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Stability of periodically switched discrete-time linear singular systems

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ABSTRACT

In this paper we present some necessary and sufficient conditions for the stability of periodically switched discrete-time linear index-1 singular system, (PSSS). In particular, it is proved that, if at least one subsystem of a PSSS is asymptotically stable, then there is a switching rule, so that the whole system is also uniformly exponentially stable. Furthermore, for a periodically switched control system with no stable subsystems, there exist a switching rule and feedback matrices, such that the obtained PSSS is uniformly exponentially stable.

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1. Introduction

In recent years, there has been a great interest in switched systems because of their importance in real life applications ranging from natural to social sciences. The reader is referred to the survey paper [11], the recent book [15], and the references cited therein for more detailed information about switched systems.

In what follows, we are interested in such periodically switched systems, whose subsystems are governed by linear singular discrete equations. Although the stability analysis and stabilization of continuous-time periodically switched systems have been studied before (see, for example, [4**?**]), to our best of knowledge, a similar research study for periodically switched discrete-time linear singular systems has not been carried out yet. Furthermore, the stability of periodically switched linear discrete-time systems in the sense of Markovian probabilities have been extensively considered (see, [6–8]). However, in this work, we limit ourselves to periodically switched systems governed by the deterministic and singular discrete-time systems. It is worth mentioning that linear singular switched systems arise in many applications, such as power electronics and systems, flight control systems, network control systems, robot manipulators, economic systems, and so forth (see, [10,13,16,17]). Consider for example, a switched system occurred in the modeling of electrical circuits, whose subsystems are described by the differential–algebraic equation (DAE) (1-3.4) in [5]. An explicit Euler scheme applied to the above mentioned DAE leads to a switched linear singular discrete-time system.

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In this paper we consider the following periodically switched singular system (PSSS):

$$E(k)x(k+1) = A(k)x(k), \tag{1}$$

where

$$(E(k), A(k)) = \begin{cases} (E_1, A_1), & lN \le k < k_1 + lN \\ (E_2, A_2), & k_1 + lN \le k < k_2 + lN \\ \ldots.. & \\ (E_\sigma, A_\sigma), & k_{\sigma-1} + lN \le k < (l+1)N. \end{cases} \qquad l = 0, 1, 2, \ldots.$$

Here, $E_i, A_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, \sigma$, are given matrices, and $x(k) \in \mathbb{R}^n$ is an unknown vector. Suppose all the matrices E_i are singular and have the same rank, i.e. $\text{rank} E_i = r < n$. Let $0 = k_0 < k_1 < k_2 < \cdots < k_{\sigma-1} < k_\sigma = N$ be given integers for determining the switching moments.

The aim of this paper is to study the stability and stabilizability of periodically switched systems involving linear singular discrete equations. The paper is organized as follows. Section 2 reviews some notions and stability results for linear nonsingular difference equations as well as for linear index-1 difference equations. Section 3 gives necessary and sufficient conditions for uniform exponential stability. Finally, Section 4 deals with sufficient conditions for stabilizing PSSS by suitable switching rules and by feedback controls.

2. Preliminaries

2.1. Stability of periodically switched linear nonsingular systems

Consider the following initial value problem for periodically switched discrete-time linear system:

$$x(k+1) = \begin{cases} A_1 x(k), & lN \le k < k_1 + lN \\ A_2 x(k), & k_1 + lN \le k < k_2 + lN \\ \ldots.. & \\ A_\sigma x(k), & k_{\sigma-1} + lN \le k < (l+1)N \end{cases} \qquad l = 0, 1, 2, \ldots, \tag{2}$$
$$x(0) = x_0$$

where $A_1, A_2, \ldots, A_\sigma \in \mathbb{R}^{n\times n}$, and $0 = k_0 < k_1 < k_2 < \cdots < k_{\sigma-1} < k_\sigma = N$, are given matrices and integers, respectively. We need to study the asymptotic behavior of the solution $x(k) \in \mathbb{R}^n$ of system (2) .

Putting

$$A(k) = \begin{cases} A_1, & lN \le k < k_1 + lN \\ A_2, & k_1 + lN \le k < k_2 + lN \\ \ldots.. & \\ A_\sigma, & k_{\sigma-1} + lN \le k < (l+1)N, \end{cases}$$

we can rewrite system (2) as:

$$\begin{aligned} &x(k+1) = A(k)x(k), \quad k \ge 0 \\ &x(0) = x_0 \end{aligned} \tag{3}$$

Evidently, system (3) has the solution $x(k) = A_{i+1}^{k-(k_i+lN)} A_i^{\Delta_i} \cdots A_1^{\Delta_1} R^l x_0$, where $R = A_\sigma^{\Delta_\sigma} \cdots A_1^{\Delta_1}$, $\Delta_j = k_j - k_{j-1}$, $j = 1, \ldots, \sigma$ and $k_i + lN \leq k < k_{i+1} + lN$.

