Analysis of Systems of Linear Delay Differential Equations Using the Matrix Lambert Function and the Laplace Transformation

Sun Yi[*a], A. Galip Ulsoy[b] and Patrick W. Nelson[c]

[*] Corresponding author. Phone +1-734-615-6568 Fax +1-734-615-0312
[a, b] Department of Mechanical Engineering, University of Michigan: syjo@umich.edu, ulsoy@umich.edu
[c] Department of Mathematics, University of Michigan: pwn@umich.edu

Abstract

An approach for the analytical solution, free and forced, to systems of delay differential equations (DDEs) has been developed using the matrix Lambert function. To generalize the Lambert function solution for scalar DDEs to systems of DDEs, we introduce a new matrix, **Q** when the coefficient matrices in a system of DDEs do not commute. The solution is in the form of an infinite series of modes written in terms of the matrix Lambert function using **Q**. The essential advantage of this approach is the similarity with the concept of the state transition matrix in linear ordinary differential equations (ODEs), enabling its use for general classes of linear delay differential equations. Examples are presented to illustrate the new approach by comparison to numerical methods. The analytical solution to systems of DDEs in terms of the Lambert function is also presented in the Laplace domain to reinforce the analogy to ODEs.

Key words: Delay differential equation, Lambert function, Stability analysis, Transition matrices, Laplace transforms.

1. Introduction

Time-delay systems are those systems in which lag time exists between the applications of input or control to the system and their resulting effect on it. They arise either as a result of inherent delays in the components of the system or as a deliberate introduction of time delay into the system for control purposes (Richard, 2003).

Delay differential equations (DDEs), also known as difference-differential equations, were initially introduced in the 18th century by Laplace and Condorcet (Gorecki *et al*., 1989). The basic theory concerning stability of systems described by equations of this type was developed by Pontryagin (1942). Important works have been written by Bellman and Cooke in 1963, Smith in 1957, Pinney in 1958, Halanay in 1966, El'sgol'c and Norkin in 1971, Myshkis in 1972, Yanushevski in 1978, Marshal in 1979, and Hale in 1977. The reader is referred to the detailed review in Gorecki *et al*., 1989.

The principal difficulty in studying delay differential equations lies in its special transcendental character. The delay operator can be expressed in the form of an infinite series. Delay equations always lead to an infinite spectrum of frequencies. The determination of this spectrum requires a corresponding determination of zeros of certain analytic functions. One of the well-known approximation methods is the Padé approximation, which results in a shortened repeating fraction for the approximation of the characteristic equation of the delay (Lam, 1993; Golub and Van Loan, 1989).

Another approach to the delay problem is to look at the entire delay spectrum. Delay differential equations are often solved using numerical methods, asymptotic solutions, and graphical tools. Several attempts have been made to find an analytical solution for delay differential equations by solving its characteristic equation under different conditions. A related study on analytic solutions of linear DDEs can be found in Falbo, 1995. A Fourier-like analysis of the existence of the solution and its properties for the nonlinear DDEs is studied by Wright, 1946. Similar approaches to linear and nonlinear DDEs are also reported by Bellman in 1963. The uniqueness of the solution and its properties for the linear DDEs with varying coefficients and solution properties for the linear DDE with asymptotically constant coefficients are also studied by Wright in 1948.

An analytic approach to obtain the complete solution of systems of delay differential equations based on the concept of Lambert function was developed by Asl and Ulsoy in 2003. In this paper, the analytical approach of Asl and Ulsoy, 2003 is extended to general systems of DDEs and non-homogeneous DDEs, and compared with the results obtained by numerical integration. The advantage of this approach lies in the fact that the form of the solution obtained is analogous to the general solution form of ordinary differential equations, and the concept of the state transition matrix in ODEs can be generalized to DDEs using the concept of the matrix Lambert function (see Table 2).

2. Homogeneous Systems

2.1. Scalar Case

First we summarize the results for the first-order scalar homogenous DDE

$$\begin{aligned} &\dot{x}(t) + ax(t) + a_d x(t-T) = 0 && t > 0 \\ &x(t) = \phi(t) && t \in [-T, 0] \end{aligned} \tag{1}$$

which has a solution in terms of the Lambert function, W_k (Asl and Ulsoy, 2003),

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{(\frac{1}{T}W_k(-a_d T e^{aT}) - a)t} \tag{2}$$

where the C_k's are determined from the preshape function, $\phi(t)$. Every function $W(h)$, such that $W(h)e^{W(h)} = h$, is called a Lambert function (Coreless *et al*, 1996).

2.2. Generalization to Systems of DDEs

The Lambert function approach can be applied to the solution of systems of DDEs in matrix-vector form,

$$\begin{aligned} &\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-T) = 0 && t > 0 \\ &\mathbf{x}(t) = \boldsymbol{\varphi}(t) && t \in [-T, 0] \end{aligned} \tag{3}$$

In the special case that coefficient matrices, $\mathbf{A_d}$ and $\mathbf{A}$ commute, the solution is given as (see Asl and Ulsoy, 2003),

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{(\frac{1}{T}\mathbf{W}_k(-\mathbf{A_d}Te^{\mathbf{A}T}) - \mathbf{A})t}\mathbf{C}_k \tag{4}$$

However, this solution (which is of the same form as Eq. (2)) is only valid when the matrices $\mathbf{A}$ and $\mathbf{A_d}$ commute, that is $\mathbf{A}\mathbf{A_d} = \mathbf{A_d}\mathbf{A}$. Therefore, the solution in Eq. (4) in terms of a matrix Lambert function $\mathbf{W}_k$ is not general. We provide here, for the first time, the solution to Eq. (3) for the general case.

First we assume a solution form for Eq. (3) as,

$$\mathbf{x}(t) = e^{\mathbf{S}t}\mathbf{x}_0 \tag{5}$$

where $\mathbf{S}$ is $n \times n$, and substitution into Eq. (3) yields,

$$(\mathbf{S} + \mathbf{A_d}e^{-\mathbf{S}T} + \mathbf{A})e^{\mathbf{S}t}\mathbf{x}_0 = \mathbf{0} \tag{6}$$

Consequently, we must have,

$$\mathbf{S} + \mathbf{A_d}e^{-\mathbf{S}T} + \mathbf{A} = \mathbf{0} \tag{7}$$

Multiply through by $Te^{\mathbf{A}T}$ and rearrange to obtain,

$$T(\mathbf{S} + \mathbf{A})e^{\mathbf{S}T}e^{\mathbf{A}T} = -\mathbf{A_d}Te^{\mathbf{A}T} \tag{8}$$

When $\mathbf{S}$ and $\mathbf{A}$ commute, we can write the solution in terms of a Lambert function, as given in Eq. (4). It is shown in the Appendix that when $\mathbf{A}$ and $\mathbf{A_d}$ commute, then $\mathbf{S}$ and $\mathbf{A}$ also commute. However, in general $\mathbf{S}$ and $\mathbf{A}$ do not commute. Thus, in general

$$T(\mathbf{S} + \mathbf{A})e^{\mathbf{S}T}e^{\mathbf{A}T} \neq T(\mathbf{S} + \mathbf{A})e^{(\mathbf{S}+\mathbf{A})T} \tag{9}$$

Consequently, to write the solution in terms of a matrix Lambert function

$$\mathbf{W}(\mathbf{H})e^{\mathbf{W}(\mathbf{H})} = \mathbf{H} \tag{10}$$

we introduce an unknown matrix $\mathbf{Q}$ that satisfies,

$$T(\mathbf{S} + \mathbf{A})e^{(\mathbf{S}+\mathbf{A})T} = -\mathbf{A_d}T\mathbf{Q} \tag{11}$$

