence of intervals of $\mathbb{R}$ such that $I_{n+1} \subset I_n$, then $\cap I_n$ is nonempty. (Also true if I_n are k-cells).
4) Every k-cell is compact.

Theorem 2.15. If $E \subset \mathbb{R}^k$, then the following are equivalent.

1) E is closed and bounded
2) E is compact
3) Every infinite subset of E has a limit point in E.

Theorem 2.16 (Weierstrass). Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Definition 2.17. A set $E \subset X$ is **connected** if E is not the union of two nonempty separated sets. Two sets, $A, B \subset X$ are **separated** if $A \cap \overline{B} = \overline{A} \cap B = \varnothing$.

Theorem 2.18. A subset E of $\mathbb{R}$ is connected if and only if it is an interval (open or closed).

3 Sequences

Given a sequence (p_n) in a metric space X and a point $p \in X$, we say (p_n) **converges to** p, written $p_n \to p$, if for any $\epsilon > 0$ there exists an integer N such that if $n \geq N$, then $d(p_n, p) < \epsilon$.

Theorem 3.1. Let (p_n) be a sequence in metric space X.

1) (p_n) converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.
2) If $p, p' \in X$ so that $p_n \to p$ and $p_n \to p'$, then $p = p'$.
3) (p_n) convergent implies (p_n) bounded.
4) If $E \subset X$ and p is a limit point of E, then there is a sequence (p_n) in E such that $p_n \to p$.

Theorem 3.2. Suppose $(s_n), (t_n)$ are complex sequences and $s_n \to s$ and $t_n \to t$. Then

1) $s_n + t_n \to s + t$
2) $cs_n \to cs$ and $c + s_n \to c + s$
3) $s_n t_n \to st$
4) $\frac{1}{s_n} \to \frac{1}{s}$ provided $s \neq 0$ and $s_n \neq 0$ for any n.

Proposition 3.3.

1) If (p_n) is a sequence in a compact metric space X, then (p_n) has a convergent subsequence.
2) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proposition 3.4. The set of subsequential limits of a sequence (p_n) in a metric space X is closed.

3.1 Cauchy Sequences

Definition 3.5. A sequence (p_n) in a metric space X is a **Cauchy sequence** if for any $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n, m \geq N$.

Definition 3.6. Let $\varnothing \neq E \subseteq X$, where X is a metric space. Define $S = \{d(p,q) : p, q \in E\}$. Then the **diameter** of E is $\sup(S)$.

Proposition 3.7. If E is a set in a metric space X, then $\operatorname{diam}(\overline{E}) = \operatorname{diam}(E)$.

Proposition 3.8. If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ and if $\operatorname{diam} K_n \to 0$, then $\cap_1^\infty K_n$ contains exactly one point.

Theorem 3.9.

1) In any metric space, every convergent sequence is Cauchy.
2) If (p_n) is Cauchy in a compact metric space X, then $p_n \to p$ for some $p \in X$.
3) In $\mathbb{R}^k$, every Cauchy sequence converges.

Definition 3.10. A sequence (s_n) of real number is

a) **monotonically increasing** if $s_n \leq s_{n+1}$;
b) **monotonically decreasing** if $s_n \geq s_{n+1}$.

Theorem 3.11. Suppose (s_n) is monotonic. Then (s_n) converges if and only if it is bounded.

3.2 Upper and Lower Limits

If (s_n) is a sequence such that for any real M there is an integer N, such that $n \geq N$ implies $s_n \geq M$ $(s_n \leq M)$, then we write $s_n \to \infty$ $(s_n \to -\infty)$.

Given a sequence (s_n), let E be the set of all subsequential limits (possibly including $\pm\infty$). Then

$$\limsup_{n\to\infty} s_n = \sup E \text{ and } \liminf_{n\to\infty} s_n = \inf E$$

Proposition 3.12. If $s^* = \limsup_{n\to\infty} s_n$, as defined above, then $s^* \in E$ and if $x > s^*$, then there exists an integer N so that $n \geq N$ implies $s_n < x$. (An analogous result holds for s_*.)

Note that if $s_n \leq t_n$ for all $n \geq N$ (N fixed), then $s_* \leq t_*$ and $s^* \leq t^*$.

3.3 Special Sequences

Theorem 3.13.

1) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
2) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
3) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
4) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
5) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

4 Series

Given a sequence $(a_n)_n$ we define the sequence $(s_n)_n$ where $s_n = \sum_{i=1}^{n} a_i$. We say the infinite series $\sum a_i$ converges if $(s_n)_n$ converges.

Theorem 4.1. $\sum a_i$ converges if and only if for every $\epsilon > 0$ there is an integer N such that $|\sum_{i=n}^{m} a_i| \leq \epsilon$ for $m \geq n \geq N$. In particular, we require $\lim_{n\to\infty} a_n = 0$.

Theorem 4.2 (Comparison Test).

(a) If $|a_n| \leq c_n$ for $n \geq N_0$ where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$, for $n \geq N_)$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proposition 4.3. If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, then the series diverges.

Theorem 4.4 (Cauchy Condensation Test). Let $(a_n)_{n\geq 1}$, $a_n \geq 0$, be a monotone decreasing sequence. Then $\sum a_i$ converges if and only if

$$\sum_{i=0}^{\infty} 2^k a_{2^k}$$

converges.

Theorem 4.5 (p-series Test). $\sum \frac{1}{n^p}$ converges if $p > 1$, otherwise it diverges.