Now define the state transition matrix $\Phi(k, h)$ for system (3) as

$$\Phi(k,h) = \begin{cases} A_{j+1}^{k-k_j-lN} A_j^{\Delta_j} \cdots A_1^{\Delta_1} R^{l-m} A_\sigma^{\Delta_\sigma} \cdots A_{i+1}^{k_{i+1}+mN-h}, \text{ if } m < l, \\ \qquad k_j + lN \leq k < k_{j+1} + lN, k_i + mN \leq h < k_{i+1} + mN, \\ A_{j+1}^{k-k_j-lN} A_j^{\Delta_j} \cdots A_{i+1}^{k_{i+1}+lN-h}, \\ \qquad \text{if } k_i + lN \leq h < k_{i+1} + lN \leq k_j + lN \leq k < k_{j+1} + lN, \\ A_{i+1}^{k-h}, \quad \text{if} \quad k_i + lN \leq h < k < k_{i+1} + lN, \\ I_n, \qquad \text{if} \quad k = h. \end{cases}.$$

Then the unique solution of system (3) is given by

$$x(k) = \Phi(k, 0)x_0, \quad k \geq 0.$$

For the reader's convenience, we recall some notions and results, see, e.g. [9,14].

Definition 2.1: System (3) is called uniformly stable if there exists a positive constant γ such that for any x_0 the corresponding solution satisfies

$$\|x(k)\| \leq \gamma \|x_0\| \quad \forall k \geq 0.$$

Definition 2.2: System (3) is called uniformly exponentially stable if there exist a positive constant γ and a constant $0 \leq \lambda < 1$ such that for any x_0 the corresponding solution satisfies

$$\|x(k)\| \leq \gamma\lambda^k \|x_0\| \quad \forall k \geq 0.$$

According to the stability theory for linear difference systems (see, [9,14]), system (3) is uniformly exponentially stable if and only if there exist a positive constant γ and a constant $0 < \lambda < 1$ such that

$$\left\| \Phi(k,j) \right\| \leq \gamma\lambda^{k-j}, \ \forall k \geq j.$$

In the case $A(k) \equiv A$ is a constant matrix, system (3) is exponentially stable if and only if all the eigenvalues of A are inside the unit circle.

Denote by $\sigma(R) := \{\lambda \in \mathbb{C} : \det(\lambda I - R) = 0\}$ and $\rho(R) := \max\{|\lambda| : \lambda \in \sigma(R)\}$ the spectrum and the spectral radius of a matrix R, respectively, and let $S(0, 1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. We begin with the following simple stability criterion for a periodically switched discrete-time linear system.

Proposition 2.3: *System (2) is uniformly exponentially stable if and only if $\sigma(R) \subset S(0, 1)$.*

The proof of Proposition 2.3 is standard and will be omitted.

2.2. Linear index-1 singular difference systems

In this subsection we recall some notions and concepts on singular difference equations, see [1–3,12]. Consider a linear singular difference system (LSDS)

$$E(k)x(k+1) = A(k)x(k) + q(k), \tag{4}$$

where $E(k), A(k) \in \mathbb{R}^{n\times n}$, $q(k) \in \mathbb{R}^n$ are given, and the matrices $E(k)$ are singular for all $k \geq 0$.

Definition 2.4 [2,3]**:** System (4) is said to be of index-1 if it satisfies two following conditions

(i) $\operatorname{rank}E(k) \equiv r < n, \quad \forall k \geq 0,$
(ii) $S(k) \bigcap \ker E(k-1) = \{0\}, \quad \forall k \geq 1,$

where $S(k) = \{\xi \in \mathbb{R}^n : A(k)\xi \in \operatorname{Im}E(k)\}$.

Hereafter, we assume that $\dim S(0) = r$ and let $E(-1) \in \mathbb{R}^{n\times n}$ be a chosen matrix, such that $\operatorname{rank}E(-1) = r$ and $\mathbb{R}^n = S(0) \bigoplus \ker E(-1)$.

Let $V(k-1) = \{s_k^1, \ldots, s_k^r, h_{k-1}^{r+1}, \ldots, h_{k-1}^n\}$ be matrices whose columns are basis vectors of $S(k)$ and $\ker E(k-1)$, and $Q = \operatorname{diag}(O_r, I_{n-r})$, $P = I_n - Q$. Here O_r is the $r \times r$ zero matrix and I_m stands for the $m \times m$ identity matrix.

Lemma 2.5 [2]: *The following assertions are equivalent*

(i) $S(k) \bigcap \ker E(k-1) = \{0\}$;
(ii) *The matrix* $G(k) = E(k) + A(k)V(k-1)QV(k)^{-1}$ *is nonsingular;*
(iii) $S(k) \bigoplus \ker E(k-1) = \mathbb{R}^n$.

With the defined above matrices $V(k)$, the matrix $Q(k) = V(k)QV(k)^{-1}$ is the so-called canonical projection onto $\ker E(k)$ along $S(k+1)$, see, [2].