Comparing Eqs. (10) and (11) we note that,

$$T(\mathbf{S} + \mathbf{A}) = \mathbf{W}(-\mathbf{A_d}T\mathbf{Q}) \tag{12}$$

Then solving Eq. (12) for $\mathbf{S}$ yields,

$$\mathbf{S} = \frac{1}{T}\mathbf{W}(-\mathbf{A_d}T\mathbf{Q}) - \mathbf{A} \tag{13}$$

Substituting Eq. (13) into (7) yields the following condition, which can be used to solve for the unknown matrix $\mathbf{Q}$

$$\mathbf{W}(-\mathbf{A_d}T\mathbf{Q})e^{\mathbf{W}(-\mathbf{A_d}T\mathbf{Q}) - \mathbf{A}T} = -\mathbf{A_d}T \tag{14}$$

Finally, the $\mathbf{Q}$ obtained from Eq. (14) can be substituted into Eq. (13) to obtain $\mathbf{S}$, and then into Eq. (5) to obtain the homogeneous solution to Eq. (3),

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k \tag{15}$$

Thus, the solution to the system of homogeneous DDEs in Eq. (3) is given by Eq. (15), where the $\mathbf{C}_k$'s are computed from a given preshape function $\mathbf{x}(t) = \boldsymbol{\varphi}(t)$, $t \in [-T, 0]$. Corresponding to each branch, k, of the Lambert function, there is a solution $\mathbf{Q}_k$ from Eq. (14) and then for $\mathbf{H}_k = -\mathbf{A_d}T\mathbf{Q}_k$, we compute, for $i = 1, 2, .. n$, the eigenvalues λ_{ki} of $\mathbf{H}_k$ and the corresponding eigenvector matrix $\mathbf{V}_k$. We can then compute the matrix Lambert function,

$$\mathbf{W}_k(\mathbf{H}_k) = \mathbf{V}_k \begin{bmatrix} W_k(\lambda_{k1}) & 0 & \cdots & 0 \\ 0 & W_k(\lambda_{k2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k(\lambda_{kn}) \end{bmatrix} \mathbf{V}_k^{-1} \tag{16}$$

and then the $\mathbf{S}_k$ corresponding to $\mathbf{W}_k$ from Eq. (13). In the many examples we have studied, Eq. (14) always has a unique solution $\mathbf{Q}$ for each branch, k. The solution is obtained numerically, for a variety of initial conditions, using the *fsolve* function in Matlab.

The following example, from Lee and Dianat, 1981, illustrates the approach and compares the results to those obtained using numerical integration

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} + \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.330 \end{bmatrix} \begin{Bmatrix} x_1(t-1) \\ x_2(t-1) \end{Bmatrix} \quad (17)$$

Table 1 shows the values, for k = -1, 0, 1, for $\mathbf{Q}_k$, $\mathbf{S}_k$, and the eigenvalues, λ_{k1} and λ_{k2}, of $\mathbf{S}_k$. One of the eigenvalues due to the principal branch ($k=0$) is closest to the imaginary axis which means that it determines the stability of the system. For the scalar case, Eq. (2), it is proven that the root obtained using the principal branch always determines the stability of the system (Shinozaki, 2003). Although such a proof is not available in the matrix-vector case, we observe the same behavior in all the examples we have considered. In this case, the value of the real part of the dominant eigenvalue is in the left half plane, and therefore the system is stable. Using the values of $\mathbf{S}_k$ from Table 1, we obtain the solution,

$$\mathbf{x}(t) = \cdots + e^{\begin{bmatrix} -0.3499-4.9801i & -1.6253+0.1459i \\ 2.4174+0.1308i & -5.1048-4.5592i \end{bmatrix} t} \mathbf{C}_{-1}$$
$$+ e^{\begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix} t} \mathbf{C}_0 + e^{\begin{bmatrix} -0.3499+4.9801i & -1.6253-0.1459i \\ 2.4174-0.1308i & -5.1048+4.5592i \end{bmatrix} t} \mathbf{C}_1 + \cdots \quad (18)$$

Table 1. Intermediate results for computing the solution in Eq. (18) for the example in Eq. (17) via the matrix Lambert function

	k= -1	k= 0	k= 1
$\mathbf{Q}_k$	$\begin{bmatrix} -18.8024+10.2243i & 6.0782+2.2661i \\ -61.1342+23.6812i & 1.0161+0.2653i \end{bmatrix}$	$\begin{bmatrix} -9.9183 & 14.2985 \\ -32.7746 & 6.5735 \end{bmatrix}$	$\begin{bmatrix} -18.8024-10.2243i & 6.0782-2.2661i \\ -61.1342-23.6812i & 1.0161-0.2653i \end{bmatrix}$
$\mathbf{S}_k$	$\begin{bmatrix} -0.3499-4.9801i & -1.6253+0.1459i \\ 2.4174+0.1308i & -5.1048-4.5592 \end{bmatrix}$	$\begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix}$	$\begin{bmatrix} -0.3499+4.9801i & -1.6253-0.1459i \\ 2.4174-0.1308i & -5.1048+4.5592 \end{bmatrix}$
λ_{ki}	$\begin{cases} -1.3990-5.0935i \\ -4.0558-4.4458i \end{cases}$	$\begin{cases} -1.0119 \\ -1.9841 \end{cases}$	$\begin{cases} -1.3990+5.0935i \\ -4.0558+4.4458i \end{cases}$

The coefficients $\mathbf{C}_k$ in Eq. (18) are determined from specified preshape functions, e.g., let

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} \phi_1(t) \\ \phi_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad t \in [-1, 0] \quad (19)$$

Thus, for a delay T, we can write the N term approximation (Asl and Ulsoy, 2003),

$$\begin{Bmatrix} \boldsymbol{\varphi}(0) \\ \boldsymbol{\varphi}(-\frac{T}{2N}) \\ \boldsymbol{\varphi}(-\frac{2T}{2N}) \\ \vdots \\ \boldsymbol{\varphi}(-T) \end{Bmatrix} = \begin{bmatrix} e^{\mathbf{S}_{-N}0} & \cdots & e^{\mathbf{S}_N 0} \\ e^{\mathbf{S}_{-N}(-\frac{T}{2N})} & \cdots & e^{\mathbf{S}_N(-\frac{T}{2N})} \\ e^{\mathbf{S}_{-N}(-\frac{2T}{2N})} & \cdots & e^{\mathbf{S}_N(-\frac{2T}{2N})} \\ \vdots & \ddots & \vdots \\ e^{\mathbf{S}_{-N}(-T)} & \cdots & e^{\mathbf{S}_N(-T)} \end{bmatrix} \begin{Bmatrix} \mathbf{C}_{-N} \\ \mathbf{C}_{-N+1} \\ \mathbf{C}_{-N+2} \\ \vdots \\ \mathbf{C}_N \end{Bmatrix},$$

$$\text{i.e.,} \quad \mathbf{\Phi}(T,N) = \mathbf{\Omega}(T,N)\mathbf{C}_k \quad (20)$$
$$\mathbf{C}_k = \lim_{N\to\infty} \{\mathbf{\Omega}^{-1}(T,N) \cdot \mathbf{\Phi}(T,N)\}_k$$

and for $T = N = 1$ in our example we obtain,

$$\mathbf{C}_{-1} = \begin{Bmatrix} 1.3663+3.9491i \\ 3.2931+9.3999i \end{Bmatrix}, \mathbf{C}_0 = \begin{Bmatrix} -1.7327+0.0000i \\ -6.5863+0.0000i \end{Bmatrix}, \quad (21)$$
$$\text{and } \mathbf{C}_1 = \begin{Bmatrix} 1.3663-3.9491i \\ 3.2931-9.3999i \end{Bmatrix}$$

The results are compared to those obtained using numerical integration in Fig. 1, and show good agreement as more branches are used, *i.e.*, as the dimension of matix, N increases.

