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# Decay function estimation for linear time delay systems via the Lambert W function

**Shiming Duan, Jun Ni and A Galip Ulsoy**

**Abstract**

The estimation of the decay function (i.e., $Ke^{\alpha t}\Phi$; see equation (2)) for time delay systems has been a long-standing problem. Most existing methods focus on dominant decay rate (i.e., $\alpha$) estimation, i.e., the estimation of the rightmost eigenvalue. Although some frequency domain approaches, such as bifurcation or finite dimensional approximation approaches are able to approximate the optimal decay rate computationally, the estimation of the factor, $K$, requires knowledge of the system trajectory over time and cannot be obtained from the frequency domain alone. The existing time domain approaches, such as matrix measure/norm or Lyapunov approaches, yield conservative estimates of decay rate. Furthermore, the factor $K$ in the Lyapunov approaches is typically not optimized.

A new Lambert W-function-based approach for estimation of the decay function for time delay systems is presented. This new approach is able to provide a closed-form solution for time delay systems in terms of an infinite series. Using this solution form, the optimal decay rate, $\alpha$, and an estimate of the corresponding factor, $K$, can be obtained. Less conservative estimates of the decay function can lead to more accurate description of the exponential behavior of time delay systems, and more effective control design based on those results. The method is illustrated with several examples, and results compare favorably with existing methods for decay function estimation.

**Keywords**

$\alpha$ stability, decay function estimation, exponential decay rate, Lambert W function, time delay systems

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## 1. Introduction

The stability of time delay systems has been a problem of recurring interest over the past several decades. Time delays exist in many practical systems in engineering, biology, chemistry, physics and ecology and can lead to effects such as oscillation, instability or inaccuracy. For highway transportation systems (Orosz et al., 2010), delays from human reactions and vehicle systems are critical for analyzing traffic flow stability and designing safe flow control algorithms. In machine tool chatter problems (Kalmar-Nagy et al., 2001; Yi et al., 2007b), inherent delays in the milling process may lead to instability and further deteriorate surface finish. The effects of time delay systems are also studied for teleoperation systems (Anderson and Spong, 1989), networked control systems (Murray, 2003), HIV pathogenesis (Yi et al., 2008) and automotive engine control systems (Cook and Powell, 1988). Besides these applications, fruitful results for stability criteria for time delay systems have been analytically derived. Detailed reviews of these techniques can be found in Gu and Niculescu (2003) and Richard (2003).

In addition to stability criteria, a quantitative description of asymptotic behavior for time delay systems is also valuable since it characterizes the transient response of these systems. Considerable work has been done toward deriving exponential bounds of the form $Ke^{\alpha t}\Phi$ (see equation (2)) for the solution of time delay systems for completely characterizing their exponential time response. Examples of such decay functions are shown in Figure 1, where the normed state trajectory of a second order time delay system is bounded by the two exponential envelope functions (dotted and

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dash-dot lines) after the system is excited by the preshape function $[1,0]^T$ over the time interval $-h \leq t \leq 0$ (delay $h = 1$ in this example). As shown in the figure, decay function 1 with better estimates of $\alpha$ and $K$ yields a closer bound to the actual trajectory than decay function 2, which is more conservative. A less conservative estimate of the decay function gives a more accurate description of the system transient behavior.

Although most of the existing literature focuses on the estimation of decay rate, $\alpha$, the estimation of $K$ is also important. With an estimate of $K$, the exact value of the boundover time can be determined, which could be useful in engineering practice. For example, the decay function can be used for dwell time control for switched time delay systems (Chiou, 2006; Kim et al., 2006; Yan and Ozbay, 2008; Zhang et al., 2007), where the factor $K$ represents the maximum energy rise during switching and $\alpha$ represents the energy decay rate. An application to a distributed cart pendulum control system with delay is given in Chen and Zhang (2010) showing that the system can still be stable if the control loop is opened with low frequency and a small time period. Some applications to engineering systems can be found in power systems (Meyer et al., 2004) and networked control systems (Dai et al., 2009; Kim et al., 2004). Less conservative estimate of the decay function will lead to more effective control design (e.g., lower bound on dwell time) for these applications.

A variety of approaches have been proposed for $\alpha$-stability, i.e., the dominant (slowest) decay rate of time delay systems. Methods based on the Bellman-Gronwall Lemma and matrix measure/norm have been proposed in (Bourlès, 1987; Lehman and Shujaee, 1994; Mori et al., 1982; Mori and Kokame, 1989; Niculescu et al., 1998). Methods based on Lypapunov's second method, such as Lyapunov-Krasovskii functional-based approaches (Kacem et al., 2009; Mondié and Kharitonov, 2005; Niculescu et al., 1998; Xu et al., 2006), Lyapunov-Razumikhin functional-based approaches (Hou and Qian, 1998, Jankovic, 2001) and Riccati equation approaches (Phat and Niamsup, 2006) have also been proposed. Methods using the Hopf Bifurcation Theorem are proposed in Engelborghs et. al. (2001), Fofana (2003), Forde and Nelson (2004), and Kalmar-Nagy et al. (2001). In Sipahi and Olgac (2003), a method based on examining one infinite cluster of roots at a time has been developed. Methods using pseudospectral and operator approximation techniques are proposed in Breda et al. (2005) and Michiels et. al. (2006).

Although fruitful results for $\alpha$-stability of time delay systems have been obtained, the methods for a complete estimation of the decay function are limited. The estimation of $K$ requires knowledge of the system trajectory over time, since such an exponential function needs to bound the states for any $t > 0$. Although it is feasible to approximate the optimal decay rate, i.e., the optimal $\alpha$, using some frequency domain approaches (e.g., finite dimensional approximation or bifurcation), these cannot provide an estimate of $K$. One needs the information in the time domain to determine both $\alpha$ and $K$ simultaneously. However, current time domain approaches, such as the aforementioned matrix measure/norm approaches and Lyapunov approaches, have inherent conservativeness, which limits their performance in obtaining an optimal estimate. For example, the estimates of $\alpha$ from these approaches can never reach the optimal value. The estimate of $K$ is also difficult to optimize in the Lyapunov approaches.

In Yi et al. (2007a), a closed-form solution for the system of delay differential equations (DDEs) in equation (1) has been derived in terms of an infinite series based on the Lambert W function. These results have recently been extended to solve eigenvalue assignment (Yi et al., 2010a,b) and stability problem (Yi et al., 2007b) for time delay systems. Using this solution form, the trajectory of systems of DDEs can be explicitly determined in terms of system parameters and preshape functions via the Lambert W function. Following these results, an optimal estimate of the decay rate $\alpha$ can be obtained. The estimation of a corresponding $K$ associated with this optimal $\alpha$ can be derived analytically by numerically evaluating the infinite series, which may be more efficient than solving the optimization problems in Lyapunov approaches.

A novel approach, based on the Lambert W function, to estimate the decay function for characterizing the exponential nature of the solution of time delay systems is presented. An optimal estimate of the decay rate, $\alpha$, and an estimate of the associated factor, $K$, are derived. In Section 2, the problem is formulated. In Section 3, the method based on the Lambert W function is presented, followed by numerical examples in Section 4. A summary and concluding remarks are provided in Section 5.