Theorem 4.6. If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \geq 1$, the series diverges. (*Proof.* Cauchy condensation, followed by p-series)

Theorem 4.7 (Root Test). Let $a = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

(a) If $a < 1$, then the series converges;

(b) if $a > 1$, the series diverges;

(c) if $a = 1$, this test is inconclusive.

Theorem 4.8 (Ratio Test). Let $r = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

(a) If $r < 1$, then the series converges.

(b) If $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for all $n \geq n_0$ for some fixed integer n_0, then the series diverges.

Theorem 4.9 (Raabe-Duhamel). Assume $a_n > 0$ for all $n \geq 0$.

1) If $\liminf\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$, then the series converges.

2) If $\limsup\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) < 1$, then the series diverges.

3) Otherwise, the test is inconclusive.

Theorem 4.10. Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \qquad \text{and} \qquad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, put $R = \infty$, if $\alpha = \infty$, put $R = 0$.) Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Theorem 4.11 (Summation by Parts). Given two sequences $(a_n), (b_n)$, let $A_n = \sum_{k=1}^n a_k$ if $n \geq 0$ and put $A_{-1} = 0$. Then for $0 \leq p \leq q$ we have

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q-1} A_k(b_k - b_{k+1}) + A_q b_q - A_{p-1} b_p.$$

Corollary 4.11.1. If the partial sums of A_n form a bounded sequence and $b_n \to 0$ is monotone decreasing, then $\sum a_n b_n$ converges.

Corollary 4.11.2. If the sequence (c_n) satisfies (1) $|c_n| \geq |c_{n+1}|$ for all $n \geq 0$, (2) c_n alternates sign, and (3) $c_n \to 0$, then $\sum c_n$ converges.

Definition 4.12. We say $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Proposition 4.13. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Note: Sums of convergent series and scalar multiples of convergent series behave as expected.

Definition 4.14. Given two series $\sum a_n$, $\sum b_n$, put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

for all $n \geq 0$. We call $\sum c_n$ the product of the two given series.

Theorem 4.15. If $\sum a_n = A$ and $\sum b_n = B$ converge and at least one converges absolutely, then the product of the two series $\sum c_n$ converges and its value is AB.

Proposition 4.16. More generally, if the product of two series converges, it will converge to the product of the limits of the two series.

4.1 Rearrangements

Theorem 4.17. Let Σa_n be a series of real numbers which converges but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. There there exists a rearrangement $\Sigma a'_n$ with partial sums s'_n such that

$$\liminf s'_n = \alpha \qquad \text{and} \qquad \limsup s'_n = \beta.$$

Proposition 4.18. If Σa_n converges absolutely, then every rearrangement converges and to the same value.

5 Continuity

Definition 5.1. Let X, Y be metric spaces; suppose $E \subset X$, $f : E \to Y$ and p is a limit point of E. Then we write $\lim_{x \to p} f(x) = q$, for some $q \in Y$, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < d_X(x, p) < \delta$$

for $x \in E$, then

$$d_Y(f(x), q) < \epsilon.$$

Proposition 5.2. Let X, Y, E, f, and p be as in definition 5.1. Then $\lim_{x \to p} f(x) = q$ if and only if $f(p_n) \to q$ for every sequence $(p_n)_{n \geq 1} \subset E$, $p_n \neq p$, with $p_n \to p$.

Proof. ($\Rightarrow$). Choose $(p_n)_{n \geq 1}$ as above. Let $\epsilon > 0$ and choose δ so that $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \epsilon$. There exists N so that for $n \geq N$, $0 < d_X(p_n, p) < \delta$. Hence for $n \geq N$,

$$d_Y(f(p_n), f(p)) < \epsilon.$$

($\Leftarrow$). Contrapositive. There exists $\epsilon > 0$ so that for all $\delta > 0$, there exists $x \in E$ so that $d_y(f(x), f(p)) \geq \epsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = \frac{1}{n}$, $n = 1, 2 \ldots$, we can form the desired sequence. ∎

Remark. Limits of functions at a point are unique (if they exist). As a corollary to proposition 5.2, we see that limits of functions have the analogous properties of sequences, as in theorem 3.2. For example, if $f(x) \to q, g(x) \to r$ as $x \to p$, then $(fg)(x) \to qr$ as $x \to p$.

Definition 5.3. Let X, Y be metric spaces, $E \subset X$, $p \in E$ and $f : E \to Y$. Then f is continuous at p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \epsilon$$

whenever $d_X(x, p) < \delta$, $x \in E$.

Remark. Note the subtle change from the definition of a limit of a function, to a function being continuous at a point. (1) f has to be defined at p, and (2) $d_X(x, p)$ can equal 0. As a consequence, if p is an isolated point of E, then f will always be continuous at p.

Remark. The composition, addition, multiplication, and division (where it is defined) of continuous functions will always be continuous.

Theorem 5.4. A mapping $f : X \to Y$, metric spaces, is continuous if and only if for every open set $O \subset Y$ we have $f^{-1}(O)$ open in X.

Proof. ($\Rightarrow$). Let $V \subset Y$ be open. Suppose $p \in X$ and $f(p) \in V$. There exists $\epsilon > 0$ so that $y \in V$ if $d_Y(y, f(p)) < \epsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ if $d_X(x, p) < \delta$. Thus $x \in f^{-1}(V)$ when $d_x(x, p) < \delta$.

($\Leftarrow$). Suppose $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X$ and $\epsilon > 0$. Let V be the set of $y \in Y$ so that $d_Y(y, f(p)) < \epsilon$. Then V is open, so $f^{-1}(V)$ must be open. Thus there exists $\delta > 0$ so that $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$. But if $x \in f^{-1}(V)$, then $f(x) \in V$, so $d_Y(f(x), f(p)) < \epsilon$. ∎

Corollary 5.4.1. A mapping $f : X \to Y$, metric spaces, is continuous if and only if for every closed set $O \subset Y$ we have $f^{-1}(O)$ closed in X.

