The intuition behind Black's formula

Derek Huang

August 7, 2019

1 Introduction

In 1973, Black and Scholes released their seminal paper on pricing European stock options. Black would extend their results in a 1976 paper applying the Black-Scholes pricing PDE methodology to pricing European options on commodity futures. However, solving the Black-Scholes PDE by means of transformation to the heat equation proffers no intuition on the form of the pricing formula. An alternate approach using the risk-neutral pricing method allows for a simpler, more direct derivation of the formula, and allows one to re-express the formula in terms of risk-neutral conditional probabilities that provide an intuitive understanding of option price dynamics. In this short article, we derive the Black formula from the risk-neutral pricing formula, and use our derivation to rewrite the formula in a way that isolates the volatility premium, the portion of the option value due to the total volatility of the underlying, defined as $\sigma\sqrt{T-t}$.

2 A conditional expectation

Black's model for futures prices under the risk-neutral measure $\mathbb{Q}$ is

$$dF(t) = \sigma F(t) dW_t \tag{2.1}$$

Here W_t is a Wiener process under $\mathbb{Q}$, and we know that $W_t \sim \mathcal{N}(0,\ t)$. By the risk-neutral pricing formula, we can write the time t value $C(F,\ t)$ of a European call option on F as the conditional expectation

$$C(F,\ t) = \mathbb{E}[D(t,\ T)\max\{F(T) - K,\ 0\} \mid \mathcal{F}_t]$$

Here $F = F(t)$ is the current futures price, T is the time of option expiration, and K is the fixed strike price of the option. $D(t,\ T)$ is a deterministic discount factor, which Black specifies as $e^{-r(T-t)}$, with r being a known, constant riskless discount rate. $\mathcal{F}_t$ is the natural filtration of the process $F(t)$. By the definition, we can rewrite the risk-neutral conditional expectation as

$$C(F,\ t) = D(t,\ T)\int_{-\infty}^{\infty} \max\{f - K,\ 0\}\varphi_\tau(f,\ T \mid \mathcal{F}_t)df = D(t,\ T)\int_K^{\infty}(f-K)\varphi_\tau(f,\ T \mid \mathcal{F}_t)df$$

Here $\varphi_\tau(F,\ T \mid \mathcal{F}_t)$ is the risk-neutral probability density for $F(T)$ conditioned on the natural filtration $\mathcal{F}_t$, where $\tau \geq T \geq t$ is a delivery date. For brevity, we drop the explicit dependence on $\mathcal{F}_t$, T and use the simpler notation $\widetilde{\varphi}_\tau(F)$ instead. Dividing by $D(t,\ T)$ and expanding the remaining integral, we have

$$\begin{aligned}\frac{1}{D(t,\ T)}C(F,\ t) &= \int_K^{\infty}(f-K)\widetilde{\varphi}_\tau(f)df = \int_K^{\infty} f\widetilde{\varphi}_\tau(f)df - K\int_K^{\infty}\widetilde{\varphi}_\tau(f)df \\ &= \int_K^{\infty} f\widetilde{\varphi}_\tau(f)df - K\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t)\end{aligned} \tag{2.2}$$

We see that the expectation can be written as the difference between a partial conditional expectation and K times a risk-neutral probability conditioned on the natural filtration $\mathcal{F}_t$. To simplify our computation of the partial expectation, we recall that the solution to (2.1) is given by the equation

$$F(t) = F_0 \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right)$$

We want to find $F(T)$ conditioned on $\mathcal{F}_t$. We set W_t to 0, and by the properties of the Wiener process, we write $W_{t\,\ldots\,T}$ as $\sqrt{T-t}\ Z$, where $Z \sim \mathcal{N}(0,\ 1)$ under $\mathbb{Q}$. Substituting F for $F(t)$, we have that

$$F(T) \mid \mathcal{F}_t = F \exp\left[-\frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T-t}\ Z\right] \tag{2.3}$$

