Pricing a Black-Scholes log contract

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1 Motivation

While browsing through the quantitative finance Stack Exchange website perhaps three months ago, I stumbled upon a question asking how to price a log-contract using the Black-Scholes PDE[1]. I had just started looking into quantitative finance at the time, and although intrigued by the question, did not understand the solution nor know how to approach it myself. The problem text is stated, almost verbatim, below:

~ The payoff of a so-called European log-contract is $\Lambda(S_T) = \log(S_T/K)$ where K is the strike price and S is a risky non-dividend Black-Scholes asset. Find the price $C(S,\ t)$ of the log-contract.

  Hint: Use the Black-Scholes PDE and give yourself the fact that $C(S,\ t)$ has the following form:

$$C(S,\ t) = a(t) + b(t)\log(S/K)$$

  Find the functions $a(t)$ and $b(t)$.

The problem nudges one towards using the Black-Scholes PDE as the starting point for the answer, but as usual, there is more than one way to approach the problem. We present in this article two ways to answer this question: first, using the PDE approach, and second, by means of a conditional expectation.

2 A PDE approach

The solution to the problem is stated to have the form

$$C(S,\ t) = a(t) + b(t)\log(S/K) \tag{2.1}$$

Any European derivative security on a Black-Scholes underlying satisfies the Black-Scholes PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{2.2}$$

We first note boundary conditions at T, the payoff date. Since at T we must have $C(S,\ T) = \log(S/K)$, this implies that $a(T) = 0$ and $b(T) = 1$. To solve (2.2), we take partial derivatives of (2.1) and get

$$\frac{\partial C}{\partial t} = \frac{da}{dt} + \frac{db}{dt}\log(S/K) \qquad \frac{\partial C}{\partial S} = \frac{1}{S}b(t) \qquad \frac{\partial^2 C}{\partial S^2} = -\frac{1}{S^2}b(t) \tag{2.3}$$

Substituting the above results and (2.1) into (2.2) and grouping terms, we have

$$\frac{da}{dt} + \left(r - \frac{1}{2}\sigma^2\right)b(t) - ra(t) + \log(S/K)\left[\frac{db}{dt} - rb(t)\right] = 0 \tag{2.4}$$

The two time-varying quantities are S and t, so $\forall t,\ S \in \mathbb{R}_{\geq 0}$, (2.4) must hold. We thus must have

$$\frac{da}{dt} + \left(r - \frac{1}{2}\sigma^2\right)b(t) - ra(t) = -\log(S/K)\left[\frac{db}{dt} - rb(t)\right]$$

[1] Link: https://quant.stackexchange.com/questions/31397/pricing-log-contract-with-black-scholes-pde

For all t, the left hand side is constant for all S, so the right hand side must also be constant for all S, even though the right hand side varies with S. This occurs only if both sides equal 0, so we can split (2.4) into

$$\frac{da}{dt} + \left(r - \frac{1}{2}\sigma^2\right) b(t) - ra(t) = 0 \tag{2.5}$$

$$\log(S/K) \left[\frac{db}{dt} - rb(t)\right] = 0 \tag{2.6}$$

We now proceed to solve (2.6). Dividing out the $\log(S/K)$ term and rearranging terms, we have

$$\frac{1}{b} db = r dt$$

Integrating from 0 to t, setting κ as the constant of integration, and exponentiating the result, we have $b(t) = e^{\kappa + rt}$. Subjecting our result to the terminal condition $b(T) = 1$, we must have that $\kappa = -rT$, so

$$b(t) = e^{-r(T-t)} \tag{2.7}$$

We now proceed to solve (2.5). Rearranging terms and multiplying by dt, we have

$$da = ra(t)dt - \left(r - \frac{1}{2}\sigma^2\right) b(t)dt \tag{2.8}$$

Using the ansatz of ae^{-rt} and substituting in (2.7), we find that

$$d(ae^{-rt}) = e^{-rt}da - rae^{-rt} = -\left(r - \frac{1}{2}\sigma^2\right) b(t)e^{-rt}dt = -\left(r - \frac{1}{2}\sigma^2\right) e^{-rT}dt$$

Integrating from 0 to t, setting κ as the constant of integration, and multiplying by e^{rt}, we have

$$a(t) = \kappa e^{rt} - \left(r - \frac{1}{2}\sigma^2\right) e^{-r(T-t)}t$$

Subjecting our result to the terminal condition $a(T) = 0$ and setting $t = T$, we must choose κ such that

$$\kappa = \left(r - \frac{1}{2}\sigma^2\right) e^{-rT}T$$

Substituting the expression for κ into our expression for $a(t)$ and factoring terms, we thus have

$$a(t) = e^{-r(T-t)} \left(r - \frac{1}{2}\sigma^2\right) (T - t) \tag{2.9}$$

We note that if we substitute (2.7) and (2.9) into (2.1) and factor by $b(t)$, we have

$$C(S,\ t) = e^{-r(T-t)} \left[\left(r - \frac{1}{2}\sigma^2\right) (T - t) + \log(S/K)\right] \tag{2.10}$$

