Controllability and Observability of Systems of Linear Delay Differential Equations via the Matrix Lambert W Function

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Abstract

During recent decades, controllability and observability of linear time delay systems have been studied, including various definitions and corresponding criteria. However, the lack of an analytical solution approach has limited the applicability of the existing theory. Recently, the solution to systems of linear delay differential equations has been derived in the form of an infinite series of modes written in terms of the matrix Lambert W function. The solution form enables one to put the results for point-wise controllability and observability of systems of delay differential equations to practical use. We derive the criteria for point-wise controllability and observability, obtain the analytical expressions for their Gramians in terms of the parameters of the system, and develop a method to approximate them for the first time using the matrix Lambert W function-based solution form.

Index Terms

controllability, observability, delay differential equations, matrix Lambert W function, Gramian.

I. Introduction

Time delay systems (TDS) arise from an inherent time delay in the components of the system or a deliberate introduction of time delay into the system for control purposes. Time delay systems can be represented by delay differential equations (DDEs), which belong to the class of functional differential equations (FDEs), and have been extensively studied (see, e.g., [13], [14], and [23]). The principal difficulty in studying DDEs results from their special transcendental character, since delay terms in characteristic equations always lead to infinite eigenspectra. For this reason, DDEs are often solved using numerical methods, asymptotic solutions, and graphical approaches. Widely used approximation methods are the rational approximations (e.g., Padé approximation), which treat an infinite-dimensional system like a finite-dimensional one [23]. However, the lack of an analytical solution form remains a major obstacle to the analysis and control of time delay systems.

During recent decades, the Lambert W function has been used to develop an approach for the solution of linear time invariant (LTI) systems of DDEs with a single delay (e.g., see [1], [32] and the references therein). The approach using the Lambert W function provides a solution form for DDEs and thus enables one to put the theoretical results on point-wise controllability and observability of TDS and their Gramians, to practical use. In this paper, the properties of controllability and observability for TDS are studied via the matrix Lambert W function approach. Using the analytical solution form in terms of the matrix Lambert W function, we derive point-wise controllability and observability criteria, and their Gramians, for LTI systems of DDEs with a single delay. Also, the results are applied to an example for illustration.

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II. Solution Using the Matrix Lambert W function

Consider a real LTI system of DDEs with a single constant delay, h

$$\begin{array}{rcll} \dot{\mathbf{x}}(t) & = & \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t) & t > 0 \\ \mathbf{x}(t) & = & \mathbf{g}(t) & t \in [-h, 0) \\ \mathbf{x}(t) & = & \mathbf{x}_0 & t = 0 \\ \mathbf{y}(t) & = & \mathbf{C}\mathbf{x}(t) & \end{array} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{A_d}$ are $n \times n$ matrices, and $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{B}$ is an $n \times r$ matrix, $\mathbf{u}(t)$, an $r \times 1$ vector, is a function representing the external excitation, and $\mathbf{g}(t)$ and $\mathbf{x}_0$ are a specified preshape function and an initial point respectively defined in the Banach space of continuous mappings [14]. The coefficient matrix $\mathbf{C}$ is $p \times n$ and $\mathbf{y}(t)$ is a $p \times 1$ measured output vector. There exist two kinds of initial conditions, $\mathbf{x}_0$ which is the value of $\mathbf{x}(t)$ at $t = 0$, and the preshape function, $\mathbf{g}(t)$ in (1) and is equal to $\mathbf{x}(t)$ on the interval $t \in [-h, 0)$. For general retarded FDEs, the existence and uniqueness of the solution are proved based upon the assumption of continuity, i.e., $\mathbf{g}(0) = \mathbf{x}_0$. However, in the specific case of the LTI system of DDEs with a single constant delay as in (1), the existence and uniqueness can be also proved without such an assumption [14], [25]. Consequently, for generality, one can assume $\mathbf{g}(0) \neq \mathbf{x}_0$ in (2). In [32] the solution to (1) was derived using the matrix Lambert W function-based approach and given by

