4

Linear Matrix Inequalities in Control

Guido Herrmann, Matthew C. Turner, and Ian Postlethwaite

Summary. This chapter gives an introduction to the use of linear matrix inequalities (LMIs) in control. LMI problems are defined and tools described for transforming matrix inequality problems into a suitable LMI-format for solution. Several examples explain the use of these fundamental tools.

4.1 Introduction to LMI Problems

The design of a controller should in general satisfy four basic requirements:

1. *Closed-Loop Stability*
2. *Robustness:* The closed-loop controller has to remain stable despite uncertainty in the mathematical plant description or disturbances.
3. *Performance:* The controller has to have certain dynamical or steady state characteristics such as rise time, overshoot, controller bandwidth or steady state error etc.
4. *Robust Performance:* The controller has to remain *well performing* and certainly stable, although disturbances and uncertainties affect the plant. [1]

These control design requirements are usually best encoded in the form of an optimization criterion subject to some constraints e.g., terminal constraints or practical constraints on plant variables. Control design optimization criteria were initially based on the idea of Linear Quadratic Gaussian Control [1, 1, 8] which was later generalized to the idea of H_2-controller design [20]. The development of small gain theory [10, 17] layed the foundations of robust H_∞-control [3, 20]. For linear systems and suitable optimization criteria, such as H_2- and H_∞, the solution to the optimization problem is readily found solving Riccati-equations [3, 19, 20]. Many of these optimal control problems can be stated in terms of linear matrix inequalities and their existence can be traced back over 100 years to the work of Lyapunov. Linear matrix inequalities are matrix inequalities which are linear (or affine) in a set of *matrix* variables.

[1] It is obvious that robust performance implies robustness and performance but it is a stronger requirement than considering robustness and performance each as a single entity.

M.C. Turner et al. (Eds.): Mathe. Methods for Robust & Nonlin. Ctrl., LNCIS 367, pp. 123-142, 2007.
springerlink.com

It is interesting to note that the equivalence between the Riccati-equation and LMI formulations of the control problem was found at an early point in the seventies [19]. This created the possibility of solving optimal control problems using LMI-methods, as the numerical LMI-optimization problem was found to be convex. However, only during the past 10-15 years has the development of sophisticated numerical routines, i.e., semi-definite programming [11,18], made it possible to solve LMIs in a reasonably efficient manner.

From a control engineering perspective, one of the main attractions of LMIs is that they can be used to solve problems which involve several matrix variables, and, moreover, different structures can be imposed on these matrix variables. Another attractive feature of LMI methods is that they are flexible, so it is often relatively straightforward to pose a variety of problems as LMI problems amenable to LMI methods. Furthermore, in many cases the use of LMIs can remove restrictions associated with conventional methods and aid their extension to more general scenarios. Often LMI methods can be applied in instances where conventional methods either fail, or struggle to find a solution [15]. In actual fact, the flexibility of LMIs has created a much wider scope for controller design. They allow the efficient consideration of H_2 and H_∞-constraints for performance, robustness and robust performance in the design of one single controller [13,14]. Thus an advantage of LMIs, also in a pedagogical sense, is that they are able to *unite* many previous results in a common framework. This can also enable one to obtain additional insight into established areas.

Some of the contents of this chapter are available as a University of Leicester Technical report [16] which draws heavily on the material contained within the book in [2] and the MATLAB®LMI control toolbox in [6]. A similar treatment can be found in [12]. In order to convey the main points, the presentation is somewhat condensed and the interested reader should consult the work in [2] for a more complete exposure to LMIs.

4.1.1 Fundamental LMI Properties

A notion central to the understanding of matrix inequalities is *definiteness*. In particular, a matrix Q is defined to be *positive definite* if

$$x^T Qx > 0 \quad \forall x \neq 0 \tag{4.1}$$

Likewise, Q is said to be *positive semi-definite* if

$$x^T Qx \geq 0 \quad \forall x \tag{4.2}$$

It is common practise to write $Q > 0$ $(Q \geq 0)$ to indicate that it is positive (semi-) definite. In particular, we are interested in positive definite matrices which are also symmetric, i.e., $Q = Q^T$. A symmetric, positive definite matrix has two key features: it is square and all of its eigenvalues are positive real. A symmetric, positive semi-definite matrix shares the first attribute, but the last is relaxed to the requirement that all of its eigenvalues are positive real or zero. A matrix $P = -Q$ is said to be *negative (semi) definite* if Q is positive (semi) definite. To indicate negative (semi) definiteness we write $P < 0$ $(P \leq 0)$. In fact, once the notation $Q > 0$ $(Q \geq 0)$ or $P > 0$ $(P \geq 0)$ is used, we usually also require Q and P to be symmetric and we will do so in the rest of this chapter, although from a mathematical point of view this is not necessary. Nevertheless, numerical solution routines for LMIs have enforced this fact as this simplifies the computational process and it is therefore a common assumption for LMIs [2].

