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An Analytical Eigenvalue Assignment of Linear Time-Delay Systems Using Lambert W Function

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***Abstract*—A new method is presented in this article to control linear time-delay systems using Lambert W function. The absence of a sufficient proof for using matrix Lambert W function in controlling time-delay systems has frustrated researchers. It is a well-established fact that unlike Lambert W Function, the matrix version still lacks the proof, which would guarantee the validity of finding the right-most eigenvalue of linear time-delay systems using this function. This paper presents a new approach in which, a system from any order is forced to use scalar Lambert W function instead of unreliable matrix version. This article studies systems with input delay and state delay respectively and for both of them, sufficient conditions for eigenvalue assignment are proposed.**

Keywords*—*Lambert W, time-delay, eigenvalue assignment.

1 Introduction

Lambert W function possesses many applications. It is useful in solving Jet fuel problem and combustion equations [1][2]. But the most important use for Lambert W function is solving linear time-delay differential equations. Many researchers have used this function to analyze and control time-delay systems [3][4]. There have been some studies in which novel controllability and observability theories have been defined for linear time-delay systems that are more useful than the conventional definitions [5]. But thus far, the most comprehensive study that has been conducted in this field is [6]. Although in [6] the corresponding authors have tried to thoroughly discuss the analysis and design of time-delay systems, there are many flaws in the methods presented in [6]. In addition to that, a few vital proofs are definitely missing. One of the major drawbacks of the methods proposed in the aforementioned study is that there is no proof that the right-most eigenvalue of time-delay system is acquired by the principal branch of the matrix Lambert W function, unlike the scalar case, for which two respective proofs are available. Another problem with the solutions presented in [6] is that for computing the feedback policy to assign the right-most eigenvalue of a system, a numerical solution is acquired. This numerical solution cannot determine whether a problem has a unique solution or not. This feature is a crucial one for solving time-delay differential equations. Even in some cases, the numerical solution could not obtain an existing solution of the problem. This kind of issues can cause serious damages to the design of the closed-loop system. Moreover, it can entirely question the validity of this approach [7][8].

The design strategy proposed in this document is certainly accurate for any time-delay system and it barely uses any numerical solutions if it does at all. The control techniques presented in this article assigns the right-most eigenvalue of a time-delay system to the desired point using scalar Lambert W function, whether the system is scalar one or not. In this paper the basic idea is that scalar Lambert W function is valid and reliable, therefore, the article uses the scalar Lambert W function instead of its matrix version, even for the multi-dimensional systems. Because of this simple but effective idea, proposed approach always assigns the right-most eigenvalue to the desired point, and if it is not feasible one would know it before any calculations, which is the second characteristic that given method possesses. This feature of the presented approach is a result of this fact that, this paper uses the ranges of different branches of the scalar Lambert W function, not the function itself. Accordingly, unlike conventional solution, proposed method can determine if it is feasible to assign the right-most eigenvalue of a particular system to the desired point or not. As indicated before, this document tries to make a connection between the scalar Lambert W function and multi-dimensional linear time-delay systems.

2 Preliminary

To understand the represented materials in the following sections, one required to possess the essential background knowledge about Lambert W function which is brought up in this section.

2.1 Lambert W function

Multivalued function that satisfies the following equation is called Lambert W function

$$W_k(z)e^{W_k(z)} = z \tag{1}$$

where $z \in \mathbf{C}$ is the independent variable and $k \in \mathbf{Z}$ denotes the number of the branch of Lambert W function. This function is a multi-valued one and there are infinite solutions for (1). There is a one-to-one correspondence between infinite branches of Lambert W function and these solutions. The definition for the matrix Lambert W function is the same as scalar one. However, matrix Lambert W function is obtained through scalar one. Consider a square matrix $H \in \mathbf{C}^{n\times n}$ which has the Jordan Canonical form of $H = Z\mathbf{J}Z^{-1}$. Assume that $\mathbf{J} = diag(J_1(\lambda_1), J_2(\lambda_2), \cdots, J_p(\lambda_p))$ where J_i are the Jordan blocks and $\lambda_i \in \mathbf{C}$ are eigenvalues of H. Then the matrix Lambert W function is defined as

$$\mathbf{W}_k(H) = Z \;\; diag\left(\mathbf{W_k}(\mathbf{J}_1), \mathbf{W_k}(\mathbf{J}_2), \cdots, \mathbf{W_k}(\mathbf{J}_p)\right) Z^{-1} \tag{2}$$

