system uncertainties are eliminated, and good tracking performance is achieved. To eliminate chatterings, we implement the boundary layer control law (3.17). Here we take $\delta = 0.01$. Good system performance is shown in Fig. 4. As can be seen from these figures, the chattering is eliminated.

V. Conclusions

In this paper, a robust control scheme for rigid robotic manipulators using the MIMO terminal sliding mode technique has been proposed. The main contributions of this paper are that an MIMO terminal sliding mode is defined, and a robust terminal sliding mode control scheme for n-link rigid robotic manipulators is developed with the result that the output tracking error can converge to zero in a finite time. In addition, the robot control systems using the proposed scheme have a strong robustness property not only because on the sliding mode, the error dynamics is insensitive to uncertain dynamics, but also because only three uncertain bounds based on the structure properties of rigid robotic manipulators are used in controller design. It has also been remarked that this scheme is more practical in the sense that the gain of the terminal sliding mode controller can be significantly reduced with respect to the ones of linear sliding mode control schemes developed in [19], where the sampling interval is nonzero. A few problems for the practical implementation of this scheme, however, have been noted. Like all other control techniques, the ideal error convergence cannot be obtained in practical control systems where sampling interval is nonzero. To implement this scheme, some nonlinear electronic hardware to deal with the nonlinear function ϵ_i^p needs to be built. The advantage that the proposed terminal sliding mode controller has smaller control gain with respect to linear sliding mode controller, however, cannot be counteracted by the above factors. In this paper, the terminal sliding mode technique is used to control only the second order robotic systems. The research on design of high-order terminal sliding mode control systems, however, is under author's investigations based on [12] and [13].

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A Common Lyapunov Function for Stable LTI Systems with Commuting A-Matrices

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***Abstract*—The paper demonstrates that a common quadratic Lyapunov function exists for all linear systems of the form $\dot{x} = A_i x$, $i = 1, 2, \cdots, N$, where the matrices A_i are asymptotically stable and commute pairwise. This in turn assures the exponential stability of a switching system $\dot{x}(t) = A(t)x(t)$ where $A(t)$ switches between the above constant matrices A_i.**

I. Introduction

In recent years, the scope of control theory is being enlarged to include intelligent control systems. One of the main features of such intelligent control systems is the systematic application of the idea of switching between different controllers [1], [2]. One of the first questions to be resolved in this context is that of the stability of the overall system.

Many of the stability problems that arise in intelligent control systems can be addressed by considering the following basic problem:

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Consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t), \quad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and the matrix $A(t)$ switches between stable matrices $A_1, A_2, \cdots, A_N$ belonging to the set $\mathcal{A} \triangleq \{A_1, A_2, \cdots, A_N\}$. We shall refer to this as the switching system. Our objective is to find necessary and sufficient conditions for the asymptotic stability of the equilibrium state of the switching system. Within this class of problems, an interesting question that arises frequently is when system (1) is exponentially stable for any arbitrary switching sequence between the elements of $\mathcal{A}$. It is well known that if the individual systems $\dot{x} = A_i x$, $i = 1, 2, \cdots, N$ share a common quadratic Lyapunov function of the form $V(x) = x^T P x$ with a negative definite time-derivative for each $i = 1, 2, \cdots, N$, then system (1) is exponentially stable for any arbitrary switching sequence. Hence, the conditions under which system (1) is exponentially stable and those under which a common quadratic Lyapunov function exists are both of interest in intelligent control.

In this note, we consider a special case of the above problem, one in which the matrices A_i commute pairwise. It is shown that a common Lyapunov function exists for this class of matrices. The main idea is given in Theorem 1, in which $\mathcal{A}$ contains only two matrices A_1 and A_2. The principal result is contained in item ii) where an explicit method of generating a common Lyapunov function is presented. This is generalized in Theorem 2 to the case where $\mathcal{A}$ contains a finite number of matrices.

The idea of switching is well known in control theory. In the past decade, switching has been used in adaptive control to assure stability in situations where stability could not be proved otherwise [3]–[6]. In [7] switching between multiple adaptive models was used to improve the transient response of adaptive control systems in a stable fashion. It seems reasonable to predict that stable switching between controllers will be used with increasing frequency to improve performance in intelligent control systems.

II. Main Results

Theorem 1: Consider the switching system (1) with $\mathcal{A} = \{A_1, A_2\}$. Assume that A_1 and A_2 are asymptotically stable matrices such that $A_1 A_2 = A_2 A_1$. Then

i) The system is exponentially stable for any arbitrary switching sequence between the elements of $\mathcal{A}$.

ii) Given a symmetric positive definite matrix P_0, let P_1 and P_2 be the unique symmetric positive definite solutions to the Lyapunov equations

$$A_1^T P_1 + P_1 A_1 = -P_0 \tag{2}$$
$$A_2^T P_2 + P_2 A_2 = -P_1. \tag{3}$$

Then the function $V(x) = x^T P_2 x$ is a common Lyapunov function for both the individual systems $\dot{x} = A_i x$, $i = 1, 2$, and hence a Lyapunov function for the switching system (1).