The basic structure of an LMI is

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{4.3}$$

where $x \in \mathbb{R}^m$ is a variable and F_0, F_i are given constant symmetric matrices.

The basic LMI problem - the *feasibility* problem - is to find x such that inequality (4.3) holds. Note that $F(x) > 0$ describes an *affine* relationship in terms of the matrix x. Normally the variable x, which we are interested in, is composed of one or many matrices whose columns have been 'stacked' as a vector. That is,

$$F(x) = F(X_1, X_2, \ldots, X_n) \tag{4.4}$$

where $X_i \in \mathbb{R}^{q_i \times p_i}$ is a matrix, $\sum_{i=1}^{n} q_i \times p_i = m$, and the columns of all the matrix variables are stacked up to form a single vector variable.
Hence, from now on, we will consider functions of the form

$$\begin{aligned} F(X_1, X_2, \ldots, X_n) &= F_0 + G_1 X_1 H_1 + G_2 X_2 H_2 + \ldots && (4.5) \\ &= F_0 + \sum_{i=1}^{n} G_i X_i H_i > 0 && (4.6) \end{aligned}$$

where F_0, G_i, H_i are given matrices and the X_i are the matrix variables which we seek.

4.1.2 Systems of LMIs

In general, we are frequently faced with LMI constraints of the form

$$\begin{aligned} F_1(X_1, \ldots, X_n) &> 0 && (4.7) \\ \vdots &> 0 && (4.8) \\ F_p(X_1, \ldots, X_n) &> 0 && (4.9) \end{aligned}$$

where

$$F_j(X_1, \ldots, X_n) = F_{0j} + \sum_{i=1}^{n} G_{ij} X_i H_{ij} \tag{4.10}$$

However, it is easily seen that, by defining $\tilde{F}_0, \tilde{G}_i, \tilde{H}_i, \tilde{X}_i$ as

$$\begin{aligned} \tilde{F}_0 &= \mathrm{diag}(F_{01}, \ldots, F_{0p}) && (4.11) \\ \tilde{G}_i &= \mathrm{diag}(G_{i1}, \ldots, G_{ip}) && (4.12) \\ \tilde{H}_i &= \mathrm{diag}(H_{i1}, \ldots, H_{ip}) && (4.13) \\ \tilde{X}_i &= \mathrm{diag}(X_i, \ldots X_i) && (4.14) \end{aligned}$$

we actually have the inequality

$$F_{big}(X_1,\ldots,X_n) := \tilde{F}_0 + \sum_{i=1}^{n} \tilde{G}_i \tilde{X}_i \tilde{H}_i > 0 \tag{4.15}$$

That is, we can represent a (big) system of LMIs as a single LMI. Therefore, we do not distinguish a single LMI from a system of LMIs; they are the same mathematical entity. We may also encounter systems of LMIs of the form:

$$F_1(X_1,\ldots,X_n) > 0 \tag{4.16}$$
$$F_2(X_1,\ldots,X_n) > F_3(X_1,\ldots,X_n). \tag{4.17}$$

Again, it is easy to see that this can be written in the same form as inequality (4.15) above. For the remainder of the chapter, we do not distinguish between LMIs which can be written as above, or those which are in the more generic form of inequality (4.15).

4.1.3 Types of LMI Problems

The term 'LMI problem' is rather vague and in fact there are several sub-groups of LMI problems. These will be described below in the same way that they are separated in the MATLAB®LMI toolbox. Note that by 'LMI problem' we normally mean solving an optimization problem or an eigenvalue problem with LMI constraints.

LMI Feasibility Problems

These are simply problems for which we seek a *feasible* solution $\{X_1,\ldots,X_n\}$ such that

$$F(X_1,\ldots,X_n) > 0 \tag{4.18}$$

We are not interested in the optimality of the solution, only in finding a solution, which may not be unique.

