The extra log

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July 12, 2019

1 Problem outline

In Beckers' [1] paper on the CEV model and using it price options on single name stocks, he makes a strange substitution. Consider the stochastic differential equation given by

$$dS(t) = \mu S(t)dt + \sigma S^{\beta}(t)dW_t \tag{1.1}$$

This is a standard Ito drift-diffusion process, with drift term $\mu S(t)$ and diffusion term $\sigma S^{\beta}(t)$. W_t is a standard Wiener process, distributed as $W_t \sim \mathcal{N}(0,\ t)$. $\mu,\ \sigma,\ \beta$ are constants, and $0 \leq \beta < 1$. β effectively skews the return distribution of this process; $\beta = 1$ for a lognormal diffusion (Black-Scholes model for underlying), $\beta = 0$ for a normal diffusion, $\beta = 1/2$ for a noncentral chi-squared (CIR process) diffusion.

Beckers makes the observation that returns $S(t+dt)/S(t)$, implicitly conditioned on $S(t)$ (level of underlying at time t, are normally distributed. We can make this clear by writing that

$$\left.\frac{S(t+dt)}{S(t)}\ \right|\ S(t) \sim \mathcal{N}\left(1 + \mu dt,\ \sigma^2 S^{2(\beta-1)}(t)dt\right) \tag{1.2}$$

Beckers is a little confusing with terminology in the paper–he drops dt when talking about instantaneous percentage returns $dS(t)/S(t)$, when you really should not. So for simplicity, we just set $dt = 1$, and have

$$\operatorname{sd}\left[\frac{S(t+1)}{S(t)}\right] = \sigma S^{\beta-1}(t)$$

Here sd[$\cdot$] is standard deviation. If we take the natural logarithm, denoted by log, we have

$$\log\left(\operatorname{sd}\left[\frac{S(t+1)}{S(t)}\right]\right) = \log\sigma + (\beta - 1)\log S(t) \tag{1.3}$$

He casts this as a regression equation, with constant $a = \log\sigma$, independent variable $S(t)$ with coefficient $b = \beta - 1$, and dependent variable $\log\left(\operatorname{sd}[S(t+1)/S(t)]\right)$. However, obviously one cannot calculate the standard deviation of a single observation. Beckers show in Appendix C that given a normal random variable $X \sim \mathcal{N}(\nu,\ \tau^2)$, where μ is sufficiently small compared to τ, it holds that

$$\mathbb{E}|X| \cong \tau\sqrt{\frac{2}{\pi}} \tag{1.4}$$

This is the expectation $|X|$, which follows the folded normal distribution. In general, $\forall\nu,\ \tau$, we have [2]

$$\mathbb{E}|X| = \tau\sqrt{\frac{2}{\pi}}e^{-\nu^2/2\tau^2} + \nu\left[1 - 2\Phi\left(-\frac{\nu}{\tau}\right)\right] \tag{1.5}$$

Here $\Phi(z)$ is the standard normal cdf. It is easy to see that in the limit $\nu \to 0$, (1.5) will become (1.4). Beckers argues that since the mean of $S(t+1)/S(t)$ is small, the approximate relation in (1.4) is justified [1]. So from (1.2), making the substitutions of $S(t+1)/S(t)$ for X and $\sigma S^{\beta-1}(t)$ for τ, we have

$$\mathbb{E}\left|\frac{S(t+1)}{S(t)}\right| \cong \sigma S^{\beta-1}(t)\sqrt{\frac{2}{\pi}} \tag{1.6}$$

Taking the natural logarithm, we therefore get something analogous to (1.3), and arrive at

$$\log\left(\mathbb{E}\left|\frac{S(t+1)}{S(t)}\right|\right) \cong \frac{1}{2}\log(2/\pi) + \log\sigma + (\beta-1)\log S(t) \tag{1.7}$$

Writing (1.7) as a regression equation, where ϵ is an error term, would give us[1]

$$\log\left|\frac{S_{t+1}}{S_t}\right| = a + b\log S_t + \epsilon \tag{1.8}$$

We subscript instead of using parentheses to emphasize that these are data points. Here a and b are

$$a = \frac{1}{2}\log(2/\pi) + \log\sigma$$
$$b = \beta - 1$$

However, compared to Beckers' result (3), he has another log of the $|S_{t+1}/S_t|$ as well [1]. This makes the left-hand sides totally different. We can ignore the extra constant in (1.7). But what gives?

