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SECOND ORDER GENERALIZED LINEAR SYSTEMS ARISING IN ANALYSIS OF FLEXIBLE BEAMS

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Abstract

The paper deals with the analysis of a one-link flexible beam using power series solution with respect to the spatial variable. The Euler-Bernoulli beam with payload mass and moment of inertia at the right end is clamped at the left end or driven by a torque applied by an actuator located at the base of the beam. The former case results in a second order state-space equation while the later in a second order singular equation. For both models an extended Leverrier-Faddeev algorithm is derived to invert the matrix polynomial $\mathbf{R}(s) = \mathbf{E}s^2 + \mathbf{A}$, where $s \in \mathbb{C}$, $\mathbf{E}$ and $\mathbf{A}$ are constant matrices and $\mathbf{E}$ max be singular. Extension of the algorithm for beams with energy dissipation yielding a matrix polynomial $\mathbf{R}(s) = \mathbf{E}s^2 + \mathbf{A}_1 s + \mathbf{A}_2$ is also given.

1. Introduction

Growing needs for advanced and precise robot manipulators in space industry and mechanically flexible constructions result in new and complicated problems of modelling, identification and control of flexible structures, i.e. flexible beams, robot arms, etc. Dealing with flexible systems one is faced with inherent infinite dimensionality of the systems, light damping, nonlinearities, influence of variable environment etc. [4,5,12]. One of the most important factors is to establish a suitable mathematical model of the system to make analysis as realistic as possible. Therefore inclusion of the dynamics of electrical devices (i.e. DC servomotors, tachogenerators, etc.) to a mechanical model may be required [12,13].

In this paper a power series method is used for the Euler-Bernoulli beam to derive a state-space or singular model of the form

$$\mathbf{E}\,\ddot{\mathbf{x}}(t) + \mathbf{A}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t),\ \mathbf{y}(t) = \mathbf{C}_1\mathbf{x}(t) + \mathbf{C}_2\dot{\mathbf{x}}(t) \quad (1)$$

where $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}_i$ are constant matrices, $\mathbf{E}$ may be singular, $\mathbf{x}$ is a generalized state vector, $\mathbf{u}$ and $\mathbf{y}$ are input and output vectors, respectively.

Derivation of the model (1) for the Euler-Bernoulli beam is based on suitable application of a power series to a partial differential equation of the system including the boundary conditions at both ends. The stucture of the coefficient matrices in (1) does generally not depend on the number of terms of a power series taken into consideration. For a flexible beam which is clamped at the left end it is possibloe to find explicit formulas for the entries of $\mathbf{E}^{-1}$, however, the most interesting case, the so-called pinned or unconstrained beam (with a motor at the base of the beam), gives the model (1) with singular $\mathbf{E}$. This means that there are algebraic relations between elements of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ or derivatives of $\mathbf{u}(t)$. Although in general, an analysis of such a system via elimination of the algebraic relations is possible, we concentrate in this paper rather on direct analysis of the matrix pencil

$$\mathbf{R}(s) = \mathbf{E}s^2 + \mathbf{A}$$

and propose an extended Leverrier-Faddeev algorithm to invert $\mathbf{R}(s)$.

The paper is organized as follows. In Section 2 the Euler-Bernoulli model of an undamped flexible beam with boundary conditions is given. Other models including those with damping mechanisme are also shortly mentioned. In Section 3 lumped parameter models in the form of (1) are derived for the clamped and pinned beams. An extension of the Leverrier-Faddeev algorithm for inversion of the matrix polynomial

$$\mathbf{E}\,s^2 + \mathbf{A}$$

is proposed in Section 4. The algorithm presented here can be considered as a hybrid between algorithms for a first order matrix polynomial

$$\mathbf{E} + \mathbf{A}$$

in [10] and an algorithm for a second order matrix polynomial

$$\mathbf{I}\,s^2 + \mathbf{A}_1 s + \mathbf{A}$$

in [1]. In Section 5 we discuss some extensions of the method of power series for beams with energy dissipation, which yield the matrix polynomials of the form

$$\mathbf{E}s^2 + \mathbf{A}_1 + \mathbf{A}_2,\ \det\mathbf{E} = 0.$$

The damping due to Voigt-Kelvin [14] is discussed in details but other damping mechanisms can be analysed in the same way.

