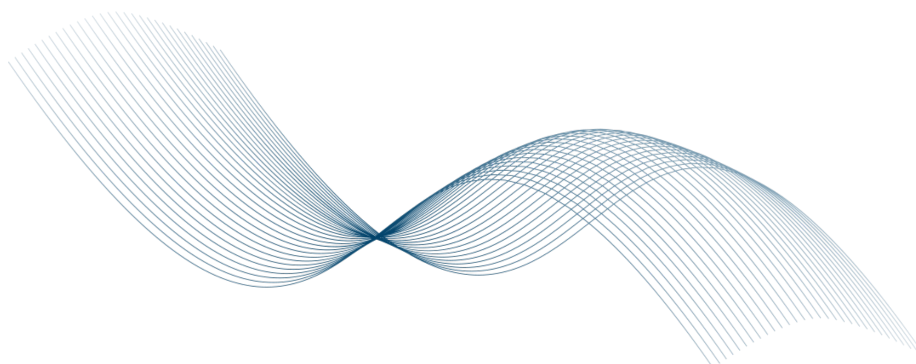
Differential Equations

Spring 2017



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1 Introduction

Differential equations are equations involving the derivatives of a function or functions. A solution to a differential equation is therefore, a function which satisfies the given equality. In this section, we'll discuss some important terminology and theorems needed to solve basic differential equations.

Definition (Order). The differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(x) \tag{1.1}$$

where a_i are functions of x and a_0 isn't identically zero is a linear differential equation of order n.

Theorem 1.1

Suppose $y_1, y_2, \ldots, y_n$ are distinct solutions to a differential equation, G. Then, any linear combination of the y_i is also a solution to G. That is, the function $c_1 y_1 + \cdots + c_n y_n$, where $c_1, \ldots, c_n \in \mathbb{R}$, also satisfies G.

Proof. This follows from the additivity and homogeneity of the differential operator. ∎

Definition (General Solution). A general solution to a differential equation of order n, is a function y containing parameters $c_1, \ldots, c_n$, from which any solution can be determined by specifying the parameters.

Example. The differential equation, $y = y'$, is a first order linear differential equation. Furthermore, it has general solutions in the form: $y = ce^t$, where $c \in \mathbb{R}$. We'll see how to find these solutions later on.

Often, we'll be given certain initial conditions when modeling with differential equations. For instance, when modeling projectile motion we typically have access to conditions such as the projectile's initial velocity and position. The following theorem, states that in order to find a specific solution to a n-th order differential equation, we first need to specify n initial conditions.

Theorem 1.2

Let $a_0, \ldots, a_n$ be continuous on the interval I. Then for $x_0 \in I$, the equation $(\star)$ with initial conditions

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_n. \end{aligned}$$

has a unique solution on I.

The following theorem guarantees the existence and uniqueness of a solution to the initial value problem, like the ones specified above.

Theorem 1.3

Let $f(x, y)$ be continuous on a rectangle $R = \{(x, y)|a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$. Suppose $\frac{\partial f}{\partial y}$ is continuous on R. Then, for any interior point $(x_0, y_0) \in R$ there exists an interval I containing x_0 as an interior point such that $y' = f(x, y)$, with initial condition $y(x_0) = y_0$, has a unique solution for $x \in I$.

We can develop a graphic representation of a first-order differential equation by using a slope field. Furthermore, there are handy techniques to help identify a slope field, given possible equations.

(i) Set $y = 0$ to look at how the derivative behaves along the x-axis, and vice versa.

(ii) The curve in the plane defined by setting $y' = 0$ should correspond to points where the slope is 0.

(iii) Additionally, setting $y' = c$, a non-zero constant gives the curve of points where the slope is that constant. The term for these are isoclines.

