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Stability of switched singular time delay systems with switching induced state jumps

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Abstract: This study addresses the problem of exponential stability for switched singular state-delayed systems with switching induced state jumps, which has not been studied up to now. The considered state delay varies in a time-varying interval. On the basis of equivalent dynamics decomposition, a model of state jump at switching instants is firstly established under an assumption that the time length between arbitrary two adjacent switches is larger than the upper bound of the state delay. Then, a sufficient condition on exponential stability of the system under the reranged dwell-time switching constraint is presented. The key idea is the design of a dwell-time-dependent generalised Lyapunov function as well as a dwell-time-dependent function with respect to algebraic variables and application of the Razumikhin approach. The obtained stability condition exploits the lower bound and the upper bound of the dwell time. In addition, it is independent of the size of the state delay and allows the delay to be a fast time-varying function. Finally, two numerical examples are given to show the efficacy of the derived result.

1 Introduction

This paper considers linear switched singular time delay systems (switched STDSs) of the form

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t - d(t)), \tag{1}$$

where $\sigma(t):\mathbb{R}^+ \to \mathscr{P} = \{1, 2, \ldots, N\}$ is a piecewise constant right continuous function of time, termed switching signal; $E_p \in \mathbb{R}^{n\times n}$, $\operatorname{rank}E_p = r < n$, $A_p \in \mathbb{R}^{n\times n}$ and $A_{dp} \in \mathbb{R}^{n\times n}$, $p \in \mathscr{P}$, are constant matrices. The motivation for studying such systems is mainly in twofold. First, STDSs, also called descriptor time delay systems or delay differential-difference equations, provide a general framework for modelling dynamical systems, because they can simultaneously describe the dynamic relationship and the static constrains between state variables, as well as the delays in the system [1]. Second, due to internal and/or external cause such as failure of a component, transitions between different operating regimes and sudden environmental disturbances, practical systems are often subjected to abrupt structure changing and parameter variations. One very effective way to model these systems is to use the switched system models [2]. Switched STDS models have found wide applications in real-world systems, including large-scale electric networks, chemical engineering systems, mechanical systems, etc. (see, e.g. [3–5]), and other dynamical systems such as switched neutral time-delay systems [6]. During the last decade, switched STDSs have become a relatively new research topic in the hybrid control community and many stability results have been reported in the literature; see, e.g. [7–13] and the references therein.

As is well known, the solutions of each individual STDS $E_p\dot{x}(t) = A_px(t) + A_{dp}x(t - d(t))$, $p \in \mathscr{P}$, exhibit finite discontinuous jumps due to incompatible initial conditions [14]. For switched STDSs, it cannot be ensured that the states before the switch always lie in the compatibility space of the subsystem after the switch; consequently, the existence of discontinuous state jumps at switching points is generally unavoidable. This is one of the fundamental differences between switched STDSs and switched normal systems with or without time delay. It should be noted that such jumps can be destablising if the jump amplitude is large enough [15]. Hence, it is essential to consider the switching induced state jumps when studying stability of switched STDSs. However, an investigation of the existing literature on switched STDSs shows that this fact has been either ignored or simply dealt with under the 'switching-impulse-free' assumption (see [7–11]).

In switched STDSs, due to the algebraic equations, the state jump value $\triangle x(t) = x(t) - x(t^-)$ at the switching time t depends not only on $x(t^-)$ but also on the past value $x(t - d(t))^-$. A major difficulty for stability analysis of the systems lies in *handling the jumps of the algebraic variables*. Like regular switched systems, the dominant method for stability analysis of switched STDSs is still the multiple Lyapunov functionals and functions approach, that is, each subsystem is assigned a time-invariant singular-type Lyapunov functional or function. However, the time-invariant Lyapunov functionals or functions are unable to characterise the jump behaviour of the algebraic variables at switching instants. Recently, The authors in [16–22] investigated the stability of switched delay-free singular systems with state jumps induced by mode switching. Liberzon and Trenn [16, 17] presented some sufficient conditions for stability of switched linear singular systems and switched non-linear singular systems, respectively. The main idea of these two works is to construct multiple Lyapunov functions on the consistency spaces of the subsystems and the existence of well fitted consistency projectors at switching instants. Ding *et al.* [18] proposed a new consistency projector based on the Drazin inverse of system matrices. It seems difficult to extend the methods in [16–18] to the case of systems with state delay. For switched linear singular systems, Zhou *et al.* [19] gave a switching induced state jump model based on the equivalent dynamics decomposition and, by applying multiple singular-type Lyapunov functions, presented some sufficient stability conditions. Such a jump model was also utilised in [20, 21]. For Markovian jump singular systems (they can be viewed as a kind of stochastic switched systems), Li *et al.* [23] studied the effect of state jump on the switching function of a sliding surface. However, in the derivation of stability results, the authors in [19–22] only considered the jumps of slow state variables. Therefore, the stability problem for switched STDSs with switching induced state jumps is still open and remains a challenge.

This paper aims to develop a new Lyapunov-based method to analyse the stability of switched STDSs with switching induced state jumps. The state delay considered varies within an interval. The main contribution of the paper is twofold: (i) based on equivalent dynamics decomposition, a model describing the state jump at switching instants is established under an assumption that the time length between two arbitrary adjacent switching instants is larger than the upper bound of the time-varying state delay; and (ii) motivated by Chen *et al.* [24], by constructing two new dwell-time-dependent functions: one is a generalised Lyapunov function, and the other is related to algebraic variables, and applying the Razumikhin approach, a sufficient condition for exponential stability of the system under the reranged dwell-time switching constraint is derived. The obtained stability condition exploits the lower bound and the upper bound of the dwell time. Moreover, it does not depend on the size of the time-varying delay and allows the delay to vary fast. To the best of our knowledge, this paper makes the first effort to establish exponential stability criterion for switched STDSs without ignoring the state jump effect at switching instants.

The rest of the paper is organised as follows. Section 2 gives the system description, problem formulation and some lemmas. In Section 3, an exponential stability criterion is derived by using the Lyapunov–Razumikhin function method. An illustrative example is given in Section 4 and conclusions are presented in Section 5.

Notation: $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the n-dimensional and $m\times n$-dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$, respectively, $\mathbb{Z}^+$ denotes the set of positive integers. Superscript 'T' represents the transpose. $\mathrm{Sym}\{A\}$ stands for $A + A^T$. The symbol $\triangleq$ denotes equality by definition. For a symmetric matrix P, $P > 0$ (≥ 0) means that P is positive definite (semi-positive definite). $\lambda_{\max}(P)(\lambda_{\min}(P))$ denotes the largest (smallest) eigenvalue of P. I is an identity matrix with appropriate dimension. For $d > 0$, $\mathscr{C}([-d,0],\mathbb{R}^n)$ denotes the space of continuous functions $\phi:[-d,0]\to\mathbb{R}^n$ with the norm $\|\phi\|_d = \sup_{-d\leq\theta\leq 0}\|\phi(\theta)\|$. $\mathscr{N}(X)$ denotes the right zero subspace of a matrix X. $\mathrm{diag}\{\dots\}$ denotes a block-diagonal matrix.

2 System description and problem formulation

2.1 System description and problem formulation

Recall system (1) with initial condition

$$\begin{cases} E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) \\ x(s) = \phi(s),\ -d_2\leq s\leq 0, \end{cases} \tag{2}$$

where $x(t)\in\mathbb{R}^n$ is the state, $d(t)$ is the time-varying state delay satisfying $0 < d_1 \leq d(t) \leq d_2$, where d_1 and d_2 are some scalars, and $\phi(s)\in\mathscr{C}([-d_2,0],\mathbb{R}^n)$ is a compatible vector valued initial function. For σ, the switching instants $0\triangleq t_0 < t_1 < \cdots t_i < t_{i+1} < \cdots$ are defined recursively as $t_{i+1} = \inf\{t > t_i : \sigma(t)\neq\sigma(t_i)\}$. Also, let $\sigma(t) = \sigma_i$, $\sigma_i\in\mathscr{P}$ for all $t\in[t_i,t_{i+1})$, $i\in\mathbb{Z}^+\cup\{0\}$. In this paper, for given positive scalars Δ_1 and Δ_2 with $\Delta_1\leq\Delta_2$, we denote by $\mathscr{S}(\Delta_1,\Delta_2)$ the set of all switching signals whose switching instants satisfying $\Delta_1\leq t_k - t_{k-1}\leq\Delta_2$ for all $k\in\mathbb{Z}^+$. According to [25, 26], $\mathscr{S}(\Delta_1,\Delta_2)$ represents the class switching signals under ranged dwell-time constraints.

From [27], it is known that the solution to the pth subsystem in (2) exists and it is unique and impulse free if the pair (E_p, A_p) is regular and impulse free. Hence, we initially assume the following assumption.

Assumption 1: For every $p\in\mathscr{P}$, (E_p, A_p) is regular and impulse free.

In continuation, it can be seen that the regularity and absence of impulses of every subsystem can be verified by the condition of Theorem 1 stated in the next section. To model the state jumps of system (2), we make the following assumptions.

Assumption 2: $\mathscr{N}(E_1) = \mathscr{N}(E_2) = \cdots = \mathscr{N}(E_N)$.

Assumption 3: $t_k - t_{k-1} > d_2$, $\forall k\in\mathbb{Z}^+$.

Remark 1: Since the singular matrix E is mode-dependent, Assumption 2 is made to find one same state-space coordinate basis for all the subsystems [28, 29]. Assumption 3 ensures that $t_k - d(t_k)$, $k\in\mathbb{Z}^+$, are non-switching points. Such an assumption is common in the literature on the systems with delayed impulses; see, e.g. [30–32].

Under Assumption 2, we can find a matrix $\Phi\in\mathbb{R}^{n\times(n-r)}$ such that $E_p\Phi = 0$, $\forall p\in\mathscr{P}$. Then, there exist a matrix $\bar{N}$ and invertible matrices M_p, $p\in\mathscr{P}$ such that $N\triangleq[\bar{N}\quad\Phi]$ is invertible, $\bar{E}_p\triangleq M_pE_pN = \mathrm{diag}\{I_r, 0\}$,

$$\begin{aligned} \bar{A}_p &\triangleq M_pA_pN = \begin{bmatrix} A_{p1} & A_{p2} \\ A_{p3} & A_{p4} \end{bmatrix}, \\ \bar{A}_{dp} &\triangleq M_pA_{dp}N = \begin{bmatrix} A_{dp1} & A_{dp2} \\ A_{dp3} & A_{dp4} \end{bmatrix}, \end{aligned} \tag{3}$$

where $A_{p1}, A_{dp1}\in\mathbb{R}^{r\times r}$, $p\in\mathscr{P}$. Let

$$\zeta(t)\triangleq N^{-1}x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}, \tag{4}$$

where $\zeta_1(t)\in\mathbb{R}^r$. Hence, system (2) can be rewritten in the following dynamics decomposition form:

$$\begin{aligned} \dot{\zeta}_1(t) &= A_{\sigma(t)1}\zeta_1(t) + A_{\sigma(t)2}\zeta_2(t) + A_{d\sigma(t)1}\zeta_1(t-d(t)) \\ &\quad + A_{d\sigma(t)2}\zeta_2(t-d(t)), \\ 0 &= A_{\sigma(t)3}\zeta_1(t) + A_{\sigma(t)4}\zeta_2(t) + A_{d\sigma(t)3}\zeta_1(t-d(t)) \\ &\quad + A_{d\sigma(t)4}\zeta_2(t-d(t)), \end{aligned} \tag{5}$$

with the initial condition $\varphi(s)\triangleq N^{-1}\phi(s) = [\varphi_1^T(s)\quad\varphi_2^T(s)]^T$, $-d_2\leq s\leq 0$, where $\varphi_1(s)\in\mathbb{R}^r$. For system (5), define the compatible initial function space $\mathscr{C}_E\triangleq\{\varphi\in\mathscr{C}_{n,d_2} : A_{\sigma(0)3}\varphi_1(0) + A_{\sigma(0)4}\varphi_2(0) + A_{d\sigma(0)3}\varphi_1(-d(0)).$ $+ A_{d\sigma(0)4}\varphi_2(-d(0)) = 0\}$

Under Assumption 1, A_{p4} is non-singular for every $p\in\mathscr{P}$ [27]. Set

$$\begin{aligned} &\tilde{A}_{p0}\triangleq -A_{p4}^{-1}A_{p3},\ \tilde{A}_{p1}\triangleq -A_{p4}^{-1}A_{dp3},\ \tilde{A}_{p2}\triangleq -A_{p4}^{-1}A_{dp4}, \\ &\mathscr{A}_{pq0}\triangleq\tilde{A}_{p0} - \tilde{A}_{q0},\ \mathscr{A}_{pq1}\triangleq\left[\tilde{A}_{p1} - \tilde{A}_{q1}\quad\tilde{A}_{p2} - \tilde{A}_{q2}\right], \end{aligned}$$

where $p, q\in\mathscr{P}$.