2. Dynamic models of a one-link flexible beam

The one-link Euler-Bernoulli flexible beam of a unit lenght is described by the following partial differential equation for the transversal deflection $w(x,t)$, $0 \leq x \leq 1$, $t \geq 0$ [5]

$$E\,I\,\frac{\partial^4 w(x,t)}{\partial x^4} + m\,\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad (2)$$

where E is the Young's modulus of elasticity, I is the moment of inertia of the beam's cross-sectorial area and m is the mass density per unit length. In the above and it what follows we assume that all parameters of the beam, hub and payload are constant in time and space.

If the beam is clamped at x=0, then

$$w(0,t) = 0 \quad (3a)$$

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0, \quad (3b)$$

while, if the beam is driven by torque T(t) applied at the base of the beam then we have (3a) and

$$I_h\left.\frac{\partial^3 w(x,t)}{\partial t^2 \partial x}\right|_{x=0} - E\,I\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0} = T(t) \quad (3c)$$

instead of (3b) [5], where I_h is the total beam's moment of inertia concentrated at the base.

The boundary conditions at the second (right) end of the beam are due to a point force f(t) and a point bending moment b(t) applied at x=1, where a payload mass m_p and a moment of inertia I_p (about the axis vertical to the surface of movements of the beam passing through the centre of mass m_p) exists

$$m_p\,\frac{\partial^2 w(1,t)}{\partial t^2} - E\,I\left.\frac{\partial^3 w(x,t)}{\partial x^3}\right|_{x=1} = f(t) \quad (4a)$$

$$I_p \left.\frac{\partial^3 w(x,t)}{\partial t^2 \partial x}\right|_{x=1} + E\ I \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=1} = b(t)\ . \tag{4b}$$

The dynamic models (2), (3a,b), (4a,b) for a clamped beam and (2), (3a,c), (4a,b) for a pinned beam do not contain any damping of the vibration, which means that no work is done on the beam by an internal attenuating mechanism. In order to include the damping terms we have to modify (2) by adding to its left hand side the term

$$2\ E\ I\ \gamma \frac{\partial w(x,t)}{\partial t}$$

in the case of uniform damping, the term

$$2\ E\ I\ \gamma \frac{\partial^3 w(x,t)}{\partial x^2 \partial t}$$

in the case of structural damping, or the term

$$2\ E\ I\ \gamma \frac{\partial^5 w(x,t)}{\partial x^4 \partial t}$$

in the case of a more realistic Voigt-Kelvin damping ($\gamma>0$). Additionally, the boundary conditions (3c), (4a,b) have to be suitably modified to include the damping terms (see [14] for details). The Voigt-Kelvin damping is analysed using the power series approach in Section 5.

3. Analysis of flexible beams using a power series method

Let us assume that the solution $w(x,t)$ of eq. (2) with suitable boundary conditions is of the following form

$$w(x,t) = \sum_{i=0}^{\infty} v_i(t) x^i / i!\ . \tag{5}$$

where $v_i(t)$, $i=0,1,\ldots$ are assumed to be twice differentiable funcitons of time. In order to derive a set of equations equivalent to the partial differential model of the beam, note that form (5) we have

$$v_i(t) = \left.\frac{\partial^i w(x,t)}{\partial x^i}\right|_{x=0}\ ,\ i=0,1,\ldots \tag{6}$$

which for boundary conditions (3a,b) gives

$$v_0(t) = v_1(t) = 0 \tag{7a}$$

and for (3a,c)