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k (t-\xi)}\mathbf{C}_k^N \mathbf{B}\mathbf{u}(\xi) d\xi \tag{2}$$

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) + \mathbf{A} \tag{3}$$

Conditions for convergence of the infinite series in (2) have been studied in [2], [3], [14], and [20]. For example, for a bounded external excitation, $\mathbf{u}(t)$, if the coefficient matrix, $\mathbf{A_d}$, is nonsingular, the infinite series converges to the solution. The coefficient $\mathbf{C}_k^I$ in (2) is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and the preshape function, $\mathbf{g}(t)$ and the initial point, $\mathbf{x}_0$, while $\mathbf{C}_k^N$ is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and does not depend on $\mathbf{g}(t)$ or $\mathbf{x}_0$. The numerical and analytical methods for computing $\mathbf{C}_k^I$ and $\mathbf{C}_k^N$ were developed respectively in [1], [33]. The following equation based on the Lambert W function is used to solve for the unknown matrix $\mathbf{Q}_k$

$$\mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) e^{\mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) + \mathbf{A}h} = \mathbf{A_d} h \tag{4}$$

The solution to Eq. (4), $\mathbf{Q}_k$, is obtained numerically, for a variety of initial conditions, e.g., using the *fsolve* function in Matlab. The matrix Lambert W function, $\mathbf{W}_k(\mathbf{H}_k)$, is complex valued, with a complex argument, $\mathbf{H}_k$, and has an infinite number of branches for $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, and satisfies the definition, [9]

$$\mathbf{W}_k(\mathbf{H}_k) e^{\mathbf{W}_k(\mathbf{H}_k)} = \mathbf{H}_k \tag{5}$$

The principal ($k = 0$) and other ($k \neq 0$) branches of the Lambert W function can be calculated analytically [9], or using commands already embedded in the various commercial software packages, such as Matlab, Maple, and Mathematica.

Note that, compared with results by other existing methods for the series expansion of solution to DDEs in [2], [3], [20], where eigenvalues are obtained from exhaustive numerical computation, the solution in terms of the Lambert W function has an analytical form expressed in terms of the parameters, $\mathbf{A}$, $\mathbf{A_d}$ and h, of the DDE in (1). Hence, one can determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution [30], [34]. Also, each eigenvalue is distinguished by k, which indicates the branch of the Lambert W function. The solution to DDEs in terms of the Lambert W function, is analogous to that of ODEs in terms of the state transition matrix as summarized in [31].

III. CONTROLLABILITY

Controllability and observability are two fundamental attributes of a dynamical system. Such properties of TDS have been explored since the 1960s and the controllability and observability Gramians for TDS were presented respectively by Weiss (1967, [29]) and Delfour *et al*. (1972, [10]) based upon assumed solution forms of the DDEs. However, application of the results with Gramians to verify controllability and observability of linear time delay systems has been difficult, due to the lack of analytical solutions to DDEs [21]. The analysis of controllability and observability based on the solution form in terms of the matrix Lambert W function are presented in this and the subsequent sections.

Depending on the nature of the problem under consideration, there exist various definitions of controllability and observability for time delay systems [23]. Among them, the concept of *point-wise controllability* of a system of DDEs, as in (1), and the related conditions were introduced in [29].

Definition 1: The system (1) is *point-wise controllable* (or equivalently, defined as *fixed-time completely controllable* in [7] or $\mathbb{R}^n$*-controllable to the origin* in [23], [28]) if, for any given initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, there exists a time t_1, $0 < t_1 < \infty$, and an admissible (i.e., measurable and bounded on a finite time interval) control segment $\mathbf{u}(t)$ for $t \in [0, t_1]$ such that $\mathbf{x}(t_1; \mathbf{g}, \mathbf{x}_0, \mathbf{u}(t)) = \mathbf{0}$ [29].
The solution form to (1) is assumed as [3]

