Some Title

And Maybe a Subtitle

Peter Illig Eli Orvis Yukihiko Segawa Nick Spinale

Carleton College

February 20th, 2018

Contents

1. Introduction
2. Simplifying the Problem
3. Newly Reducible Third Iterates, Part 1
4. Newly Reducible Third Iterates, Part 2

A Title

Contents of the slide

Roots of $f^n(x)$

$f(x) = (x-\gamma)^2 + \gamma + m$

- The roots of $f(x)$ are $\gamma \pm \sqrt{-m-\gamma}$
- If α is a root of $f^n(x)$, then $\gamma \pm \sqrt{\alpha - m - \gamma}$ are roots of $f^{n+1}(x)$

Observation

For $n > 0$, the roots of $f^n(x)$ are, with n radicals:

$$\gamma \pm \sqrt{-m \pm \sqrt{-m \pm \sqrt{-m \pm \ldots \sqrt{-m-\gamma}}}}$$

Roots of $f^n(x)$

Observation

For $n > 0$, the roots of $f^n(x)$ are, with n radicals:

$$\gamma \pm \sqrt{-m \pm \sqrt{-m \pm \sqrt{-m \pm \ldots \sqrt{-m - \gamma}}}}$$

For notational convenience, define $\beta : \Sigma^* \to \mathbb{C}$ where

$$\begin{aligned} \beta_\epsilon &= -\gamma \\ \beta_{0s} &= \sqrt{-m + \beta_s} \\ \beta_{1s} &= -\sqrt{-m + \beta_s} \end{aligned}$$

For $n > 0$, the roots of $f^n(x)$ are exactly $\{ \gamma + \beta_s \mid s \in \Sigma^n \}$.

Newly Reducible Third Iterates Part 1

Let

- $p_1(x) = a + b(x-\gamma) + c(x-\gamma)^2 + d(x-\gamma)^3 + (x-\gamma)^4$
- $p_2(x) = a - b(x-\gamma) + c(x-\gamma)^2 - d(x-\gamma)^3 + (x-\gamma)^4$

If $f^3 = p_1(p_2)$, then

$$\begin{aligned} \gamma + m^4 + 2m^3 + m^2 + m &= a^2 \\ 4m^3 + 4m^2 &= 2ac - b^2 \\ 6m^2 + 2m &= 2a - 2bd + c^2 \\ 4m &= 2c - d^2 \end{aligned}$$

Newly Reducible Third Iterates Part 1

Every newly reducible third iterate is a rational point on this surface!

Simplifying the System

We can simplify this system of equations using linear substitutions and the quadratic formula to get

$$\gamma = \pm\,\beta\left(-\frac{3d^6}{16} - \frac{d^4 m}{2} - \frac{d^2 m^2}{2} - \frac{d^2 m}{2}\right) + \frac{17d^8}{64} + \frac{5m}{4}d^6 + \frac{11d^4}{4}m^2 + \frac{7m}{4}d^4 + 2d^2m^3 + 2d^2m^2 - m$$

Simplifying the System

We can simplify this system of equations using linear substitutions and the quadratic formula to get

$$\gamma = \pm\beta\left(-\frac{3d^6}{16} - \frac{d^4 m}{2} - \frac{d^2 m^2}{2} - \frac{d^2 m}{2}\right) + \frac{17d^8}{64} + \frac{5m}{4}d^6 + \frac{11d^4}{4}m^2 + \frac{7m}{4}d^4 + 2d^2m^3 + 2d^2m^2 - m$$

Note that every expression in this formula is a rational function of d, m, and β

All about β

Let's look at β

$$\beta = \sqrt{2d^4 + 8d^2m + 16m^2 + 16m}$$

All about β

Let's look at β

$$\beta = \sqrt{2d^4 + 8d^2m + 16m^2 + 16m}$$

Letting $\beta = y$ and $d = s$ we have the surface

$$S: \; y^2 = 2s^4 + 8ms^2 + 16m^2 + 16m$$

The Curves C_m

We want to explore the surface S, by considering the curve C_m that results from a fixed value of m.

The Curves C_m

We want to explore the surface S, by considering the curve C_m that results from a fixed value of m.

$$C_{m_0}: \; y^2 = 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0$$

The Curves C_m

We want to explore the surface S, by considering the curve C_m that results from a fixed value of m.

$$C_{m_0} : \; y^2 = 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0$$

Some questions to consider:

- What are the roots of C_m?