## 2. Problem formulation

Consider the continuous linear time invariant (LTI) homogeneous time delay system (TDS):

$$\begin{aligned} &\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) = 0, \quad t > 0 \\ &\mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(t) = \mathbf{g}(t) \quad \text{for} \quad t \in [-h, 0) \end{aligned} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{A_d}$ are $n \times n$ coefficient matrices, $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{g}(t)$ is an $n \times 1$ preshape function, $t$ is time, and $h$ is a constant scalar time delay. A discontinuity is permitted at $t = 0$ when $\mathbf{g}(0^-) \neq \mathbf{x}(0) = \mathbf{x}_0$. The goal is to find an upper bound for the decay rate, which is referred

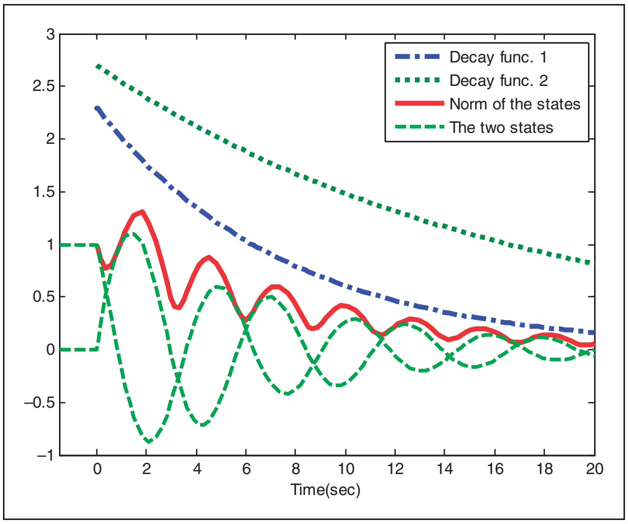


**Figure 1.** An example of decay functions for a second order time delay system $\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) = 0$ with $\mathbf{A} = [0\ 1; -1.25\ -1]$, $\mathbf{A_d} = [-0.1\ 0.6; 0.2\ 0]$, $h = 1$ and $\mathbf{g}(t) = [0\ 1]^T$ for $t \leq 0$.

to as $\alpha$-stability, as well as an upper bound for the factor $K$, such that the norm of the states is bounded:

$$\left\|\mathbf{x}(t)\right\| \leq Ke^{\alpha t}\Phi(h, t_0) \tag{2}$$

where $\Phi(h, t_0) = \sup_{t_0-h \leq t \leq t_0}\{\left\|\mathbf{x}(t)\right\|\}$ and $\|\cdot\|$ denotes the 2-norm. The conditions for the existence of $K$ and $\alpha$ have been discussed in Hale and Verduyn-Lunel (1993).

It is important to point out that $K$ and $\alpha$ are paired and cannot be estimated separately. Although there exist some frequency domain approaches to approximate the optimal $\alpha$, an estimate of the corresponding $K$ is not provided in those approaches. For a given $\alpha$, it is not feasible to obtain the $K$ using simulations, because the preshape function $\mathbf{g}(t)$ and the initial condition $\mathbf{x}_0$ cannot be uniquely determined knowing only the bound of $\left\|\mathbf{x}(t)\right\|$ for $t \in [-h, 0]$. The difficulty is to find an envelope function that bounds the norm of the states for any time $t > 0$ and for any possible preshape functions and initial conditions.

When $h = 0$, one has $\mathbf{A_d} = 0$ and the delay differential equation (DDE) in equation (1) reduces to an ordinary differential equation (ODE), whose decay function is

$$\left\|\mathbf{x}(t)\right\| \leq e^{\mu(\mathbf{A})t}\left\|\mathbf{x}(0)\right\| \tag{3}$$

where $\mu(\mathbf{A}) = \lim_{\theta \to 0^+} \frac{\|\mathbf{I}+\theta\mathbf{A}\|-1}{\theta}$ is the matrix measure (Hale and Verduyn-Lunel, 1993).

In the presence of time delay, the problem becomes much more complex since the trajectory of time delay systems depends not only on the initial states $\mathbf{x}_0$, but also on the preshape function $\mathbf{g}(t)$. Existing approaches lead to an estimate of the decay function with significant conservativeness. For example, the result from the matrix norm approach (Hale and Verduyn-Lunel, 1993) yields the estimates

$$K = 1 + \|\mathbf{A_d}\|h, \alpha = \|\mathbf{A}\| + \|\mathbf{A_d}\| \tag{4}$$

where the estimate of $\alpha$ can only be positive. For the matrix measure approaches (Lehman and Shujaee, 1994; Niculescu et al., 1998), $K$ is fixed to be 1, which renders the estimation of $\alpha$ very conservative. For example, consider the trajectory of the system shown in Figure 1. If $K$ equals 1, the decay function has a value of 1 at $t = 0$. Then $\alpha$ must be positive for the decay function to bound the peak of the normed state trajectory at $t = 2$ s. However, the optimal $\alpha$ is obviously a negative number.

Alternatively, one can apply Lyapunov approaches to solve the problem computationally. Using the classical Lyapunov-Krasovskii methods, the estimates of $K$ and $\alpha$ can be obtained as

$$K = \sqrt{\frac{c_2}{c_1}}, \alpha = -c_3 \tag{5}$$

assuming the existence of positive constant scalars $c_1$, $c_2$, $c_3$ such that

$$c_1\left\|\mathbf{x}(t)\right\|^2 \leq V(\mathbf{x}_t) \leq c_2\|\mathbf{x}_t\|^2 \tag{6}$$

and

$$\dot{V}(\mathbf{x}_t) \leq -2c_3\left\|\mathbf{x}(t)\right\|^2 \tag{7}$$

where $\mathbf{x}_t$ denotes the segment of $\{\mathbf{x}(t+\theta)|\theta \in [-h, 0]\}$ and $V(\cdot)$ is a Lyapunov-Krasovskii functional. Because of the inherent conservativeness of the Lyapunov approaches, the decay rate estimate, $c_3$, will not be able to reach the optimal value. The estimate of $K$, i.e., $\sqrt{\frac{c_2}{c_1}}$, is also difficult to optimize.

From the solution form in Yi et al. (2007a), the relationship between the trajectory of the system in equation (1) and the preshape function as well as initial conditions can be analytically determined in terms of an infinite Lambert W function series. We intend to overcome, or reduce, the inherent conservativeness in matrix measure/norm approaches and Lyapunov approaches and investigate the decay function estimation problem from a new and different point of view by applying the Lambert W function approach.

## 3. Main results

Consider the homogenous matrix DDE in equation (1). The solution of (1) can be written as (Yi et al., 2006),

$$x(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I \tag{8}$$

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q}_k) - \mathbf{A} \tag{9}$$

and $\mathbf{Q}_k$ is solved by the following condition

$$\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q}_k)e^{\mathbf{A}e^{\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q})-\mathbf{A}h}} = -\mathbf{A_d}h \tag{10}$$

Here $\mathbf{S}_k$ and $\mathbf{Q}_k$ are $n \times n$ matrices. $\mathbf{C}_k^I$ are $n \times 1$ vectors and are determined by the preshape function $\mathbf{g}(t)$ and $\mathbf{x}_0$, by using either one of two different approaches (Yi et al., 2006, 2007b). The matrix Lambert W function, $\mathbf{W}_k(\cdot)$, is a complex valued function with a complex matrix argument $\mathbf{z}$ and an infinite number of branches denoted by $k$, where $k = -\infty, \ldots, -1, 0, 1, \ldots, \infty$ and satisfies $\mathbf{W}_k(\mathbf{z})e^{\mathbf{W}_k(\mathbf{z})} = \mathbf{z}$ for all branches. Note that $\mathbf{W}_k$ can readily be evaluated using functions available in standard software packages such as Matlab or Mathematica. The conditions for the existence and uniqueness of the solution in equation (9) are discussed in (Bellman and Cooke, 1963).

