International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

Algebraic and transfer-function criteria of fixed-time controllability of delay-differential systems†

A. K. CHOUDHURY [a]
[a] Goddard Space Flight Center , Greenbelt, Md, 20771, U.S.A.
Published online: 24 Oct 2007.

INT. J. CONTROL, 1972, VOL. 16, NO. 6, 1073–1082

Algebraic and transfer-function criteria of fixed-time controllability of delay-differential systems†

A. K. CHOUDHURY‡
Goddard Space Flight Center, Greenbelt, Md., 20771, U.S.A.

[Received 2 December 1971]

Sufficient algebraic and transfer-function criteria of fixed-time controllability of linear time-invariant delay-differential system of the form

$$\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t),$$
$$x(t) \in R^n, \quad u(t) \in R^m$$

are given. It is shown that these criteria are also necessary if the system is pointwise complete. It is known that if (1) rank $B=1$, (2) rank $B=n$, (3) $n=2$, the above system is always point-wise complete and in these cases the algebraic and transfer function criteria of fixed-time controllability are both necessary and sufficient.

1. Introduction

Weiss (1967) introduced the concept of fixed-time controllability of a delay-differential system, and he obtained a sufficient condition of fixed-time controllability, expressed in terms of the kernel function of the delay-differential system. He showed that the sufficient condition is also a necessary condition of fixed-time controllability of the delay-differential system if the system is assumed to be point-wise complete.

In this paper we shall show that the criterion obtained by Weiss is equivalent to algebraic and transfer-function criteria, which are sufficient for fixed-time controllability of a delay-differential system, and these criteria are also necessary if the system is point-wise complete. These criteria are easier to verify than the kernel function criteria obtained by Weiss. Earlier work on the controllability of delay-differential systems was done by Kirillova and Curacova (1967) and they obtained necessary conditions (which are not sufficient) and sufficient conditions (which are not necessary) of complete controllability in a different sense. Recently, the N.A.S.C. of controlling any solution of a linear time-invariant delay-differential system to a terminal function has been obtained by Popov (1970), and the result is expressed in terms of the transfer function. An algebraic N.A.S.C. for the usual (point-wise as well as functional) concepts of complete controllability is still not known.

2. Definitions and notations

Consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t), \quad t > 0, \tag{1}$$

where

$$x(t) \in R^n, \quad u(t) \in R^m.$$

† Communicated by the Author.

‡ Now at the General Electric Company, Space Division, Beltsville, Maryland 20705.

R^n and R^m are Euclidean spaces of dimensions n and m. A, B and C are constant matrices of dimensions $n \times n$, $n \times n$ and $n \times m$ respectively, and A^T denotes the transpose of the matrix A. h is a positive real number. $u(t)$ is a piece-wise continuous control function and $u_{[t_0, t_1]}$ denotes the control function in the closed interval $[t_0, t_1]$, i.e.

$$u_{[t_0, t_1]} = \{u(t) \; ; \; t \in [t_0, t_1]\}.$$

The piece-wise continuous control function $u(t)$ will be called an admissible control function. The initial function space is $\beta = C([-h, 0] \to R^n)$ (the space of continuous functions mapping $[-h, 0]$ into R^n). The solution of (1) exists and is unique for $t > 0$, if one specifies an initial function $x(t) = g(t)$; for $t \in [-h, 0]$, where $g \in \beta$. We shall denote by $x(t \; ; \; g, u_{[0, t]})$ the solution of eqn. (1) at time t which corresponds to the initial condition g, and the control function $u_{[0, t]}$. We introduce the kernel matrix $K(t-s)$ which occurs in the general solution of eqn. (1) expressed as

$$\begin{aligned} x(t \; ; \; g, u_{[0, t]}) &= x(t \; ; \; g, 0) + \int_0^t K(t-s)Cu(s)\, ds \\ &= K(t)g(0) + \int_{-h}^0 K(t-s-h)Bg(s)\, ds + \int_0^t K(t-s)Cu(s)\, ds, \end{aligned} \tag{2}$$