For notational simplicity, we set $\theta = T - t$. To simplify the partial expectation, we make note that $F(T)$ conditioned on $\mathcal{F}_t$ is a function of a standard normal random variable Z. Therefore, using (2.3), we can rewrite the partial expectation in terms of the standard normal density function $\phi(z)$ instead to get

$$\int_K^\infty f\widetilde{\varphi}_\tau(f)df = F\int_\zeta^\infty e^{-\frac{1}{2}\sigma^2\theta+\sigma\sqrt{\theta}z}\phi(z)dz = F\int_\zeta^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2\theta+\sigma\sqrt{\theta}z-\frac{1}{2}z^2}dz \tag{2.4}$$

The last step follows from the definition of the standard normal density function $\phi(z)$, given below as

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

Since we are changing our density function and also the random variable we are integrating, we need to change our threshold from K to ζ. We find ζ by solving the equation

$$K = F\exp\left(-\frac{1}{2}\sigma^2\theta + \sigma\sqrt{\theta}\ \zeta\right)$$

This relates ζ to K through the relation between $F(T) \mid \mathcal{F}_t$ and Z, and we see that ζ is

$$\zeta = \frac{\log(K/F) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}} \tag{2.5}$$

Here log is the natural logarithm. From (2.4), we see that the exponentiated polynomial is a square, so

$$\int_K^\infty f\widetilde{\varphi}_\tau(f)df = F\int_\zeta^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{\theta})^2}dz = F\mathbb{Q}(\chi \geq \zeta)$$

Here we have $\chi \sim \mathcal{N}(\sigma\sqrt{\theta},\ 1)$, as we see that the function in our integral is another normal density. Since $\chi = Z + \sigma\sqrt{\theta}$, we can rewrite $\mathbb{Q}(\chi \geq \zeta)$ in terms of Z, and substituting for ζ the result in (2.5), we have

$$\begin{aligned}\mathbb{Q}(\chi \geq \zeta) &= \mathbb{Q}\left(Z + \sigma\sqrt{\theta} \geq \frac{\log(K/F) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) = \mathbb{Q}\left(Z \geq \frac{\log(K/F) - \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right)\\ &= \mathbb{Q}\left(Z \leq \frac{\log(F/K) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) = \Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right)\end{aligned} \tag{2.6}$$

Here the second to last step follows from the symmetry of Z around 0, and the function $\Phi(z)$ is the standard normal cdf. Using our result from (2.6), and replacing θ with $T-t$, we have that

$$\int_K^\infty f\widetilde{\varphi}_\tau(f)df = F\Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \tag{2.7}$$

This is the first normal cdf term in Black's formula. Moving on to the risk-neutral conditional probability $\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t)$ we defined in (2.2), we again recall our expression for $F(T) \mid \mathcal{F}_t$ in (2.3). Substituting θ for $T-t$ as before, we can rewrite $\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t)$ in terms of Z, where again $Z \sim \mathcal{N}(0,\ 1)$, as

$$\begin{aligned}\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t) &= \mathbb{Q}\left(Fe^{-\frac{1}{2}\sigma^2\theta+\sigma\sqrt{\theta}Z} \geq K\right) = \mathbb{Q}\left(Z \geq \frac{\log(K/F) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right)\\ &= \mathbb{Q}\left(Z \leq \frac{\log(F/K) - \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) = \Phi\left(\frac{\log(F/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\end{aligned} \tag{2.8}$$

Again we replace θ with $T-t$, and finally can complete our derivation. Substituting our results from (2.7) and (2.8) back into (2.2) and multiplying by $D(t,\ T) = e^{-r(T-t)}$, we get exactly Black's formula, given by

$$C(F,\ t) = D(t,\ T)\left[F\Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - K\Phi\left(\frac{\log(F/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right] \tag{2.9}$$

3 The volatility premium

The main benefit of deriving Black's formula from the risk-neutral pricing formula is that one can re-express the normal cdf terms as risk-neutral probabilities in terms of $F(T)$, which is the inverse of what was done when deriving the formula. Since we have already shown the relationship between the second normal cdf term and its corresponding $\mathbb{Q}$ probability in (2.8), we work on the first normal cdf term. From (2.3), we have

$$Z = \frac{\log(F(T)/F) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \,\bigg|\, \mathcal{F}_t \sim \mathcal{N}(0,\ 1) \tag{3.1}$$

