Bài tập tuần 5

Bài 20

1) $y = x\sqrt{1+x^2}$

$y' = \sqrt{1+x^2} + x\dfrac{2x}{2\sqrt{1+x^2}} = \dfrac{1+x^2+x^2}{\sqrt{1+x^2}} = \dfrac{1+2x^2}{\sqrt{1+x^2}}$

$y'' = \dfrac{4x\sqrt{1+x^2} - (1+2x^2)\frac{x}{\sqrt{1+x^2}}}{(\sqrt{1+x^2})^2} = \dfrac{4x(x^2+1) - x - 2x^3}{(1+x^2)\sqrt{1+x^2}} = \dfrac{2x^3+3x}{(1+x^2)\sqrt{1+x^2}}$

2) $y = \dfrac{x}{\sqrt{1-x^2}}$

$y' = \dfrac{\sqrt{1-x^2} + x\frac{x}{\sqrt{1-x^2}}}{1-x^2} = \dfrac{1-x^2+x^2}{(1-x^2)\sqrt{1-x^2}} = \dfrac{1}{(1-x^2)^{3/2}} = (1-x^2)^{-\frac{3}{2}}$

$y'' = -\dfrac{3}{2}(1-x^2)'(1-x^2)^{-\frac{3}{2}-1} = -\dfrac{3}{2}(-2x)\dfrac{1}{(1-x^2)^{\frac{5}{2}}} = \dfrac{3x}{(1-x^2)^{\frac{5}{2}}}$

3) $y = e^{-x^2}$

$y' = -2x\,e^{-x^2}$ $\qquad y'' = -2(e^{-x^2} - 2x^2 e^{-x^2}) = 2e^{-x^2}(2x^2-1)$

4) $y = \ln f(x)$

$y' = \dfrac{f'(x)}{f(x)}$ $\qquad y'' = \dfrac{f''(x)f(x) - f'(x)f'(x)}{f^2(x)} = \dfrac{f''(x)f(x) - f'^2(x)}{f^2(x)}$

Bài 21

1) $x_t = 2t - t^2 \rightarrow x'_t = 2 - 2t = 2(1-t)$

$y_t = 3t - t^3 \rightarrow y'_t = 3 - 3t^2 = 3(1-t^2) = 3(1-t)(1+t)$

$y'_x = \dfrac{y'_t}{x'_t} = \dfrac{3(1-t)(1+t)}{2(1-t)} = \dfrac{3}{2}(1+t)$

$y''_{xx} = \dfrac{(y'_x)'}{x'_t} = \dfrac{3}{2}(1+t)'\dfrac{1}{2(1-t)} = \dfrac{3}{4}\dfrac{1}{(1-t)}$

2) $x_t = a\cos t \rightarrow x'_t = -a\sin t$

$y_t = a\sin t \rightarrow y'_t = a\cos t$

$y'_x = \dfrac{y'_t}{x'_t} = \dfrac{a\cos t}{-a\sin t} = -\cot t$

$y''_{xx} = \dfrac{(y'_x)'}{x'_t} = \dfrac{(-\cot t)'}{-a\sin t} = \dfrac{-1}{a\sin^3 t}$

B 3) $x_t = a(t - \sin t) \Rightarrow x'_t = a(1 - \cos t) = 2a\sin^2\frac{t}{2}$

$y_t = a(1 - \cos t) \Rightarrow y'_t = a\sin t = 2a\sin\frac{t}{2}\cos\frac{t}{2}$

$$y'_x = \frac{y'_t}{x'_t} = \frac{2a\sin\frac{t}{2}\cos\frac{t}{2}}{2a\sin^2\frac{t}{2}} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \cot\frac{t}{2}$$

$$y''_{xx} = \frac{(y'_x)'}{x'_t} = \frac{(\cot\frac{t}{2})'}{2a\sin^2\frac{t}{2}} = \frac{-1}{2\sin^2\frac{t}{2}\cdot 2a\sin^2\frac{t}{2}} = \frac{-1}{4\sin^4\frac{t}{2}}$$

Bài 22)