The results in Yi et al. (2006) are extended here to derive a general solution (see Appendix A) to show that the solution of (1) can be written as

$$\mathbf{x}(t) = \underbrace{\sum_{k=-\infty}^{\infty}\left\{e^{\mathbf{S}_k t}\left(\sum_{j=1}^{n}\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\right)\mathbf{x}_0\right\}}_{P_1} - \underbrace{\sum_{k=-\infty}^{\infty}\left\{e^{\mathbf{S}_k t}\sum_{j=1}^{n}(\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{A_d}\mathbf{G}(\lambda_{kj}))\right\}}_{P_2} \tag{11}$$

where

$$\lambda_{kj} = eig(\mathbf{S}_k), \quad j = 1, 2, \ldots, n \tag{12}$$

$$\mathbf{G}(\lambda_{kj}) = \int_0^h e^{-\lambda_{kj}\tau}\mathbf{g}(\tau - h)d\tau \tag{13}$$

$$\mathbf{L}_{kj}^I = \lim_{s\to\lambda_{kj}}\left\{\frac{\frac{\partial}{\partial s}\prod_{j=1}^{n}(s-\lambda_{kj})}{\frac{\partial}{\partial s}\det(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})}adj(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})\right\} \tag{14}$$

$$\tilde{\mathbf{R}}_k^{I+} = \begin{bmatrix}\mathbf{T}_{k1}^I & \mathbf{T}_{k2}^I & \ldots & \mathbf{T}_{kn}^I\end{bmatrix} = \tilde{\mathbf{R}}_k^{I*}(\tilde{\mathbf{R}}_k^I\tilde{\mathbf{R}}_k^{I*})^{-1} \tag{15}$$

$$\mathbf{R}_{kj}^I = adj(\lambda_{kj}\mathbf{I} - \mathbf{S}_k), \quad \tilde{\mathbf{R}}_k^I = \begin{bmatrix}\mathbf{R}_{k1}^I \\ \mathbf{R}_{k2}^I \\ \mathbf{R}_{kn}^I\end{bmatrix} \tag{16}$$

Note that $\tilde{\mathbf{R}}_k^{I+}$ is the $n \times n^2$ Moore-Penrose Generalized Inverse, $\tilde{\mathbf{R}}_k^{I*}$ is the $n \times n^2$ conjugate transpose of $\tilde{\mathbf{R}}_k^I$ and $\mathbf{T}_{kj}^I$ is the $j$th square block of $\tilde{\mathbf{R}}_k^{I+}$.

**Theorem 1:** If there exist scalars $\alpha$, $K_1$, $K_2$, $K_3$ and $K_4$ such that

$$\alpha = \max\{\mathrm{Re}(eig(\mathbf{S}_{-m})), \ldots, \mathrm{Re}(eig(\mathbf{S}_0)), \ldots, \mathrm{Re}(eig(\mathbf{S}_m))\} \tag{17}$$

$$K_1 = \sup_{0\le t<h}\left\|e^{(-\mathbf{A}-\alpha\mathbf{I})t}\right\| \tag{18}$$

$$K_2 = \lim_{N\to\infty}\left\{\sup_{t\ge h}\left\|\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\sum_{j=1}^{n}\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\right\}\right\|\right\} \tag{19}$$

$$K_3 = \sup_{0\le t<h}\int_0^t\left\|e^{(-\mathbf{A}-\alpha\mathbf{I})t+\mathbf{A}\tau}\mathbf{A_d}\right\|d\tau \tag{20}$$

$$K_4 = \lim_{N\to\infty}\left\{\sup_{t\ge h}\int_0^h\left\|\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\sum_{j=1}^{n}(\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{A_d}e^{\lambda_{ki}\tau})\right\}\right\|d\tau\right\} \tag{21}$$

where $m = \mathrm{nullity}(\mathbf{A_d})$ and $eig(\mathbf{S}_i)$ are the eigenvalues of $\mathbf{S}_i$. Then, the trajectories of equation (1) are bounded by the exponential function $\|\mathbf{x}(t)\| \le Ke^{\alpha t}\Phi(h)$ for any time $t > 0$, where $\Phi(h) = \sup_{-h\le t\le 0}\{\|\mathbf{x}(t)\|\}$ and $K = \max(K_1, K_2) + \max(K_3, K_4)$.

**Proof of Theorem 1**

*Estimation of Decay Rate α*

For the scalar case of (1), the rightmost eigenvalue can be determined by the principal branch, i.e., $k = 0$, of the scalar Lambert W function (Shinozaki and Mori, 2006). Such a proof can readily be extended to the matrix case when $\mathbf{A}$ and $\mathbf{A_d}$ in eqquation (1) commute (Jarlebring and Damm, 2007). No such proof is currently available for the general case of matrix DDEs. However, in all the examples considered in the literature, it has been observed that the rightmost eigenvalue is obtained using the first $m$ branches, where $m$ is the nullity of $\mathbf{A_d}$ (Yi et al., 2010c). We state this here as a conjecture:

$$\max\{\mathrm{Re}(eig(\mathbf{S}_{-m})), \ldots, \mathrm{Re}(eig(\mathbf{S}_0)), \ldots, \mathrm{Re}(eig(\mathbf{S}_m))\} \ge \max\{\mathrm{Re}(eig(\mathbf{S}_i))\}, \quad \forall i \tag{22}$$

where $m = Nullity(\mathbf{A}_d)$ and $eig(\mathbf{S}_i)$ are the eigenvalues of $\mathbf{S}_i$. Thus, based on the above conjecture, the optimal decay rate for matrix DDEs can be calculated as equation (17).

*Estimation of Factor K*

Having determined $\alpha$ using equation (17), one must then determine the factor $K$ such that $\|x(t)\| \leq Ke^{\alpha t}\Phi$, where $\Phi = \sup_{-h\leq t\leq 0}\|x(t)\|$. Taking the norm of both sides of equation (11) yields

$$\|\mathbf{x}(t)\| \leq \|\mathbf{P}_1(t)\| + \|\mathbf{P}_2(t)\| \tag{23}$$

where $\mathbf{P}_1(t)$ and $\mathbf{P}_2(t)$ have been defined in equation (11). The use of the inequality in equation (23) will introduce conservativeness in our results. Since the envelope function should bound any possible trajectory, one must separate $\|\mathbf{x}_0\|$ and $\|\mathbf{g}(\cdot)\|$ out from the infinite series but without affecting convergence.

First, note that for $t \in [0, h)$, the state $\mathbf{x}(t-h)$ in matrix DDEs is solely determined by the preshape function. Thus, for this period, the homogeneous matrix DDE function can be treated as a matrix ODE with an input from the preshape function:

$$\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) = -\mathbf{A_d}\mathbf{g}(t-h) \tag{24}$$

Therefore, for $t \in (0, h)$, $\mathbf{P}_1(t)$ equals the free response of (24) with $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{P}_2(t)$ can be treated as the forced response of equation (24) with the input $\mathbf{g}(t-h)$. Thus, for this period, the bounds for $\|\mathbf{P_1}(t)\|$ and $\|\mathbf{P}_2(t)\|$ can be obtained as,

$$\|\mathbf{P_1}(t)\| \leq \|e^{(-\mathbf{A}-\alpha\mathbf{I})t}\mathbf{x}_0\|e^{\alpha t} \leq K_1 e^{\alpha t}\Phi, t \in [0, h) \tag{25}$$

$$\begin{aligned}\|\mathbf{P}_2(t)\| &\leq \int_0^t \|-e^{-\mathbf{A}(t-\tau)}\mathbf{A_d}\|\cdot\|\mathbf{g}(\tau-h)\|d\tau \\ &\leq \int_0^t \|-e^{(-\mathbf{A}-\alpha\mathbf{I})t+\mathbf{A}\tau}\mathbf{A_d}\|\cdot\|\mathbf{g}(\tau-h)\|e^{\alpha t}d\tau \\ &\leq K_3 e^{\alpha t}\Phi, \quad t \in [0, h)\end{aligned} \tag{26}$$

where $K_1$ and $K_3$ are defined in equations (18) and (20) respectively.

