Some Title

And Maybe a Subtitle

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Contents

A Title

Contents of the slide

Roots of $f^n(x)$

$$f(x) = (x-\gamma)^2 + \gamma + m$$

- The roots of $f(x)$ are $\gamma \pm \sqrt{-m-\gamma}$
- If α is a root of $f^n(x)$, then $\gamma \pm \sqrt{\alpha - m - \gamma}$ are roots of $f^{n+1}(x)$

Observation

For $n > 0$, the roots of $f^n(x)$ are, with n radicals:

$$\gamma \pm \sqrt{-m \pm \sqrt{-m \pm \sqrt{-m \pm \dots \sqrt{-m-\gamma}}}}$$

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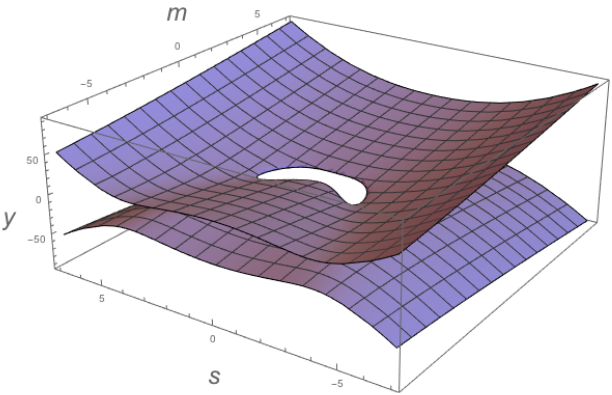
For notational convenience, define $\beta : \Sigma^* \to \mathbb{C}$ where

$$\begin{aligned} \beta_\epsilon &= -\gamma \\ \beta_{0s} &= \sqrt{-m + \beta_s} \\ \beta_{1s} &= -\sqrt{-m + \beta_s} \end{aligned}$$

For $n > 0$, the roots of $f^n(x)$ are exactly $\{ \gamma + \beta_s \mid s \in \Sigma^n \}$.

A New Perspective

$$S : y^2 = 2s^4 + 8ms^2 + 16m^2 + 16m$$



A New Perspective

So far,

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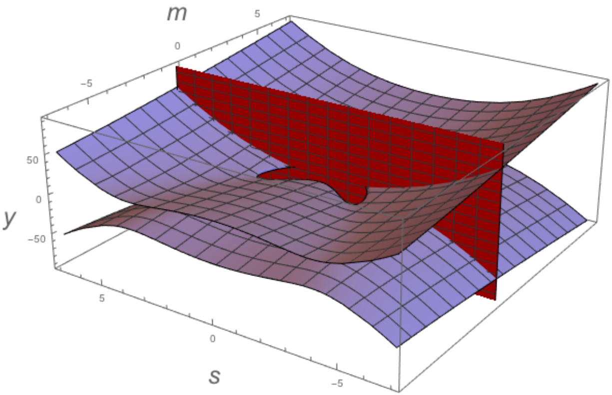
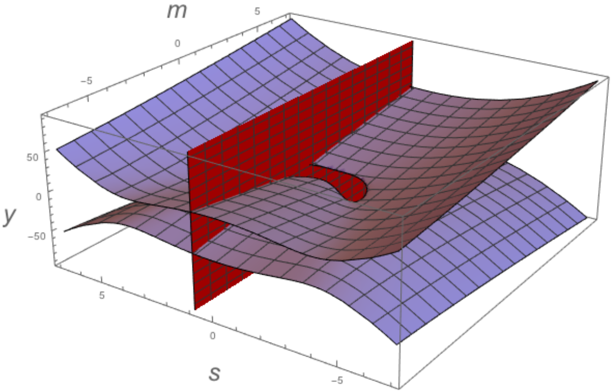


