Complex Analysis

Spring 2019

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1 Complex-Valued Functions

Let $i = \sqrt{-1}$. We define the set $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. Let $z = a + bi$ be a complex number. We define the norm or modulus of z, denoted $|z|$, to be the quantity $\sqrt{a^2 + b^2}$. The complex conjugate of z is $\overline{z} = a - bi$. Note $|z|^2 = z\overline{z}$. We write $\Re(z) = a$ to denote the real part of z and $\Im(z) = b$ to denote the imaginary part of z. The argument of z is $\arg(z) = \{\theta_0 + 2\pi k : k \in \mathbb{Z}\}$, where θ_0 satisfies $\sin\theta_0 = \frac{b}{|z|}$ and $\cos\theta_0 = \frac{a}{|z|}$. The principal argument of z, denoted $\mathrm{Arg}(z)$, is the angle $\theta \in \arg(z)$ which lies in the interval $[-\pi, \pi)$.

The complex exponential is defined as follows:

$$e^z = e^a(\cos(b) + i\sin(b)).$$

Many trigonometric identities can be derived from this definition, e.g. DeMoirve's formula: $(\cos x + i\sin x)^n = \cos nx + i\sin nx$.

The polar form of $z \neq 0$ is $z = re^{i\theta}$ where $\theta \in \arg(z)$ and $r = |z|$. The n-th roots of z are

$$\sqrt[n]{r}e^{i(\theta+2k\pi)/m}, \quad 0 \leq k \leq m-1.$$

The n-th roots of unity are denoted $\omega_i = e^{2\pi i/n}$ for $0 \leq i \leq n-1$.

Definition.

- The **open disc** of radius ρ centered at $z_0 \in \mathbb{C}$ is $\{z \in \mathbb{C} : |z - z_0| < \rho\}$.
- A set S is **open** if every point of S is in the interior of S.
- Let $w_1, \ldots, w_{n+1} \in \mathbb{C}$ and for $k = 1, \ldots, n$ let ℓ_k be the line segment connecting w_k and w_{k+1}. Then $\ell_1, \ldots, \ell_k$ for a **polygonal path** joining w_1 and w_{n+1}. An open set S is **connected** if every pair of points is joined by a polygon path that lies entirely in S.
- An open, connected set is a **domain**.
- Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is said to be a **boundary point** of S if every neighborhood of z_0 contains at least one point of S and one point of S^C.

- We say S is **closed** if it contains all of its boundary points.
- We say S is **bounded** if there exists $x \in S$ and $r > 0$ such that $|s - x| < r$ for all $s \in S$.
- A **region** is a domain with some, all, or none of its boundary.

2 Differentiability

Let f be a complex-valued function. Write $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, for $x, y \in \mathbb{R}$ and u, v are real-valued functions.

Definition. We say f is **analytic** in an open set G if it is differentiable at every point in G. Saying f is analytic at a point z_0 means f is analytic on an open neighborhood of z_0. If f is analytic on all of $\mathbb{C}$ then we say f is **entire**.

A point z_0 is a **singular point** if f is not analytic at z_0 but there exist a sequence $(z_n)_{n\geq 1}$ converging to z_0 such that f is analytic at every z_n.

2.1 Cauchy-Riemann Equations

Proposition 2.1. Differentiability at z_0 implies the Cauchy-Riemann equations hold at z_0.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{2.1}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.2}$$

Proof.

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta x, \Delta y \to 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0))}{\Delta x + i\Delta y} \tag{2.3}$$

Taking $\Delta y = 0$, the RHS of equation 2.1 becomes $\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)$. Taking $\Delta x = 0$, the RHS of equation 2.3 becomes $-i\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$. Equating the real and imaginary parts yields equations 2.1 and 2.2. ∎

Theorem 2.2

Let $f(z) = u(x, y) + iv(x, y)$ be a complex-valued function. Suppose the first order partial derivatives of u and v exist on an open neighborhood of z_0, are continuous at z_0, and the Cauchy-Riemann equations hold at z_0. Then f is differentiable at z_0.

Proof. Let

$$A = \underbrace{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)}_{A_1} + \underbrace{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}_{A_2}.$$

Define B analogously using v. Then $\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{\Delta z\to 0}\frac{A+iB}{\Delta z}$.