Theorem 5.5. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Proof. Let $\{O_i\}_{i \in \mathcal{I}}$ be an open cover of $f(X)$. Since f is continuous, $f^{-1}(O_i)$ is open in X. Note $\bigcup_{i \in \mathcal{I}} f^{-1}(O_i)$ is an open cover of X. Since X is compact, there exist finitely many $i_1, \ldots, i_k$ so that $X \subset f^{-1}(O_{i_1}) \cup \ldots \cup f^{-1}(O_{i_k})$. But then

$$f(X) \subset O_{i_1} \cup \ldots \cup O_{i_k}.$$

∎

Corollary 5.5.1. (*Extreme Value Theorem*). With the setup of theorem 5.5, there exist points $p, q \in X$ such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$; in other words, f attains it maximum and minimum at p and q, respectively.

Proof. $f(X)$ is compact and thus closed and bounded. Hence $f(X)$ contains $\sup f(X)$ and $\inf f(X)$. ∎

5.1 Uniform Continuity

Definition 5.6. Let $f : X \to Y$, for metric spaces X, Y. We say f is uniformly continuous on X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_y(f(p), f(q)) < \epsilon$$

for all $p, q \in X$ with $d_X(p, q) < \delta$.

Theorem 5.7. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof. Let $\epsilon > 0$ be fixed. For all $x \in X$, there exists $\delta > 0$ so that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$. Here δ is dependent on x and ϵ.

Note $X = \bigcup_{x \in X} B(x, \frac{\delta}{2})$, where the $\delta = \delta(x, \epsilon)$ is chosen as above. By compactness, we can write

$$X = \bigcup_{i=1}^{n} B(x_i, \frac{\delta_i}{2}),$$

where $\delta_i = \delta(x_i, \epsilon)$. Take $\delta = \min_{i \in \{1,\ldots,n\}} \frac{\delta_i}{2}$. Suppose $d_X(x, y) < \delta$, then $x \in B(x_i, \frac{\delta_i}{2})$ for some i.

$$d_X(x, x_i) < \frac{\delta_i}{2} < \delta_i$$

and

$$d_X(x_i, y) < \frac{\delta_i}{2} + \delta < \delta_i.$$

Therefore,

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(Y)) < 2\epsilon.$$

∎

Theorem 5.8. Let X, Y be metric spaces. Suppose $f : X \to Y$ is continuous. Let $A \subset X$ be connected. Then $f(A)$ is a connected subset of Y.

Proof. Suppose $f(A)$ is not connected. Then $f(A) = B \cup C$ where $B, C \neq \varnothing$ and $B \cap \overline{C} = \overline{B} \cap C = \varnothing$. Define

$$D = \{x \in A : f(x) \in B\}$$
$$E = \{x \in A : f(x) \in C\}.$$

Note $A = D \cup E$ and $D, E \neq \varnothing$. If $x \in \overline{D}$ then there exist $(x_n)_{n \geq 1} \subset D$ such that $x_n \to x$, where $x \in D$. Then $(f(x_n))_{n \geq 1} \subset B$ tending to $f(x)$. So $f(x) \in \overline{B}$. Thus $f(x) \notin C$, so $x \notin E$. Hence $\overline{D} \cap E = \varnothing$. The rest of the argument is analogous. ∎

Corollary 5.8.1. (*Intermediate Value Theorem*). Let $f : I \to \mathbb{R}$. Suppose $a, b \in I$ with $a < b$. Then for any y between $f(a)$ and $f(b)$ there exists $c \in (a, b)$ so that $f(c) = y$.

Proof. By theorem 5.8, if $A = [a, b]$, then $f(A)$ is an interval. ∎

The converse of the above does not hold. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

for some $c \in [0, 1]$, has the intermediate value property, but is not continuous.

5.2 Discontinuities

Definition 5.9. Let $f : (a, b) \to \mathbb{R}$. Suppose $a \leq x < b$. We write

$$f(x+) = q$$

if $f(x_n) \to q$ for all sequences $(x_n)_{n\geq 1} \subset (x, b)$ with $x_n \to x$. This is the *right-hand limit* of f at x.

Similarly, to define the left-hand limit $f(x-)$ we restrict our sequences to (a, x) for $a < x \leq b$.

Definition 5.10. Suppose $f : (a, b) \to \mathbb{R}$ is discontinuous at a point x.

- If $f(x+)$ and $f(x-)$ exists. Then f is said to have a simple discontinuity (or a *discontinuity of the first kind*) at x. Either $f(x+) \neq f(x-)$ or $f(x+) = f(x-)$ but $f(x+) \neq f(x)$.
- A *discontinuity of the second kind* is when either $f(x+)$ or $f(x-)$ does not exist.

5.3 Monotonicity

Definition 5.11. Let $f : (a, b) \to \mathbb{R}$. We say f is monotonically increasing on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, then we say f is monotonically decreasing.

Theorem 5.12. Let f be monotonically increasing. Then $f(x+)$ and $f(x-)$ exist for any $x \in (a, b)$. Moreover,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t). \tag{5.1}$$

Furthermore, if $a < x < y < b$ then

$$f(x+) \leq f(y-).$$

An analogous result holds for f monotone decreasing.