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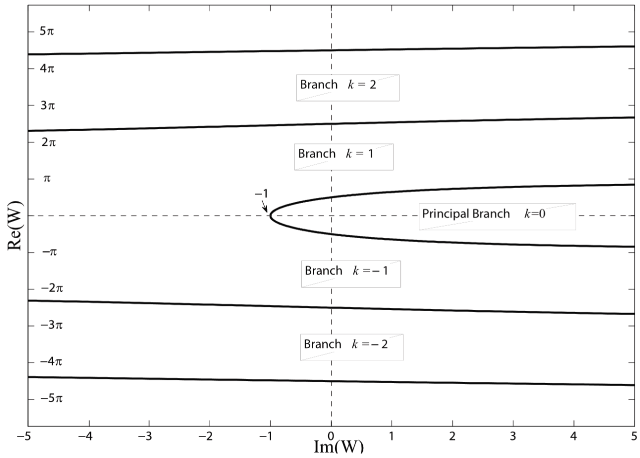

Fig. 1. *Range of the different branches of Lambert W function [6]. It is clear that real part of the principal branch of Lambert W function is always greater than* -1

where matrix $\mathbf{W}_k(\mathbf{J}_i)$ is shown in (3)

$$\begin{bmatrix} W_k(\lambda_i) & W_k'(\lambda_i) & \cdots & \frac{1}{(m-1)!}W_k^{(m-1)}(\lambda_i) \\ 0 & W_k(\lambda_i) & \cdots & \frac{1}{(m-2)!}W_k^{(m-2)}(\lambda_i) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k(\lambda_i) \end{bmatrix} \quad (3)$$

One the most important characteristics of Lambert W function used in this paper is the range of the Lambert W function which is presented in Fig.1. From this figure, it is evident that real part of the principal branch($k = 0$) of Lambert W function cannot be smaller than -1. This feature is the cornerstone for determining the limitations of many design synthesis that are conducted for time-delay systems particularly by Lambert W function [9][10]. A key attribute of Lambert W function is that for the same input of z, the real part of the $W_0(z)$ is bigger than the real part of the other branches. To put it another way

$$\forall z \in \mathbf{C} \quad , \quad Re(W_0(z)) \geq Re(W_{\pm 1}(z)) \geq Re(W_{\pm 2}(z)) \cdots \quad (4)$$

2.2 Conventional Methods

Consider the following linear time-delay system

$$\dot{x} = ax(t) + a_d x(t-\tau) \quad (5)$$

where $a, a_d \in \mathbf{R}$ and τ represents delay. The characteristic equation of (5) can be written as

$$s - a - a_d e^{-\tau s} = 0 \quad (6)$$

which can be written out by using Lambert W function [1]

$$s_k = \tfrac{1}{\tau} W_k(\tau a_d e^{-\tau a}) + a \quad (7)$$

therefore, using (7) one can evaluate all the eigenvalues of system (5), especially the right-most eigenvalue which is equal to s_0 [1]. For higher order systems the calculations differ, although the basics of the both cases are the same. Consider

$$\dot{x} = Ax(t) + A_d x(t-\tau) \quad (8)$$

where A and A_d are square matrices. To find the eigenvalues of system (8), the following algorithm is proposed in [6]

1) Compute matrix Q_k from the subsequent equation

$$\mathbf{W}_k(\tau A_d Q_k) e^{\mathbf{W}_k(\tau A_d Q_k) + \tau A} = \tau A_d \quad (9)$$

2) Obtain $\mathbf{S}_k$ using the resulted Q_k in (9) from

$$\mathbf{S}_k = \frac{1}{\tau}\mathbf{W}_k(\tau A_d Q_k) + A \quad (10)$$

It is stated that the right-most eigenvalue of $\mathbf{S}_0$ is the right-most eigenvalue of the system, although this statement is based on observation and does not have any analytical proof. [7]. Besides, since equation (9) is solved numerically sometimes finding its solution is not an easy or even feasible task for some cases. The most comprehensive design method using Lambert W function is discussed in [6]. Consider the following closed-loop system

$$\dot{x} = (A + BK)x(t) + (A_d + B)x(t-\tau) \quad (11)$$

First, select a desired matrix $\mathbf{S}_0$ for system (11). Then try to evaluate Q_k in (9) where $A \equiv A + BK$. Notice that here Q_k is a matrix function of unknown matrices K. Solve (10) numerically to find K and Q_k simultaneously. This synthesis has the weaknesses of the above-mentioned stability analysis and some other extra flaws too. For instance, the feasibility of the selected S_0 never comes to light.

