IMA Journal of Mathematical Control and Information Advance Access published June 28, 2016

IMA Journal of Mathematical Control and Information (2016) **32**, 1–12
doi: 10.1093/imamci/dnw022

The null controllability of nonlinear discrete control systems with degeneracy

XUE-LI TAN
School of Mathematics, Jilin University, Changchun, 130012, People's Republic of China

AND

YONG LI*
School of Mathematics and Statistics, and Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, 130024, People's Republic of China
*Corresponding author: liyongmath@163.com

[Received on 15 August 2015; revised on 23 December 2015; accepted on 03 May 2016]

This article mainly concerns the null controllability of autonomous and non-autonomous nonlinear discrete control systems. By using Cayley–Hamilton theorem and Brouwer's fixed point theorem under a condition on the nonlinearity, sufficient and necessary conditions of null controllability for nonlinear discrete control systems are presented. In our approach, the linear part of systems under consideration might admit some degeneracy. In addition, applications are given to illustrate the obtained results.

Keywords: nonlinear discrete control system; degeneracy, null controllability; Cayley–Hamilton theorem; Brouwer's fixed point theorem.

1. Introduction

Controllability is one of the most important issues in mathematical control theory. The idea of controllability was put forward in the 1960s. It is well known that a necessary and sufficient condition of controllability for linear autonomous discrete control system

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

is

$$rank[B\ AB\ \cdots\ A^{n-1}B] = n,$$

where $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^p$, A is an $n \times n$ constant matrix, B is an $n \times p$ constant matrix (Kalman, 1963; Wonham, 1985), and rank denotes the rank of matrix, $\mathbf{R}^l$ is the usual l-dimensional Euclidean space with norm $\|\cdot\|$. The similar questions for nonlinear systems also developed rapidly. Since the 1970s, Lie algebra methods and other powerful tools of differentiable manifold theory have been developed to study controllability of nonlinear systems (Sussmann & Jurdevic, 1972; Hermann & Krener, 1977; Isidori, 1995; Jurdjecic, 1997). In recent years, controllability has played an important role in developments and applications of mathematical control theory. There are many different kinds of definitions on controllability, such as local controllability, completely controllability, small controllability, near controllability and null controllability (Tie *et al.*, 2010, 2013, 2014; Tie & Cai, 2011a,b; Tie, 2014a,b). Additionally, the controllability of discrete-time polynomial systems is studied (Kawano & Ohtsuka, 2013, 2016). In (Zhao & Sun, 2013), by characterizing the infinitesimal principle in terms of the sequences of codistribution and distribution, explicit criteria for the local observability and local accessibility of the system are derived, respectively. However, in those studies, the linear part of systems must be nonsingular, that is, it requires that $rank[B\,AB\ \cdots A^{n-1}B] = n$.

In this article, we consider the null controllability of the following nonlinear autonomous discrete control system

$$x(k+1) = Ax(k) + Bu(k) + f(x(k)), \tag{2}$$

where $k \in \mathbf{N}$ (the set of all natural numbers), $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^p$, A is an $n \times n$ constant matrix, B is an $n \times p$ constant matrix and $f : \mathbf{R}^n \to \mathbf{R}^n$ is the nonlinear part of x. The non-autonomous form is as follows

$$x(k+1) = A(k)x(k) + B(k)u(k) + f(k, x(k)), \tag{3}$$

where $k \in \mathbf{N}$, $x(k) \in \mathbf{R}^n$, $u(k) \in \mathbf{R}^p$, $A(k)$ is an $n \times n$ matrix, $B(k)$ is an $n \times p$ matrix and $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is the nonlinear part of x.

Compared with system (1), system (2) increases the nonlinear part of x. We will study the null controllability of such nonlinear discrete control systems. At the same time, we do not require that

$$rank[B\,AB\ \cdots\ A^{n-1}B] = n,$$

that is the linear part of systems might be degenerate. To establish sufficient and necessary conditions of null controllability for the systems, we suppose that the following conditions hold:
(1) $f : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and bounded in x;
(2) $Imf \subseteq Span[B\,AB\ \cdots\ A^{n-1}B]$.
Here, Imf denotes the image of mapping f, and $Span[C]$ denotes the subspace spanned by all column vectors of matrix C.

