Some matrix exercises

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1 Computer exercises for the lab

Make sure you are comfortable with the matrix operators in R. Also, be very very sure you can subscript matrices, i.e. use things such as [,1] to select column 1, [,-1] to select everything *except* column 1, and [,2:3] to select columns 2 and 3.

1. Find (where possible) the determinants and the inverse of the following matrices:
   $C_1 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix}$ and $C_3 = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$
   Very briefly comment on the magnitude of the difference between C_2^{-1} and C_3^{-1} given the only difference between C_2 and C_2 amounts to a difference of 0.000001 in the bottom right position.

```
> A <- matrix(c(4,4,4,4),2,2)
> A
     [,1] [,2]
[1,]    4    4
[2,]    4    4
> det(A)
[1] 0
> solve(A)
Error in solve.default(A) : Lapack routine dgesv: system is exactly singular
> B <- matrix(c(4,4.001,4.001,4.002),2,2)
> B
      [,1]  [,2]
[1,] 4.000 4.001
[2,] 4.001 4.002
> det(B)
[1] -1e-06
> solve(B)
         [,1]     [,2]
[1,] -4002000  4001000
```

```
[2,]  4001000 -4000000
> C <- matrix(c(4,4.001,4.001,4.002001),2,2)
> C
      [,1]     [,2]
[1,] 4.000 4.001000
[2,] 4.001 4.002001
> det(C)
[1] 3e-06
> solve(C)
         [,1]     [,2]
[1,]  1334000 -1333667
[2,] -1333667  1333333
```

Note that A is singular, the determinant is zero and it can't be inverted. Also note that the inverses of B and C are very very different - but this is something of a pathological example

2. Matrix partitioning. Consider Sterling's financial data held in the R object LifeCycleSavings (see `?LifeCycleSavings`). To make life a little easier, reorder the columns using `X <- LifeCycleSavings[,c(2,3,1,4,5)]`.

   - Find the correlation matrix of `X` (longhand, using the centering matrix), call this matrix `R`

     ```
     > data(LifeCycleSavings)
     > X <- LifeCycleSavings[,c(2,3,1,4,5)]
     > R <- cor(X)
     > R
                pop15       pop75         sr        dpi        ddpi
     pop15  1.00000000 -0.90847871 -0.4555381 -0.7561881 -0.04782569
     pop75 -0.90847871  1.00000000  0.3165211  0.7869995  0.02532138
     sr    -0.45553809  0.31652112  1.0000000  0.2203589  0.30478716
     dpi   -0.75618810  0.78699951  0.2203589  1.0000000 -0.12948552
     ddpi  -0.04782569  0.02532138  0.3047872 -0.1294855  1.00000000
     ```

   - Partition $R = cov(X)$ following the scheme below such that $\boldsymbol{R_{11}}$ is a 2×2 matrix containing the covariance of `pop15` and `pop75`, and $\boldsymbol{R_{22}}$ contains the covariance of `sr`, `dpi` and `ddpi`

     $$\boldsymbol{R} = \left(\begin{array}{c|c} \boldsymbol{R_{11}} & \boldsymbol{R_{12}} \\ \hline \boldsymbol{R_{21}} & \boldsymbol{R_{22}} \end{array} \right)$$

     You should find for example that $\boldsymbol{R_{11}}$ is given by:

     ```
     > R11 <- R[1:2,1:2]
     > R12 <- R[1:2,3:5]
     > R21 <- R[3:5,1:2]
     > R22 <- R[3:5,3:5]
     ```

	pop15	pop75
pop15	83.75	−10.73
pop75	−10.73	1.67

```
> R11
          pop15      pop75
pop15  1.0000000 -0.9084787
pop75 -0.9084787  1.0000000
> R12
              sr        dpi        ddpi
pop15 -0.4555381 -0.7561881 -0.04782569
pop75  0.3165211  0.7869995  0.02532138
```

- Find the matrix $\boldsymbol{A}$, where:

$$\boldsymbol{A} = \boldsymbol{R_{22}^{-1} R_{21} R_{11}^{-1} R_{12}}$$

```
A <- solve(R22) %*% R21 %*% solve(R11) %*% R12
> A
                sr        dpi         ddpi
sr    0.2082835957 0.13567828  0.028218990
dpi   0.2428243102 0.60736349  0.019316733
ddpi -0.0001528633 0.06307685 -0.001930969
```