2 Resolution

Make the observation that the distribution of $[S(t+1) - S(t)]/S(t)$, implicitly conditioned on $S(t)$, is

$$\left.\frac{S(t+1)-S(t)}{S(t)}\ \right|\ S(t) \sim \mathcal{N}\left(\mu,\ \sigma^2 S^{2(\beta-1)}(t)\right)$$

Here we set $dt = 1$. Note how $[S(t+1) - S(t)]/S(t)$ has the same standard deviation as $S(t+1)/S(t)$. In finance, it is very common to approximate instantaneous returns and period returns with log returns, especially because of the nice additive property that log returns provide. Basically, the following set of approximations are typically treated as the same most of the time, except in cases where it matters:

$$\frac{dS(t)}{S(t)} \approx \log\frac{S(t+1)}{S(t)} \approx \frac{S(t+1)-S(t)}{S(t)} \tag{2.1}$$

Implicitly, they are also assumed to be distributed almost the same, although this is a non-technical way of putting it. So following this line of thought, we substitute $S(t+1)/S(t)$ in (1.3) with $[S(t+1) - S(t)]/S(t)$ instead, as we have shown that they have the same standard deviation. Thus, (1.7) becomes

$$\log\left(\mathbb{E}\left|\frac{S(t+1)-S(t)}{S(t)}\right|\right) \cong \frac{1}{2}\log(2/\pi) + \log\sigma + (\beta-1)\log S(t)$$

But if from (2.1) we then substitute $[S(t+1) - S(t)]/S(t)$ with $\log S(t+1) - \log S(t)$, we have

$$\log\left(\mathbb{E}\left|\log\frac{S(t+1)}{S(t)}\right|\right) \cong \frac{1}{2}\log(2/\pi) + \log\sigma + (\beta-1)\log S(t)$$

Therefore, our expression in (1.8) becomes

$$\log\left|\log\frac{S(t+1)}{S(t)}\right| = a + b\log S_t + \epsilon \tag{2.2}$$

Where a, b, and ϵ are the same as defined for (1.8). This corresponds to Beckers' result (3).

[1] Note that by Jensen's inequality, since the log function is concave, the specification in (1.8) would lead to the estimator

$$\hat{y}(S_t \mid \beta) = \mathbb{E}\log\left|\frac{S_{t+1}}{S_t}\right| \le \log\left(\mathbb{E}\left|\frac{S_{t+1}}{S_t}\right|\right)$$

This introduces significant bias into the estimate, but we don't really care since we are regressing only to estimate β. Our estimation problem is just a error minimization problem, for example mean squared error, given by

$$\beta = \arg\min_{\beta}\left\{\frac{1}{N-1}\sum_{k=1}^{N-1}\left[\log\left|\frac{S_{k+1}}{S_k}\right| - \hat{y}(S_t \mid \beta)\right]^2\right\}$$

Here each $S_1,\ \ldots\ S_N$ is the price of a single stock S, indexed in time order.

3 Conclusion

Hopefully you made it this far. Anyways, the moral of the story is that when you get stuck, not to quickly look for help but to spend some time thinking about it on your own, as you may have some prior knowledge that may resolve the issue. I still think Becker shouldn't have made this arbitrary substitution without explaining it. Also I may still be wrong; this is the most intuitive explanation I can come up with.

References

[1] Beckers, S. (1980). The Constant Elasticity of Variance Model and Its Implications For Option Pricing. *Journal of Finance*, *35*(3), 661-673.

[2] Beneki, C., Hassani, H., Tsagris, M. (2014). On the Folded Normal Distribution.