3 Main Results

Materials presented in this section are divided into two subcategories. First systems with input delay are examined and then systems with delayed states are studied. Each of these subcategories is presented in scalar and matrix versions. Although matrix version includes scalar case study, the scalar version helps to understand the process easily and illustrate how scalar Lambert W function is used for the systems with higher orders.

3.1 Input Delay

In general, having a delay in the input of a system is a much more demanding challenge in comparison with other time-delay cases. This section tries to control linear time-delay systems with input delay.

Scalar Case: Consider the following system

$$\dot{x}(t) = ax(t) + bkx(t-\tau) \quad (12)$$

where a and b are real numbers. Then k for the desired right-most eigenvalue s_0 can be computed by

$$k = \frac{(s_0 - a)}{b} e^{\tau s_0} \quad (13)$$

Notice that for $s_0 \in \mathbf{R}$, desired right-most eigenvalue can only acquire values inside the set $[-\frac{1}{\tau} + a, \infty)$ because the range of the principle branch of Lambert W function that is obvious from fig1. The output of W_0 cannot be lower than -1. Therefore, the presented limitation always holds true. If one tries to place the right-most eigenvalue of system (12) at any point outside the determined set using (13), an eigenvalue of the system would be put at that point but it is not the

TABLE 1. RESULTS OF THE RIGHT-MOST EIGENVALUE ASSIGNMENTS FOR SYSTEM (12) WHERE $a = -1$, $b = 2$ AND $\tau = 1$

Desired s_0	k	Actual s_0	In or Out of feasible range
-6	-0.0062	-1.0349	Out
-4	-0.0275	-1.1786	Out
-2	-0.0677	-2	Boundry Point
-1.5	-0.0558	-1.5	In
-0.5	0.1516	-0.5	In

right-most eigenvalue of the system. For instance, in system (12) consider $a = -1$, $b = 2$ and $\tau = 1$. Accordingly, the restriction $s_0 \in [-2, \infty)$ should always be fulfilled, so that the eigenvalue assignment can be implemented correctly. In table1 it can be easily seen that for feasible right-most eigenvalues feedback k places s_0 at desired points, while for s_0 outside of the determined range it fails to do so.

Matrix Case: In this section, the concept illustrated in scalar case is expanded for high order systems with input delay. Consider system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) \\ u(t) &= Kx(t) \end{aligned} \tag{14}$$

where A is a $\mathbf{R}^{n\times n}$ square matrix and B and K are $\mathbf{R}^{n\times n}$. The characteristic equation of the system is

$$S_0 = A + BKe^{-\tau S_0} \tag{15}$$

The right-most eigenvalue of system (14) can be placed in the left half plane if matrix A is Hurwitz. If matrix A is not Hurwitz then system (14) can be stabilized for $\tau \le \frac{1}{max(real(eig(A)))}$, where $eig(A)$ are the eigenvalues of A.

Proof: The right-most eigenvalue of system (14) can be obtained from S_0

$$S_0 = \frac{1}{\tau}\mathbf{W}_0(\tau BKQ_0) + A \tag{16}$$

In order to determine all the feasible S_0 for (16) a set of conditions should be imposed. To determine such a set this document uses Jordan form of matrix $(S_0 - A)$. If this matrix has a real and diagonal Jordan form then due to (2) and (3), one can interact with scalar Lambert W function instead of the matrix version. By taking all aforementioned statements the following conditions are set

$$\begin{aligned} &1)\ \ Jordan(S_0 - A) \ \text{ should be real-diagonal} \\ &2)\ \ eig(S_0 - A) > \tfrac{-1}{\tau} \end{aligned} \tag{17}$$

where $jordan(S_0 - A)$ denotes the Jordan form of $S_0 - A$. To prove the mentioned statements all the possible forms of the matrix A would be examined but since the procedure of the proof is identical for all cases of A, only the most involving case, in which A has repeated complex eigenvalues will be discussed. assume that A has repeated complex eigenvalues. in that case, it is obvious that it can be represented as $A = Q_A J Q_A^{-1}$ where J would have the following structure [11].