For $t \in [h, \infty)$, the Lambert W function method is applied. Note that,

$$\begin{aligned}\|\mathbf{P}_1(t)\| &= \lim_{N\to\infty}\left\|\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\sum_{j=1}^{n}\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{x}_0\right\}\right\|e^{\alpha t} \\ &\leq \lim_{N\to\infty}\left\{\sup_{t\geq h}\left\|\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\sum_{j=1}^{n}\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\right\}\right\|\right\} \\ &\quad e^{\alpha t}\|\mathbf{x}_0\|, t \in [h, \infty)\end{aligned} \tag{27}$$

Thus, if $K_2$ in equation (19) exists, $\|\mathbf{P}_1(t)\| \leq K_2 e^{\alpha t}\Phi$ for $t \in [h, \infty)$ will be satisfied since $\Phi = \sup_{-h\leq t\leq 0}\|\mathbf{x}(t)\| \geq \|\mathbf{x}(0)\|$. Similarly,

$$\begin{aligned}\|\mathbf{P}_2(t)\| &= \lim_{N\to\infty}\left\{\left\|\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\right.\right.\right. \\ &\qquad \left.\left.\left.\sum_{j=1}^{n}(\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{A_d}\int_0^h e^{\lambda_{ki}\tau}\mathbf{g}(\tau-h)d\tau)\right\}\right\|\right. \\ &\leq \lim_{N\to\infty}\left\|\int_0^h\left\{\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\sum_{j=1}^{n}(\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{A_d}e^{\lambda_{ki}\tau})\right\}\right\}\right. \\ &\qquad \left.\times\,\mathbf{g}(\tau-h)d\tau\right\|e^{\alpha t}, t \in [h, \infty)\end{aligned} \tag{28}$$

Switch the sequence of integration and obtaining the norm, and move $\|\mathbf{g}(t-h)\|$ outside of the integration:

$$\begin{aligned}\|\mathbf{P}_2(t)\| \leq &\lim_{N\to\infty} \\ &\left\{\sup_{t\geq h}\int_0^h\left\|\sum_{k=-N}^{N}\left\{e^{(\mathbf{S}_k-\alpha\mathbf{I})t}\sum_{j=1}^{n}(\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{A_d}e^{\lambda_{ki}\tau})\right\}\right\|d\tau\right. \\ &\qquad \left.\times\,\|\mathbf{g}(\tau-h)\|\right\}\end{aligned} \tag{29}$$

Thus, if $K_4$ in (21) exists, then $\|\mathbf{P}_2(t)\| \leq K_4 e^{\alpha t}\Phi$ for $t \in [h, \infty)$ will hold since $\Phi = \sup_{-h\leq t\leq 0}\|\mathbf{x}(t)\| \geq \|\mathbf{g}(t-h)\|$ for $t \in [0, h)$.

Hence the proof.

**Remark 1:** The Lambert W function approach provides a solution in terms of infinite series. The feasibility of the approach depends on the convergence of the series. The proof for the convergence of such a Lambert W function series is currently not available. However, one can still evaluate the series numerically to obtain the estimate of $K$. The procedure is demonstrated in the numerical examples in Section 4.

**Remark 2:** It has been observed that, although the Lambert W function series may converge slowly at $t = 0^+$, the convergence speed increases quickly when $t$ becomes larger. Since the DDE can be treated as an ODE for $t \in [0, h)$, the Lambert W function approach is applied for $t \geq h$ to achieve better convergence.

**Remark 3:** Since the envelope function needs to bound any possible trajectories, one must separate $\|\mathbf{x}_0\|$ and $\|\mathbf{g}(\cdot)\|$ out from the infinite series but without affecting convergence. Thus, in equation (29) the sequence of integration and obtaining the norm is switched before moving $\|\mathbf{g}(t-h)\|$ outside of the integration.

**Remark 4:** Note that the estimates of $K_1$, $K_2$, $K_3$ and $K_4$ are obtained directly based on the solution of the system and no conservativeness is introduced following the proposed procedure. However, the use of the solution form in equation (11) accommodates the discontinuity at $t=0$ (i.e., $\mathbf{g}(0^-) \neq \mathbf{x}(0) = \mathbf{x}_0$) but introduces conservativeness when the triangle inequality equation (23) is applied. When such a discontinuity is considered, the estimate of $K$ from our approach is the optimal.

Theorem 1 gives the result for general systems of DDEs. For the scalar case, the results can be further simplified. Consider the scalar version of equation (1):

$$\begin{aligned} &\dot{x}(t) + ax(t) + a_d x(t-h) = 0, \quad t > 0 \\ &x(t) = g(t), \quad t \in [-h, 0); \quad x(0) = x_0, \quad t = 0 \end{aligned} \tag{30}$$

where $a$, $a_d$, $h$ are all scalar constants, $t$ is time, and $x(t)$, $g(t)$ are scalar functions.

**Corollary 1:** If there exist scalar $\alpha$, $K_1$, $K_2$, $K_3$ and $K_4$ such that

$$\alpha = \mathrm{Re}\left[\frac{W_0(-a_d h e^{ah})}{h} - a\right] \tag{31}$$

$$K_1 = \sup_{0 \le t < h} \left\| e^{(-a-\alpha \mathbf{I})t} \right\| = \begin{cases} e^{(-a-\alpha)h}, & -a > \alpha \\ 1, & -a \le \alpha \end{cases} \tag{32}$$

$$K_2 = \lim_{N \to \infty} \left\{ \sup_{t \ge h} \left\| \sum_{k=-N}^{N} \frac{e^{(S_k - \alpha)t}}{1 - a_d h e^{-S_k h}} \right\| \right\} \tag{33}$$

$$K_3 = \sup_{0 \le t < h} \int_0^t \left\| e^{(-a-\alpha)t + a\tau} a_d \right\| d\tau = \left\| \frac{a_d (1 - e^{-ah}) e^{-\alpha h}}{a} \right\| \tag{34}$$

$$K_4 = \lim_{N \to \infty} \left\{ \sup_{t \ge h} \int_0^h \left\| \sum_{k=-N}^{N} \frac{a_d e^{-S_k \tau}}{1 - a_d h e^{-S_k h}} e^{(S_k - \alpha)t} \right\| d\tau \right\} \tag{35}$$

Then, the trajectories of equation (30) are bounded by the exponential function $\|x(t)\| \le K e^{\alpha t} \Phi(h)$ for any time $t > 0$, where $\Phi(h) = \sup_{-h \le t \le 0}\{\|x(t)\|\}$.

**Proof of Corollary 1** The solution to equation (30) can be written in terms of the Lambert W function, $W_k$ (Asl and Ulsoy, 2003), as:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k^I e^{S_k t}, \quad S_k = \frac{1}{h} W_k(-a_d h e^{ah}) - a \tag{36}$$

Following the Laplace transformation based method in Yi et al. (2006) for determining $C_k^I$ in equation (36) gives

$$C_K^I = \frac{x_0 - a_d \int_0^h e^{-S_k t} g(t) dt}{1 - a_d h e^{-S_k h}} \tag{37}$$

Note that the free response $x(t)$, using equations (36) and (37), can be separated into two parts:

$$x(t) = \underbrace{\sum_{k=-\infty}^{\infty} \frac{x_0 e^{S_k t}}{1 - a_d h e^{-S_k h}}}_{P_1(t)} - \underbrace{\sum_{k=-\infty}^{\infty} \frac{a_d \int_0^h e^{-S_k \tau} g(\tau - h) d\tau}{1 - a_d h e^{-S_k h}} e^{S_k t}}_{P_2(t)} \tag{38}$$

One can then follow a similar procedure as for the proof of Theorem 1 to complete the proof for the scalar case.