Since the partials exist on some open set G, by the MVT, there exists x^* between x_0 and $x_0+\Delta x$ such that

$$A_1=\frac{\partial u}{\partial x}(x^*,y_0+\Delta y)\Delta x.$$

As $\Delta z\to 0$ so do Δx and Δy. Hence $x^*\to x_0$. So by continuity at z_0,

$$A_1=\Delta x\left(\frac{\partial u}{\partial x}(x_0,y_0+\Delta y)+\epsilon_1\right)$$

where $\epsilon_1\to 0$ as $\Delta z\to 0$. We get analogous expressions for A_2,B_1,B_2. Then

$$\begin{aligned}\frac{A+iB}{\Delta z}&=\Delta x\frac{\left(\frac{\partial u}{\partial x}(x_0,y_0+\Delta y)+\epsilon_1+i\frac{\partial v}{\partial x}(x_0,y_0+\Delta y)+i\epsilon_3\right)}{\Delta z}\\&+\Delta y\frac{\left(\frac{\partial u}{\partial y}(x_0,y_0)+\epsilon_2+i\frac{\partial v}{\partial y}(x_0,y_0)+i\epsilon_4\right)}{\Delta z}.\end{aligned}\tag{2.4}$$

We may factor i out of the Δy term and apply CR equations to get

$$\begin{aligned}\frac{A+iB}{\Delta z}&=\Delta x\frac{\left(\frac{\partial u}{\partial x}(x_0,y_0+\Delta y)+\epsilon_1+i\frac{\partial v}{\partial x}(x_0,y_0+\Delta y)+i\epsilon_3\right)}{\Delta z}\\&+i\Delta y\frac{\left(i\frac{\partial v}{\partial x}(x_0,y_0)-i\epsilon_2+\frac{\partial u}{\partial x}(x_0,y_0)+\epsilon_4\right)}{\Delta z}.\end{aligned}\tag{2.5}$$

Note $\frac{\partial u}{\partial x}(x_0,y_0+\Delta y)\to\frac{\partial u}{\partial x}(x_0,y_0)$ as $\Delta z\to 0$ by the continuity of u at z_0. Similarly for v.

$$\frac{A+iB}{\Delta z}=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)+\frac{\Delta x(\epsilon_1+i\epsilon_3)+\Delta y(-i\epsilon_2+\epsilon_4)}{\Delta z}.\tag{2.6}$$

The rightmost term above clearly goes to 0 as $\Delta\to 0$. Hence the limit exists so f is differentiable at z_0 and $f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)$. By CR, we also have $f'(z_0)=\frac{\partial v}{\partial y}(x_0,y_0)-i\frac{\partial u}{\partial y}(x_0,y_0)$.

∎

Proposition 2.3. If f is analytic in a domain D and $f'\equiv 0$ on D then f is constant on D.

Proof. Since $f'\equiv 0$ on D, by CR

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\equiv 0.$$

Equating the real and imaginary parts gives $\frac{\partial u}{\partial x}\equiv 0$ and $\frac{\partial v}{\partial x}\equiv 0$. Similarly, $\frac{\partial u}{\partial y}\equiv\frac{\partial v}{\partial y}\equiv 0$. Thus u and v are constant. So $f=u+iv$ is constant on D. ∎

2.2 Harmonic Functions

Definition. A real-valued function $\phi(x,y)$ is **harmonic** in a domain D if all of its second order partials are continuous in D and ϕ satisfies

$$\nabla^2\phi:=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\equiv 0$$

on D.

Proposition 2.4. If f is analytic in a domain D then u and v are harmonic.

Proof. We use without proof that if f is analytic then u and v have continuous partials of all orders. Then since $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \equiv \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$, by CR

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \equiv -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right),$$

which shows v is harmonic. Showing u is harmonic is proved similarly. ∎

Definition. Given a harmonic function $u(x, y)$, we can construct a harmonic function $v(x, y)$ such that $u + iv$ is analytic. Such a function is called the **harmonic conjugate** of u.

Example. Let $u(x, y) = xy - x + y$. We want to produce $v(x, y)$ satisfying the CR equations. Thus

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y - 1.$$

Integrating with respect to y gives $v(x, y) = y^2/2 - y + \psi(x)$ where ψ is some real-valued function in x. Furthermore, $\psi'(x) = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x - 1$. Integrating both sides gives,

$$\psi(x) = -x^2 - x + c_0$$

for some $c_0 \in \mathbb{R}$. Then

$$v(x, y) = y^2/2 - y - x^2/2 - x + c_0.$$

It follows that $f = u + iv$ is entire.

3 Elementary Functions

Theorem 3.1

Every non-constant polynomial with complex coefficients has at least one root in $\mathbb{C}$.

Example. Rewrite the polynomial $p(z) = z^3 + 2z + i$ in terms of powers of $z - i$. *Hint: Use the Taylor series expansion centered at $z = i$.*

Proposition 3.2. If $R(z) = \frac{p(z)}{b_n(z-\beta_1)^{d_1}\ldots(z-\beta_r)^{d_r}}$, where the β_i are distinct *poles* of the rational function R and $p(z)$ is a polynomial in $\mathbb{C}[x]$, then R has a partial fraction decomposition into the form

$$R(z) = \frac{1}{b_n}\sum_{i=1}^{r}\sum_{j=1}^{d_i}\frac{A_{d_i-j}^{(i)}}{(z-\beta)^j}$$

where the $A_p^q \in \mathbb{C}$.

Example. Let $R(z) = \frac{4z+4}{z(z-1)(z-2)^2} = \frac{A_0^{(1)}}{z} + \frac{A_0^{(2)}}{z-1} + \frac{A_0^{(3)}}{(z-2)^2} + \frac{A_1^{(3)}}{z-2}$.