The purpose of this article is to establish sufficient and necessary conditions guaranteeing the null controllability of nonlinear discrete control systems via Cayley–Hamilton theorem and Brouwer's fixed point theorem. In our approach, the linear part of systems under consideration might admit some degeneracy. To our knowledge, the study in this aspect is still rare. Hence, our results provide a way to deal with the null controllability of nonlinear discrete control systems with some degeneracy, especially the systems that the linear parts are not controllable.

2. Main Results

By system (2), we have

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) + f(x(0)), \\ x(2) &= Ax(1) + Bu(1) + f(x(1)) \end{aligned}$$

$$
\begin{aligned}
&= A^2x(0) + ABu(0) + Bu(1) + Af(x(0)) + f(x(1)),\\
&\vdots\\
x(k) &= Ax(k-1) + Bu(k-1) + f(x(k-1)) = \cdots\\
&= A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i) + \sum_{i=0}^{k-1} A^{k-1-i}f(x(i)).
\end{aligned}
$$

DEFINITION 1 If, for any $x(0) \in \mathbf{R}^n$, there exists $N \in \mathbf{N}$ and there exist $u(0)$, $u(1),\cdots,u(N-1)$ such that

$$
\begin{aligned}
x(N) &= A^N x(0) + \sum_{i=0}^{N-1} A^{N-1-i}Bu(i) + \sum_{i=0}^{N-1} A^{N-1-i}f(x(i))\\
&= 0, \qquad (4)
\end{aligned}
$$

then nonlinear autonomous discrete control system (2) is said to be null controllable.

Rewrite eq. (4) as the vector form

$$
\begin{pmatrix} B\, AB\, \cdots\, A^{N-1}B \end{pmatrix} \begin{pmatrix} u(N-1)\\ u(N-2)\\ \vdots\\ u(0) \end{pmatrix} = -A^N x(0) - \sum_{i=0}^{N-1} A^{N-1-i} f(x(i)). \qquad (5)
$$

We have the following.

THEOREM 1 Suppose that f is continuous, bounded in x and satisfies

$$
Imf \subseteq Span[B\, AB\, \cdots\, A^{n-1}B].
$$

Then the nonlinear autonomous discrete control system (2) is null controllable if and only if there exists $k \in \mathbf{N}$, such that

$$
Span[A^k] \subseteq Span[B\, AB \cdots A^{n-1}B].
$$

Applying Theorem 1 to the case of non-degenerate nonlinear systems, we have:

COROLLARY 1 Suppose that f is continuous and bounded in x. If $rank[B\, AB\, \cdots\, A^{n-1}B] = n$, then nonlinear autonomous discrete control system (2) is null controllable.

We also use a similar approach to nonlinear non-autonomous forms.

By system (3), we have

$$
\begin{aligned}
x(h+1) &= A(h)x(h) + B(h)u(h) + f(h, x(h)),\\
x(h+2) &= A(h+1)x(h+1) + B(h+1)u(h+1) + f(h+1, x(h+1))\\
&= \Phi(h+2,h)x(h) + \sum_{i=h}^{h+1} \Phi(h+2,i+1)B(i)u(i) + \sum_{i=h}^{h+1} \Phi(h+2,i+1)f(i,x(i)),\\
&\ \vdots\\
x(h+k) &= A(h+k-1)x(h+k-1) + B(h+k-1)u(h+k-1) + f(h+k-1, x(h+k-1))\\
&= \Phi(h+k,h)x(h) + \sum_{i=h}^{h+k-1} \Phi(h+k,i+1)B(i)u(i) + \sum_{i=h}^{h+k-1} \Phi(h+k,i+1)f(i,x(i)),
\end{aligned}
$$

where the state-transition matrix $\Phi(k,h)$ $(k \geq h)$ is given by

$$
\begin{cases}
\Phi(k,h) = A(k-1)A(k-2)\cdots A(h),\\
\Phi(k,k) = I.
\end{cases}
$$

DEFINITION 2 If, for any $x(h) \in \mathbf{R}^n$, there exists $N \in \mathbf{N}$ and there exist $u(h)$, $u(h+1),\cdots,u(h+N-1)$ such that

$$
x(h+N) = \Phi(h+N,h)x(h) + \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)B(i)u(i) + \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)f(i,x(i)) = 0, \tag{6}
$$

then nonlinear non-autonomous discrete control system (3) is said to be null controllable in step h.