where $x(t \; ; \; g, 0)$ is the solution of the homogeneous equation

$$\dot{x}(t) = Ax(t) + Bx(t-h) \tag{3}$$

corresponding to the initial function $g \in \beta$. $K(t-s)$ satisfies the following equations (Bellman and Cooke 1963) :

$$\frac{\partial K}{\partial s} = -K(t-s)A - K(t-(s+h))B, \quad 0 \leqslant s \leqslant t-h, \tag{4}$$

$$\frac{\partial K}{\partial s} = -K(t-s)A, \quad t-h \leqslant s \leqslant t. \tag{5}$$

$K(0) = I$ (the identity matrix of appropriate dimensions).

Definition

The system (1) is said to be fixed-time completely controllable if there exists a number $t_1 > 0$, such that for every $g \in \beta$, there exists a piece-wise continuous control segment $u_{[0, t]}$ (depending on g) such that

$$x(t_1 \; ; \; g, u_{[0, t_1]}) = 0 \quad \text{(Weiss 1967)}.$$

In order to obtain a necessary condition of fixed-time complete controllability of the delay-differential system, Weiss introduced the following concept of point-wise completeness of the system (3).

Definition

The system (3) is said to be point-wise complete at time $t_1 > 0$, if for all $y \in R^n$, there exists a $g \in \beta$, such that

$$x(t_1 \; ; \; g, 0) = y.$$

We introduce a matrix Q of the form

$$Q=[Q_1{}^1C,\ Q_1{}^2C,\ Q_2{}^2C,\ Q_1{}^3C,\ Q_2{}^3C,\ Q_3{}^3C,\ \dots\ Q_1{}^nC,\ Q_2{}^nC,\ \dots,\ Q_n{}^nC], \tag{6}$$

where

$$Q_1{}^1=I, \quad \text{and} \quad Q_i{}^k=0 \quad \text{for} \quad i=0 \quad \text{or} \quad i>k \tag{7}$$

and

$$Q_i{}^{r+1}=AQ_i{}^r+BQ_{i-1}\,. \tag{8}$$

The $Q_i{}^r$ above are the same as in Kirillova and Curacova (1967). We shall denote by f.t.c.c. the property of fixed-time complete controllability.

3. Algebraic criterion of fixed-time complete controllability

In this section we shall obtain an algebraic criterion which is sufficient for fixed-time controllability of the delay-differential system and if the system is point-wise complete, the criterion is also necessary†. Let us now show the equivalence of the following chain of implications :

(c_1) Rank $Q=n$.

(c_2) For all $t_1>nh$,

$$\operatorname{rank} \int_0^{t_1} K(t_1-s)CC^{\mathrm{T}}K^{\mathrm{T}}(t_1-s)\,ds=n.$$

(c_3) There does not exist any n-vector $d\neq 0$, such that the variable

$$\eta(t)\triangleq d^{\mathrm{T}}x(t\ ;\ g,\ u_{[0,\ t]}) \tag{9}$$

satisfies an equation of the form

$$\sum_{j=0}^{N}\ \sum_{i=0}^{N-j} c_{ji}\eta^i(t-jh)=0, \quad t>(N-1)h \tag{10}$$

for every solution $x(t\ ;\ g)$ of eqn. (1), $g\in\beta$, where c_{ji} are constants and $\max|c_{ji}|\neq 0$, and N is a positive integer, and $\eta^i(t)=\dfrac{d^i}{dti}(\eta(t))$.

Theorem 1. $(c_1)\Rightarrow(c_2)\Rightarrow(c_3)\Rightarrow(c_1)$.

Proof of $(c_1)\Rightarrow(c_2)$.

Same as in Kirillova and Curacova (1967) and Weiss (1967).

Proof of $(c_2)\Rightarrow(c_3)$.

We shall show that if both non (c_3) and (c_2) are supposed to be true, we obtain a contradiction. Non (c_3) stands for the negation of Property (c_3).