- To compute $A_0^{(1)}$, we multiply both sides by z and take $z = 0$. This gives $A_0^{(1)} = -1$.
- To compute $A_0^{(2)}$, we multiply both sides by $z-1$ and take $z = 1$. This gives $A_0^{(2)} = 8$.
- To compute $A_0^{(3)}$, we multiply both sides by $(z-2)^2$ and take $z = 2$. This gives $A_0^{(3)} = 6$.
- To compute $A_1^{(3)}$, we multiply both sides by $(z-2)^2$ and differentiate before taking $z = 2$. This gives $A_1^{(3)} = -7$.

3.1 Complex Exponential

Proposition 3.3.

a) $e^z = 1$ if and only if $z = 2\pi ik$, $k \in \mathbb{Z}$.

b) $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 + 2\pi ik$, $k \in \mathbb{Z}$.

Note exp is $2\pi i$ periodic.

3.2 Trig Functions

For $z \in \mathbb{C}$,

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}} \tag{3.1}$$

3.2.1 Hyperbolic Trig

$$\boxed{\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}} \tag{3.2}$$

Some basic identities:

- $\sin(iz) = i\sinh(z)$.
- $\cos(iz) = \cosh(z)$.
- $\cosh^2(z) - \sinh^2(z) = 1$.
- $\frac{\mathrm{d}\sinh(z)}{\mathrm{d}z} = \cosh(z)$ and $\frac{\mathrm{d}\cosh(z)}{\mathrm{d}z} = \sinh(z)$.

3.3 Complex Logarithm

If $z \neq 0$, we define $\log z$ to be the set

$$\{\log_{\mathbb{R}} |z| + i \arg(z)\} = \{\ln_{\mathbb{R}} |z| + i \operatorname{Arg}(z) + 2\pi i k\}$$

The **principal valued logarithm** is

$$\operatorname{Log} z = \log_{\mathbb{R}} |z| + i \operatorname{Arg}(z).$$

Proposition 3.4. $\operatorname{Log} z$ is analytic in the domain $D^* = \mathbb{C} \backslash \{\text{non-positive real axis}\}$. It is discontinuous on the non-positive real axis. The derivative of $\operatorname{Log} z$ is $\frac{1}{z}$ wherever it exists.

Corollary. $\operatorname{Re}(\operatorname{Log} z) = \operatorname{Log} |z|$ is harmonic for $z \neq 0$ and $\operatorname{Im}(\operatorname{Log} z) = \operatorname{Arg} z$ is harmonic in D^*.

Definition. $F(z)$ is a **branch** of a multi-valued function $f(z)$ in a domain D if $F(z)$ is single-valued and continuous in D and has the property that for each $z \in D$, the value of $F(z)$ is one of the values obtained by $f(z)$.

3.4 Powers

For any $\alpha \in \mathbb{C}$ and $z \neq 0$ we define $z^\alpha := e^{\alpha \log(z)}$.

- If α is an integer, z^α is single-valued.
- If $\alpha = \frac{m}{n}$, $(m, n) = 1$ is rational, then z^α is multi-valued. There are precisely n values of z^α.
- Otherwise, z^α has infinitely-many values.

Since e^z is entire and $\operatorname{Log} z$ is analytic on D^*. It follows that z^α is analytic on D^* and $\frac{\mathrm{d}z^\alpha}{\mathrm{d}z} = \frac{\alpha z^\alpha}{z}$.

3.5 Inverse Trig

Suppose $z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$. Then

$$e^{iw} - 2iz - e^{-iw} = 0.$$

Multiplying across by e^{iw} and putting $\chi = e^{iw}$, we have

$$\chi^2 - 2iz\chi - 1 = 0.$$

So

$$\chi = 1 \pm \sqrt{1 - z^2}.$$

Thus

$$\arcsin z = w = -i \log(iz \pm \sqrt{1 - z^2}).$$

Similarly,

- $\arccos z = -i \log(z + \sqrt{z^2 - 1})$.
- $\arctan z = \frac{1}{2} i \log\left(\frac{i+z}{i-z}\right)$.
- $\operatorname{arcsinh} z = \log(z + \sqrt{z^2 + 1})$.

4 Complex Integration

Definition. A **path** in $\mathbb{C}$ is a function $z : [a, b] \to \mathbb{C}$. For each t, $a \leq t \leq b$, we can write $z(t) = x(t) + iy(t) \leftrightarrow (x(t), y(t))$.

A subset $\gamma \subseteq \mathbb{C}$ is a **smooth arc** if it is the range of some continuous complex-valued function $z(t)$, $a \leq t \leq b$, such that

a) $z(t)$ is continuously differentiable on $[a, b]$;

b) $z'(t) \neq 0$ for any $t \in [a, b]$;

c) $z(t)$ is one-to-one on $[a, b]$.