Interchanging $T-t$ and θ as usual, we can re-express the first normal cdf term in (2.9) using (3.1) as

$$\begin{aligned}\Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) &= \mathbb{Q}\left(Z \leq \frac{\log(F/K) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) = \mathbb{Q}\left(Z \geq \frac{\log(K/F) - \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) \\ &= \mathbb{Q}\left(\frac{\log(F(T)/F) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}} \geq \frac{\log(K/F) - \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}} \,\bigg|\, \mathcal{F}_t\right) \\ &= \mathbb{Q}\left(F(T) \geq Ke^{-\sigma^2(T-t)} \,\Big|\, \mathcal{F}_t\right)\end{aligned} \tag{3.2}$$

Substituting our results from (2.8) and (3.2) into the Black formula given in (2.9), we have

$$C(F,\ t) = D(t,\ T)\left[F\mathbb{Q}\left(F(T) \geq Ke^{-\sigma^2(T-t)} \,\Big|\, \mathcal{F}_t\right) - K\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t)\right] \tag{3.3}$$

The intuition here is that under the Black-Scholes model, one can express the value of a European call option as the discounted difference between the current value of the underlying, here the futures price F, and the strike K, each weighted by a risk-neutral probability of exercise. One can observe that the first exercise probability $\mathbb{Q}(F(T) \geq Ke^{-\sigma^2(T-t)})$ has the strike weighted by the total variance $\sigma^2(T-t)$, such that ceteris paribus, if σ increases, $C(F,\ t)$ increases. This expresses the fact that as volatility increases, ceteris paribus, the value of an option increases. We can quantify this by separating out the volatility premium[1], as

$$\mathbb{Q}\left(F(T) \geq Ke^{-\sigma^2(T-t)} \,\Big|\, \mathcal{F}_t\right) = \Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) = \mathbb{Q}\left(Z \leq -\zeta + \sigma\sqrt{\theta}\right) = \int_{-\infty}^{-\zeta+\sigma\sqrt{\theta}} \phi(z)dz$$

$$\mathbb{Q}\left(F(T) \geq K \mid \mathcal{F}_t\right) = \Phi\left(\frac{\log(F/K) - \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}}\right) = \mathbb{Q}(Z \leq -\zeta) = \int_{-\infty}^{-\zeta} \phi(z)dz$$

Here we have ζ as defined in (2.5), $\theta = T-t$, and $\phi(z)$ is the normal pdf. We can thus see that

$$\mathbb{Q}\left(F(T) \geq Ke^{-\sigma^2(T-t)} \,\Big|\, \mathcal{F}_t\right) = \mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t) + \int_{-\zeta}^{-\zeta+\sigma\sqrt{T-t}} \phi(z)dz$$

We can therefore define the volatility premium as

$$\lambda_\sigma(F,\ \theta) = \int_{-\zeta}^{-\zeta+\sigma\sqrt{\theta}} \phi(z)dz, \text{ where } -\zeta = \frac{\log(F/K) - \frac{1}{2}\sigma^2\theta}{\sigma\sqrt{\theta}} \tag{3.4}$$

Here we emphasize the time-dependent parameters only, as all other parameters are constants. We know that $\lambda_\sigma \geq 0$ as by definition of a density function, $\phi(z) \geq 0$, and because $\sigma > 0$ and $\theta \geq 0$. Using our new results in (3.4), we can now rewrite (3.3) with only $\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t)$ while incorporating λ_σ to arrive at

$$C(F,\ t) = D(t,\ T)\left[\mathbb{Q}(F(T) \geq K \mid \mathcal{F}_t)(F-K) + F\int_{-\zeta}^{-\zeta+\sigma\sqrt{T-t}} \phi(z)dz\right] \tag{3.5}$$

We have now expressed $C(F,\ t)$ as a sum of the difference $F-K$ between the underlying and the strike weighted by the risk-neutral exercise probability $\mathbb{Q}(F(T) \geq K)$ and the volatility premium weighted by F.

[1]Here we use the term 'volatility' to refer to the total volatility, i.e. the instantaneous diffusion coefficient σ multiplied by $\sqrt{T-t}$, the square root of the time to maturity. The option time premium is implicitly included in this quantity.