iii) For a given choice of the matrix P_0, the matrices A_1 and A_2 can be chosen in any order in (2), (3) to yield the same solution P_2, i.e., if

$$A_2^T P_3 + P_3 A_2 = -P_0 \tag{4}$$

then

$$A_1^T P_2 + P_2 A_1 = -P_3. \tag{5}$$

iv) The matrix P_2 can also be expressed in integral form as

$$P_2 = \int_0^\infty e^{A_2^T t}\left[\int_0^\infty e^{A_1^T \tau} P_0 e^{A_1 \tau}\, d\tau\right] e^{A_2 t}\, dt$$
$$= \int_0^\infty e^{A_1^T t}\left[\int_0^\infty e^{A_2^T \tau} P_0 e^{A_2 \tau}\, d\tau\right] e^{A_1 t}\, dt.$$

Proof:

i) This can be proved directly by using the fact that if A_1 and A_2 commute, then $e^{A_1 t} e^{A_2 \tau} = e^{A_2 \tau} e^{A_1 t}$, for all t, τ. Exponential stability also follows from ii) below.

ii) Let $V(x) = x^T P_2 x$. If $\dot{x} = A_2 x$, using (3) the derivative of V along the trajectories of this system is seen to be $\dot{V} = x^T(A_2^T P_2 + P_2 A_2)x = -x^T P_1 x < 0$, showing that V is a Lyapunov function for this system. Now, the derivative of V along the trajectories of the system $\dot{x} = A_1 x$ is given by $\dot{V} = x^T(A_1^T P_2 + P_2 A_1)x$. It remains to show that this is negative definite. Substituting for P_1 from (3) into (2) and using the commutativity of A_1 and A_2, we get

$$P_0 = A_1^T(A_2^T P_2 + P_2 A_2) + (A_2^T P_2 + P_2 A_2)A_1$$
$$= A_2^T(A_1^T P_2 + P_2 A_1) + (A_1^T P_2 + P_2 A_1)A_2. \tag{6}$$

Since A_2 is stable and $P_0 > 0$, this shows that $A_1^T P_2 + P_2 A_1 < 0$, as required.

Finally, the derivative of V along the trajectories of the switching system (1) is given by

$$\dot{V} = x^T(A^T(t)P_2 + P_2 A(t))x = \begin{cases} x^T(A_2^T P_2 + P_2 A_2)x = -x^T P_1 x < 0, & A(t) = A_2 \\ x^T(A_1^T P_2 + P_2 A_1)x < 0, & A(t) = A_1 \end{cases}$$

and hence V is also a Lyapunov function for the switching system.

iii) Since P_3 is the solution to the Lyapunov equation (4), it is positive definite. Hence there is a unique positive definite solution P_4 to the Lyapunov equation

$$A_1^T P_4 + P_4 A_1 = -P_3. \tag{7}$$

The statement will be proved if we show that $P_2 = P_4$. Using (4) and (7) and the commutativity of A_1 and A_2, it follows that

$$P_0 = A_1^T(A_2^T P_4 + P_4 A_2) + (A_2^T P_4 + P_4 A_2)A_1. \tag{8}$$

From (6) and (8), $A_2^T P_2 + P_2 A_2$ and $A_2^T P_4 + P_4 A_2$ are the unique solutions of the same Lyapunov equation. Hence $A_2^T(P_2 - P_4) + (P_2 - P_4)A_2 = 0$. Since A_2 is stable, $P_2 = P_4$.

iv) The solution to the Lyapunov equation (2) is known to be $P_1 = \int_0^\infty e^{A_1^T \tau} P_0 e^{A_1 \tau}\, d\tau$. Hence the solution P_2 to the Lyapunov equation (3) is $\int_0^\infty e^{A_2^T t}[\int_0^\infty e^{A_1^T \tau} P_0 e^{A_1 \tau}\, d\tau] e^{A_2 t}\, dt$. The second form for P_2 can be derived similarly using the second characterization of P_2 given by (4) and (5). ■

Theorem 2: Consider the switching system (1) with $\mathcal{A} = \{A_1, A_2, \cdots, A_N\}$, where the matrices A_i are asymptotically stable and commute pairwise. Then

i) The system is exponentially stable for any arbitrary switching sequence between the elements of $\mathcal{A}$.

ii) Given a symmetric positive definite matrix P_0, let $P_1, P_2, \cdots, P_N$ be the unique symmetric positive definite solutions to the Lyapunov equations

$$A_i^T P_i + P_i A_i = -P_{i-1}, \quad i = 1, 2, \cdots, N. \tag{9}$$

Then the function $V(x) = x^T P_N x$ is a common Lyapunov function for each of the individual systems $\dot{x} = A_i x$, $i = 1, 2, \cdots, N$, and hence a Lyapunov function for the switching system (1).

iii) For a given choice of the matrix P_0, the matrices $A_1, \cdots, A_N$ can be chosen in any order in (9) to yield the same solution P_N.

iv) The matrix P_N can also be expressed in integral form as

$$P_N = \int_0^\infty e^{A_N^T t_N} \cdots \left[\int_0^\infty e^{A_2^T t_2}\left[\int_0^\infty e^{A_1^T t_1} P_0 e^{A_1 t_1}\, dt_1\right] \cdot e^{A_2 t_2}\, dt_2\right] \cdots e^{A_N t_N}\, dt_N$$

where, as in part iii) above, the order in which the matrices $A_1, A_2, \cdots, A_N$ appear can be replaced by any permutation.