- Find the matrix $\boldsymbol{B}$ where:

$$\boldsymbol{B} = \boldsymbol{R_{11}^{-1} R_{12} R_{22}^{-1} R_{21}}$$

```
> B <- solve(R11) %*% R12 %*% solve(R22) %*% R21
> B
          pop15      pop75
pop15  0.4471770 -0.3095406
pop75 -0.2362827  0.3665391
```

- Are $\boldsymbol{A}$ and $\boldsymbol{B}$ symmetric? What is the difference between symmetric and asymmetric matrices in terms of their eigenvalues and eigenvectors?
  Note that they are both asymmetric matrices, it just so happens for these particular matrices that the eigenvalues are positive and the eigenvectors are real. This isn't always the case for asymmetric matrices!
- Find the eigenvalues and eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$ then find the square roots of the eigenvalues.

```
> eigen(A)
$values
```

```
[1]  6.802894e-01  1.334267e-01 -3.516940e-19

$vectors
            [,1]       [,2]        [,3]
[1,] -0.28005745 -0.8743085 -0.15327638
[2,] -0.95591190  0.4395680  0.02986593
[3,] -0.08831912  0.2058267  0.98773194

> eigen(B)
$values
[1] 0.6802894 0.1334267

$vectors
           [,1]      [,2]
[1,]  0.7988131 0.7023149
[2,] -0.6015793 0.7118664

> sqrt(eigen(A)$values)
[1] 0.8247966 0.3652762       NaN
> sqrt(eigen(B)$values)
[1] 0.8247966 0.3652762
```

- Do you notice any similarities between the first two eigenvalues from either matrix?
  Note that the square roots of the eigen values are identical.

3. Revisit the `wines` data in the `Flury` package. Consider only Y1, Y5, Y6, Y8 and Y9, use matrix algebra to find the means, correlation and covariance of these data. Compare the eigenvalues and eigenvectors, and the determinants and inverse you get from the covariance matrix and the correlation matrix.

   Part of this is straightforward application of algebra covered in notes. Remainder follows, eigenvalues and eigenvectors or corrlation matrix and covariance matrix are very very different

```
> eigen(cor(X))
$values
[1] 1.9816214 1.5230931 0.7636275 0.4807668 0.2508912

$vectors
           [,1]       [,2]       [,3]         [,4]        [,5]
[1,] -0.2777688  0.6861076 -0.1803411 -0.008575869  0.64769164
[2,] -0.5561121 -0.1831683  0.2691167  0.763621722  0.04058118
[3,] -0.4812999 -0.3654121  0.4287131 -0.604767532  0.29203728
[4,] -0.5632681  0.3692665 -0.1701835 -0.225349857 -0.68310049
```

```
[5,] -0.2542899 -0.4752074 -0.8260121 -0.016792704  0.16412387

> eigen(cov(X))
$values
[1] 2330.125258  814.467485  545.200401    3.584549    1.403159

$vectors
            [,1]         [,2]        [,3]         [,4]         [,5]
[1,] -0.05483952  0.735862357  0.67335086  0.007708220  0.045148845
[2,]  0.50030899  0.604193500 -0.61998120 -0.013591715  0.008928181
[3,]  0.86393142 -0.303609773  0.40160604 -0.008022325  0.009582822
[4,]  0.01284463  0.035080362  0.02616497  0.195179471 -0.979706462
[5,]  0.01187679 -0.006876112 -0.01580852  0.980610251  0.194846828
```

2 Consolidation Exercises

You should complete these exercises over the next week. You are guaranteed to meet some simple matrix arithmetic in the exam! We will briefly go through the answers in class next week. Don't rely on memorising model solutions.