1) $y = \frac{x^2}{1-x}$ $\quad y^{(8)} = ?$

$$y = \frac{1}{1-x} - (x+1)$$

$$y^{(8)} = \left(\frac{1}{1-x}\right)^{8} - (x+1)^8 = (-1)^8 \cdot \frac{8!}{(1-x)^{8+1}} + 0 = \frac{8!}{(1-x)^9}$$

2) $y = \frac{1+x}{\sqrt{1-x}} = (1+x)(1-x)^{-\frac{1}{2}}$, $y^{(100)} = ?$

$f(x) = 1+x$

$g(x) = (1-x)^{-\frac{1}{2}}$

$$(f.g)^{100} = C^0_{100} f^{(0)} g^{(100)} + C^1_{100} f^{(1)} g^{(99)} + C^2_{100} f^{(2)} g^{(98)} + \dots + C^{100}_{100} f^{(100)} g^{(100)}$$

$$= C^0_{100} f^{(0)} g^{(100)} + C^1_{100} f^{(1)} g^{(99)} + 0$$

Có $\left((1-x)^{-\frac{1}{2}}\right)^{(n)} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2} - n + 1\right)(1-x)^{-\frac{1}{2}-n}$

$$= \frac{(-1)^n}{2^n}[1.3.5.\dots(\cancel{2}-2n)] \frac{1}{\sqrt{1-x}(1-x)^n}$$

$$= \frac{1.3.\dots(\cancel{2}\,2n)(2n-1)\cancel{(-1)^{n+1}}}{2^n\sqrt{1-x}\,(1-x)^n}$$

$$y^{(100)} = (1+x)\frac{(1.3.5\dots199)}{2^{100}\sqrt{1-x}\,(1-x)^{100}} + \frac{100.(1.3.5\dots197)}{2^{99}\sqrt{1-x}\,(1-x)^{99}}$$

3) $y = x^2 e^{2x} \qquad y^{(20)} = ?$

$f(x) = x^2 \qquad g(x) = e^{2x}.$

$(f.g)^{20} = C_{20}^{0} f^{(0)} g^{(20)} + C_{20}^{1} f^{(1)} . g^{(19)} + C_{20}^{2} f^{(2)} g^{(18)} + \dots + C_{20}^{20} f^{(20)} g^{(0)}$

$= C_{20}^{0} f^{(0)} g^{(20)} + C_{20}^{1} f^{(1)} g^{(19)} + C_{20}^{2} f^{(2)} g^{(18)} + 0$

$= x^2 . 2^{20} . e^{2x} + \cancel{20.2^{19}.2}\ 20.2x.2^{19}.e^{2x} + 190.2.2^{18} e^{2x}$

$= e^{2x}.2^{20}(x^2 \cancel{e^{2x}} + 20x + 95)$

4) $y = x^2 \sin 2x \qquad y^{(50)} = ?$

$f(x) = x^2 \qquad g(x) = \sin 2x.$

$(f.g)^{(50)} = C_{50}^{0} f^{(0)} g^{(50)} + C_{50}^{1} f^{(1)} g^{(49)} + C_{50}^{2} f^{(2)} g^{(48)} + \dots + C_{50}^{50} f^{(50)} g^{(0)}$

$= C_{50}^{0} f^{(0)} g^{(50)} + C_{50}^{1} f^{(1)} g^{(49)} + C_{50}^{2} g^{(2)} g^{(48)} + 0$

$= x^2\, 2^{50} . \sin(2x + \frac{50\pi}{2}) + 50.2x.2^{49} \sin(2x + \frac{49\pi}{2})$

$+ 1225 . 2 . 2^{48} \sin(2x + \frac{48\pi}{2})$

$= x^2 2^{50} \sin(2x + \pi) + 50x.2^{50} \sin(2x + \frac{\pi}{2}) + \frac{1225}{2} . 2^{50} \sin 2x.$

$= -x^2 . 2^{50} \sin(2x) + 50x.2^{50} \cos 2x + \frac{1225}{2} . 2^{50} \sin 2x.$

$= 2^{50}(-x^2 \sin 2x + 50x \cos 2x + \frac{1225}{2} \sin 2x)$

Bài 3.