$$\mathbf{x}(t) \equiv \mathbf{x}(t; \mathbf{g}, \mathbf{x}_0, \mathbf{u}) = \mathbf{M}(t; \mathbf{g}, \mathbf{x}_0) + \int_0^t \mathbf{K}(\xi, t)\mathbf{B}\mathbf{u}(\xi)d\xi, \tag{6}$$

where $\mathbf{M}(t; \mathbf{g}, \mathbf{x}_0)$ is the free solution to Eq. (1) and $\mathbf{K}(\xi, t)$ is the kernel function for Eq. (1). Then using the kernel $\mathbf{K}(\xi, t)$ in (6), the condition for point-wise controllability was derived in [29] with the following definition.

Definition 2: A system (1) is *point-wise complete* at time t_1 if, for all $\mathbf{x}_1 \in \mathbb{R}^n$, there exist initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, such that $\mathbf{x}(t_1; \mathbf{g}, \mathbf{x}_0, \mathbf{0}) = \mathbf{x}_1$, where $\mathbf{x}(t; \mathbf{g}, \mathbf{x}_0, \mathbf{0})$ is a solution of (1) starting at time $t = 0$ [8].
The conditions for point-wise completeness are presented in [8], [21], [24]. For example, all 2×2 DDEs or DDEs with a nonsingular coefficient, $\mathbf{A_d}$, are point-wise complete.

Even though the equations to obtain the kernel function in (6) were presented in [3] and [21] the lack of the knowledge of solution to the systems of DDEs has prevented the evaluation and application of the results in [29]. This has prompted many authors to develop algebraic controllability criteria in terms of systems matrices [5], [7], [16], [28]. Other definitions of controllability, which belong in different classifications, such as spectral controllability, have alternatively been provided [22]. For definitions and conditions of various types of controllability and comparisons, refer to [21], [23].

Using the matrix Lambert W function, however, the linear time-invariant system with a single delay can be solved as in (2) and, thus, the kernel function used in the condition for point-wise controllability can be derived. The kernel function $\mathbf{K}(\xi, t_1)$ is obtained, by comparing (6) with (2), as

$$\mathbf{K}(\xi, t_1) \equiv \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \tag{7}$$

Therefore, we can express the controllability Gramian, and state the following main result, for controllability of the systems of DDEs in (1)

Theorem 1: If a system (1) is point-wise complete, there exists a control which results in *point-wise controllability* in finite time of the solution of (1) for any initial conditions $\mathbf{g}$ and $\mathbf{x}_0$, if and only if

$$\operatorname{rank}\left[\mathcal{C}(0, t_1) \equiv \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \mathbf{B}\mathbf{B}^T \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi\right] = n \tag{8}$$

where $\mathcal{C}(0, t_1)$ is the *controllability Gramian* of the system of DDEs and ^{T} indicates transpose.

Proof: Sufficiency In (2), in order to transfer $\mathbf{x}(t)$ to $\mathbf{0}$ at t_1, substitute an input obtained with the inverse of the controllability Gramian in (8)

$$\mathbf{u}(t) = -\mathbf{B}^T \left\{\mathbf{K}(t,t_1)\right\}^T \mathcal{C}^{-1}(0,t_1)\mathbf{M}(t_1;\mathbf{g},\mathbf{x}_0) \tag{9}$$

where $\mathbf{M}$ is the free solution to (1), and comparing (6) with (2) yields

$$\mathbf{M}(t_1;\mathbf{g},\mathbf{x}_0) \equiv \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-0)}\mathbf{C}_k^I \tag{10}$$

then $\mathbf{x}(t_1) = \mathbf{0}$.