Let $P(k) = I - Q(k)$. Then, the following relations hold:

$P(k) = G(k)^{-1}E(k)$, $P(k)G(k)^{-1}A(k) = P(k)G(k)^{-1}A(k)P(k-1)$ $Q(k)G(k)^{-1}A(k) = Q(k)G(k)^{-1}A(k)P(k-1) + V(k)QV(k-1)^{-1}$. Multiplying both sides of system (4) by $P(k)G(k)^{-1}$ and $Q(k)G(k)^{-1}$ respectively, we can decouple this system into subsystems

$$P(k)x(k+1) = P(k)G(k)^{-1}A(k)P(k-1)x(k) + P(k)G(k)^{-1}q(k); \tag{5}$$

$$\begin{aligned} &V(k-1)QV(k)^{-1}G(k)^{-1}A(k)P(k-1)x(k) + Q(k-1)x(k) \\ &\quad = -V(k-1)QV(k)^{-1}G(k)^{-1}q(k). \end{aligned} \tag{6}$$

Putting $y(k) = P(k-1)x(k)$, $z(k) = Q(k-1)x(k)$, we can rewrite the above subsystems as

$$\begin{aligned} y(k+1) &= P(k)G(k)^{-1}A(k)y(k) + P(k)G(k)^{-1}q(k); \\ z(k) &= -V(k-1)QV(k)^{-1}G(k)^{-1}A(k)y(k) - V(k-1)QV(k)^{-1}G(k)^{-1}q(k). \end{aligned} \tag{7}$$

Hence,

$$\begin{aligned} x(k) &= y(k) + z(k) \\ &= (I_n - V(k-1)QV(k)^{-1}G(k)^{-1}A(k))y(k) - V(k-1)QV(k)^{-1}G(k)^{-1}q(k), \end{aligned} \tag{8}$$

where $y(k)$ solves the inherent ordinary difference Equation (7). Thus, if system (4) is of index-1, then for given $y(0) = P(-1)x(0) \in \operatorname{Im}P(-1)$, we can compute $y(k)$ and $x(k)$ by (7) and (8), respectively. As in the DAEs case, we need only to initialize the $P(-1)$-

component of $x(0)$, i.e.

$$P(-1)(x(0)-x_0)=0.$$

This initial condition is equivalent to the condition

$$E(-1)(x(0)-x_0)=0, \tag{9}$$

which is independent of the choice of the projection $P(-1)$. According to [2,3], the initial value problem (4) and (9) has a unique solution.

Definition 2.6 [1]: System (4) is called E-uniformly stable if there exists a positive constant γ such that for any $x_0 \in \mathbb{R}^n$, the corresponding solution satisfies

$$\|x(k)\| \le \gamma \|E(-1)x_0\| .$$

Definition 2.7: System (4) is called E-uniformly exponentially stable if there exist a positive constant γ and $0 \le \lambda < 1$ such that for any $x_0 \in \mathbb{R}^n$, the corresponding solution satisfies

$$\|x(k)\| \le \gamma\lambda^k \|E(-1)x_0\| .$$

Next consider a linear system, obtained from (4) via scaling and transforming variables, namely, the following system

$$\bar{E}(k)\bar{x}(k+1)=\bar{A}(k)\bar{x}(k)+\bar{q}(k),$$

where $\bar{E}(k)=F(k)E(k)H(k)$, $\bar{A}(k)=F(k)A(k)H(k-1)$, $\bar{q}(k)=P(k)q(k)$, and the matrices $F(k),H(k)$ are nonsingular. Here $F(k)$ are scaling matrices, while the transformations of variables are defined by $x(k)=H(k-1)\bar{x}(k)$. Since $\bar{S}(k)\bigcap \mathrm{Ker}\bar{E}(k)=H(k-1)(S(k)\bigcap \mathrm{KerE}(k-1))$, the index-1 property of LSDS is invariant under scaling and linear transformations.

Theorem 2.8 [2,3]: *Every index-1 LSDS can be reduced to the Kronecker normal form*

$$\mathrm{diag}(I_r,O_{n-r})\bar{x}(k+1)=\mathrm{diag}(W(k),I_{n-r})\bar{x}(k)+\bar{q}(k),$$

where $W(k)$ is a certain $r\times r$ matrix.

3. Stability analysis of singular periodically switched systems

In this section we provide E-stability analysis for singular discrete periodically switched systems. We begin with the following definition.

Definition 3.1: System (1) is called a PSSS of index-1, if it satisfies the following conditions:

(i) $\mathrm{rank}E_i=r<n, i=1,2,\dots,\sigma$;
(ii) $S_i\bigcap \ker E_{i-1}=\{0\}, k=1,2,\dots,\sigma, E_0=E_\sigma$;
(iii) if $\Delta_i:=k_i-k_{i-1}\ge 2$ then $S_i\bigcap \ker E_i=\{0\}$;

where $S_i=\{\xi\in\mathbb{R}^n : A_i\xi\in \mathrm{Im}E_i\}$.