Throughout the paper, let t^- and t^+ denote the time immediately before and after the instant t, respectively. Also, assume $x(t) = x(t^+)$, i.e. the solutions of system (2) are right continuous. For system (5) with $\varphi\in\mathscr{C}_E$, at every switching instant t_k, $k\in\mathbb{Z}^+$, it is clear that $\zeta_2(t_k^-) = \tilde{A}_{\sigma(t_k^-)0}\zeta_1(t_k^-) + \tilde{A}_{\sigma(t_k^-)1}\zeta_1(t_k - d(t_k))^- + \tilde{A}_{\sigma(t_k^-)2}\zeta_2(t_k - d(t_k))^-$ and $\zeta_2(t_k) = \tilde{A}_{\sigma(t_k)0}\zeta_1(t_k) + \tilde{A}_{\sigma(t_k)1}\zeta_1(t_k - d(t_k)) + \tilde{A}_{\sigma(t_k)2}\zeta_2(t_k - d(t_k))$. Note that $\zeta_1(t_k) = \zeta_1(t_k^-)$ and $\zeta_1(t_k - d(t_k)) = \zeta_1(t_k - d(t_k))^-$. By Assumption 3 it can be readily verified that $\zeta_2(t_k - d(t_k)) = \zeta_2(t_k - d(t_k))^-$. Hence, we have that for all $k\in\mathbb{Z}^+$,

$$\begin{aligned} &\zeta_2(t_k) - \zeta_2(t_k^-) \\ &\quad= (\tilde{A}_{\sigma_k0} - \tilde{A}_{\sigma_{k-1}0})\zeta_1(t_k^-) \\ &\qquad + \left[\tilde{A}_{\sigma_k1} - \tilde{A}_{\sigma_{k-1}1}\quad\tilde{A}_{\sigma_k2} - \tilde{A}_{\sigma_{k-1}2}\right]\zeta(t_k - d(t_k))^- \\ &\quad= \mathscr{A}_{\sigma_k\sigma_{k-1}0}\zeta_1(t_k^-) + \mathscr{A}_{\sigma_k\sigma_{k-1}1}\zeta(t_k - d(t_k))^-. \end{aligned} \tag{6}$$

Remark 2: Equation (6) describes the switching induced state jumps in the dynamics decomposition form system (5) under the condition in Assumption 3. The jumps only occur in the algebraic variables. In addition, at every switching instant, the jump value relies on not only the value of differential variables immediately before the switching but also on the past state value of the system. These are caused by a special bilayer feature of switched singular time delay systems. It should be pointed out that such delayed state jumps are much more difficult to be tackled than the delay-free state jumps. Using (4) and (6), it is easy to obtain the following switching induced state jump equation of the system (2):

$$\begin{aligned} x(t_k) = {} & N\begin{bmatrix} I_r & 0 \\ \mathscr{A}_{\sigma_k\sigma_{k-1}0} & I \end{bmatrix} N^{-1}x(t_k^-) + N \\ & \times \begin{bmatrix} 0 & 0 \\ \tilde{A}_{\sigma_k 1} - \tilde{A}_{\sigma_{k-1}1} & \tilde{A}_{\sigma_k 2} - \tilde{A}_{\sigma_{k-1}2} \end{bmatrix} \\ & \times N^{-1}x(t_k - d(t_k))^-, \ k \in \mathbb{Z}^+, \end{aligned}$$

which can be regarded as a delayed version of the consistency projectors proposed in [16, 17].

Definition 1: System (2) is said to be exponentially stable if there exist positive scalars χ, c and γ such that, for any compatible initial condition ϕ with $\| \phi \|_{d_2} \le \chi$, the solution $x(t)$ satisfies

$$\| x(t) \| \le c e^{-\gamma t}, \ t \ge 0.$$

The objective of this work is to establish a sufficient condition for exponential stability of the system (2) under the class switching signals $\sigma \in \mathcal{S}(\Delta_1, \Delta_2)$.

2.2 Lemmas

Set

$$t^0 \triangleq t, \ t^j \triangleq t^{j-1} - d(t^{j-1}), \ j \in \mathbb{Z}^+. \tag{7}$$

The following lemmas are given for later development.

Lemma 1: Consider system (5) with compatible initial condition $\varphi \in \mathscr{C}_E$ and assume that A_{p4} is non-singular for every $p \in \mathscr{P}$. Then, we have the following estimate for solutions of the system:

$$\| \zeta(t) \| \le \eta, \ t \in [-d_2, d_2], \tag{8}$$

where $\eta \triangleq \eta_1 e^{d_2\eta_2} + \max\{ \| \varphi_2 \|_{d_2}, \eta_3 + \eta_4\eta_1 e^{d_2\eta_2} \}$,

$$\begin{aligned} \eta_l \triangleq {} & \max_{p \in \mathscr{P}} \{\eta_{lp}\}, \ l = 1,2,3,4, \\ \eta_{1p} \triangleq {} & \| \varphi_1 \|_{d_2} + \| \tilde{A}_{p2}(\tilde{A}_{p2})^{K-2}\tilde{A}_{p1} \| \ \| \varphi_1 \|_{d_2} \\ & + \| \tilde{A}_{p2}(\tilde{A}_{p2})^{K-1} \| \ \| \varphi_2 \|_{d_2}, \\ \eta_{2p} \triangleq {} & \| \tilde{A}_{p0} \| + \| \tilde{A}_{p1} + \tilde{A}_{p2}\tilde{A}_{p0} \| \\ & + \cdots + \| \tilde{A}_{p2}((\tilde{A}_{p2})^{K-4}\tilde{A}_{p1} + (\tilde{A}_{p2})^{K-3}\tilde{A}_{p1}) \| \\ & + \| \tilde{A}_{p2}((\tilde{A}_{p2})^{K-3}\tilde{A}_{p1} + (\tilde{A}_{p2})^{K-2}\tilde{A}_{p0}) \|, \\ \eta_{3p} \triangleq {} & \| \tilde{A}_{p2}^{K-1} \| \ \| \tilde{A}_{p1} \| \ \| \varphi_1 \|_{d_2} + \| \tilde{A}_{p2}^{K} \| \ \| \varphi_2 \|_{d_2}, \\ \eta_{4p} \triangleq {} & \| \tilde{A}_{p0} \| \sum_{j=0}^{K-1} \| \tilde{A}_{p2}^{j} \| + \| \tilde{A}_{p1} \| \sum_{j=0}^{K-2} \| \tilde{A}_{p2}^{j} \|, \end{aligned}$$

with $\tilde{A}_{p0} \triangleq A_{p1} - A_{p2}A_{p4}^{-1}A_{p3}$, $\tilde{A}_{p1} \triangleq A_{dp1} - A_{p2}A_{p4}^{-1}A_{dp3}$, $\tilde{A}_{p2} \triangleq A_{dp2} - A_{p2}A_{p4}^{-1}A_{dp4}$, and K is a positive integer such that $t^{K-1} > 0$ and $-d_2 < t^K \le 0$.

Proof: See the Appendix. □

Lemma 2: Let $\tilde{V}_i : [\tilde{t} - d_2, \infty] \to [0, \infty)$, $i = 1, 2$, be piecewise continuous functions. Assume the following inequalities hold:

$$\tilde{V}_2(t) \le q_0\tilde{V}_1(t) + q_1\tilde{V}_1(t - d(t)) + q_2\tilde{V}_2(t - d(t)), \ t \ge \tilde{t}_0, \tag{9}$$

$$\tilde{V}_1(t) \le H_1, \ t \in [\tilde{t}_0 - d_2, T], \tag{10}$$

$$\tilde{V}_2(t) \le H_2, \ t \in [\tilde{t}_0 - d_2, \tilde{t}_0], \tag{11}$$

where q_0, q_1, q_2, H_1, H_2, $\tilde{t}_0$, T are non-negative scalars, $q_2 \in (0, 1)$ and $T \ge \tilde{t}_0$. Then,

$$\tilde{V}_2(t) \le \frac{q_0 + q_1}{1 - q_2}H_1 + H_2, \quad t \in [\tilde{t}_0 - d_2, T]. \tag{12}$$

Proof: See the Appendix. □

3 Exponential stability analysis

For an arbitrary switching sequence $\{t_k\}$ generated by $\sigma(t) \in \mathcal{S}(\Delta_1, \Delta_2)$ and $t \ge -d_2$, define

$$\begin{aligned} \varrho(t) & \triangleq \frac{1}{t_k - t_{k-1}}, \ t \in [t_{k-1}, t_k), \ k \in \mathbb{Z}^+, \\ \rho_1(t) & \triangleq \begin{cases} (t - t_{k-1})\varrho(t), & t \in [t_{k-1}, t_k), \ k \in \mathbb{Z}^+, \\ 1, & t \in [-d_2, 0). \end{cases} \end{aligned}$$

Because $1/\Delta_2 \le \varrho(t) \le 1/\Delta_1$ for all $t \ge 0$, there exists a function $\bar{\rho}_2(t) : [0, +\infty) \to [0, 1]$ such that

$$\varrho(t) = \frac{1}{\Delta_1}(1 - \bar{\rho}_2(t)) + \frac{1}{\Delta_2}\bar{\rho}_2(t). \tag{13}$$

Set $\bar{\rho}_1(t) \triangleq 1 - \bar{\rho}_2(t)$, $\rho_2(t) \triangleq 1 - \rho_1(t)$, and $\tilde{\rho}_i(t) \triangleq \rho_i(t - d(t))$, $\forall t \ge 0$, $i = 1, 2$.

Now, the following theorem gives an exponential stability criterion for system (2).

