A Mathematical Justification for Joining BAΨ

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March 2019

1 Introduction

Here we attempt to present, under a few potentially plausible assumptions, a mathematical model of the time-varying state of one's life denoted by S, and how one's entry and continued participation in one's local Beta Alpha Psi (BAΨ) chapter can increase the expectation return on S. For the sake of brevity, and because this is a fun example, the assumptions will not be rigorously explained or defended. If you wish to skip most of the mathematics and read only the conclusion and closing remarks, please skip to section 4. Step-by-step work for some of the results used can be found in section 6, the article's appendix.

Invest in something that earns real alpha.

2 A three-factor model for $S(t)$

We will suppose that $S(t)$, the time-dependent state of one's life, follows the dynamics given by the stochastic differential equation

$$dS(t) = [\mu(t) + \alpha(t)]S(t)dt + \sqrt{\nu(t)}S(t)dW_1 \tag{2.1}$$

Here, $\mu(t)$ is the time-dependent, deterministic drift of returns on S, defined by

$$\mu(t) = \gamma(t) + \lambda(t)\sin(\pi(t)t) \tag{2.2}$$

$\gamma(t)$, $\lambda(t)$, and $\pi(t)$ are unknown but deterministic functions of time where $\forall t$, $\gamma(t) \geq 0$, $0 \leq \lambda(t) \leq 1$, and $0 < \pi(t) \leq 1$. This is a very generic formulation of $\mu(t)$, and we need not worry about the exact choices of $\gamma(t)$, $\lambda(t)$, $\pi(t)$ as long as they are all smooth, real-valued functions of time. One can interpret $\mu(t)$ as the changes in life controllable by one's own choices and actions.

In contrast to our deterministic definition of $\mu(t)$, we choose $\nu(t)$ to be stochastic, defining $\nu(t)$ as the stochastic variance of returns on $S(t)$ described by

$$d\nu(t) = \kappa[\xi - \nu(t)]dt + \eta\sqrt{\nu(t)}dW_2 \tag{2.3}$$

Here we have chosen $\nu(t)$ to be a a mean-reverting stochastic process of the Cox-Ingersoll-Ross kind. κ would be our constant speed of mean-reversion to ξ, the mean of variation in returns on $S(t)$, with η as the constant volatility of $\nu(t)$. Again, we do not have to worry about the exact choices for these parameters, as long as we have that $\kappa > 0$, $\xi > 0$, $\eta > 0$ as a general guideline. One can interpret $\sqrt{\nu(t)}$ as a factor scaling the changes in life that are caused by events beyond our control. Our choice to make $\nu(t)$ stochastic reflects how this factor also randomly changes, as some periods in life are more peaceful, others more turbulent. We also have W_1 and W_2 as two Wiener processes, so as usual, $W_1 \sim \mathcal{N}(0,\ t)$, $W_2 \sim \mathcal{N}(0,\ t)$. We also define W_1 and W_2 as having an expected constant correlation ρ defined by the relation

$$\rho dt = \mathbb{E}[dW_1 dW_2] \tag{2.4}$$

Here $\mathbb{E}[\ \cdot\ ]$ is the expectation operator. One can interpret ρ as quantifying the extent of how positive changes in life are often followed by other positive changes, while negative changes are often followed by other negative changes. Finally, we define the currently unexplained function of time $\alpha(t)$ as being described by

$$d\alpha(t) = \theta \left[\frac{1}{4} \psi \log(\sqrt{\beta + 1}) - \alpha(t) \right] dt + \zeta(\sqrt{\beta + 1})^{-1} dW_3 \tag{2.5}$$

Here θ is the speed of mean reversion of $\alpha(t)$ to a constant $\psi \log(\sqrt{\beta + 1})/4$, while $\zeta(\sqrt{\beta + 1})^{-1}$ is the volatility of $\alpha(t)$. β and ψ are two currently unspecified real-valued constants. We have W_3 as a third Wiener process that is uncorrelated with W_1 and W_2, where $W_3 \sim \mathcal{N}(0,\ t)$. We need not worry about our choices for θ and ζ, as long as we have $\zeta > 0$ and $\theta > 0$.

We now have a three-factor model of sufficient generality to describe the dynamics of $S(t)$. Admittedly, our model looks a bit complicated, but we will see later that it has properties that make it quite easy to prove our main point.

3 Defining β, $\alpha(t)$, and ψ

In the previous section, we did not specify exactly what $\alpha(t)$ or the two parameters β and ψ represented in our three-factor model for $S(t)$. We also did not specify, as we did with the other parameters in the model, what values β and ψ are limited to taking. Here we will state clearly what these quantities represent and any conditions on what values they make take.