$$v_0(t) = 0\ ,$$

$$I_h \frac{d^2 v_i(t)}{dt^2} - E\ I\ v_2(t) = T(t). \tag{7b}$$

The remaining equations are found from (2) by calculating the higher order spatial derivatives at x=0 and taking into account (7a) or (7b). This yields

$$E\ I\ v_{i+1}(t) = -m \frac{d^2 v_i(t)}{dt^2}\ , \tag{8}$$

where $i \in N_1 \equiv \{2,3,6,7,\ldots\}$ for (7a) and $i \in N_2 \{1,2,3,5,6,7,\ldots\}$ for (7b). The absence of numbers $\{4,8,12,\ldots\}$ for both cases is due to (8) and the fact that $v_0(t)=0$. Additionally, the terms with numbers $\{1,5,9,\ldots\}$ vanish in the case of a clamped beam since $v_1(t)=0$. Equations (4a,b) are now equivalent to

$$m_p \sum_{i \in N} \frac{d^2 v_i(t)}{dt^2} \frac{1}{i!} - E\ I \sum_{i \in N''} v_i(t) \frac{1}{(i-3)!} = f(t), \tag{9a}$$

$$I_p \sum_{i \in N} \frac{d^2 v_i(t)}{dt^2} \frac{1}{(i-1)!} + E\ I \sum_{i \in N'} v_i(t) \frac{1}{(i-2)!} = b(t), \tag{9b}$$

where $N=N_1$, $N'=N_1$, $N''=N_1-\{2\}$ for clamped beams and $N=N_2$, $N'=N_2-[1]$, $N''=N_2-\{1,2\}$ for pinned beams.

Equations (7a), (8) and (9a,b) for clamped beams and (7b), (8) and (9a,b) for pinned beams can in both cases be written in the generalized form

$$\mathbf{E}\ \ddot{\mathbf{v}}(t) + \mathbf{A}\ \mathbf{v}(t) = \mathbf{B}\ \mathbf{u}(t), \tag{10}$$

where

$$\left.\begin{aligned} \mathbf{v}(t) &= [v_2(t),\ v_3(t),\ v_6(t),\ v_8(t),\ \ldots]', \\ \mathbf{u}(t) &= [f(t),\ b(t)]' \end{aligned}\right\} \tag{11a}$$

$$\left.\begin{aligned} \mathbf{v}(t) &= [v_1(t), v_2(t), v_3(t), v_5(t), v_6(t), v_7(t), \ldots]' \\ \mathbf{u}(t) &= [T(t),\ f(t),\ b(t)]', \end{aligned}\right\} \tag{11b}$$

for pinned beam. Matrices **E**, **A** and **B** are given in Appendix.

These structures are given assuming that the number of terms in (5), e.g. components of the generalized vector $\mathbf{v}(t)$, equals 2,4,6,... (k=3,7,11,... see (11a)) for a clamped beam and 3,6,9,... (k=3,7,11,...,see (11b)) for a pinned beam.

Note that the present approach results in a lumped parameter model (10) with det $\mathbf{E} \neq 0$ for a clamped beam and det $\mathbf{E} = 0$ for a pinned beam. It is possible to compute $\mathbf{E}^{-1}$ for a clamped beam (see Appendix for explicit formulas for the entries of $\mathbf{E}^{-1}$, $\mathbf{E}^{-1}\mathbf{A}$ and $\mathbf{E}^{-1}\mathbf{B}$) and to use the second order state-space model

$$\ddot{\mathbf{v}}(t) + \mathbf{E}^{-1}\mathbf{A}\ \mathbf{v}(t) = \mathbf{E}^{-1}\mathbf{B}\ \mathbf{u}(t)$$

for further analysis, identification or control of a clamped beam. However, for a pinned beam the situation is much more complicated. Although the rank of **E** is in this case only 1 less the dimension of $\mathbf{v}(t)$, the model arising after elimination of one component of $\mathbf{v}(t)$, i.e. $v_1(t)$ from (7b) is still not minimal, since (7b) and (8) for i=1 give an algebraic relation

