Pricing a Black-Scholes log contract

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1 Motivation

While browsing through the quantitative finance Stack Exchange website perhaps a couple months ago, I stumbled upon a question asking how to price a log-contract using the Black-Scholes PDE[1]. I had just started looking into quantitative finance at the time, and although intrigued by the question, did not understand the solution nor know how to approach it myself. The problem text is stated, almost verbatim, below:

∼ The payoff of a so-called European log-contract is $\Lambda(S_T) = \log(S_T/K)$ where K is the strike price and S is a risky non-dividend Black-Scholes asset. Find the price $C(S,\ t)$ of the log-contract.

Hint: Use the Black-Scholes PDE and give yourself the fact that $C(S,\ t)$ has the following form:

$$C(S,\ t) = a(t) + b(t)\log(S/K)$$

Find the functions $a(t)$ and $b(t)$.

The problem nudges one towards using the Black-Scholes PDE as the starting point for the answer, but as usual, there is more than one way to approach the problem. We present in this article an alternative approach to answering this question by means of the risk-neutral pricing formula to get our solution.

2 A conditional expectation

The risk-neutral pricing formula states that the arbitrage price of a European contingent claim $V(S,\ t)$ expiring at time T, where $t \leq T$ is the current time and S is the current level of the underlying $S(t)$, is

$$V(S,\ t) = \mathbb{E}[D(t,\ T)\Lambda(S(T)) \mid \mathcal{F}_t] \tag{2.1}$$

Here $D(t,\ T)$ is the discount factor for the interval $[t,\ T)$, $\mathcal{F}_t$ is the natural filtration at time t, and the expectation $\mathbb{E}[\ \cdot\ ]$ is under the risk-neutral measure $\mathbb{Q}$. The price of our Black-Scholes log-contract is thus

$$C(S,\ t) = \mathbb{E}\left[e^{-r(T-t)}\log(S(T)/K)\ \middle|\ \mathcal{F}_t\right] = e^{-r\theta}\int_{-\infty}^{\infty}\log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds \tag{2.2}$$

Here we used the definition of conditional expectation. We have $D(t,\ T) = e^{-r(T-t)}$ under the Black-Scholes model, where r is the continuously compounded risk-free discount rate, and $\varphi(s,\ T \mid \mathcal{F}_t)$ is the pdf of $S(T)$ conditional on $\mathcal{F}_t$. Simplifying the integral in the above expression, we have

$$\int_{-\infty}^{\infty}\log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds = \int_{-\infty}^{\infty}\log(s)\varphi(s,\ T \mid \mathcal{F}_t)ds - \log K\int_{-\infty}^{\infty}\varphi(s,\ T \mid \mathcal{F}_t)ds$$

Since the integral of a real-valued density function over $\mathbb{R}$ must equal 1, we thus have

$$\int_{-\infty}^{\infty}\log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds = \int_{-\infty}^{\infty}\log(s)\varphi(s,\ T \mid \mathcal{F}_t)ds - \log K \tag{2.3}$$

[1]Link: https://quant.stackexchange.com/questions/31397/pricing-log-contract-with-black-scholes-pde

In order to proceed further, we must determine the form of $\varphi(s,\ T \mid \mathcal{F}_t)$ if we wish to proceed further. Under the Black-Scholes model, we know that under $\mathbb{Q}$, the underlying stock $S(t)$ follows the risk-neutral dynamics

$$dS(t) = rS(t)dt + \sigma S(t)dW_t \tag{2.4}$$

Here σ is the constant instantaneous standard deviation of returns and dW_t is a differential change in a Wiener process. Solving (2.4) for $S(T)$ conditional on $\mathcal{F}_t$, so $S(t) = S$, and using the fact that since $W_t \sim \mathcal{N}(0,\ t)$, we can write $W_{t\ \dots\ T}$ as $\sqrt{T-t}\ Z$, with $Z \sim \mathcal{N}(0,\ 1)$, and thus have

$$S(T) \mid \mathcal{F}_t = S \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\ Z\right] \tag{2.5}$$

We can therefore instead integrate over $\mathbb{R}$ using $\phi(z)$, the standard normal pdf, instead of using $\varphi(s,\ T \mid \mathcal{F}_t)$. Substituting our expression for $S(T) \mid \mathcal{F}_t$ in (2.5) into (2.3), and setting $\theta = T - t$ for brevity, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \log(s/K)\varphi(s,\ T \mid \mathcal{F}_t)ds &= \int_{-\infty}^{\infty} \left[\log S + \left(r - \frac{1}{2}\sigma^2\right)\theta + \sigma\sqrt{\theta}\ z\right]\phi(z)dz - \log K \\
&= \left[\log S + \left(r - \frac{1}{2}\sigma^2\right)\theta\right]\int_{-\infty}^{\infty}\phi(z)dz + \sigma\sqrt{\theta}\int_{-\infty}^{\infty} z\phi(z)dz - \log K \qquad (2.6) \\
&= \log S + \left(r - \frac{1}{2}\sigma^2\right)\theta - \log K
\end{aligned}$$

The last step follows from the fact that the integral of a real-valued density function over $\mathbb{R}$ must equal 1, and that by the definition of expectation and by fact that $Z \sim \mathcal{N}(0,\ 1)$, we have that

$$\int_{-\infty}^{\infty} z\phi(z)dz = \mathbb{E}[Z] = 0$$

Substituting our results in (2.6) back into (2.2), replacing θ with $T - t$, and rearranging terms, we have

$$C(S,\ t) = e^{-r(T-t)}\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \log(S/K)\right] \tag{2.7}$$

It thus follows that we have $a(t)$ and $b(t)$ given by

$$\begin{aligned}
a(t) &= e^{-r(T-t)}\left(r - \frac{1}{2}\sigma^2\right)(T-t) \\
b(t) &= e^{-r(T-t)}
\end{aligned}$$

We have therefore answered the question without a single PDE, by first taking a conditional expectation to get the final expression for $C(S,\ t)$, and then using the form of $C(S,\ t)$ to extract $a(t)$ and $b(t)$.