Theorem 1: Consider system (2) with $\sigma \in \mathcal{S}(\Delta_1, \Delta_2)$ and let Assumptions 2 and 3 hold. For given positive scalars α_1, α_2, μ, q_0, q_1, q_2, λ_0, ε_1 and ε_2, satisfying $q_2\bar{\mu}/\underline{\mu} < 1$, where $\underline{\mu} \triangleq \min\{1, \mu\}$ and $\bar{\mu} \triangleq \max\{1, \mu\}$, if there exists matrices

$$P_{\underline{i}i} \triangleq \begin{bmatrix} P_{\underline{i}i1} & 0 \\ P_{\underline{i}i2} & P_{\underline{i}i3} \end{bmatrix}$$

with $P_{\underline{i}i1} > 0$, $Q_{\underline{i}i} > 0$, $\forall \underline{i} \in \mathscr{P}$, $i = 1, 2$, such that the following LMIs hold:

$$\begin{bmatrix} \Lambda_{\underline{i}ih} & P_{\underline{i}i}^T\bar{A}_{d\underline{i}} \\ * & -(\underline{\mu}/\bar{\mu})\mathscr{P}_{\underline{j}j} \end{bmatrix} < 0, \ \forall \underline{i}, \underline{j} \in \mathscr{P}, \ i, j, h = 1, 2, \tag{14}$$

$$\begin{aligned} \mathscr{A}_{\underline{i}3}^T Q_{\underline{i}i} \mathscr{A}_{\underline{i}3} \le \operatorname{diag}\{q_0 P_{\underline{i}i1}, q_1 P_{\underline{j}j1}, q_2 Q_{\underline{j}j}\}, \\ \forall \underline{i}, \underline{j} \in \mathscr{P}, \ i, j = 1, 2, \end{aligned} \tag{15}$$

$$P_{\underline{i}21} \le \mu P_{\underline{j}11}, \ \forall \underline{i}, \underline{j} \in \mathscr{P}, \ \underline{i} \ne \underline{j} \tag{16}$$

$$\lambda_0 I \le P_{\underline{i}i1}, \ \lambda_0 I \le Q_{\underline{i}i}, \ \forall \underline{i} \in \mathscr{P}, \ i = 1, 2, \tag{17}$$

$$\begin{aligned} \begin{bmatrix} -\nu\mu Q_{\underline{i}1} & Q_{\underline{j}2} & 0 & 0 \\ * & -Q_{\underline{j}2} & Q_{\underline{j}2}\mathscr{A}_{\underline{j}\underline{i}0} & Q_{\underline{j}2}\mathscr{A}_{\underline{j}\underline{i}1} \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \\ \forall \underline{i}, \underline{j} \in \mathscr{P}, \ \underline{i} \ne \underline{j}, \end{aligned} \tag{18}$$

where $\mathscr{P}_{\underline{j}j} \triangleq \operatorname{diag}\{\alpha_1 P_{\underline{j}j1}, \alpha_2 Q_{\underline{j}j}\}$, $\mathscr{A}_{\underline{i}2} \triangleq \begin{bmatrix} \tilde{A}_{\underline{i}0} & \tilde{A}_{\underline{i}1} & \tilde{A}_{\underline{i}1} \end{bmatrix}$,

$$\Lambda_{\underline{i}ih} \triangleq \left(\frac{\ln\mu}{\triangle_h} + \alpha_1 + \left(\frac{q_0 + q_1\bar{\mu}/\mu}{1 - q_2\bar{\mu}/\underline{\mu}} + 1\right)\alpha_2\right)\bar{E}_{\underline{i}}P_{\underline{i}i}$$
$$+\frac{1}{\triangle_h}\bar{E}_{\underline{i}}(P_{\underline{i}1} - P_{\underline{i}2}) + P_{\underline{i}i}^T\bar{A}_{\underline{i}} + \bar{A}_{\underline{i}}^T P_{\underline{i}i},$$
$$\nu \triangleq 1 - \left(\frac{\varepsilon_1}{\mu\lambda_0} + \frac{\varepsilon_2}{\underline{\mu}\lambda_0}\left(\frac{q_0 + q_1\bar{\mu}/\mu}{1 - q_2\bar{\mu}/\underline{\mu}} + 2\right)\right)\frac{1}{\frac{q_0 + q_1\bar{\mu}/\mu}{1 - q_2\bar{\mu}/\underline{\mu}} + 1}.$$

then system (2) is exponential stable under any switching signal $\sigma \in \mathcal{S}(\triangle_1, \triangle_2)$.

Proof: By Schur complement, (15) implies $\Lambda_{\underline{i}ih} < 0$, $\forall \underline{i} \in \mathcal{P}$. Substituting $\bar{E}_{\underline{i}}$, $P_{\underline{i}i}$ and $\bar{A}_{\underline{i}}$ into it yields

$$\begin{bmatrix} \star & \star \\ \star & P_{\underline{i}i3}^T A_{\underline{i}4} + A_{\underline{i}4}^T P_{\underline{i}i3} \end{bmatrix} < 0,\ \forall \underline{i} \in \mathcal{P},$$

which implies that $A_{\underline{i}4}$ is invertible for every $\underline{i} \in \mathcal{I}$, where $\star$ denotes the matrix which is not relevant to the discussion. By Xu *et al.* [27], the pair $(\bar{E}_{\underline{i}}, \bar{A}_{\underline{i}})$ is regular and impulse and then is the pair $(E_{\underline{i}}, A_{\underline{i}})$. Therefore, as described in Section 2.1, we know the stability of the system (2) is equivalent to the stability of the following impulsive switched singular time delay systems:

$$\begin{cases} \bar{E}_{\sigma(t)}\dot{\zeta}(t) = \bar{A}_{\sigma(t)}\zeta(t) + \bar{A}_{d\sigma(t)}\zeta(t - d(t)),\ t \neq t_k, \\ \zeta_2(t_k) = \zeta_2(t_k^-) + \mathcal{A}_{\sigma_k\sigma_{k-1}0}\zeta_1(t_k^-) \\ \qquad + \mathcal{A}_{\sigma_k\sigma_{k-1}1}\zeta(t_k - d(t_k))^-,\ t = t_k,\ k \in \mathbb{Z}^+, \\ \zeta(s) = \varphi(s),\ -d_2 \le s \le 0. \end{cases} \tag{19}$$

From (14) and (18), there exists a scalar $\beta > 0$ such that $\tilde{q}_2 \triangleq q_2 e^{\beta d_2}\bar{\mu}/\underline{\mu} < 1$ and the following LMIs hold:

$$\tilde{\Omega}_{\underline{i}j\underline{i}jh} \triangleq \begin{bmatrix} \tilde{\Lambda}_{\underline{i}ih} & P_{\underline{i}i}^T\bar{A}_{d\underline{i}} \\ * & -e^{-\beta d_2}(\underline{\mu}/\bar{\mu})\mathcal{P}_{\underline{j}j} \end{bmatrix} < 0, \quad \forall \underline{i}, \underline{j} \in \mathcal{P},\ i, j, h = 1, 2, \tag{20}$$

$$\begin{bmatrix} -\tilde{\nu}\mu Q_{\underline{i}1} & Q_{\underline{j}2} & 0 & 0 \\ * & -Q_{\underline{j}2} & Q_{\underline{j}2}\mathcal{A}_{\underline{j}\underline{i}0} & Q_{\underline{j}2}\mathcal{A}_{\underline{j}\underline{i}1} \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad \forall \underline{i}, \underline{j} \in \mathcal{P},\ \underline{i} \neq \underline{j} \tag{21}$$

where

$$\tilde{\Lambda}_{\underline{i}ih} \triangleq \left(\frac{\ln\mu}{\triangle_h} + \beta + \tilde{\alpha}_1\right)\bar{E}_{\underline{i}}P_{\underline{i}i} + \frac{1}{\triangle_h}\bar{E}_{\underline{i}}(P_{\underline{i}1} - P_{\underline{i}2})$$
$$+P_{\underline{i}i}^T\bar{A}_{\underline{i}} + \bar{A}_{\underline{i}}^T P_{\underline{i}i},$$
$$\tilde{\nu} \triangleq 1 - \left(\frac{\varepsilon_1}{\mu\lambda_0} + \frac{\varepsilon_2 e^{\beta d_2}}{\underline{\mu}\lambda_0}\left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 2\right)\right)\frac{1}{\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 1},$$

with

$$\tilde{\alpha}_1 \triangleq \alpha_1 + \left(\frac{q_0 + q_1 e^{\beta d_2}\bar{\mu}/\underline{\mu}}{1 - \tilde{q}_2} + 1\right)\alpha_2.$$

Choose the following dwell-time-dependent Lyapunov function for system (19):

$$W_{\sigma(t)1}(t, \zeta(t)) \triangleq e^{\beta(t - d_2)}\mu^{\rho_1(t)}V_{\sigma(t)1}(t, \zeta(t)), \tag{22}$$

where $V_{\sigma(t)1}(t, \zeta(t)) \triangleq \sum_{i=1}^2 \rho_i(t)\zeta^T(t)\bar{E}_{\sigma(t)}P_{\sigma(t)i}\zeta(t)$. We also select the following dwell-time-dependent function:

$$W_{\sigma(t)2}(t, \zeta_2(t)) \triangleq e^{\beta(t - d_2)}\mu^{\rho_1(t)}V_{\sigma(t)2}(t, \zeta_2(t)), \tag{23}$$

where $V_{\sigma(t)2}(t, \zeta_2(t)) \triangleq \sum_{i=1}^2 \rho_i(t)\zeta_2^T(t)Q_{\sigma(t)i}\zeta_2(t)$. Pick an arbitrary positive scalar ϵ and set $\bar{\mu}_\epsilon \triangleq \bar{\mu} + \epsilon$. By Lemma 1, we have that for any $t \in [-d_2, d_2]$,

$$W_{\underline{i}i}(t, x(t)) \le \bar{\mu}\lambda_1\eta^2 < \bar{\mu}_\epsilon\lambda_1\eta^2,\ \forall \underline{i} \in \mathcal{P},\ i = 1, 2, \tag{24}$$

where $\lambda_1 \triangleq \max\{\lambda_{\max}(P_{\underline{i}i1}), \lambda_{\max}(S_{\underline{i}i}), \underline{i} \in \mathcal{P}, i = 1, 2\}$. In the sequel, for notational simplicity, we write $W_{\sigma(t)1}(t, \zeta(t))$, $W_{\sigma(t)2}(t, \zeta_2(t))$, $V_{\sigma(t)1}(t, \zeta(t))$ and $V_{\sigma(t)2}(t, \zeta_2(t))$ as $W_{\sigma(t)1}(t)$, $W_{\sigma(t)2}(t)$, $V_{\sigma(t)1}(t)$ and $V_{\sigma(t)2}(t)$, respectively.

In what follows, we will prove

$$W_{\sigma(t)1}(t) < \bar{\mu}_\epsilon\lambda_1\eta^2,\ \forall t \in [-d_2, t_k),\ k \in \mathbb{Z}^+, \tag{25}$$

$$W_{\sigma(t)2}(t) \le \left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 1\right)\bar{\mu}_\epsilon\lambda_1\eta^2,\ \forall t \in [-d_2, t_k),\ k \in \mathbb{Z}^+, \tag{26}$$

by mathematical induction method.