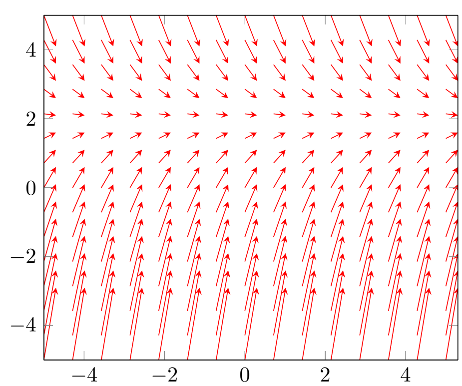

Figure 1: Direction field of $y' = 2 - y$

2 Methods of Solution

2.1 Separation of Variables

We can apply the method of separation of variables to an equations of the form, $u(x) = w(x)y'$. First, assuming $w \neq 0$, divide both sides by w. Then integrate to find general solutions.

> **Example.** Recall the differential equation from example 1.1, this is $y = y'$. Alternatively, we can write, $y' = \frac{dy}{dx}$. Separating the variables we obtain, $dx = \frac{dy}{y}$. Then,
>
> $$\begin{aligned} \int \frac{dy}{y} &= \int dx \\ \ln|y| &= x + c \\ y &= e^{x+c} \\ y &= Ce^x, \end{aligned}$$
>
> as expected.

2.2 Integrating Factors

We can find solutions to first order linear differential equations of the form: $a(x)y' + b(x)y = r(x)$, using the integrating factors method. The algorithm is as follows. First, divide both sides by $a(x)$ to obtain an equation of the form, $y' + p(x)y = q(x)$.

Then, try to find a function $I(x)$ such that if both sides are multiplied by I the LHS simplifies to $\frac{d}{dx}(yI(x))$. Notice that by the product rule, we must have $I'(x) = p(x)I(x)$; therefore, we should choose $I(x) = e^{\int p(x)dx}$. Now, once our differential equation is in the aforementioned form we can attempt to integrate both sides, then dividing both sides of the equation by $I(x)$ to isolate y.

> **Example.** Consider $\frac{dy}{dx} + xy = 5x$. Multiplying by $I(x)$ we obtain, $I(x)\frac{dy}{dx} + I(x)xy = I(x)5x$. Therefore, $I(x) = e^{x^2/2}$, so our equation simplifies to $\frac{d}{dx}(e^{x^2/2}y) = 5xe^{x^2/2}$. Integrating both sides, then isolating y, we have our generic solution, $y = 5 + Ce^{-x^2/2}$.

3 Modeling

> **Example.** Let r_1 be the inflow rate in lt/min into a vat of a chemical mixture with concentration c_1. Assume the concentration in the vat is always

homogeneous. Let r_2 be the outflow rate. Suppose, initially, the vat has v_0 liters of fluid with A_0 grams of the chemical. Determine $c_2(t) = \frac{A(t)}{V(t)}$ the concentration of the chemical at time t.

Solution. Note $\frac{dV}{dt} = r_1 - r_2$, thus $V(t) = (r_1 - r_2)t + v_0$. Next, $\frac{dA}{dt} = r_1c_2 - r_2c_2(t) = r_1c_1 - r_2\frac{A(t)}{(r_1-r_2)t+v_0}$. Notice that we should use integrating factors.

We choose

$$I(x) = \exp\left(\frac{r_2}{r_1 - r_2}\ln|(r_1 - r_2)t + v_0|\right) = |(r_1 - r_2)t + v_0|^{\frac{r_2}{r_1-r_2}}.$$

Thus our equation simplifies to $\frac{d}{dt}(I(t)A(t)) = r_1c_1I(t)$. Then we integrate both sides to obtain $A(t)I(t) = c_1((r_1 - r_2)t + v_0)^{\frac{r_1}{r_1-r_2}} + K$. Thus, $A(t) = c_1((r_1 - r_2)t + v_0) + \frac{K}{I(t)}$. Then, we can use our initial condition to determine K, and finalize our solution.

4 Eigenvalues

Definition (Linear Transformation). Let V and W be vector spaces over a field $\mathbb{F}$. A function, $T : V \to W$ is linear if

- $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
- $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{F}$.