1) $y = \frac{1}{x(1-x)} = \frac{x+1-x}{x(1-x)} = \frac{1}{1-x} + \frac{1}{x}.$

$y^{(n)} = \left(\frac{1}{1-x}\right)^{(n)} + \left(\frac{1}{x}\right)^{(n)} = \frac{n!}{(1-x)^{n+1}} + \frac{(-1)^n . n!}{x^{n+1}}.$

2) $y = \frac{1}{x^2-3x+2} = \frac{1}{(1-x)(2-x)} = \frac{2-x-(1-x)}{(1-x)(2-x)} = \frac{1}{1-x} \quad \frac{1}{2-x}$

$y^{(n)} = \left(\frac{1}{1-x}\right)^{(n)} - \left(\frac{1}{2-x}\right)^{(n)} = \frac{n!}{(1-x)^{n+1}} - \frac{n!}{(2-x)^{n+1}}$

3) $y = \dfrac{x}{\sqrt[3]{1+x}} = x(1+x)^{-\frac{1}{3}}$

$f(x) = x \qquad g(x) = (1+x)^{-\frac{1}{3}}$

$(f.g)^{(n)} = C_n^0 f^{(0)} g^{(n)} + C_n^1 f^{(1)} g^{(n-1)} + \dots + C_n^n f^{(n)} g^{(0)}$

$= C_n^0 f^{(0)} g^{(n)} + C_n^1 f^{(1)} g^{(n-1)} + 0 = x\left((1+x)^{-\frac{1}{3}}\right)^{(n)} + n.\left((1+x)^{-\frac{1}{3}}\right)^{n-1}$

có $\left((1+x)^{-\frac{1}{3}}\right)^{(n)} = -\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)\dots\left(-\frac{(3n-2)}{3}\right)\cdot\frac{1}{(1+x)^{n+\frac{1}{3}}}$

$= \frac{(-1)^{n+1}}{3^n}(1.4\dots(3n-2))\cdot\frac{1}{(1+x)^{n+\frac{1}{3}}}$

$\left((1+x)^{-\frac{1}{3}}\right)^{(n-1)} = \frac{(-1)^n}{3^{n-1}}(1.4\dots(3n-5))\,\frac{1}{(1+x)^{n+\frac{1}{3}}}$

có $y^{(n)} = x.\frac{(-1)^{n+1}}{3^n}(1.4\dots(3n-5)(3n-2))\,\frac{1}{(1+x)^{n+\frac{1}{3}}} +$

$n.\frac{(-1)^n}{3^n}.3.(1.4\dots(3n-5))\,\frac{1}{(1+x)^{n+\frac{1}{3}}}$

$= \frac{(-1)^n}{3^n}\cdot\frac{1}{(1+x)^{n+\frac{1}{3}}}.(1.4\dots(3n-5)).\left[(3n-2).(-1)x + 3n\right]$

$= \frac{(-1)^n}{3^n}\cdot\frac{1}{(1+x)^{n+\frac{1}{3}}}(1.4\dots(3n-5))\left[2x + 3nx + 3n\right]$

$= \frac{(-1)^n}{3^n}\cdot\frac{1}{(1+x)^{n+\frac{1}{3}}}(1.4\dots(3n-5))\left[2x + 3n(x+1)\right]$

Bài tập bổ sung

2.40 $f(x) = \dfrac{1}{x^2-3x+2}$ $\quad f^{(2008)}(0)$.

$$\cancel{\left(f(x) = \frac{1}{(1-x)(1-2)} = \frac{1}{x-2} + \frac{1}{x-1}\right.}$$

$$f(x) = \frac{1}{(1-x)(2-x)} = \frac{2-x-(1-x)}{(1-x)(2-x)} = \frac{1}{1-x} - \frac{1}{2-x}.$$

$$f^{(n)}(x) = \left(\frac{1}{1-x}\right)^{(n)} - \left(\frac{1}{2-x}\right)^{(n)} = \frac{n!}{(1-x)^{n+1}} - \frac{n!}{(2-x)^{n+1}}.$$

$$f^{(2008)}(0) = \frac{2008!}{(1-0)^{2008+1}} - \frac{2008!}{(2-0)^{2008+1}} = \frac{2008!}{1} - \frac{2008!}{2^{2009}}$$

