Analysis and Control of Time Delay Systems Using the LambertWDDE Toolbox

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Abstract. This chapter provides an overview of the Lambert W function approach. The approach has been developed for analysis and control of linear time-invariant time delay systems with a single known delay. A solution in the time-domain is given in terms of an infinite series, with the important characteristic that truncating the series provides a dominant solution in terms of the rightmost eigenvalues. A solution via the Lambert W function approach is first presented for systems of order one, then extended to higher order systems using the matrix Lambert W function. Free and forced solutions are used to investigate key properties of time-delay systems, such as stability, controllability and observability. Through eigenvalue assignment, feedback controllers and state-observers are designed. All of these can be achieved using the Lambert W function-based framework. The use of the MATLAB-based open source software in the LambertWDDE Toolbox is also introduced using numerical examples.

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1 Introduction

Time delay systems (TDS) arise in numerous natural and engineered systems, such as processes with transport delays, traffic flow problems, biological systems, teleoperation and many others. The literature on TDS is quite extensive, and includes several excellent books and review papers, e.g. [2, 11, 12, 14, 17, 19, 20]. This chapter focuses on a specific and recently developed approach, based on the classical Lambert W function [5], for analysis and control of linear time invariant TDS with a single known delay [27].

1.1 Motivation and Background

Consider a typical n^{th}-order system of linear time invariant (LTI) ordinary differential equations (ODEs), without delay, in standard state equation form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{1}$$

the closed-form free and forced solutions can be obtained using the concepts of a state transition matrix and a convolution integral [4]. As the system of ODEs in Eq. (1) has a finite spectrum, the stability can be determined by examining the locations of the finite number of eigenvalues in the s-plane.

The solutions are also used to derive the controllability and observability Gramians. Then, controllability and observability can be determined by the rank of the controllability and observability matrices, respectively. If the system is controllable, a closed-loop controller can be designed by a variety of methods, including state feedback control and eigenvalue assignment. Similarly, if a system is observable, then a state estimator, or observer, can be designed, e.g., using eigenvalue assignment.

These systematic steps for analysis and control are standard for LTI systems of ODEs as in (1), because the closed-form solutions to Eq. (1) are obtained analytically. However, unlike ODEs, these steps are often difficult to achieve for LTI TDS due to lack of analytical time-domain solutions to delay differential equations. Main difficulty is due to their infinite spectrum arising from the delays. Recently methods, based on the Lambert W function, have been proposed, developed and demonstrated for the analysis and control of LTI TDS, which enable the analysis and control design steps outlined above to be applied in a manner analogous to LTI systems of ODEs [1, 28]. The key characteristic of the method is that the solution is given in terms of an infinite eigenvalue expansion, based on the branches of the Lambert W function, such that truncating the series always yields a finite dimensional representation in terms of the rightmost (i.e., dominant) eigenvalues.

There are numerous natural and engineered systems where time delays are significant (e.g., biological systems, economic models, supply chains, traffic flow, teleoperation, networked control systems, automotive control systems) [19]. Thus, benefits from the extension of the system analysis and control tools, which are standard for ODEs, to systems described by DDEs can be substantial.

1.2 Purpose and Scope

This chapter is intended to provide a succinct overview of the Lambert W function approach to the analysis and control of LTI TDS with a single known delay and introduction, via simple numerical examples, to the use of the open source software LambertWDDE Toolbox, which is available for downloading from the web [7]. Additional examples can be found at the same website [7] and numerous applications of the method can be found in the references cited (e.g., [8,21,24–27,32]). Note that a preliminary version of this chapter has been presented in [33].