1. Which of the following are orthogonal to each other:

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} 6 \\ 7 \\ 1 \\ -2 \end{pmatrix} \boldsymbol{z} = \begin{pmatrix} 5 \\ -4 \\ 5 \\ 7 \end{pmatrix}$$

Normalise each of the two orthogonal vectors.

$$\boldsymbol{x}^T\boldsymbol{y} = \begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \times 6+ & -2 \times 7+ & 3 \times 1+ & -4 \times -2 \end{pmatrix} = 3$$

i.e. not orthogonal

$$\boldsymbol{x}^T\boldsymbol{z} = \begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \times 5+ & -2 \times -4+ & 3 \times 5+ & -4 \times 7 \end{pmatrix} = 0$$

i.e. orthogonal

$$\boldsymbol{y}^T\boldsymbol{z} = \begin{pmatrix} 6 & 7 & 1 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 6\times 5+ & 7\times -4+ & 1\times 5+ & -2\times 7 \end{pmatrix} = -7$$

i.e. not orthogonal

$$\begin{aligned} \|\boldsymbol{x}\| &= \sqrt{1+4+9+16} = \sqrt{30} \\ \|\boldsymbol{y}\| &= \sqrt{36+49+1+4} = \sqrt{90} \\ \|\boldsymbol{z}\| &= \sqrt{25+16+25+49} = \sqrt{115} \end{aligned}$$

Normalise the orthogonal vector, divide by the length of the vector:

$$\boldsymbol{y}_{norm} = \begin{pmatrix} 6/\sqrt{90} \\ 7/\sqrt{90} \\ 1/\sqrt{90} \\ 2/\sqrt{90} \end{pmatrix}$$

2. Find vectors which are orthogonal to:

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \boldsymbol{v} = \begin{pmatrix} 2 \\ 4 \\ -1 \\ 2 \end{pmatrix}$$

Any w_1 and w_2 that satisfies $w_1 + 3w_2 = 0$ will be orthogonal to $\boldsymbol{u}$, e.g. $w = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. For $\boldsymbol{v}$ we require $2x_1 + 4x_2 - x_3 + 2x_4 = 0$, which can be solved with $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

3. Find vectors which are orthonormal to:

$$\boldsymbol{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$$

If $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, then $\boldsymbol{x}^T\boldsymbol{u} = \frac{1}{\sqrt{2}}u_1 - \frac{1}{\sqrt{2}}u_3$, so $\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is orthogonal, and we only require to normalise it. $\|\boldsymbol{u}\| = \sqrt{1+0+1} = \sqrt{2}$. Consequently, $\boldsymbol{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ has length 1 and is orthogonal.

In a similar way, we find $\begin{pmatrix} 0 \\ 0 \\ 5 \\ -1 \end{pmatrix}$, a vector which has length $\sqrt{25+1}$, so the orthogonal vector of unit length is given by: $\begin{pmatrix} 0 \\ 0 \\ 5/\sqrt{26} \\ -1/\sqrt{26} \end{pmatrix}$

4. What are the determinants of:

$$(a) \begin{pmatrix} 1 & 3 \\ 6 & 4 \end{pmatrix} (b) \begin{pmatrix} 3 & 1 & 6 \\ 7 & 4 & 5 \\ 2 & -7 & 1 \end{pmatrix}$$

Solutions are (a) -14, (b) -222.

5. Invert the following matrices:

$$(a) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} (b) \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} (c) \begin{pmatrix} 3 & 2 & -1 \\ 1 & 4 & 7 \\ 0 & 4 & 2 \end{pmatrix} (d) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -1 \\ 3 & 1 & -1 \end{pmatrix}$$

Solutions are (a) $\begin{pmatrix} 0.333 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.111 \end{pmatrix}$, note the determinant is 108

(b) $\begin{pmatrix} 0.714 & -0.429 \\ -0.143 & 0.286 \end{pmatrix}$ (with a determinant of 7),

(c) $\begin{pmatrix} 0.294 & 0.118 & -0.265 \\ 0.029 & -0.088 & 0.325 \\ -0.059 & 0.176 & -0.147 \end{pmatrix}$ (with a determinant of -68) and

(d) $\begin{pmatrix} 0.222 & -0.111 & 0.333 \\ 0.056 & 0.222 & -0.167 \\ 0.722 & -0.111 & -0.l67 \end{pmatrix}$ (with a determinant -18)

6. Find eigenvalues and corresponding eigen vectors for the following matrices:

$$\boldsymbol{a} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \boldsymbol{b} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \boldsymbol{c} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \boldsymbol{d} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\boldsymbol{e} = \begin{pmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{f} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{g} = \begin{pmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{pmatrix}$$

(a)