Let $V_{i,j} = \{s_i^1, \ldots, s_i^r, h_j^{r+1}, \ldots, h_j^n\}$ be matrices, whose columns are basis vectors of S_i and $\ker E_j$, and $Q = \mathrm{diag}(O_r, I_{n-r})$.We define the matrices $V(k)$, $G(k)$ as follows

$$V(k) = \begin{cases} V_{i,i-1}, & \text{if } \ k = k_{i-1} + lN \\ V_{i,i}, & \text{if } \ k_{i-1} + 1 + lN \le k < k_i + lN, \Delta_i \ge 2 \end{cases};$$

$$G(k) = \begin{cases} E_i + A_i V_{i,i-1} Q V_{i,i}^{-1}, & \text{if } \ k = k_{i-1} + lN, \Delta_i \ge 2 \\ E_i + A_i V_{i,i-1} Q V_{i+1,i}^{-1}, & \text{if } \ k = k_{i-1} + lN, \Delta_i = 1 \\ E_i + A_i V_{i,i} Q V_{i,i}^{-1}, & \text{if } \ k_{i-1} + 1 + lN \le k < k_i - 2 + lN, \Delta_i > 2 \\ E_i + A_i V_{i,i} Q V_{i+1,i}^{-1}, & \text{if } \ k = k_i - 1 + lN, \Delta_i \ge 2 \end{cases}.$$

System (1) has the following properties:

(1) $S_i \bigoplus \ker E_{i-1} = \mathbb{R}^n, \quad i = 1, 2, \ldots, \sigma$. If $\Delta_i > 1$ then $S_i \bigoplus \ker E_i = \mathbb{R}^n$;
(2) $G(k)$ are nonsingular $\forall k = 0, 1, 2, \ldots$.

Using Kronecker transformations

$$\begin{aligned} \overline{E}(k) &= V(k)^{-1} G(k)^{-1} E(k) V(k); \\ \overline{A}(k) &= V(k)^{-1} G(k)^{-1} A(k) V(k-1); \\ \overline{x}(k) &= V(k-1)^{-1} x(k), \end{aligned}$$

we can reduce system (1) to

$$\begin{pmatrix} I_r & O \\ O & O_{n-r} \end{pmatrix} \overline{x}(k+1) = \begin{pmatrix} W(k) & O \\ O & I_{n-r} \end{pmatrix} \overline{x}(k), \tag{10}$$

with $\overline{E}(k) = \begin{pmatrix} I_r & O \\ O & O_{n-r} \end{pmatrix}$, $\overline{A}(k) = \begin{pmatrix} W(k) & O \\ O & I_{n-r} \end{pmatrix}$.

Put

$$\overline{x}(k) := \begin{pmatrix} v(k) \\ w(k) \end{pmatrix},$$

where $v(k) \in \mathbb{R}^r$, $w(k) \in \mathbb{R}^{n-r}$.

Thus, system (10) is equivalent to the following systems

$$\begin{cases} v(k+1) = W(k) v(k) \\ w(k) = 0 \end{cases}. \tag{11}$$

Introduce the matrix function

$$H(\lambda) = \lambda E_\sigma - A_\sigma G(N-2)^{-1} A_\sigma \ldots G(0)^{-1} A_1,$$

and its characteristic function

$$p_H(\lambda) = \det H(\lambda).$$

Then we define the spectrum of the matrix function H and the spectrum of the couple $\{E_i, A_i\}$ as $\sigma(H) = \{\lambda \in \mathbb{C} : p_H(\lambda) = 0\}$, and $\sigma(E_i, A_i) = \{\lambda \in \mathbb{C} : \det(\lambda E_i - A_i) = 0\}$, respectively. From now on, we consider the initial-value problem (IVP) (1) and (9) with $E(-1) = E_0 = E_\sigma$.

Theorem 3.2: *System (1) is E-uniformly exponentially stable if and only if the spectrum $\sigma(H)$ is inside the unit circle.*

Proof: Define

$$\overline{H}(\lambda) = \lambda\overline{E}(N-1) - \overline{A}(N-1)\overline{A}(N-2)\dots\overline{A}(0),$$

with the characteristic function

$$p_{\overline{H}}(\lambda) = \det\overline{H}(\lambda) = \det(\lambda I_r - R),$$

where $R = W(N-1)W(N-2)\cdots W(0)$.

By definition, $\sigma(\overline{H})$ is the spectrum of R. It means that system (11) is uniformly exponentially stable if and only if $\sigma(\overline{H}) \subset S(0,1)$.

On the other hand, we have

$$\begin{aligned}\overline{H}(\lambda) &= \lambda\overline{E}(N-1) - \overline{A}(N-1)\overline{A}(N-2)\cdots\overline{A}(0)\\ &= \lambda\overline{E}(N-1) - V(N-1)^{-1}G(N-1)^{-1}A(N-1)G(N-2)^{-1}A(N-2)\dots\\ &\quad G(0)^{-1}A(0)V(-1)\\ &= V(N-1)^{-1}G(N-1)^{-1}(\lambda E_\sigma - A_\sigma G(N-2)^{-1}A_\sigma\cdots G(0)^{-1}A_1)V(-1)\\ &= V(N-1)^{-1}G(N-1)^{-1}H(\lambda)V(-1).\end{aligned}$$

This shows that $\sigma(\overline{H}) = \sigma(H)$, i.e. system (11) is uniformly exponentially stable if and only if the spectrum $\sigma(H)$ is included in $S(0,1)$.