The key step in this approach, which allows the Lambert function approach to be used in Eq. (3), is the introduction, in Eq. (11), of the unknown matrix $\mathbf{Q}_k$, and the use of Eq. (14) to solve for $\mathbf{Q}_k$.

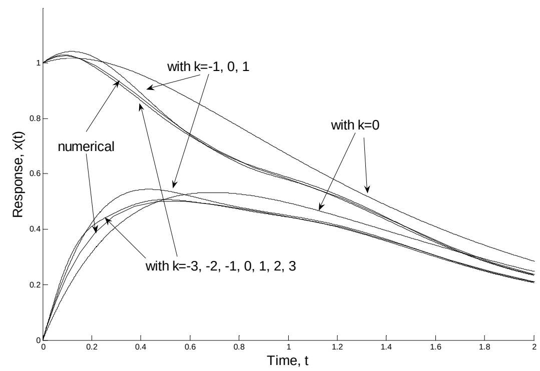

Fig. 1 Comparison for example in Eq.(17) of results from numerical integration vs. Eqs. (18) and (21) with one, three and seven terms. With more branches the results show better agreement.

3. Non-homogeneous (forced) Systems

3.1. Scalar Case

First we start with the non-homogeneous version of the scalar DDE in Eq.(1)

$$\begin{aligned} &\dot{x}(t) + ax(t) + a_d x(t-T) = bu(t) && t > 0 \\ &x(t) = \phi(t) && t \in [-T, 0] \end{aligned} \quad (22)$$

where $u(t)$ is a continuous function representing the external excitation. In Malek-zavarei and Jamshidi, 1987, the authors present the forced solution to Eq. (22) as,

$$x(t) = \int_0^t \Psi(t,\xi) u(\xi) d\xi \quad (23)$$

where the following conditions for $\Psi(t,\xi)$ must be satisfied

$$\begin{aligned} a)\ \frac{\partial}{\partial \xi}\Psi(t,\xi) &= a\Psi(t,\xi), && t-T \le \xi < t \\ &= a\Psi(t,\xi) + a_d \Psi(t,\xi+T), && \xi < t-T \end{aligned} \quad (24)$$
$$b)\ \Psi(t,t) = 1$$
$$c)\ \Psi(t,\xi) = 0 \quad \text{for} \quad \xi > t$$

Malek-zavarei and Jamshidi, 1987 do not indicate how to compute the fundamental function $\Psi(t,\xi)$. Here we present an approach based upon the Lambert function to do so. First a $\Psi(t,\xi)$ which satisfies the first condition in Eq. (24) is

$$\Psi(t,\xi)=e^{-a(t-\xi)} \tag{25}$$

A $\Psi(t,\xi)$ satisfying the second condition in Eq. (24) can be obtained using the Lambert function and confirmed by substitution as,

$$\Psi(t,\xi)_k=e^{(\frac{1}{T}W_k(-a_dTe^{aT})-a)(t-\xi)}, \tag{26}$$

where $k=-\infty\ldots\infty$

There are an infinite number of solutions for the branches of the Lambert function. Therefore the complete solution can be written in terms of the summation

$$\Psi(t,\xi)_k=\sum_{k=-\infty}^{\infty}C'_k e^{(\frac{1}{T}W_k(-a_dTe^{aT})-a)(t-\xi)} \tag{27}$$

Thus, the fundamental function is

$$\begin{aligned} a)\ \Psi(t,\xi)&=e^{-a(t-\xi)}, && t-T\le\xi\le t\\ &=\sum_{k=-\infty}^{\infty}C'_k e^{(\frac{1}{T}W_k(-a_dTe^{aT})-a)(t-\xi)}, && \xi<t-T \\ b)\ \Psi(t,\xi)&=0 \quad \text{for } \xi>t \end{aligned} \tag{28}$$

Consequently, the forced solution is obtained as,

Case I. $t\le T$

$$x(t)=\int_0^t\Psi(t,\xi)bu(\xi)d\xi=\int_0^t e^{-a(t-\xi)}bu(\xi)d\xi \tag{29}$$

Case II. $t>T$

$$x(t)=\int_0^{t-T}\sum_{k=-\infty}^{\infty}C'_k e^{S_k(t-\xi)}bu(\xi)d\xi+\int_{t-T}^{t}e^{-a(t-\xi)}bu(\xi)d\xi \tag{30}$$

where, $S_k=\dfrac{1}{T}W_k(-a_dTe^{aT})-a$

Though the calculation is dependent on *u(t)*, the C_k' can be computed using the functions in (29) and (30):

$$\underbrace{\begin{bmatrix}\sigma(T)\\ \sigma(T-\frac{T}{2N})\\ \sigma(T-\frac{2T}{2N})\\ \vdots\\ \sigma(0)\end{bmatrix}}_{\boldsymbol{\sigma}}=\underbrace{\begin{bmatrix}\eta_{-N}(T) & \cdots & \eta_N(T)\\ \eta_{-N}(T-\frac{T}{2N}) & \cdots & \eta_N(T-\frac{T}{2N})\\ \eta_{-N}(T-\frac{2T}{2N}) & \cdots & \eta_N(T-\frac{2T}{2N})\\ \vdots & \ddots & \vdots\\ \eta_{-N}(0) & \cdots & \eta_N(0)\end{bmatrix}}_{\boldsymbol{\eta}(T,N)}\begin{bmatrix}C'_{-N}\\ C'_{-N+1}\\ C'_{-N+2}\\ \vdots\\ C'_N\end{bmatrix}+\underbrace{\begin{bmatrix}\delta(T)\\ \delta(T-\frac{T}{2N})\\ \delta(T-\frac{2T}{2N})\\ \vdots\\ \delta(0)\end{bmatrix}}_{\boldsymbol{\delta}} \tag{31}$$

where

$$\begin{aligned}\sigma(t)&=\int_0^t e^{-a(t-\xi)}bu(\xi)d\xi\\ \eta(t)&=\int_0^{t-T}e^{S_k(t-\xi)}bu(\xi)d\xi\\ \delta(t)&=\int_{t-T}^{t}e^{-a(t-\xi)}bu(\xi)d\xi\end{aligned} \tag{32}$$

Consequently the C_k' can be represented as

$$C'_K=\lim_{N\to\infty}\{\boldsymbol{\eta}^{-1}(T,N)\times(\boldsymbol{\sigma}-\boldsymbol{\delta})\}_k \tag{33}$$

According to Eqs. (31)-(33), we can express the forced solution in Eqs. (29)-(30) as one equation (detailed derivation is in the Appendix)

$$x(t)=\int_0^t\sum_{k=-\infty}^{\infty}C'_k e^{S_k(t-\xi)}bu(\xi)d\xi \tag{34}$$

Hence, the solution becomes

$$x(t)=\underbrace{\sum_{k=-\infty}^{\infty}C_k e^{S_k t}}_{\text{free}}+\underbrace{\int_0^t\sum_{k=-\infty}^{\infty}C'_k e^{S_k(t-\xi)}bu(\xi)d\xi}_{\text{forced}} \tag{35}$$

As seen in Eq. (35), the total solution of DDEs using the Lambert function has a similiar form to that of ODEs. (Refer to Table 2). The coefficients C_k depend on the initial conditions and the preshape function, but the C_k' do not. Note that the C_k' are determined only by *a*, *b*, a_d and the delay time *T* in Eq. (22). See Eq. (A6) in the Appendix.

