Expectations of normal cdfs

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1 Motivation

This article is based off of a problem I originally had, that had many interesting answers on stats.stackexchange.com. Specifically, the problem involves taking the expectation of a function of a normal random variable, where the function in question is a normal cumulative distribution function (cdf) of a differently distributed normal random variable. The solution we seek is an analytical form, where we do not have to do complicated integrals or series approxmiations, etc. Here we will denote the cdf of a random variable $X \sim \mathcal{N}(\mu,\ \sigma^2)$ with

$$F(x \mid \mu,\ \sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Here $\Phi(x)$ is the cdf of a standard normal random variable. In this article, we will pursue a generalized version of the problem, and show how different interpretations of the same problem lead to the same result, which can be conveniently and succinctly expressed as a different normal cdf.

2 Solutions

Suppose we have a normal random variable $X \sim \mathcal{N}(\mu,\ \sigma^2)$, and we wish to find $\mathbb{E}[F(Y \mid \mu,\ \sigma)]$, where $Y \sim \mathcal{N}(\nu,\ \tau^2)$. We can rewrite the problem as

$$\mathbb{E}[F(Y \mid \mu,\ \sigma)] = \mathbb{E}\left[\Phi\left(\frac{Y-\mu}{\sigma}\right)\right]$$

We first note that by the definition of a normal cdf, we can write

$$F(k \mid \mu,\ \sigma) = \Phi\left(\frac{k-\mu}{\sigma}\right) = \mathbb{P}(X \leq Y \mid Y = k)$$

Therefore, by the law of total probability, we can rewrite the expectation as

$$\mathbb{E}[F(Y \mid \mu,\ \sigma)] = \int_{-\infty}^{\infty} \Phi\left(\frac{y-\mu}{\sigma}\right) \phi_Y(y)dy = \int_{-\infty}^{\infty} \mathbb{P}(X \leq Y \mid Y = y)\phi_Y(y)dy$$

$$= \mathbb{P}(X \leq Y)$$

Here $\phi_Y(y)$ is the density of Y. Since we see that $X - Y \sim \mathcal{N}(\mu - \nu,\ \sigma^2 + \tau^2)$, we can rewrite this using the standard normal cdf, and therefore we have

$$\mathbb{P}(X \leq Y) = \mathbb{P}(X - Y \leq 0) = \mathbb{P}\left(\frac{(X - Y) - (\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}} \leq \frac{-(\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}}\right)$$
$$= \Phi\left(\frac{-(\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}}\right)$$

Alternatively, we could just express the probability with the cdf of $X - Y$, so

$$\mathbb{P}(X \leq Y) = \mathbb{P}(X - Y \leq 0) = F(0 \mid \mu - \nu,\ \sqrt{\sigma^2 + \tau^2})$$

We can even restrict ourselves to using only standard normal density and cumulative distribution functions, observing that we can rewrite Y as a new normal random variable $\tau Z + \nu$, where $Z \sim \mathcal{N}(0,\ 1)$. We thus have

$$\Phi\left(\frac{k - \mu}{\sigma}\right) = \Phi\left(\frac{(\tau z + \nu) - \mu}{\sigma}\right) = \mathbb{P}\left(X \leq \tau Z + \nu \;\middle|\; Z = z,\ z = \frac{k - \nu}{\tau}\right)$$

We can therefore use the standard normal density function, $\phi(x)$, to take the expectation. Integrating over $\phi(x)$, we have

$$\mathbb{E}[F(Y \mid \mu,\ \sigma)] = \mathbb{E}[F(\tau Z + \nu \mid \mu,\ \sigma)] = \int_{-\infty}^{\infty} \mathbb{P}(X \leq \tau Z + \nu \mid Z = z)\phi(z)dz$$
$$= \mathbb{P}(X \leq \tau Z + \nu)$$

Since we have that $X - (\tau Z + \nu) \sim \mathcal{N}(\mu - \nu,\ \sigma^2 + \tau^2)$, we can again rewrite this with the standard normal cdf or with the cdf of $X - (\tau Z + \nu)$, resulting in

$$\mathbb{P}(X \leq \tau Z + \nu) = \mathbb{P}\left(\frac{X - (\tau Z + \nu) - (\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}} \leq \frac{-(\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}}\right) = \Phi\left(\frac{-(\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}}\right)$$