A subset $\gamma \subseteq \mathbb{C}$ is a **smooth closed curve** if it is the range of some continuous complex-valued function $z(t)$, $a \leq t \leq b$, such that

a) $z(t)$ is continuously differentiable on $[a, b]$;

b) $z'(t) \neq 0$ for any $t \in [a, b]$;

c) $z(t)$ is one-to-one on $[a, b)$ and $z(a) = z(b)$ and $z'(a) = z'(b)$.

A **contour** Γ is either a single point or a finite sequence of directed smooth curves $(\gamma_1, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for $1 \leq k \leq n-1$. We write $\Gamma = \gamma_1 + \ldots + \gamma_n$. We say Γ is a **closed contour** if its initial and terminal points coincide. It is a **simple closed contour** if $z(t)$ is one-to-one on $(a, b]$.

Theorem 4.1

(Jordan Curve Theorem). Any simple closed contour separates the plane into two domains each having the contour as its boundary. One of then, the interior, is bounded, the other, the exterior, is unbounded.

Definition. We say Γ has **positive orientation** if the interior lies to the left of Γ (from the perspective of an observer walking on Γ in the direction of its orientation). Otherwise Γ has **negative orientation**.

Definition.

- Let $f(z)$ be defined on a directed smooth curve γ with initial point α and terminal point β. For $n \geq 1$, define a **partition** $\mathcal{P}_n$ of γ to be finitely-many points $z_0, \ldots, z_m$ such that $z_0 = \alpha$, $z_m = \beta$ and z_{k-1} precedes z_k for each $k = 1, \ldots, m$. The **mesh** of a partition $\mathcal{P}_n$, denoted $\mu(\cap P)$, is $\max_{1 \leq k \leq n} \text{arc-length}_\gamma(z_{k-1}, z_k)$.

- Let $\Delta z_k = z_k - z_{k-1}$. Choose $c_1, \ldots, c_m$ to be any points on γ such that c_k is on the arc from z_{k-1} to z_k. The **Riemann sum** of f corresponding to partition $\mathcal{P}_n$ is

$$S(\mathcal{P}_n) = \sum_{k=1}^{m} f(c_k)\Delta z_k.$$

We say f is **integratable** along γ if there exists $L \in \mathbb{C}$, such that for every sequence of partitions $(\cap P_i)$, we have

$$\lim_{n\to\infty} S(\mathcal{P}_n) = 0 \text{ whenever } \lim_{n\to\infty} \mu(\mathcal{P}_n) = 0.$$

We write

$$L = \int_\gamma f(z)\mathrm{d}z.$$

Basic properties from real-integrals also hold:

- $\int_\gamma (cf(z) \pm g(z))\mathrm{d}z = c\int_\gamma f(z)\mathrm{d}z \pm \int_\gamma g(z)\mathrm{d}z$.
- $\int_{-\gamma} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z$ where $-\gamma$ denotes γ with the reversed orientation.
- If f is continuous on γ, then f is integratable on γ.

Theorem 4.2

Let γ be a directed smooth curve, f be continuous along γ with admissible parameterization $z(t)$, $a \leq t \leq b$. Then

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

Proof idea.

- Given a partition $\mathcal{P}_n = \{z_0, \ldots, z_m\}$ of γ we can write $z_i = z(t_i)$ where $a = t_0 < t_1 < \ldots < t_m = b$ by the continuity of z.
- Since z is continuously differentiable on $[a, b]$, $\Delta z_k = z(t_k) - z(t_{k-1}) \approx z'(t_k)\Delta t_k$. Applying this approximation in the Riemann sum, gives the result.

Example. Let C be the circle $|z - z_0| = r$ with clockwise orientation. Let $f(z) = (z - z_0)^n$. Let $I_n = \int_C f(z)\mathrm{d}z$. For each $n \in \mathbb{Z}$, compute I_n.

Parameterize C via $\gamma(t) = re^{it} + z_0$, $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} I_n &= \int_0^{2\pi} (re^{it})^n \cdot ire^{it}\mathrm{d}t \\ &= ir^{n+1}\int_0^{2\pi} e^{it(n+1)} \\ &= 0, \text{ if } n \geq -1. \end{aligned} \tag{4.1}$$

If $n = -1$, then the $I_n = 2\pi i$.

Definition. Let Γ be a contour consisting of directed smooth curves $\gamma_1, \ldots, \gamma_n$ and let f be continuous on Γ. The **contour integral** of f along Γ is

$$\int_\gamma f(z)\mathrm{d}z = \sum_{i=1}^n \int_{\gamma_i} f(z)\mathrm{d}z.$$

Proposition 4.3. Let f be continuous on a smooth, directed curve γ and $|f(z)| \leq M$ for all $z \in \gamma$. Then

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq M|\gamma|.$$

4.1 Path Independence

Theorem 4.4

Suppose $f(z)$ is continuous in a domain D and has anti-derivative $F(z)$ throughout D. Then for any contour $\Gamma \subseteq D$ with initial point I and terminal point T, we have

$$\int_\gamma f(z)\mathrm{d}z = F(T) - F(I).$$

Corollary. Let be $f(z)$ as above and Γ be a closed contour. Then $\int_\gamma f(z)\mathrm{d}z = 0$.