Let us define $\alpha(t)$ as the extra return one can attain by being an active member of one's local collegiate BAΨ chapter. We have defined $\alpha(t)$ as being governed by (2.5), and upon inspection, we can see that β and ψ directly affect the dynamics of $\alpha(t)$ over time. β we will define as the number of people one meets and builds relationships with upon becoming a member of BAΨ, so we must necessarily impose the restriction that $\beta \in \{0\} \cup \mathbb{N}$. ψ we will define as the percentage of time one spends on BAΨ-related activities, where we have $\psi \in [0,\ 1]$. One may also notice that the stochastic differential equation we have chosen to describe $\alpha(t)$ is a modified version of an Orstein-Uhlenbeck process—this choice will become significant later on when we calculate $\mathbb{E}[dS/S(t)]$, expected the percentage return on $S(t)$, and how it is affected by $\alpha(t)$.

4 BAΨ's real α

What we wish to show is that given the model for $S(t)$ we defined in section 2, we want to show that our expectation return on life, $\mathbb{E}[dS/S(t)]$, necessarily increases upon joining BAΨ and being an active member. Given the form of $S(t)$ defined in (2.1) and suppressing the time dependence of $dS(t)$, we have

$$\mathbb{E}\left[\frac{dS}{S(t)}\right] = \mu(t)dt + \mathbb{E}[\alpha(t)]dt$$

Since $\alpha(t)$ is an Ornstein-Uhlenbeck process, the analytical solution for $\alpha(t)$ is

$$\alpha(t) = \alpha_0 e^{-\theta t} + \frac{1}{4}\psi \log\left(\sqrt{\beta+1}\right)(1 - e^{-\theta t}) + \zeta(\sqrt{\beta+1})^{-1}\int_0^t e^{-\theta(t-\tau)}dW_3 \tag{4.1}$$

We also know the expectation $\mathbb{E}[\alpha(t)]$ and variance $\mathrm{Var}[\alpha(t)]$ of $\alpha(t)$, given by

$$\mathbb{E}[\alpha(t)] = \alpha_0 e^{-\theta t} + \frac{1}{4}\psi \log\left(\sqrt{\beta+1}\right)(1 - e^{-\theta t}) \tag{4.2}$$

$$\mathrm{Var}[\alpha(t)] = \frac{\zeta^2}{2\theta}(\beta+1)^{-1}(1 - e^{-2\theta t}) \tag{4.3}$$

We can reasonably say that if one joins BAΨ at time $t = 0$, there is some accumulated benefit $\alpha_0 > 0$ acquired during the candidate process. However, over time, the efforts one puts into BAΨ, which control the magnitude of the parameters β and ψ, $\alpha(t)$ will fluctuate but continually revert to $\psi \log(\sqrt{\beta+1})/4$, which is the long-term mean of $\alpha(t)$. Since we know $\mathbb{E}[\alpha(t)]$, we thus have

$$\mathbb{E}\left[\frac{dS}{S(t)}\right] = \left[\mu(t) + \alpha_0 e^{-\theta t} + \frac{1}{4}\psi \log\left(\sqrt{\beta+1}\right)(1 - e^{-\theta t})\right] dt \tag{4.4}$$

Since we have $\alpha_0 > 0$, $\psi \in [0,\ 1]$, $\beta \in \{0\} \cup \mathbb{N}$, we can guarantee that $\forall t \in \mathbb{R}$, $t \geq 0$, $\mathbb{E}[dS/S(t)] \geq \mu(t)dt$. If $\psi > 0$ and $\beta > 0$, then $\forall t \in \mathbb{R}$, $t \geq 0$, we have that $\mathbb{E}[dS/S(t)] > \mu(t)dt$. That is, given the restrictions on what values our defined parameters can take, as long as $\psi > 0$ and $\beta > 0$, the expectation return on $S(t)$ is guaranteed to be greater than just $\mu(t)dt$. And the greater ψ and β are, the greater our expectation return. Interestingly, we also see that as β increases, not only does $\mathbb{E}[dS/S(t)]$ increase, but $\mathrm{Var}[\alpha(t)]$ also decreases, which means the distribution of $\alpha(t)$ is more tightly distributed around $\mathbb{E}[\alpha(t)]$.

Therefore, not only does one's expected return on the state of one's life $S(t)$ increase with meaningful involvement in BAΨ, but the more people one meets and forms relationships with in BAΨ, the more the mean return $\mathbb{E}[\alpha(t)]$ from BAΨ increases, and the more closely the actual return $\alpha(t)$ is distributed around $\mathbb{E}[\alpha(t)]$. From a rational standpoint, if one wants to maximize the expectation return on life $\mathbb{E}[dS/S(t)]$, then the decision to join BAΨ should be an easy one.