Definition (Eigenvalues & Eigenvectors). Let T be a linear transformation. An eigenvector, is vector v such that $Tv = \lambda v$ for some scalar λ, called the eigenvalue.

Eigenvectors are special because they are vectors which are simply scaled under the linear transformation, and thus are parallel to the original vector. Suppose T is a linear mapping with representative matrix A. If v is an eigenvector of T, then $Av = \lambda v = \lambda Iv$ for some scalar λ. Therefore, we have $(A - \lambda I)v = 0$. If the matrix $A - \lambda I$ is invertible then its null space is trivial and we have no eigenvectors. Thus, we must solve for λ such that $A - \lambda I$ is singular.

Definition (Characteristic Polynomial). The characteristic polynomial of a linear transformation T with representative matrix A, is defined to be

$$p_A(z) = \det(zI - A) = \prod(z - \lambda_i)^{\alpha_i}$$

where λ_i are the roots of p_A with multiplicities α_i.

Hence by finding the roots of the characteristic polynomial of a linear transformation, we obtain the eigenvalues of that linear transformation. Using these eigenvalues, we can then solve for the eigenvectors.

Definition (Eigenspace). Let $A \in M_n(\mathbb{R})$. For any eigenvalue λ_i of A define the eigenspace of λ_i to be

$$E_i = \{\vec{v} \in \mathbb{C}^n | A\vec{v} = \lambda_i \vec{v}\} = \text{null}(A - \lambda_i I).$$

The dimension of E_i is the geometric multiplicity of λ_i.

Theorem 4.1

Let λ_i be an eigenvalue of A with multiplicity m_i and corresponding eigenspace E_i. Then

(i) E_i is a subspace of $\mathbb{C}^n$.

(ii) $\dim(E_i) \leqslant m_i$.

We say a matrix $A \in M_n(\mathbb{R})$ if it has n linearly independent eigenvectors, otherwise it is defective. These vectors correspond to an eigenbasis of $\mathbb{C}^n$. Equivalently, A is non-defective if it has n distinct eigenvalues or the geometric multiplicity of each eigenvalue equals its algebraic multiplicity.

5 Linear Differential Equations

Definition (Auxillary Equation). Given linear homogeneous differential equation with constant coefficients

$$a_0 y^{(n)} + \cdots + a_{n-1} y' + a_n y = 0, \tag{5.1}$$

it can be seen that e^{rx} is a solution to (5.1) if r is a root of

$$a_0 r^n + \cdots + a_{n-1} r + a_n = 0. \tag{5.2}$$

We call (5.2) the auxiliary equation to (5.1).

Note that if the root r_i of the auxiliary equation has multiplicity m_i, then $e^{r_i x}, xe^{r_i x}, \ldots, x^{m_i - 1} e^{r_i x}$ are solutions to (5.1). And if $r_j = a + bi$ is a root with $b \neq 0$, then $e^{ax}(\cos(bx) + \sin(bx))$ is a solution.

5.1 Non-homogeneous Differential Equations

This is a differential equation of the form

$$Ly = a_0 y^{(n)} + \cdots + a_{n-1} y' + a_n y = F(x). \tag{5.3}$$

To solve these types of differential equations, we find the solution set to the corresponding homogeneous differential equation and to that, add a particular solution, y_p, of (5.3). In particular, y_p can be found using the method of annihilators.

5.1.1 Annihilators

The annihilator of $F(x)$ in (5.3) is the minimal polynomial $P(D)$ such that $P(D)F(x) = 0$. Then, the particular solution y_p satisfies $Ly_p = F(x)$ and $P(D)y_p = 0$.

If $A_1(D)$ is the annihilator of $p(x)$ and $A_2(D)$ is the annihilator of $q(x)$, then $A_1(D)A_2(D)$ is the annihilator of $p(x)q(x)$. See Table 1 for useful annihilators to common functions. Note that $p(x)$ is assumed to be a polynomial function of degree k.