† It can be shown (Popov, private communication) that under the following circumstances (1) $B=bc^{\mathrm{T}}$, b and c are n-vectors, (2) rank $B=n$, (3) $n=2$, the system (3) is always point-wise complete and in these cases the algebraic and transfer criteria are both necessary and sufficient conditions of fixed-time complete controllability (Choudhury 1972).

Since non (c_3) is supposed to be true, there exists a non-zero n-vector d such that the variable

$$\eta(t) = d^{\mathrm{T}}x(t\ ;\ g,\ u_{[0,\ t]})$$

satisfies eqn. (10). Equation (10) shows that $\eta(t)$ does not depend on the control function $u(t)$ for $t > (N-1)h$ and $\eta(t)$ depends on the control function only through the initial function of the above equation defined for

$$-h \leqslant t \leqslant (N-1)h.$$

But from eqn. (2), we have

$$\begin{aligned}\eta(t) &= d^{\mathrm{T}}x(t\ ;\ g,\ u_{[0,\ t]}) \\ &= d^{\mathrm{T}}x(t\ ;\ g,\ 0) + d^{\mathrm{T}} \int_0^{(N-1)h} K(t-s)Cu(s)\,ds \\ &\qquad + d^{\mathrm{T}} \int_{(N-1)h}^{t} K(t-s)Cu(s)\,ds. \qquad (11)\end{aligned}$$

Since (c_2) is supposed to be true, and the system is time-invariant, we must have

$$d^{\mathrm{T}}K(t_1 - s)C \neq 0, \quad t_1 \geqslant nh, \quad \forall s \in [0,\ t_1]. \qquad (12)$$

Taking into account eqn. (12) and the fact that we can choose $u(t)$, $t > (N-1)h$ as we please, we see that we can make the influence of the last term

$$d^{\mathrm{T}} \int_{(N-1)h}^{t} K(t-s)Cu(s)\,ds$$

appearing in eqn. (11) on $\eta(t)$ non-zero and hence a contradiction.

Proof of $(c_3) \Rightarrow (c_1)$.

We shall prove that $(c_3) \Rightarrow (c_1)$ by showing that non $(c_1) \Rightarrow$ non (c_3). Non (c_1) means that there exists a non-zero n-vector d such that

$$d^{\mathrm{T}}Q_i{}^kC = 0, \quad k = 1,\ 2,\ 3,\ \ldots,\ n, \quad i = 1,\ 2,\ 3,\ \ldots,\ K. \qquad (13)$$

We have the following equations for $\eta(t)$ and its derivatives obtained by differentiating eqn. (9) successively, and using eqns. (1), (7), (8), (13). We assume that $t > (N-1)h$, so that all the variables are well defined :

$$\eta(t) = d^{\mathrm{T}}Q_1{}^1x(t), \qquad (14)$$

$$\eta^{(1)}(t) = d^{\mathrm{T}}Q_1{}^2x(t) + d^{\mathrm{T}}Q_2{}^2x(t-h), \qquad (15)$$

$$\eta^{(2)}(t) = d^{\mathrm{T}}Q_1{}^3x(t) + d^{\mathrm{T}}Q_2{}^3x(t-h) + d^{\mathrm{T}}Q_3{}^3x(t-2h) \qquad (16)$$

and in general

$$\eta^{(N)}(t) = d^{\mathrm{T}}Q_1{}^{N+1}x(t) + d^{\mathrm{T}}Q_2{}^{N+1}x(t-h) + \ \ldots\ + d^{\mathrm{T}}Q_{N+1}{}^{N+1}x(t-Nh). \qquad (17)$$