Example 4.1. (Determining stability of a linear system). Consider an autonomous linear system

$$\dot{x} = Ax \tag{4.19}$$

then the Lyapunov LMI problem for proving asymptotic stability of this system is to find a $P > 0$ such that

$$A^T P + PA < 0 \tag{4.20}$$

This is obviously an LMI feasibility problem in $P > 0$. However, given any $P > 0$ which satisfies this, it is obvious that any matrix from the set

$$\mathscr{P} = \{\beta P : \ \text{scalar}\ \beta > 0\} \tag{4.21}$$

also solves the problem. In fact, as will be seen later, the matrix P forms part of a Lyapunov function for the linear system.

Linear Objective Minimization Problems

These problems are also called eigenvalue problems. They involve the minimization (or maximization) of some *linear scalar* function, $\alpha(.)$, of the matrix variables, subject to LMI constraints:

$$\min \alpha(X_1,\ldots,X_n) \quad \text{s.t.} \tag{4.22}$$
$$F(X_1,\ldots,X_n) > 0 \tag{4.23}$$

where we have used the abbreviation 's.t.' to mean 'such that'. In this case, we are therefore trying to optimize some quantity whilst ensuring some LMI constraints are satisfied.

Example 4.2. (Calculating the $\mathscr{H}_\infty$ norm of a linear system). Consider a linear system

$$\dot{x} = Ax + Bw \tag{4.24}$$
$$z = Cx + Dw \tag{4.25}$$

then the problem of finding the $\mathscr{H}_\infty$ norm of the transfer function matrix T_{zw} from w to z is equivalent to the following optimization procedure (see, for example the work in [5]):

$$\min \gamma \quad \text{s.t.} \tag{4.26}$$
$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$
$$P > 0 \tag{4.27}$$

Note that although $\gamma > 0$ is unique, the uniqueness of $P > 0$ is, in general, not guaranteed.

Generalized Eigenvalue Problems

The generalized eigenvalue problem, or GEVP, is slightly different to the preceding problem in the sense that the objective of the optimization problem is not actually convex, but *quasi-convex*. However, the methods used to solve such problems are similar. Specifically a GEVP is formulated as

$$\min \lambda \quad \text{s.t.} \tag{4.28}$$
$$F_1(X_1,\ldots,X_n) + \lambda F_2(X_1,\ldots,X_n) < 0 \tag{4.29}$$
$$F_2(X_1,\ldots,X_n) < 0 \tag{4.30}$$
$$F_3(X_1,\ldots,X_n) < 0 \tag{4.31}$$

The first two lines are equivalent to minimizing the largest 'generalized' eigenvalue of the matrix pencil $F_1(X_1,\ldots,X_n) + \lambda F_2(X_1,\ldots,X_n)$. In some cases, a GEVP problem can be reduced to a linear objective minimization problem, through an appropriate change of variables.

Example 4.3. (Bounding the decay rate of a linear system). A good example of a GEVP is given in [2]. Given a stable linear system $\dot{x} = Ax$, the decay rate is the largest α such that

$$\|x(t)\| \leq \exp(-\alpha t)\beta\|x(0)\| \quad \forall x(t) \tag{4.32}$$

where β is a constant. If we choose $V(x) = x^T Px > 0$ as a Lyapunov function for the system and ensure that $\dot{V}(x) \leq -2\alpha V(x)$ it is easily shown that the system will have a decay rate of at least α. Hence the problem of finding the decay rate could be posed as the optimization problem

$$\min -\alpha \quad \text{s.t.} \tag{4.33}$$
$$A^T P + PA + 2\alpha P < 0 \tag{4.34}$$
$$-P < 0 \tag{4.35}$$

This problem is a GEVP with the functions

$$F_1(P) := A^T P + PA \tag{4.36}$$
$$F_2(P) := -2P \tag{4.37}$$
$$F_3(P) := -I \tag{4.38}$$

4.2 Tricks in LMI Problems

Although many control problems can be cast as LMI problems, a substantial number of these need to be manipulated before they are in a suitable LMI problem format. Fortunately, there are a number of common tools or 'tricks' which can be used to transform problems into suitable LMI forms. Some of the more useful ones are described below.