Function	Annihilator
x^k	D^{k+1}
Ae^{rx}	$D - r$
$A\cos\omega x + B\sin\omega x$	$D^2 + \omega^2$
$p(x)e^{rx}$	$(D - r)^{k+1}$

Table 1: List of Annihilators.

5.2 Oscillations

Recall that Hooke's law states that the restoring spring force, F_s is proportional to displacement from the equilibrium position L_0, i.e. $F_s = -kx$. Furthermore, the air resistance is proportional to the velocity of the oscillating object.

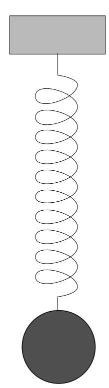

Figure 2: Spring with length l with mass m.

By Newton's 2nd law we have that $mx''(t) = -cx'(t) - kx(t) + F(t)$, where $F(t)$ is some external force. Rearranging, we derive a constant coefficient nonhomogeneous linear differential equation describing the motion of the oscillating system.

$$x''(t) + \frac{c}{m}x'(t) + \frac{k}{m}x(t) = \frac{1}{m}F(t) \tag{5.4}$$

5.2.1 Free Oscillations, No damping.

This implies that $F(t) = 0$ and $c = 0$. So our equation reduces to $x''(t)+\frac{k}{m}x(t) = 0$, and $x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$. Or more simply,

$$x(t) = \sqrt{c_1^2 + c_2^2}\cos\left(\sqrt{\frac{k}{m}}t + \theta\right)$$

where $\theta = \arctan(c_1/c_2)$. Furthermore, it has period $T = 2\pi\sqrt{\frac{m}{k}}$.

5.2.2 Forced Oscillations

Suppose an external force, $F(t) = F_0\cos(\omega t)$ acts on an oscillating system with no damping. Then, $x''(t)+\frac{k}{m}y = F(t)$. Since the annihilator of $F(t)$ is $D^2+\omega^2$, we need to solve, $(D^2 + \omega^2)(D^2 + \frac{k}{m}) = 0$.

- Case I. ($\omega^2 \neq \frac{k}{m}$) The basis of our solutions set is therefore: $\cos(\sqrt{\frac{k}{m}}t), \sin(\sqrt{\frac{k}{m}}t), \alpha\cos(\omega t)$, where $\alpha(\frac{k}{m} - \omega^2) = F_0$.
- Case II. ($\omega^2 = \frac{k}{m}$) Note that this means that the angular frequency is equal to the resonant frequency. In this case, our basis is $\cos(\sqrt{\frac{k}{m}}t), \sin(\sqrt{\frac{k}{m}}t), \alpha t\cos(\sqrt{\frac{k}{m}}t)$, where $\alpha = \frac{1}{2}F_0\sqrt{\frac{m}{k}}$.

5.2.3 Damping

Similarly, we can derive a basis for a solution set where damping exists, i.e. where air resistance is not negligible.

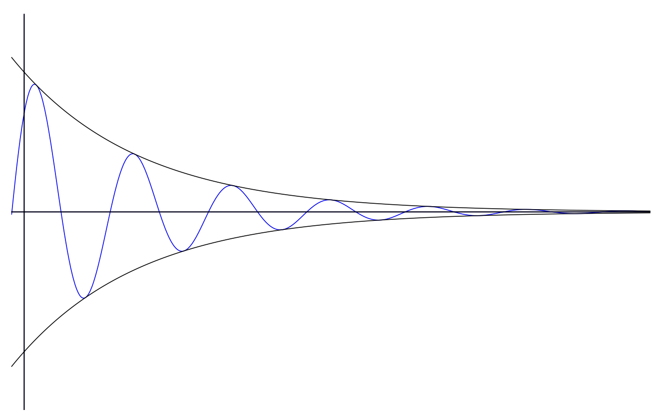

Figure 3: Dampened sinusoidal wave: $y(t) = e^{-t}(\sin(8t) + \cos(8t))$.

6 Systems of Differential Equations

Example. Convert $y'' + 2y' + 3y = 0$ to a system of differential equations.