Again, we can just express the probability with the cdf of $X - (\tau Z + \nu)$, so

$$\mathbb{P}(X \leq \tau Z + \nu) = \mathbb{P}\left(X - (\tau Z + \nu) \leq 0\right) = F(0 \mid \mu - \nu,\ \sqrt{\sigma^2 + \tau^2})$$

We can even rewrite Y in terms of another random variable X' that has the same distribution as X, denoted as $X' \stackrel{d}{=} X$ so we can integrate over the density function of X, $\phi_X(x)$. We write Y in terms of X' to see that

$$Y = \frac{\tau(X' - \mu)}{\sigma} + \nu \sim \mathcal{N}(\nu,\ \tau^2)$$

Using the standard normal cdf, we can then observe that

$$\Phi\left(\frac{k - \mu}{\sigma}\right) = \Phi\left(\frac{\tau\sigma^{-1}(X' - \mu) + \nu - \mu}{\sigma}\right)$$

$$= \mathbb{P}\left(X \leq \frac{\tau(X'-\mu)}{\sigma} + \nu \;\middle|\; X' = x',\ x' = \frac{\sigma(k-\nu)}{\tau} + \mu\right)$$

We then take the expectation by integrating over the density of X, which is the same as the density of X', to get the expression

$$\mathbb{E}[F(Y \mid \mu,\ \sigma)] = \mathbb{E}\left[F\left(\frac{\tau(X'-\mu)}{\sigma} + \nu \;\middle|\; \mu,\ \sigma\right)\right]$$

$$= \int_{-\infty}^{\infty} \mathbb{P}\left(X \leq \frac{\tau(X'-\mu)}{\sigma} + \nu \;\middle|\; X' = x'\right) \phi_X(x')dx'$$

$$= \mathbb{P}\left(X \leq \frac{\tau(X'-\mu)}{\sigma} + \nu\right) = \mathbb{P}(\sigma X - \tau X' \leq -\tau\mu + \sigma\nu)$$

We note that $\sigma X - \tau X' \sim \mathcal{N}(\mu\sigma - \mu\tau,\ \sigma^4 + \sigma^2\tau^2)$, so by standardizing $\sigma X - \tau X'$, the expression $\mathbb{P}(\sigma X - \tau X' \leq -\tau\mu + \sigma\nu)$ becomes

$$\mathbb{P}\left(\frac{(\sigma X - \tau X') - (\mu\sigma - \mu\tau)}{\sqrt{\sigma^4 + \sigma^2\tau^2}} \leq \frac{-\tau\mu + \sigma\nu - (\mu\sigma - \mu\tau)}{\sqrt{\sigma^4 + \sigma^2\tau^2}}\right)$$

We simplify the expression and rewrite with the standard normal cdf to see that

$$\mathbb{P}(\sigma X - \tau X' \leq -\tau\mu + \sigma\nu) = \mathbb{P}\left(\frac{(\sigma X - \tau X') - (\mu\sigma - \mu\tau)}{\sqrt{\sigma^4 + \sigma^2\tau^2}} \leq \frac{\nu - \mu}{\sqrt{\sigma^2 + \tau^2}}\right)$$

$$= \Phi\left(\frac{-(\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}}\right)$$

We can then equivalently express this in terms of the cdf of $\sigma X - \tau X'$, so

$$\mathbb{P}\left(X \leq \frac{\tau(X'-\mu)}{\sigma} + \nu\right) = \mathbb{P}(\sigma X - \tau X' \leq -\tau\mu + \sigma\nu)$$

$$= F(-\tau\mu + \sigma\nu \mid \mu\sigma - \mu\tau,\ \sigma\sqrt{\sigma^2 + \tau^2})$$

Finally, we can even change the cdf in the expectation to the cdf of Y. We introduce a new normal random variable Y' with the same distribution as Y. Since we want to find a number γ such that we have

$$F(k \mid \mu,\ \sigma) = \Phi\left(\frac{k-\mu}{\sigma}\right) = \Phi\left(\frac{\gamma(k) - \nu}{\tau}\right) = F(\gamma(k) \mid \nu,\ \tau)$$