Proof. Let $A = \sup\{f(t) : t < x\}$. By the definition of the supremum, for all $\epsilon > 0$, there exists $t' < x$ so that $f(t') > A - \epsilon$. By monotonicity, for all $t \in (t', x)$, we have $A - \epsilon < f(t') \leq f(t) \leq A$. Choosing $\delta = x - t'$ we have $f(x-) = A$. The case of $f(x+)$ is similar. Hence (5.1) follows by monotonicity.

Further, if $x < y$, then

$$f(x+) = \inf_{t>x} f(t) \leq \inf_{x<t<\frac{x+y}{2}} f(t) \leq f\left(\frac{x+y}{2}\right) \tag{5.2}$$

$$f(y-) = \sup_{t<y} f(t) \geq \sup_{\frac{x+y}{2}<t<y} f(t) \geq f\left(\frac{x+y}{2}\right). \tag{5.3}$$

∎

Remark. Note by theorem 5.8, a monotone function cannot have discontinuities of the 2nd kind.

Theorem 5.13. Let f be monotonic on (a,b). Then f has at most countably many discontinuities on (a,b).

Proof. WLOG f is increasing. Let E be the set of all points in (a,b) at which f is discontinuous. Then $x \in E$ implies $f(x-) < f(x+)$. Define $r : E \to \mathbb{Q}$ so that $r(x) = q$ for some $q \in \mathbb{Q}$ where

$$f(x-) < q < f(x+).$$

If $x, y \in E$ and $x < y$, then $r(x) < f(x+) \leq f(y-) < r(y)$. Hence $r(x) \neq r(y)$, so r is injective. Thus $|E| \leq |\mathbb{Q}|$. ■

5.4 Limits at Infinity

Definition 5.14. The neighborhoods of ∞ are (c, ∞) for $c \in \mathbb{R}$. The neighborhoods of $-\infty$ are defined similarly.

Let $f : E \subset \mathbb{R} \to \mathbb{R}$. We say the $\lim_{t \in x} f(t) = A$ if for every neighborhood V of A, there exists a neighborhood U of x with $U \cap E \neq \varnothing$ so that $f(U \cap E) \subset V$.

6 Differentiability

Definition 6.1. Let $f : [a,b] \to \mathbb{R}$ and take $x \in [a,b]$. We say f is differentiable at x if

$$f'(x) := \lim_{\substack{t \to x \\ t \in [a,b] \setminus \{x\}}} \frac{f(t) - f(x)}{t - x}$$

exists and is finite. We say f is differentiable on $[a,b]$ if it is differentiable at all points in $[a,b]$. On an open interval, (a,b), $f(a)$ and $f(b)$ are undefined.

Proposition 6.2. Let $f : [a,b] \to \mathbb{R}$. If f is differentiable at $x \in [a,b]$, then f is continuous at x.

Proof. Let $t \to x$. Then

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

■

Warning. Continuity does not implies differentiability.

Properties.

- Linearity of the derivative.
- Product and quotient rules.
- Chain rule.

Definition 6.3. Let f be a real-valued function defined on an interval $I \subset \mathbb{R}$. We say a point x_0, interior to I is a local extremum of f if there exists a neighborhood V of x_0, $V \subset I$ such that either $\sup_V f = f(x_0)$ or $\inf_V f = f(x_0)$.

Theorem 6.4. Let x_0 be a local extremum, f be differentiable at x_0. Then $f'(x_0) = 0$.

Proof. Let x_0 be a local maximum. Then there exists $V = (x_0 - \delta, x_0 + \delta) \subset I$ such that $\sup_V f = f(x_0)$. Hence

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x < x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{x \to x_0 \\ x > x_0}} \frac{f(x) - f(x_0)}{x - x_0}.$$

However, the first limit is at least 0 and the second is at most 0. The conclusion follows. ∎

Theorem 6.5. Suppose f is a real, diff. function on $[a, b]$ and $f'(a) < \lambda < f'(b)$. Then there exists $x \in (a, b)$ so that $f'(x) = \lambda$.

Proof. Put $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so there exists $t_1 \in (a, b)$ so that $g(t_1) < g(a)$ and $g'(b) > 0$ so there exists $t_2 \in (a, b)$ so that $g(t_2) < g(b)$. Hence g attains a minimum on $[a, b]$ at some $x \in (a, b)$. Hence $g'(x) = 0$, so $f'(x) = \lambda$. ∎

Warning. The last theorem shows that if a function is differentiable, then its derivative has the IV property. However, differentiability does *not* imply the continuity of the derivative, nor does it imply the derivative is differentiable.

Theorem 6.6 (Rolle). Let f be a real continuous function on $[a, b]$, f diff. on (a, b). Suppose $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. f continuous implies there exist $x_0, y_0 \in [a, b]$ are which f achieves a max, min, respectively. If $f(x_0) = f(y_0)$, then f is constant on $[a, b]$ so $f' = 0$. Otherwise $f(x_0) > f(y_0)$, so either x_0 or y_0 is in (a, b). So we have a critical point. ∎

Theorem 6.7 (Mean Value Theorem). If f, g are continuous real functions on $[a, b]$ which are differentiable on (a, b), then there exists $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Proof. Take $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$. Note $h(a) = h(b)$. By Rolle, there exists $x \in (a, b)$, so that $h'(x) = 0$. ∎

Corollary 6.7.1. If f is a real continuous function on $[a, b]$ and is differentiable on (a, b) then there exists a $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(x).$$