Replacing t by $(t-h)$ in the first N of the above equations, we obtain the following equations :

$$\eta(t-h) = d^{\mathrm{T}}Q_1{}^1x(t-h), \qquad (18)$$

$$\eta^{(1)}(t-h) = d^{\mathrm{T}}Q_1{}^2x(t-h) + d^{\mathrm{T}}Q_2{}^2x(t-2h), \qquad (19)$$

$$\eta^{(2)}(t-h)=d^{\mathrm{T}}Q_1{}^3x(t-h)+d^{\mathrm{T}}Q_2{}^3x(t-2h)+d^{\mathrm{T}}Q_3{}^3x(t-3h), \tag{20}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\eta^{(N-1)}(t-h)=d^{\mathrm{T}}Q_1{}^Nx(t-h)+d^{\mathrm{T}}Q_2{}^Nx(t-2h)+\ \ldots\ +d^{\mathrm{T}}Q_N{}^Nx(t-Nh). \tag{21}$$

Repeating the same process successively we obtain $(N-1)$ new equations and then $(N-2)$ new equations and so on until we get the following last two stages of this process :

$$\eta(t-(N-1)h)=d^{\mathrm{T}}Q_1{}^1x(t-(N-1)h), \tag{22}$$

$$\eta^{(1)}(t-(N-1)h)=d^{\mathrm{T}}Q_1{}^2x(t-(N-1)h)+d^{\mathrm{T}}Q_2{}^2x(t-Nh), \tag{23}$$

$$\eta(t-Nh)=d^{\mathrm{T}}Q_1{}^1x(t-Nh), \quad t>(N-1)h. \tag{24}$$

The above $(N+1)+N+(N-1)+\ \ldots\ +2+1=(N+2)(N+1)/2$ equations can be written in the matrix form $y=D\tilde{x}(t)$, where

$$\tilde{x}(t)=\begin{pmatrix} x(t) \\ x(t-h) \\ x(t-2h) \\ \vdots \\ x(t-Nh) \end{pmatrix},$$

$$D=\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1N(n+1)} \\ d_{21} & d_{22} & \cdots & d_{2N(n+1)} \\ d_{(N+1)[1+(N/2)]1} & d_{(N+1)[1+(N/2)]2} & \cdots & d_{(N+1)[1+(N/2)]N(n+1)} \end{pmatrix}$$

and

$$y^{\mathrm{T}}(t)=(\eta(t),\eta^{(1)}(t),\ \ldots,\ \eta^{N}(t)\ ;\ \eta(t-h),\eta^{(1)}(t-h),\ \ldots,\ \eta^{(N-1)}(t-h)\ ;$$
$$\ldots\ ;\ \eta(t-(N-1)h),\eta^{(1)}(t-(N-1)h)\ ;\ \eta(t-Nh)).$$

Since D is an $(N+1)(1+N/2)\times n(N+1)$ matrix, there exists an N such that $(N+1)(1+N/2)>n(N+1)$ and for such an N there exists a non-zero $(N+1)(1+N/2)$-vector

$$c^{\mathrm{T}}=(c_{00},c_{01},c_{02},\ \ldots,\ c_{0N}\ ;\ c_{10},c_{11},c_{12},\ \ldots,\ c_{1N-1}\ ;\ c_{21},c_{22},,$$
$$\ldots c_{2N-2}\ ;\ \ldots,\ c_{N0})$$

such that

$$c^{\mathrm{T}}D\tilde{x}(t)=c^{\mathrm{T}}y(t)=0$$

or

$$\sum_{j=0}^{N}\sum_{i=0}^{N-j}c_{ji}\eta^{(i)}(t-jh)=0, \quad t>(N-1)h$$

which is non (c_3) and hence proof of $(c_3)\Rightarrow(c_1)$ is complete.

Theorem 2

A sufficient condition of fixed-time complete controllability of the system (1) is

$$\operatorname{rank} Q=n.$$

The above condition is also necessary if the system is point-wise complete.

Proof of theorem 2.

This follows by combining Theorem 1 above and Lemma 2 of Weiss (1967).

4. Transfer function criterion of f.t.c.c.

In this section we obtain the transfer function criterion of f.t.c.c. The criterion is sufficient for fixed-time complete controllability and is necessary if the system (1) is point-wise complete†. The transfer function version is particularly interesting as it shows the close connection existing between the N.A.S.C. of f.t.c.c. and that of controlling any solution of (1) to a final target, recently obtained by Popov (1970).