## 4. Numerical examples

In this section, one scalar example and one matrix example are provided to demonstrate the effectiveness of the proposed approach.

*Example 1(Scalar DDE)*: Consider the scalar DDE in (30) with $a = a_d = h = 1$(Yi et al., 2006):

$$\dot{x}(t) + x(t) + x(t-1) = 0, \quad t > 0 \tag{39}$$

Note that the exact value of $g(t)$ and $x_0$ is not needed here but their supremum is known as $\Phi(h) = \sup_{-h \le t \le 0}\{\|x(t)\|\}$. The obtained decay function applies to any $g(t)$ and $x_0$ for the system.

From equation (31), the rightmost pole is found to be:

$$\alpha = \mathrm{Re}\left[\frac{W_0(-a_d h e^{ah})}{h} - a\right] = -0.605 \tag{40}$$

Thus, the decay rate $\alpha = -0.605$ is obtained. Next, equations (32), (33), (34) and (35) are used to calculate $K_1$, $K_2$, $K_3$ and $K_4$ respectively. To facilitate the process, define

$$J_1(t) = e^{(-a-\alpha)t}, \quad t \in [0, h) \tag{41}$$

$$J_2(N, t) = \left\| \sum_{k=-N}^{N} \frac{e^{(S_k - \alpha)t}}{1 - a_d h e^{-S_k h}} \right\|, \quad t \in [h, \infty) \tag{42}$$

$$J_3(t) = \left\| \frac{a_d (1 - e^{-ah}) e^{-\alpha h}}{a} \right\|, \quad t \in [0, h) \tag{43}$$

$$J_4(N, t) = \int_0^h \left\{ \left\| \sum_{k=-N}^{N} \frac{a_d e^{-S_k \tau}}{1 - a_d h e^{-S_k h}} e^{(S_k - \alpha)t} \right\| \right\} d\tau, \quad t \in [h, \infty) \tag{44}$$

and note that $K_1 = \sup_{0 \leq t < h} J_1(t)$, $K_2 = \lim_{N\to\infty} \{\sup_{t \geq h} J_2(N,t)\}$, $K_3 = \sup_{0 \leq t < h} J_3(t)$ and $K_4 = \lim_{N\to\infty} \{\sup_{t \geq h} J_4(N,t)\}$. For this example, $K_1 = J_1(0) = 1$ and $K_3 = J_3(h) = 1.1576$ are obtained. To estimate $K_2$, $J_2(N,t)$ in equation (42) must be evaluated for $t \geq h$ with a sufficiently large $N$. First, note that $J_2(N,t)$ approaches a constant amplitude for large $t$ since $\max\{\mathrm{Re}(\mathbf{S}_k - \alpha)\} \leq 0$ holds for any branch. Thus, it is always sufficient to examine the first several periods (e.g., $0 \leq t \leq 5h$ here) to obtain its maximum value. Second, it has been observed that the convergence of $J_2(N,t)$ w.r.t. $N$ is much faster when $t$ becomes larger. For example, here when $t > 1.5$, $J_2(N,t)$ is very close to the final trajectory for $N \geq 10$. It is favorable to find the location of the peak with a large $N$ (e.g., $N = 10$ is sufficient here) first and then evaluate of $J_2(N,t)$ at this specific location with increased $N$ for a better accuracy, if necessary.

Due to limited space, only the convergence for the worst case (i.e., $t = h = 1$) is provided here in Figure 2. Here, we take $N = 50$ and obtain $K_2 = 0.9$ from Figure 3. Also note that $J_2(N,t)$ with $N = 50$, $t = h$ is very close to $J_1(t)$ with $t = h^-$, showing good consistency of the two estimates.

The convergence of $J_4(N,t)$ at $t = 1$ is shown in Figure 4. $K_4$ is selected by picking the maximum value along the trajectory of $J_4(N,t)$ with a sufficiently large number of branches $N$ (e.g., $N = 50$ here) in Figure 5. It can been seen that $J_4(N,t)$ with $N = 50$, $t = h$ also coincides with $J_3(t)$ with $t = h^-$.

Consequently, one obtains

$$K_1 = 1, \quad K_2 = 0.9, \quad K_3 = 1.1576, \quad K_4 = 1.16$$

and the $K$ factor is then estimated as

$$K = \max(K_1, K_2) + \max(K_3, K_4) = 2.16$$

The decay function parameters, obtained by the methods in Hale and Verduyn-Lunel (1993) and Mondié and Kharitonov (2005), and by using the proposed method, are compared in Table 1. The decay rate, $\alpha$, is significantly improved over the methods in Hale and Verduyn-Lunel (1993) and Mondié and Kharitonov (2005). For estimating the factor, $K$, the proposed approach reaches a more conservative result in this example because we use the triangle inequality to separate $P_1$ and $P_2$ when taking the norm. Also note that the singularity at $t = 0$ (i.e., if $g(0) \neq x_0$) is considered in our approach. Such a singularity cannot be tolerated by the Lyapunov function based approaches, (e.g. Mondié and Kharitonov, 2005), since it renders the Lyapunov functions not continuously differentiable at $t = 0^+$. Although the estimate of $K$ using the Lambert W function approach is larger, the exponential decay function using this new approach gives a