2 Theory, Examples and Numerical Simulation

2.1 Lambert W Function

By definition [5, 10, 15], every function $W(s)$ that satisfies:

$$W(s)e^{W(s)} = s \tag{2}$$

is called a Lambert W function. The Lambert W function, with complex argument s, is a complex valued function with infinite branches, $k = 0, \pm 1, \pm 2, .., \pm\infty$, where s is either a scalar (i.e., scalar Lambert W function) or a matrix (i.e., matrix Lambert W function). The scalar Lambert W function is available as an embedded function in many computational software systems, e.g., see the function *lambertw* in MATLAB. The matrix Lambert W function [27] can be obtained using a similarity transformation and can be readily evaluated using the LambertWDDE Toolbox [7]. These functions are useful in combinatorics (e.g., the enumeration of trees) as well as relativity and quantum mechanics. They can be used to solve various equations involving exponentials (e.g. the maxima of the Planck, Bose-Einstein, and Fermi-Dirac distributions) as well as in the solution of delay differential equations as discussed here.

2.2 Scalar Case

Consider the first-order TDS [1]:

$$\dot{x}(t) = ax(t) + a_d x(t-h) + bu(t) \tag{3}$$

with constant known parameters a, b and a_d, and where h is the constant known delay. The initial condition $x(t = 0) = x_0$ and preshape function $x(t) = g(t)$ for

$-h \leq t < 0$, must be specified. The Lambert W function is applied to solve the transcendental characteristic equation of Eq. (3), which can be written as:

$$(s-a)e^{sh} = a_d \tag{4}$$

Multiplying both sides of Eq. (4) by he^{-ah} yields:

$$h(s-a)e^{h(s-a)} = a_d h e^{-ah} \tag{5}$$

Based on the definition of the Lambert W function in Eq. (5) it is clear that

$$W(a_d h e^{-ah})e^{W(a_d h e^{-ah})} = a_d h e^{-ah} \tag{6}$$

Comparing Eqs. (5) and (6)

$$h(s-a) = W(a_d h e^{-ah}) \tag{7}$$

Thus, the solution of the characteristic equation in Eq. (4) can be written in terms of the Lambert W function as:

$$s = \frac{1}{h}W(a_d h e^{-ah}) + a \tag{8}$$

The infinite spectrum of the scalar DDE in (3) is, thus, obtained using the infinite branches of the Lambert W function, and is given explicitly in terms of parameters a, a_d and h of the system. The roots of the characteristic equation (4), for $k = 0, \pm 1, \pm 2, .., \pm\infty$, are:

$$s_k = \frac{1}{h}W_k(a_d h e^{-ah}) + a \tag{9}$$

Furthermore, for Eq. (3) stability is determined by the rightmost eigenvalue in the s-plane, which has been shown in [18] to be obtained using only the principal (i.e., $k = 0$) branch of the Lambert W function. Consequently, to ensure stability, it is not necessary to check the other eigenvalues in the infinite spectrum.

2.3 Example 1 - Spectrum and Series Expansion in the Scalar Case

For $a = -1$, $a_d = 0.5$, and $h = 1$ the characteristic roots are obtained using Eq. (9) and the function *lambertw* in MATLAB, and are plotted in Fig. 1. It can also be shown [28] that the total (i.e., free plus forced) solution to Eq. (3), can be represented in terms of an infinite series based on the eigenvalues in Eq. (9) as:

$$x(t) = \sum_{k=-\infty}^{+\infty} e^{s_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{+\infty} e^{s_k(t-\eta)} C_k^N b u(\eta) \mathrm{d}\eta \tag{10}$$

where

$$C_k^I = \frac{x_0 + a_d e^{-s_k h} \int_0^h e^{-s_k t} g(t-h)\mathrm{d}t}{1 + a_d h e^{-s_k h}} \tag{11}$$

and

$$C_k^N = \frac{1}{1 + a_d h e^{-s_k h}} \tag{12}$$

Note that the coefficients C_k^I are determined from the preshape function $g(t)$ and the initial state x_0, and the coefficients C_k^N are determined only in terms of the system parameters a, a_d and h. Thus, the total solution in Eq. (10) can be viewed as the sum of the free and forced solutions. Conditions for convergence of such a series solution form are discussed in [2]. A very practically important and useful aspect of this particular series representation of the solution for $x(t)$ is that truncating the series, e.g., $k = 0, \pm 1, \pm 2, .., \pm n$, yields an approximation of the solution in terms of the $(2n+1)$ rightmost, or most dominant, eigenvalues. Consequently, a simple finite dimensional approximation of the system accurately represents its dynamics.