Solution. Let $z = y'$, then

$$y' = z$$
$$z' = -2z - 3y$$

Or equivalently, in matrix notation,

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{6.1}$$

Definition (System of Differential Equations). A general system of n differential equations is given by

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) + b_1(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) + b_n(t) \end{aligned} \tag{6.2}$$

This can be rewritten in matrix notation as $\vec{x}'(t) = A\vec{x}(t) + \vec{b}(t)$ as in (6.1).

Note that (6.2) is homogeneous if and only if $b_1(t) = \cdots = b_n(t) = 0$. Solving homogeneous systems is rather straightforward.

Example. The system

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1 - 2x_2$$

can be rewritten as $(D-1)x_1 - 2x_2 = 0$ and $-2x_1 + (D-2)x_2 = 0$. To solve the system, we can start by eliminating x_1 to get $\frac{1}{2}[(D-1)(D+2)-2]x_2 = 0$. Thus, $x_2 = c_1e^{2t} + c_2e^{-3t}$. Then, substituting back into our original system we can find x_1.

Definition. The set of all n-dimensional vector functions on an interval I is denoted $V_n(I)$. Note that this set is a vector space.

Proposition 6.1. The basis of the solution set to the linear system, $\vec{x}'(t) = A\vec{x}(t)$ is $\{e^{\lambda_1}\vec{v}_1, \ldots, e^{\lambda_n}\vec{v}_n\}$ where λ_i are eigenvalues of A with associated eigenvectors v_i.

If $\vec{x}(t) = e^{\lambda t}\vec{v}$, where λ is an eigenvalue of the matrix A, with eigenvector v, then $\vec{x}'(t) = \lambda e^{\lambda t}\vec{v} = \lambda\vec{x}(t)$, and

$$\begin{aligned} A\vec{x}(t) &= Ae^{\lambda t}\vec{v} \\ &= e^{\lambda t}A\vec{v} \\ &= \lambda x(t). \end{aligned}$$

Hence, $\vec{x}'(t) = A\vec{x}(t)$.

Theorem 6.2

Consider the system

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{b}(t) \tag{6.3}$$

If $A(t)$ and $\vec{b}(t)$ are continuous on I, then (6.3) has a unique solution on I.

Theorem 6.3

The set of all solutions $\vec{x}(t)$ to

$$\vec{x}'(t) = A(t)\vec{x}(t) \tag{6.4}$$

where $A(t)$ is continuous on an interval I, is an n-dimensional vector space.

Proof. Let S be the set of solutions to (6.4). Clearly, S is nonempty since $\vec{x}(t) = \vec{0}$ is a solution. Let $c \in \mathbb{R}$ and $\vec{x}(t)$ be a solution, then multiplying both sides of (6.4) by c we have $(c\vec{x}(t))' = A(t)(c\vec{x}(t))$. Furthermore, sums of solutions are also solutions.

The proof that S is n-dimensional is left to the reader. ∎

Proposition 6.4. Let $u(t)$ and $v(t)$ be real-valued functions. If

$$\begin{aligned} w_1(t) &= u(t) + iv(t) \\ w_2(t) &= u(t) - iv(t) \end{aligned}$$

are solutions to (6.4), then u and v are solutions to (6.4).

This is obvious because u and v are a linear combination of w_1, w_2, or explicitly

$$\begin{aligned} u(t) &= \tfrac{1}{2}\left(w_1(t) + w_2(t)\right) \\ v(t) &= \tfrac{1}{2i}\left(w_1(t) - w_2(t)\right). \end{aligned}$$

Example. Solve the following system.

$$\vec{x}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{x}(t).$$

Upon inspection we determine that the coefficient matrix has eigenvalues $\lambda_1 = i, \lambda_2 = -i$ and eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Thus our general solution is of the form

$$\vec{x}(t) = c_1 e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Then applying Proposition 6.4, we can decomplexify our answer to obtain

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix},$$

which does indeed satisfy our original system.