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} &= (1-\lambda)(3-\lambda) - 8 = 0 \\ &= 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0 \\ &= \lambda^2 - 4\lambda - 5 = 0 \\ &= (\lambda - 5)(\lambda + 1) = 0 \\ &= \lambda = -1, 5 \end{aligned}$$

Now, assuming $\lambda = -1$:

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \boldsymbol{A} + \boldsymbol{I} = \begin{pmatrix} 1 & 4 \\ 2+3 & \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix})$$

So that:

$$(\boldsymbol{A} + \boldsymbol{I})\boldsymbol{e} = 0 = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So two solutions are given by:

$$\begin{aligned} 2e_1 + 4e_2 = 0 \\ 2e_1 + 4e_2 = 0 \end{aligned}$$

which are the same(!) and have *a* solution $e_1 = 2, e_2 = -1$, so $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector. Given the length of this vector ($\sqrt{2^2 + -1^2}$), we have a normalised eigenvector as:

$$\begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

Also, we might like to consider the case where $\lambda = 5$:

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \boldsymbol{A} - 5\boldsymbol{I} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}$$

So that:

$$(\boldsymbol{A} - 5\boldsymbol{I})\boldsymbol{e} = 0 = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So two solutions are given by:

$$\begin{aligned} -4e_1 + 4e_2 &= 0 \\ 2e_1 - 2e_2 &= 0 \end{aligned}$$

which are the same(!) and have *a* solution $e_1 = 1, e_2 = 1$, so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector. Given the length of this vector ($\sqrt{1^2 + 1^2}$), we have a normalised eigenvector as:

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

(b) In a similar way, you should find that the eigenvalues are $4, -1$. Possible eigenvectors are $\begin{pmatrix} -0.555 \\ -0.832 \end{pmatrix}$ and $\begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$

(c) Eigenvalues are $6, 1$. Possible eigenvectors are $\begin{pmatrix} -0.447 \\ 0.894 \end{pmatrix}$ and $\begin{pmatrix} 0.894 \\ 0.447 \end{pmatrix}$

(d) Eigenvalues are $6, 1$. Possible eigenvectors are $\begin{pmatrix} 0.447 \\ 0.894 \end{pmatrix}$ and $\begin{pmatrix} 0.894 \\ -0.447 \end{pmatrix}$

How interesting!

(e)

$$\begin{aligned}
&= \begin{vmatrix} 1-\lambda & 4 & 0 \\ 4 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 4\begin{vmatrix} 4 & 0 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 4 & 1-\lambda \\ 0 & 0 \end{vmatrix} \\
&= (1-\lambda)^3 - 4 \times 4(1-\lambda) \\
&= (1-\lambda)^3 - 16(1-\lambda) \\
&= (1-\lambda)\big((1-\lambda)^2 - 16\big) \\
&= (1-\lambda)\big(1 - 2\lambda + \lambda^2 - 16\big) \\
&= (1-\lambda)(\lambda^2 - 2\lambda - 15) \\
&= (1-\lambda)(\lambda + 3)(\lambda - 5)
\end{aligned}$$

So that eigenvalues are $5, 1, -3$.

With $\lambda = 5$:

$$\begin{pmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which has a solution at $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$ is a normalised eigenvector.

If $\lambda = 1$ we have:

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which has a solution at $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ which is already normalised.

(f) Eigenvalues $9, 4, 1$, with possible eigenvectors: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(g) Eigenvalues $18, 9, 9$, which possible eigenvectors: $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$

which, given length $\sqrt{2^2 + -2^2 + 1^2} = 3$ for example, normalise as: $\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$

7. Convert the following covariance matrix (you've seen it earlier) to a correlation matrix, calculate the eigenvalues and eigenvectors and verify that the eigen vectors are orthogonal.

$$\boldsymbol{g} = \begin{pmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{pmatrix}$$

Form $\boldsymbol{D} = diag(13, 13, 10)$. We need to find $\boldsymbol{D}^{-\frac{1}{2}}$ which is trival for diagonal matrices, i.e. $\boldsymbol{D}^{-\frac{1}{2}} = diag(1/\sqrt{13}, 1/\sqrt{13}, 1/\sqrt{10})$.