2.4 Example 2 - Scalar Case Approximation Response

For $a = -1$, $a_d = 0.5$, and $h = 1$ one can obtain the values of s_k using Eq. (9) and the function *lambertw* as in Ex. 1, and the values of C_k^I and C_k^N using Eqs. (11)-(12), where $x_0 = 1$ and $g(t) = 1$ for $-h \le t < 0$. These are given in Table 1. Fig. 2 shows the total response to $u(t) = sin(t)$ and a comparison between the Lambert W function-based method (using the 7 terms in Table 1) and a numerical solution (using the function $dde23$ in MATLAB). The two plots are essentially indistinguishable.

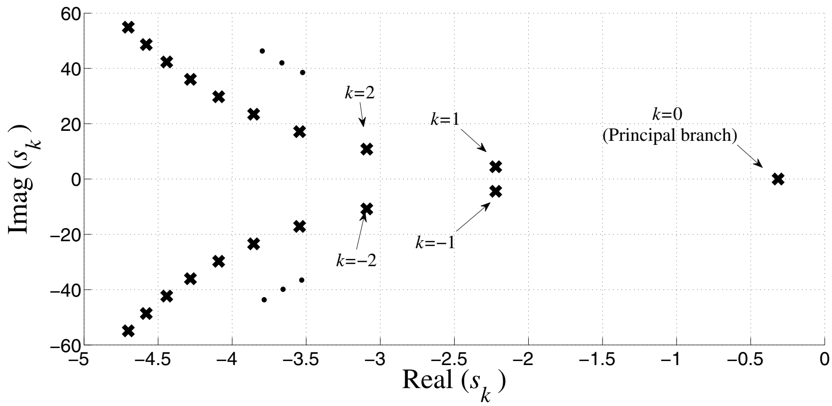

Fig. 1 Eigenvalue of Eq. (3) when $a = -1$, $a_d = 0.5$, and $h = 1$. The rightmost eigenvalue is for $k = 0$, the next pair are for $k = \pm 1$, the next for $k = \pm 2$, to $k = \pm 9$.

Table 1 The eigenvalues and coefficients in the solution for Ex. 2

k	s_k	C_k^I	C_k^N
0	-0.3149	0.9422	0.5934
±1	$-2.2211 \pm 4.4442i$	$0.0197 \pm 0.0111i$	$-0.0112 \pm 0.2245i$
±2	$-3.0915 \pm 10.8044i$	$0.0038 \pm 0.0015i$	$-0.0093 \pm 0.0916i$
±3	$-3.545 \pm 17.1313i$	$0.0016 \pm 0.0005i$	$-0.0052 \pm 0.0579i$

2.5 General Case

The approach presented in the previous section has been generalized in [27] to LTI TDS of the form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned} \tag{13}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ is the input vector, $\mathbf{y}(t)$ is the output vector, $\mathbf{A}$, $\mathbf{A_d}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are coefficient matrices, and h is the constant known scalar delay. The initial condition $\mathbf{x}(t=0) = \mathbf{x}_0$ and preshape function $\mathbf{x}(t) = \mathbf{g}(t)$ for $-h \leq t < 0$, must also be specified. The total solution for the states is now given as:

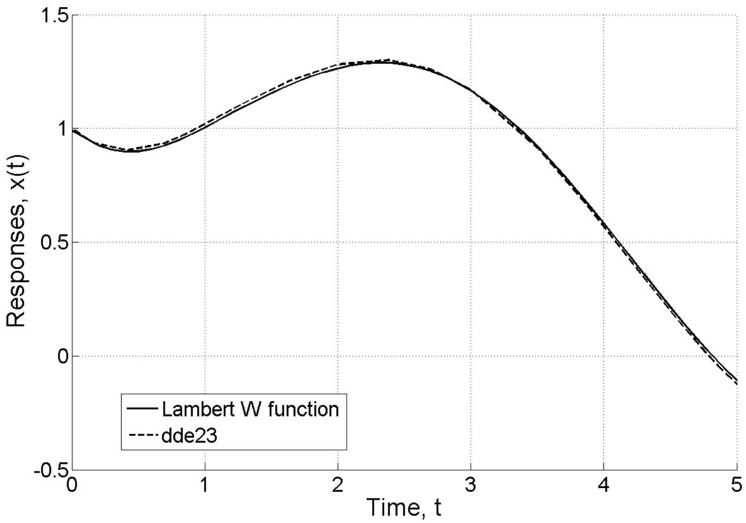

Fig. 2 Total response to $u(t) = sin(t)$, with $x_0 = 1$ and $g(t) = 1$ for $-h \leq t < 0$, and comparison between the 7-term (see Table 1) Lambert W function-based method and the numerical method (function $dde23$ in MATLAB). Parameters are $a = -1$, $a_d = 0.5$, and $h = 1$.

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\eta)}\mathbf{C}_k^N \mathbf{B}\mathbf{u}(\eta)d\eta \tag{14}$$

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where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A}_d h \mathbf{Q}_k) + \mathbf{A} \tag{15}$$

and $\mathbf{Q}_k$ is obtained from numerical solution (e.g., using *fsolve* in MATLAB) of:

$$\mathbf{W}_k(\mathbf{A}_d h \mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A}_d h \mathbf{Q}_k)+\mathbf{A}h} = \mathbf{A}_d h \tag{16}$$

This generalization is dependent on the introduction of a matrix Lambert W function, $\mathbf{W}_k$, as described in [27]. The quantities $\mathbf{Q}_k$, $\mathbf{W}_k$, $\mathbf{S}_k$, $\mathbf{C}_k^I$ and $\mathbf{C}_k^N$ in Eqs. (14)-(16) can all be computed using the software in the LambertWDDE Toolbox in terms of given h, $\mathbf{A}$, $\mathbf{A}_d$, $\mathbf{g}(t)$, $\mathbf{x}_0$, $\mathbf{B}$ and $\mathbf{u}(t)$. The main functions of the LambertWDDE Toolbox [7] are summarized in Table 2.

Table 2 Main functions of the LambertWDDE Toolbox [7]

Name	Description
lambertw_matrix	Calculate matrix Lambert W function
find_Sk	Find $\mathbf{S}_k$ and $\mathbf{Q}_k$ for a given branch
find_CI	Calculate $\mathbf{C}_I$ under specific initial conditions for a given branch
find_CN	Calculate $\mathbf{C}_N$ for a given branch
pwcont_test	Controllability test for DDEs
pwobs_test	Observability test for DDEs
cont_gramian_dde	Controllability Gramian for DDEs
obser_gramian_dde	Observability Gramian for DDEs
place_dde	Rightmost eigenvalue assignment for DDEs
stabilityradius_dde	Calculate stability radius for DDEs
examples	Lists examples for using this toolbox; each cell is a short example and can be evaluated separately (Ctr+Enter)

2.6 *Example 3 - General Case Approximation*

To obtain $\mathbf{S}_k$ for a particular branch, k, one needs to solve Eq. (16) for $\mathbf{Q}_k$ first, then substitute the result into Eq. (15) to obtain $\mathbf{S}_k$. The steps are carried out in the function *find_Sk*. Note that, the matrix Lambert W function $\mathbf{W}_k(\cdot)$ in Eq. (15) and (16) is calculated using the function *lambertw_matrix*, which is based on a Jordan canonical form transformation. To better understand the whole process, an example is provided here. Given

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}; \mathbf{A_d} = \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.33 \end{bmatrix}; h = 1$$