The Curves C_m

We want to explore the surface S, by considering the curve C_m that results from a fixed value of m.

$$C_{m_0} : \; y^2 = 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0$$

Some questions to consider:

- What are the roots of C_m?
- What is the genus of C_m?

The Curves C_m

We want to explore the surface S, by considering the curve C_m that results from a fixed value of m.

$$C_{m_0} : \; y^2 = 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0$$

Some questions to consider:

- What are the roots of C_m?
- What is the genus of C_m?
- Does C_m have rational points? If so, how many?

The roots of C_m

What do the roots of C_m tell us?

The roots of C_m

What do the roots of C_m tell us?
If C_m has repeated roots, it's a conic! Otherwise, it's an elliptic curve.

The roots of C_m

What do the roots of C_m tell us?
If C_m has repeated roots, it's a conic! Otherwise, it's an elliptic curve.
So when does C_m have repeated roots?

The roots of C_m

$$\begin{aligned} C_{m_0}:\ y^2 &= 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0 \\ &= 2(s^2)^2 + 8s^2(m_0) + 16(m_0)^2 + 16m_0 \end{aligned}$$

Using the quadratic formula we get that C_m has a repeated root if and only $\sqrt{-2m - m^2} = 0$ or $2(-m \pm \sqrt{-2m - m^2}) = 0$. This happens when $m \in \{0, -1, -2\}$.

The roots of C_m

$$\begin{aligned} C_{m_0} : \ y^2 &= 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0 \\ &= 2(s^2)^2 + 8s^2(m_0) + 16(m_0)^2 + 16m_0 \end{aligned}$$

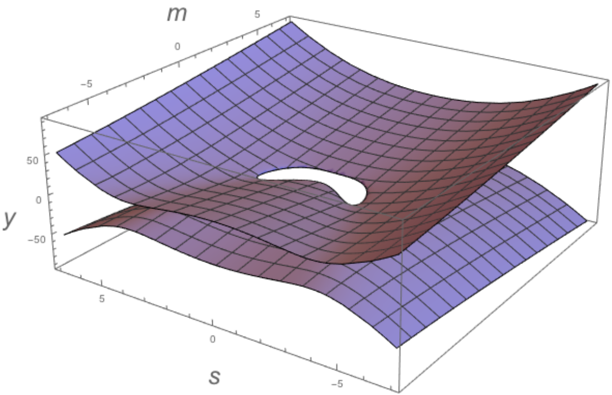
The roots of C_m

$$\begin{aligned} C_{m_0}: \; y^2 &= 2s^4 + 8s^2(m_0) + 16(m_0)^2 + 16m_0 \\ &= 2(s^2)^2 + 8s^2(m_0) + 16(m_0)^2 + 16m_0 \end{aligned}$$

Using the quadratic formula we get that C_m has a repeated root if and only $\sqrt{-2m - m^2} = 0$ or $2(-m \pm \sqrt{-2m - m^2}) = 0$. This happens when $m \in \{0, -1, -2\}$.

A New Perspective

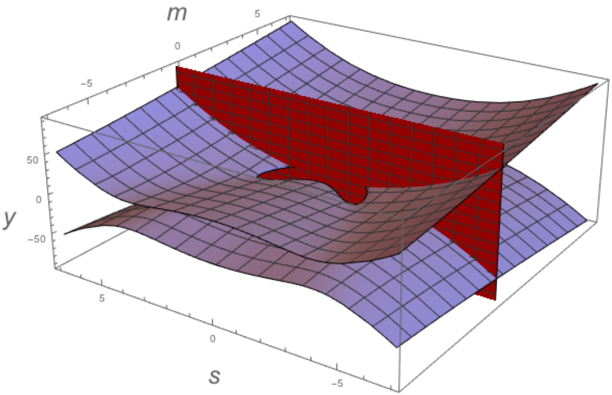
$$S : y^2 = 2s^4 + 8ms^2 + 16m^2 + 16m$$