We do some algebra to show that

$$\gamma(k) = \frac{\tau(k-\mu)}{\sigma} + \nu$$

Therefore, with $Y' \stackrel{d}{=} Y$, we can now rewrite $F(k \mid \mu,\ \sigma)$ as

$$F(k \mid \mu,\ \sigma) = F\left(\gamma(k) \mid \nu,\ \tau\right) = \mathbb{P}\left(Y \leq \frac{\tau(Y'-\mu)}{\sigma} + \nu \;\middle|\; Y' = k\right)$$

Therefore, integrating over the density of Y when taking the expectation and again using the law of total probability, we have

$$\mathbb{E}[F(Y \mid \mu,\ \sigma)] = \mathbb{E}\left[F\left(\frac{\tau(Y'-\mu)}{\sigma} + \nu \,\middle|\, \nu,\ \tau\right)\right]$$

$$= \int_{-\infty}^{\infty} \mathbb{P}\left(Y \leq \frac{\tau(Y'-\mu)}{\sigma} + \nu \,\middle|\, Y' = y'\right) \phi_Y(y')dy'$$

$$= \mathbb{P}\left(Y \leq \frac{\tau(Y'-\mu)}{\sigma} + \nu\right) = \mathbb{P}(\sigma Y - \tau Y' \leq -\tau\mu + \sigma\nu)$$

We note that $\sigma Y - \tau Y' \sim \mathcal{N}(\nu\sigma - \nu\tau,\ \tau^4 + \tau^2\sigma^2)$, so by standardizing $\sigma Y - \tau Y'$, the expression $\mathbb{P}(\sigma Y - \tau Y' \leq -\tau\mu + \sigma\nu)$ becomes

$$\mathbb{P}\left(\frac{(\sigma Y - \tau Y') - (\nu\sigma - \nu\tau)}{\sqrt{\tau^4 + \tau^2\sigma^2}} \leq \frac{-\tau\mu + \sigma\nu - (\nu\sigma - \nu\tau)}{\sqrt{\tau^4 + \tau^2\sigma^2}}\right)$$

We simplify the expression and rewrite with the standard normal cdf to see that

$$\mathbb{P}(\sigma Y - \tau Y' \leq -\tau\mu + \sigma\nu) = \mathbb{P}\left(\frac{(\sigma Y - \tau Y') - (\nu\sigma - \nu\tau)}{\sqrt{\tau^4 + \tau^2\sigma^2}} \leq \frac{-\mu + \nu}{\sqrt{\sigma^2 + \tau^2}}\right)$$

$$= \Phi\left(\frac{-(\mu - \nu)}{\sqrt{\sigma^2 + \tau^2}}\right)$$

Again, we can then express this in terms of the cdf of $\sigma X - \tau X'$, so

$$= \mathbb{P}\left(Y \leq \frac{\tau(Y'-\mu)}{\sigma} + \nu\right) = \mathbb{P}(\sigma Y - \tau Y' \leq -\tau\mu + \sigma\nu)$$

$$F(-\tau\mu + \sigma\nu \mid \nu\sigma - \nu\tau,\ \tau\sqrt{\tau^2 + \sigma^2})$$

3 Conclusions

We find that by appropriately introducing new random variables, rewriting random variables as combinations of other random variables, and by relying on the law of total probability, we can avoid computing $\mathbb{E}[F(Y \mid \mu,\ \sigma)]$. We instead recast the problem as finding the probability $\mathbb{P}(W \leq \delta)$, where W is some linear combination of normal random variables and δ is an appropriately chosen constant. We showed that given appropriate choices of δ, we could have W be combinations of X, Y, standard normal Z, and X', Y' such that $X' \stackrel{d}{=} X$, $Y' \stackrel{d}{=} Y$, and still arrive at the same answer, which we could conveniently express with either $F_W(\delta \mid \mu_W,\ \sigma_W)$, the cumulative distribution function of W, where $W \sim \mathcal{N}(\mu_W,\ \sigma_W^2)$, or with the standard normal cdf $\Phi(\varepsilon)$, where ε is a constant that we found to be equal to $-(\mu - \nu)/\sqrt{\sigma^2 + \tau^2}$. The solutions presented here are an example of how to recast new problems in the form of familiar problems, avoiding messy computation or approximations in favor of simple expressions.