Necessity Given any $\mathbf{g}$ and $\mathbf{x}_0$, suppose there exist $t_1 > 0$ and a control $\mathbf{u}_{[0,t_1]}$ such that $\mathbf{x}(t_1) = \mathbf{0}$, but (8) does not hold. The latter implies that there exists a non-zero vector $\mathbf{x}_1 \in \Re^n$ such that $\mathbf{x}_1^T\mathbf{K}(t,t_1)\mathbf{B} = \mathbf{0},\ 0 \le t \le t_1$ due to the following fact. Let $\mathbf{F}$ be an $n \times p$ matrix. Define

$$\mathcal{P}_{(t_1,t_2)} \equiv \int_{t_1}^{t_2} \mathbf{F}(t)\mathbf{F}^T(t)dt \tag{11}$$

Then the rows of $\mathbf{F}$ are linearly independent on $[t_1,t_2]$ if and only if the $n \times n$ constant matrix $\mathcal{P}_{(t_1,t_2)}$ is nonsingular [6]. Then, from (6),

$$\mathbf{x}_1^T\mathbf{x}(t_1) = \mathbf{x}_1^T\mathbf{M}(t_1;\mathbf{g},\mathbf{x}_0) + \int_0^{t_1} \mathbf{x}_1^T\mathbf{K}(\xi,t_1)\mathbf{B}\mathbf{u}(\xi)d\xi \tag{12}$$

and $\mathbf{0} = \mathbf{x}_1^T\mathbf{M}(t_1;\mathbf{g},\mathbf{x}_0)$. By hypothesis, however, $\mathbf{g}$ and $\mathbf{x}_0$ can be chosen such that $\mathbf{M}(t_1;\mathbf{g},\mathbf{x}_0) = \mathbf{x}_1$. Then $\mathbf{x}_1^T\mathbf{x}_1 = \mathbf{0}$ which contradicts the assumption that $\mathbf{x}_1 \neq \mathbf{0}$ ■

In the ODE case, the input computed using the controllability Gramian will use the minimal energy in transferring $(\mathbf{x}_0, 0)$ to $(\mathbf{0}, t_1)$ [6]. Even though the proof is omitted due to space limitations, using (8), one can prove that such a result is also available for DDE's in a similar way to the ODE case in [6]. That is, the input defined in (9) consumes the *minimal* amount of energy, among all the $\mathbf{u}$'s that can transfer $(\mathbf{x}_0, 0)$ to $(\mathbf{0}, t_1)$.

With Theorem 1 and (11), assuming that the system (1) is point-wise complete, we can conclude

Corollary 1: The system in (1) is *point-wise controllable* if and only if all rows of

$$\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{B} \tag{13}$$

are linearly independent on $[0,\infty)$.

The Laplace transform of (13) is [33]

$$\mathfrak{L}\left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{B}\right\} = \left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1}\mathbf{B} \tag{14}$$

Since the Laplace transform is a one-to-one linear operator we then obtained the following corollary.

Corollary 2: The system in (1) is *point-wise controllable* if and only if all rows of

$$\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1}\mathbf{B} \tag{15}$$

are linearly independent, over the field of complex numbers except at the roots of the characteristic equation of Eq. (1).

In systems of ODEs, if the state variable $\mathbf{x}(t)$ is forced to zero at $t = t_1$, it stays at zero on $[t_1,\infty)$. However, because the system of DDEs in (1) has a *delayed term* in its equation, even though all the individual state variables are zero at $t = t_1$ they can become non-zero again after t_1. Therefore, additional definitions of controllability for systems of DDEs [29] for functional, not point-wise, types of controllability are available in [23], [29].

Remark 1: We have presented some examples in [34] that show that if the system of DDEs is point-wise controllable, it is possible to design the linear feedback controllers via rightmost eigenvalue assignment for the system in Eq. (1); otherwise, it is not. This paper presents the theoretical foundation for establish point-wise controllability. To date there is no general theory for DDEs, as there is for ODEs, that controllability is required for eigenvalue assignment by linear feedback [25], [26].