5 Closing remarks

We have proven that joining BAΨ and being an active member guarantees, on expectation, a higher average return on one's state of life $S(t)$ than if one did not join BAΨ and become an active member. Interestingly, we also found that as β increases, $\mathrm{Var}[\alpha(t)]$ decreases, meaning that the actual value of $\alpha(t)$ will more closely distributed around its expectation $\mathbb{E}[\alpha(t)]$. Therefore, only does your expected return on $S(t)$ increase with involvement in BAΨ, but the more people you meet and form meaningful relationships with, the less the random noise component of $\alpha(t)$ affects your expected return $\mathbb{E}[dS/S(t)]$.

Of course, the choice to join a professional organization like BAΨ is a highly personal one, and some will find during the candidate process that perhaps BAΨ is not the best fit for them. And despite the generality of the model, it is evident that we are trying to quantify the state of one's life, which perhaps cannot be modeled by the preference-free model that has been laid out in this article. To avoid the problem of quantifying the exact numerical value of $S(t)$, we choose to think only in terms of percentage returns on $S(t)$, which are unitless and allow us to discard any exact measurement of $S(t)$. However, the author hopes that after reading this article, one may be more inclined to join BAΨ simply knowing that there are members dedicated enough to find a mathematical justification for why their organization is worth joining.

The author welcomes all comments and inquiries, and will try his best to give satisfactory answers when possible. Opinions expressed in this paper are entirely the author's own, and are not endorsed by any local chapter of BAΨ.

6 Appendix

In section 2, we defined $\alpha(t)$ as an Orstein-Uhlenbeck process with constant parameters. Since analytical solutions for an Orstein-Uhlenbeck process $X(t)$, its expectation $\mathbb{E}[X(t)]$, and its variance $\mathrm{Var}[X(t)]$ already exist, we could easily substitute in the different parameters we used in our definition of $\alpha(t)$ to get the appropriate analytical solutions for $\alpha(t)$, $\mathbb{E}[\alpha(t)]$, and $\mathrm{Var}[\alpha(t)]$. However, we can also these analytical solutions using stochastic calculus tools such as Ito's lemma and Ito's isometry, which we will do in this appendix. We will also show why the quantity $\mathbb{E}[dS/S(t)]$ is of the form presented in section 4.

6.0.1 Deriving $\alpha(t)$

Here we will derive the analytical solution to $\alpha(t)$. Recalling (2.5), we have

$$d\alpha(t) = \theta\left[\frac{1}{4}\psi\log(\sqrt{\beta+1}) - \alpha(t)\right]dt + \zeta(\sqrt{\beta+1})^{-1}dW_3$$

We simplify our computational steps by setting

$$A = \frac{1}{4}\psi\log(\sqrt{\beta+1}) \tag{6.1}$$

$$B = \zeta(\sqrt{\beta+1})^{-1} \tag{6.2}$$

Now our equation for $d\alpha(t)$ looks like an Ornstein-Uhlenbeck process given by

$$d\alpha(t) = \theta[A - \alpha(t)]dt + BdW_3$$

If we choose the function $\alpha(t)e^{\theta t}$, we have by Ito's lemma

$$\begin{aligned} d(\alpha(t)e^{\theta t}) &= e^{\theta t}d\alpha(t) + \theta\alpha(t)e^{\theta t}dt \\ &= e^{\theta t}\theta[A - \alpha(t)]dt + e^{\theta t}BdW_3 + \theta\alpha(t)e^{\theta t}dt \\ &= e^{\theta t}\theta Adt + e^{\theta t}BdW_3 \end{aligned}$$

Integrating both sides from time 0 to time t, we have

$$\begin{aligned} \alpha(t)e^{\theta t} &= C + A\int_0^t \theta e^{\theta\tau}d\tau + B\int_0^t e^{\theta\tau}dW_3 \\ &= \alpha_0 + A(e^{\theta t} - 1) + B\int_0^t e^{\theta\tau}dW_3 \end{aligned}$$

Here we set the constant of integration to α_0 and solve the deterministic definite integral. Dividing both sides by $e^{\theta t}$ and substituting in (6.1) and (6.2), we have

$$\alpha(t) = \alpha_0 e^{-\theta t} + \frac{1}{4}\psi\log(\sqrt{\beta+1})(1 - e^{-\theta t}) + \zeta(\sqrt{\beta+1})^{-1}\int_0^t e^{-\theta(t-\tau)}dW_3$$

This is exactly the form of $\alpha(t)$ in (4.1).