3 A conditional expectation

The risk-neutral pricing formula states that the arbitrage price of a European contingent claim $V(S,\ t)$ expiring at time T, where $t \leq T$ is the current time and S is the current level of the underlying $S(t)$, is

$$V(S,\ t) = \mathbb{E}[D(t,\ T)\Lambda(S(T)) \mid \mathcal{F}_t] \tag{3.1}$$

Here $D(t,\ T)$ is the discount factor for the interval $[t,\ T)$, $\mathcal{F}_t$ is the natural filtration at time t, and the expectation $\mathbb{E}[\ \cdot\ ]$ is under the risk-neutral measure $\mathbb{Q}$. The price of our Black-Scholes log-contract is thus

$$C(S,\ t) = \mathbb{E}\left[e^{-r(T-t)} \log(S(T)/K) \;\middle|\; \mathcal{F}_t\right] = e^{-r(T-t)} \int_{-\infty}^{\infty} \log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds \tag{3.2}$$

Here we used the definition of conditional expectation. We have $D(t,\ T) = e^{-r(T-t)}$ under the Black-Scholes model, where r is the continuously compounded risk-free discount rate, and $\varphi(s,\ T \mid \mathcal{F}_t)$ is the pdf of $S(T)$ conditional on $\mathcal{F}_t$. Simplifying the integral found in (3.2), we have

$$\int_{-\infty}^{\infty} \log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds = \int_{-\infty}^{\infty} \log(s)\varphi(s,\ T \mid \mathcal{F}_t)ds - \log K \int_{-\infty}^{\infty} \varphi(s,\ T \mid \mathcal{F}_t)ds$$

Since the integral of a real-valued density function over $\mathbb{R}$ must equal 1, we thus have

$$\int_{-\infty}^{\infty} \log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds = \int_{-\infty}^{\infty} \log(s)\varphi(s,\ T \mid \mathcal{F}_t)ds - \log K \tag{3.3}$$

In order to proceed further, we must determine the form of $\varphi(s,\ T \mid \mathcal{F}_t)$. Under the Black-Scholes model, we know that under $\mathbb{Q}$, the underlying stock $S(t)$ has the risk-neutral dynamics

$$dS(t) = rS(t)dt + \sigma S(t)dW_t \tag{3.4}$$

Here σ is the constant instantaneous standard deviation of returns and dW_t is a differential change in a Wiener process. Solving (3.4) for $S(T)$ conditional on $\mathcal{F}_t$, so $S(t) = S$, and using the fact that since $W_t \sim \mathcal{N}(0,\ t)$, we can write $W_{t\ \dots\ T}$ as $\sqrt{T-t}\ Z$, with $Z \sim \mathcal{N}(0,\ 1)$, and thus have

$$S(T) \mid \mathcal{F}_t = S \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\ Z\right] \tag{3.5}$$

We can therefore instead integrate over $\mathbb{R}$ using $\phi(z)$, the standard normal pdf, instead of using $\varphi(s,\ T \mid \mathcal{F}_t)$. Substituting our expression for $S(T) \mid \mathcal{F}_t$ in (3.5) into (3.3), and setting $\theta = T - t$ for brevity, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds &= \int_{-\infty}^{\infty} \left[\log S + \left(r - \frac{1}{2}\sigma^2\right)\theta + \sigma\sqrt{\theta}\ z\right]\phi(z)dz - \log K \\ &= \left[\log S + \left(r - \frac{1}{2}\sigma^2\right)\theta\right]\int_{-\infty}^{\infty} \phi(z)dz + \sigma\sqrt{\theta}\int_{-\infty}^{\infty} z\phi(z)dz - \log K \\ &= \log S + \left(r - \frac{1}{2}\sigma^2\right)\theta - \log K \end{aligned} \tag{3.6}$$

The last step follows from the fact that the integral of a real-valued density function over $\mathbb{R}$ must equal 1, and that by the definition of expectation and by the fact that $Z \sim \mathcal{N}(0,\ 1)$, we have that

$$\int_{-\infty}^{\infty} z\phi(z)dz = \mathbb{E}[Z] = 0$$

Substituting our results in (3.6) back into (3.2), replacing θ with $T - t$, and rearranging terms, we have

$$C(S,\ t) = e^{-r(T-t)}\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \log(S/K)\right] \tag{3.7}$$

Our result in (3.7) is the same as in (2.10). It also follows that we have $a(t)$ and $b(t)$ given by

$$\begin{aligned} a(t) &= e^{-r(T-t)}\left(r - \frac{1}{2}\sigma^2\right)(T-t) \\ b(t) &= e^{-r(T-t)} \end{aligned}$$

We see that solving the Black-Scholes PDE and taking the conditional expectation of the payoff at T give us the same results for $a(t)$ and $b(t)$. The difference between the two approaches is that by solving the PDE, we find $a(t)$ and $b(t)$ directly before finding (3.7). But by taking a conditional expectation of the payoff by using the risk-neutral pricing formula, we first find (3.7) and then use the result to extract $a(t)$ and $b(t)$.