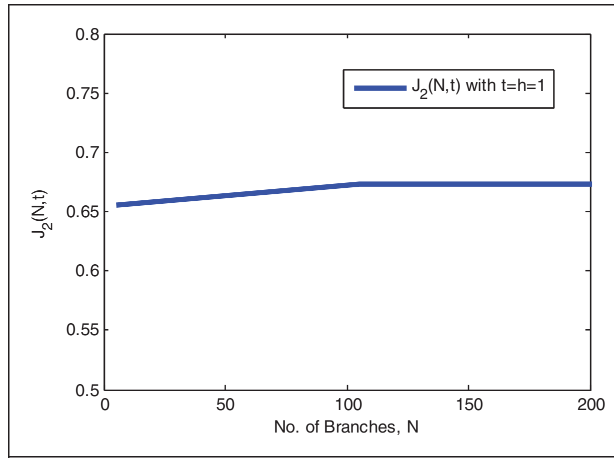


**Figure 2.** Convergence of $J_2(N,t)$ at $t = h = 1$ for Example 1.

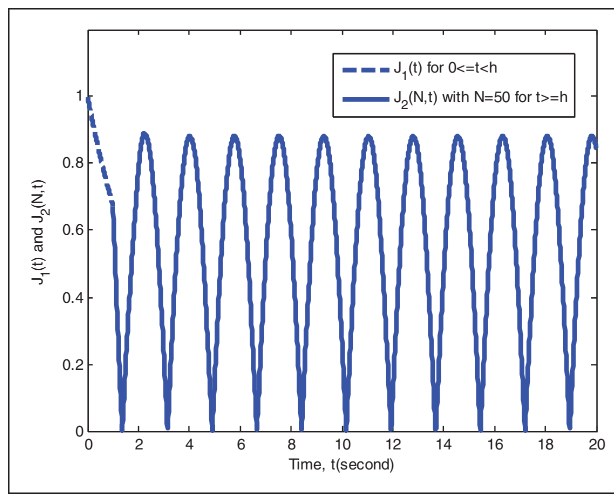


**Figure 3.** The functions $J_1(t)$ and $J_2(N,t)$ with $N = 50$ for Example 1.

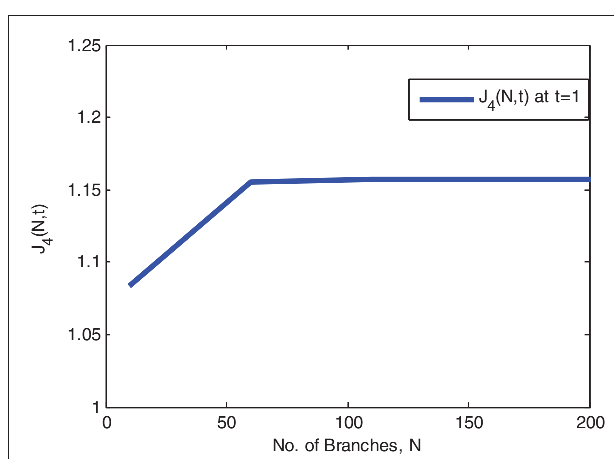


**Figure 4.** Convergence of $J_4(N,t)$ at $t = h = 1$ for Example 1.

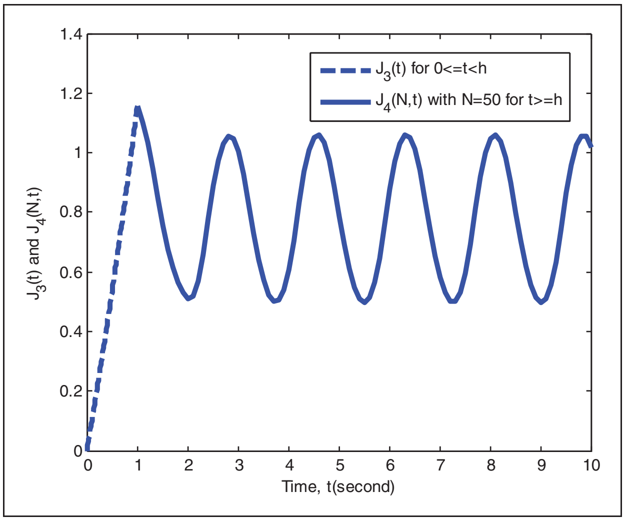


**Figure 5.** The functions $J_3(t)$ and $J_4(N,t)$ with $N=50$ for Example 1.

**Table 1.** Comparison of results for Example 1

| | Factor, $K$ | Decay rate, $\alpha$ |
|---|---|---|
| Matrix Measure Approach (Hale and Verduyn-Lunel, 1993) | 2 | 2 |
| Lyapunov Approach (Mondié and Kharatinov, 2005) | 1.414 | −0.42 |
| Corollary 1 | 2.16 | −0.605 |

better estimate when $t$ becomes larger as the function decays exponentially.

**Remark 5:** The decay rate obtained in our proposed method is the optimal, which cannot be obtained using Lyapunov approaches and matrix measure approaches due to their conservativeness. Although only selected other approaches are compared here, the conservativeness is inherent in Lyapunov approaches and matrix measure approaches.

***Example 2* (*Matrix DDEs*):** Consider the example (Yi et al., 2006):

$$\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) = 0, \quad t > 0$$

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}; \quad \mathbf{A_d} = \begin{bmatrix} -1.66 & 0.697 \\ -0.93 & 0.33 \end{bmatrix}; \; h = 1 \tag{45}$$

First, the Lambert W approach proposed in Yi et al. (2006) is used to analyze the spectrum of this matrix system and locate the rightmost pole. For this example, $m = Nullity(\mathbf{A}_d) = 0$ and the rightmost eigenvalue of the system can be obtained from the principal ($k=0$) branch. Thus,

$$\begin{aligned}\alpha &= \max\{\mathrm{Re}(eig(\mathbf{S}_0))\} \\ &= \max\left\{\mathrm{Re}(eig(\frac{1}{h}\mathbf{W}_0(-\mathbf{A}_d h\mathbf{Q}_0) - \mathbf{A}))\right\} \\ &= -1.0119\end{aligned} \tag{46}$$

After the decay rate is obtained, the right-hand side of equations (18), (19), (20) and (21) are evaluated numerically to calculate $K_1$, $K_2$, $K_3$ and $K_4$ respectively. Similarly, define

$$J_1(t) = \left\| e^{(-\mathbf{A}-\alpha\mathbf{I})t} \right\|, \quad t \in [0, h) \tag{47}$$

$$J_2(N,t) = \left\| \sum_{k=-N}^{N} \left\{ e^{(\mathbf{S}_k - \alpha\mathbf{I})t} \sum_{j=1}^{n} \mathbf{T}_{kj}^I \mathbf{L}_{kj}^I \right\} \right\|, \quad t \in [h, \infty) \tag{48}$$

$$J_3(t) = \int_0^t \left\| e^{(-\mathbf{A}-\alpha\mathbf{I})t + \mathbf{A}\tau} \mathbf{A_d} \right\| d\tau, \quad t \in [0, h) \tag{49}$$

$$J_4(N,t) = \int_0^h \left\| \sum_{k=-N}^{N} \left\{ e^{(\mathbf{S}_k - \alpha\mathbf{I})t} \sum_{j=1}^{n} (\mathbf{T}_{kj}^I \mathbf{L}_{kj}^I \mathbf{A_d} e^{\lambda_{ki}\tau}) \right\} \right\| d\tau, \quad t \in [h, \infty)$$

Note that $K_1 = \sup_{0 \le t < h} J_1(t)$, $K_2 = \lim_{N\to\infty}\{\sup_{t \ge h} J_2(N,t)\}$, $K_3 = \sup_{0 \le t < h} J_3(t)$ and $K_4 = \lim\limits_{N\to\infty}\{\sup_{t \ge h} J_4(N,t)\}$.

As in the scalar case, $J_2(N,t)$ in equation (48) also converges to a certain trajectory as $N$ increases for the matrix case, as shown in Figure 6. Thus, $K_1$ is obtained by evaluating $J_1(t)$ for $0 \le t < h$ and $K_2$ is obtained by taking the maximum value of $J_2(N,t)$ for $t \ge h$ with a sufficiently large number of branches $N$ (i.e., $N=50$ here) as shown in Figure 7.

A similar procedure can be applied to obtain $K_3$ and $K_4$ as illustrated in Figures 8 and 9. As a result, one obtains

$$K_1 = 1.076, \quad K_2 = 1.9, \quad K_3 = 1.89 \quad K_4 = 1.9$$

and the factor, $K$, is determined as

$$K = \max(K_1, K_2) + \max(K_3, K_4) = 3.8$$

Again, the decay function estimated by the methods in Mondié and Kharitonov (2005) and Hale and Verduyn-Lunel (1993) are compared with the proposed method in Table 2.

In Example 2, the decay rate obtained using the proposed method is the optimal value of $\alpha$ and shows significant improvement over the other time domain methods.