6.0.2 Deriving $\mathbb{E}[\alpha(t)]$

Now that we have the closed form solution to $\alpha(t)$, it is fairly straightforward for us to find $\mathbb{E}[\alpha(t)]$. Taking the expectation of $\alpha(t)$, we have

$$\begin{aligned}\mathbb{E}[\alpha(t)] = \alpha_0 e^{-\theta t} + \frac{1}{4}\psi \log(\sqrt{\beta+1})(1-e^{-\theta t}) \\ + \zeta(\sqrt{\beta+1})^{-1}\ \mathbb{E}\left[\int_0^t e^{-\theta(t-\tau)} dW_3\right]\end{aligned}$$

To find the expectation of the Ito integral, we can approximate it discretely as

$$\mathbb{E}\left[\int_0^t e^{-\theta(t-\tau)} dW_3\right] \approx \mathbb{E}\left[\sum_{\tau=0}^{t-\delta t} e^{-\theta(t-\tau)}(W_{3(\tau+\delta t)} - W_{3(\tau)})\right]$$

Here δt represents a very small but nonzero quantity of time. We also have that $\forall \tau_i \in [0,\ t)$, $i \in \mathbb{N}$, $\tau_{i+1} = \tau_i + \delta t$. By the property of the expectation operator's linear additivity and because the term $e^{-\theta(t-\tau)}$ is deterministic, we can write

$$\mathbb{E}\left[\int_0^t e^{-\theta(t-\tau)} dW_3\right] \approx \sum_{\tau=0}^{t-\delta t} e^{-\theta(t-\tau)}\mathbb{E}\left[(W_{3(\tau+\delta t)} - W_{3(\tau)})\right]$$

Let us recall the properties of the Wiener process, which are

1. $W_0 = 0$ a.s.
2. W_t is continuous in t with probability 1.
3. W_t has stationary, independent increments.
4. Each increment is Gaussian, i.e. $W_{t+dt} - W_t \sim \mathcal{N}(0,\ dt)$.

We therefore know that $\mathbb{E}\left[W_{3(t+\delta t)} - W_{3(t)}\right] = \mathbb{E}\left[dW_3\right] = 0$. Therefore,

$$\mathbb{E}\left[\int_0^t e^{-\theta(t-\tau)} dW_3\right] = 0$$

In general, any Ito integral with a Wiener process as an integrator and a bounded, deterministic function as the integrand has an expectation of zero, since all the increments of a Wiener process have an expectation of zero. We therefore have the expectation $\mathbb{E}[\alpha(t)]$ as just

$$\mathbb{E}[\alpha(t)] = \alpha_0 e^{-\theta t} + \frac{1}{4}\psi \log(\sqrt{\beta+1})(1-e^{-\theta t})$$

This is exactly the form of $\mathbb{E}[\alpha(t)]$ in (4.2).

6.0.3 Deriving $\mathrm{Var}[\alpha(t)]$

Now that we have the analytical form of $\mathbb{E}[\alpha(t)]$, we can find $\mathrm{Var}[\alpha(t)]$, although the calculation will be bit more involved. We begin with the identity

$$\mathrm{Var}[\alpha(t)] = \mathbb{E}[\alpha(t)^2] - \mathbb{E}[\alpha(t)]^2$$

Recalling (4.1) and (4.2), substituting (6.1) and (6.2), and expanding, we have

$$\begin{aligned}
\mathrm{Var}[\alpha(t)] = {\alpha_0}^2 e^{-2\theta t} &+ 2\alpha_0 e^{-\theta t} A(1-e^{-\theta t}) + 2\alpha_0 B \;\mathbb{E}\left[\int_0^t e^{-\theta(2t-\tau)} dW_3\right] \\
&+ A^2(1-e^{-\theta t})^2 + 2AB(1-e^{-\theta t}) \;\mathbb{E}\left[\int_0^t e^{-\theta(t-\tau)} dW_3\right] \\
&+ B^2 \;\mathbb{E}\left[\left(\int_0^t e^{-\theta(t-\tau)} dW_3\right)^2\right] - {\alpha_0}^2 e^{-2\theta t} - 2\alpha_0 e^{-\theta t} A(1-e^{-\theta t}) \\
&- A^2(1-e^{-\theta t})^2
\end{aligned}$$