2.41 $f(x) = \dfrac{1}{x^2}$. Tính $f^{(2008)}(1)$

$$f(x) = x^{-2}.$$

$$f^{(2008)}(x) = -2\,(-2-1)(-2-2)\ldots(-2-2008+1)\,x^{-2-2008}$$

$$= (-2)(-3)(-4)\ldots(-2009).\,x^{-2-2008}$$

$$f^{(2008)}(1) = (-2)(-3)(-4)\ldots(-2009).\,1^{-2010}$$

$$= [illegible]\,1)\ \ 2009!$$

2.42 $f(x) = \ln\dfrac{x+1}{x-1}$. $\quad f^{(2008)}(2)$.

$$f(x) = \ln(x+1) - \ln(x-1)$$

$$f^{(2008)}(x) = (\ln(x+1))^{(2008)} - (\ln(x-1))^{(2008)}$$

$$= (-1)^{2008-1}\frac{(2008-1)!}{(x+1)^{2008}} - (-1)^{2008-1}\frac{(2008-1)!}{(x-1)^{2008}}$$

$$= -\frac{2007!}{(x+1)^{2008}} + \frac{2007!}{(x-1)^{2008}}$$

$$f^{(2008)}(2) = \frac{2007!}{(2-1)^{2008}} - \frac{2007!}{(2+1)^{2008}} = 2007!\left(1 - \frac{1}{3^{2008}}\right)$$

2.43, $f(x) = \frac{1+x}{1-x}$ $\quad f^{(n)}(0) = ?$

$$f(x) = -1 + \frac{2}{1-x}$$

$$f^{(n)}(0) = \cancel{}(-1)^{(n)} + \left(\frac{2}{1-x}\right)^{(n)} = 0 + 2.\left(\frac{1}{1-x}\right)^{(n)}$$

$$= 2.\frac{(+1)^n . n!}{(1-x)^{n+1}} = \frac{2n!}{1} = 2n!$$

2.44 $f(x) = \ln\frac{1}{1-x}$ $\quad f^{(n)}(0) = ?$

$$f(x) = \ln(1) - \ln(1-x)$$

$$f^{(n)}(x) = (\ln(1))^{(n)} - \ln(1-x)^{(n)} = 0 - (-1)^{(n)-1}\frac{(n-1)!}{(1-x)^n}(-1)^n$$

$$= (-1)^{2n}\frac{(n-1)!}{(1-x)^n} \Rightarrow f^{(n)}(0) = (-1)^{2n}.(n-1)!$$

2.45. $y''(x) = ?$ $\quad y(x) = (1+x)^{\frac{1}{x}}$

$$\ln y(x) = \frac{1}{x}\ln(1+x).$$

Đạo hàm 2 vế

$$\frac{y'(x)}{y(x)} = \frac{-1}{x^2}\ln(1+x) + \frac{1}{x}\frac{1}{x+1} = \frac{1}{x^2+x} - \frac{1}{x^2}\ln(1+x)$$

$$y'(x) = \left(\frac{1}{x^2+x} - \frac{1}{x^2}\ln(1+x)\right)y(x)$$

$$y''(x) = \left(\frac{1}{x^2+x} - \frac{1}{x^2}\ln(1+x)\right)'y(x) + \left(\frac{1}{x^2+x} - \frac{1}{x}\ln(1+x)\right)y'(x)$$

$$= \underbrace{\left(\frac{1}{x^2+x} - \frac{1}{x^2}\ln(1+x)\right)'}_{I}y(x) + \left(\frac{1}{x^2+x} - \frac{1}{x}\ln(1+x)\right)^2.y(x)$$