For $k = 1$, it is clear that (25) and (26) hold for $t \in [-d_2, d_2]$. For $t \in (d_2, t_1)$, if (25) does not hold, then there exists a time t on (d_2, t_1) such that $W_{\sigma(t)1}(t) \ge \bar{\mu}_\epsilon\lambda_1\eta^2$. Let $t_1^* \triangleq \inf\{t \in (d_2, t_1): W_{\sigma(t)1}(t) \ge \bar{\mu}_\epsilon\lambda_1\eta^2\}$. Since $W_{\sigma(t)1}(t)$ is continuous on $(0, t_1)$, it is readily seen that $t_1^* \in (d_2, t_1)$,

$$W_{\sigma(t_1^*)1}(t_1^*) \triangleq \bar{\mu}_\epsilon\lambda_1\eta^2,\ W_{\sigma(t)1}(t) < \bar{\mu}_\epsilon\lambda_1\eta^2,\ \forall t \in [-d_2, t_1^*), \tag{27}$$

$$\dot{W}_{\sigma(t_1^*)1}(t_1^*) \ge 0. \tag{28}$$

Noting $W_{\sigma(t_1^*)1}(t_1^*) \ge W_{\sigma(t_1^* - d(t_1^*))1}(t_1^* - d(t_1^*))$ and $\mu^{\rho_1(t_1^* - d(t_1^*))}/\mu^{\rho_1(t_1^*)} \ge \underline{\mu}/\bar{\mu}$, we have that for any $\alpha_i > 0$, $i = 1, 2$,

$$\begin{aligned} 0 &\le \alpha_1(W_{\sigma(t_1^*)1}(t_1^*) - W_{\sigma(t_1^* - d(t_1^*))1}(t_1^* - d(t_1^*))) \\ &\quad -\alpha_2 W_{\sigma(t_1^* - d(t_1^*))2}(t_1^* - d(t_1^*)) \\ &\quad +\alpha_2 W_{\sigma(t_1^* - d(t_1^*))2}(t_1^* - d(t_1^*)) \\ &\le e^{\beta(t_1^* - \bar{d})}\mu^{\rho_1(t_1^*)}\sum_{i,j=1}^{2}\rho_i(t_1^*)\tilde{\rho}_j(t_1^*)\Big(\alpha_1\zeta^T(t_1^*)\bar{E}_{\sigma_0}P_{\sigma_0 i}\zeta(t_1^*) \\ &\quad -e^{-\beta\bar{d}}(\underline{\mu}/\bar{\mu})\zeta^T(t_1^* - d(t_1^*))\mathrm{diag}\{\alpha_1 P_{\sigma_0 j1}, \alpha_2 Q_{\sigma_0 j}\} \\ &\quad \times\zeta(t_1^* - d(t_1^*))\Big) + \alpha_2 W_{\sigma_0 2}(t_1^* - d(t_1^*)). \end{aligned} \tag{29}$$

By the second equation in (5) and (15), it is easy to verify that for any $t \in [d_2, t_1)$, $V_{\sigma(t)2}(t) = V_{\sigma_0 2}(t)$, and

$$\begin{aligned} V_{\sigma_0 2}(t) &= \sum_{i,j=1}^{2}\rho_i(t)\tilde{\rho}_j(t)\zeta_2^T(t)Q_{\sigma_0 i}\zeta_2(t) \\ &= \sum_{i,j=1}^{2}\rho_i(t)\tilde{\rho}_j(t)\xi_1^T(t)\mathcal{A}_{\sigma_0 3}^T Q_{\sigma_0 i}\mathcal{A}_{\sigma_0 3}\xi_1(t) \\ &\le \sum_{i,j=1}^{2}\rho_i(t)\tilde{\rho}_j(t)\xi_1^T(t)\mathrm{diag}\{q_0 P_{\sigma_0 i1}, q_1 P_{\sigma_0 j1}, \\ &\qquad q_2 Q_{\sigma_0 j}\}\xi_1(t) \\ &= q_0 V_{\sigma_0 1}(t) + q_1 V_{\sigma_0 1}(t - d(t)) + q_2 V_{\sigma_0 2}(t - d(t)), \end{aligned}$$

where $\xi_1(t) \triangleq [\zeta_1^T(t) \quad \zeta_1^T(t - d(t)) \quad \zeta_2^T(t - d(t))]^T$. Hence, for any $t \in [d_2, t_1)$,

$$W_{\sigma_0 2}(t) \le q_0 W_{\sigma_0 1}(t) + \tilde{q}_1 W_{\sigma_0 1}(t - d(t)) + \tilde{q}_2 W_{\sigma_0 2}(t - d(t)),$$

where $\tilde{q}_i \triangleq q_i e^{\beta d_2}\bar{\mu}/\underline{\mu}$, $i = 1, 2$. In view of the inequality in (27) and (24), by applying Lemma 2 with $\tilde{t}_0 = d_2$ and $T = t_1^*$, we have that for any $t \in [0, t_1^*]$,

$$W_{\sigma_0 2}(t) \le \left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2}\right)\bar{\mu}_e \lambda_1 \eta^2 + \bar{\mu}_e \lambda_1 \eta^2 = \left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 1\right) W_{\sigma(t_1^*)1}(t_1^*) .$$

Combining the above inequality with (29) yields

$$\begin{aligned} 0 \le\ & e^{\beta(t_1^* - d_2)} \mu^{\rho_1^*(t)} \sum_{i,j=1}^{2} \rho_i(t_1^*) \tilde{\rho}_j(t_1^*) \Big(\tilde{\alpha}_1 \zeta^T(t_1^*) \bar{E}_{\sigma_0} P_{\sigma_0 i} \zeta(t_1^*) \\ & - e^{-\beta d_2} (\underline{\mu}/\bar{\mu}) \zeta^T(t_1^* - d(t_1^*)) \mathrm{diag}\{\alpha_1 P_{\sigma_0 j 1}, \alpha_2 Q_{\sigma_0 j}\} \\ & \times \zeta(t_1^* - d(t_1^*))\Big) . \end{aligned} \tag{30}$$

When $t \in (0, t_1)$, taking the derivative of $W_{\sigma(t)1}(t)$ along the trajectories of system (13) gives

$$\begin{aligned} \dot{W}_{\sigma(t)1}(t) =\ & e^{\beta(t - d_2)} \sum_{i=1}^{2} \rho_i(t) \Big\{ \zeta^T(t) [(\varrho(t)) \ln \mu + \beta) \\ & \times \bar{E}_{\sigma_0} P_{\sigma_0 i} + \bar{E}_{\sigma_0} (P_{\sigma_0 1} - P_{\sigma_0 2}) \varrho(t)] \zeta(t) \\ & + 2\zeta^T(t) P_{\sigma_0 i}^T (\bar{A}_{\sigma_0} \zeta(t) + \bar{A}_{d\sigma_0} \zeta(t - d(t))) \Big\} . \end{aligned}$$

Substituting (13) into the above equation and adding the right side of $\le$ in (30) yields

$$\begin{aligned} \dot{W}_{\sigma(t_1^*)1}(t_1^*) \le\ & e^{\beta(t_1^* - d_2)} \sum_{i,j,h=1}^{2} \rho_i(t_1^*) \tilde{\rho}_j(t_1^*) \bar{\rho}_h(t_1^*) \\ & \times \xi_2^T(t_1^*) \tilde{\Omega}_{\sigma_0 \sigma_0 ijh} \xi_2(t_1^*), \end{aligned} \tag{31}$$

where $\xi_2(t) \triangleq [\zeta^T(t) \quad \zeta^T(t - d(t))]^T$. By (20), we have $\dot{W}_{\sigma(t_1^*)1}(t_1^*) < 0$, which is a contradiction of (28). Hence, $W_{\sigma(t)1}(t) < \bar{\mu}_e \lambda_1 \eta^2$ holds for any $t \in (\bar{d}, t_1)$, and then (25) holds for $k = 1$. Using Lemma 2 with $\tilde{t}_0 = d_2$ and $T = t_1^-$, we have $W_{\sigma(t)2}(t) \le (((q_0 + \tilde{q}_1)/(1 - \tilde{q}_2)) + 1)\bar{\mu}_e \eta^2$, $\forall t \in [0, t_1)$. So, (26) is true for $k = 1$.

Assume that (25) and (26) hold for some $k = m > 1$. In the sequel, we will prove they also hold for $k = m + 1$. The rest of the proof comprises two steps. We first derive the upper bounds of $W_{\sigma(t_m)1}(t_m)$ and $W_{\sigma(t_m)2}(t_m)$. Then, we calculate the upper bounds of $W_{\sigma(t)1}(t)$ and $W_{\sigma(t)2}(t)$ for $t \in (t_m, t_{m+1})$.

Step 1: Bounds of $W_{\sigma(t_m)1}(t_m)$ and $W_{\sigma(t_m)2}(t_m)$. From (25) with $k = m$ and (22), it follows that

$$\begin{aligned} \bar{\mu}_e \lambda_1 \eta^2 >\ & W_{\sigma(t_m^-)1}(t_m^-) \\ =\ & e^{\beta(t_m - d_2)} \mu \zeta_1^T(t_m^-) P_{\sigma_{m-1} 11} \zeta_1(t_m^-) . \end{aligned} \tag{32}$$

Using (16) and (22), we have

$$\begin{aligned} W_{\sigma(t_m)1}(t_m) =\ & e^{\beta(t_m - d_2)} \zeta_1^T(t_m) P_{\sigma_m 21} \zeta_1(t_m) \\ \le\ & e^{\beta(t_m - d_2)} \mu \zeta_1^T(t_m) P_{\sigma_{m-1} 11} \zeta_1(t_m) \\ <\ & \bar{\mu}_e \lambda_1 \eta^2 . \end{aligned} \tag{33}$$

By (32) and (17), it is easy to get

$$\zeta_1^T(t_m^-) \zeta_1(t_m^-) \le \frac{\bar{\mu}_e \lambda_1 \eta^2}{\mu \lambda_0} e^{-\beta(t_m - d_2)} . \tag{34}$$

On the other hand, by (22) and (23) with $t = (t_m - d(t_m))^-$, (25) and (26) with $k = m$ and (18), one can verify that $\zeta_1^T(t_m - d(t_m))^- \zeta_1(t_m - d(t_m))^- \le e^{\beta d_2} (\bar{\mu}_e \lambda_1 / \underline{\mu} \lambda_0) \eta^2 e^{-\beta(t_m - d_2)}$ and

$$\begin{aligned} \zeta_2^T(t_m - d(t_m))^- \zeta_2(t_m - d(t_m))^- \le\ & e^{\beta d_2} \Big(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} \\ & + 1\Big) \frac{\bar{\mu}_e \lambda_1}{\underline{\mu} \lambda_0} \eta^2 e^{-\beta(t_m - d_2)} . \end{aligned}$$

Hence, by $\zeta^T(\cdot)\zeta(\cdot) = \zeta_1^T(\cdot)\zeta_1(\cdot) + \zeta_2^T(\cdot)\zeta_2(\cdot)$, we have

$$\begin{aligned} & \zeta^T(t_m - d(t_m))^- \zeta(t_m - d(t_m))^- \\ & \quad \le e^{\beta d_2} \left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 2\right) \frac{\bar{\mu}_e \lambda_1}{\underline{\mu} \lambda_0} \eta^2 e^{-\beta(t_m - d_2)} . \end{aligned} \tag{35}$$

Pre- and post-multiplying (21) with $\underline{i} = \sigma_{m-1}$, $\underline{j} = \sigma_m$ by $\mathrm{diag}\{\zeta_2^T(t_m^-), I, I, I\}$ and its transpose, respectively, and in view of $V_{\sigma(t_m^-)2}(t_m^-) = \zeta_2^T(t_m^-) Q_{\sigma_{m-1} 1} \zeta_2(t_m^-)$, we have

$$\begin{bmatrix} -\tilde{\nu} \mu V_{\sigma(t_m^-)2}(t_m^-) & \zeta_2^T(t_m^-) Q_{\sigma_m 2} & 0 & 0 \\ * & -Q_{\sigma_m 2} & Q_{\sigma_m 2} \mathscr{A}_{\sigma_m \sigma_{m-1} 0} & Q_{\sigma_m 2} \mathscr{A}_{\sigma_m \sigma_{m-1} 1} \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0. \tag{36}$$

From (26) with $k = m$, it follows that $\mu V_{\sigma(t_m^-)2}(t_m^-) \le (((q_0 + \tilde{q}_1)/(1 - \tilde{q}_2)) + 1)\bar{\mu}_e \lambda_1 \eta^2 e^{-\beta(t_m - d_2)}$. Thus, (36) guarantees

$$\begin{bmatrix} \Gamma & \zeta_2^T(t_m^-) Q_{\sigma_m 2} & 0 & 0 \\ * & -Q_{\sigma_m 2} & Q_{\sigma_m 2} \mathscr{A}_{\sigma_m \sigma_{m-1} 0} & Q_{\sigma_m 2} \mathscr{A}_{\sigma_m \sigma_{m-1} 1} \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Gamma = \Bigg(& -\left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 1\right) + \frac{\varepsilon_1}{\mu \lambda_0} + \frac{\varepsilon_2 e^{\beta d_2}}{\underline{\mu} \lambda_0} \left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 2\right) \\ & \Bigg) \bar{\mu}_e \lambda_1 \eta^2 e^{-\beta(t_m - d_2)} . \end{aligned}$$