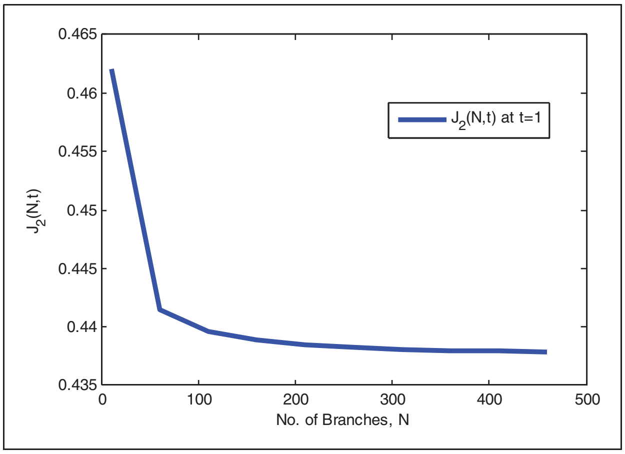


**Figure 6.** Convergence of $J_2(N, t)$ at $t = h = 1$ for Example 2.

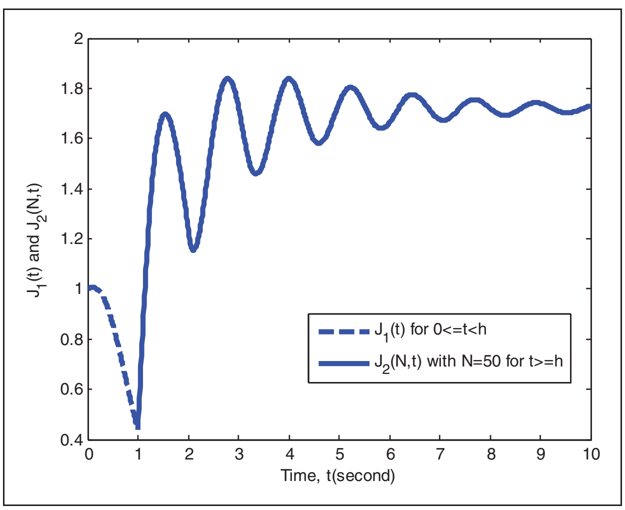


**Figure 7.** The functions $J_1(t)$ and $J_2(N, t)$ with $N = 50$ for Example 2.

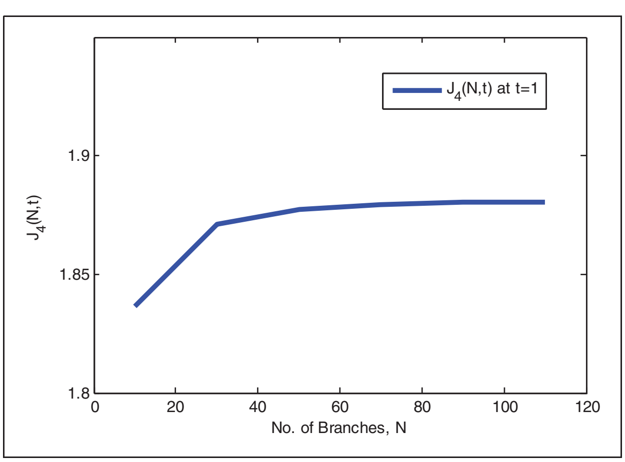


**Figure 8.** The convergence of $J_4(N, t)$ at $t = h = 1$ for Example 2.

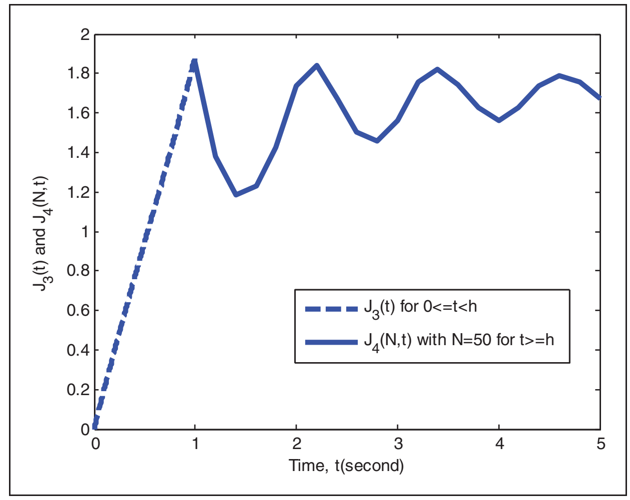


**Figure 9.** The functions $J_3(t)$ and $J_4(N, t)$ with $N = 50$ for Example 2.

**Table 2.** Comparison of results for Example 2

| | Factor, $K$ | Decay rate, $\alpha$ |
|---|---|---|
| Matrix Measure Approach (Hale and Verduyn-Lunel, 1993) | 8.0192 | 3.0525 |
| Lyapunov Approach (Mondié and Kharatinov, 2005) | 9.33 | −0.9071 |
| Theorem 1 | 3.8 | −1.0119 |

The result for the factor $K$ from our approach is also significantly less conservative than the other methods considered. For Lyapunov approaches, the increase of system order leads to a dramatic increase in the dimension of the corresponding optimization problem, which results in more conservativeness. Further, the estimate of $K$ is not typically optimized in Lyapunov function approaches. For the Lambert W function approach, the problem is tackled by evaluating the explicit series, not formulating it as an optimization problem.

## 5. Concluding remarks

A Lambert W function-based approach for the estimation of the decay function for linear time delay systems is presented. From the proposed approach, the optimal estimate of decay rate, $\alpha$, can be determined analytically. The constant factor $K$ is obtained using an infinite Lambert W function series and is typically less conservative compared with other approaches for matrix DDEs. A less conservative estimate of the decay function leads not only to a more accurate description of the exponential behavior of time delay systems, but also to more effective control designs based on those results. The results from our approach are explicitly expressed in terms of Lambert W function series. Convergence properties of the Lambert W function remains an important topic for future research. A general proof for determining the rightmost poles of time delay systems with a finite (few) number of terms from the infinite series is also of future interest. An emerging field of interest for this work is time-periodic delay systems (Insperger and Stepan, 2002, 2010), where the delay and the time-periodic system parameter together makes the decay function estimation challenging.

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## Appendix A. Solution for Matrix DDEs

Consider the linear time invariant matrix DDE in (1). As shown in (Yi et al., 2006), one way to determine the $\mathbf{C}_k^I$ is to Laplace transform the matrix DDE in (1):

$$s\mathbf{I}\mathbf{X}(s)-\mathbf{x}_0+\mathbf{A}\mathbf{X}(s)+\mathbf{A_d}e^{-sh}\mathbf{X}(s)+\mathbf{A_d}\mathbf{G}(s)=0 \quad \text{(A.1)}$$

Note that,

$$\begin{aligned}&\int_0^\infty e^{-st}\mathbf{X}(t-h)dt\\ &=\int_0^h e^{-st}\mathbf{X}(t-h)dt+\int_h^\infty e^{-st}\mathbf{X}(t-h)dt\\ &=\int_0^h e^{-st}\mathbf{g}(t-h)dt+\int_0^\infty e^{-s(t+h)}\mathbf{X}(t)dt\\ &=\mathbf{G}(s)+e^{-sh}\mathbf{X}(s)\end{aligned} \quad \text{(A.2)}$$

Therefore,

$$\mathbf{X}(s)=(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})^{-1}(\mathbf{x}_0-\mathbf{A_d}\mathbf{G}(s)) \quad \text{(A.3)}$$

The solution in (8) can also be Laplace transformed to obtain

$$\mathbf{X}(s)=\sum_{k=-\infty}^{\infty}(s\mathbf{I}-\mathbf{S}_k)^{-1}\mathbf{C}_k^I \quad \text{(A.4)}$$

Equating the expressions in (A.3) and (A.4) yields

$$\begin{aligned}&\frac{adj(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})}{\det(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})}(\mathbf{x}_0-\mathbf{A_d}\mathbf{G}(s))\\ &\quad=\sum_{k=-\infty}^{\infty}(s\mathbf{I}-\mathbf{S}_k)^{-1}\mathbf{C}_k^I\end{aligned} \quad \text{(A.5)}$$

where $adj(\cdot)$ denotes the adjugate matrix.