Cancelling terms, we can simplify the above expression to

$$\begin{aligned}
\mathrm{Var}[\alpha(t)] = 2\alpha_0 B \;\mathbb{E}\left[\int_0^t e^{-\theta(2t-\tau)} dW_3\right] &+ 2AB(1-e^{-\theta t}) \;\mathbb{E}\left[\int_0^t e^{-\theta(t-\tau)} dW_3\right] \\
&+ B^2 \;\mathbb{E}\left[\left(\int_0^t e^{-\theta(t-\tau)} dW_3\right)^2\right]
\end{aligned} \tag{6.3}$$

The first two Ito integrals in our expression for $\mathrm{Var}[\alpha(t)]$ both involve bounded, deterministic functions being integrated by the Wiener process W_3, so each integral has an expectation of 0. By Ito's isometry, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t e^{-\theta(t-\tau)} dW_3\right)^2\right] = \mathbb{E}\left[\int_0^t e^{-2\theta(t-\tau)} dt\right] &= \frac{e^{-2\theta t}}{2\theta}(e^{2\theta t} - 1) \\
&= \frac{1}{2\theta}(1 - e^{-2\theta t})
\end{aligned} \tag{6.4}$$

Substituting (6.4) and (6.2) back into (6.3), we arrive at

$$\mathrm{Var}[\alpha(t)] = \frac{\zeta^2}{2\theta}(\beta+1)^{-1}(1 - e^{-2\theta t})$$

This is exactly the form of $\mathrm{Var}[\alpha(t)]$ in (4.3).

6.0.4 Deriving $\mathbb{E}[dS/S(t)]$

Here we will show how $\mathbb{E}[dS/S(t)]$ is of the form shown in section 4. Recalling our definition of $dS(t)$ in (2.1), we have the differential equation

$$dS(t) = [\mu(t) + \alpha(t)]S(t)dt + \sqrt{\nu(t)}S(t)dW_1$$

Dividing by $S(t)$ and suppressing the time dependence of $dS(t)$, we have

$$\frac{dS}{S(t)} = [\mu(t) + \alpha(t)]dt + \sqrt{\nu(t)}dW_1$$

To make our analysis simpler, we can discretize the above equation as

$$\frac{S(t+\delta t) - S(t)}{S(t)} = [\mu(t) + \alpha(t)]\delta t + \sqrt{\nu(t)}(W_{1(t+\delta t)} - W_{1(t)}) \tag{6.5}$$

Again we have δt as a very small but nonzero discrete quantity of time. We can see that to calculate the expectation $\mathbb{E}[dS/S(t)]$, we will have to take the expectation of $\sqrt{\nu(t)}dW_1$, an expectation of two stochastic quantities. But if we recall that the Wiener process is defined as having stationary, independent increments $W_{t+dt} - W_t \sim \mathcal{N}(0,\ dt)$, then by definition $\nu(t)$ and the increment $W_{1(t+\delta t)} - W_{1(t)}$ are independent. Therefore, taking the expectation of the discretized form $\mathbb{E}[dS/S(t)]$, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{S(t+\delta t) - S(t)}{S(t)}\right] &= \mathbb{E}[\mu(t) + \alpha(t)]\delta t + \mathbb{E}\left[\sqrt{\nu(t)}(W_{1(t+\delta t)} - W_{1(t)})\right] \\
&= \mu(t)\delta t + \mathbb{E}[\alpha(t)]\delta t + \mathbb{E}\left[\sqrt{\nu(t)}\right]\mathbb{E}\left[W_{1(t+\delta t)} - W_{1(t)}\right] \\
&= \mu(t)\delta t + \mathbb{E}[\alpha(t)]\delta t + \mathbb{E}\left[\sqrt{\nu(t)}\right]\cdot 0 \\
&= \mu(t)\delta t + \mathbb{E}[\alpha(t)]\delta t
\end{aligned}$$

We can write $\mathbb{E}\left[\sqrt{\nu(t)}(W_{1(t+\delta t)} - W_{1(t)})\right]$ as $\mathbb{E}\left[\sqrt{\nu(t)}\right]\mathbb{E}\left[W_{1(t+\delta t)} - W_{1(t)}\right]$ only because $\nu(t)$ and $W_{1(t+\delta t)} - W_{1(t)}$ are independent. Here we also use the fact that $W_{1(t+\delta t)} - W_{1(t)} \sim \mathcal{N}(0,\ \delta t)$, so $\mathbb{E}\left[W_{1(t+\delta t)} - W_{1(t)}\right] = 0$. The continuous time analogue to the expectation of the discretized $dS/S(t)$ is therefore

$$\mathbb{E}\left[\frac{dS}{S(t)}\right] = \mu(t)dt + \mathbb{E}[\alpha(t)]dt$$

This is exactly the form of $\mathbb{E}[dS/S(t)]$ in section 4.