By (34) and (35), the above inequality yields

$$\begin{bmatrix} \tilde{\Gamma} & \zeta_2^T(t_m^-) Q_{\sigma_m 2} & 0 & 0 \\ * & -Q_{\sigma_m 2} & Q_{\sigma_m 2} \mathscr{A}_{\sigma_m \sigma_{m-1} 0} & Q_{\sigma_m 2} \mathscr{A}_{\sigma_m \sigma_{m-1} 1} \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{37}$$

where $\tilde{\Gamma} = -(((q_0 + \tilde{q}_1)/(1 - \tilde{q}_2)) + 1)\bar{\mu}_e \lambda_1 \eta^2 e^{-\beta(t_m - d_2)} + \varepsilon_1 \zeta_1^T(t_m^-) \zeta_1(t_m^-) + \varepsilon_2 \zeta^T(t_m - d(t_m))^- \zeta(t_m - d(t_m))^-$. By Schur complement, (37) is equivalent to

$$\begin{aligned}&\begin{bmatrix}-\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2 e^{-\beta(t_m-d_2)} & \zeta_2^T(t_m^-)Q_{\sigma_m 2}\\ * & -Q_{\sigma_m 2}\end{bmatrix}\\&+\varepsilon_1\begin{bmatrix}\zeta_1^T(t_m^-)\\0\end{bmatrix}\begin{bmatrix}\zeta_1(t_m^-) & 0\end{bmatrix}\\&+\varepsilon_1^{-1}\begin{bmatrix}0\\Q_{\sigma_m 2}\mathscr{A}_{\sigma_m\sigma_{m-1}0}\end{bmatrix}\begin{bmatrix}0 & \mathscr{A}^T_{\sigma_m\sigma_{m-1}0}Q_{\sigma_m 2}\end{bmatrix}\\&+\varepsilon_2\begin{bmatrix}\zeta^T(t_m-d(t_m))^-)\\0\end{bmatrix}\begin{bmatrix}\zeta(t_m-d(t_m))^- & 0\end{bmatrix}\\&+\varepsilon_2^{-1}\begin{bmatrix}0\\Q_{\sigma_m 2}\mathscr{A}_{\sigma_m\sigma_{m-1}1}\end{bmatrix}\begin{bmatrix}0 & \mathscr{A}^T_{\sigma_m 2}Q_{\sigma_m\sigma_{m-1}1}\end{bmatrix}<0.\end{aligned}\tag{38}$$

Note that for any scalars $\varepsilon_i>0$, $i=1,2$,

$$\begin{aligned}&\mathrm{Sym}\left\{\begin{bmatrix}\zeta_1^T(t_m^-)\\0\end{bmatrix}\begin{bmatrix}0 & \mathscr{A}^T_{\sigma_m\sigma_{m-1}0}Q_{\sigma_m 2}\end{bmatrix}\right\}\\&+\mathrm{Sym}\left\{\begin{bmatrix}\zeta^T(t_m-d(t_m))^-)\\0\end{bmatrix}\begin{bmatrix}0 & \mathscr{A}^T_{\sigma_m\sigma_{m-1}1}Q_{\sigma_m 2}\end{bmatrix}\right\}\\&\le\varepsilon_1\begin{bmatrix}\zeta_1^T(t_m^-)\\0\end{bmatrix}\begin{bmatrix}\zeta_1(t_m^-) & 0\end{bmatrix}\\&+\varepsilon_1^{-1}\begin{bmatrix}0\\Q_{\sigma_m 2}\mathscr{A}_{\sigma_m\sigma_{m-1}0}\end{bmatrix}\begin{bmatrix}0 & \mathscr{A}^T_{\sigma_m\sigma_{m-1}0}Q_{\sigma_m 2}\end{bmatrix}\\&+\varepsilon_2\begin{bmatrix}\zeta^T(t_m-d(t_m))^-)\\0\end{bmatrix}\begin{bmatrix}\zeta(t_m-d(t_m))^- & 0\end{bmatrix}\\&+\varepsilon_2^{-1}\begin{bmatrix}0\\Q_{\sigma_m 2}\mathscr{A}_{\sigma_m\sigma_{m-1}1}\end{bmatrix}\begin{bmatrix}0 & \mathscr{A}^T_{\sigma_m\sigma_{m-1}2}Q_{\sigma_m 1}\end{bmatrix},\end{aligned}\tag{39}$$

and by (6),

$$\begin{aligned}\zeta_2(t_m)=\zeta_2(t_m^-)+\mathscr{A}_{\sigma_m\sigma_{m-1}0}\zeta_1(t_m^-)\\+\mathscr{A}_{\sigma_m\sigma_{m-1}1}\zeta(t_m-d(t_m))^-.\end{aligned}\tag{40}$$

Then, it follows from (36)–(40) that

$$\begin{bmatrix}-\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2 e^{-\beta(t_m-d_2)} & \zeta_2^T(t_m)Q_{\sigma_m 2}\\ * & -Q_{\sigma_m 2}\end{bmatrix}<0.$$

Using Schur complement again and recalling $W_{\sigma(t_m)2}(t_m)=e^{\beta(t_m-d_2)}\zeta_2^T(t_m)Q_{\sigma_m 2}\zeta_2(t_m)$, we have

$$W_{\sigma(t_m)2}(t_m)\le\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2.\tag{41}$$

Step 2: Bounds of $W_{\sigma(t)1}(t)$ and $W_{\sigma(t)2}(t)$ for $t\in(t_m,t_{m+1})$. Suppose (25) does not hold for $k=m+1$. Let $t_m^*\triangleq\inf\{t\in[t_m,t_{m+1}):W_{\sigma(t)1}(t)\ge\bar\mu_\epsilon\lambda_1\varrho_0^2\}$. By (33), it is readily seen that $t_m^*\in(t_m,t_{m+1})$, $W_{\sigma(t_m^*)1}(t_m^*)\triangleq\bar\mu_\epsilon\lambda_1\eta^2$, $W_{\sigma(t)1}(t)<\bar\mu_\epsilon\lambda_1\eta^2$ for $t\in[-d_2,t_m^*)$, and $\dot W_{\sigma(t_m^*)1}(t_m^*)\ge 0$. Then, for any $\alpha_i>0$, $i=1,2$, the following relations hold:

$$\begin{aligned}0\le{}&\alpha_1(W_{\sigma(t_m^*)1}(t_m^*)-W_{\sigma(t_m^*-d(t_m^*))1}(t_m^*-d(t_m^*)))\\&-\alpha_2W_{\sigma(t_m^*-d(t_m^*))2}(t_m^*-d(t_m^*))\\&+\alpha_2W_{\sigma(t_m^*-d(t_m^*))2}(t_m^*-d(t_m^*))\\&\le e^{\beta(t_m^*-d_2)}\mu^{\rho_1(t_m^*)}\sum_{i,j=1}^{2}\rho_i(t_m^*)\tilde\rho_j(t_m^*)\Big(\alpha_1x^T(t_m^*)\\&\times\bar E_{\sigma_m}P_{\sigma_m i}x(t_m^*)-e^{-\beta d_2}(\underline{\mu}/\bar\mu)\zeta^T(t_m^*-d(t_m^*))\\&\times\mathrm{diag}\{\alpha_1P_{\sigma(t_m^*-d(t_m^*))j1},\alpha_2Q_{\sigma(t_m^*-d(t_m^*))j}\}\\&\times\zeta(t_m^*-d(t_m^*))\Big)+\alpha_2W_{\sigma(t_m^*-d(t_m^*))2}(t_m^*-d(t_m^*)).\end{aligned}$$

In what follows, we calculate the upper bound of $\dot W_{\sigma(t_m^*)1}(t_m^*)$ for the following three cases:

Case 1: $t_m^*-d(t_m^*)<t_m$. By Assumption 3, we know $\sigma(t_m^*-d(t_m^*))=\sigma_{m-1}$. It follows from (26) with $k=m$ that $W_{\sigma(t_m^*-d(t_m^*))2}(t_m^*-d(t_m^*))\le(((q_0+\tilde q_1)/(1-\tilde q_2))+1)\bar\mu_\epsilon\lambda_1\eta^2=((.$ Then, $(q_0+\tilde q_1)/(1-\tilde q_2)+1)W_{\sigma(t_m^*)1}(t_m^*,x(t_m^*))$
using an argument similar to the derivation of (31), we can obtain

$$\begin{aligned}\dot W_{\sigma(t_m^*)1}(t_m^*)\le e^{\beta(t_m^*-d_2)}\sum_{i,j,h=1}^{2}\rho_i(t_m^*)\tilde\rho_j(t_m^*)\bar\rho_h(t_m^*)\\\times\xi_2^T(t_m^*)\tilde\Omega_{\sigma_m\sigma_{m-1}ijh}\xi_2(t_m^*).\end{aligned}\tag{42}$$

Case 2: $t_m^*-d(t_m^*)=t_m$. For this case, $\sigma(t_m^*-d(t_m^*))=\sigma_m$. By (41), we have $W_{\sigma(t_m^*-d(t_m^*))2}(t_m^*-d(t_m^*))\le(((q_0+\tilde q_1)/(1-\tilde q_2))+1)\bar\mu_\epsilon\lambda_1\eta^2$. Then, following a similar line of the derivation of (31), we have

$$\begin{aligned}\dot W_{\sigma(t_m^*)1}(t_m^*)\le e^{\beta(t_m^*-d_2)}\sum_{i,j,h=1}^{2}\rho_i(t_m^*)\tilde\rho_j(t_m^*)\bar\rho_h(t_m^*)\\\times\xi_2^T(t_m^*)\tilde\Omega_{\sigma_m\sigma_m ijh}\xi_2(t_m^*).\end{aligned}\tag{43}$$

Case 3: $t_m^*-d(t_m^*)>t_m$. For this case, $\sigma(t_m^*-d(t_m^*))=\sigma_m$. Let $\bar t^0\triangleq t_m^*-d(t_m^*)$, $\bar t^i\triangleq\bar t^{i-1}-d(\bar t^{i-1})$, $i\in\mathbb{Z}^+$. Then, there exists a positive integer $\bar K$ such that $\bar t^{\bar K-1}>t_m$ and $\bar t^{\bar K}\le t_m$. By using (15) and the definitions of $V_{\sigma(t)1}(t)$ and $V_{\sigma(t)2}(t)$, it is easy to get $V_{\sigma_m 2}(\bar t^{\bar K-1})\le q_0V_{\sigma_m 1}(\bar t^{\bar K-1})+q_1V_{\sigma_{m-1}1}(\bar t^{\bar K})+q_2V_{\sigma_{m-1}2}(\bar t^{\bar K})$. Note that $W_{\sigma(t)1}(t)<\bar\mu_\epsilon\lambda_1\eta^2$ for $t\in[-\bar d,t_m^*)$. In view of (26) with $k=m$ and noting $0<\tilde q_2<1$, we have

$$\begin{aligned}&W_{\sigma_m 2}(\bar t^{\bar K-1})\\&\le q_0W_{\sigma_m 1}(\bar t^{\bar K-1})+\tilde q_1W_{\sigma_{m-1}1}(\bar t^{\bar K})+\tilde q_2W_{\sigma_{m-1}2}(\bar t^{\bar K})\\&\le(q_0+\tilde q_1)\bar\mu_\epsilon\lambda_1\eta^2+\tilde q_2\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2\\&<\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2.\end{aligned}$$

For $t=\bar t^{\bar K-2}$, by (15), similarly, one can obtain $V_{\sigma_m 2}(\bar t^{\bar K-2})\le q_0V_{\sigma_m 1}(\bar t^{\bar K-1})+q_1V_{\sigma_m 1}(\bar t^{\bar K-1})+q_2V_{\sigma_m 2}(\bar t^{\bar K-1})$. Then,

$$\begin{aligned}&W_{\sigma_m 2}(\bar t^{\bar K-2})\\&\le q_0W_{\sigma_m 1}(\bar t^{\bar K-2})+\tilde q_1W_{\sigma_m 1}(\bar t^{\bar K-1})+\tilde q_2W_{\sigma_m 2}(\bar t^{\bar K-1})\\&\le(q_0+\tilde q_1)\bar\mu_\epsilon\lambda_1\eta^2+\tilde q_2\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2\\&<\left(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1\right)\bar\mu_\epsilon\lambda_1\eta^2.\end{aligned}$$

Repeating the above procedure, we have $W_{\sigma_m 2}(\bar t^0)=W_{\sigma_m 2}(t_m^*-d(t_m^*))<(\frac{q_0+\tilde q_1}{1-\tilde q_2}+1)\bar\mu_\epsilon\lambda_1\eta^2$. Therefore, (43) still holds.