Besides, if system (11) is uniformly exponentially stable then there exist a finite positive constant $\gamma > 0$ and $0 \le \lambda < 1$ such that for any $v_0 \in \mathbb{R}^r$, the corresponding solution satisfies

$$\|v(k)\| \le \gamma\lambda^k\|v_0\|, \tag{12}$$

where $v(0) = v_0$ is an initial value in $\mathbb{R}^r$. Using the relation $\|v(k)\| = \left\|(v(k)^T,0)^T\right\|$, we can rewrite (12) as

$$\left\|(v(k)^T,0)^T\right\| \le \gamma\lambda^k\left\|(v(0)^T,0)^T\right\|. \tag{13}$$

Since the corresponding solution of system (1) is $x(k) = V(k-1)(v(k)^T,0)^T$, we have

$$\left\|V(k-1)^{-1}x(k)\right\| \le \gamma\lambda^k\left\|V(-1)^{-1}x(0)\right\|. \tag{14}$$

Choosing $\mu = \max_{k=0,\dots,N-1}\|V(k)\|$, $\bar{\mu} = \max_{k=0,\dots,N-1}\left\|V(k)^{-1}\right\|$, we get

$$\begin{aligned}\|x(k)\| &= \left\|V(k-1)V(k-1)^{-1}x(k)\right\|\\ &\le \|V(k-1)\|\left\|V(k-1)^{-1}x(k)\right\|\\ &\le \mu\gamma\lambda^k\left\|V(-1)^{-1}x(0)\right\|\\ &\le \mu\bar{\mu}\gamma\lambda^k\|x(0)\|.\end{aligned}$$

On the other hand $x(0) = V(-1)(v(0)^T,0)^T = V(-1)\mathrm{diag}(I_r,0)(v(0)^T,0)^T = V(-1)(I_n - Q)V(-1)^{-1}x(0) = P(-1)x(0)$, using the relation $P(-1) = G(-1)^{-1}E(-1)$ we find

$$\begin{aligned} \|x(k)\| &\leq \mu\bar{\mu}\gamma\lambda^k \left\|G(-1)^{-1}E(-1)x(0)\right\| \\ &\leq \mu\bar{\mu}\left\|G(-1)^{-1}\right\|\gamma\lambda^k \left\|E(-1)x(0)\right\| \\ &= \mu\bar{\mu}\left\|G(-1)^{-1}\right\|\gamma\lambda^k \left\|E(-1)x_0\right\|. \end{aligned} \tag{15}$$

Putting $\bar{\gamma} := \mu\bar{\mu}\left\|G(-1)^{-1}\right\|\gamma$, from (15) we have

$$\|x(k)\| \leq \bar{\gamma}\lambda^k \left\|E(-1)x_0\right\|.$$

The last relation shows that the solution of the IVP (1) and (9) is E-uniformly exponentially stable. Theorem 3.2 is completely proved. □

In the following examples, we will use the Euclidean norms of vectors and matrices.

Example 3.3: Consider the PSSS (1) with $N = 3$, $\Delta_1 = \Delta_2 = \Delta_3 = 1$,

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$E_2 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$E_3 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As $\operatorname{rank}E_1 = \operatorname{rank}E_2 = \operatorname{rank}E_3 = 2$, $\ker E_1 = \ker E_2 = \ker E_3 = \operatorname{span}\{(0,0,1)^T\}$, $S_1 = S_2 = S_3 = \operatorname{span}\{(1,0,0)^T, (0,1,0)^T\}$, the PSSS is of index-1. Clearly, $V(k) = I_3, \forall k \geq 0$, $Q = \operatorname{diag}(0, I_1)$, hence $G(k) = E(k) + A(k)Q$. A simple calculation shows that $H(\lambda) = \begin{pmatrix} \lambda - \frac{1}{2} & \lambda & 0 \\ 3\lambda - \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\sigma(H) = \left\{\frac{1}{2}, -\frac{1}{2}\right\} \subset S(0,1)$, and the matrices $W_1 = W_3 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $W_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$. Thus, we obtain the following estimates:

$$\begin{aligned} \|v(3k)\| &= \left\|(W_3W_2W_1)^k v(0)\right\| &&\leq 2\left(\tfrac{1}{\sqrt[3]{2}}\right)^{3k} \|v(0)\| \\ \|v(3k+1)\| &= \left\|W_1(W_3W_2W_1)^k v(0)\right\| &&\leq 4\sqrt[3]{2}\left(\tfrac{1}{\sqrt[3]{2}}\right)^{3k+1} \|v(0)\| \\ \|v(3k+2)\| &= \left\|W_2W_1(W_3W_2W_1)^k v(0)\right\| &&\leq 2\sqrt[3]{4}\left(\tfrac{1}{\sqrt[3]{2}}\right)^{3k+2} \|v(0)\|. \end{aligned}$$