A New Perspective

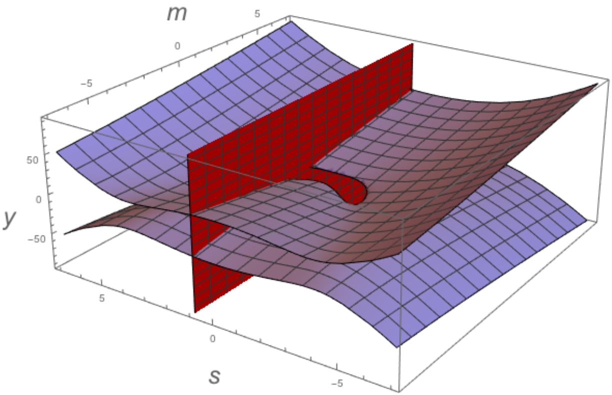
So far,

$$S : y^2 = 2s^4 + 8ms^2 + 16m^2 + 16m$$



A New Perspective

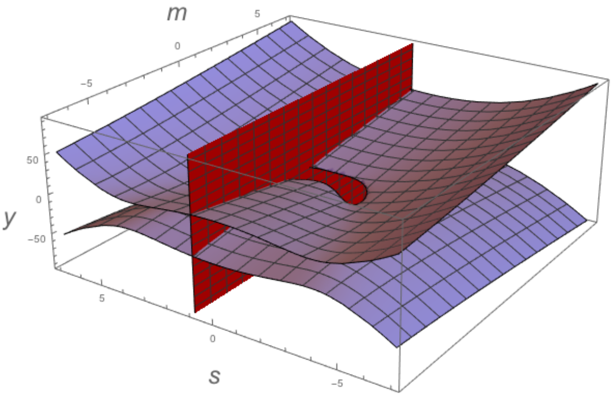
$$S : y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$



A New Perspective

This is a conic!

$$y^2 = am^2 + bm + c$$



Rational Projection

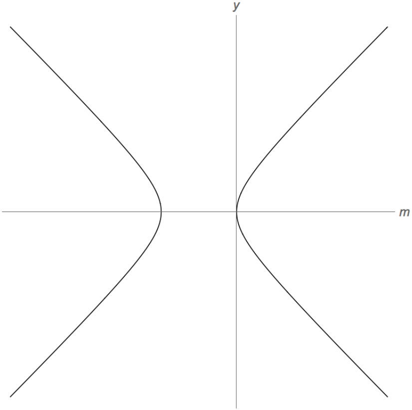
For any s, we have

$$y^2 = 16m^2 + (16 + 8s^2)m + 2s^4.$$

Rational Projection

For any s, we have

$$y^2 = 16m^2 + (16 + 8s^2)m + 2s^4.$$



Example: $s = 0$

Rational Projection

$$S : y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$

Rational Projection

$$S : y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$

Observation

*We're only looking for **rational** solutions.*

Rational Projection

$$S : y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$

Observation

*We're only looking for **rational** solutions.*

$$\text{Let } y = \frac{Y}{Z} \text{ and } m = \frac{M}{Z}.$$

Rational Projection

$$S : y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$

Observation

*We're only looking for **rational** solutions.*

$$\text{Let } y = \frac{Y}{Z} \text{ and } m = \frac{M}{Z}.$$

Definition

*The **homogeneous form** of S is*

$$S : Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4 Z^2.$$

Rational Projection

If $Z = 0$...

Rational Projection

If $Z = 0$...

$$Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4Z^2$$

Rational Projection

If $Z = 0$...

$$Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4 Z^2$$

$$Y^2 = 16M^2 + \cancel{(8s^2 + 16)MZ} + \cancel{2s^4 Z^2}$$

Rational Projection

If $Z = 0$...

$$Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4 Z^2$$

$$Y^2 = 16M^2 + \cancel{(8s^2 + 16)MZ} + \cancel{2s^4 Z^2}$$

$$Y^2 = 16M^2$$

Rational Projection

If $Z = 0$...

$$Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4Z^2$$

$$Y^2 = 16M^2 + \cancel{(8s^2 + 16)MZ} + \cancel{2s^4Z^2}$$

$$Y^2 = 16M^2$$

$$Y = \pm 4M$$

Rational Projection

If $Z = 0$...

