# Discretization problems in Monte Carlo

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## 1 Introduction

The simulation of stochastic processes is a very important topic, as when the process gets complicated enough, it may no longer have an analytical solution. Therefore, in fields such as finance, where a potentially multi-factor stochastic process is needed as an underlying to price some derivative security, Monte Carlo simulation is preferred, as it gives more flexibility than a binomial tree and is easily parallelizable. However, the inevitable discretization that occurs introduces numerical errors and problems into simulation. Prominent examples involve stochastic volatility models, which often end up calibrated with a relatively large constant scaling their diffusion term. The large diffusion term results in larger jumps between discrete time steps, and models that are supposed to not take negative values may do so silently. We conduct a quick exploration.

## 2 Stochastic volatility models

Stochastic volatility models were introduced to try and capture the dynamics of the volatility skew/smile, with some models also incorporating a mean reversion factor in an attempt to also capture the term structure of volatility. We will quickly explore two common models, the Heston model and the stochastic $\alpha$, $\beta$, $\rho$ or SABR model, and show the probability that the variance $\nu(t)$ in the Heston model or volatility $\alpha(t)$ will drop below 0, conditional on the value at the previous time step, is nonzero.

### 2.1 The Heston model

Steven L. Heston's stochastic volatility model for the risk-neutral dynamics of a spot asset is given by

$$dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)dW_t^{(1)}$$

$$d\nu(t) = \kappa[\theta - \nu(t)]dt + \eta\sqrt{\nu(t)}dW_t^{(2)}$$

$$\mathbb{E}[dW_t^{(1)}dW_t^{(2)}] = \rho dt$$

Here $S(t)$ is the process for the spot asset, $r$ is the risk-free short rate, $\nu(t)$ is the variance process, $\kappa$ is the speed of mean reversion of $\nu(t)$, $\theta$ is the long term mean of the variance process, $\eta$ is the volatility of the variance factor, and $\rho$ is a constant instantaneous correlation between the two Wiener processes $W_t^{(1)}$ and $W_t^{(2)}$. More formally, we can write the correlation as $W_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2}Z_t$, where $Z_t$ is another independent Wiener process. Discretizing the variance process in the Euler-Maruyama style, we have

$$\nu_{t+\delta t} = \nu_t + \kappa(\theta - \nu_t)\delta t + \eta\sqrt{\nu_t}Z_2\sqrt{\delta t}$$

Here $\delta t$ is a small discrete time step, and $Z_2$ is a standard normal random variable with correlation $\rho$ to $Z_1$, another standard random variable that drives the $S(t)$ process. We know $Z_2$ is standard normal because

$$Z_2 = \rho Z_1 + \sqrt{1-\rho^2}Z \sim \mathcal{N}(0,\ \rho^2 + 1 - \rho^2) \stackrel{d}{=} \mathcal{N}(0,\ 1)$$

Here the random variable $Z$ is also a standard normal random variable. We can therefore see that the distribution of $\nu_{t+\delta t}$ conditional on $\nu_t > 0$ is given by

$$\nu_{t+\delta t} \mid \nu_t \sim \mathcal{N}\left(\kappa\theta\delta t + \nu_t(1-\kappa\delta t),\ \eta^2\nu_t\delta t\right)$$

Therefore, we can write the conditional probability of $\nu_{t+\delta t}$ becoming negative given $\nu_t > 0$ as

$$\mathbb{P}(\nu_{t+\delta t} < 0 \mid \nu_t > 0) = \Phi\left(\frac{-\kappa\theta\delta t + \nu_t(\kappa\delta t - 1)}{\eta\sqrt{\nu_t \delta t}}\right)$$

Here $\Phi(x)$ is the cdf of a standard normal random variable. To keep this probability small, it would be ideal to have a relatively large $\kappa$ and $\theta$, while a smaller $\eta$. Empirically, however, model calibration typically results in relatively large value for $\eta$, so $\mathbb{P}(\nu_{t+\delta t} < 0 \mid \nu_t > 0)$ is no longer trivial, and an actual concern during simulation. Note that when $\kappa = 0$, i.e. we set the variance to be a martingale square root process, we have

$$\mathbb{P}(\nu_{t+\delta t} < 0 \mid \nu_t > 0,\ \kappa = 0) = \Phi\left(\frac{-\nu_t}{\eta\sqrt{\nu_t \delta t}}\right) = \Phi\left(-\frac{1}{\eta}\sqrt{\frac{\nu_t}{\delta t}}\right)$$

## 2.2 The SABR model

The stochastic volatility model by Patrick S. Hagan et al. for the modeling of forward prices is given by

$$dF(t) = \alpha(t) F^{\beta}(t) dW_t^{(1)}$$

$$d\alpha(t) = \nu\alpha(t) dW_t^{(2)}$$

$$\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$$

Here $F(t)$ is the process for the forward, $\alpha(t)$ is the volatility process, $\beta \in [0,\ 1]$ is a skew factor applied to the diffusion of $F(t)$, $\nu$ is the volatility of $\alpha(t)$, and $\rho$ is a constant instantaneous correlation between the two Wiener processes $W_t^{(1)}$ and $W_t^{(2)}$. Note that the skew factor $\beta$ is commonly set a priori based on practitioner beliefs about the market the model is being used for. The common choices of $\beta = 0$ would correspond to normal diffusion for $F(t)$, while $\beta = 1/2$ would correspond to noncentral chi-squared diffusion for $F(t)$, and $\beta = 1$ would correspond to lognormal diffusion for $F(t)$. Discretizing $\alpha(t)$, we have

$$\alpha_{t+\delta t} = \alpha_t + \nu\alpha_t Z_2\sqrt{\delta t} = \alpha_t(1 + \nu Z_2\sqrt{\delta t})$$

Here $Z_2$ is a standard normal random variable with correlation $\rho$ to $Z_1$, the driving factor for the $F(t)$ process. We can therefore see that the distribution of $\alpha_{t+\delta t}$ conditional on $\alpha_t > 0$ is therefore

$$\alpha_{t+\delta t} \mid \alpha_t \sim \mathcal{N}\left(\alpha_t,\ \nu^2\alpha_t^2\delta t\right)$$

We can therefore write the conditional probability of $\alpha_{t+\delta t}$ becoming negative given $\alpha_t > 0$ as

$$\mathbb{P}(\alpha_{t+\delta t} < 0 \mid \alpha_t > 0) = \Phi\left(\frac{-\alpha_t}{\nu\alpha_t\sqrt{\delta t}}\right) = \Phi\left(-\frac{1}{\nu\sqrt{\delta t}}\right)$$

Notice here that the value of $\alpha_t$ does not affect the probability of $\alpha_{t+\delta t}$ becoming negative, as long as $\alpha_t > 0$. Again, calibration typically results in relatively high values for $\nu$, especially for shorter-maturity options with more pronounced smiles. As with the Heston model, this may result in $\alpha(t)$ taking negative values silently.

## 2.3 Conclusion

Through this short example, we can see how when discretizing stochastic processes for Monte Carlo simulation, various numerical problems can silently creep in. We use two stochastic volatility models as examples of stochastic processes that are defined to be nonnegative $\forall t \in [0,\ \infty)$, but have a nonzero probability of dropping below 0 when discretized, conditional on the prior realized process value. And depending on the magnitude of the parameters used, this conditional probability may be a real concern, especially with models such as the Heston model where functions of the volatility process have a domain restricted to $\mathbb{R}_{\geq 0}$.