Theorem 4.5

Let f be continuous in a domain D. The following are equivalent:

a) f has an antiderivative throughout D.

b) The integral of f around any loop in D is 0.

c) If Γ_1 and Γ_2 are any two contours in D with the same initial and terminal points, then $\int_{\Gamma_1} f(z)\mathrm{d}z = \int_{\Gamma_2} f(z)\mathrm{d}z$.

4.2 Cauchy's Theorem

Definition. The loop Γ_0 is **continuously deformable** (or homotopic) to the loop Γ_1 if there exists $z(s,t)$, continuous on the unit square, $0 \leq s, t \leq 1$ such that

- for each fixed s, $z(s,t)$ parameterizes a loop in D
- $z(0,t)$ parameterizes Γ_0 and $z(1,t)$ parameterizes Γ_1.

Observe that

- if $z(s,t)$ deforms Γ_0 into Γ_1, then $z(1-s,t)$ deforms Γ_1 into Γ_0.
- In a domain D, if Γ_0, Γ_1 can be deformed into points, then Γ_0 can be deformed into Γ_1.
- If Γ_0 can be deformed into a point, so can $-\Gamma_0$.

Definition. A domain D is **simply connected** if every loop in D can be continuously deformed into a point.

Theorem 4.6: Deformation Invariance

Let f be analytic in a domain D containing loops Γ_0, Γ_1. If $\Gamma_0 \simeq \Gamma_1$, then

$$\int_{\Gamma_0} f(z)\mathrm{d}z = \int_{\Gamma_1} f(z)\mathrm{d}z.$$

The book proves a weaker theorem which assumes f' is continuous (needed to apply Liebniz's rule) and the deformation from Γ_0 to Γ_1 has continuous 2nd order partial derivatives (so that the 2nd order partials are symmetric).

Corollary. (Cauchy's Theorem). If f is analytic in a simply connected domain D, then $\int_\Gamma f(z)\mathrm{d}z = 0$ for any loop $\Gamma \subseteq D$.

Proof. By deformation invariance, since Γ is homotopic to a point, the claim is immediate. ∎

Corollary. In a simply connected domain, an analytic function has an antiderivative. Thus, its contour integrals are path independent and its loop integrals vanish.

4.3 Cauchy's Integral Formula

Theorem 4.7

Let Γ be a simple, closed, positively oriented contour. If F is analytic in some simply connected domain D containing Γ and if z_0 is any point inside Γ, then

$$f(z_0) = \int_\Gamma \frac{1}{2\pi i}\frac{f(z)}{z - z_0}\mathrm{d}z.$$

Proof. The function $\frac{f(z)}{z-z_0}$ has a simple pole at z_0, and hence is analytic in $D\backslash\{z_0\}$. So we can deform Γ into a positively oriented circle $C_r : |z - z_0| = r$ contained in D. Thus

$$\int_\Gamma \frac{f(z)}{z - z_0}\mathrm{d}z = \int_{C_r} \frac{f(z_0)}{z - z_0}\mathrm{d}z + \int_{c_r} \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z = 2\pi i f(z_0) + \lim_{r\to 0}\int_{c_r} \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z.$$

Let $M_r = \max\{|f(z) - f(z_0) : x \in C_r\}$. By the continuity of f, $M_r \to 0$ as $z \to z_0$. Hence

$$\lim_{r\to 0}\left|\int_{c_r} \frac{f(z) - f(z_0)}{z - z_0}\mathrm{d}z\right| \leq \lim_{r\to 0}\frac{M_r}{r}(2\pi r) = 0.$$

∎

Remark. By rearranging CIF, we can evaluate contour integrals by evaluating f at a single point in Γ, assuming all hypothesis above are met.

Theorem 4.8

Let g be continuous on a contour Γ and for z not on Γ, set $G(z) = \int_\Gamma \frac{g(\zeta)}{(\zeta - z)^n}\mathrm{d}\zeta$, $n \geq 1$. Then $G(z)$ is analytic at each point not in Γ and

$$G'(z) = n\int_\Gamma \frac{g(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta.$$

Corollary. With the same setup as theorem 4.7, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta.$$

Corollary. If f is analytic in a domain D, then all of its antiderivatives exist and are analytic in D.

Corollary. If $f = u + iv$ is analytic in a domain D, all the partial derivatives of u, v exist and are continuous in D.

Theorem 4.9: Morera

If f is continuous in a domain D and the integral of f over any closed contour in D is 0, then f has an antiderivative F throughout D. Since F is analytic in D, F has derivatives of all orders, hence f is analytic and has analytic derivatives of all orders.

4.4 Bounds on Analytic Functions

Theorem 4.10

Let f be analytic inside and on a circle $C_r : |z - z_0| = r$. If $|f(z)| \leq M$ for all z on C_r, then

$$|f^{(n)}(z_0)| \leq \frac{n! \cdot M}{r^n}, \qquad n \geq 1.$$

Proof.