A New Perspective

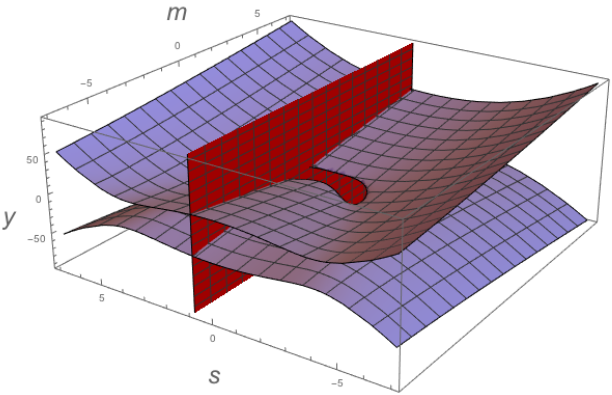
$$S : y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$



A New Perspective

This is a conic!

$$y^2 = am^2 + bm + c$$



Rational Projection

For any s, we have

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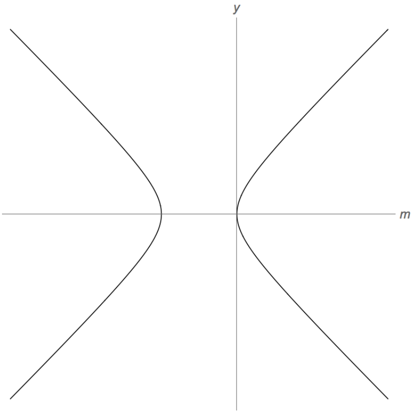

Example: $s = 0$

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Observation

*We're only looking for **rational** solutions.*

$$\text{Let } y = \frac{Y}{Z} \text{ and } m = \frac{M}{Z}.$$

Definition

*The **homogeneous form** of S is*

$$S : Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4Z^2.$$

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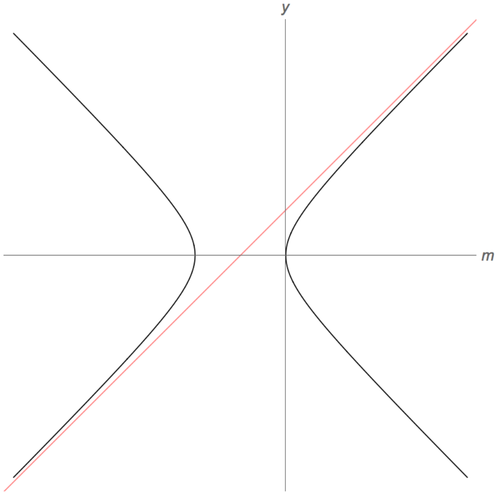
Observation

The point $[M : Y : Z] = [1 : 4 : 0]$ *is a solution to the homogeneous form of* S.

Rational Projection

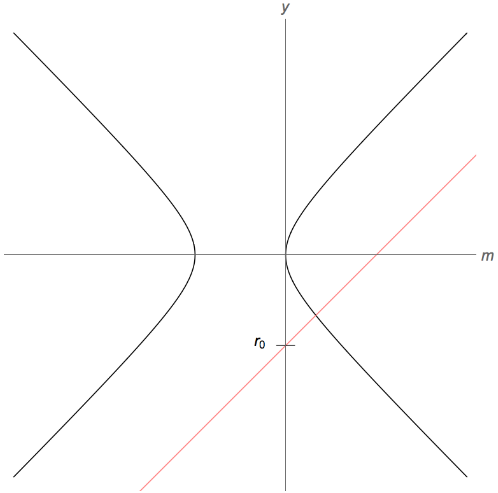
Geometrically, this is a line with slope 4.