$$I_h \frac{EI}{m} v_5(t) + E\ I\ v_2(t) = -\ T(t), \tag{12}$$

and from (8) for i=5 we get

$$m \frac{d^2 v_5(t)}{dt^2} = -\ E\ I\ v_9(t). \tag{13}$$

Therefore, if the truncated series (5) contains at least the term $v_9(t)$, e.g. k=11, 15, 19,... in Appendix, then $v_5(t)$ can also be excluded from eq. (10). It can be proved that in general the following relation holds

$$I_h \left[-\frac{EI}{m}\right]^{[i/4]+1} v_{i+3}(t) - EI \left[-\frac{EI}{m}\right]^{[i/4]} v_i(t) = T^{(2[i/4])}(t), \tag{14}$$

where $[i/4]$ is the greatest integer smaller than i/4 and an upper index of T(t) denotes the order of differentiation.

It is not very convenient to use (14) for reducing the dimension of equation (10) since (14) introduces derivatives of T(t) after reduction. Instead we prefer to use the original equation (10) even if, as in the case of clamped beam, **E** is nonsingular. One of the important issues before any control strategy is applied may be the computation of the transfer function of the flexible system. Usually the transfer functions from the torque to tip position w(1,t), tip velocity dw(1,t)/dt or rigid angle of rotation $\partial w(x,t)/\partial x$ for x=0 with f(t)=b(t)=0 are used [4]. The main problem in computation of the transfer function in the present approach is inversion of the matrix pencil $\mathbf{R}(s) \equiv \mathbf{E}\ s^2 + \mathbf{A}$. For this purpose we present in the next section an extended Leverrier-Faddeev algorithm which can be considered as a hybrid between algorithms for the first order matrix pencil

$R(s)=Es+A$ [8,10] and second order pencil $R(s)=Is^2+A_1s+A_2$ [1]. The present algorithm is general and can be used for any system such as (10), for example, for second order singular systems arising in electric power systems analysis [2,3].

4. Leverrier-Faddeev algorithm for the second order generalized systems

Consider the nxn regular matrix polynomial $R(s)=Es^2+A$ with any E (singular or nonsingular). The simultaneous computation of the adjoint matrix

$$(Es^2 + A)_{ad}$$

and the determinant $|Es^2 + A|$ can be done using the following formula

$$R^{-1}(s) = -\frac{1}{a_n(s)}\left[R^{n-1}(s)+a_1(s)R^{n-2}(s)+...+a_{n-z}(s)R(s)+a_{n-1}(s)I\right]$$

$$= -\frac{1}{\sum\limits_{i=0(2)}^{2n} a_{2n,2n-i}s^{2n-i}}\left[\sum_{i=0(2)}^{2n-2} s^{2n-2-i}\,R_{2n-2-i}\right], \quad (15)$$

where

$$R^p(s) = s^{2p}R_{2p,2p}+s^{2p-2}R_{2p,2p-2}+...+s^2R_{2p,2}+R_{2p,0}, \quad (16)$$

$p=1,2,...,n-1$; $R^0(s)\equiv R_{0,0}=I$ and

$$a_j(s) = s^{2j}a_{2j,2j}+s^{2j-2}a_{2j,2j-2}+...+s^2a_{2j,2}+a_{2j,0} \quad (17)$$

with $j=1,2,...,n$; $a_0(s)=a_{0,0}=1$.