Theorem 6.8. Suppose f is diff. on (a, b). Then

- If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotone increasing (decreasing if the inequality is reversed).
- If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Proof. Let $x, y \in (a, b)$, $x > y$, then

$$f(x) - f(y) = (x - y)f'(c)$$

for some c between x, y. ∎

Theorem 6.9 (L'Hopital). Suppose f, g are real differentiable in (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty <\leq a < b \leq \infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a$$

and either $f(x), g(x) \to 0$ as $x \to a$ or $g(x) \to \infty$ as $x \to a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A.$$

Proof. Assume $-\infty \leq A < \infty$. Pick $q > A$. Then for some $c \in (a, b)$, we have

$$\frac{f'(x)}{g'(x)} < r$$

for all $a < x < c$. If $a < x < y < c$, then by MVT there exists $t \in (x, y)$ so that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r. \tag{6.1}$$

If f, g both tend to 0, then letting $x \to a$, we have

$$\frac{f(y)}{g(y)} \leq r < q.$$

Suppose g tends to ∞. Fix x. Then there exists $c_1 \in (a, x)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $x \in (a, c_1)$ (as g goes to ∞). Rearranging (6.1),

$$\frac{f(y)}{g(y)} < \frac{f(x)}{g(y)} + r\left(1 + \frac{g(x)}{g(y)}\right)$$

and letting $y \to a$, we can pick $c_2 \in (a, c_2)$ so

$$\frac{f(y)}{g(y)} < q.$$

Similarly, if $-\infty < A \leq \infty$, we can pick $p < A$ and a c_3 so

$$p < \frac{f(y)}{g(y)}$$

for all $a < x < c_3$. The result follows. ∎

7 Integration

7.1 Riemann-Stieljes

Let f be a bounded, real-valued function defined on $[a,b]$. Let $\alpha : [a,b] \to \mathbb{R}$ be monotone increasing. Define

$$U(f,\mathcal{P},\alpha) = \sum_{i=1}^{n} (\alpha(x_i) - \alpha(x_{i-1}) \sup_{[x_{i-1},x_i]} f$$

$$L(f,\mathcal{P},\alpha) = \sum_{i=1}^{n} (\alpha(x_i) - \alpha(x_{i-1}) \inf_{[x_{i-1},x_i]} f$$

where $\mathcal{P} = \{[x_i, x_{i+1}]\}_{i=0}^{n-1}$, $a = x_0 \leq x_1 \ldots \cdots \leq x_n = b$ is some partition of $[a,b]$. Note, since f is bounded these are well-defined. We write

$$\int_{\overline{a}}^{b} f \mathrm{d}\alpha = \inf_{\mathcal{P}} U(f,\mathcal{P},\alpha)$$

and

$$\int_{a}^{\overline{b}} f \mathrm{d}\alpha = \sup_{\mathcal{P}} L(f,\mathcal{P},\alpha).$$

We may drop the f and α from the parameter list when it is clear from context.

Definition 7.1. We say f is integratable if $\int_a^{\overline{b}} f \mathrm{d}\alpha = \int_{\overline{a}}^{b} f \mathrm{d}\alpha$. We write $f \in \mathcal{R}(\alpha)$ if f is Riemann-Stieljes integratable with respect to α.

Proposition 7.2. Let $P_1 \subset P_2$ be two partitions. Then

$$L(P_1) \leq L(P_2) \leq U(P_2) \leq U(P_1)$$

Proof. Consider the case where P_1 and P_2 are identical except the interval $[x_{i-1}, x_i]$ in P_1 is split into $[x_{i-1}, \tilde{x_i}]$ and $[\tilde{x_i}, x_i]$ in P_2. Then the difference between $U(P_1)$ and $U(P_2)$ is

$$(\alpha(x_i) - \alpha(x_{i-1})) \left(\sup_{[x_{i-1},x_i]} f \right) - (\alpha(x_i) - \alpha(\tilde{x_i})) \left(\sup_{[\tilde{x_i},x_i]} f \right) - (\alpha(\tilde{x_i}) - \alpha(x_{i-1})) \left(\sup_{[x_{i-1},\tilde{x_i}]} f \right).$$

Since the supremum f on $[x_{i-1}, \tilde{x_i}]$ and $[\tilde{x_i}, x_i]$ is less than or equal to the supremum of f on $[x_{i-1}, x_i]$, we see the difference is non-negative. By induction this shows, $U(P_1) \geq U(P_2)$. We prove $L(P_1) \leq L(P_2)$ similarly. ∎

For arbitrary partitions P, Q, we have

$$L(Q) \leq L(P \cup Q) \leq U(P \cup Q) \leq U(P)$$

since for any fixed partition we obviously have $L(P) \leq U(P)$. Thus $L(Q) \leq \inf_{\mathcal{P}} U(\mathcal{P}) = \int_a^{\overline{b}} f \mathrm{d}\alpha$. Applying the supremum to the right-hand side (since Q is arbitrary), we have

$$L(Q) \leq \int_{\overline{a}}^{b} f \mathrm{d}\alpha \leq \int_{a}^{\overline{b}} f \mathrm{d}\alpha \leq U(P). \tag{7.1}$$

Proposition 7.3. f is integratable if and only if for all $\epsilon > 0$ there exists a partition P_ϵ of $[a,b]$ such that $U(P_\epsilon) - L(P_\epsilon) < \epsilon$.