Example

Consider Eq. (22) with a unit step function excitation:

$$\begin{cases}bu(t)=1 & t>0\\ bu(t)=0 & t\in[-T,0]\end{cases} \tag{36}$$

and with $a_d=1$, $a=1$ and $T=1$ as in Asl and Ulsoy, 2003. Then, the forced solution (see Fig. 2)

$$x(t)=\int_0^t C'_k e^{S_k(t-\xi)}bu(\xi)d\xi=\sum_{k=-\infty}^{\infty}\frac{C'_k}{S_k}e^{S_k t}+0.5bu(t) \tag{37}$$

and the complete solution is the summation of the free solution in Eq. (2) and the forced solution, that is

$$\begin{aligned}x(t)&=x_{\text{free}}(t)+x_{\text{forced}}(t)\\ &=\sum_{k=-\infty}^{\infty}C_k e^{S_k t}+\sum_{k=-\infty}^{\infty}\frac{C'_k}{S_k}e^{S_k t}+0.5bu(t)\\ &\text{where, } S_k=\frac{1}{T}W_k(-a_dTe^{aT})-a=W_k(-e)-1\end{aligned} \tag{38}$$

The results are shown in Fig. 3 for the preshape function $\phi=1$, and compared to those obtained by numerical integration.

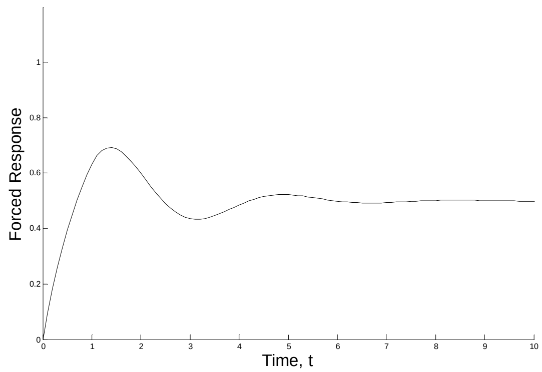

Fig. 2. Response of only forced solution in Eq. (37) with bu(t) defined by (36). Parameters are $a = a_d = T = 1$.

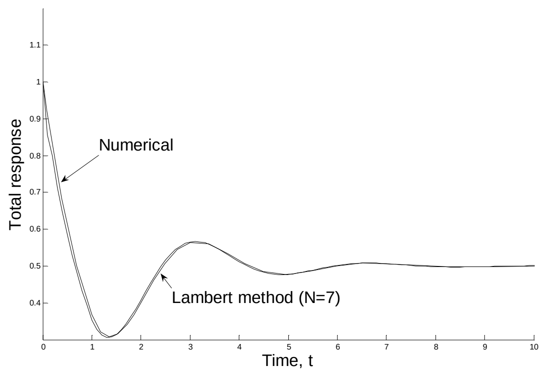

Fig. 3. Total forced response in Eq. (38) and comparison with numerical integration. They show good agreement. Parameters are $a = a_d = T = 1$.

Fig. 3 shows that the solution in Eq. (38) converges for the external force of unit step input.

As a second example, consider the forcing input

$$\begin{cases} u(t) = \cos(t) & t > 0 \\ u(t) = 0 & t \in [-T, 0] \end{cases} \tag{39}$$

The total response is shown in Fig. 4 for the preshape function $\phi = 1$, and compared to the result obtained by numerical integration. The agreement is excellent.

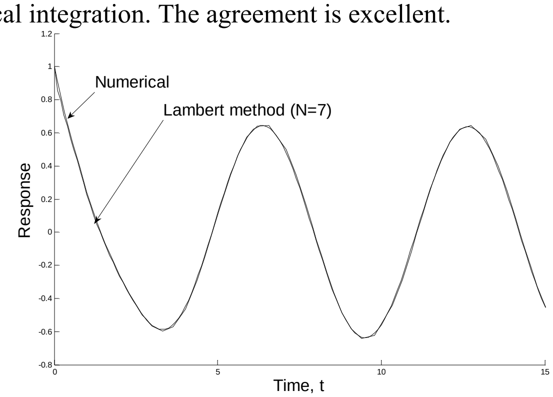

Fig. 4. Total forced response and comparison between the new method and the numerical method. The agreement is excellent. Parameter values are given in fig. 3 with $u(t)$ defined by equation (39).

3.2. Generalization to System of DDEs

The non-homogeneous matrix form of the delay differential equation in Eq. (3) can be written as

$$\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-T) = \mathbf{B}\mathbf{u}(t) \tag{40}$$

where $\mathbf{B}$ is an $n \times r$ matrix, and $\mathbf{u}(t)$ is a $r \times 1$ vector. The forced solution can be derived from Eqs. (29)-(30) as,

Case I. $t \le T$

$$\mathbf{x}(t) = \int_0^t \mathbf{\Psi}(t,\xi)\mathbf{B}\mathbf{u}(\xi)d\xi = \int_0^t e^{-\mathbf{A}(t-\xi)}\mathbf{B}\mathbf{u}(\xi)d\xi \tag{41}$$

Case II. $t > T$

$$\mathbf{x}(t) = \int_0^t \mathbf{\Psi}(t,\xi)\mathbf{B}\mathbf{u}(\xi)d\xi$$

$$= \int_0^{t-T} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}'_k\,\mathbf{B}\mathbf{u}(\xi)d\xi + \int_{t-T}^{t} e^{-\mathbf{A}(t-\xi)}\mathbf{B}\mathbf{u}(\xi)d\xi \tag{42}$$

$$\text{where, } \mathbf{S}_k = \frac{1}{T}\mathbf{W}_k(-\mathbf{A_d}T\mathbf{Q}) - \mathbf{A}$$

In Eq. (42), $\mathbf{C}'_k$ is a coefficient matrix of $n \times n$ dimension and can be calculated in the same way as Eq. (31). Like the scalar case in the previous section, Eqs. (41)-(42) are combined as

$$\mathbf{x}(t) = \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}'_k\,\mathbf{B}\mathbf{u}(\xi)d\xi \tag{43}$$

And the total solution is

$$\mathbf{x}(t) = \underbrace{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k}_{\text{free}} + \underbrace{\int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}'_k\,\mathbf{B}\mathbf{u}(\xi)d\xi}_{\text{forced}} \tag{44}$$

<u>Example</u>

Consider the system of DDEs with an external excitation:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.330 \end{bmatrix}\begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, t > 0$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ t \in [-1, 0] \tag{45}$$

Then the forced solution to Eq. (45) is obtained from Eq. (44), and the complete solution is shown in Fig. 5 when the input is a unit step function (u_1=u_2=1).

$$\int_0^\infty e^{-st}x(t-T)dt = \int_0^T e^{-st}x(t-T)dt + \int_T^\infty e^{-st}x(t-T)dt$$
$$= \int_0^T e^{-st}\phi(t)dt + \int_0^\infty e^{-s(t+T)}x(t)dt = \int_0^\infty e^{-st}\phi(t)dt + e^{-sT}\int_0^\infty e^{-st}x(t)dt$$
$$= \Phi(s) + e^{-sT}X(s) \tag{47}$$