The correlation matrix is then given by:

$$\begin{pmatrix} 1/\sqrt{13} & 0 & 0 \\ 0 & 1/\sqrt{13} & 0 \\ 0 & 0 & 1/\sqrt{10} \end{pmatrix} \begin{pmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{13} & 0 & 0 \\ 0 & 1/\sqrt{13} & 0 \\ 0 & 0 & 1/\sqrt{10} \end{pmatrix} =$$

$$\begin{pmatrix} 13/(\sqrt{13}\sqrt{13}) & -4/(\sqrt{13}\sqrt{13}) & 2/(\sqrt{13}\sqrt{10}) \\ -4/(\sqrt{13}\sqrt{13}) & 13/(\sqrt{13}\sqrt{13}) & -2/(\sqrt{13}\sqrt{10}) \\ 2/(\sqrt{13}\sqrt{10}) & -2/(\sqrt{13}\sqrt{10}) & 10/(\sqrt{10}\sqrt{10}) \end{pmatrix}.$$

This has eigenvalues $1.446, 0.862, 0.692$ with possible eigenvectors $\begin{pmatrix} 0.618 \\ -0.618 \\ 0.486 \end{pmatrix}$, $\begin{pmatrix} 0.344 \\ -0.344 \\ -0.874 \end{pmatrix}$, $\begin{pmatrix} 0.707 \\ 0.707 \\ 0 \end{pmatrix}$. This is rather a difficult example to work by hand. Do check in particular that you understand the underlying matrix operations, for example look to see if the matrix procedures for converting from correlation to covariance matrices (and vice versa) are actually quite familiar operations.

8. If $\boldsymbol{X} = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ 4 & 2 \end{pmatrix}$, use matrix procedures to find $cov(\boldsymbol{X})$ and $cor(\boldsymbol{X})$. What is a sum of squares and crossproducts matrix? What are the eigenvalues and eigenvectors of $cov(\boldsymbol{X})$ and $cor(\boldsymbol{X})$?

   If we first form $\boldsymbol{Z}$ by mean-centering $\boldsymbol{X}$, we find $\boldsymbol{Z} = \begin{pmatrix} 2 - 2\frac{1}{3} & 6 - 3\frac{2}{3} \\ 1 - 2\frac{1}{3} & 3 - 3\frac{2}{3} \\ 4 - 2\frac{1}{3} & 2 - 3\frac{2}{3} \end{pmatrix}$. We then need to form the sum of squares and cross-product matrix $\boldsymbol{SSCP} = \boldsymbol{Z}^T\boldsymbol{Z} = \begin{pmatrix} -\frac{1}{3} & -1\frac{1}{3} & 1\frac{2}{3} \\ 2\frac{1}{3} & -\frac{2}{3} & -1\frac{2}{3} \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & 2\frac{1}{3} \\ -1\frac{1}{3} & -\frac{2}{3} \\ 1\frac{2}{3} & -1\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 4\frac{2}{3} & -2\frac{2}{3} \\ -2\frac{2}{3}8\frac{2}{3} & \end{pmatrix}$.

   The covariance is estimated by $\frac{1}{n-1}\boldsymbol{SSCP} = \frac{1}{2}\begin{pmatrix} 4\frac{2}{3} & -2\frac{2}{3} \\ -2\frac{2}{3}8\frac{2}{3} & \end{pmatrix} = \begin{pmatrix} 2\frac{1}{3} & -1\frac{1}{3} \\ -1\frac{1}{3}4\frac{1}{3} & \end{pmatrix}$.

   Eigenvalues of the covariance matrix are $\lambda = 5, 1\frac{2}{3}$, and possibly eigenvectors include $\begin{pmatrix} -0.4472 \\ 0.8944 \end{pmatrix}$, $\begin{pmatrix} -0.8944 \\ -0.4472 \end{pmatrix}$.

   The correlation matrix can be estimated as $\begin{pmatrix} 1 & -0.4193 \\ -0.4193 & 1 \end{pmatrix}$, with eigenvalues $\lambda = 1.4193, 0.5807$, and possible eigenvectors: $\begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix}$, $\begin{pmatrix} -0.7071 \\ -0.7071 \end{pmatrix}$.

9. Find $\dfrac{\begin{vmatrix} 2 & 4 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 9 \\ 4 & 5 \end{vmatrix}}$.

This can be simply shown to be $\frac{14-8}{5-36} = -0.1935$.