In (15) only even values of i must be taken into account. Moreover, the constant matrices $R_{2p,2p-l}$, $l=0,2,...,22p$, and coefficients $a_{2j,2j-r}$, $r=0,2,...,2j$ are determined by

$$R_{2p,2p-l} = R_{2p-2,2p-l}A + R_{2p-2,2p-l-2}E, \quad (18a)$$

and

$$a_{2j,2j-r} = -\frac{1}{j}\,\text{trace}\left[R_{2j,2j-r} + \sum_{\substack{s+h=2j\\ t+g=2j-r\\ t\leq s,\ g\leq h}} a_{s,t}R_{h,g}\right] \quad (18b)$$

Finally, the matrices R_{2n-2-i}, $i=0,2,...,2n-2$, are obtained with

$$R_{2n-2-i} = R_{2n-2,2n-2-i} + \sum_{\substack{s+h=2n-2\\ t+g=2n-2-i\\ t\leq s,\ g\leq h}} a_{s,t}R_{h,g} \quad . \quad (19)$$

Equations (15)-(19) form an extended Leverrier-Faddeev algorithm [6] for inversion of the matrix polynomial

$$R(s) = Es^2 + A.$$

Note that this algorithm is recursive with matrix multiplications and additions only.

The derivation of formulas (15)-1(19) can be accomplished using the Cayley-Hamilton theorem and a basic Leverrier-Faddeev algorithm for the 2nx2n matrix pencil of the first order sytem, as follows

$$\det\left(\begin{bmatrix}I & 0\\ 0 & E\end{bmatrix}s + \begin{bmatrix}0 & -I\\ A & 0\end{bmatrix}\right) = \det[Es^2 + A] = \det R(s).$$

It is interesting to notice that the recursive equations (16)-(19) can be visualized in a diagram shown in Fig. 1. Matrix $s^{2k-l}R_{2k,2k-l}$ is at node (2k,2k-l), at the intersection of the kth horizontal and lth vertical lines, $k=0,2,...,2n$; $i=0,2,...,2k$. Matrix $R_{2k,2k-l}$ follows directly (see (18a)) from the two matrices at neighbouring upper nodes as shown in Fig. 2. Similary, from (18b) we see that the coefficients $a_{2j,2j-r}$ is obtained via trace operation of a sum of matrices, multiplied by appropriate coefficients, at the nodes directly above the node (2j,2j-r). For example, if j=3 and r=2, we have

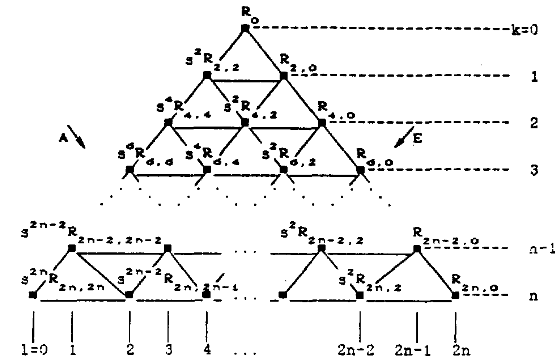

Fig. 1. Diagram of the Leverrier-Faddeev algorithm

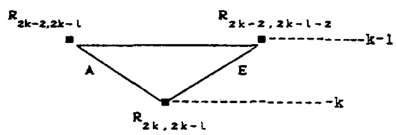

Fig. 2. Computation of the matrix $R_{2k,2k-l}$ for beams without damping

$$a_{6,4} = -\frac{1}{3}\,\text{trace}[R_{6,4}+a_{2,0}R_{4,4}+a_{2,2}R_{4,2}+a_{4,2}R_{2,2}+a_{4,4}R_{2,0}]$$

and (21a)

$$R_{2,0} = A,\ R_{2,2} = E,\ a_{2,0} = -\text{trace}\ R_{2,0},\ a_{2,2}=-\text{trace}\ R_{2,2}\ ,\ ... \quad (21b)$$

Therefore all operations for this algorithm are easy to be coded for computer calculations.