For the principal branch, $k = 0$, one obtains:

$$\mathbf{S}_0 = \begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix}$$

The eigenvalues of which are -1.0119 and -1.9841. Next, using *find_CI* and *find_CN*, one can obtain the coefficients for the series solution in Eq. (14). For example, if $u(t) = 0$, $\mathbf{g}(t) = 0$, and we have an abrupt change at $t = 0$ to $\mathbf{x}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, we can obtain (using *find_CI*) the coefficients for the free response for $k = 0$ as:

$$\mathbf{C}_0^I = \begin{bmatrix} 0.2635 \\ 0.4290 \end{bmatrix}$$

Thus, the single branch approximation, for $k = 0$, for the free response is:

$$\mathbf{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = e^{\begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix} t} \begin{Bmatrix} 0.2635 \\ 0.4290 \end{Bmatrix}$$

Note that in MATLAB the matrix exponential is evaluated using the function *expm*, not the scalar exponential function *exp*. To improve the approximation, this process can be repeated for additional branches, k, then an approximate series solution can be obtained using Eq. (14) with a finite number of k (see Fig. 3). For example, including the branches $k = \pm 1$ gives the additional complex conjugate $\mathbf{S}_k$ matrices:

$$\mathbf{S}_{-1,+1} = \begin{bmatrix} -0.399 \pm 4.980i & -1.6253 \pm 0.1459i \\ 2.4174 \pm 0.1308i & -5.1048 \pm 4.5592i \end{bmatrix}$$

with complex conjugate coefficients for the free response for $k = \pm 1$ as:

$$\mathbf{C}_{-1,+1}^I = \begin{bmatrix} 0.0909 \pm 0.1457i \\ 0.0435 \pm 0.1938i \end{bmatrix}$$

2.7 Observability and Controllability

In [25], the criteria for point-wise controllability and observability have been derived as follows.

Point-wise Controllability: The system of DDEs in Eq. (13) is point-wise controllable if, for any given initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, there exists a time t_1, $0 < t_1 < \infty$, and an admissible (i.e., measurable and bounded on a finite time interval) control segment $\mathbf{u}(t)$ for such that $\mathbf{x}(t_1; \mathbf{g}, \mathbf{x}_0, \mathbf{u}(t)) = 0$ [22]. For the scalar DDE in Eq. (3) it is point-wise controllable if and only if, for all s not at the poles of the system, $s - a - a_d e^{-sh} \neq 0$; similarly for Eq. (13) one must have linearly independent rows of $(s\mathbf{I} - \mathbf{A} - \mathbf{A}_d e^{-sh})^{-1}\mathbf{B}$. Furthermore, the controllability Gramian $\mathbf{C}(0, t_1) = \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\eta)} \mathbf{C}_k^N \mathbf{B}\mathbf{B}^T \{e^{\mathbf{S}_k(t_1-\eta)} \mathbf{C}_k^N\}^T d\eta$ for Eq. (13) must be full rank [27].

Point-wise Observability: The system of DDEs in Eq. (13) is point-wise observable in $[0,t_1]$ if the initial point $\mathbf{x}_0$ can be uniquely determined from the knowledge of $\mathbf{u}(t)$, $\mathbf{g}(t)$, and $\mathbf{y}(t)$ [6]. For the scalar DDE in Eq. (3), it is point-wise observable if and only if, for all s not at the poles of the system; similarly, for Eq. (13) one must have linearly independent columns of $\mathbf{C}(s\mathbf{I}-\mathbf{A}-\mathbf{A}_d e^{-sh})^{-1}$. Furthermore, the observability Gramian $\mathbf{O}(0,t_1)=\int_0^{t_1}\sum_{k=-\infty}^{\infty}\{e^{\mathbf{S}_k(t_1-\eta)}\mathbf{C}_k^N\}^T\mathbf{C}^T\mathbf{C}e^{\mathbf{S}_k(t_1-\eta)}\mathbf{C}_k^N d\eta$ for Eq. (13) must be full rank [27].