IV. Observability

Consider the system in (1). If one knows the initial conditions, $\mathbf{g}(t)$ and $\mathbf{x}_0$, then one can know all state variables for any time using the solution in (2) to the systems of DDEs. As seen in (2), however, the main obstacle is the fact that the free solution does not have the form of just the product of initial conditions and the transition matrix, in contrast to the ODE case. Then, we introduce a concept of point-wise observability for systems of DDEs, which is different from that of observability for systems of ODEs.

Definition 3: The system of (1) is *point-wise observable*, (or equivalently, *observable* as in [10]) in $[0, t_1]$ if the initial point $\mathbf{x}_0$ can be uniquely determined from the knowledge of $\mathbf{u}(t)$, $\mathbf{g}(t)$, and $\mathbf{y}(t)$ [10]. This concept was introduced by Gabasov (1972) for purely mathematical reasons. However, disturbances which can be approximated by Dirac distributions cause the system response to be approximatable by jumps in the trajectory response [19]. For such cases, the concept of point-wise observability has been used in analyzing singularly perturbed delay system, where the perturbation is very small but can not be ignored (see, e.g., [12], [17]).

Just as in the case of controllability, the lack of analytical solutions of the systems of DDEs has prevented the evaluation and application of the above condition. Unlike controllability, the development of algebraic conditions for the investigation of the observability of TDS has not received much attention [21]. Bhat and Koivo [4] used spectral decomposition to decompose the state space into a finite-dimensional and a complementary part. In [18], Olbrot presented various types of observability of TDS and corresponding algebraic conditions. For a detailed study, refer to [18], [21].

Applying the kernel function in (7) to the observability Gramian defined symbolically in [10], one can present the following condition for observability for systems of DDEs. Here the system of (1) is assumed to be *point-wise complete*.

Theorem 2: The system of (1) is *point-wise observable* if and only if

$$\operatorname{rank}\left[\mathcal{O}(0,t_1) \equiv \operatorname{rank}\int_0^{t_1}\left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T \mathbf{C}^T\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi\right] = n \tag{16}$$

where $\mathcal{O}(0,t_1)$ is the *observability Gramian* of the system of DDEs.
With Theorem 2 and (11), we can conclude

Corollary 3: The system of (1) is *point-wise observable* if and only if all columns of the matrix

$$\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N \tag{17}$$

are linearly independent.
Since the Laplace transform is a one-to-one linear operator we then obtained the following corollary.

Corollary 4: The system of (1) is *point-wise observable* if and only if all columns of the matrix

$$\mathbf{C}\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1} \tag{18}$$

are linearly independent except at the roots of the characteristic equation of Eq. (1).

Proof: The proofs are essentially similar to those of controllability in Section III and are omitted for brevity. ■