Example

Consider the clamped beam (eqs. (2), (3a,b), (4a,b)) with E and A given by (A.1) for k=7, $EI=12.00374(Nm^2)$, $m=0.2458664(kg/m)$, $m_p=0.02(kg)$, $I_p=0.000008(kgm^2)$. Applying the above given algorithm we use (18a) for computation of $R_{2p,2p-l}$ and (18b) for computation of $a_{2p,2p-l}$; $p=1,2,3,4$; $l=0,2,...,2p$ with $R_{2,0}=A$, $R_{2,2}=E$ (note that n=4 in (15)). Finally, using (19) we obtain from (15) the following result, where $a_4(s)$ and all R_i, i=0,2,4,6 are divided by $a_{8,8}=\det E$ = 2.755732E-07 (see (17)).

$$R^{-1}(s) = -\frac{1}{a_4(s)}[s^6R_6 + s^4R_4 + s^2R_2 + R_0]$$

$$a_4(s) = s^8+1.829283E+0.7s^6+1.560465E+12s^4+1.762437E+16s^2$$
$$+7.792249E+18$$

$$R_6 = -\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -1800 & -480 & 5040 & -720\\ 10080 & 2520 & -30240 & 5040\end{bmatrix}$$

$$\mathbf{R}_4 = -\begin{bmatrix} 1.838071E+07 & 2.343466E+04 & -2.460639E+05 & 3.515199E+04 \\ -4.921278E+05 & 1.816980E+07 & 1.476384E+06 & -2.460639E+05 \\ -2.168271E+10 & -6.524366E+09 & 4.528555E+10 & 1.202717E+08 \\ 1.062359E+11 & 3.042751E+10 & -2.266933E+11 & -4.515912E+08 \end{bmatrix}$$

$$\mathbf{R}_2 = -\begin{bmatrix} 2.619063E+12 & 3.185340E+11 & -2.210940E+12 & -5.871929E+09 \\ -5.186670E+12 & 7.492692E+10 & 1.106767E+13 & 2.204768E+10 \\ -4.743248E+13 & -1.089066E+14 & -2.658312E+14 & -1.064051E+11 \\ 5.446362E+14 & 4.971402E+14 & 2.658315E+14 & 3.515178E+07 \end{bmatrix}$$

$$\mathbf{R}_0 = -\begin{bmatrix} 1.994013E+16 & 5.317062E+15 & 1.297847E+16 & 5.194930E+12 \\ -2.659034E+16 & -6.647106E+15 & -1.297848E+16 & -1.716187E+09 \\ 1.596044E+17 & -1.050361E+09 & 1.156370E+10 & -1.784777E+09 \\ 3.110570E+10 & 1.596044E+17 & -8.689543E+10 & 1.245001E+10 \end{bmatrix}$$

One can check that the same result follows directly from the inversion of $\mathbf{E}s^2+\mathbf{A}$.

5. Beams with energy dissipation

Only a slightly modified Leverrier-Faddev algorithm of Section 4 may be used for the Euler-Bernoulli beam with damping. If, for example, we modify eq. (2) to include the Voigt-Kelvin damping mechanism as follows [12,14]

$$E\,I\,\frac{\partial^4 w(x,t)}{\partial x^4} + 2\gamma\,E\,I\,\frac{\partial^5 w(x,t)}{\partial x^4 \partial t} + m\,\frac{\partial^2 w(x,t)}{\partial t^2} = 0 \qquad (23)$$

and add additioal terms

$$-2\gamma\,E\,I\,\partial^4 w(x,t)/\partial^3 x \partial t\Big|_{x=1},$$

$$2\gamma\,E\,I\,\partial^3 w(x,t)/\partial x^2 \partial t\Big|_{x=1} \quad \text{and}$$

$$-2\gamma\,E\,I\,\partial^3 w(x,t)/\partial x^2 \partial t\Big|_{x=0}$$

to the left hand side of eqs. (4a,b) and (3c) respectively, then the generalized second order model (10) will contain a damping term $2\gamma\mathbf{A}\dot{\mathbf{v}}(t)$, as follows

$$\mathbf{E}\,\ddot{\mathbf{v}}(t) + 2\gamma\,\mathbf{A}\,\dot{\mathbf{v}}(t) + \mathbf{A}\,\mathbf{v}(t) = \mathbf{B}\,\mathbf{u}(t). \qquad (24)$$