$$Có\ I = \left(\frac{-2x-1}{(x^2+x)^2} - \left(\frac{-2}{x^3}\ln(1+x) + \frac{1}{x^2}\frac{1}{x+1}\right)\right.$$

$$= \frac{-2x-1}{x^2(x+1)^2} + \frac{2}{x^3}\ln(1+x) - \frac{1}{x^2(x+1)}$$

$$= \frac{(-2x-1)x + 2\ln(1+x)\cdot(x+1)^2 - x(x+1)}{x^3(x+1)^2}$$

$$= \frac{-2x^2-x+2\ln(1+x)\cdot(x+1)^2-x^2-x}{x^3(x+1)^2} = \frac{-3x^2-2x+2\ln(1+x)(1+x)^2}{x^3(x+1)^2}$$

$$y''(x) = y(x)\left[\frac{-3x^2-2x+2(1+x)^2\ln(1+x)}{x^3(x+1)^2} + \left(\frac{1-(x+1)\ln(1+x)}{x(x+1)}\right)^2\right]$$

$$= y(x)\cdot\left[\frac{-3x^2-2x+2(1+x^2)\ln(1+x)}{x} + (1-(x+1)\ln(1+x))^2\right]\cdot\frac{1}{x^2(x+1)^2}$$

$$= \left[\frac{-3x^2-2x+2(1+x^2)\ln(1+x)}{x} + (1-(x+1)\ln(1+x))^2\right]\frac{(1+x)^{\frac{1}{x}}}{x^2(x+1)^2}$$

2.89

$g(x) = \sin x\cdot(x^2+1)$ $\qquad g^{(20)}(x)$.

$f(x) = x^2+1$

$g(x) = \sin x$.

$$(f\cdot g)^{(20)} = C_{20}^0 f^{(0)}g^{(20)} + C_{20}^1 f^{(1)}g^{(19)} + C_{20}^2 f^{(2)}g^{(18)} + \dots + C_{20}^{19} f^{(19)}g^{(1)} + C_{20}^{20} f^{(20)}g^{(0)}$$

$$= C_{20}^0 f^{(0)}g^{(20)} + C_{20}^1 f^{(1)}g^{(19)} + C_{20}^2 f^{(2)}g^{(18)} + 0$$

$$= \cancel{(x^2+1)}\ (x^2+1)\left(\sin x + \frac{20\pi}{2}\right)\cdot 1^{20} + 20\cdot 2x\cdot 1^{19}\sin\left(x+\frac{19\pi}{2}\right)$$

$$+ 190\cdot 2\cdot 1^{18}\sin\left(x+\frac{18\pi}{2}\right)$$

$$= (x^2+1)\sin x + 40x\cos x - 380\sin x$$

$$= (x^2-379)\sin x + 40x\cos x$$

Một số bài tập trong đề thi các năm trước

Câu 2: $y(x) = (x^3 - 2x + 5)\cos(3x)$ $\quad y^{(2016)}(\pi)$

$f(x) = (x^3 - 2x + 5)$

$g(x) = \cos 3x$

$(f \cdot g)^{(2016)} = C^0_{2016} f^{(0)} g^{(2016)} + C^1_{2016} f^{(1)} g^{(2015)} + C^2_{2016} f^{(2)} g^{(2014)} + C^3_{2016} f^{(3)} g^{(2013)}$

$+ \dots + C^{2016}_{2016} f^{(2016)} g^{(0)}$

$= C^0_{2016} g^{(0)} g^{(2016)} + C^1_{2016} f^{(1)} g^{(2016)} + C^2_{2016} f^{(2)} g^{(2014)} + C^3_{2016} f^{(3)} g^{(2013)} + 0$

$= (x^3 - 2x + 5) \cdot 3^{2016} \cos\left(3x + \frac{2016\pi}{2}\right) + 2016(3x^2 - 2)\, 3^{2015} \cos\left(3x + \frac{2015\pi}{2}\right)$

$+ 2016 \cdot \frac{2015}{2} \cdot (6x)\, 3^{2014} \cdot \cos\left(3x + \frac{2014\pi}{2}\right) + C^3_{2016}\, 6 \cdot 3^{2013} \cos\left(3x + \frac{2013\pi}{2}\right)$