$$Y^2 = 16M^2 + (8s^2 + 16)MZ + 2s^4Z^2$$

$$Y^2 = 16M^2 + \cancel{(8s^2 + 16)MZ} + \cancel{2s^4Z^2}$$

$$Y^2 = 16M^2$$

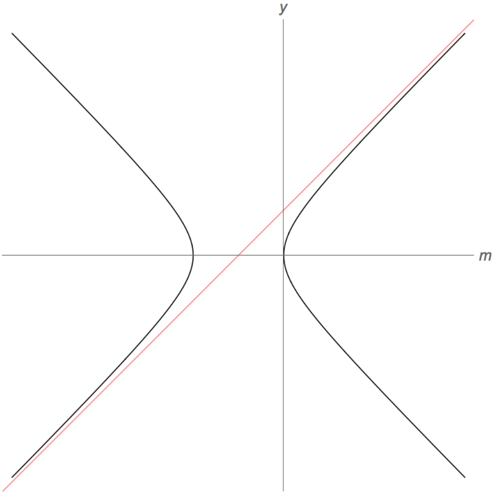
$$Y = \pm 4M$$

Observation

The point $[M : Y : Z] = [1 : 4 : 0]$ *is a solution to the homogeneous form of* S.

Rational Projection

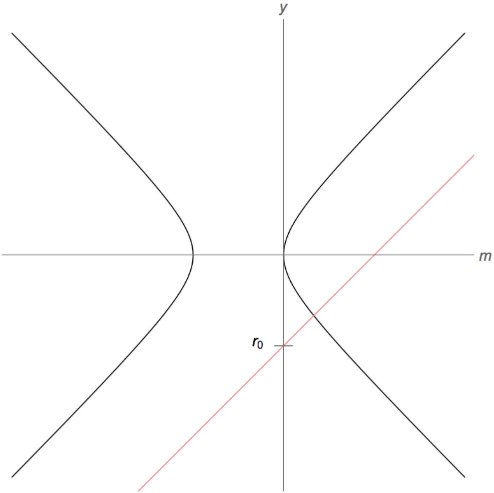
Geometrically, this is a line with slope 4.



Example: $s = 0$ and $y = 4m + 2$

Rational Projection

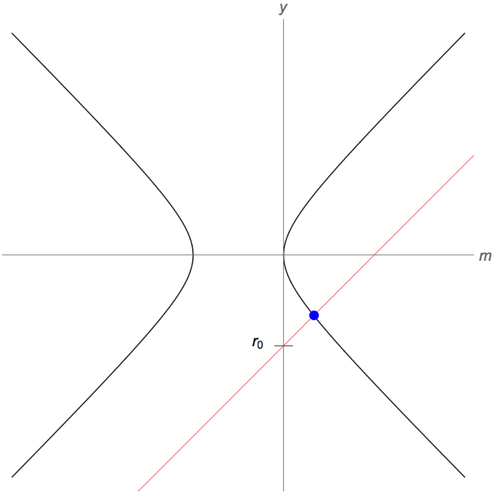
To project from the point at infinity, take any line with slope 4.



Example: $s = 0$ and $y = 4m + r_0$

Rational Projection

This intersects S at a rational point:



Example: $s = 0$ and $y = 4m + r_0$

Rational Projection

Solving for this intersection point gives

$$m = \frac{2s^4 - r_0^2}{8r}$$
$$\text{and } y = \frac{-4 + r^2 - 4s^2 + s^4}{2r}$$

where $r = \left(r_0 - s^2 - 2\right)$.

Rational Projection

Solving for this intersection point gives

$$m = \frac{2s^4 - r_0^2}{8r}$$
$$\text{and } y = \frac{-4 + r^2 - 4s^2 + s^4}{2r}$$

where $r = \left(r_0 - s^2 - 2\right)$.
So for every rational r and s, we get rational m and y such that

$$y^2 = 16m^2 + (16 + 8s^2)m + 2s^4$$

Rational Projection

Definition

We define this projection as

$$\phi(r,s) = (m(r,s), y(r,s))$$

where

$$m(r,s) = \frac{-4 - 4r - r^2 - 4s^2 - 2rs^2 + s^4}{8r},$$

$$y(r,s) = \frac{-4 + r^2 - 4s^2 + s^4}{2r}$$

Rational Projection

This gives us a value for m. Defining f requires m and γ. Luckily, we've already seen an equation for γ.

Rational Projection

This gives us a value for m. Defining f requires m and γ. Luckily, we've already seen an equation for γ.