4.2.1 Change of Variables

Many control problems can be posed in the form of a set of nonlinear matrix inequalities; that is, the inequalities are nonlinear in the matrix variables we seek. However by defining new variables it is sometimes possible to 'linearise' the nonlinear inequalities, hence making them solvable by LMI methods.

Example 4.4. (State feedback control synthesis problem). Consider the problem of finding a matrix $F \in \mathbb{R}^{m\times n}$ such that the matrix $A + BF \in \mathbb{R}^{n\times n}$ has all of its eigenvalues in the open left-half complex plane. By the theory of Lyapunov equations (see [20]), this is equivalent to finding a matrix F and a positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that the following inequality holds

$$(A + BF)^T P + P(A + BF) < 0 \tag{4.39}$$

or

$$A^T P + PA + F^T B^T P + PBF < 0 \tag{4.40}$$

This problem is not in LMI form due to the terms which contain products of F and P these terms are 'nonlinear' and as there are products of two variables they are said to be 'bilinear'. If we multiply on either side of equation (4.40) by $Q := P^{-1}$ (which does not change the definiteness of the expression since $\mathrm{rank}(P) = \mathrm{rank}(Q) = n$) we obtain

$$QA^T + AQ + QF^T B^T + BFQ < 0 \tag{4.41}$$

This is a new matrix inequality in the variables $Q > 0$ and F, but it is still nonlinear. To rectify this, we simply define a second new variable $L = FQ$ giving

$$QA^T + AQ + L^T B^T + BL < 0 \tag{4.42}$$

We now have an LMI feasibility problem in the new variables $Q > 0$ and $L \in \mathbb{R}^{m\times n}$. Once this LMI has been solved we can recover a suitable state-feedback matrix as $F = LQ^{-1}$ and our Lyapunov function as $P = Q^{-1}$. Hence, by making a change of variables we have obtained an LMI from a nonlinear matrix inequality.

The key facts to consider when making a change of variables is the assurance that the original variables can be recovered and that they are not over-determined.

4.2.2 Congruence Transformation

For a given positive definite matrix $Q \in \mathbb{R}^{n\times n}$, we know that, for another real matrix $W \in \mathbb{R}^{n\times n}$ such that $\mathrm{rank}(W) = n$, the following inequality holds

$$WQW^T > 0 \tag{4.43}$$

In other words, *definiteness* of a matrix is invariant under pre and post-multiplication by a full rank real matrix, and its transpose, respectively. The process of transforming $Q > 0$ into equation (4.43) using a real full rank matrix is called a 'congruence transformation'. It is very useful for 'removing' bilinear terms in matrix inequalities and is often used, in conjunction with a change of variables, to make a bilinear matrix inequality *linear*. Often W is chosen to have a diagonal structure.

Example 4.5. (Making a bilinear matrix inequality linear). Consider

$$Q = \begin{bmatrix} A^T P + PA & PBF + C^T V \\ \star & -2V \end{bmatrix} < 0 \tag{4.44}$$

where the matrices $P > 0, V > 0$ and F (definiteness not specified) are the matrix variables and the remaining matrices are constant. The $\star$ in the bottom left entry of the matrix denotes the term required to make the expression symmetric and will be used frequently hereafter. Notice that this inequality is bilinear in the variables P and F which occur in the (1,2) and (2,1) elements of the matrix $Q \in \mathbb{R}^{(n+p)\times(n+p)}$. However, if we choose the matrix

$$W = \begin{bmatrix} P^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{4.45}$$

which is full rank ($\mathrm{rank}(W) = n+m$) by virtue of the inverses of P and V (which exist as the matrices are positive definite), then calculating WQW^T gives

$$WQW^T = \begin{bmatrix} P^{-1}A^T + AP^{-1} & BFV^{-1} + P^{-1}C^T \\ \star & -2V^{-1} \end{bmatrix} < 0 \tag{4.46}$$

Hence, in the new variables $X = P^{-1}$, $U = V^{-1}$ and $L = FV^{-1}$ we have a *linear* matrix inequality

$$WQW^T = \begin{bmatrix} XA^T + AX & BL + XC^T \\ \star & -2U \end{bmatrix} \tag{4.47}$$

Notice that the original variables can be recovered by inverting X and U and computing $F = LU^{-1}$

4.2.3 Schur Complement

The main use of the Schur complement is to transform quadratic matrix inequalities into linear matrix inequalities, or at least as a step in this direction. Schur's formula says that the following statements are equivalent:

i.