$$J = \begin{bmatrix} \alpha + j\beta & 1 & 0 & 0 \\ 0 & \alpha + j\beta & 0 & 0 \\ 0 & 0 & \alpha - j\beta & 1 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} \tag{18}$$

to have a real-diagonal $jordan(S - A)$ the following matrix S is proposed for $S_0 = Q_A S Q_A^{-1}$

$$S = \begin{bmatrix} s_\alpha + j\beta & 1 & 0 & 0 \\ 0 & s_\alpha + j\beta & 0 & 0 \\ 0 & 0 & s_\alpha - j\beta & 1 \\ 0 & 0 & 0 & s_\alpha - j\beta \end{bmatrix} \tag{19}$$

This specific block structure makes $(S-J)$ a real-diagonal matrix and since $S_0 - A = Q_A(S-J)Q_A^{-1}$ then $jordan(S_0 - A)$ would be a real-diagonal matrix. This means matrix Lambert W function can be manipulated through scalar Lambert W function. This feature indicates that the eigenvalues of $(S_0 - A)$ all would be real-valued, which is the ideal scenario in this design method. It is obvious that matrix $(S - J) = (s_\alpha - \alpha)I_{4\times 4}$ where I is the identity matrix. Since matrix $S - J$ is diagonal with real entries, then $jordan(S_0 - A)$ will be real and diagonal too. Accordingly, matrix S_0 with the proposed structure would satisfy the conditions in (17) and its largest eigenvalue will be the right-most eigenvalue of the system (14) if the following holds true

$$s_\alpha - \alpha > \frac{-1}{\tau} \tag{20}$$

From (20), It is obvious that if A is Hurwitz ($\alpha < 0$), then system can be stabilized regardless of τ and if not, system can be stabilized for $\tau \le \frac{1}{max(real(eig(A)))}$. Note that the same proof could be conducted for real diagonal J where the corresponding $S = Q_A^{-1} S_0 Q_A$ would be real diagonal too. ■

3.2 Delay in States

This part discusses LTD systems with delay in their states. Like the previous segment, this section will be presented in two parts, scalar and matrix version.

Scalar Case : Consider a system with following dynamics

$$\begin{aligned} \dot{x}(t) &= ax(t) + a_d x(t-\tau) + bu(t) \\ u(t) &= kx(t) \end{aligned} \tag{21}$$

where a, a_d and b are real numbers. The characteristic equation of the system is

$$s - a - bk - a_d e^{-\tau s} = 0 \tag{22}$$

and the right-most eigenvalue of system can be derived from the following equation.

$$s_0 = \frac{1}{\tau} w(a_d e^{-\tau(a+bk)}) + a + bk \tag{23}$$

Then for feasible desired real right-most eigenvalue s_0 the feedback k can be computed by

$$s_0 \in \mathbf{R} \ \ , k = \frac{s_0 - a - a_d e^{-\tau s_0}}{b} \tag{24}$$

Due to the range of the principle branch of Lambert W function, s_0 is feasible only when inequality $s_0 - a - bk > \frac{-1}{\tau}$

TABLE 2. RESULTS OF THE RIGHT-MOST EIGENVALUE ASSIGNMENTS FOR SYSTEM (21) WITH $a = 1$, $b = 2$, $a_d = -3$ AND $\tau = 0.2$

Desired S_0	K	Actual S_0	In or Out of feasible range
-7	2.0828	3.7479	Out
-5	1.0774	0.3674	Out
-2.5541	0.7229	-2.5541	Boundry Point
-2	0.7377	-2	In
-1	0.8321	-1	In

is satisfied. Since there are two variables in this inequality, one of them will be eliminated. Replace term bk in $s_0 - a - bk > \frac{-1}{\tau}$ with its equivalent from (22). In that case, the condition will change into $a_d e^{-\tau s_0} > \frac{-1}{\tau}$. This means that if a_d is a positive number then s_0 can acquire any desired value, which means that right-most eigenvalue of the system (21) can be place at any desired point on the real axis and if a_d is a negative number then s_0 should acquire the values that satisfies $a_d e^{-\tau s_0} > \frac{-1}{\tau}$. For instance assume that in (21) $a = 1$, $b = 2$, $a_d = -3$ and $\tau = 0.2$ which requires that inequality $s_0 < -2.5541$ be correct. This fact is very conspicuous in table2