There is no need to change anything in the formulas given in the Appendix and only a slight modification is required to include the damping in the Leverrier-Faddeev algorithm. We have to modify eqs. (15) - (17) to include odd terms and eq. (18a) now takes the following form for $l=0,1,2,\ldots,2p-1,1p$

$$\mathbf{R}_{2p,2p-l} = \mathbf{R}_{2p-2,2p-l}\mathbf{A} + 2\gamma\mathbf{R}_{2p-2,2p-l-1}\mathbf{A} + \mathbf{R}_{2p-2,2p-l-2}\mathbf{E}. \qquad (25)$$

Simultaneously, the sum of matrices in the square brackets in (18b) contains also the odd terms for computation of $a_{2j,2j-r}$, $r=0,1,2,\ldots,2j-1,2j$. These modifications can be easily introduced in the diagram in Fig. 1. All we need is to make odd numbered nodes on horizontal lines as shown in Fig. 3. This is obvious when one compares (18a) and Fig. 2 with (25) and Fig. 3.

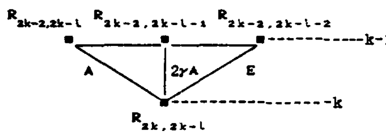

Fig. 3. Computation of the matrix $\mathbf{R}_{2k,2k-l}$ for beams with damping

Conclusions

We discussed some issues of modelling the clamped and pinned one-link flexible beam using power series approach. The generalized Leverrier-Faddeev algorithm presented here can be used for both beams no matter if **E** is singular or not. There is only a small modification which is necessary to be included in the algorithm when the damping of the vibration is taken into account. Perhaps the most interesting fact is that the present approach seems to be applicable for modelling of beams with nonuniform parameters if there exist spatial derivatives of the parameters EI=EI(x) and m=m(x) of suitable order. This case is generally not easy to be analysed and the conventional eigenmode analysis [12,14] can not be applied.

The result obtained by the present method for nonuniform beams will be given in a forthcoming paper.

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Appendix

1. General structures of **E**, **A**, **B** and $\mathbf{E}^{-1}$, $\mathbf{E}^{-1}\mathbf{A}$, $\mathbf{E}^{-1}\mathbf{B}$ for the clamped structure

$$\mathbf{E} = \left[\begin{array}{cccccc|c} \multicolumn{6}{c|}{I_{\alpha\times\alpha}} & 0_{\alpha\times 2} \\ \hline \frac{1}{2!} & \frac{1}{3!} & \frac{1}{6!} & \frac{1}{7!} & \cdots & \frac{1}{(k-1)!} & \frac{1}{k!} \\ 1 & \frac{1}{2!} & \frac{1}{5!} & \frac{1}{6!} & \cdots & \frac{1}{(k-2)!} & \frac{1}{(k-1)!} \end{array}\right],$$

$$\mathbf{A} = \left[\begin{array}{cccccc} \multicolumn{2}{c|}{0_{\alpha\times 2}} & \multicolumn{4}{c}{\frac{EI}{m} I_{\alpha\times\alpha}} \\ \hline 0 & -\frac{EI}{m_p} & -\frac{EI}{3!m_p} & -\frac{EI}{4!m_p} & \cdots & -\frac{EI}{(k-3)!m_p} \\ \frac{EI}{I_p} & \frac{EI}{I_p} & \frac{EI}{4!I_p} & \frac{EI}{5!I_p} & \cdots & \frac{EI}{(k-2)!I_p} \end{array}\right] \equiv \left[\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array}\right] \tag{A.1}$$