4 Analyzing the vol premium

From our result in (3.5), it is implied that the value of European call options struck at the money is the discounted volatility premium weighted by $F = K$. We make this explicit below by writing

$$C(K,\ t) = D(t,\ T)K \int_{-\zeta'}^{-\zeta' + \sigma\sqrt{T-t}} \phi(z)dz = D(t,\ T)K \int_{-\zeta'}^{\zeta'} \phi(z)dz, \text{ where } -\zeta' = -\frac{1}{2}\sigma\sqrt{T-t}$$

However, since the greatest difference between the price of a European call option under the Black model and its intrinsic value is greatest at the money, one might be interested in how λ_σ changes as we vary the time t futures price F. Because we already showed that the value of at the money Black call is completely determined by F and λ_σ, our intuition tells us that a plot of λ_σ against F is most likely concave, with one maximum at $F = K$. Taking the partial derivative $\partial\lambda_\sigma/\partial F$, we thus have

$$\frac{\partial \lambda_\sigma}{\partial F} = \frac{\partial}{\partial F}\int_{-\zeta}^{-\zeta+\sigma\sqrt{\theta}} \phi(z)dz = \frac{\partial}{\partial F}\left[\Phi\left(-\zeta + \sigma\sqrt{\theta}\right) - \Phi(-\zeta)\right] = \phi\left(-\zeta + \sigma\sqrt{\theta}\right)\frac{1}{F\sigma\sqrt{\theta}} - \phi(-\zeta)\frac{1}{F\sigma\sqrt{\theta}}$$

Factoring the expression above and rearranging terms, we simply have

$$\frac{\partial \lambda_\sigma}{\partial F} = \frac{1}{F\sigma\sqrt{\theta}}\left[\phi\left(-\zeta + \sigma\sqrt{\theta}\right) - \phi(-\zeta)\right] \tag{4.1}$$

We consider the cases $F < K$, $F = K$, and $F > K$ to roughly understand how λ_σ changes as F changes.

$F < K$: In this case, assuming $\sigma\sqrt{\theta} > 0$, $\phi\left(-\zeta + \sigma\sqrt{\theta}\right) > \phi(-\zeta)$. We can thus conclude that

$$\frac{\partial \lambda_\sigma}{\partial F}\bigg|_{F<K} > 0$$

$F = K$: In this case, we see that $-\zeta = -\zeta' = -\frac{1}{2}\sigma\sqrt{\theta}$, so $-\zeta' + \sigma\sqrt{\theta} = \zeta'$. Due to the symmetry of $\phi(z)$ around 0, we have that $\phi(\zeta') - \phi(-\zeta') = 0$, so we conclude that

$$\frac{\partial \lambda_\sigma}{\partial F}\bigg|_{F=K} = 0$$

$F > K$: In this case, assuming $\sigma\sqrt{\theta} > 0$, $\phi\left(-\zeta + \sigma\sqrt{\theta}\right) < \phi(-\zeta)$. We can thus conclude that

$$\frac{\partial \lambda_\sigma}{\partial F}\bigg|_{F>K} < 0$$

This confirms our intuition that λ_σ is greatest when $F = K$. Plots of λ_σ over F and $\partial\lambda_\sigma/\partial F$ over F in Figures 1 and 2 show more complex dynamics, but still confirm our results. Interestingly, because of the lognormal dynamics assumed in the Black model, the plots of λ_σ and $\partial\lambda_\sigma/\partial F$ over F are positively skewed.

5 Conclusion

In this article, we derived the Black formula by means of conditional expectation, which allowed us to re-express the formula in terms of risk-neutral exercise probabilities and isolate the volatility premium embedded in an option's price. We were able to avoid constructing the risk-free portfolio and solving the resulting PDE as outlined in the original Black-Scholes (1973) and Black (1976) papers. The probabilistic approach may be preferred for its financial intuition over the PDE approach, which requires a change of variable into dimensionless groups, thus temporarily causing the equation to lose all financial meaning.

6 Figures

All figures were plotted in R.