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\left|\int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\right| \leq \frac{n! \cdot M}{r^n}$$

∎

Theorem 4.11

The only bounded, entire functions are constant.

Proof. Suppose f is entire and $|f(z)| \leq M$ for all $z \in \mathbb{C}$. If $z_0 \in \mathbb{C}$, then for all $r > 0$, f is analytic in C_r are

$$f'(z_0) \leq \frac{M}{r}.$$

Taking $r \to \infty$ gives $f'(z_0) = 0$. ∎

Corollary. Every non-constant polynomial in $\mathbb{C}[X]$ has a root in $\mathbb{C}$.

Proof. Let $P(z) = a_n z^n + \ldots + a_0$, $a_n \neq 0$. Suppose P has no roots. Let $f(z) = \frac{1}{P(z)}$. Show f is bounded, hence constant. Thus P is constant. ∎

Suppose f is analytic in and on a circle $C_R : |z - z_0| = R$. Parameterizing $C_r : z(t) = z_0 + Re^{it}$, $0 \leq t \leq 2\pi$, by CIF,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) \mathrm{d}t.$$

That is, the value of f at the center of a disk, is the average value on its boundary.

Lemma. Suppose f is analytic in a disk centered at z_0 and the maximum value of the modulus of f over the disk is $|f(z_0)|$. Then $|f(z)|$ is constant in D.

Proof. Suppose f is analytic in D and the maximum modulus occurs at z_0 and $|f|$ isn't constant. Then there is $z_1 \in D$ such that $|f(z_1)| < |f(z_0)|$. Let C_r be the circle centered at z_0, through z_1. For all $z \in C_r$, $|f(z)| \leq |f(z_0)|$ and by continuity, $|f(z)| < |f(z_0)|$ holds in some section of C_r containing z_1. Hence

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| \mathrm{d}t < |f(z_0)|.$$

∎

Theorem 4.12: Maximum Modulus Principle

If F is analytic in any domain D and $|f(z)|$ achieves in maximum value at a point in D, then f is constant.

Proof. We'll show $|f|$ is constant in D, hence by analyticity, f is constant. Suppose $|f|$ weren't constant, say there is $z_1 \in D$ such that $|f(z_1)| < |f(z_0)|$. Let γ be a path from z_0 to z_1. There exists a first point w such that

a) For all z preceding w on γ, $|f(z)| = |f(z_0)|$.

b) There are y on γ arbitrarily close to w such that $|f(z)| < |f(z_0)|$.

Since f is continuous, $|f(w)| = |f(z_0)|$. Choose a disk centered at w in D. By (a), f has a maximum value on D at w. By the previous lemma, f is constant on D, contradicting (b). ∎

Corollary. If f is analytic in a bounded domain and continuous on its boundary, then $|f|$ attains its maximum value on the boundary.

5 Sequences and Series

Definition. Let $S_n = \sum_{j=0}^{m} c_j$. If S_n has a limit S as $n \to \infty$, we say the series $\sum_{j=0}^{\infty} c_j$ **converges** and we write $\sum_{j=0}^{\infty} c_j = S$. Otherwise, we say $\sum_{j=0}^{\infty} c_j$ **diverges**.

Proposition 5.1. Let $c \in \mathbb{C}$. The series $\sum_{j=0}^{\infty} c^j$ converges to $\frac{1}{1-c}$ if $|c| < 1$ and diverges otherwise.

Proposition 5.2.

a) If $\sum c_j$ converges, then $c_j \to 0$ as $j \to \infty$.

b) If $|c_j| \leq M_j$ for $j \geq J$, then if $\sum M_j$ converges, so does $\sum c_j$.

c) If $\lim_{j\to\infty} \left|\frac{c_{j+1}}{c_j}\right| = L$, then $\sum c_j$ converges if $L < 1$ and diverges if $L > 1$.

Definition. A sequence of function (f_n) defined on a set T **converges pointwise** to f if for all $z \in T$, $\lim_{n\to\infty} f_n(z) = f(z)$. We say (f_n) **converges uniformly** to f on T if for all $\epsilon > 0$, there exists $N \geq 0$ such that

$$|f_n(z) - f(z)| < \epsilon \qquad \text{for all } n \geq N, z \in T.$$

Definition. If f is analytic at z_0, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!}$$

is the **Taylor series** for f at z_0.

Theorem 5.3: Taylor

If f is analytic in a disk $D : |z - z_0| < R$, then the Taylor series for f around z_0 converges to f for all $z \in D$ and for any closed disk contained in D, this convergence if uniform.

Taylor series work naturally with function addition, subtraction and scalar multiplication. The Taylor series of the product of two analytic functions is the Cauchy product of their respective Taylor series.

Proposition 5.4. If f is analytic at z_0, the Taylor series for f' is given by the term-wise differentiation of the Taylor series for f. Both converge on the same disk.