Determining the $\mathbf{C}_k^I$ in (A.5) is analogous to determining the residue in a typical partial fraction expansion. To calculate $\mathbf{C}_k^I$ for a particular branch $k=q$, i.e., to obtain $\mathbf{C}_q^I$, both sides of (A.5) are multiplied by the expression $\prod_{i=1}^{n}(s-\lambda_{qi})$ and the resulting expression is evaluated at $s=\lambda_{qj}, j=1,2,\ldots,n$, where $\lambda_{qj}=eig(\mathbf{S}_q)$. Thus,

$$\begin{aligned}&\frac{adj(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})}{\det(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})}(\mathbf{x}_0-\mathbf{A_d}\mathbf{G}(s))\prod_{i=1}^{n}(s-\lambda_{qi})\\ &=\sum_{k=-\infty}^{\infty}\left\{\frac{\prod_{i=1}^{n}(s-\lambda_{qi})}{s\mathbf{I}-\mathbf{S}_k}\mathbf{C}_k^I\right\}\end{aligned} \quad \text{(A.6)}$$

Therefore, substitution of $\lambda_{qj}$ for $s$ will make $\frac{\prod_{i=1}^{n}(s-\lambda_{qi})}{s\mathbf{I}-\mathbf{S}_k}=0$ except for $k=q$ since $\lambda_{qj}=eig(\mathbf{S}_q)$.

However, on the left side of the equation, one obtains the indeterminate form

$$\frac{\lim_{s\to\lambda_{qj}} \prod_{i=1}^{n} (s-\lambda_{qi})}{\lim_{s\to\lambda_{qj}} \det(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})} = \frac{0}{0} \tag{A.7}$$

This can be resolved by applying L' Hospital's rule. Assume, any two eigenvalues from different branches are distinct:

$$\lambda_{mj} \neq \lambda_{nj}, \quad \text{where} \quad m \neq n, \quad j = 1, 2, \ldots, n \tag{A.8}$$

On the right-hand side of (A.6),

$$\lim_{s\to\lambda_{qj}} \sum_{k=-\infty}^{\infty} \left\{ \frac{\prod_{i=1}^{n}(s-\lambda_{qi})}{s\mathbf{I}-\mathbf{S}_k} \mathbf{C}_k^I \right\} = \lim_{s\to\lambda_{qj}} \frac{\det(s\mathbf{I}-\mathbf{S}_q)}{s\mathbf{I}-\mathbf{S}_q}\mathbf{C}_k^I$$
$$= \lim_{s\to\lambda_{qj}} adj(s\mathbf{I}-\mathbf{S}_q)\mathbf{C}_k^I \tag{A.9}$$

Thus, one can then rewrite (A.6) as

$$\lim_{s\to\lambda_{kj}} \left\{ \frac{\frac{\partial}{\partial s}\prod_{i=1}^{n}(s-\lambda_{ki})}{\frac{\partial}{\partial s}\det(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})} \times adj(s\mathbf{I}+\mathbf{A}+\mathbf{A_d}e^{-sh})(\mathbf{x}_0-\mathbf{A_d}\mathbf{G}(s)) \right\}$$
$$= adj(\lambda_{qj}\mathbf{I}-\mathbf{S}_q)\mathbf{C}_k^I \tag{A.10}$$

and define $\mathbf{L}_{qj}^I$ and $\mathbf{R}_{qj}^I$ as in (14) and (15). Then (A.10) can be simplified as,

$$\mathbf{L}_{qj}^I(\mathbf{x}_0-\mathbf{A_d}\mathbf{G}(\lambda_{qj})) = \mathbf{R}_{qj}^I\mathbf{C}_q^I, \ \text{for } j=1,\ldots,n \tag{A.11}$$

When $\mathbf{S}_q$ has distinct eigenvalues, or repeated eigenvalues with geometric multiplicity of one, one can show that

$$rank(\lambda_{qj}\mathbf{I}-\mathbf{S}_q) = n-1 \text{ for } i=1,\ldots,n \tag{A.12}$$

When $rank(\lambda_{qj}\mathbf{I}-\mathbf{S}_q) < n-1$, which means some states associated with the repeated eigenvalues are decoupled, one can separate these states in to independent new DDEs and solve the problems individually.

Also, one can show that (Bernstein, 2005):

$$rank(\lambda_{qj}\mathbf{I}-\mathbf{S}_q) = n-1 \Leftrightarrow rank(\text{adj}(\lambda_{qj}\mathbf{I}-\mathbf{S}_q)) = 1 \tag{A.13}$$

which means that each equation in (A.11) has only one linearly independent row among n rows. However, n equations from (A.11) will provide n linearly independent rows, making the solution of $\mathbf{C}_q^I$ unique.

Grouping (i.e., stacking) all n equations in (A.11) into a single matrix equation:

$$\tilde{\mathbf{L}}_q^I = \tilde{\mathbf{R}}_q^I\mathbf{C}_q^I \tag{A.14}$$

where

$$\tilde{\mathbf{L}}_q^I = \begin{bmatrix} \mathbf{L}_{q1}^I \\ \mathbf{L}_{q2}^I \\ \vdots \\ \mathbf{L}_{qn}^I \end{bmatrix}\mathbf{x}_0 + \begin{bmatrix} -\mathbf{L}_{q1}^I\mathbf{A_d}\mathbf{G}(\lambda_{q1})) \\ -\mathbf{L}_{q2}^I\mathbf{A_d}\mathbf{G}(\lambda_{q2})) \\ \vdots \\ -\mathbf{L}_{qn}^I\mathbf{A_d}\mathbf{G}(\lambda_{qn})) \end{bmatrix}, \tilde{\mathbf{R}}_q^I = \begin{bmatrix} \mathbf{R}_{q1}^I \\ \mathbf{R}_{q2}^I \\ \\ \mathbf{R}_{qn}^I \end{bmatrix} \tag{A.15}$$

The $\mathbf{C}_q^I$ can be calculated by using the Moore-Penrose Generalized Inverse:

$$\mathbf{C}_q^I = \tilde{\mathbf{R}}_q^{I+}\tilde{\mathbf{L}}_q^I = \tilde{\mathbf{R}}_q^{I+}\begin{bmatrix} \mathbf{L}_{q1}^I \\ \mathbf{L}_{q2}^I \\ \vdots \\ \mathbf{L}_{qn}^I \end{bmatrix}\mathbf{x}_0 + \tilde{\mathbf{R}}_q^{I+}\begin{bmatrix} -\mathbf{L}_{q1}^I\mathbf{A_d}\mathbf{G}(\lambda_{q1})) \\ -\mathbf{L}_{q2}^I\mathbf{A_d}\mathbf{G}(\lambda_{q2})) \\ \vdots \\ -\mathbf{L}_{qn}^I\mathbf{A_d}\mathbf{G}(\lambda_{qn})) \end{bmatrix}$$
$$= \sum_{j=1}^{n}\mathbf{T}_{qj}^I\mathbf{L}_{qj}^I\mathbf{x}_0 - \sum_{j=1}^{n}(\mathbf{T}_{qj}^I\mathbf{L}_{qj}^I\mathbf{A_d}\mathbf{G}(\lambda_{qj})) \tag{A.16}$$

where $\mathbf{T}_{qj}^I$, $\tilde{\mathbf{R}}_q^{I+}$ and is defined in (15).

Note that $rank(\tilde{\mathbf{R}}_q^I) = n$ (full column rank), thus the $\mathbf{C}_q^I$ obtained from (A.16) using the generalized pseudo inverse will be unique. Therefore, the solution of (1), in (8), can be expressed as:

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty}\left\{e^{\mathbf{S}_k t}\left(\sum_{j=1}^{n}\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\right)\mathbf{x}_0\right\} - \sum_{k=-\infty}^{\infty}\left\{e^{\mathbf{S}_k t}\sum_{j=1}^{n}(\mathbf{T}_{kj}^I\mathbf{L}_{kj}^I\mathbf{A_d}\mathbf{G}(\lambda_{kj}))\right\} \tag{A.17}$$