Choosing $\gamma_1 = \max\{2, 4\sqrt[3]{2}, 2\sqrt[3]{4}\} = 4\sqrt[3]{2}$, $\lambda = \frac{1}{\sqrt[3]{2}}$, we find $\|v(k)\| \leq \gamma_1\lambda^k \|v(0)\|$. Furthermore, using the relations $x(k) = V(k)^{-1}(v(k)^T, 0)^T = (v(k)^T, 0)^T$, $x(0) = V(-1)^{-1}(v(0)^T, 0)^T = P(-1)x(0) = G(-1)^{-1}E(-1)x(0)$ and choosing $E(-1) = E_\sigma$ we have

$$\begin{aligned} \|x(k)\| &\leq \gamma_1\lambda^k \left\|G(-1)^{-1}E(-1)x(0)\right\| \\ &\leq 2\gamma_1\lambda^k \left\|E(-1)x(0)\right\| \\ &\leq \gamma\lambda^k \left\|E(-1)x_0\right\|, \end{aligned}$$

where $\gamma = 4\gamma_1, E(-1)x(0) = E(-1)x_0, x_0 \in \mathbb{R}^3$. Thus, the IVP (1) and (9) is E-uniformly exponentially stable.

4. Stabilization of singular periodically switched systems

In this section we consider a problem of stabilizing periodically switched systems by determining suitable activation durations Δ_i and/or by feedback controls.

Firstly, we show that if at least one of the subsystems, say $\{E_i, A_i\}$, is asymptotically stable, then by choosing the activation duration Δ_i sufficiently large compared to those of the remaining subsystems, we will get an asymptotically stable periodically switched system.

Theorem 4.1: *Assume that at least one of the spectra $\sigma(E_j, A_j)$, $j = 1, \dots, \sigma$ is included in $S(0,1)$. Then, there exist a sufficiently large $N > 0$ and activation durations $\Delta_1, \dots, \Delta_\sigma$ such that system (1) is E-uniformly exponentially stable.*

Proof: We assume that $\sigma(E_i, A_i) \subset S(0,1)$, where $1 \leq i \leq \sigma$ is a fixed integer. Put $\Delta_i = k_i - k_{i-1}$, $\tau = k_{i-1} + 2$ and let $\Delta_i \geq 3$, then we have $V(\tau) = V(\tau - 1)$ because $V(\tau)$ and $V(\tau-1)$ are matrices whose columns are basis vectors of S_i and $\ker E_i$. We will show that the spectrum of $W(\tau)$ is included in $S(0,1)$. From the relations

$$\begin{aligned} &\det(\lambda I_r - W(\tau)) = 0 \\ \Leftrightarrow \quad &\det(\lambda \overline{E}(\tau) - \overline{A}(\tau)) = 0 \\ \Leftrightarrow \det(\lambda V(\tau)^{-1}G(\tau)^{-1}E(\tau)V(\tau) - V(\tau)^{-1}G(\tau)^{-1}&A(\tau)V(\tau-1)) = 0 \\ \Leftrightarrow \quad &\det(\lambda E(\tau) - A(\tau)) = 0, \end{aligned}$$

it implies that $\sigma(W(\tau)) \equiv \sigma(E_i, A_i)$.

Further, observe that

$$\begin{aligned} \|R\| &= \|W(N-1)\cdots W(k_i)\cdots W(0)\| \\ &= \left\| W(N-1)\cdots W(k_i)W(\tau)^{\Delta_i - 2}W(k_{i-1})\cdots W(0) \right\| \\ &\leq \|W(N-1)\cdots W(k_i)\| \left\| W(\tau)^{\Delta_i-2} \right\| \|W(k_{i-1})\cdots W(0)\|. \end{aligned}$$

Choosing $\Delta_1 = \Delta_2 = \cdots = \Delta_{i-1} = \Delta_{i+1} = \cdots = \Delta_N = 1$, then the number of the matrices $W(N-1), \dots, W(k_i), W(k_{i-1}), \dots, W(0)$ is not changed, hence there exists $M > 0$ such that

$$\|W(N-1)\cdots W(k_i)\| \, \|W(k_{i-1})\cdots W(0)\| \leq M.$$

Since $\sigma(W(\tau)) \subset S(0,1)$, there exists a sufficiently large Δ_i such that

$$\left\| W(\tau)^{\Delta_i - 2} \right\| < \frac{1}{M}.$$

The last relation shows that

$$\|R\| < 1,$$

hence, $\rho(R) \leq ||R|| < 1$, which yields the E-uniform exponential stability of systems (1). □

Example 4.2: Consider system (1) with $\sigma = 3$

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$E_2 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$E_3 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly,
$$\sigma(E_1, A_1) = \{-1, 2\}, \sigma(E_2, A_2) = \{\frac{1}{2}, -\frac{1}{2}\}, \sigma(E_3, A_3) = \{1, -1\},$$
and only $\sigma(E_2, A_2) \subset S(0, 1)$. According to Theorem 4.1, for sufficiently large Δ_2, the resulting system is uniformly exponentially stable. Indeed, choosing $\Delta_2 = 4$, $\Delta_1 = \Delta_3 = 1$, we have $H(\lambda) = \begin{pmatrix} \lambda - \frac{3}{8} & \lambda - \frac{3}{16} & 0 \\ 3\lambda - \frac{1}{8} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\sigma(H) = \left\{\frac{3}{16}, \frac{1}{24}\right\} \subset S(0, 1)$. Besides, it is not difficult to get the estimate $\|v(k)\| \leq 3\sqrt{5}(\frac{1}{\sqrt[12]{3}})^k \|v(0)\|$. Putting $E(-1) = E_3$, $\gamma = 3\sqrt{5}\left\|G(-1)^{-1}\right\| = \sqrt{55}$, $\lambda = \frac{1}{\sqrt[12]{3}}$ we have

$$\|x(k)\| \leq \gamma\lambda^k \|E(-1)x_0\|,$$

where x_0 is an arbitrary vector in $\mathbb{R}^3$. Thus, system (1) is E-uniformly exponentially stable.