Then, the transform of Eq. (1) can be written as

$$\begin{aligned} &sX(s) - x(0) + a_d e^{-sT}X(s) + a_d\Phi(s) + aX(s) \\ &= (s + a_d e^{-sT} + a)X(s) - x(0) + a_d\Phi(s) = 0 \end{aligned} \tag{48}$$

Therefore,

$$X(s) = \frac{x(0) - a_d\Phi(s)}{s + a_d e^{-sT} + a} \tag{49}$$

On the other hand, the solution obtained by the Lambert method is Eq. (4), which we can write as

$$x(t) = \sum_{k=-\infty}^{\infty} e^{(\frac{1}{T}W_k(-a_dTe^{aT})-a)t}C_k \xrightarrow{\text{Transformation}}$$
$$X(s) = \cdots + \frac{C_{-2}}{s-S_{-2}} + \frac{C_{-1}}{s-S_{-1}} + \frac{C_0}{s-S_0} + \frac{C_1}{s-S_1} + \frac{C_2}{s-S_2} + \cdots = \sum_{k=-\infty}^{\infty}\frac{C_k}{s-S_k} \tag{50}$$

where,

$$S_k = \frac{1}{T}W_k(-a_dTe^{aT}) - a \tag{51}$$

We can reduce the fractions to a common denominator,

$$X(s) = \sum_{k=-\infty}^{\infty}\frac{C_k n_k(s)}{d(s)} \tag{52}$$

By defining,

$$d(s) \equiv \prod_{k=-\infty}^{\infty}(s-S_k) = \cdots(s-S_{-2})(s-S_{-1})(s-S_0)(s-S_1)(s-S_2)\cdots \tag{53}$$

$$n_k(s) = \frac{d(s)}{(s-S_k)} = \cdots(s-S_{k-2})(s-S_{k-1})(s-S_{k+1})(s-S_{k+2})\cdots \tag{54}$$

where, $n_k(s)$ and $d(s)$ are polynomials in the Laplace variable, s. Now, it is possible to compare Eq. (52) with Eq. (49). Doing so we obtain

$$X(s) = \sum_{k=-\infty}^{\infty}\frac{C_k n_k(s)}{d(s)} = \frac{x(0) - a_d\Phi(s)}{s + a_d e^{-sT} + a} \tag{55}$$

Comparing denominators of both sides in Eq. (55) yields

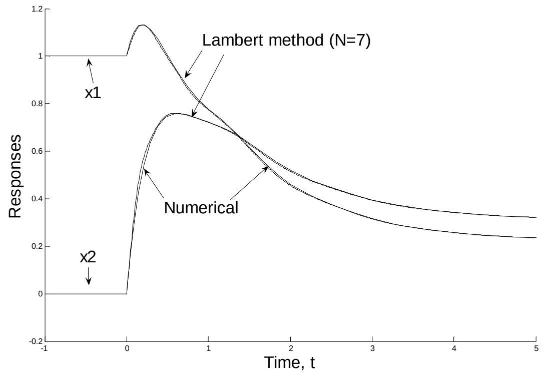

Fig. 5. Total response of Eq. (45) and comparison of the new method with the numerical method.

If we replace the external force with the trigonometric function,

$$\mathbf{Bu} = \begin{bmatrix}\cos(t)\\ 0\end{bmatrix} \tag{46}$$

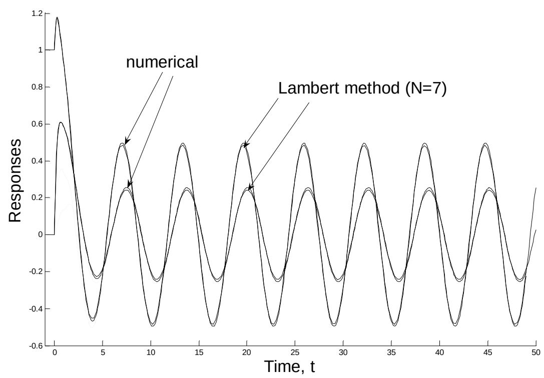

Fig. 6. Total forced response of Eq. (45) when external force is trigonometric (see Eq. (46)) and comparison of the new method with the numerical method.

The results are shown in Figure 6, and the differences between our new method with seven terms and the numerical integration are essentially indistinguishable.

4. Approach using the Laplace Transformation

4.1. Homogeneous Case

Consider the scalar homogeneous DDE in Eq. (1). Because $x(t)$ generally has a non-zero preshape function $\phi(t)$ as seen in Eq. (1), it can be transformed as,

$$d(s) \equiv \prod_{k=-\infty}^{\infty}(s-S_k) = J(s+a_d e^{-sT}+a) \tag{56}$$

and

$$\sum_{k=-\infty}^{\infty} C_k \cdot n_k(s) = J\{x(0)-a_d\Phi(s)\} \tag{57}$$

where J is an undetermined polynomial in s.

According to Eq. (54), $n_k(s)$ has the interesting and useful property,

$$\begin{aligned} n_k(s=S_l) &= 0 && \text{when } k \neq l \\ &= \cdots(S_l - S_{k-2})(S_l - S_{k-1})(S_l - S_{k+1})(S_l - S_{k+2})\cdots && \text{when } k = l \end{aligned} \tag{58}$$

With this property, we can compute C_k from Eq. (57) by substituting S_k for s, that is,

$$\begin{aligned} &\vdots \\ C_0 &= \frac{J\{x(0)-a_d\Phi(s)\}}{n_0(S_0)} = \frac{J(S_0)\{x(0)-a_d\Phi(s)\}}{\cdots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)(S_0-S_3)\cdots} \\ C_1 &= \frac{J\{x(0)-a_d\Phi(s)\}}{n_1(S_1)} = \frac{J(S_1)\{x(0)-a_d\Phi(s)\}}{\cdots(S_1-S_{-1})(S_1-S_0)(S_1-S_2)(S_1-S_3)(S_1-S_4)\cdots} \\ &\vdots \end{aligned} \tag{59}$$

where J is calculated from Eq. (56) as

$$\prod_{k=-\infty}^{\infty}(s-S_k) = J(s+a_d e^{-sT}+a) \quad \Rightarrow \quad J(s) = \frac{\prod_{k=-\infty}^{\infty}(s-S_k)}{(s+a_d e^{-sT}+a)} \tag{60}$$

Therefore,

$$\begin{aligned} J(S_0) &= \lim_{s\to S_0}\frac{\prod_{k=-\infty}^{\infty}(s-S_k)}{(s+a_d e^{-sT}+a)} = \lim_{s\to S_0}\frac{\frac{\partial}{\partial s}\prod_{k=-\infty}^{\infty}(s-S_k)}{\frac{\partial}{\partial s}(s+a_d e^{-sT}+a)} = \lim_{s\to S_0}\frac{\frac{\partial}{\partial s}\prod_{k=-\infty}^{\infty}(s-S_k)}{1-a_d T e^{-sT}} \\ &= \frac{\cdots(S_0-S_{-2})(S_0-S_{-1})(S_0-S_1)(S_0-S_2)\cdots}{1-a_d T e^{-S_0 T}} \end{aligned} \tag{61}$$