2.8 Example 4 - Piecewise Observability and Controllability

Consider Eq. (13), with $\mathbf{A}$, $\mathbf{A}_d$ and h as given in Ex. 3, and

$$\mathbf{B}=\left\{\begin{matrix}1\\0\end{matrix}\right\} \quad \text{and} \quad \mathbf{C}=\begin{bmatrix}0 & 1\end{bmatrix}$$

The function *pwcontr_test* can be used to establish that the system is piecewise controllable. It examines the rank of the matrix $(s\mathbf{I}-\mathbf{A}-\mathbf{A}_d e^{-sh})^{-1}\mathbf{B}$ to determine if the system is piecewise controllable, and will display the conclusion on the screen. The Piecewise observability can be established using the function *pwobs_test* in a similar way. Furthermore, the controllability and observability Gramians over a specific time interval can be approximately computed, for $k=n$ branches, using the functions *contr_gramian_dde* and *obs_gramian_dde* respectively.

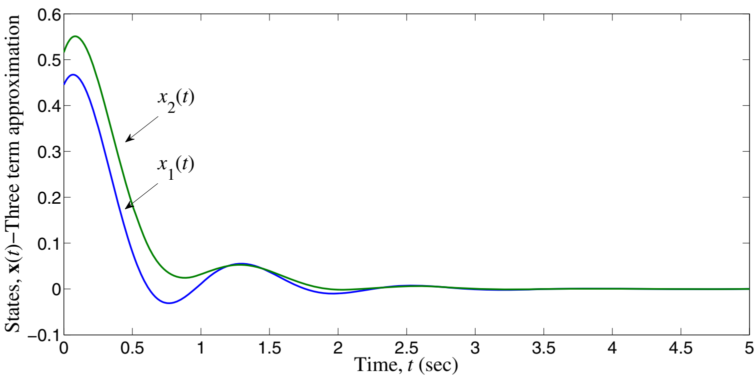

Fig. 3 Approximate (3-term) free response for the system in Ex. 3

2.9 Placement of Dominant Poles

In [28], a method for eigenvalue assignment via Lambert W function was proposed. Consider the scalar LTI TDS in Eq. (3) with the generalized feedback structure

$$u(t)=-Kx(t)-K_d x(t-h)$$

The closed-loop system becomes

$$\dot{x}(t) = (a - bK)x(t) + (a_d - bK_d)x(t-h)$$

One can use the Lambert W function approach to assign the rightmost eigenvalues of the system. The procedure for selecting the gains K and K_d can be described as:

1. Select desired rightmost eigenvalue $\lambda_{desired}$
2. Set initial gains $K = K_0$ and $K_d = K_{d0}$
3. while $\lambda(S0_{\_new} - \lambda_{desired}) > tolerance$
4. Select $K = K_{new}$ and $K_d = K_{d\_new}$
5. Let $a_{new} = (a - bK_{new})$, $\quad a_{d\_new} = (a_d - bK_{d\_new})$, calculate $S0_{\_new} = \frac{1}{h}W_0(a_{d\_new}he^{-a_{new}h}) + a_{new}$.
6. End

Due to the range limitation of the branches of the Lambert W function, the rightmost poles cannot be assigned to any arbitrary location in the s-plane. For scalar time-delay systems, as in Eq. (3), this can be easily seen by examining the principal branch [27].

$$\mathrm{Re}\{\frac{1}{h}W_0(a_d he^{-ah}) + a\} \geq \mathrm{Re}\{-\frac{1}{h} + a\} \geq -\frac{1}{h} + a$$

since $\mathrm{Re}\{W_0(H)\} \geq -1$. Thus the rightmost pole cannot be less than $-\frac{1}{h} + a$. For the matrix case, similar constraints apply but the relationship becomes more complicated. This feasibility constraint has to be considered in the design process (e.g., in the selection of $\lambda_{desired}$) for the method to succeed. The generalization of this approach to systems of DDEs, as in Eq. (13), is presented in [27, 28] and applied to both controller and observer design problems.