Definition. A series of the form $\sum_{j=0}^{\infty} a_j z^j$ is a **power series** about z_0.

Proposition 5.5. For any power series, $\sum a_j z^j$, there is a real number R, $0 \leq R \leq \infty$ such that R depends only on the a_j and

a) The series converges if $|z - z_0| < R$;

b) Converges uniformly in any closed sub-disk of $|z - z_0| < R$;

c) If $|z - z_0| > R$, then the series diverges.

Lemma. Let (f_n) be a sequence of continuous function converging uniformly to f on $T \subseteq \mathbb{C}$. Then f is continuous on T.

Proposition 5.6. Let (f_n) be a sequence of continuous functions on a set $T \subseteq \mathbb{C}$ containing a contour Γ and suppose f_n converge uniformly to f on T. Then

$$\int_\Gamma f_n(z)\mathrm{d}z \xrightarrow{n\to\infty} \int_\Gamma f(z)\mathrm{d}z.$$

Proof. Let $\epsilon > 0$ and let $\ell = |\Gamma|$. By uniform convergence, there exists an N such that $|f_n(z) - f(z)| < \frac{\epsilon}{\ell}$ for all $n \geq N$ and z on Γ. Then $\left|\int_\Gamma f_n(z) - f(z)\mathrm{d}z\right| < \epsilon$. ∎

Theorem 5.7

Let (f_n) be a sequence of functions, analytic in a simply connected domain D and uniformly convergent to f in D. Then f is analytic in D.

Proof. Let Γ be a loop in D. Then

$$\int_\Gamma f(z)\mathrm{d}z = \int_\Gamma \lim_{n\to\infty} f_n(z)\mathrm{d}z = \lim_{n\to\infty}\int_\Gamma f_n(z)\mathrm{d}z = 0.$$

Note f is continuous, since f_n are continuous and uniformly convergent to f, so by Morera's theorem, we're done. ∎

Corollary. Applying this to $S_n = \sum_{j=0}^n a_j(z-z_0)^j$, we see that a power series converges to an analytic function inside its radius of convergence. In particular, if $S_n \to f$, then $a_j = f^{(j)}(z_0)/j!$. Moreover, for any contour Γ lies inside the disk of convergence, we can integrate f over Γ, by integrating the power series termwise.

5.1 Laurent Series

Definition. A series of the form $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$ is called a **Laurent series**.

Theorem 5.8

Let f be analytic in the annulus $r < |z-z_0| < R$. Then the Laurent series of f is given by:

$$\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$$

where

$$a_j = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{j+1}}\mathrm{d}\zeta$$

with C a positively oriented, simple, closed curve containing z_0, converges to f pointwise on the annulus. The convergence is uniform on any closed subannulus.

Proposition 5.9. Let $\sum_{k=-\infty}^{\infty} c_k(z-z_0)^k$ be any Laurent series such that

a) $\sum_{k=0}^{\infty} c_k(z-z_0)^k$ converges for $|z-z_0| < R$

b) $\sum_{k=0}^{\infty} c_{-k}(z-z_0)^{-k}$ converges for $|z-z_0| > r$

c) $r < R$,

then there exists a function f, analytic on $r < |z-z_0| < R$ whose Laurent series is given as above.

Example. Laurent series of $f(z) = \frac{1}{(z-1)(z-2)}$ in (a) $|z| < 1$; (b) $1 < |z| < 2$; (c) $|z| > 2$.

a) Note $f(z) = \frac{1}{z-2} - \frac{1}{z-1}$. Using standard expansions of geometric series the Laurent series is

$$\sum_{k=0}^{\infty}\left(1-\frac{1}{2^{k+1}}\right)z^k.$$

This is justified as the Taylor series for $g(z)=\frac{1}{z-2}$ converges to $g(z)$ for $|z|<2$ and the Taylor series for $h(z)=\frac{1}{z-1}$ converges to $h(z)$ for $|z|<1$.

b) Now, the Taylor series for $h(z)$ no longer converges to $h(z)$ in this domain. However, if we write $h(z)=\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$, then the Taylor series for $\frac{1}{1-\frac{1}{z}}$ converges to $\frac{1}{1-\frac{1}{z}}$ for $|z|>1$. Hence the Laurent series is

$$-\sum_{k=0}^{\infty}\left(\frac{z^k}{2^{k+1}}-\frac{1}{z^{k+1}}\right).$$

c) Applying a similar inversion technique as in (b), we have

$$\sum_{k=0}^{\infty}\frac{2^k-1}{z^{k+1}}.$$

5.2 Singularities & Zeros

Definition. A point z_0 is a **zero of order** m if f is analytic at z_0 and f and its first $m-1$ derivative vanish at z_0 but $f^{(m)}(z_0)\neq 0$. A zero of order 1 is a **simple zero**.

