Some matrix exercises

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September 29, 2007

1 Computer exercises for the lab

Make sure you are comfortable with the matrix operators in R. Also, be very very sure you can subscript matrices, i.e. use things such as [,1] to select column 1, [,-1] to select everything *except* column 1, and [,2:3] to select columns 2 and 3.

1. Find (where possible) the determinants and the inverse of the following matrices:
   $C_1 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, $C_2 = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{bmatrix}$ and $C_3 = \begin{bmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$
   Very briefly comment on the magnitude of the difference between C_2^{-1} and C_3^{-1} given the only difference between C_2 and C_2 amounts to a difference of 0.000001 in the bottom right position.

2. Matrix partitioning. Consider Sterling's financial data held in the R object LifeCycleSavings (see `?LifeCycleSavings`). To make life a little easier, reorder the columns using `X <- LifeCycleSavings[,c(2,3,1,4,5)]`.

   - Find the correlation matrix of `X` (longhand, using the centering matrix), call this matrix `R`
   - Partition $R = cov(X)$ following the scheme below such that $\boldsymbol{R_{11}}$ is a 2×2 matrix containing the covariance of `pop15` and `pop75`, and $\boldsymbol{R_{22}}$ contains the covariance of `sr`, `dpi` and `ddpi`

   $$\boldsymbol{R} = \left(\begin{array}{c|c} \boldsymbol{R_{11}} & \boldsymbol{R_{12}} \\ \hline \boldsymbol{R_{21}} & \boldsymbol{R_{22}} \end{array} \right)$$

   You should find for example that $\boldsymbol{R_{11}}$ is given by:

	pop15	pop75
pop15	83.75	−10.73
pop75	−10.73	1.67

- Find the matrix $\boldsymbol{A}$, where:

$$\boldsymbol{A} = \boldsymbol{R_{22}^{-1} R_{21} R_{11}^{-1} R_{12}}$$

- Find the matrix $\boldsymbol{B}$ where:

$$\boldsymbol{B} = \boldsymbol{R_{11}^{-1} R_{12} R_{22}^{-1} R_{21}}$$

- Are $\boldsymbol{A}$ and $\boldsymbol{B}$ symmetric? What is the difference between symmetric and asymmetric matrices in terms of their eigenvalues and eigenvectors?

- Find the eigenvalues and eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$ then find the square roots of the eigenvalues.
- Do you notice any similarities between the first two eigenvalues from either matrix?

3. Revisit the `wines` data in the `Flury` package. Consider only Y1, Y5, Y6, Y8 and Y9, use matrix algebra to find the means, correlation and covariance of these data. Compare the eigenvalues and eigenvectors, and the determinants and inverse you get from the covariance matrix and the correlation matrix.

2 Consolidation Exercises

You should complete these exercises over the next week. You are guaranteed to meet some simple matrix arithmetic in the exam! We will briefly go through the answers in class next week. Don't rely on memorising model solutions.

1. Which of the following are orthogonal to each other:

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} 6 \\ 7 \\ 1 \\ -2 \end{pmatrix} \boldsymbol{z} = \begin{pmatrix} 5 \\ -4 \\ 5 \\ 7 \end{pmatrix}$$

   Normalise each of the two orthogonal vectors.

2. Find vectors which are orthogonal to:

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \boldsymbol{v} = \begin{pmatrix} 2 \\ 4 \\ -1 \\ 2 \end{pmatrix}$$

3. Find vectors which are orthonormal to:

$$\boldsymbol{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{6} \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$$

4. What are the determinants of:

$$(a) \begin{pmatrix} 1 & 3 \\ 6 & 4 \end{pmatrix} (b) \begin{pmatrix} 3 & 1 & 6 \\ 7 & 4 & 5 \\ 2 & -7 & 1 \end{pmatrix}$$

5. Invert the following matrices:

$$(a) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} (b) \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} (c) \begin{pmatrix} 3 & 2 & -1 \\ 1 & 4 & 7 \\ 0 & 4 & 2 \end{pmatrix} (d) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -1 \\ 3 & 1 & -1 \end{pmatrix}$$

6. Find eigenvalues and corresponding eigen vectors for the following matrices:

$$\boldsymbol{a} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \boldsymbol{b} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \boldsymbol{c} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \boldsymbol{d} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\boldsymbol{e} = \begin{pmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{f} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{g} = \begin{pmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{pmatrix}$$

7. Convert the following covariance matrix (you've seen it earlier) to a correlation matrix, calculate the eigenvalues and eigenvectors and verify that the eigen vectors are orthogonal.

$$\boldsymbol{g} = \begin{pmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{pmatrix}$$

8. If $\boldsymbol{X} = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ 4 & 2 \end{pmatrix}$, use matrix procedures to find $cov(\boldsymbol{X})$ and $cor(\boldsymbol{X})$. What is a sum of squares and crossproducts matrix? What are the eigenvalues and eigenvectors of $cov(\boldsymbol{X})$ and $cor(\boldsymbol{X})$?

9. Find $\frac{\begin{vmatrix} 2 & 4 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 9 \\ 4 & 5 \end{vmatrix}}$.