Remark 1: In some cases, although all the spectra $\sigma(E_j, A_j), j = 1, \ldots, \sigma$ are not inside the unit circle, the resulting periodically switched system may still be stable.

Consider, for example, system (1) with $\sigma = 2$ and

$$E_1 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$
$$E_2 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

Obviously, $\sigma(E_1, A_1), \sigma(E_2, A_2)$ are not inside the unit circle. Choosing $N = 4$, $\Delta_1 = \Delta_2 = 2$, we get $H(\lambda) = \begin{pmatrix} \lambda - \frac{4}{9} & \lambda - \frac{19}{81} & 0 \\ 3\lambda - \frac{4}{3} & -\frac{1}{9} & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and $\sigma(H) = \left\{\frac{4}{9}, \frac{16}{81}\right\}$. Using the same argument as in example 3.3, we find $\|x(k)\| \leq 30(\sqrt[4]{\frac{2}{3}})^k \|E(-1)x_0\|$, which yields the E-uniform exponential stability of the resulting system.

Next we will stabilize periodically switched systems by feedback controls. Consider a control system

$$E(k)x(k+1) = A(k)x(k) + B(k)u(k), \tag{16}$$

where

$$(E(k), A(k), B(k)) = \begin{cases} (E_1, A_1, B_1), & lN \le k < k_1 + lN \\ (E_2, A_2, B_2), & k_1 + lN \le k < k_2 + lN \\ \ldots\ldots \\ (E_\sigma, A_\sigma, B_\sigma), & k_{\sigma-1} + lN \le k < (l+1)N. \end{cases}$$

Here $E_i, A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 1, 2, \ldots, \sigma$, are given matrices, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is a control vector, and $0 = k_0 < k_1 < k_2 < \cdots < k_{\sigma-1} < k_\sigma = N$ are integers.

Together with (16), we consider the corresponding system

$$E(k)x(k+1) = A(k)x(k), \tag{17}$$

We find the feedback control as

$$u(k) = D(k)x(k), \tag{18}$$

where $D(k) \in \mathbb{R}^{m\times n}$. State feedback (18) and system (16) form the closed-loop system

$$E(k)x(k+1) = (A(k) + B(k)D(k))x(k). \tag{19}$$

Theorem 4.3: *Assume that system (17) is of index 1 and there exists a triple (E_i, A_i, B_i) such that $\operatorname{rank}(\lambda E_i - A_i, B_i) = n, \forall \lambda \in \mathbb{C}, |\lambda| \ge 1$. Then there exist a sufficiently large number $N > 0$, the activation durations $\Delta_1, \Delta_2, \ldots, \Delta_\sigma$, and the feedback matrices $D_1, D_2, \ldots, D_\sigma$ such that system (16)with the feedback (18) is E-uniformly exponentially stable.*

Proof: Using Kronecker transformation for system (16), and putting

$$\overline{B}(k) = V(k)^{-1}G(k)^{-1}B(k) \equiv \begin{pmatrix} B1(k) \\ B2(k) \end{pmatrix},$$

we can rewrite system (16) as follows

$$\begin{cases} \overline{x1}(k+1) = W(k)\overline{x1}(k) + B1(k)u(k) \\ \overline{x2}(k) = -B2(k)u(k). \end{cases}$$

Choosing $\Delta_i \ge 4$, and $\tau = k_i + 2$, we find

$$\begin{aligned} \operatorname{rank}(\lambda E_i - A_i, B_i) &= \operatorname{rank}(\lambda E(\tau) - A(\tau), B(\tau)) \\ &= \operatorname{rank}(\lambda \overline{E}(\tau) - \overline{A}(\tau), \overline{B}(\tau)) \\ &= \operatorname{rank}\begin{pmatrix} \lambda I_r - W(\tau) & 0 & B1(\tau) \\ 0 & -I_{n-r} & B2(\tau) \end{pmatrix} \\ &= \operatorname{rank}(\lambda I_r - W(\tau), B1(\tau)) + n - r \\ &= n, \quad \forall \lambda \in \mathbb{C}, |\lambda| \ge 1. \end{aligned}$$

From the last relations, we get $\operatorname{rank}(\lambda I_r - W(\tau), B1(\tau)) = r, \forall\lambda \in \mathbb{C}, |\lambda| \geq 1$. Hence, we can choose a matrix $D1 \in \mathbb{R}^{m\times r}$ such that $\sigma(W(\tau) + B1(\tau)D1) \subset S(0,1)$ (cf. [5, p. 246]). Then the feedback matrix can be constructed as

$$D(k) = \begin{cases} (D1,0)V(\tau)^{-1}, & \text{if } \tau + lN \leq k \leq k_{i+1} - 2 + lN \\ 0, & \text{otherwise} \end{cases}.$$