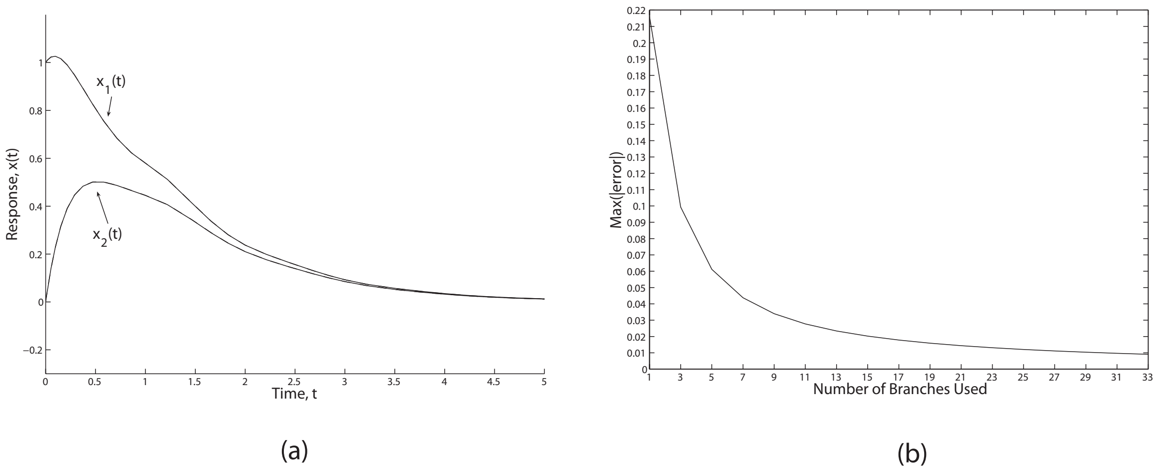

Fig. 1. Response of the example in (19) obtained by the matrix Lambert W function approach with 33 terms (a) and the maximum of errors between the response by the solution form in (2) and the numerically obtained one (b) corresponding to the number of branches used for the response. The errors continue to be reduced as more terms in the series solution are included.

Remark 2: As in the case of point-wise controllability, for point-wise observable systems of DDEs, a linear asymptotic observer can be designed via rightmost eigenvalue assignment as shown by examples in [34].

In the case that $\mathbf{g}(t)$ is unknown, if $\mathbf{g}(t)$, as well as $\mathbf{x}_0$, can be determined uniquely from a knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$, the system of (1) is termed *absolutely observable* (or *strongly observable* in [10]). For a detailed explanation of the definition of *absolute observability* and the corresponding conditions, the reader is referred to [10].

V. Illustrative Example

Consider a system of DDEs (1) with parameters, from [19],

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}, \quad \mathbf{A_d} = \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.330 \end{bmatrix}, \quad h = 1 \tag{19}$$

The response, using the solution form in (2), is depicted in Fig. 1-(a) when $\mathbf{g}(t) = \{\ 1 \quad 0\ \}^T$ and $\mathbf{x}_0 = \{\ 1 \quad 0\ \}^T$. The solution (2) has the form of an infinite series of modes written in terms of the matrix Lambert W function. Even though it is not practically feasible to add all the infinite terms of the series in (2), it can be approximated by a finite number of terms. For example, in Fig. 1-(a), 33 branches ($k = -16, \ldots, 16$) of the Lambert W function are used. As one adds terms, the errors between the response (from Eq. (2)) and a solution obtained numerically (using *dde23* in Matlab) continue to be reduced and validate the convergence of the solution in Eq. (2) (see Fig. 1-(b)).

Using the criterion in [8] (also presented in Section III), the system in (19) is point-wise complete. For $\mathbf{B}$=$[\ 1 \quad 0\ ]^T$, we can compute the controllability Gramian $\mathcal{C}(0, t_1)$ in (8). Then in order for the system (19) to be point-wise controllable, $\mathcal{C}(0, t_1)$ should have full rank. This means that the determinant of the matrix is non-zero. That is,

$$\det |\mathcal{C}(0, t_1)| \neq 0 \tag{20}$$

Computing the determinant of the matrix for an increasing number of branches yields the result in Fig. 2. As more branches are included, the solution in (2) converges (see Fig. 1), so do the kernel function in (7) and the controllability Gramian in (8). Figure 2 shows that the determinant converges to a non-zero value, which implies that the system is point-wise controllable.

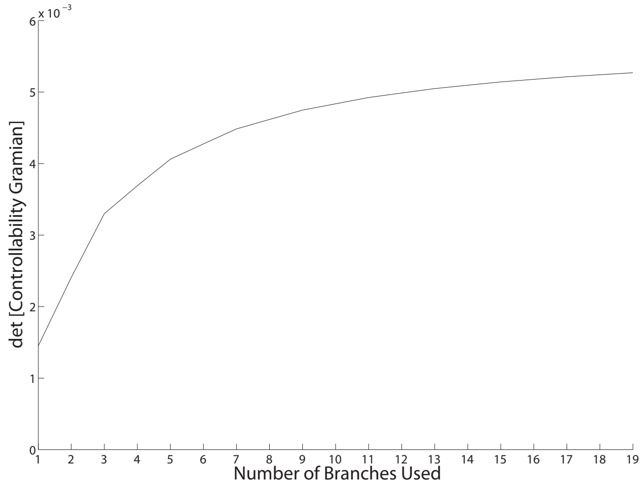

Fig. 2. Determinant of the controllability Gramian versus branches. As more branches are included, the value of determinant converges to a non-zero value.