Rewrite eq. (6) as the vector form

$$
\begin{pmatrix} B(h+N-1) & \Phi(h+N,h+N-1)B(h+N-2) & \cdots & \Phi(h+N,h+1)B(h) \end{pmatrix}
\begin{pmatrix} u(h+N-1)\\ u(h+N-2)\\ \vdots \\ u(h) \end{pmatrix}
$$

$$
= -\Phi(h+N,h)x(h) - \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)f(i,x(i)). \tag{7}
$$

Let

$$
\Psi(i) := \Big(B(i) \ \ \Phi(i+1,i)B(i-1)\cdots\Phi(i+1,h+1)B(h)\Big) \ \ i \in \mathbf{N}, \ \ i \geq h.
$$

Then eq. (7) is equivalent to

$$\Psi(h+N-1)\begin{pmatrix} u(h+N-1) \\ u(h+N-2) \\ \vdots \\ u(h) \end{pmatrix} = -\Phi(h+N,h)x(h) - \sum_{i=h}^{h+N-1} \Phi(h+N,i+1)f(i,x(i)). \tag{8}$$

For non-autonomous system (3), we have similar result as follows.

THEOREM 2 Let $f(i,x(i))$ be continuous and bounded in x for each $i \in [h, h+N-1]$. If there exists $N \in \mathbf{N}$, such that

$$Span[\Phi(h+N,h)] \subseteq Span[\Psi(h+N-1)],$$

$$f(i,x(i)) \in Span[\Psi(i)],\ i \in \mathbf{N},\ h \leq i \leq h+N-1,$$

then the non-autonomous discrete control system (3) is null controllable in step h.

3. Proof of Main results

First, let us introduce and prove some preliminaries.

THEOREM 3 (Cayley-Hamilton (Gantmacher, 1960; Lancaster, 1974)) Let $\mathbf{C}^{n\times m}$ be the set of complex $n \times m$ matrices and $A \in \mathbf{C}^{n\times n}$. Let

$$\varphi(\lambda) = det(\lambda I_n - A) = \sum_{i=0}^{n} a_i \lambda^i \ \ (a_n = 1)$$

be the characteristic polynomial of A, where I_n is the $n \times n$ identity matrix. Then every square matrix satisfies its own characteristic equation, i.e.

$$\varphi(A) = \sum_{i=0}^{n} a_i A^i = 0,$$

where 0 is the zero matrix.

COROLLARY 2 The $n \times n$ matrix A to the power of k $(k \geq n)$ satisfies

$$A^k = \sum_{i=0}^{n-1} b_i A^i,$$

where $b_i (i = 0, 1, \ldots, n-1)$ are constants.

LEMMA 1 For any $k \in \mathbf{N}$ ($k \geq n$), the following formula holds:

$$Span[B\,AB \cdots A^{k-1}B] = Span[B\,AB \cdots A^{n-1}B].$$

LEMMA 2 If there exists $k \in \mathbf{N}$ ($k \leq n$) such that

$$Span[A^k] \subseteq Span[B\,AB \cdots A^{n-1}B],$$

then for any $x(0) \in \mathbf{R}^n$,

$$A^n x(0) \in Span[B\,AB \cdots A^{n-1}B].$$

Proof. If there exists $k \in \mathbf{N}$ ($k \leq n$) such that

$$Span[A^k] \subseteq Span[B\,AB \cdots A^{n-1}B],$$

then for any $x(0) \in \mathbf{R}^n$,

$$A^k x(0) \in Span[B\,AB \cdots A^{n-1}B].$$

Thereby, there exist $u(0)$, $u(1)$,$\cdots$,$u(n-1)$ such that

$$A^k x(0) = \begin{pmatrix} B\,AB & \cdots & A^{n-1}B \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix},$$