Observe that if f has a zero of order m at z_0, the Taylor series expansion of f at z_0 is $(z-z_0)^m\sum_{k=0}a_{k+m}(z-z_0)^k$. The series $\sum_{k=0}a_{k+m}(z-z_0)^k$ converges to an analytic function $g(z)$ wherever the original series converged to $f(z)$. So we can write $f(z)=(z-z_0)^m g(z)$ where g is analytic in some neighborhood of z_0.

Corollary. If f is analytic at z_0 such that $f(z_0)=0$, then either f is identically 0 in a neighborhood of z_0 or there is a punctured disk about z_0 in which f is nonzero.

Proof. Suppose $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ is the Taylor series of f about z_0, convergent to f on a circular neighborhood N of z_0. If f is non-zero in N, then there is some smallest m such that $a_m\neq 0$ but $a_j=0$ for $j<m$. We can write $f(z)=(z-z_0)^m g(z)$ where $g(z_0)=a_m\neq 0$ and is analytic at z_0. Since $g(z_0)$ is nonzero, by continuity, there is some neighborhood of z_0 contained in N on which g is nonzero, i.e. f has no other zeros in this neighborhood. ■

Definition. Let f have an isolated singularity at z_0 and let

$$\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$$

be the Laurent series of f which converges to f in some annulus about z_0.

a) If $a_j=0$ for $j<0$, we say z_0 is a **removable singularity**.

b) If $a_{-m}=-$ for some $m>0$ but $a_j=0$ for $j<-m$, we say z_0 is a **pole of order** m. For $m=1$, we call it a **simple pole**.

c) If $a_j\neq 0$ for infinitely-many negative j's, then z_0 is an **essential singularity**.

Example. $\frac{\sin(z)}{z}$ has a removable singularity at $z = 0$. $\frac{e^z}{z^5}$ has a pole of order 5 at $z = 0$. $e^{1/(z-1)}$ has an essential singularity at $z = 1$.

Lemma. If f has a removable singularity at z_0 then

a) f is bounded in some punctured neighborhood of z_0,

b) f has a finite limit as $z \to z_0$, and by redefining f at z_0, we can make it analytic at z_0.

Proof. Let $h(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j$ be the Laurent series of f. So $f(z) = h(z)$ for all z on some punctured disk about z_0 and $h(z_0) = a_0$ (redefine $f(z_0) = a_0 = \lim_{z\to z_0} f(z)$). Then h is analytic on some circular neighborhood of z_0 and is bounded on a closed disk containing z_0. ∎

Lemma. If f has a pole of order m at z_0 then $|(z - z_0)^\ell f(z)| \to \infty$ for $\ell < m$ as $z \to z_0$. For $\ell = m$, $(z - z_0)^m f(z)$ has a removable singularity at z_0.

Lemma. A function f has a pole of order m at z_0 iff in some punctured neighborhood of z_0, $f(z) = \frac{g(z)}{(z-z_0)^m}$ where g is analytic at z_0 and $g(z_0) \neq 0$.

Remark. The only singularities of rational functions are removable or poles. If f has a zero of order m at z_0, then $1/f$ has a pole of order m at z_0. If f has a pole of order m at z_0, then $1/f$ has a removable singularity at z_0 and if we define $(1/f)(z_0) = 0$, this is a zero of order m.

Theorem 5.10: Picard

A function with an essential singularity assumes every complex number with possible 1 exception as a value in every neighborhood of this singularity.

5.3 Residue Theory

If f has an isolated singularity at z_0, then the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent series expansion about z_0 is the **residue of f at** z_0, denoted $\mathrm{Res}(f; z_0)$ or $\mathrm{Res}(z_0)$.

If $f(z)$ has a pole of order n at $z = z_0$, then

$$\boxed{\mathrm{Res}(z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}}(z - z_0)^n f(z).} \tag{5.1}$$

This formula can be used to compute the constants in PFDs of a rational function.

Theorem 5.11: Cauchy Residue

If Γ is a simple closed, positively-oriented contour and f is analytic inside and on Γ except at $z_1, \ldots, z_n$ inside Γ then

$$\oint_\Gamma f(z)\mathrm{d}z = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(z_i).$$

6 Integrals

6.1 Trig Integrals

Suppose $U(x,y)$ is a rational function with real coefficients. Suppose we want to calculate $\int_0^{2\pi} U(\cos\theta, \sin\theta)\mathrm{d}\theta$. Parameterizing the unit circle as $z = e^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we can write $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}(z+z^{-1})$ and $\sin\theta = \frac{1}{2i}(z - z^{-1})$, hence

$$\int_0^{2\pi} U(\cos\theta, \sin\theta)\mathrm{d}\theta = \int_{|z|=1} U\left(\frac{1}{2}(z+z^{-1}), \tfrac{1}{2i}(z-z^{-1})\right) \tfrac{1}{iz}\mathrm{d}z$$

which can be compute using residue theory.

6.2 Improper Integrals

For any $f \in C(-\infty, \infty)$, the **Cauchy principal value** of the integral of f over $(-\infty, \infty)$ is

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\mathrm{d}x := \lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(x)\mathrm{d}x.$$