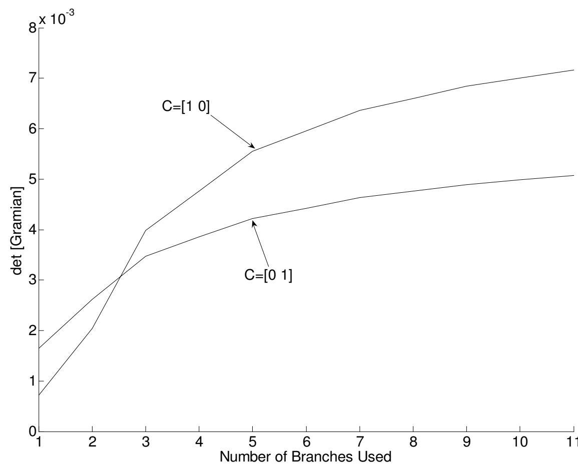

Fig. 3. Determinant of Observability Gramian when $\mathbf{C} = [0 \ \ 1]$ and $\mathbf{C} = [1 \ \ 0]$. As the number of branches used increases, the value of the determinant in case of $\mathbf{C} = [1 \ \ 0]$ tends to converge to a higher value than for $\mathbf{C} = [0 \ \ 1]$.

Even though a system satisfies the algebraic criteria already provided in previous work, such as [18], [21], in cases where the determinant of the observability Gramian in (16) is smaller than a specific value, then it is not practical to design an observer as the gains in the observer can become unrealistically high. Comparing the determinant of the observability Gramian corresponding to the system in (19), we can obtain a practical assessment. For example, the determinants of the observability Gramian for (19) when $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ are compared in Fig. 3 with $t_1 = 4$. As the number of branches used increases, the value of the determinant in case of $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ tends to converge to a higher value than the case of $\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

From the results in Figures 2 and 3, although a formal study of truncation errors is needed, we observe the convergence of the Gramians as the number of terms in the series is increased. When the convergence conditions, which are explained in Section 2, and also in the cited references, are satisfied, then the series expansion of the solution in Eq. (2) converges. The controllability Gramian in Eq. (8) and the observability

TABLE I

COMPARISON OF THE CRITERIA FOR CONTROLLABILITY AND OBSERVABILITY FOR THE SYSTEMS OF ODES AND DDES

ODEs	DDEs
Controllability	Point-Wise Controllability
$\mathcal{C}_{(0,t_1)} \equiv \int_0^{t_1} e^{\mathbf{A}(t_1-\xi)}\mathbf{B}\mathbf{B}^T \left\{e^{\mathbf{A}(t_1-\xi)}\right\}^T d\xi$	$\mathcal{C}_{(0,t_1)} \equiv \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \mathbf{B}\mathbf{B}^T \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi$
$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$	$\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\mathbf{B}$
$e^{\mathbf{A}(t-0)}\mathbf{B}$	$\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{B}$
Observability	Point-Wise Observability
$\mathcal{O}_{(0,t_1)} \equiv \int_0^{t_1} \left\{e^{\mathbf{A}(\xi-0)}\right\}^T \mathbf{C}^T\mathbf{C}e^{\mathbf{A}(\xi-0)} d\xi$	$\mathcal{O}_{(0,t_1)} \equiv \int_0^{t_1} \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T \mathbf{C}^T\mathbf{C} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi$
$\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}$	$\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}$
$\mathbf{C}e^{\mathbf{A}(t-0)}$	$\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N$

Gramian in Eq. (16) are the integrals of products of the kernel (Eq. (7)) and constant matrices (**B** and **C**) over a finite interval. Thus, the convergence of the Gramians is also assured under the same conditions.