thus

$$\begin{aligned} A^n x(0) &= A^{n-k} A^k x(0) \\ &= A^{n-k} \begin{pmatrix} B\,AB & \cdots & A^{n-1}B \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix} \\ &= \begin{pmatrix} A^{n-k}B\,A^{n-k+1}B & \cdots & A^{2n-k-1}B \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}. \end{aligned}$$

It follows from Cayley–Hamilton theorem and Corollary 2 that

$$Span[A^{n-k}B\,A^{n-k+1}B \cdots A^{2n-k-1}B] \subseteq Span[B\,AB \cdots A^{n-1}B].$$

Therefore,

$$A^n x(0) \in Span[B\,AB \cdots A^{n-1}B].$$

The following lemma is crucial to our arguments. □

LEMMA 3 Suppose that f is continuous, bounded in x and satisfies $Imf \subseteq Span[B\,AB\ \cdots\ A^{n-1}B]$. If there exists $k \in \mathbf{N}$ $(k \geq n)$ such that for any $x(0) \in \mathbf{R}^n$,

$$A^k x(0) \in Span[B\,AB \cdots A^{n-1}B],$$

then the nonlinear equations in v

$$\begin{pmatrix} B\,AB & \cdots & A^{k-1}B \end{pmatrix} v = -A^k x(0) - \sum_{i=0}^{k-1} A^{k-1-i} f(x(i)) \tag{9}$$

has at least one solution, where $x(i)$ satisfies eq. (2) with $v = (u(0), u(1), \ldots, u(k-1))^T$.

Proof. Let $f^i_{x(0),v}$ be defined by $f^i_{x(0),v} := f(x(i))$. Because

$$Imf \subseteq Span[B\,AB \cdots A^{n-1}B],$$

we have

$$A^{k-1-i} f^i_{x(0),v} \in Span[B\,AB \cdots A^{n-1}B].$$

By Lemma 1, we have

$$A^{k-1-i} f^i_{x(0),v} \in Span[B\,AB \cdots A^{k-1}B].$$

Let

$$g : \mathbf{R}^{kp} \to \mathbf{R}^n,$$
$$g(x) = (B\,AB \cdots A^{k-1}B)x.$$

Let $Y = Img \subseteq \mathbf{R}^n$, which is a linear closed subspace. Thus, $g : \mathbf{R}^{kp}/kerg \to Y$ is one–one. Here $kerg$ denotes the kernel of mapping g and is the set of all elements x of $\mathbf{R}^{kp}$ for which $g(x) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in $\mathbf{R}^n$. Hence, $h := (g\,|_{\mathbf{R}^{kp}/kerg})^{-1}$ exists and it is bounded. Then there exists only one vector $\xi^i_{x(0),v} \in \mathbf{R}^{kp}/kerg$ such that

$$g(\xi^i_{x(0),v}) = (B\,AB \cdots A^{k-1}B)\xi^i_{x(0),v} = A^{k-1-i} f^i_{x(0),v}.$$

that is, $\xi^i_{x(0),v} = h(A^{k-1-i} f^i_{x(0),v})$. Since f is bounded in x, $f^i_{x(0),v}$ is bounded uniformly in $x(0)$ and v, h is bounded, and hence $\xi^i_{x(0),v}$ is bounded. Notice that

$$A^k x(0) \in Span[B\,AB \cdots A^{n-1}B] = Span[B\,AB \cdots A^{k-1}B].$$

There exists only one vector $\eta_{x(0)} \in \mathbf{R}^{kp}/kerg$, such that

$$A^k x(0) = (B\, AB \cdots A^{k-1}B)\eta_{x(0)}.$$

If $x(0)$ is given, then $\eta_{x(0)}$ is bounded. Consequently, eq. (9) is equivalent to

$$\begin{aligned}\left(\, B\, AB\ \cdots\ A^{k-1}B\, \right) v &= -\left(\, B\, AB\ \cdots\ A^{k-1}B\, \right)\eta_{x(0)} - \sum_{i=0}^{k-1}\left(\, B\, AB\ \cdots\ A^{k-1}B\, \right)\xi^i_{x(0),v}\\ &= \left(\, B\, AB\ \cdots\ A^{k-1}B\, \right)\left(-\eta_{x(0)} - \sum_{i=0}^{k-1}\xi^i_{x(0),v}\right).\end{aligned}$$