Proof. The reverse direction is immediate by (7.1). Suppose f is integratable. Then

$$\sup_{\mathcal{P}} L(\mathcal{P}) = \underline{\int_a^b} f \mathrm{d}\alpha = \overline{\int_a^b} f \mathrm{d}\alpha = \inf_{\mathcal{P}} U(\mathcal{P}).$$

We can choose P_ϵ^1 so that

$$\overline{\int_a^b} f \mathrm{d}\alpha \leq U(P_\epsilon^1) < \overline{\int_a^b} f \mathrm{d}\alpha + \frac{\epsilon}{2}$$

and P_ϵ^2 so that

$$\underline{\int_a^b} f \mathrm{d}\alpha - \frac{\epsilon}{2} \leq U(P_\epsilon^2) < \underline{\int_a^b} f \mathrm{d}\alpha$$

so that $U(P_\epsilon^1) - L(P_\epsilon^2) < \epsilon$. Then $P_\epsilon = P_\epsilon^1 \cup P_\epsilon^2$ is the desired partition. ∎

Proposition 7.4. Let f be integratable, $\epsilon > 0$. Choose P_ϵ such that $U(P_\epsilon) - L(P_\epsilon) < \epsilon$. Take $t_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$. Then

$$\left| \int_a^b f \mathrm{d}\alpha - \sum_{i=1}^n f(t_i)(\alpha(x_i) - \alpha(x_{i-1})) \right| < \epsilon. \tag{7.2}$$

Proof. We have $L(P_\epsilon) \leq \int_a^b f \mathrm{d}\alpha \leq U(P_\epsilon)$. Let $\Delta(i) = \alpha(x_i) - \alpha(x_{i-1})$. Then

$$L(P_\epsilon) = \sum \left(\inf_{[x_{i-1}, x_i]} f \right) \Delta(i) \leq \sum f(t_i) \Delta(i) \leq \sum \left(\sup_{[x_{i-1}, x_i]} f \right) \Delta(i) \leq U(P_\epsilon), \tag{7.3}$$

since $\inf_{[x_{i-1}, x_i]} f \leq t_i \leq \sup_{[x_{i-1}, x_i]} f$. ∎

Theorem 7.5. f continuous implies $f \in \mathcal{R}(\alpha)$.

Proof. f continuous on $[a, b]$ implies f is uniformly continuous on $[a, b]$. Choose $\delta > 0$ that ensures $\forall \epsilon > 0$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Choose a partition P, so that $x_i - x_{i-1} = \frac{\delta}{2}$ for all $1 \leq i < n$. Then

$$\begin{aligned} U(P) - L(P) &= \sum_{i=1}^n \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \big(\alpha(x_i) - \alpha(x_{i-1}) \big) \\ &\leq \epsilon \sum_{i=1}^n (\alpha(x_i) - \alpha(x_{i-1})) \\ &= \epsilon(\alpha(b) - \alpha(a)). \end{aligned} \tag{7.4}$$

We can substitute ϵ, since f attains its max/min on each interval and by choice of δ, the difference is $< \epsilon$. ∎

Theorem 7.6. f monotone and α additionally continuous, then $f \in \mathcal{R}(\alpha)$.

Proof. Assume f is monotone increasing. Choose a partition P, so that $\alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b)-\alpha(a)}{n}$ (we can do this since α has IVP prop).

$$\begin{aligned} U(P) - L(P) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}) \big(\alpha(x_i) - \alpha(x_{i-1})\big) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum (f(x_i) - f(x_{i-1})) \\ &= \frac{(\alpha(b) - \alpha(a))(f(b) - f(a))}{n}. \end{aligned} \tag{7.5}$$

■

Theorem 7.7. f is continuous with the exception of a finite set $A \subset [a, b]$, α continuous at the points of A, then $f \in \mathcal{R}(\alpha)$.

Proof. Let $\epsilon > 0$. Let $M = \sup |f|$. Since α is continuous at the points of A, we may cover A by finitely many disjoint intervals $[u_i, v_i]$, so that the sum of $\alpha(v_j) - \alpha(u_j)$ is less than ϵ and every point of A is interior to some $[u_i, v_i]$. Removing the (u_i, v_j) from $[a, b]$, we are left with a compact set, on which f is uniformly continuous. Choose $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$, $x, y \in [a, b] \backslash \cup (u_i, v_i)$. We form a partition $P = \{x_0, x_1, \ldots, x_n\}$ where the $u_j, v_j \in P$, no point of any (u_i, v_i) is in P and if x_{i-1} is not some u_j, then $\Delta(x_i) < \delta$. ■

Proposition 7.8.

1) $f \in \mathcal{R}(\alpha)$; g continuous, $g \circ f$ well-defined, then $g \circ f \in \mathcal{R}(\alpha)$.
2) $f_1, f_2 \in \mathcal{R}(\alpha)$; $\lambda_1, \lambda_2 \in \mathbb{R}$; then $\lambda_1 f_1 + \lambda_2 f_2 \in \mathcal{R}(\alpha)$ and
$$\textstyle\int (\lambda_1 f_1 + \lambda_2 f_2) \mathrm{d}\alpha = \lambda_1 \int f \mathrm{d}\alpha + \lambda_2 \int d \mathrm{d}\alpha.$$
3) $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$; $\lambda_1, \lambda_2 > 0$; then
$$\textstyle\int f \mathrm{d}(\lambda_1 \alpha_1 + \lambda_2 \alpha_2) = \lambda_1 \int f \mathrm{d}\alpha_1 + \lambda_2 \int f \mathrm{d}\alpha_2.$$
4) $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $f \mathcal{R}(\alpha)$ on $[a, c]$ and on $[c, b]$, for any $c \in (a, b)$ and
$$\textstyle\int_a^b f \mathrm{d}\alpha = \int_a^c f \mathrm{d}\alpha + \int_c^b f \mathrm{d}\alpha.$$
5) $f, g \in \mathcal{R}(\alpha)$ then $fg \in \mathcal{R}(\alpha)$.
6) $f_1 < f_2$, then $\int f_1 \mathrm{d}\alpha \leq \int f_2 \mathrm{d}\alpha$ and $\left|\int_a^b f \mathrm{d}\alpha\right| \leq \int_a^b |f| \mathrm{d}\alpha$.