By (20), we have $\dot{W}_{\sigma(t_m^*)1}(t_m^*) < 0$ for the above three cases. This is a contradiction of $\dot{W}_{\sigma(t_m^*)1}(t_m^*) \geq 0$. Thus, (25) holds for $k = m + 1$. We proceed to derive the upper bound of $W_{\sigma(t)2}(t)$ for $t \in (t_m, t_{m+1})$. For any $t \in (t_m, t_{m+1})$, let $\tilde{t}^0 \triangleq t$, $\tilde{t}^i \triangleq \tilde{t}^{i-1} - d(\tilde{t}^{i-1})$, $i \in \mathbb{Z}^+$. Similar to the analysis in *Case 3*, we can obtain

$$W_{\sigma(t)2}(t) \leq \left(\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 1\right)\bar{\mu}_\epsilon \lambda_1 \eta^2, \quad \forall t \in (t_m, t_{m+1}),$$

which shows that (26) holds for $k = m + 1$. Thus, by the induction method, (25) and (26) hold for all $k \in \mathbb{Z}^+$.

Now, recalling (25) and (26) and according to the arbitrariness of ϵ, we have $\| \zeta_i(t) \| \leq L_i \eta e^{-\frac{\beta}{2}t}$, $\forall t \geq 0$, where

$$L_1 \triangleq \sqrt{\frac{\lambda_1}{\lambda_0 \min\{1, \mu^{-1}\}}} \text{ and } L_2 \triangleq L_1 \sqrt{\frac{q_0 + \tilde{q}_1}{1 - \tilde{q}_2} + 1}.$$

By (4), we finally have

$$\begin{aligned} \| x(t) \| &= \sqrt{\zeta(t)^T N^T N \zeta(t)} \\ &\leq \sqrt{\| N \|^2 (\| \zeta_1(t) \|^2 + \| \zeta_2(t) \|^2)} \\ &\leq \| N \| \sqrt{L_1^2 + L_2^2}\, \eta e^{-\frac{\beta}{2}t}, \ \forall t \geq 0. \end{aligned} \tag{44}$$

This completes the proof. □

Remark 3: Theorem 1 provides the relationships between the speed of switching (characterised by Δ_1 and Δ_2), the delay and stability of the switched system (2). From the LMIs (20) and (21), it can be seen that smaller Δ_1, Δ_2 and β are favourable for the feasibility of the equalities. However, a smaller Δ_2 means the switching cannot be too slow, and a smaller β degrades stability performance of the switched system. On the other hand, the definition of $\tilde{q}_2$ makes it clear that a smaller d_2 permits a larger β and vice versa, which indicates that there is a trade-off between the upper bound on the delay and stability performance of the switched system.

Remark 4: The stability condition presented in Theorem 1 is derived via the Razumikhin approach with the dwell-time-dependent Lyapunov function (22) and the dwell-time-dependent function (23). The two dwell-time-dependent functions have three important features as follows: (i) when $P_{\sigma(t)1} = P_{\sigma(t)2}$ and $Q_{\sigma(t)1} = Q_{\sigma(t)2}$, they reduce to normal (mode-dependent) multiple Lyapunov function $\bar{W}_{\sigma(t)1}(\zeta(t)) = e^{\beta(t-d_2)}\zeta^T(t)\bar{E}_{\sigma(t)}P_{\sigma(t)}\zeta(t)$ and mode-dependent function $\bar{W}_{\sigma(t)2}(\zeta_2(t)) = e^{\beta(t-d_2)}\zeta_2^T(t)Q_{\sigma(t)}\zeta_2(t)$, respectively; (ii) when $P_{\sigma(t)1} \neq P_{\sigma(t)2}$ and $Q_{\sigma(t)1} \neq Q_{\sigma(t)2}$, both the functions are discontinuous at switching instants. As described in Section 2, the state trajectories of the switched singular delay system (2) jump at switching instants. Therefore, in comparison with the mode-dependent functions $\bar{W}_{\sigma(t)1}(\zeta(t))$ and $\bar{W}_{\sigma(t)2}(\zeta_2(t))$, the dwell-time-dependent functions $W_{\sigma(t)1}(\zeta(t))$ and $W_{\sigma(t)2}(\zeta_2(t))$ are more proper to characterise the state jump behaviour of switched singular delay systems; and (iii) due to the insertion of the term $\varrho(t)$, the derived stability condition makes full use of the upper bounds of switching intervals. Finally, we would like to stress that the mode-dependent functions $\bar{W}_{\sigma(t)1}(\zeta(t))$ and $\bar{W}_{\sigma(t)2}(\zeta_2(t))$ cannot be used to prove Theorem 1. The reason is given as follows. To get $W_{\sigma(t_m)2}(t_m) \leq (((q_0 + \tilde{q}_1)/(1 - \tilde{q}_2)) + 1)\bar{\mu}_\epsilon \lambda_1 \eta^2$, the LMIs (19) will become the following:

$$\begin{bmatrix} -\nu Q_{\underline{i}} & Q_{\underline{j}} & 0 & 0 \\ * & -Q_{\underline{j}} & Q_{\underline{j}}\mathscr{A}_{\underline{j}\underline{i}1} & Q_{\underline{j}}\mathscr{A}_{\underline{j}\underline{i}2} \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \ \forall \underline{i}, \underline{j} \in \mathscr{P}, \underline{i} \neq \underline{j},$$

which implies

$$\begin{bmatrix} -\nu Q_{\underline{i}} & Q_{\underline{j}} \\ * & -Q_{\underline{j}} \end{bmatrix} < 0,$$

or equivalently, $Q_{\underline{j}} < \nu Q_{\underline{i}}$, $\forall \underline{i}, \underline{j} \in \mathscr{P}$, $\underline{i} \neq \underline{j}$. This requires $\nu \geq 1$. However, ν is less than 1 in Theorem 1.

Remark 5: For time delay systems, there are two main direct Lyapunov approaches for stability analysis: Lyapunov–Krasovskii approach and Lyapunov–Razumikhin approach. The former is capable of providing delay-independent and delay-dependent stability conditions, which are decided by specific forms of the constructed Lyapunov functionals. On the other hand, the latter is simpler to use and yields delay-independent stability conditions. Moreover, so far only the Lyapunov–Razumikhin approach can establish delay-independent stability conditions for systems with fast-varying delays. For more details on these two approaches, we refer readers to [33]. The stability condition presented in Theorem 1 does not depend on the size of the state delay, that is, it is delay-independent. Moreover, it imposes no constraint on the derivative of the delay. Nevertheless, delay-dependent conditions are generally of less conservatism than the delay-independent ones. For switched singular time delay systems without state jumps effect, in order to obtain the delay-dependent results, one can introduce generalised Lyapunov–Krasovskii functionals and convergence of the differential variables is easily verified. Then, by establishing an iterative equation in terms of the differential variables, convergence of the algebraic variables can be deduced from the differential ones [27, 34]. However, when considering the switching induced state jumps as shown in (6), such a method is unworkable because, at each switching instant, the jump at the algebraic variables is related to parameters of the subsystems before and after the switching and it is hard to acquire a bound on the jump value. How to obtain delay-dependent stability conditions for switched singular time-delay systems with switching induced state jumps is a technically challenging problem.

Remark 6: Note that for $W_{\sigma(t)1}(t, \zeta(t))$ and $W_{\sigma(t)2}(t, \zeta_2(t))$ in (22) and (23), the corresponding Lyapunov functions in each switching interval are a convex combination of two mode-dependent constant matrices. This may lead to conservatism of stability condition. In order to reduce the conservatism, a practicable method is to introduce the discretised Lyapunov functions [35]. To this end, we need divide the switching interval $[t_{k-1}, t_k)$ into N_0 subintervals $[t_{k,\kappa-1}, t_{k,\kappa})$ with equal length, where $t_{k,0} = t_{k-1}$, $t_{k,N_0} = t_k$, $k \in \mathbb{Z}^+$. Define

$$\rho_{10}(t) \triangleq \begin{cases} \dfrac{t - t_{k,\kappa-1}}{t_{k,\kappa} - t_{k,\kappa-1}}, \ t \in [t_{k,\kappa-1}, t_\kappa), \\ \qquad\qquad \kappa = 1, 2, \ldots, N_0, \ k \in \mathbb{Z}^+, \\ 0, \qquad\quad t \in [-\bar{d}, 0). \end{cases}$$

Then, we can construct the following dwell-time-dependent discretised Lyapunov functions for system (19):

$$\begin{aligned} \tilde{W}_{\sigma(t)1}(t, \zeta(t)) &\triangleq e^{\beta(t-d_2)}\omega(t)\zeta^T(t)\tilde{P}_{\sigma(t)}\zeta(t), \\ \tilde{W}_{\sigma(t)2}(t, \zeta(t)) &\triangleq e^{\beta(t-d_2)}\omega(t)\zeta_2^T(t)\tilde{Q}_{\sigma(t)}\zeta_2(t), \end{aligned}$$

where

$$\begin{aligned} \omega(t) &\triangleq \prod_{f=1}^{\kappa-1} \mu_f \mu_\kappa^{\rho_{10}(t)}, \\ \tilde{P}_{\sigma(t)} &\triangleq \rho_{10}(t)P_{\sigma(t)\kappa} + (1 - \rho_{10}(t))P_{\sigma(t)\kappa-1}, \\ \tilde{Q}_{\sigma(t)} &\triangleq \rho_{10}(t)Q_{\sigma(t)\kappa} + (1 - \rho_{10}(t))Q_{\sigma(t)\kappa-1}, \end{aligned}$$

for $t \in [t_{k,\kappa-1}, t_k)$, $\kappa = 1, 2, \ldots, N_0$, $k \in \mathbb{Z}^+$, in which $\mu_\kappa > 0$, $\theta = 1, 2, \ldots, N_0$,

$$P_{\underline{i}\bar{\kappa}} \triangleq \begin{bmatrix} P_{\underline{i}\bar{\kappa}1} & 0 \\ P_{\underline{i}\bar{\kappa}2} & P_{\underline{i}\bar{\kappa}3} \end{bmatrix}$$

with $P_{\underline{i}\bar{\kappa}1} > 0$, $\underline{i} \in \mathscr{P}$, $\bar{\kappa} = 0, 1, \ldots, N_0$, $Q_{\underline{i}\bar{\kappa}} > 0$, $\underline{i} \in \mathscr{P}$, $\bar{\kappa} = 0, 1, \ldots, N_0$. Generally speaking, increasing N_0 is favourable for deducing conservatism of the obtained stability conditions. However, this will lead to a marked increase in computational burden since the number of decision variables and the dimensions of involved matrix inequalities will increase sharply.