If there exists $v \in \mathbf{R}^{kp}/kerg$ such that

$$v = -\eta_{x(0)} - \sum_{i=0}^{k-1}\xi^i_{x(0),v} \tag{10}$$

holds, then eq. (9) has at least one solution.

Next, we prove the existence of the solution of eq. (10).

Given $x(0) \in \mathbf{R}^n$, there must exist real numbers η^*, ξ^*, such that

$$\left\|\eta_{x(0)}\right\| \le \eta^* < \infty, \quad \left\|\xi^i_{x(0),v}\right\| \le \xi^* < \infty, \forall v \in \mathbf{R}^{kp}/kerg,$$

therefore,

$$\|v\| = \left\|-\eta_{x(0)} - \sum_{i=0}^{k-1}\xi^i_{x(0),v}\right\| \le \left\|\eta_{x(0)}\right\| + \sum_{i=0}^{k-1}\left\|\xi^i_{x(0),v}\right\| \le \eta^* + k\xi^* < \infty.$$

Let $D := \{v \in \mathbf{R}^{kp}/kerg : \|v\| \le \eta^* + k\xi^*\}$. We define $F : D \to \mathbf{R}^{kp}$ by

$$Fv = -\eta_{x(0)} - \sum_{i=0}^{k-1}\xi^i_{x(0),v}.$$

By means of ξ^* and the definition of D, we see that $F : D \to D$. Note that F is continuous. Hence F has a fixed point in D by Brouwer's fixed point theorem, i.e. there exists $v \in D$, such that eq. (10) holds. Consequently, eq. (9) has at least one solution. □

Proof of Theorem 1. First, we prove necessity. If system (2) is null controllable, then for any $x(0) \in \mathbf{R}^n$, there exist $u(0), u(1),\cdots,u(N-1)$, such that eq. (5) holds. Thereby,

$$-A^N x(0) - \sum_{i=0}^{N-1} A^{N-1-i} f(x(i)) \in Span[B\, AB\ \cdots\ A^{N-1}B].$$

It follows from Cayley–Hamilton theorem and Corollary 2 that

$$Span[B\ AB\ \cdots\ A^{N-1}B] \subseteq Span[B\ AB\ \cdots\ A^{n-1}B].$$

And

$$Imf \subseteq Span[B\ AB\ \cdots\ A^{n-1}B],$$

thus

$$A^N x(0) \in Span[B\ AB\ \cdots\ A^{n-1}B].$$

Moreover, $x(0)$ is arbitrary, therefore

$$Span[A^N] \subseteq Span[B\ AB\ \cdots\ A^{n-1}B].$$

Let $k = N$. Thus the necessity is proved.

Next, we prove sufficiency. If there exists $k \in \mathbf{N}$, such that

$$Span[A^k] \subseteq Span[B\ AB \cdots A^{n-1}B],$$

then it follows from Lemma 2 that there exists $j \in \mathbf{N}$ $(j \geq n)$, such that

$$A^j x(0) \in Span[B\ AB \cdots A^{n-1}B].$$

From Lemma 3, we deduce that there exist $u(0), u(1), \cdots, u(j-1)$, such that

$$x(j) = A^j x(0) + \sum_{i=0}^{j-1} A^{j-1-i} Bu(i) + \sum_{i=0}^{j-1} A^{j-1-i} f(x(i)) = 0.$$

Therefore, nonlinear autonomous discrete control system (2) is null controllable. The sufficiency is proved. □