Theorem 3

In order that the system (1) be fixed-time completely controllable, it is sufficient that there does not exist any non-zero n-vector d such that

$$d^{\mathrm{T}}G(s)=0$$

for all s except a denumerable set, where $G(s)$ is given by

$$G(s)=[sI-A-\exp(-sh)B]^{-1}C.$$

Let (c_4) and (c_5) denote the following properties :

(c_4) : There does not exist a non-zero n-vector d, such that

$$d^{\mathrm{T}}G(s)=0$$

for all s except a denumerable set,

(c_5) : There does not exist a non-zero n-vector d, such that

$$d^{\mathrm{T}}r(s,z)=0$$

for all s, z such that $\det(sI-A-zB)\neq 0$, where $r(s,z)$ is given by

$$sr(s,z)=Ar(s,z)+Bzr(s,z)+C. \tag{25}$$

We observe that

$$r[s,\exp(-sh)]=G(s).$$

We prove Theorem 3 by showing the following chain of implications.

Theorem 4. $(c_2)\Rightarrow(c_4)\Rightarrow(c_5)\Rightarrow c_1$.

We prove that $(c_2)\Rightarrow(c_4)$ by showing that non $(c_4)\Rightarrow$non (c_2). Non (c_4) implies that there exists a non-zero n-vector d such that

$$d^{\mathrm{T}}r(s,z)=0 \quad \text{for all} \quad s,z \tag{26}$$

such that $\det(sI-A-zB)\neq 0$. Multiplying both sides of eqn. (25) by s^{q-1}

† See footnote on page 1075.

and replacing $sr(s, z)$ by $(A+zB)r(s, z)+C$ in the right-hand side successively, we obtain, using eqn. (26), for any positive integer q,

$$d^{\mathrm{T}}s^{q}r(s, z)=d^{\mathrm{T}}[(A+zB)^{q}r(s, z)+(A+zB)^{q-1}C+s(A+zB)^{q-2}C + s^{2}(A+zB)^{q-3}C+\ \dots\ +s^{q-2}(A+zB)C+s^{q-1}C]=0. \tag{27}$$

Dividing eqn. (27) by s^{q-1}, and letting $s\to\infty$, it follows that

$$d^{\mathrm{T}}C=0, \quad \text{since} \quad r(s, z)\to 0 \quad \text{as} \quad s\to\infty. \tag{28}$$

Using eqn. (28) and dividing eqn. (27) by s^{q-2}, it follows as before by letting $s\to\infty$, that

$$d^{\mathrm{T}}(A+zB)C=0. \tag{29}$$

Using the same reasoning successively, we obtain

$$d^{\mathrm{T}}(A+zB)^{i-1}C=0, \quad i=1, 2, 3, \dots, q. \tag{30}$$

Equation (30) implies that $d^{\mathrm{T}}Q=0$, which is non (c_2).

Proof of $(c_4)\Rightarrow(c_5)$.

We prove that $(c_4)\Rightarrow(c_5)$, by showing that non $(c_5)\Rightarrow$non (c_4). Non (c_5) implies that there exist a non-zero n-vector d such that

$$d^{\mathrm{T}}r[s, \exp(-sh)]=d^{\mathrm{T}}G(s)=0$$

for all s except a denumerable set. $d^{\mathrm{T}}G(s)$ can be expressed as

$$d^{\mathrm{T}}G(s)=d^{\mathrm{T}}(sI-A-\exp(-sh)B)^{-1}C = \frac{P[s, \exp(-sh)]}{\det(sI-A-\exp(-sh)B)} \tag{31}$$

where $P[s, \exp(-sh)]$ is a row vector whose elements can be expressed as

$$P_i[s, \exp(-sh)]=\sum_{k} P_{ik}(s)\exp(-ksh),\ P_{ik}(s)$$

are polynomials in s of degrees at most $(n-1)$, $i=1, 2, 3, \dots, m$ and k is finite. Now one can see that $P_i[s, \exp(-sh)]=0$, for all s except a denumerable set implies that $P_{ik}(s)=0$, for all s except a denumerable set. This shows that $P(s, z)=0$, and thus

$$d^{\mathrm{T}}G(s, z)=\frac{P(s, z)}{\det(sI-A-zB)}=0$$

which is non (c_5).