4 Numerical examples

In this section, we demonstrate the above stability condition by two examples.

Example 1: Consider system (2) with two modes (i.e. $\mathscr{P} = \{1, 2\}$) as follows:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -1.4 & 0.85 & 1.5 \\ 0.1 & -0.75 & -1.1 \\ -0.4 & -1.0 & 1.25 \end{bmatrix}, A_{d1} = \begin{bmatrix} -0.1 & -1.2 & 0.3 \\ -0.1 & 0.7 & -0.2 \\ 0.1 & -0.1 & 0.27 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.4 & 1.2 & 1.4 \\ -0.1 & -1.2 & -0.9 \\ -0.5 & -1.2 & 1.55 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.1 & -0.1 & -0.2 \\ 0.1 & -0.4 & 0.35 \end{bmatrix}.$$

It can be checked that Assumptions 1 and 2 hold. Choose

$$\Phi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and pick

$$M_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we obtain $\bar{E}_1 = \bar{E}_2 = \mathrm{diag}\{I_2, 0\}$,

$$\bar{A}_1 = \begin{bmatrix} -0.85 & -0.1 & 1.1 \\ -1.4 & -1.3 & 0.4 \\ 0.6 & -0.4 & 1.25 \end{bmatrix}, \bar{A}_{d1} = \begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.3 & -0.2 & 0.1 \\ 0.2 & 0.1 & 0.27 \end{bmatrix},$$
$$\bar{A}_2 = \begin{bmatrix} -1.1 & 0.1 & 0.9 \\ -1.5 & -1.5 & 0.5 \\ 0.7 & -0.5 & 1.55 \end{bmatrix}, \bar{A}_{d2} = \begin{bmatrix} -0.2 & -0.1 & 0.2 \\ 0.1 & 0.2 & 0.1 \\ 0.5 & 0.1 & 0.35 \end{bmatrix}.$$

Choose $\alpha_1 = 1.5$, $\alpha_2 = 0.05$, $\mu = 1$, $q_0 = 1.1$, $q_1 = 0.1$, $q_2 = 0.15$, $\lambda_0 = 0.0001$, $\varepsilon_1 = 0.000015$ and $\varepsilon_2 = 0.000025$. For different $\triangle_1$, by solving LMIs (14)–(18) in Theorem 1, we calculate the maximal value of $\triangle_2$, which are listed in Table 1. Given the switching signal $\sigma(t) \in \mathscr{S}(0.3, 0.3)$, the activating order of subsystems is periodic, i.e. $1 \to 2 \to 1 \to \cdots$. The state trajectories of the switched system with a compatible initial condition

$$x(0) = \begin{bmatrix} -1 \\ 2 \\ 1.25 \end{bmatrix}$$

and delay $d(t) = 0.15 + 0.05\sin(10t)$ as well as corresponding switching signal $\sigma(t)$ are shown in Fig. 1. From the figure, it can be clearly seen that the trajectories converge to zero despite the abrupt jumps of x_3 at switching instants.

Table 1 Maximal value of Δ_2 for different Δ_1

Δ_1	0.05	0.1	0.15	0.17	0.2
Δ_2	0.296	0.307	0.312	0.286	0.274

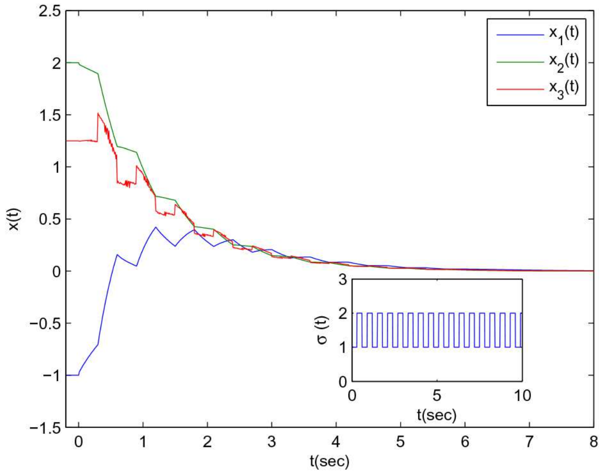

Fig. 1 *State responses and switching signal*

Example 2: Consider the time delay electrical circuit given in [36], which is described by

$$E\dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t), \tag{45}$$

where $E = \mathrm{diag}\{L, C, 0\}$, $A_d = \mathrm{diag}\{0.5, 0.5, 0\}$,

$$A = \begin{bmatrix} -R & -1 & 1 \\ 0 & -1/G & 0 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

As in [36], let $L = 3$, $C = 2$ and $G = 1$. In this paper, we assume R belongs to a set of $\{1.5, 2\}$. Then, we obtain the following switched singular time delay system:

$$E\dot{x}(t) = \hat{A}_{\sigma(t)}x(t) + A_d x(t-d) + Bu(t), \tag{46}$$

where $\sigma(t) \in \{1, 2\}$, $E = \mathrm{diag}\{3, 2, 0\}$,

$$\hat{A}_1 = \begin{bmatrix} -1.5 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} -2 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

For each subsystem $(E, \hat{A}_{\underline{i}}, A_d, B)$, $\underline{i} = 1, 2$, we design a state feedback controller $u(t) = K_{\underline{i}}x(t)$ such that the closed-loop subsystem is regular, impulse free and stable, and the feedback gains are $K_1 = [2.763 \;\; -0.122 \;\; 6.023]$ and $K_2 = [4.130 \;\; -0.270 \;\; 5.704]$. Then, the closed-loop dynamics of system (46) has the form of (2) with $E_1 = E_2 = \mathrm{diag}\{3, 2, 0\}$, $A_{d1} = A_{d2} = A_d$, $d(t) = d$, (see equation below). Choose

$$\Phi = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and set

$$M_1 = M_2 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have $\bar{E}_1 = \bar{E}_2 = \mathrm{diag}\{I_2, 0\}$, $\bar{A}_{d1} = \bar{A}_{d2} = \mathrm{diag}\{0.1677, 0.25, 0\}$,

$$A_1 = \begin{bmatrix} -1.5 & -1 & 1 \\ 2.763 & -1.122 & 6.023 \\ -1.763 & 0.122 & -6.023 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -1 & 1 \\ 4.13 & -1.27 & 5.7 \\ -3.13 & 0.27 & -5.7 \end{bmatrix}.$$

$$\bar{A}_1 = \begin{bmatrix} -0.5 & -0.3333 & 0.3333 \\ 1.3815 & -0.561 & 3.0115 \\ -1.763 & 0.1220 & -6.023 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} -0.6667 & -0.3333 & 0.3333 \\ 2.065 & -0.635 & 2.85 \\ -3.13 & 0.27 & -5.7 \end{bmatrix}.$$

Pick $\triangle_1 = 0.05$. By solving LMIs (14)–(18) in Theorem 1 with the choice of $\alpha_1 = 1.15$, $\alpha_2 = 0.02$, $\mu = 1$, $q_0 = 1.5$, $q_1 = 0.1$, $q_2 = 0.15$, $\lambda_0 = 0.0001$, $\varepsilon_1 = 0.00001$ and $\varepsilon_2 = 0.00002$, we can find that the maximal value of $\triangle_2$ such that the LMIs are feasible, is $\triangle_2 = 0.326$. Given the switching signal $\sigma(t) \in \mathcal{S}(0.3, 0.3)$, the activating order of subsystems is periodic, i.e. $1 \to 2 \to 1 \to \cdots$. Let $d = 0.2$. The state trajectories of the switched system with a compatible initial condition $x(0) = \begin{bmatrix} -3.5 \\ 1.2 \\ 1 \end{bmatrix}$ and corresponding switching signal $\sigma(t)$ are shown in Fig. 2. From Fig. 2, we can see that the states converge to zero despite the abrupt jumps of x_3 at switching instants.

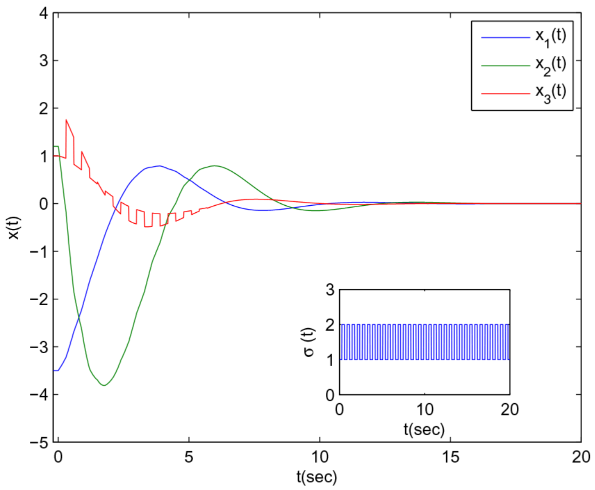

Fig. 2 *State responses and switching signal*

5 Conclusion

In this paper, an exponential stability criterion has been established for switched singular state-delayed systems without ignoring the state jumps induced by mode switching under dwell-time constraints. The main idea behind the derivation of the criterion is the application of Razumikhin approach and the design of a dwell-time-dependent generalised Lyapunov function as well as a dwell-time-dependent function with respect to the algebraic variables, which are discontinuous at switching instants. The future work will involve deriving stability criterion for the system under study with larger state delays, i.e. the upper bound of the state delay is allowed to be larger than the dwell time. Emphasis will also be put on the establishment of delay-dependent stability conditions.

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8 Appendix

Proof of Lemma 1: As mentioned earlier, for any $t \in [0, d_2)$, the σ_0th subsystem in (5) is activated. Since A_{σ_04} is non-singular, the subsystem can be rewritten as

$$\begin{aligned}\dot{\zeta}_1(t) &= \tilde{A}_{\sigma_00}\zeta_1(t) + \tilde{A}_{\sigma_01}\zeta_1(t-d(t)) + \tilde{A}_{\sigma_02}\zeta_2(t-d(t)),\\ \zeta_2(t) &= \tilde{A}_{\sigma_00}\zeta_1(t) + \tilde{A}_{\sigma_01}\zeta_1(t-d(t)) + \tilde{A}_{\sigma_02}\zeta_2(t-d(t)).\end{aligned} \tag{47}$$

Then, from (7) and the first equation in (47), it follows that

$$\dot{\zeta}_1(t) = \dot{\zeta}_1(t^0) = \tilde{A}_{\sigma_00}\zeta_1(t^0) + \tilde{A}_{\sigma_01}\zeta_1(t^1) + \tilde{A}_{\sigma_02}\zeta_2(t^1). \tag{48}$$