Proof of Theorem 2. If

$$Span[\Phi(h+N,h)] \subseteq Span[\Psi(h+N-1)],$$

then for any $x(h) \in \mathbf{R}^n$,

$$\Phi(h+N,h)x(h) \in Span[\Psi(h+N-1)],$$

thus, there exists only one vector $\eta_{x(h)} \in \mathbf{R}^{Np}/ker\Psi(h+N-1)$, such that

$$\Phi(h+N,h)x(h) = \Psi(h+N-1)\eta_{x(h)}.$$

Given $x(h) \in \mathbf{R}^n$, there must be bounded for $\eta_{x(h)}$. And by

$$f(i,x(i)) \in Span[\Psi(i)]\ i \in \mathbf{N},\ h \le i \le h+N-1,$$

we deduce that

$$\Phi(h+N,i+1)f(i,x(i)) \in Span[\Psi(h+N-1)].$$

Moreover, since $f(i,x(i))$ is bounded in x for each $i \in [h,h+N-1]$ following the proof of Lemma 3, we obtain that there exists only one vector $\xi^i_{x(h),u} \in \mathbf{R}^{Np}/ker\Psi(h+N-1)$, for any $i \in [h,h+N-1]$, $i \in \mathbf{N}$, such that

$$\Phi(h+N,i+1)f(i,x(i)) = \Psi(h+N-1)\xi^i_{x(h),u},$$

and $\xi^i_{x(h),u}$ is bounded uniformly in $x(h)$ and u. Consequently, the problem whether the non-autonomous discrete control system (3) is null controllable in step h is equivalent to the existence of the solution of the following nonlinear equations in $u \in \mathbf{R}^{Np}$:

$$\begin{aligned}\Psi(h+N-1)u &= -\Psi(h+n-1)\eta_{x(h)} - \sum_{i=h}^{h+N-1}\Psi(h+N-1)\xi^i_{x(h),u}\\ &= \Psi(h+N-1)\Big(-\eta_{x(h)} - \sum_{i=h}^{h+N-1}\Psi(h+N-1)\xi^i_{x(h),u}\Big). \end{aligned} \tag{11}$$

If there exists $u \in \mathbf{R}^{Np}/ker\Psi(h+N-1)$ such that

$$u = -\eta_{x(h)} - \sum_{i=h}^{h+N-1}\xi^i_{x(h),u} \tag{12}$$

holds, then eq. (11) has at least one solution. A similar proof of the existence of solutions of eq. (10) in Lemma 3 shows that eq. (12) has at least one solution. Therefore, system (3) is null controllable in step h. □

4. Applications

In this section, we illustrate some applications by examples.

EXAMPLE 1 Consider the controllability of the following nonlinear autonomous discrete control system:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1)\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix} x_1(k) \\ x_2(k)\end{pmatrix} + \begin{pmatrix} 1 \\ 0\end{pmatrix}u(k) + \begin{pmatrix} 1+\sin[x_1^2(k)] + 9\cos^2[x_2(k)] \\ 0\end{pmatrix}. \tag{13}$$

The discrete control system (13) shows that $n = 2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the nonlinear part is continuous and bounded. We deduce that

$$(B\ AB) = Span \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Imf \subseteq Span[B\ AB] \quad Span[A] \subseteq Span[B\ AB].$$

Hence, by Theorem 1, discrete control system (13) is null controllable.

EXAMPLE 2 Consider the controllability of the following nonlinear autonomous discrete control system:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + \begin{pmatrix} 0 \\ 5\cos[x_1(k)x_2(k)] \end{pmatrix}. \tag{14}$$

Discrete control system (14) shows that $n = 2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the nonlinear part is continuous and bounded. Notice that

$$(B\ AB) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Imf \subseteq Span[B\ AB],$$

but for any $k \in \mathbf{N}$,

$$Span[A^k] = Span[A] \nsubseteq Span[B\ AB].$$

Hence, by Theorem 1, the discrete control system (14) is not controllable.