Proof of $(c_5)\Rightarrow(c_1)$.

Finally we prove that $(c_5)\Rightarrow(c_1)$ by showing that non $(c_1)\Rightarrow$non (c_5). Non (c_1) means that the variable

$$\eta(t)=d^{\mathrm{T}}x(t\ ;\ g, u_{[0,\ t]}), \quad d\neq 0$$

satisfies an equation of the form

$$\sum_{j=0}^{N} \sum_{i=0}^{N-j} c_{ji}\eta^{(i)}(t-jh)=0, \quad t>(N-1)h,$$

where c_{ji} are constants and max $|c_{ji}| \neq 0$, for every solution of eqn. (1). This implies that

$$\sum_{j=0}^{N} \sum_{i=0}^{N-j} c_{ji}d^{\mathrm{T}}x^{(i)}(t-jh)=0 \tag{32}$$

for every pair of functions $x(t)$, $u(t)$ satisfying eqn. (1), and in particular for $x(t)=G(s)\exp(-sh)E_m$, $u(t)=\exp(-sh)E_m$, (E_m is an m-dimensional unit vector), for all s except $\det(sI-A-\exp(-sh)B)=0$. This gives us that

$$\sum_{j=0}^{N} \sum_{i=0}^{N-j} c_{ji}s^i \exp(-jsh)\, d^{\mathrm{T}}G(s)=0, \tag{33}$$

except a denumerable set.

Equation (33) shows that there exists a non-zero n-vector d, such that $d^{\mathrm{T}}G(s)=0$, except the set of points which are the zeros of the functions

$$\sum_{j=0}^{N} \sum_{i=0}^{N-j} c_{ji}s^i \exp(-jsh)$$

and $\det(sI-A-\exp(-sh)B)$. This set is denumerable (Bellman and Cooke 1963), and therefore

$$(c_5)\Rightarrow(c_1).$$

It is well known that in the case of systems without delay of the form

$$\dot{x}(t)=Ax(t)+Bu(t),$$

where A and B are constant matrices, the N.A.S.C. of complete controllability is equivalent to the condition that there is no constant n-vector $d \neq 0$, such that $d^{\mathrm{T}}H(s)=0$, where $H(s)$ is the transfer function of the above system given by $H(s)=(sI-A)^{-1}B$. Popov (1970) has shown that the N.A.S.C. of the stronger property of controlling any initial function of (1) belonging to $[-h, 0]$ to a terminal function is equivalent to the more restrictive condition. There is no polynomial vector $d(s)\neq 0$, such that

$$d^{\mathrm{T}}(s)G(s)=0.$$

Thus, a great deal of uniformity of different criteria of controllability is obtained when one expresses these criteria in terms of the transfer function of the systems.

Acknowledgment

The author gratefully acknowledges the helpful suggestions and patience of Professor V. M. Popov and Dr. L. Weiss during numerous discussions.

References

Bellman, R., and Cooke, K. L., 1963, *Differential Difference Equations* (New York : Academic Press).

Choudhury, A. K., 1970, *Proc. Fourth Annual Princeton Conference on Information Science and Systems*, p. 184 ; 1972, *Int. J. Control* (in the press).

Kirillova, F. M., and Curakova, S. V., 1967, *Diff. Urav*, **3**, 456 (English translation, *Differential Equations*).

Popov, V. M., 1970, University of Maryland Technical Report, R–70–08.

Weiss, L., 1967, *J. SIAM Control*, **5**, 575.