It is not difficult to show that the closed-loop system (19) is of index-1 and can be rewritten as

$$v(k+1) = \begin{cases} (W(\tau) + B1(\tau)D1)v(k), & \text{if } \tau + lN \leq k \leq k_{i+1} - 2 + lN \\ W(k)v(k), & \text{otherwise} \end{cases}, \tag{20}$$

$$w(k) = \begin{cases} B2(\tau)D1v(k), & \text{if } \tau + lN \leq k \leq k_{i+1} - 2 + lN \\ 0, & \text{otherwise} \end{cases}.$$

Since $\sigma(W(\tau)+B1(\tau)D1) \subset S(0,1)$, there exist a sufficiently large N and the activation durations $\Delta_1, \ldots, \Delta_\sigma$ such that system (20) is uniformly exponentially stable, i.e. there exist $\gamma > 0$ and $0 \leq \lambda < 1$ such that

$$\|v(k)\| \leq \gamma\lambda^k \|v(0)\|.$$

Putting

$$M(k) = \begin{cases} B2(\tau)D1, & \text{if } \quad \tau + lN \leq k \leq k_{i+1} - 2 + lN \\ 0, & \text{otherwise} \end{cases},$$

we get $w(k) = M(k)v(k)$. Thus the solution of system (16) with the feedback (18) has the form $x(k) = V(k-1)(v(k)^T, (M(k)V(k))^T)^T$. Clearly,

$$\begin{aligned} \|x(k)\| &= \left\| V(k-1)V(k-1)^{-1}x(k) \right\| \\ &= \left\| V(k-1)(v(k)^T, w(k)^T)^T \right\| \\ &= \left\| V(k-1)\begin{pmatrix} I_r & 0 \\ M(k) & 0 \end{pmatrix}\begin{pmatrix} v(k) \\ 0 \end{pmatrix} \right\| \\ &\leq \mu\omega\gamma\lambda^k \|v(0)\|, \end{aligned}$$

where $\mu = \max_{k=0,\ldots,N-1} \|v(k)\|$, $\omega = \max_{k=0,\ldots,N-1} \left\| \begin{pmatrix} I_r & 0 \\ M(k) & 0 \end{pmatrix} \right\|$.

On the other hand, noting that

$$\begin{aligned} (v(0)^T, 0)^T &= \operatorname{diag}(I_r, 0)(v(0)^T, (M(0)v(0))^T)^T \\ &= V(-1)^{-1}G(-1)^{-1}E(-1)x(0), \end{aligned}$$

we conclude

$$\begin{aligned} \|x(k)\| &\leq \mu\omega\gamma\lambda^k \left\| V(-1)^{-1}G(-1)^{-1}E(-1)x(0) \right\| \\ &\leq \Gamma\lambda^k \|E(-1)x_0\|, \end{aligned}$$

where $\Gamma = \mu\omega\gamma \left\| V(-1)^{-1}G(-1)^{-1} \right\|$. The last relation shows that the solution of system (16) with the feedback (18) is E-uniformly exponentially stable. Theorem 4.3 is proved. $\square$

Example 4.4: Consider system (16) with $\sigma = 2$, and

$$E_1 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix};$$
$$E_2 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

A simple computation shows that $\operatorname{rank}(\lambda E_1 - A_1, B_1) = 3, \ \forall \lambda \in \mathbb{C}$. Choosing $N = 9, \Delta_1 = 8, \Delta_2 = 1$, and the matrices

$$D_1 = \begin{pmatrix} -\frac{5}{2} & -\frac{5}{6} & 0 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix},$$

we obtain the following systems

$$v(k+1) = \begin{cases} \begin{pmatrix} 2 & 1 \\ \frac{-5}{2} & \frac{-7}{6} \end{pmatrix} v(k), & \text{if} \quad 5l \le k \le 7 + 5l \\ \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ 0 & \frac{-4}{3} \end{pmatrix} v(k), & \text{if} \quad k = 8 + 5l \end{cases},$$
$$w(k) = \begin{cases} -\begin{pmatrix} \frac{5}{2} & \frac{5}{6} \end{pmatrix} v(k), & \text{if} \quad 5l \le k \le 7 + 5l \\ 0, & \text{if} \quad k = 8 + 5l \end{cases}.$$

It is not difficult to get the estimate $\|v(k)\| \le 15(\frac{1}{\sqrt[9]{5}})^k \|v(0)\|$ and $\|x(k)\| \le 300(\frac{1}{\sqrt[9]{5}})^k$ $\|E(-1)x_0\|$, which ensures the E-uniformly exponential stability of the corresponding solution.

5. Concluding remarks

In this paper, some necessary and sufficient conditions for the stability and stabilizability of periodically switched discrete-time linear index-1 singular systems are studied. Illustrative examples are also considered. An extension to periodically switched discrete-time nonlinear singular systems may be of interest.

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