Matrix Case: Consider the following LTD system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t) \\ u(t) &= Kx(t) \end{aligned} \tag{25}$$

where its characteristic equation is

$$S_0 = A + BK + A_d e^{-\tau S_0} \tag{26}$$

and its right-most eigenvalue can be derived from

$$S_0 = \tfrac{1}{\tau}\mathbf{W}_0(\tau A_d Q_0) + A + Bk \tag{27}$$

Right-most eigenvalue of system (25) can be placed at desired feasible points regardless of the value of τ if $0 < eig(A_d)$. If inequality $0 < eig(A_d)$ does not hold true, then the system can be stabilized for a determined amount of delay.

Proof: It is obvious that to deal with scalar Lambert W function instead of the matrix version the output of the function must have real and diagonal Jordan form and because of the same reasoning used in section *3.1* the following conditions are imposed

$$\begin{aligned} &1)\ \ Jordan(S_0 - A - BK) = J \ \text{ should be real-diagonal} \\ &2)\ \ eig(S_0 - A - BK) > \tfrac{-1}{\tau} \end{aligned} \tag{28}$$

Since there are two variables K and S_0 in the conditions and dealing with both simultaneously is hard, one of them must be eliminated from the equation and that will be carried out by replacing the equivalent of term BK from equation (26) into the (28), which would result in

$$\begin{aligned} &1)\ \ Jordan(A_d e^{-\tau S_0}) \ \text{ should be real-diagonal} \\ &2)\ \ eig(A_d e^{-\tau S_0}) > \tfrac{-1}{\tau} \end{aligned} \tag{29}$$

Now one can satisfy the conditions in (29) by choosing adequate S_0 and after that feedback K can be found from (26). Like section 3.1, there are four different class of matrix A_d. The case in which A has real-distinct eigenvalues can be merge into the case where it has real-repeated eigenvalues. Consequently this case will not be discussed directly.

Part 1: assume that A_d has real-repeated eigenvalues which means it will have Jordan decomposition of $A_d = Q_{A_d} J_1 Q_{A_d}^{-1}$, where

$$J_1 = \begin{bmatrix} \lambda_p & c_1 & 0 & \cdots & 0 \\ 0 & \lambda_p & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p & c_{n-1} \\ 0 & 0 & 0 & \cdots & \lambda_p \end{bmatrix}, c_i \in \{0,1\} \tag{30}$$

by assuming $S_0 = Q_{A_d} S Q_{A_d}^{-1}$ where $S = diag\{s_1, s_2, \cdots, s_n\}$, $J_1 e^{-\tau S}$ would be an upper triangular matrix and

$$\begin{aligned} eig(J_1) \times eig(e^{-\tau S}) &= eig(J_1 e^{-\tau S}) \\ &= eig(A_d e^{-\tau S_0}) \end{aligned} \tag{31}$$

The multiplication presented below can make it easy for readers to comprehend (31)

$$J_1 e^{-\tau S} = \begin{bmatrix} \lambda_p e^{-\tau s_1} & c_1 e^{-\tau s_2} & \cdots & 0 \\ 0 & \lambda_p e^{-\tau s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_{n-1} e^{-\tau s_n} \\ 0 & 0 & \cdots & \lambda_p e^{-\tau s_n} \end{bmatrix} \tag{32}$$

Note that $J_1 e^{-\tau S}$ has real and distinct eigenvalues which means $A_d e^{-\tau S_0}$ would have real-diagonal Jordan form and from (32) it can be concluded that for $\lambda_p > 0$, s_i can acquire any desired value to satisfy (29).