If we substitute the above result into Eq. (58), we obtain

$$C_k = \frac{x(0)-a_d\Phi(S_k)}{1-a_d T e^{-S_k T}} \tag{62}$$

4.2. Nonhomogeneous Case

With the above results, let us consider the scalar non-homogeneous case in Eq. (22). The Laplace tranform of Eq. (22) is,

$$\begin{aligned} &sX(s)-x(0)+a_d e^{-sT}X(s)+a_d\Phi(s)+aX(s) \\ &=(s+a_d e^{-sT}+a)X(s)-x(0)+a_d\Phi(s)=U(s) \end{aligned} \tag{63}$$

Therefore, the transform of the response, $x(t)$ is

$$X(s) = \underbrace{\frac{x(0)-a_d\Phi(s)}{s+a_d e^{-sT}+a}}_{\text{free}} + \underbrace{\frac{bU(s)}{s+a_d e^{-sT}+a}}_{\text{forced}} \tag{64}$$

Using the transform of the convolution property of the Laplace transformation, the inverse of the forced term in Eq. (64) is:

$$\frac{bU(s)}{s+a_d e^{-sT}+a} = \frac{1}{s+a_d e^{-sT}+a}bU(s) \overset{\text{inverse transform}}{\Rightarrow} \int_0^t e^{S_k(t-\xi)}C_k' bu(\xi)d\xi \tag{65}$$

where,

$$C_k' = \frac{1}{1-a_d T e^{-S_k T}} \tag{66}$$

Therefore, the total solution to Eq. (22) is

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} e^{S_k t}C_k + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\xi)}C_k' bu(\xi)d\xi \\ \text{and } S_k &= \frac{1}{T}W_k(-a_d T e^{aT})-a \end{aligned} \tag{67}$$

Note that C_k *is* dependent on the initial condition and preshape function, but C_k' is not, and

$$\frac{1}{s+a_d e^{-sT}+a} \overset{\text{inverse transform}}{\Rightarrow} \int_0^t e^{S_k(t-\xi)}C_k' bu(\xi)d\xi \tag{68}$$

4.3. Generalization to Systems of DDEs

For the system of DDEs in Eq. (3), if we take the Laplace transform of both sides we find,

$$s\mathbf{X}(s)-\mathbf{x}_0+\mathbf{A}\mathbf{X}(s)+\mathbf{A_d}e^{-sT}\mathbf{X}(s)+\mathbf{A_d}\mathbf{\Phi}(s)=\mathbf{B}\mathbf{U}(s) \tag{69}$$

Solving for the unknown $\mathbf{X}(s)$ yields

$$\begin{aligned} &\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)\mathbf{X}(s)=\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(s)+\mathbf{B}\mathbf{U}(s) \\ &\mathbf{X}(s)=\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1}\left\{\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(s)+\mathbf{B}\mathbf{U}(s)\right\} \end{aligned} \tag{70}$$

Therefore,

$$\mathbf{x}(t)=\underbrace{\mathcal{L}^{-1}\left[\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1}\left\{\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(s)\right\}\right]}_{\text{free solution}}+\underbrace{\mathcal{L}^{-1}\left[\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1}\left\{\mathbf{BU}(s)\right\}\right]}_{\text{forced solution}} \tag{71}$$

On the other hand, we have the free solution to Eq. (3) as Eq. (15). The solution in Eq. (15) can be transformed as

$$\mathbf{Y}(s)=\sum_{k=-\infty}^{\infty}(s\mathbf{I}-\mathbf{S}_k)^{-1}\mathbf{C}_k = (s\mathbf{I}-\mathbf{S}_0)^{-1}\mathbf{C}_0+(s\mathbf{I}-\mathbf{S}_{-1})^{-1}\mathbf{C}_{-1}+(s\mathbf{I}-\mathbf{S}_1)^{-1}\mathbf{C}_1+\cdots \tag{72}$$

Comparing Eq. (72) with the free solution part in Eq. (71) yields

$$\begin{aligned}&\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1}\left\{\mathbf{y}_0-\mathbf{A_d}\mathbf{\Phi}(s)\right\}=\sum_{k=-\infty}^{\infty}(s\mathbf{I}-\mathbf{S}_k)^{-1}\mathbf{C}_k\\&=(s\mathbf{I}-\mathbf{S}_0)^{-1}\mathbf{C}_0+(s\mathbf{I}-\mathbf{S}_{-1})^{-1}\mathbf{C}_{-1}+(s\mathbf{I}-\mathbf{S}_1)^{-1}\mathbf{C}_1+(s\mathbf{I}-\mathbf{S}_{-2})^{-1}\mathbf{C}_{-2}+(s\mathbf{I}-\mathbf{S}_2)^{-1}\mathbf{C}_2\cdots\end{aligned} \tag{73}$$

Eq. (73) provides the condition for calculating $\mathbf{C}_k$. Here we provide a 2×2 example.

$$\mathbf{A}=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix},\quad \mathbf{A_d}=\begin{bmatrix}a_{d1} & a_{d2}\\ a_{d3} & a_{d4}\end{bmatrix} \tag{74}$$

we can write the term in Eq. (73) as

$$(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT})=\begin{bmatrix}s+a_1+a_{d1}e^{-sT} & a_2+a_{d2}e^{-sT}\\ a_3+a_{d3}e^{-sT} & s+a_4+a_{d4}e^{-sT}\end{bmatrix} \tag{75}$$

And the inverse of the matrix term in Eq. (75) is

$$(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT})^{-1}=\frac{1}{\Delta(s)}\begin{bmatrix}s+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & s+a_1+a_{d1}e^{-sT}\end{bmatrix} \tag{76}$$

where $\Delta(s)$ is same as the characteristic equation of Eq.(3), that is

$$\begin{aligned}\Delta(s)=s^2+&\left\{a_1+a_4+(a_{d1}+a_{d4})e^{-sT}\right\}s+(a_1a_4-a_2a_3)+\\&(a_1a_{d4}+a_4a_{d1}-a_2a_{d3}-a_3a_{d2})e^{-sT}+(a_{d1}a_{d4}-a_{d2}a_{d3})e^{-2sT}\end{aligned} \tag{77}$$

In the right side in Eq. (73),

$$(s\mathbf{I}-\mathbf{S}_k)=\left(s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}p_{k1}&p_{k2}\\p_{k3}&p_{k4}\end{bmatrix}\right)=\left(s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\mathbf{V}_k\begin{bmatrix}\lambda_{k1}&0\\0&\lambda_{k2}\end{bmatrix}\mathbf{V}_k^{-1}\right) \tag{78}$$

$$\text{, where }\ \mathbf{S}_k=\begin{bmatrix}p_{k1}&p_{k2}\\p_{k3}&p_{k4}\end{bmatrix}$$

And the inverse is

$$(s\mathbf{I}-\mathbf{S}_k)^{-1}=\frac{1}{(s-\lambda_{k1})(s-\lambda_{k2})}\left(\begin{bmatrix}s-p_{k4}&p_{k2}\\p_{k3}&s-p_{k1}\end{bmatrix}\right) \tag{79}$$

With the results of Eq. (76) and Eq. (79), we can find the coefficients $\mathbf{C}_k$ in Eq. (15). For example, to obtain the coefficient of the principal branch, $\mathbf{C}_0$ from Eq. (73),

$$\begin{aligned}&\frac{1}{\Delta(s)}\begin{bmatrix}s+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & s+a_1+a_{d1}e^{-sT}\end{bmatrix}\left\{\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(s)\right\}\\&=\frac{1}{(s-\lambda_{01})(s-\lambda_{02})}\left(\begin{bmatrix}s-p_{04}&p_{02}\\p_{03}&s-p_{01}\end{bmatrix}\right)\mathbf{C}_0+(s\mathbf{I}-\mathbf{S}_{-1})^{-1}\mathbf{C}_{-1}+(s\mathbf{I}-\mathbf{S}_1)^{-1}\mathbf{C}_1+\cdots\end{aligned} \tag{80}$$