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0$$

ii.

$$\Phi_{22} < 0$$
$$\Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^T < 0$$

A non-strict form involving a Moore-Penrose pseudo inverse also exists if Φ is only negative semi-definite; see [2].

Example 4.6. (Making a quadratic inequality linear). Consider the LQR-type matrix inequality (Riccati inequality)

$$A^T P + PA - PBR^{-1}B^T P + Q < 0 \tag{4.48}$$

where $P > 0$ is the matrix variable and the other matrices are constant with $Q, R > 0$. This inequality can be used to minimize the cost function

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt \tag{4.49}$$

for the computation of the state feedback controller $u = Kx$ controlling

$$\dot{x} = Ax + Bu$$

with an initial condition of $x(0) = x_0$. We know that the solution to this problem is

$$K = -R^{-1}B^T\tilde{P}, \qquad A^T\tilde{P} + \tilde{P}A - \tilde{P}BR^{-1}B^T\tilde{P} + Q = 0$$

and

$$J = \min_u \int_0^\infty (x^T Qx + u^T Ru)dt = x_0^T\tilde{P}x_0$$

However, the solution can be also obtained from a linear matrix inequality problem. Using a congruence transformation for $S = P^{-1}$, it follows from (4.48)

$$SA^T + AS - BR^{-1}B^T + SQS < 0 \tag{4.50}$$

We can now define

$$\Phi_{11} := SA^T + AS - BR^{-1}B^T \tag{4.51}$$
$$\Phi_{12} := S \tag{4.52}$$
$$\Phi_{22} := -Q^{-1} \tag{4.53}$$

and use the Schur complement identities. Thus, we can transform our Riccati inequality into

$$\begin{bmatrix} SA^T + AS - BR^{-1}B^T & S \\ \star & -Q^{-1} \end{bmatrix} < 0 \tag{4.54}$$

In other words, we have transformed a quadratic matrix inequality into a linear matrix inequality. The target is to minimize $x_0^T P x_0$ subject to (4.54). Alternatively, it is possible to minimize σ subject to $\sigma > x_0^T P x_0$ and (4.54). Using the Schur complement again for $x_0^T P x_0 - \sigma < 0$, it follows for $\Phi_{11} := -\sigma$, $\Phi_{12} := x_0^T$ and $\Phi_{22} := -S$:

$$\begin{bmatrix} -\sigma & x_0^T \\ x_0 & -S \end{bmatrix} < 0$$

Hence, the alternative solution to the optimization problem is given by the following LMI-problem:

$$\begin{aligned} \min \sigma \quad & \text{s.t.} \\ \begin{bmatrix} -\sigma & x_0^T \\ x_0 & -S \end{bmatrix} & < 0 \\ \begin{bmatrix} SA^T + AS - BR^{-1}B^T & S \\ \star & -Q^{-1} \end{bmatrix} & < 0 \end{aligned}$$

for which the optimal controller is given by $K = -R^{-1}B^T S^{-1}$.

4.2.4 The S-Procedure

The S-procedure is essentially a method which enables one to combine several quadratic inequalities into one single inequality (generally with some conservatism). There are many instances in control engineering when we would like to ensure that a single quadratic function of $x \in \mathbb{R}^m$ is such that

$$F_0(x) \leq 0 \quad F_0(x) := x^T A_0 x + 2b_0 x + c_0 \tag{4.55}$$

whenever certain other quadratic functions are positive semi-definite i.e., when

$$F_i(x) \geq 0 \quad F_i(x) := x^T A_i x + 2b_0 x + c_0, \quad i \in \{1, 2, \ldots, q\} \tag{4.56}$$

To illustrate the S-procedure, consider $i = 1$, for simplicity. That is, we would like to ensure $F_0(x) \leq 0$ for all x such that $F_1(x) \geq 0$. Now, if there exists a positive definite (or zero) scalar, τ, such that

$$F_{aug}(x) := F_0(x) + \tau F_1(x) \leq 0 \quad \forall x \quad s.t. F_1(x) \geq 0 \tag{4.57}$$

it follows that our goal is achieved. To see this, note that $F_{aug}(x) \leq 0$ implies that $F_0(x) \leq 0$ if $\tau F_1(x) \geq 0$ because $F_0(x) \leq F_{aug}(x)$ if $F_1(x) \geq 0$. Thus, extending this idea to q inequality constraints we have that

$$F_0(x) \leq 0 \quad \text{whenever} \quad F_i(x) \geq 0 \tag{4.58}$$

holds if

$$F_0(x) + \sum_{i=1}^{q} \tau_i F_i(x) \leq 0, \quad \tau_i \geq 0 \tag{4.59}$$

In general the S-procedure is conservative; inequality (4.59) implies inequality (4.58), but not vice versa.[2] The usefullness of the S-procedure is in the possibility of including the τ_i's as variables in an LMI problem.