Theorem 7.9.

1) Let $\alpha = \begin{cases} 0 & a \leq x < s \\ 1 & s \leq x \leq b \end{cases}$. Let f be bounded on $[a, b]$ and continuous as s. Then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f \mathrm{d}\alpha = f(s)$.

2) Let α be differentiable with $\alpha' \in \mathcal{R}$ on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$ and
$$\int_a^b f \mathrm{d}\alpha = \int_a^b f\alpha' \mathrm{d}x.$$

3) (Change of variable). $\varphi : [A, B] \to [a, b]$, monotone increasing and bijective. If $f \in \mathcal{R}(\alpha)$, then $f \circ \varphi \in \mathcal{R}(\alpha \circ \varphi)$ and
$$\int_a^b f \mathrm{d}\alpha = \int_A^B f \circ \varphi \mathrm{d}(\alpha \circ \varphi).$$

Theorem 7.10. Let f be a real-valued function on $[a, b]$, Riemann integratable. Define $F : [a, b] \to \mathbb{R}$ be $F(x) = \int_a^x f(t)\mathrm{d}t$. Then

1) F is continuous;

2) if f is continuous at x_0, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof.
$$|F(x) - F(y)| \leq \int_{\min\{x,y\}}^{\max\{x,y\}} |f(t)|\mathrm{d}t \leq \sup_{[a,b]} |f||x - y|.$$

Using the substitution $t = x_0 + hs$, $s \in [0, 1]$, we have
$$\left|\frac{\int_{x_0}^{x_0+h} f(t)\mathrm{d}t}{h} - f(x_0)\right| = \left|\frac{\int_0^1 f(t)h\mathrm{d}s}{h} - f(x_0)\right| = \left|\int_0^1 (f(x_0 + hs) - f(x_0))\mathrm{d}s\right|$$
$$\leq \int_0^1 |f(x_0 + hs) - f(x_0)|\mathrm{d}s.$$

By continuity, for h sufficiently small, the RHS can be made less than ϵ. ■

See Rudin for other statement of FToC as well as integration by parts.

8 Sequences of Functions

Definition 8.1. Let $(f_n)_{n\geq1}$ be a sequence of functions $E \subset \mathbb{R} \to \mathbb{R}$. Suppose for all $x \in E$ that $(f_n(x))_{n\geq1}$ converges. Write $f(x) = \lim_{n\to\infty} f_n(x)$. We say f_n converges pointwise to f on E.

Definition 8.2. We say $(f_n)_{n\geq1}$ converges uniformly on E to f if for every $\epsilon > 0$, there exists $N \geq 1$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

Proposition 8.3. $(f_n)_{n\geq1}$ converges uniformly on E if and only if for all $\epsilon > 0$, there exist $N \geq 1$

such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in E$.

Proposition 8.4. Suppose $f_n \xrightarrow[E]{\text{pointwise}} f$. Let $M_n = \sup_E |f_n(x) - f(x)|$. Then $f_n \xrightarrow[E]{\text{unif}} f$ if and only if $M_n \to 0$ as $n \to \infty$.

Proposition 8.5. Suppose $|f_n(x)| \leq M_n$ for all $x \in E$, $n \geq 1$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof. Apply Cauchy criterion with proposition 8.3. ∎

Theorem 8.6. Suppose $f_n \xrightarrow[E]{\text{unif}} f$. Let $x \in E'$. Suppose $\lim_{t \to x} f_n(t) = A_n$. Then (A_n) converges and $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

Proof. Let $\epsilon > 0$. By uniform convergence, there exists N such that for $m, n \geq N$, $t \in E$,

$$|f_n(t) - f_m(t)| < \epsilon.$$

Taking $t \to x$, we obtain that (A_n) is Cauchy, hence convergent.

Now

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

For t sufficiently close to x, $|f_n(t) - A_n| < \epsilon/3$. For n sufficiently large, $|f(t) - f_n(t)| < \epsilon/3$, by unif. conv. and $|A_n - A| < \epsilon/3$, by convergence. Hence $|f(t) - A| < \epsilon$. ∎

Corollary 8.6.1. If $(f_n)_{n \geq 1}$ are continuous on E and $f_n \xrightarrow[E]{\text{unif}} f$, then f is continuous on E.

Proof. Let $x \in E \cap E'$. By theorem 8.6, $\lim\limits_{y \to x} f(x) = \lim\limits_{n \to \infty} \lim\limits_{y \to x} f_n(y) = \lim\limits_{n \to \infty} f_n(x) = f(x)$. ∎

Theorem 8.7. Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and $f_n \xrightarrow[{[a,b]}]{\text{unif}} f$. Then $f \in \mathcal{R}(\alpha)$ and

$$\int_a^b f \mathrm{d}\alpha = \lim_{n \to \infty} \int_a^b f_n \mathrm{d}\alpha.$$

Proof. Let $I = [a, b]$. Let $m_n = \sup_I |f_n(x) - f(x)|$. So $f_n - m_n \leq f \leq f_n + m_n$. Hence

$$\textstyle\int_I (f_n - m_n) \leq \int_- f \leq \int^- f \leq \int_I (f_n + m_n). \tag{8.1}$$

So the difference between the upper and lower integrals is at most $2m_n(\alpha(b) - \alpha(a))$. But $m_n \to 0$ as $n \to \infty$. Hence the upper/lower integrals are equal, i.e. $f \in \mathcal{R}(\alpha)$. By applying (8.1) again,

$$\left| \int_I f \mathrm{d}\alpha - \int f_n \mathrm{d}\alpha \right| \leq m_n(\alpha(b) - \alpha(a)).$$

∎

Corollary 8.7.1. If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f(x) = \sum f_n(x)$ converges uniformly on $[a,b]$, then $\int_a^b f \mathrm{d}\alpha = \sum \int_a^b f_n \mathrm{d}\alpha$.