Definition

$$\gamma(r,s) = \pm\beta\left(-\frac{3s^6}{16} - \frac{s^4 m}{2} - \frac{s^2 m^2}{2} - \frac{s^2 m}{2}\right) + \frac{17s^8}{64} + \frac{5m}{4}s^6 + \frac{11s^4}{4}m^2 + \frac{7m}{4}s^4 + 2s^2m^3 + 2s^2m^2 - m$$

where $m = m(r,s)$ is given by our projection.

Rational Projection

Example

If $r = 1$ and $s = 1$,

$$\phi(r, s) = (m(r, s), y(r, s)) = \left(-\frac{7}{4}, 3\right)$$

and

$$\gamma(r, s) = \frac{1}{2}.$$

Rational Projection

Example

If $r = 1$ and $s = 1$,

$$\phi(r,s) = (m(r,s), y(r,s)) = \left(-\frac{7}{4}, 3\right)$$

and

$$\gamma(r,s) = \frac{1}{2}.$$

This gives the polynomial

$$\begin{aligned} f(x) &= \left(x - \frac{1}{2}\right)^2 + \frac{1}{2} - \frac{7}{4} \\ &= x^2 - x - 1. \end{aligned}$$

Rational Projection

Example

If $r = 1$ and $s = 1$,

$$\phi(r, s) = (m(r, s), y(r, s)) = \left(-\frac{7}{4}, 3\right)$$

and

$$\gamma(r, s) = \frac{1}{2}.$$

This gives the polynomial

$$\begin{aligned} f(x) &= \left(x - \frac{1}{2}\right)^2 + \frac{1}{2} - \frac{7}{4} \\ &= x^2 - x - 1. \end{aligned}$$

This is the polynomial for the golden ratio!

Rational Projection

- Every newly reducible f^3 gives a point on S.

Rational Projection

- Every newly reducible f^3 gives a point on S.
- So by choosing all (r, s), we get all newly reducible f^3.

Rational Projection

- Every newly reducible f^3 gives a point on S.
- So by choosing all (r, s), we get all newly reducible f^3.
- However, we will also get some that are not **newly** reducible.

Rational Projection

- Every newly reducible f^3 gives a point on S.
- So by choosing all (r, s), we get all newly reducible f^3.
- However, we will also get some that are not **newly** reducible.
- How can we ensure that we get a newly reducible f^3?

Finding Newly Reducible Third Iterates

Recall that

$$f \text{ is reducible } \Leftrightarrow -m-\gamma \text{ is a square,}$$
$$\text{and } f^2 \text{ is newly reducible } \Leftrightarrow 2(-m \pm \sqrt{m^2+m+\gamma}) \text{ is a square.}$$

So if we have a point on S and neither $-m-\gamma$ nor $m^2+m+\gamma$ is a square, f^3 is newly reducible.

Finding Newly Reducible Third Iterates

$$-m-\gamma = \frac{1}{256r^2}s^2\left(r^2-2(r+2)s^2+s^4-4\right)^2\left(4+2r-s^2\right)$$

$$\begin{aligned} m^2+m+\gamma &= \frac{1}{256r^2}\left(r-s^2+2\right)^2(16+16r+4r^2+32s^2+32rs^2 \\ &\qquad +4r^2s^2-2r^3s^2+8s^4+12rs^4 \\ &\qquad +5r^2s^4-8s^6-4rs^6+s^8). \end{aligned}$$

Finding Newly Reducible Third Iterates

$$(4+2r-s^2)$$

$$(16+16r+4r^2+32s^2+32rs^2+4r^2s^2-2r^3s^2+8s^4+12rs^4$$
$$+5r^2s^4-8s^6-4rs^6+s^8)$$

Finding Newly Reducible Third Iterates

$$(4 + 2r - s^2)$$

$$(16 + 16r + 4r^2 + 32s^2 + 32rs^2 + 4r^2s^2 \underline{-2r^3s^2} + 8s^4 + 12rs^4 \\ +5r^2s^4 - 8s^6 - 4rs^6 + s^8)$$

Finding Newly Reducible Third Iterates

$$(4 + 2r - s^2)$$

$$(16 + 16r + 4r^2 + 32s^2 + 32rs^2 + 4r^2s^2 \underline{-2r^3s^2} + 8s^4 + 12rs^4 \\ +5r^2s^4 - 8s^6 - 4rs^6 + s^8)$$

Let r be "big enough"