fix $\gamma = 1 + \sqrt{5} + \sqrt[3]{2}$. then γ is formed out of applying addition and root taking on integers. it is however, a nontrivial number algebraically. for example, its addends $\alpha = 1 + \sqrt{5}$ and $\beta = \sqrt[3]{2}$ are simple to understand by the following equations $(\alpha - 1)^2 = \alpha^2 - 2\alpha + 1 = 5$ and $\beta^3 = 2$.

problem what equation does γ solve?

magic solution let us consider the principal vector $v = (1, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{5}\sqrt[3]{2}, \sqrt{5}\sqrt[3]{4})$ as well as the product γv.
it is evident each of the coordinates of γv can be given as linear integer combinations of the coordinates of v.
explicitly we have

$$\gamma v = (1 + \sqrt{5} + \sqrt[3]{2})(1, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{4}, \sqrt{5}\sqrt[3]{2}, \sqrt{5}\sqrt[3]{4}) = v \begin{pmatrix} 1 & 5 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

it follows that γ is an eigenvalue of this integer matrix - a root of its characteristic polynomial.
explicitly,

$$\det(\gamma I - M) = \gamma^6 - 6\gamma^5 + 36\gamma^3 + 12\gamma^2 - 168\gamma + 4 = 0$$

motivation to understand the powers of $\gamma = 1 + \sqrt{5} + \sqrt[3]{2}$, it is easier to decompose into simple terms $a = 1$, $b = \sqrt{5}$, $c = \sqrt[3]{2}$, $\gamma = a + b + c$ and consider the products $\gamma a, \gamma b, \gamma c$.
clearly we have $\gamma a = a + b + c$. but $\gamma b = b + 5a + \sqrt{5}\sqrt[3]{2}$, $\gamma c = c + \sqrt{5}\sqrt[3]{2} + \sqrt[3]{4}$ form a problem - we cannot express these linearly in terms of a, b, c. so we introduce the terms $d = \sqrt[3]{4}$, $e = \sqrt{5}\sqrt[3]{2}$.
thus $\gamma b = b + 5a + e$, $\gamma c = c + e + d$. to complete our multiplication table, we need γd and γe.
introducing $f = \sqrt{5}\sqrt[3]{4}$ we have $\gamma d = d + 2a + f$, $\gamma e = e + 5c + f$.
finally, we have a completed table with $\gamma f = f + 5d + 2b$.

exercises

- find a polynomial of degree 3 with integer coefficients for which $1 + \sqrt[3]{2} + \sqrt[3]{4}$ is a root.
- find a polynomial of degree 4 with integer coefficients for which $\sqrt[4]{3} - 2\sqrt{3}$ is a root.