Proof: The proof proceeds along the same lines as in Theorem 1 and hence only the basic steps are given below.

1) Follows from part ii) of the theorem.

2) If $\dot{x} = A_i x$, the derivative of V along the trajectories of this system is given by $\dot{V} = x^T(A_i^T P_N + P_N A_i)x$ and hence we have to show that $A_i^T P_N + P_N A_i < 0$ for $i = 1, 2, \cdots, N$.

For this purpose, define the matrices $P_{ij} \triangleq A_i^T P_j + P_j A_i$. If we show that $P_{ij} < 0$ for $i = 1, 2, \cdots, N$, $j = i, i+1, \cdots, N$, then the result follows by choosing $j = N$ for each i. Hence, let $i \in \{1, 2, \cdots, N\}$. From (9), $P_{ii} = -P_{i-1} < 0$. Now assume that $P_{ij} < 0$ for some $j \in \{i, i+1, \cdots, N-1\}$. Then, using (9), the commutativity of A_i and A_{j+1} and the stability of A_{j+1} we find that $A_{j+1}^T P_{i,\,j+1} + P_{i,\,j+1} A_{j+1} = -(A_i^T P_j + P_j A_i) = -P_{ij} > 0$, showing that $P_{i,\,j+1} < 0$ also, proving the claim by induction.

3) This can be shown as in Theorem 1 by taking some permutation of $A_1, A_2, \cdots, A_N$ in (9), assuming that the resulting solution, say R_N, is not equal to P_N, and using the uniqueness of solutions to the Lyapunov equations to show that $R_N = P_N$.

4) This follows as before by starting with the integral expression for the solution P_1 to the first Lyapunov equation in (9) and successively substituting the expression for P_i in the expression for P_{i+1}. ∎

Comment 1: Theorem 1 asserts that commutativity assures the existence of a common Lyapunov function. The converse does not hold in general. For example, if $A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, then $V(x) = x^T x$ is a common Lyapunov function, but A_1 and A_2 do not commute.

Comment 2: The above results can be extended to the case of discrete-time systems of the form $x(k+1) = A_i x(k)$ where the matrices A_i are discrete-time asymptotically stable and commute pairwise. In this case, (9) is modified as

$$A_i^T P_i A_i - P_i = -P_{i-1}, \quad i = 1, 2, \cdots, N$$

and the common Lyapunov function is $V(x) = x^T P_N x$.

Comment 3: If $V(x) = x^T P x$ is a common Lyapunov function for A_1 and A_2, then it is also a Lyapunov function for any positive linear combination of A_1 and A_2, i.e., for matrices of the form $\alpha A_1 + \beta A_2$ where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

Comment 4: If $V(x) = x^T P x$ is a common Lyapunov function for A_1 and A_2, then there exist neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$ around A_1 and A_2, respectively, such that V remains a Lyapunov function for any pair of matrices B_1 and B_2 belonging to the set $\mathcal{N}_1 \cup \mathcal{N}_2$.

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An Analytical Method of Estimating the Domain of Attraction for Polynomial Differential Equations

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***Abstract*—A system of nth-order differential equations with polynomial right-hand sides is considered, and a simple analytical method of estimating the domain of attraction is developed. The method involves an ordinary quadratic Lyapunov function $v = X^T Q(Y) X$ with a certain parameter vector Y from the same state space. All linear factors in the expression for $\dot{v}$ are bounded in the domain $v \leq 1$, and the derivative is bounded by a quadratic function, the negativeness of which determines the restrictions for Y. The domain of attraction is estimated through a simple scaling of the obtained area or through its nonlinear transformation with optimization.**

The method allows for obtaining domains (for example, with infinite volume) that are comparable with ones obtained by complicated computational procedures. A set of examples is presented.

I. Introduction

We will consider the following system of differential equations

$$\dot{X} = AX + g(X) \tag{1}$$

where $X \in R^n$, A is a square Hurwitz's matrix, $g(X)$ is a polynomial vector-function $R^n \to R^n$ without linear terms. If $g(X) = \sum_{i=1}^n x_i A_i X$, the system is called quadratic [1].

Equation (1) covers a wide class of practically important systems. They are studied in mathematics [2] in connection with the Riccati equations theory. Many electrical systems such as induction machines in the $d-q$ frame [3] are described by quadratic equations. Polynomial systems also arise as a result of the Taylor series technique applied to a more general class of nonlinear systems.

In many cases such systems are not globally stable, and the domain of attraction must be estimated. Most modern approaches use computational methods [4]. Thus, the volume of the domain of attraction can be maximized using numerical optimization techniques [5].

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