EXAMPLE 3 Consider the controllability of the following nonlinear autonomous discrete control system:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} \ln[5 + \sin x_1(k)] \\ 1 \\ 0 \end{pmatrix}. \tag{15}$$

Discrete control system (15) shows that $n = 3$, $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,$B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and the nonlinear part is continuous and bounded. We deduce that

$$(B\ AB\ A^2B) = Span \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Imf \subseteq Span[B\ AB\ A^2B], \quad Span[A] \subseteq Span[B\ AB\ A^2B].$$

Hence, by Theorem 1, the discrete control system (15) is null controllable.

5. Conclusion

In this article, the null controllability of nonlinear discrete control systems is studied. Sufficient and necessary conditions of null controllability for the systems are presented via Cayley–Hamilton theorem and Brouwer's fixed point theorem. In our results, the linear part of systems might be degenerate, that is $rank(B, AB, \ldots, A^{n-1}B) < n$. In Examples 1 and 2, the matrix $(B, AB, \ldots, A^{n-1}B)$ is singular. In addition, applications are given to illustrate the obtained results of this article. Future work should more in-depth study controllability of nonlinear discrete control systems and find more general sufficient and necessary conditions.

Funding

National Basic Research Program of China (Grant No. 2013CB834100); National Natural Science Foundation of China (Grant Nos. 11171132, and 11571065).

References

Gantmacher, F. R. (1960) *The Theory of Matrices*. New York: Chelsea Publishing Company.

Hermann, R. & Krener, A. J. (1977) Nonlinear controllability and observability. *IEEE Trans. Automat. Contr.*, **AC-22**, 728–740.

Isidori, A. (1995) *Nonlinear Control Systems, 3rd edn.* New York: Springer.

Jurdjecic, V. (1997) *Geometric Control Theory.* New York: Cambridge University Press.

Kalman, R. E. (1963) Mathematical description of linear dynamical systems. *SIAM J. Control Optim.*, **1**, 152–192.

Kawano, Y. & Ohtsuka, T. (2013) Simple sufficient conditions for reachability of discretetime polynomial systems. *IEEE Trans. Automat. Contr.*, **58**, 3203–3206.

Kawano, Y. & Ohtsuka, T. (2016) Commutative algebraic methods for controllability of discrete-time polynomial systems. *Int. J. Contr.*, **89**, 343–351.

Lancaster, P. (1974) *Theory of matrices*. New York:Academic Press.

Sussmann, H. & Jurdevic, V. (1972) Controllability of nonlinear systems. *J. Diff. Equations*, **12**, 95–116.

Tie L. (2014a) On near-controllability, nearly controllable subspaces, and near-controllability index of a class of discrete-time bilinear systems: a root locus approach. *SIAM J. Control Optim.*, **52**, 1142–1165.

Tie L. (2014b) On Small-controllability and controllability of a class of nonlinear systems. *Int. J. Contr.*, **87**, 2167–2175.

Tie L. & Cai K. (2011a) On near-controllability and stabilizability of a class of discrete-time bilinear systems. *Syst. Control Lett.*, **60**, 650–657.

Tie L. & Cai K. (2011b) On near-controllability of nonlinear control systems. Control Conference (CCC), 2011 30th Chinese, 1416(1), 131–136.

Tie L., Cai K. & Lin Y. (2010) On uncontrollable discrete-time bilinear systems which are "nearly" controllable. *IEEE Trans. Automat. Contr.*, **55**, 2853–2858.

Tie L., Cai K. & Lin Y. (2013) A linear algebraic criterion for controllability of both continuous-time and discrete-time symmetric bilinear systems. *J. Franklin Inst.*, **350**, 898–910.

Tie L., Cai K. & Lin Y. (2014) On small-controllability of a class of Hammerstein-nonlinear systems—an implicit function approach. *Int. J. Syst. Sci.*, **45**, 1991–1998.

Wonham, W. M. (1985) *Linear Multivariable Control: A Geometric Approach, 3rd edn.* New York: Springer.

Zhao, S. & Sun, J. (2013) A geometric method for observability and accessibility of discrete impulsive nonlinear systems. *Int. J. Syst. Sci.*, **44**, 1522–1532.