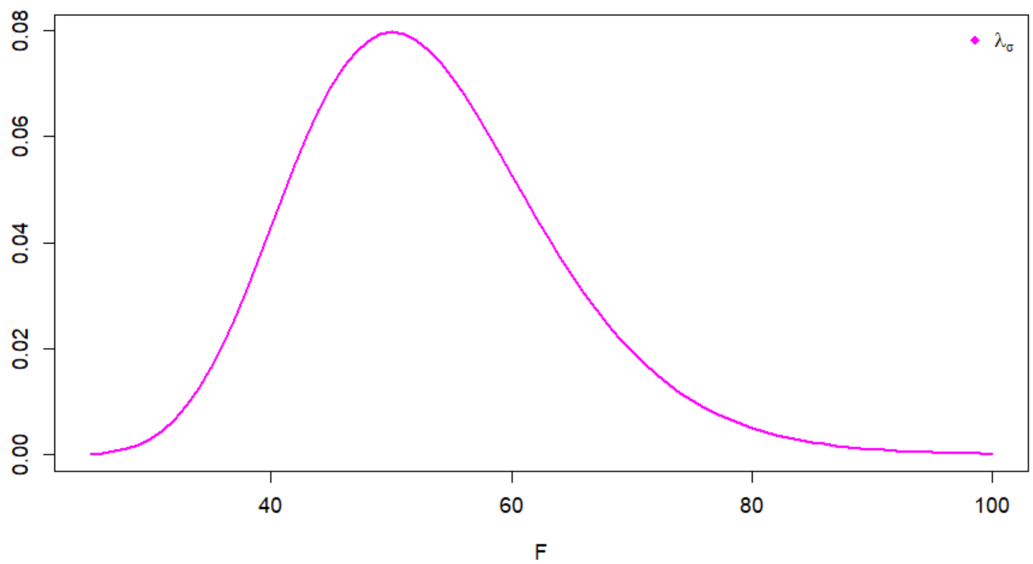

Figure 1: Plot of λ_σ over F with arbitrary parameters $\sigma = 0.2$, $K = 50$, $T - t = 1$.

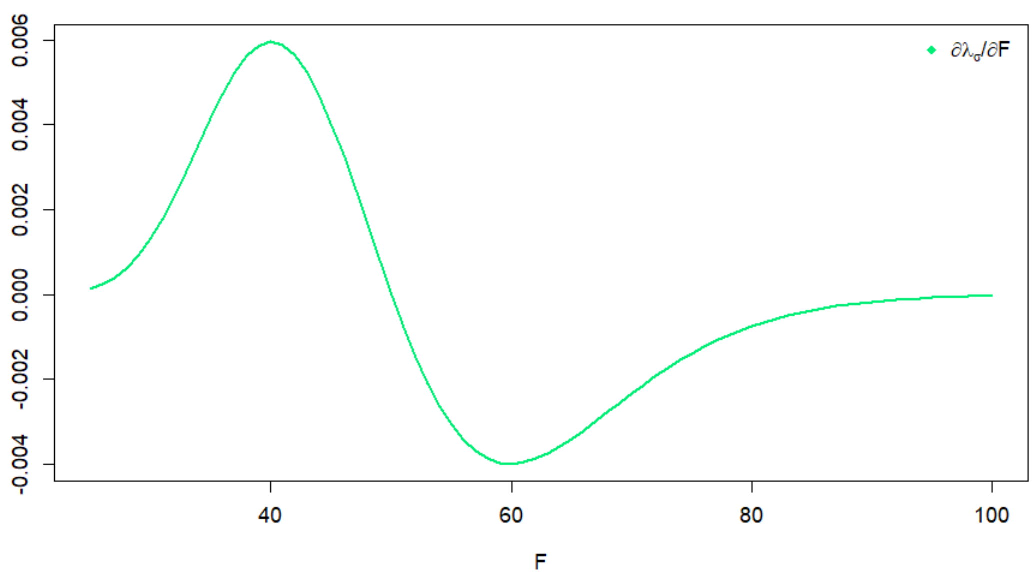

Figure 2: Plot of $\partial\lambda_\sigma/\partial F$ over F with arbitrary parameters $\sigma = 0.2$, $K = 50$, $T - t = 1$.