If we multiply the both sides by $(s-\lambda_{01})(s-\lambda_{02})$

$$\begin{aligned}&\frac{(s-\lambda_{01})(s-\lambda_{02})}{\Delta(s)}\begin{bmatrix}s+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & s+a_1+a_{d1}e^{-sT}\end{bmatrix}\left\{\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(s)\right\}\\&=\left(\begin{bmatrix}s-p_{04}&p_{02}\\p_{03}&s-p_{01}\end{bmatrix}\right)\mathbf{C}_0+(s-\lambda_{01})(s-\lambda_{02})(s\mathbf{I}-\mathbf{S}_{-1})^{-1}\mathbf{C}_{-1}+(s-\lambda_{01})(s-\lambda_{02})(s\mathbf{I}-\mathbf{S}_1)^{-1}\mathbf{C}_1+\cdots\end{aligned} \tag{81}$$

Then, substitution of λ_{01} for s in Eq. (81) makes the other terms on the right hand side zero except the first term. However, as in the scalar case, because $\Delta(s)$ is the characteristic equation, we again encounter a problem

$$\frac{(s-\lambda_{01})(s-\lambda_{02})}{\Delta(s)}=\frac{0}{0} \tag{82}$$

This problem can be resolved by application of L'Hopital's rule.

$$\begin{aligned}&\lim_{s\to\lambda_{01}}\frac{\frac{\partial}{\partial s}(s-\lambda_{01})(s-\lambda_{02})}{\frac{\partial}{\partial s}\Delta(s)}\begin{bmatrix}\lambda_{01}+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & \lambda_{01}+a_1+a_{d1}e^{-sT}\end{bmatrix}\left\{\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(\lambda_{01})\right\}\\&=\left(\begin{bmatrix}\lambda_{01}-p_{04}&p_{02}\\p_{03}&\lambda_{01}-p_{01}\end{bmatrix}\right)\mathbf{C}_0\end{aligned} \tag{83}$$

$$\begin{aligned}&\lim_{s\to\lambda_{02}}\frac{\frac{\partial}{\partial s}(s-\lambda_{01})(s-\lambda_{02})}{\frac{\partial}{\partial s}\Delta(s)}\begin{bmatrix}\lambda_{02}+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & \lambda_{02}+a_1+a_{d1}e^{-sT}\end{bmatrix}\left\{\mathbf{x}_0-\mathbf{A_d}\mathbf{\Phi}(\lambda_{02})\right\}\\&=\left(\begin{bmatrix}\lambda_{02}-p_{04}&p_{02}\\p_{03}&\lambda_{02}-p_{01}\end{bmatrix}\right)\mathbf{C}_0\end{aligned} \tag{84}$$

where $\mathbf{C}_0$ is computed from equations (83) and (84).

With the result in the previous section, continued from Eq. (71) and using the convolution property in the Laplace transform, we can write

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty}e^{\mathbf{S}_k t}\mathbf{C}_k+\int_0^t\sum_{k=-\infty}^{\infty}e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^{'}\mathbf{Bu}(\xi)d\xi \tag{85}$$

where

$$\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1} \xrightarrow{\text{Inverse Transform}} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^{'} \tag{86}$$

The coefficients $\mathbf{C}_k^{'}$ are computed from Eq. (87) as

$$\begin{aligned}\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1} &= \sum_{k=-\infty}^{\infty}(s\mathbf{I}-\mathbf{S}_k)^{-1}\mathbf{C}_k^{'}\\ &=(s\mathbf{I}-\mathbf{S}_0)^{-1}\mathbf{C}_0^{'}+(s\mathbf{I}-\mathbf{S}_{-1})^{-1}\mathbf{C}_{-1}^{'}+(s\mathbf{I}-\mathbf{S}_1)^{-1}\mathbf{C}_1^{'}+(s\mathbf{I}-\mathbf{S}_{-2})^{-1}\mathbf{C}_{-2}^{'}+(s\mathbf{I}-\mathbf{S}_2)^{-1}\mathbf{C}_2^{'}\cdots\end{aligned} \tag{87}$$

Here we again consider the 2×2 case as an example. Using the same notation as Eq. (83)-(84) for the general branch, without the terms related to initial condition and preshape function, first we should make s approach λ_{k1}, then,

$$\lim_{s\to\lambda_{k1}} \frac{\frac{\partial}{\partial s}(s-\lambda_{k1})(s-\lambda_{k2})}{\frac{\partial}{\partial s}\Delta(s)}\begin{bmatrix}\lambda_{k1}+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & \lambda_{k1}+a_1+a_{d1}e^{-sT}\end{bmatrix}=\left(\begin{bmatrix}\lambda_{k1}-p_{k4} & p_{k2}\\ p_{k3} & \lambda_{k1}-p_{k1}\end{bmatrix}\right)\mathbf{C}_k^{'} \tag{88}$$

And using another eigenvalue λ_{k1}, we obtain the second equation.

$$\lim_{s\to\lambda_{k2}} \frac{\frac{\partial}{\partial s}(s-\lambda_{k1})(s-\lambda_{k2})}{\frac{\partial}{\partial s}\Delta(s)}\begin{bmatrix}\lambda_{k2}+a_4+a_{d4}e^{-sT} & -(a_2+a_{d2}e^{-sT})\\ -(a_3+a_{d3}e^{-sT}) & \lambda_{k2}+a_1+a_{d1}e^{-sT}\end{bmatrix}=\left(\begin{bmatrix}\lambda_{k2}-p_{k4} & p_{k2}\\ p_{k3} & \lambda_{k2}-p_{k1}\end{bmatrix}\right)\mathbf{C}_k^{'} \tag{89}$$

With two equations (Eq. (88) and Eq.(89)) for $\mathbf{C}_k^{'}$, we can compute $\mathbf{C}_k^{'}$. In the case of higher order systems of DDEs, this same approach can also be applied.

5. Concluding Remark

In this paper, the Lambert function approach for analyzing linear delay differential equations in Asl and Ulsoy, 2003 is extended to general systems of DDEs and to non-homogeneous systems. To generalize the approach, we introduce a new matrix, **Q** when the coefficient matrices in the system of DDEs do not commute. The solution obtained in terms of the Lambert function using **Q** is in a form analogous to the state transition matrix in systems of linear ordinary differential equations. Free and forced responses for several cases of DDEs are presented in the paper based on this new solution approach and compared with those obtained by numerical integration. The agreement is excellent as the number of branches used in the Lambert function increases.

To provide a closed form solution to systems of linear DDEs, in a form similar to systems of ordinary differential equations, is the essential advantage of the new analytical approach presented here. Table 2 summarizes the comparison between them.

The concept of the state transition matrix in ODEs can be generalized to DDEs using the matrix Lambert function. This suggests that some analyses used in systems of ODEs, based on the concept of the state transition matrix, can potentially be extended to systems of DDEs. For example, the approach presented based on the matrix Lambert function, may be useful in controller design via eigenvalue assignment for systems of DDEs. Similarly, concepts of observability, controllability, state estimator design and modal decomposition of systems of DDEs may be tractable. The analytical approach using the matrix Lambert function for 'time-varying' DDEs based on Floquet theory is already being investigated. These, and others, are all potential topics for future research, which can build upon the foundation presented in this paper.