The presented results agree with those obtained using existing algebraic methods. However, using the method of Gramians developed in this paper, we can acquire additional information. The controllability and observability Gramians indicate how controllable and observable the corresponding states are [15], while the algebraic conditions for controllability/observability tell only whether a system is controllable/observable or not. Therefore, with the conditions using Gramian concepts, we can determine how the change in some specific parameters of the system or the delay time, h, affect the controllability and observability of the system via the changes in the Gramians.

Using the Gramians presented in the previous sections, the concept of the balanced realization, in which the controllability Gramian and observability Gramian of a system are equal and diagonal, can be extended to systems of DDEs. For systems of ODEs, the balanced realization has been studied because of its desirable properties such as good error bounds, computational simplicity, stability, and its close connection to robust multi-variable control [27]. However, for systems of DDEs, results on balanced realizations have been lacking. Here, the developed Gramians are applied to the problem of the balanced realization for systems of DDEs for the first time. Let $\mathbf{T}$ be a nonsingular state transformation, then $\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$. The corresponding effect on the Gramians is

$$\hat{\mathcal{C}}(0,t_1) = \mathbf{T}\mathcal{C}(0,t_1)\mathbf{T}^T, \quad \hat{\mathcal{O}}(0,t_1) = \mathbf{T}^{-T}\mathcal{O}(0,t_1)\mathbf{T}^{-1} \tag{21}$$

Thus, $\hat{\mathcal{C}}(0,t_1)$ and $\hat{\mathcal{O}}(0,t_1)$ can be made equal and diagonal with the aid of a suitably chosen matrix $\mathbf{T}$. In the numerical example in (19), when $\mathbf{B} = [\ 1 \quad 0\ ]^T$ and $\mathbf{C} = [\ 1 \quad 0\ ]$, the $\mathcal{C}(0,t_1)$ and $\mathcal{O}(0,t_1 = 4)$ are respectively

$$\mathcal{C}(0,t_1) = \begin{bmatrix} 0.2992 & 0.1079 \\ 0.1079 & 0.0554 \end{bmatrix}, \quad \mathcal{O}(0,t_1) = \begin{bmatrix} 0.2992 & -0.1484 \\ -0.1484 & 0.0975 \end{bmatrix} \tag{22}$$

when computed using 11 branches of the matrix Lambert W function. In this case, using the result in (21), the transformation

$$\mathbf{T} \;=\; \begin{bmatrix} -0.3929 & 1.1910 \\ 1.0880 & -0.5054 \end{bmatrix} \tag{23}$$

makes the the Gramians *balanced*, i.e., equal to each other and diagonalized,

$$\hat{\mathcal{C}}(0,t_1) \;=\; \hat{\mathcal{O}}(0,t_1) \;=\begin{bmatrix} 0.0238 & 0.0000 \\ 0.0000 & 0.2497 \end{bmatrix} \tag{24}$$

Future research is needed to establish conditions for the existence of the transformation **T** to achieve a balanced realization for DDEs, and to study its convergence as the number of branches used in the Lambert W function solution increases.

VI. Conclusions and Future work

The controllability and observability of linear systems of DDEs is studied using the solution form based on the matrix Lambert W function. The necessary and sufficient conditions for point-wise controllability and observability are derived based on the solution of DDEs. The analytical expressions of Gramians are obtained and approximated for application to real systems with time delay. Using Gramian concepts, one can figure out how the change in some specific parameters of the system or the delay time, h, affect the controllability and observability of the system via the changes in the Gramians. Also, for the first time, for systems of DDEs, the balanced realization is investigated in the time domain as in the case of ODEs. An example is presented to demonstrate the theoretical results.

Based upon the results presented, extension of well-established control design concepts for systems of ODEs to systems of DDEs appears feasible. For example, the design of feedback controllers and observers for DDEs can be developed in a manner analogous to ODEs via eigenvalue assignment [34], [35].

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