Using (7) and the second equation in (47), it is easy to obtain $\zeta_2(t^1) = \tilde{A}_{\sigma_00}\zeta_1(t^1) + \tilde{A}_{\sigma_01}\zeta_1(t^2) + \tilde{A}_{\sigma_02}\zeta_2(t^2)$. Substituting it into (48) yields
$\dot{\zeta}_1(t) = \dot{\zeta}_1(t^0) = \tilde{A}_{\sigma_00}\zeta_1(t^0) + (\tilde{A}_{\sigma_01} + \tilde{A}_{\sigma_02}\tilde{A}_{\sigma_00})\zeta_1(t^1) + \tilde{A}_{\sigma_02}\tilde{A}_{\sigma_01}\zeta_1(t^2).$
$+ \tilde{A}_{\sigma_02}\tilde{A}_{\sigma_02}\zeta_2(t^2)$
Repeating the above process and noting $t^j < t^{j-1}$, we can conclude that there exists a positive finite integer K such that

$$\begin{aligned}\dot{\zeta}_1(t) &= \dot{\zeta}_1(t^0)\\ &= \tilde{A}_{\sigma_00}\zeta_1(t^0) + (\tilde{A}_{\sigma_01} + \tilde{A}_{\sigma_02}\tilde{A}_{\sigma_00})\zeta_1(t^1)\\ &\quad + \cdots + \tilde{A}_{\sigma_02}((\tilde{A}_{\sigma_02})^{K-4}\tilde{A}_{\sigma_01} + (\tilde{A}_{\sigma_02})^{K-3}\tilde{A}_{\sigma_00})\zeta_1(t^{K-2})\\ &\quad + \tilde{A}_{\sigma_02}((\tilde{A}_{\sigma_02})^{K-3}\tilde{A}_{\sigma_01} + (\tilde{A}_{\sigma_02})^{K-2}\tilde{A}_{\sigma_00})\zeta_1(t^{K-1})\\ &\quad + \tilde{A}_{\sigma_02}(\tilde{A}_{\sigma_02})^{K-2}\tilde{A}_{\sigma_01}\zeta_1(t^K)\\ &\quad + \tilde{A}_{\sigma_02}(\tilde{A}_{\sigma_02})^{K-1}\zeta_2(t^K)\end{aligned} \tag{49}$$

and $t^K \in (-d_2, 0]$. Similarly, by iterating the second equation in (47), $\zeta_2(t)$ can be rewritten as

$$\begin{aligned}\zeta_2(t) &= (\tilde{A}_{\sigma_02})^{K-1}\tilde{A}_{\sigma_01}\zeta_1(t^K) + (\tilde{A}_{\sigma_02})^K\zeta_2(t^K)\\ &\quad + \sum_{j=0}^{K-1}(\tilde{A}_{\sigma_02})^j\tilde{A}_{\sigma_00}\zeta_1(t^j)\\ &\quad + \sum_{j=0}^{K-2}(\tilde{A}_{\sigma_02})^j\tilde{A}_{\sigma_01}\zeta_1(t^{j+1}).\end{aligned} \tag{50}$$

Define $\theta^0 \triangleq \theta$, $\theta^i \triangleq \theta^{i-1} - d(\theta^{i-1})$, $i \in \mathbb{Z}^+$. From (49), it follows that

$$\begin{aligned}\|\zeta_1(t)\| = \|&\zeta_1(0) + \tilde{A}_{\sigma_02}(\tilde{A}_{\sigma_02})^{K-2}\tilde{A}_{\sigma_01}\zeta_1(t^K)\\ &+ \tilde{A}_{\sigma_02}(\tilde{A}_{\sigma_02})^{K-1}\zeta_2(t^K)\\ &+ \int_0^t [\tilde{A}_{\sigma_00}\zeta_1(\theta^0) + (\tilde{A}_{\sigma_01} + \tilde{A}_{\sigma_02}\tilde{A}_{\sigma_00})\zeta_1(\theta^1)\\ &+ \cdots + \tilde{A}_{\sigma_02}((\tilde{A}_{\sigma_02})^{K-4}\tilde{A}_{\sigma_01} + (\tilde{A}_{\sigma_02})^{K-3}\tilde{A}_{\sigma_01})\\ &\times \zeta_1(\theta^{K-2}) + \tilde{A}_{\sigma_02}((\tilde{A}_{\sigma_02})^{K-3}\tilde{A}_{\sigma_01}\\ &+ (\tilde{A}_{\sigma_02})^{K-2}\tilde{A}_{\sigma_00})\zeta_1(\theta^{K-1})]\,\mathrm{d}\theta\|.\end{aligned} \tag{51}$$

Set

$$\bar{\zeta}_1(t) \triangleq \sup_{\vartheta \in [-d_2, t]} \|\zeta_1(\vartheta)\|, \ \forall t \in [0, d_2].$$

Since $\|\zeta_1(\theta^0)\| \le \bar{\zeta}_1(\theta)$, $\|\zeta_1(\theta^1)\| \le \bar{\zeta}_1(\theta)$, …, $\|\zeta_1(\theta^{K-1})\| \le \bar{\zeta}_1(\theta)$, $\forall\theta^0 \in [0, t]$, we have from (51) that

$$\begin{aligned}\|\zeta_1(t)\| \le\ &\|\varphi_1\|_{d_2} + \|\tilde{A}_{\sigma_02}(\tilde{A}_{\sigma_02})^{K-2}\tilde{A}_{\sigma_01}\|\ \|\varphi_1\|_{d_2}\\ &+ \|\tilde{A}_{\sigma_02}(\tilde{A}_{\sigma_02})^{K-1}\|\ \|\varphi_2\|_{d_2}\\ &+ \int_0^t [\ \|\tilde{A}_{\sigma_00}\|\ \bar{\zeta}_1(\theta) + \|\tilde{A}_{\sigma_01} + \tilde{A}_{\sigma_02}\tilde{A}_{\sigma_00}\|\ \bar{\zeta}_1(\theta)\\ &+ \cdots + \|\tilde{A}_{\sigma_02}((\tilde{A}_{\sigma_02})^{K-4}\tilde{A}_{\sigma_01}\\ &+ (\tilde{A}_{\sigma_02})^{K-3}\tilde{A}_{\sigma_01})\|\ \bar{\zeta}_1(\theta)\\ &+ \|\tilde{A}_{\sigma_02}((\tilde{A}_{\sigma_02})^{K-3}\tilde{A}_{\sigma_01}\\ &+ (\tilde{A}_{\sigma_02})^{K-2}\tilde{A}_{\sigma_00})\|\ \bar{\zeta}_1(\theta)]\,\mathrm{d}\theta\\ =\ &\eta_{1\sigma_0} + \int_0^t \eta_{2\sigma_0}\bar{\zeta}_1(\theta)\,\mathrm{d}\theta.\end{aligned} \tag{52}$$

Note that for all $\vartheta \in [0, t]$, we have

$$\|\zeta_1(\vartheta)\| \le \eta_{1\sigma_0} + \int_0^\vartheta \eta_{2\sigma_0}\bar{\zeta}_1(\theta)\,\mathrm{d}\theta \le \eta_{1\sigma_0} + \int_0^t \eta_{2\sigma_0}\bar{\zeta}_1(\theta)\,\mathrm{d}\theta,$$

where the second inequality comes from the increase of the non-negative function $\bar{\zeta}_1(t)$. Hence,

$$\begin{aligned}\bar{\zeta}_1(t) &= \sup_{\vartheta \in [-d_2, t]} \|\zeta_1(\vartheta)\|\\ &\le \max\left\{\sup_{\vartheta \in [-d_2, 0]} \|\zeta_1(\vartheta)\|,\ \sup_{\vartheta \in [0, t]} \|\zeta_1(\vartheta)\|\right\}\\ &\le \max\left\{\|\varphi_1\|_{d_2},\ \eta_{1\sigma_0} + \int_0^t \eta_{2\sigma_0}\bar{\zeta}_1(\theta)\,\mathrm{d}\theta\right\}\\ &= \eta_{1\sigma_0} + \int_0^t \eta_{2\sigma_0}\bar{\zeta}_1(\theta)\,\mathrm{d}\theta.\end{aligned} \tag{53}$$

Applying the Gronwall–Bellman lemma, we obtain from (53) that for any $t \in [0,\ d_2]$

$$\bar{\zeta}_1(t) \le \eta_{1\sigma_0}e^{d_2\eta_{2\sigma_0}}, \ \forall t \in [0, d_2]. \tag{54}$$

On the other hand, from (50), it follows that

$$\begin{aligned}\|\zeta_2(t)\| \le\ &\|\tilde{A}_{\sigma_02}^{K-1}\|\ \|\tilde{A}_{\sigma_01}\|\ \|\varphi_1\|_{\bar{d}} + \|\tilde{A}_{\sigma_02}^K\|\ \|\varphi_2\|_{\bar{d}}\\ &+ \|\tilde{A}_{\sigma_00}\| \sum_{j=0}^{K-1} \|\tilde{A}_{\sigma_02}^j\|\ \|\zeta_1(t^j)\|\\ &+ \|\tilde{A}_{\sigma_01}\| \sum_{j=0}^{K-2} \|\tilde{A}_{\sigma_02}^j\|\ \|\zeta_1(t^{j+1})\|.\end{aligned}$$

Consequently,

$$\sup_{\vartheta \in [0, d_2]} \| \zeta_2(\vartheta) \| \leq \| \tilde{A}_{\sigma_0 2}^{K-1} \| \| \tilde{A}_{\sigma_0 1} \| \| \varphi_1 \|_{d_2} + \| \tilde{A}_{\sigma_0 2}^{K} \| \| \varphi$$

$$+ \left(\| \tilde{A}_{\sigma_0 0} \| \sum_{j=0}^{K-1} \| \tilde{A}_{\sigma_0 2}^{j} \| + \| \tilde{A}_{\sigma_0 1} \| \sum_{j=0}^{K-2} \| \tilde{A}_{\sigma_0 2}^{j} \| \right) \eta_1 e^{d_2 \eta_2}$$

$$= \eta_{3\sigma_0} + \eta_{4\sigma_0} \eta_{1\sigma_0} e^{d_2 \eta_{2\sigma_0}}.$$

Therefore,

$$\begin{aligned} &\sup_{\vartheta \in [-d_2, d_2]} \| \zeta_2(\vartheta) \| \\ &\leq \max \left\{ \sup_{\vartheta \in [-d_2, 0]} \| \zeta_2(\vartheta) \|, \sup_{\vartheta \in [0, d_2]} \| \zeta_2(\vartheta) \| \right\} \\ &\leq \max \left\{ \| \varphi_2 \|_{d_2}, \eta_{3\sigma_0} + \eta_{4\sigma_0} \eta_{1\sigma_0} e^{d_2 \eta_{2\sigma_0}} \right\}. \end{aligned} \quad (55)$$

By (54) and (55), we finally have

$$\begin{aligned} \sup_{\vartheta \in [-d_2, d_2]} \| \zeta(\vartheta) \| &\leq \sup_{\vartheta \in [-d_2, d_2]} \| \zeta_1(\vartheta) \| + \sup_{\vartheta \in [-d_2, d_2]} \| \zeta_2(\\ &\leq \eta_{1\sigma_0} e^{d_2 \eta_{2\sigma_0}} + \max \{ \| \varphi_2 \|_{d_2}, \\ &\quad \eta_{3\sigma_0} + \eta_{4\sigma_0} \eta_1 e^{d_2 \eta_{2\sigma_0}} \}, \end{aligned}$$

which completes the proof. □

Proof of Lemma 2: It is readily seen that (9) holds for $t \in [\tilde{t}_0 - d_2, \tilde{t}_0]$. The remaining proof is similar to that of Lemma 4 in [24]. For any $t \in (\tilde{t}_0, T]$, according to (7), there exists a positive integer K such that $t^K \leq \tilde{t}_0$ and $t^{K-1} > \tilde{t}_0$. Iterating K times (9) yields

$$\tilde{V}_2(t) \leq \sum_{i=0}^{K-1} q_0 q_2^i \tilde{V}_1(t^i) + \sum_{i=1}^{K} q_1 q_2^{i-1} \tilde{V}_1(t^i) + q_2^K \tilde{V}_2(t^K).$$

In view of (10) and (11), $\tilde{V}_2(t)$ can be bounded by

$$\tilde{V}_2(t) \leq H_1(q_0 + q_1) \sum_{i=0}^{K-1} q_2^i + q_2^K H_2 \leq \frac{q_0 + q_1}{1 - q_2} H_1 + q_2^K H_2.$$

Since $q_2 \in (0, 1)$ and $T \geq \tilde{t}_0$, (12) follows from the above inequality. This completes the proof. □