2.10 Example 5 - Rightmost Eigenvalue Assignment

For $a = -1, a_d = 0.5, b = 1$, and $h = 1$ the rightmost eigenvalue can be assigned to any value > -2. Here we consider $\lambda_{desired} = -1.5$, and use the function *place_dde* to obtain the controller gains:

$$K = 1.1378, K_d = 0.3576$$

Thus, the closed-loop LTI TDS becomes:

$$\dot{x}(t) = -2.1378x(t) + 0.1424x(t-1)$$

and the rightmost eigenvalue can be found, using $k = 0$ in the function *lambertw*, as in Ex. 1, to now be located at -1.4998 as desired.

2.11 Robust Control and Time Domain Specifications

The assignment of rightmost eigenvalues for LTI TDS can also be used for observer design, and extended to robust design in the presence of structured model uncertainties. Since the response of the LTI TDS is dominated by the rightmost eigenvalues, approximate specification of time domain characteristics (e.g., settling time, overshoot) can also be achieved [29]. The Toolbox function *stabilityradius_dde* can be used to calculate the stability radius for DDEs as described in [13, 27, 29].

2.12 Decay Function for TDS

Accurate estimation of the decay function for time delay systems has been a long-standing problem, which has recently been addressed using the Lambert W function based approach [8]. The goal is to find a tight upper bound for the decay rate, which is referred to as α-stability, as well as an upper bound for the factor K, such that the norm of the states is bounded:

$$\|\mathbf{x}(t)\| \leq Ke^{\alpha t}\Phi(h,t_0) \tag{17}$$

where $\Phi(h,t_0) = \sup_{t_0-h\leq t\leq t_0} \{\|\mathbf{x}(t)\|\}$ and $\|\cdot\|$ denotes the 2-norm. Based on the solution form in Eq. (14) in terms of the Lambert W function, an optimal estimate of α can be obtained. The estimate of the factor K is also less conservative especially for the matrix case. A less conservative estimate of the decay function leads to a more accurate description of the transient response, and more efficient control strategies based on the decay model [8].

2.13 Example 6 - Factor and Decay Rate

Consider the system in Eq. (13) with the same coefficients as in Ex. 3. From Eq. (15), with $k = 0$, the rightmost pole is found to be:

$$\begin{aligned}\alpha &= \max\{\mathrm{Re}(eig(\mathbf{S}_0))\}\\ &= \max\left\{\mathrm{Re}(eig(\tfrac{1}{h}\mathbf{W}_0(-\mathbf{A}_d h\mathbf{Q}_0) - \mathbf{A}))\right\} = -1.012\end{aligned}$$

Thus the exact decay rate is obtained and, using the solution in Eq. (14) and the approach in [8], one can also obtain a bound on K. As shown in Table 3, the results are less conservative when compared to other methods [8].

Table 3 Comparison of results for Ex. 6

	Factor, K	Decay Rate, α
Matrix Measure Approach [12]	8.019	3.053
Lyapunov Approach [16]	9.33	-0.907
Lambert-W Approach [8]	3.8	-1.012

3 Concluding Remarks

A summary of the recently developed Lambert W function approach for analysis and control of LTI TDS with a constant delay was provided in this chapter. For more details, readers are referred to the publications cited here (e.g., [27]) and the web [7]. Several numerical examples are given to illustrate the use of the *lambertw* function in MATLAB, as well as other useful functions available in the open source LambertWDDE Toolbox software package for LTI DDEs [7].

The proposed approach can be used, just as for systems of LTI ODEs as in Eq. (1), for a variety of important analysis and control tasks for LTI DDEs, such as free and forced solutions, stability, observability and controllability, controller and observer design via assignment of dominant eigenvalues, robust stability, determination of the decay function, etc. The open source software in the LambertWDDE Toolbox, as well as the accompanying documentation and examples [7], we hope will make the Lambert W function based approach more accessible and useful for those interested in applications that are well modeled as LTI TDS with a single constant delay. Numerous applications of the method (e.g., machine tool chatter, engine control, HIV dynamics, decay function estimation, DC motor control, PID control and robust control) can also be found in the references [8, 21, 24–28, 32]. Besides the LambertWDDE Toolbox, other useful software for time-delay systems based on a variety of algorithms is also available for downloading from the web [3, 9, 23].

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