Part 2: Now assume that A_d has distinct complex eigenvalues. As a result of that Jordan form of matrix $A_d = Q_{A_d} J_2 Q_{A_d}^{-1}$ would have the following representation

$$J_2 = \begin{bmatrix} \alpha_1 + j\beta_1 & 0 & 0 & 0 \\ 0 & \alpha_1 - j\beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 + j\beta_2 & 0 \\ 0 & 0 & 0 & \alpha_2 - j\beta_2 \end{bmatrix} \tag{33}$$

by assuming $S_0 = Q_{A_d} S Q_{A_d}^{-1}$, matrix S is proposed

$$S = \begin{bmatrix} s_1 + j\theta_1 & 0 & 0 & 0 \\ 0 & s_1 - j\theta_1 & 0 & 0 \\ 0 & 0 & s_2 + j\theta_2 & 0 \\ 0 & 0 & 0 & s_2 - j\theta_2 \end{bmatrix} \tag{34}$$

which makes $J_2 e^{-\tau S}$ have the following set-up

$$\begin{bmatrix} \chi_1 + j\Omega_1 & 0 & 0 & 0 \\ 0 & \chi_1 - j\Omega_2 & 0 & 0 \\ 0 & 0 & \chi_2 + j\Omega_1 & 0 \\ 0 & 0 & 0 & \chi_2 - j\Omega_2 \end{bmatrix} \tag{35}$$

where

$$\begin{aligned} \chi_i &= e^{-\tau s_i}(\alpha_i \cos \tau\theta_i + \beta_i \sin \tau\theta_i) \\ \Omega_i &= e^{-\tau s_i}(\beta_i \cos \tau\theta_i - \alpha_i \sin \tau\theta_i) \end{aligned} \tag{36}$$

To satisfy the conditions in (28) the eigenvalues of (35) must be on the real axis ($\Omega_i = 0$) and all of them must be in the range of the principle branch of Lambert W Function ($\chi_i > \frac{-1}{\tau}$). Following conditions guarantee to do so

$$\begin{aligned} \tan \tau\theta_i = \frac{\beta_i}{\alpha_i} \\ e^{-\tau s_i}(\frac{\alpha_i^2+\beta_i^2}{\alpha_i}) \geq \frac{-1}{\tau} \end{aligned} \tag{37}$$

Since both $\pm\beta_i$ and $\pm\theta_i$ have appeared in conjugate terms in the equations, considering that β_i and θ_i are positive numbers will not affect the logic in the proofs. Moreover, the designer would determine a value for θ. Taking all into the consideration, from (37) It is obvious that by assuming $0 < \tau\theta_i < \frac{\pi}{2}$ for $0 < \alpha_i$ and presuming $\frac{\pi}{2} < \tau\theta_i < \pi$ when $0 > \alpha_i$ is true, conditions in (37) are satisfied for any real value of s_i. Thus, by choosing right value for θ_i, the real part of the right-most eigenvalue of the system can acquire any desired real number.

Part 3: In this segment consider that matrix $A_d = Q_{A_d} J_3 Q_{A_d}^{-1}$ has repetitive complex eigenvalues. In that case its Jordan form will have the following structure.

$$J_3 = \begin{bmatrix} \alpha e^{j\beta} & 1 & 0 & 0 \\ 0 & \alpha e^{j\beta} & 0 & 0 \\ 0 & 0 & \alpha e^{-j\beta} & 1 \\ 0 & 0 & 0 & \alpha e^{-j\beta} \end{bmatrix} \tag{38}$$

where $\beta \notin \{0, \pi\}$. In this case, a different approach is adopted because the conventional methods. The proposed S_0 would be

$$S_0 = \frac{-1}{\tau} \ln \frac{A_d^{-1}}{h} \tag{39}$$

where the logarithm and A_d^{-1} exist since $\alpha e^{\pm j\beta} \notin \mathbf{R}^-$ and $\alpha \neq 0$ and also $h \in \mathbf{R}$ [12], which satisfies

$$0 < h\alpha < 1 \tag{40}$$

This condition makes S_0 a Hurwitz matrix since from (39) it can be shown that $eig(S_0) = \frac{\ln(\alpha h) \pm j\beta}{\tau}$. Moreover, S_0 has to fulfill (29). Considering (39) the condition in (29) changes into the following inequality

$$-1 < \frac{\tau}{h} \tag{41}$$

By choosing adequate h which satisfies (40) and (41) simultaneously, S_0 is evaluated and the system is stabilised. Note that since α is always a positive number, due to (40) h will be positive too and consequently system will be stable for any value of τ. ■

Although in this paper the four possibilities for matrices A and A_d went under study respectively, the reasoning used for Jordan forms can easily expand to Jordan blocks. Which means that proposed approach can be used on the systems that are compositions of four mentioned cases. For instance, in section 3.2 assume that $A_d = Q_{A_d} J_c Q_{A_d}^{-1}$ where

$$J_c = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} \tag{42}$$

the corresponding matrix $S = Q^{-1} S_0 Q$ for this particular system will have the following pattern

$$S_c = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 + j\theta & 0 \\ 0 & 0 & 0 & s_3 - j\theta \end{bmatrix} \tag{43}$$

which under the above-mentioned circumstances it would place the right-most eigenvalue of the time-delay system at desired point.