Example: $s = 0$ and $y = 4m + 2$

Rational Projection

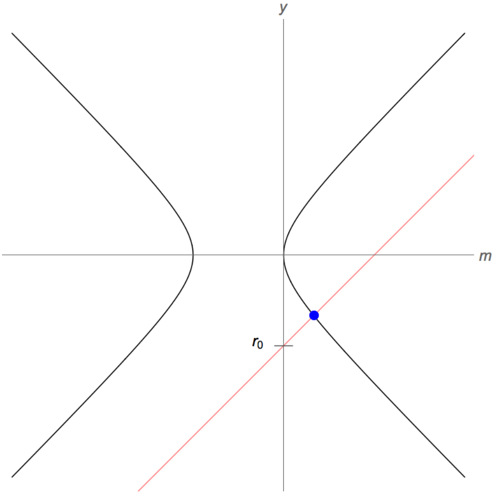
To project from the point at infinity, take any line with slope 4.



Example: $s = 0$ and $y = 4m + r_0$

Rational Projection

This intersects S at a rational point:



Example: $s = 0$ and $y = 4m + r_0$

Rational Projection

Solving for this intersection point gives

$$m = \frac{2s^4 - r_0^2}{8r}$$

$$\text{and } y = \frac{-4 + r^2 - 4s^2 + s^4}{2r}$$

where $r = \left(r_0 - s^2 - 2\right)$.

Rational Projection

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where $r = \left(r_0 - s^2 - 2\right)$.
So for every rational r and s, we get rational m and y such that

$$y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$

Rational Projection

Definition

We define this projection as

$$\phi(r,s) = (m(r,s), y(r,s))$$

where

$$m(r,s) = \frac{-4 - 4r - r^2 - 4s^2 - 2rs^2 + s^4}{8r},$$

$$y(r,s) = \frac{-4 + r^2 - 4s^2 + s^4}{2r}$$

Rational Projection

This gives us a value for m. Defining f requires m and γ. Luckily, we've already seen an equation for γ.

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Definition

$$\gamma(r,s) = \pm\beta\left(-\frac{3s^6}{16} - \frac{s^4 m}{2} - \frac{s^2 m^2}{2} - \frac{s^2 m}{2}\right) + \frac{17s^8}{64} + \frac{5m}{4}s^6 + \frac{11s^4}{4}m^2 + \frac{7m}{4}s^4 + 2s^2m^3 + 2s^2m^2 - m$$

where $m = m(r,s)$ is given by our projection.

Rational Projection

Example

If $r = 1$ and $s = 1$,

$$\phi(r,s) = (m(r,s), y(r,s)) = \left(-\frac{7}{4}, 3\right)$$

and

$$\gamma(r,s) = \frac{1}{2}.$$

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This gives the polynomial

$$\begin{aligned} f(x) &= \left(x - \frac{1}{2}\right)^2 + \frac{1}{2} - \frac{7}{4} \\ &= x^2 - x - 1. \end{aligned}$$

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This is the polynomial for the golden ratio!

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- So by choosing all (r, s), we get all newly reducible f^3.
- However, we will also get some that are not **newly** reducible.
- How can we ensure that we get a newly reducible f^3?

Finding Newly Reducible Third Iterates

Recall that

$$\begin{aligned} f \text{ is reducible } &\Leftrightarrow -m-\gamma \text{ is a square,} \\ \text{and } f^2 \text{ is newly reducible } &\Leftrightarrow 2(-m \pm \sqrt{m^2+m+\gamma}) \text{ is a square.} \end{aligned}$$

So if we have a point on S and neither $-m-\gamma$ nor $m^2+m+\gamma$ is a square, f^3 is newly reducible.

Finding Newly Reducible Third Iterates

$$-m-\gamma=\frac{1}{256r^2}s^2\left(r^2-2(r+2)s^2+s^4-4\right)^2\left(4+2r-s^2\right)$$

$$\begin{aligned} m^2+m+\gamma \quad = \quad & \frac{1}{256r^2}\left(r-s^2+2\right)^2(16+16r+4r^2+32s^2+32rs^2 \\ & +4r^2s^2-2r^3s^2+8s^4+12rs^4 \\ & +5r^2s^4-8s^6-4rs^6+s^8). \end{aligned}$$

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$$(4+2r-s^2)$$

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Finding Newly Reducible Third Iterates

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Let r be "big enough"