Table 2. Comparison of Equations and Solutions between ODEs and DDEs

ODEs	DDEs
Scalar Case	
$\dot{x}(t)+ax(t)=bu(t)$	$\dot{x}(t)+ax(t)+a_d x(t-T)=bu(t),\ t>0$; $x(t)=\phi(t),\quad t\in[-T,0]$
$x(t)=e^{-at}x_0+\int_0^t e^{-a(t-\xi)}bu(\xi)d\xi$	$x(t)=\sum_{k=-\infty}^{\infty}e^{S_k t}C_k+\int_0^t\sum_{k=-\infty}^{\infty}e^{S_k(t-\xi)}C_k^{'}bu(\xi)d\xi$; $S_k=\frac{1}{T}W_k(-a_dTe^{aT})-a$
$X(s)=\underbrace{\frac{x_0}{s+a}}_{\text{free}}+\underbrace{\frac{bU(s)}{s+a}}_{\text{forced}}$	$X(s)=\underbrace{\frac{x_0-a_d\Phi(s)}{s+a_de^{-sT}+a}}_{\text{free}}+\underbrace{\frac{bU(s)}{s+a_de^{-sT}+a}}_{\text{forced}}$
Matrix-vector Case	
$\dot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)=\mathbf{B}\mathbf{u}(t)$	$\dot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)+\mathbf{A_d}\mathbf{x}(t-T)=\mathbf{B}\mathbf{u}(t),\ t>0$; $\mathbf{x}(t)=\boldsymbol{\varphi}(t)\quad t\in[-T,0]$
$\mathbf{x}(t)=e^{-\mathbf{A}t}\mathbf{x}_0+\int_0^t e^{-\mathbf{A}(t-\xi)}\mathbf{B}\mathbf{u}(\xi)d\xi$	$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty}e^{\mathbf{S}_k t}\mathbf{C}_k+\int_0^t\sum_{k=-\infty}^{\infty}e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^{'}\mathbf{B}\mathbf{u}(\xi)d\xi$; $\mathbf{S}_k=\frac{1}{T}\mathbf{W}_k(-\mathbf{A_d}T\mathbf{Q})-\mathbf{A}$
$\mathbf{X}(s)=\underbrace{\left(s\mathbf{I}+\mathbf{A}\right)^{-1}\mathbf{x}_0}_{\text{free}}+\underbrace{\left(s\mathbf{I}+\mathbf{A}\right)^{-1}\{\mathbf{B}\mathbf{U}(s)\}}_{\text{forced}}$	$\mathbf{X}(s)=\underbrace{\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1}\{\mathbf{x}_0-\mathbf{A_d}\boldsymbol{\Phi}(s)\}}_{\text{free}}+\underbrace{\left(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sT}\right)^{-1}\{\mathbf{B}\mathbf{U}(s)\}}_{\text{forced}}$

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Appendix

Commutation of Matrices A and S in Eq. (9)

In general,

$$e^{\mathbf{X}}e^{\mathbf{Y}} = e^{\mathbf{X}+\mathbf{Y}} \tag{A1}$$

is only true, when the matrices **X** and **Y** commute (i.e., **XY** = **YX**). As noted previously, the solution in Eq. (4) for the system in Eq. (2) is only valid when the matrices **S** and $\mathbf{A_d}$ commute, and in general they do not (see Eqs. (8) and (9)). Here we note that when the matrices $\mathbf{A_d}$ and **A** in Eq. (3) commute, then **S** and **A** will commute, and the solution in Eq. (4) becomes valid.

From Eq. (13) we note that **S** can be expressed in terms of a polynomial function of the matrices $\mathbf{A_d}$ and **A**, since both the exponential and Lambert functions are represented as such polynomial series. In general if two matrices **X** and **Y** commute, and the matrix functions **f**(**X**) and **g**(**Y**) can be expressed in a polynomial series form, *i.e*.,

$$\mathbf{f}(\mathbf{X}) = \sum_{k=k_0}^{k_1} p_k \mathbf{X}^k, \quad \mathbf{g}(\mathbf{Y}) = \sum_{k=k_0}^{k_1} q_k \mathbf{Y}^k \tag{A2}$$

where p_k and q_k are arbitrary coefficients, then

$$\mathbf{f}(\mathbf{X})\mathbf{g}(\mathbf{Y}) = \mathbf{g}(\mathbf{Y})\mathbf{f}(\mathbf{X}) \tag{A3}$$

Consequently, if $\mathbf{A_d}$ and **A** commute, then **S** and **A** commute, and Eq. (4) is valid.

Reduction of Eqs. (29)-(30) to Eq. (35)

The Eq. (31) means that for $t \in [0,T]$,

$$\int_0^t e^{-a(t-\xi)} bu(\xi)d\xi = \sum_{k=-\infty}^{\infty} \int_0^{t-T} C_k^{'} e^{S_k(t-\xi)} bu(\xi)d\xi + \int_{t-T}^{t} e^{-a(t-\xi)} bu(\xi)d\xi \tag{A4}$$

Continued from Eq. (A4),

$$\begin{aligned}
&\int_0^t e^{-a(t-\xi)} bu(\xi)d\xi - \int_{t-T}^{t} e^{-a(t-\xi)} bu(\xi)d\xi = \sum_{k=-\infty}^{\infty} \int_0^{t-T} C_k^{'} e^{S_k(t-\xi)} bu(\xi)d\xi \\
&\Rightarrow \int_0^{t-T} e^{-a(t-\xi)} bu(\xi)d\xi = \int_0^{t-T} \sum_{k=-\infty}^{\infty} C_k^{'} e^{S_k(t-\xi)} bu(\xi)d\xi \\
&\Rightarrow \int_0^{t-T} e^{-a(t-\xi)} bu(\xi)d\xi - \int_0^{t-T} \sum_{k=-\infty}^{\infty} C_k^{'} e^{S_k(t-\xi)} bu(\xi)d\xi = 0 \\
&\Rightarrow \int_0^{t-T} \left\{ e^{-a(t-\xi)} - \sum_{k=-\infty}^{\infty} C_k^{'} e^{S_k(t-\xi)} \right\} bu(\xi)d\xi = 0 \\
&\overset{(g=t-\xi)}{\Rightarrow} \int_t^T \left\{ e^{-ag} - \sum_{k=-\infty}^{\infty} C_k^{'} e^{S_k g} \right\} bu(t-g)dg = 0, \quad \text{for } \forall t \in [0,T]
\end{aligned} \tag{A5}$$

For the last equation in Eq. (A5) to hold, for any value of $t \in [0,T]$, we can write it as

$$\int_t^T \left\{ e^{-ag} - \sum_{k=-\infty}^{\infty} C_k^{'} e^{S_k g} \right\} bu(t-g)dg = 0, \quad \text{for } \forall t \in [0,T] \tag{A6}$$

$$\Rightarrow e^{-ag} = \sum_{k=-\infty}^{\infty} C_k^{'} e^{S_k g}, \text{ where } g \in [0,T]$$

If the result in Eq. (A6) is applied, Eqs. (29)-(30) are combined into Eq. (34).

The result in Eq. (A6) means that when $t \in [0,T]$ the response of DDEs is same as that of ODEs. Because the external excitation starts from t=0, there is no forced response before the starting time. Therefore, in DDEs, the delayed term does not have any effect on the response during the interval, $t \in [0,T]$.