4 Numerical Example

Example 1: Consider the following time-delay system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau) \\ u(t) &= Kx(t) \end{aligned} \tag{44}$$

where A=[0 1; -0.1 1], B=[0 -2; 0.5 1] and τ =1. The open-loop system is unstable with two eigenvalues located at $\lambda_1 = 0.112$ and $\lambda_2 = 0.887$. Using the proposed method, feedback $k = [-0.2173 \; -2.5488 \; ; \; 0.1708 \; 0.3109]$ is evaluated; which, puts all the eigenvalues of the closed-loop system on the left half plane. The real parts of the twenty right-most eigenvalues of system can be found in fig2. As it is obvious in fig2, in this method the best case scenario happening at the exact point where $S_0 = S_{-1}$, which is natural due to the structure that is proposed in this article to implement the feedback policy for systems in section *3.1*.

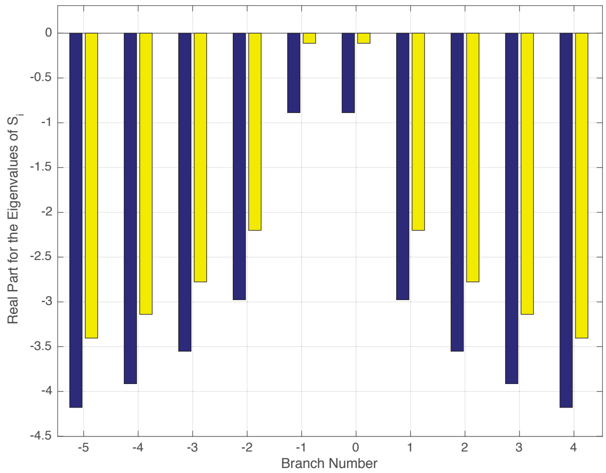

Fig. 2. Real parts of the right-most Eigenvalues for the closed-loop system in example 1

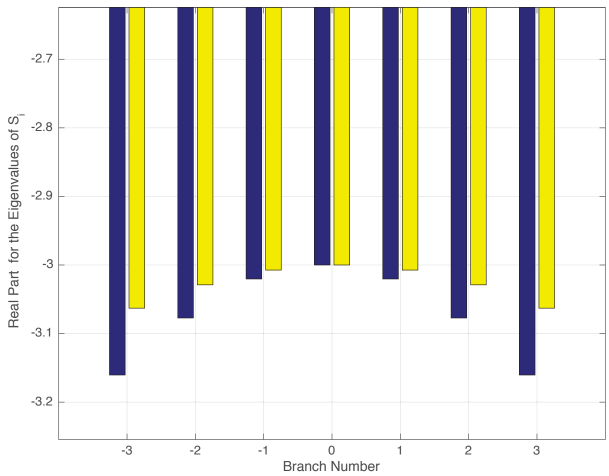

Fig. 3. Real parts of the right-most Eigenvalues of closed-loop system in example 2

Example 2: Consider the following system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t) \\ u(t) &= Kx(t) \end{aligned} \tag{45}$$

where A=[1.1 -0.1732 ; -0.06 1], A_d=[3.6 -1.25 ; 1.9 0.35] and B=[0 -1.1;1 2]. The right-most eigenvalues of open-loop system are $\lambda_1 = 1.5450$ and $\lambda_2 = 1.4399$. Considering matrices in the example desired right-most eigenvalues -3 is feasible for this particular system. Therefore after following the given structure in section *3.2* matrix K=[-177.0260 34.9339; 69.4618 -22.9819] is evaluated and as it can be seen in fig.3 the right-most eigenvalues are placed at the desired points.

5 Conclusion

The analytical and more reliable method provided in this article engages matrix Lambert W function through the scalar Lambert W function. Therefore, what it lacks in being conservative, compensates in being definitive. The basics of this paper are two novel ideas; using the matrix Lambert W range, instead of the function itself and manipulating and determining adequate matrix S_0 alongside matrix K.

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