$\cancel{= (x^3 - 2x + 5)\, 3^{2016} \cdot \cos(3x) + 2016(3x^2 - 2)\, 3^{2015} \cos 3x}$
$\cancel{- 2016 \cdot \frac{2015}{2} \cdot 6x \cdot 3^{2014} \sin(3x)}$

$= (x^3 - 2x + 5)\, 3^{2016} \cos 3x + 2016(3x^2 - 2)\, 3^{2015} \sin 3x - C^2_{2016}\, 6x\, 3^{2014} \cos 3x$
$- C^3_{2016}\, 6\, 3^{2013} \sin 3x$

$\cancel{= (\pi^3 - 2\pi} \Rightarrow y^{(2016)}(\pi) = (\pi^3 - 2\pi + 5)\, 3^{2016} (-1) + 0 - C^2_{2016}\, 6\pi \cdot 3^{2014} (-1)$
$+ 0$

$= 3^{2016} \left[C^2_{2016} \frac{2\pi}{3} - (\pi^3 - 2\pi + 5) \right]$

Câu 3: $y(x) = (3x^2+1)\sin^2 x$ $\quad y^{(2017)}(\pi)$

$f(x) = 3x^2+1 \quad g(x) = \sin^2 x$

$$(f\cdot g)^{(2017)} = C_{2017}^0 f^{(0)} g^{(2017)} + C_{2017}^1 f^{(1)} g^{(2016)} + C_{2017}^2 f^{(2)} f^{(2015)} + \dots + C_{2017}^{2017} f^{(2017)} g^{(0)}$$

$$= C_{2017}^0 f^{(0)} g^{(2017)} + C_{2017}^1 f^{(1)} g^{(2016)} + C_{2017}^2 f^{(2)} g^{(2015)} + 0$$

Xét $g(x) = \sin^2 x = \frac{1}{2}(1-\cos 2x)$

$g'(x) = \sin 2x = -\cos\left(2x+\frac{\pi}{2}\right)$

$g''(x) = 2\sin\left(2x+\frac{\pi}{2}\right) = 2\cos(2x+\pi)$

$g'''(x) = -4\sin(2x+\pi) = -2^2\cos\left(2x+\frac{3\pi}{2}\right)$

$\Rightarrow g^{(n)}(x) = (-2)^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$

$$y^{(2017)}(x) = C_{2017}^0 (3x^2+1).(-2)^{2017-1}\cos\left(2x+\frac{2017\pi}{2}\right)$$

$$+ C_{2017}^1\, 6x\,.(-2)^{2016-1}\cos\left(2x+\frac{2016\pi}{2}\right) + C_{2017}^2\, 6\,.(-2)^{2015-1}\cos\left(2x+\frac{2015\pi}{2}\right)$$

$$= (3x^2+1).2^{2016}\cos\left(2x+\frac{2017\pi}{2}\right) + 2017.6x.2^{2015}.(-1)^{2015}\cos\left(2x+\frac{2016\pi}{2}\right)$$
$$+ C_{2017}^2\, 6.\,2^{2014}\cos\left(2x+\frac{2015\pi}{2}\right)$$

$$y^{(2017)}(\pi) = 0 - 2017.6\pi.2^{2015} + 0 = -2017.6\pi.2^{2015}$$

Câu 4: Tính đạo hàm $y''(x)$ $\begin{cases} x = t-\sin t \\ y = 1-\cos t \end{cases}$

làm rất tốt

$x' = 1-\cos t = 2\sin^2\frac{t}{2}$

$y' = \sin t = 2\sin\frac{t}{2}\cos\frac{t}{2}$

$$y'(x) = \frac{y'}{x'} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \cot\frac{t}{2}$$

$$y''(x) = \frac{(y'(x))'}{x'} = \frac{(\cot\frac{t}{2})'}{2\sin^2\frac{t}{2}} = \frac{-1}{2.2\sin^4\frac{t}{2}} = \frac{-1}{4\sin^4\frac{t}{2}}$$