$$\mathbf{B} = \left[\begin{array}{cc} \multicolumn{2}{c}{0_{\alpha\times 2}} \\ \hline \frac{1}{m_p} & 0 \\ 0 & \frac{1}{I_p} \end{array}\right], \quad \mathbf{E}^{-1} = \left[\begin{array}{c|cc} I_{\alpha\times\alpha} & \multicolumn{2}{c}{0_{\alpha\times 2}} \\ \hline e'_a & k! & -(k-1)! \\ e'_b & -k!(k-1) & k! \end{array}\right]$$

$$e'_a = -(k-1)! \left[\begin{array}{cccccc} \frac{k-2}{2!} & \frac{k-3}{3!} & \frac{k-6}{6!} & \frac{k-7}{7!} & \cdots & \frac{4}{(k-4)!} \end{array}\right]$$

$$e'_b = k! \left[\begin{array}{cccccc} \frac{k-3}{2!} & \frac{k-4}{3!} & \frac{k-7}{6!} & \frac{k-8}{7!} & \cdots & \frac{3}{(k-4)!} \end{array}\right]$$

$$\mathbf{E}^{-1}\mathbf{A} = \left[\begin{array}{cccccc} \multicolumn{6}{c}{\alpha_1} \\ \hline -a_2 & -a_3 & -a_6 & -a_7 & \cdots & -a_k \\ b_2 & b_3 & b_6 & b_7 & \cdots & b_k \end{array}\right] \tag{A.2}$$

$$a_i = (k-1)!\, EI \left[\frac{k-i+4}{(i-1)!m} + \frac{k}{(i-3)!m_p} + \frac{1}{(i-2)!I_p}\right]$$

$$b_i = k!\, EI \left[\frac{k-i+3}{(i-4)!m} + \frac{k-1}{(i-3)!m_p} + \frac{1}{(i-2)!I_p}\right]$$

$$\mathbf{B} = \left[\begin{array}{cc} \multicolumn{2}{c}{0_{\alpha\times 2}} \\ \hline \frac{k!}{m_p} & -\frac{(k-1)!}{I_p} \\ -\frac{k!(k-1)}{m_p} & \frac{k!}{I_p} \end{array}\right] \tag{A.3}$$

$\alpha=0,2,4,\ldots$ for $k=3,7,11,\ldots$, respectively

2. Structures for **E**, **A**, **B** for the pinned beam

$$\mathbf{E} = \left[\begin{array}{ccccccc} \multicolumn{4}{c|}{I_{\alpha\times\alpha}} & \multicolumn{3}{c}{0_{\alpha\times 3}} \\ \hline 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{5!} & \cdots & \frac{1}{(k-1)!} & \frac{1}{k!} \\ 1 & 1 & \frac{1}{2!} & \frac{1}{4!} & \cdots & \frac{1}{(k-2)!} & \frac{1}{(k-1)!} \end{array}\right],$$

$$\mathbf{A} = \left[\begin{array}{cccccc} \multicolumn{3}{c|}{0_{\alpha\times 3}} & \multicolumn{3}{c}{\frac{EI}{m} I_{\alpha\times\alpha}} \\ \hline 0 & \frac{EI}{I_h} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\frac{EI}{m_p} & -\frac{EI}{2!m_p} & \cdots & -\frac{EI}{(k-3)!m_p} \\ 0 & \frac{EI}{I_p} & \frac{EI}{I_p} & \frac{EI}{3!I_p} & \cdots & \frac{EI}{(k-2)!I_p} \end{array}\right]$$

$$\mathbf{B} = \left[\begin{array}{ccc} \multicolumn{3}{c}{0_{\alpha\times 3}} \\ \hline \frac{1}{I_h} & 0 & 0 \\ 0 & \frac{1}{m_p} & 0 \\ 0 & 0 & \frac{1}{I_p} \end{array}\right]$$

$\alpha=0,3,6,\ldots$ for $k=3,7,11,\ldots$,respectively