Theorem 8.8. Suppose $(f_n)_{n\geq 1}$ are differentiable on $[a,b]$ and $(f_n(x_0))$ converges for some $x_0 \in [a,b]$. If (f_n') converges uniformly on $[a,b]$, then $f_n \xrightarrow[{[a,b]}]{\text{unif}} f$, then

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Proof. Let $\epsilon > 0$. Choose N so that for all $m,n \geq N$, $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$ and $|f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)}$ for $t \in [a,b]$. Applying the MVT, to $f_n - f_m$,

$$|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| \leq \frac{|x-t|\epsilon}{2(b-a)} \leq \frac{\epsilon}{2} \tag{8.2}$$

for any $x,t \in E$ and $m,n \geq N$. Now using the inequality

$$|(f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|,$$

we have $|(f_n(x) - f_m(x)| < \epsilon$ for all $m,n \geq N$, so that f_n converges uniformly on $[a,b]$. Put $f(x) = \lim_{n\to\infty} f_n(x)$. Fix $x \in [a,b]$ and set

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x} \qquad \phi(t) = \frac{f(t) - f(x)}{t-x} \tag{8.3}$$

$t \in [a,b]$, $t \neq x$. By assumption, $\lim_{t\to x} \phi_n(t) = f_n'(x)$. By (8.2),

$$|\phi_n(t) - \phi_n(x))| \leq \frac{\epsilon}{2(b-a)}$$

so (ϕ_n) converges uniformly for $t \neq x$. Since f_n converges uniformly to f, we note

$$\lim_{n\to\infty} \phi_n(t) = \phi(t)$$

for $t \in [a,b]$, $t \neq x$. By theorem 8.6,

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x).$$

∎

Theorem 8.9 (Dini). Let $(f_n)_{n\geq 1}, f : E \subseteq \mathbb{R} \to \mathbb{R}$ with E compact.

1) $(f_n)_{n\geq 1}, f$ are continuous;
2) $f_n \xrightarrow[E]{\text{point}} f$;
3) (f_n) is monotone decreasing, i.e. $f_n(x) \geq f_{n+1}(x)$ for all $x \in E$.

Then $f_n \xrightarrow[E]{\text{unif}} f$;

Proof. See Rudin chapter 7. ■

Theorem 8.10 (Weierstrass). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists a sequence $(P_n)_{n\geq 1}$ of polynomials in $\mathbb{R}[x]$ such that $P_n \xrightarrow[{[a,b]}]{\text{unif}} f$.

In particular,

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

These are called the *Bernstein polynomials*.

9 Convexity

Definition Let f be a real-valued function on a set I. We say f is *convex* if

$$f(\lambda x + (1 - \lambda y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x < y$, $x, y \in I$ and $\lambda \in [0, 1]$.

Theorem 9.1. If f is a convex function on I and $x < y < z$, then

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Proof. Write

$$y = \frac{z-y}{z-x} x + \frac{y-x}{z-x} z.$$

Let $\lambda = \frac{z-y}{z-x}$, so $1-\lambda = \frac{y-x}{z-x}$ and $\lambda \in (0, 1)$. Then by convexity, we have $f(y) \leq \lambda f(x) + (1-\lambda) f(z)$. Hence

$$f(y) \leq \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z).$$

Rearrange (add $-f(z)(z - x)$ to both sides) gives the RHS of the desired inequality. To get the LHS we initially negate the above inequality (add $f(x)(z - x)$ to both sides) and rearrange. ■

Theorem 9.2. f convex on an open interval implies f continuous.

Proof. Let $I = (a, b)$. Let $a < u < v < w < s < b$. We observe the following inequalities:

$$f(v) \leq f(u) + \tfrac{f(w)-f(u)}{w-u}(v - u) \tag{9.1}$$

$$f(w) \leq f(v) + \tfrac{f(s)-f(w)}{s-w}(w - v) \tag{9.2}$$

Upon rearranging,

$$\frac{f(v)-f(u)}{v-u}(w-u)+f(u)\leq f(w)\leq f(v)+\frac{f(s)-f(v)}{s-v}(w-v).$$

Equivalently,

$$f(v)\left(\frac{w-u}{v-u}\right)+f(u)\left(1-\frac{w-u}{v-u}\right)\leq f(w)\leq f(v)+(f(s)-f(v))\frac{w-v}{s-v}.$$

Let $(v_n)_{n\geq 1}\subset(u,w)$ be a sequence converging to w on the left. Then by squeeze theorem, we have $\lim_{n\to\infty}f(v_n)=f(w)$. But (v_n) was an arbitrary, hence $\lim_{x\to w^-}f(x)=f(w)$. On the other hand, using the secant lines from v to s and from w to t, we can bound $f(s)$ and then take (s_n) to approach w on the right. ∎

Theorem 9.3. f convex, twice differentiable on an open interval if and only if $f''(x)\geq 0$. (It can also be shown f convex, differentiable is equivalent to f' increasing).