Example 4.7. (Combining quadratic constraints to yield an LMI.) An instructive example, taken from [2], Chapter 2, involves finding a matrix variable $P > 0$ such that

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} < 0 \tag{4.60}$$

whenever $x \neq 0$ and z satisfy the constraint

$$z^T z \leq x^T C^T C x \tag{4.61}$$

Note that inequality (4.61) is equivalent to

$$(x^T C^T C x - z^T z) \geq 0 \tag{4.62}$$

or

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} C^T C & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0 \tag{4.63}$$

The two quadratic constraints (4.60) and (4.63) can thus be combined with the S-procedure to yield the LMI

$$\begin{bmatrix} A^T P + PA + \tau C^T C & PB \\ B^T P & -\tau I \end{bmatrix} < 0 \tag{4.64}$$

in the variables $P > 0$ and $\tau \geq 0$.

4.2.5 The Projection Lemma and Finsler's Lemma

In some types of control problems, particularly those seeking dynamic controllers, we encounter inequalities of the form

$$\Psi(X) + G(X)\Lambda H^T(X) + H(X)\Lambda^T G^T(X) < 0 \tag{4.65}$$

where X and Λ are the matrix variables and $\Psi(.), G(.), H(.)$ are (normally affine) functions of X but not of Λ. Λ is an unstructured matrix variable i.e., all element values of Λ are not constrained and they are to be considered part of the unstructured matrix variable.[3] In [5], it is proved that inequality (4.65) is satisfied, for some X and Λ, if and only if

[2] Equivalence is only guaranteed for $i = 1$ (see [2]).

[3] It is for instance not permitted to constrain Λ to be sparse, symmetric or diagonal.

$$\begin{cases} W_{G(X)}^T \Psi(X) W_{G(X)} < 0 \\ W_{H(X)}^T \Psi(X) W_{H(X)} < 0 \end{cases} \tag{4.66}$$

where $W_{G(X)}$ and $W_{H(X)}$ are matrices with columns which form bases for the null spaces of $G(X)$ and $H(X)$ respectively. Alternatively, $W_{G(X)}$ and $W_{H(X)}$ are sometimes called *orthogonal complements* of $G(X)$ and $H(X)$ respectively. Note that

$$W_{G(X)} G(X) = 0 \quad W_{H(X)} H(X) = 0. \tag{4.67}$$

The main point of this result (referred to as Gahinet and Apkarian's projection lemma) is that it enables one to transform a matrix inequality which is a, not necessarily linear, function of *two* variables, X and Λ, into two inequalities which are just functions of *one* variable. This has two useful consequences

(i) It can facilitate the derivation of an LMI.
(ii) There are less variables for computation.

In [4], it is also proved that inequality (4.65) was equivalent to two inequalities

$$\begin{cases} \Psi(X) - \sigma G(X) G(X)^T < 0 \\ \Psi(X) - \sigma H(X) H(X)^T < 0 \end{cases} \tag{4.68}$$

for some real σ. In other words, inequalities (4.66) and (4.68) are equivalent. This result is often referred to as *Finsler's Lemma*.

Example 4.8. (State feedback control synthesis problem revisited). Consider again the state feedback synthesis problem of finding $P > 0$ and F such that

$$(A + BF)^T P + P(A + BF) < 0 \tag{4.69}$$

Using the change of variables described earlier in Section 4.2.1 we can change this problem into that of finding $Q > 0$ and L such that

$$QA^T + AQ + L^T B^T + BL < 0 \tag{4.70}$$

If we choose to eliminate the variable L using the projection lemma we get

$$\begin{cases} W_B^T (AQ + QA^T) W_B < 0, \quad Q > 0 \\ W_I^T (AQ + QA^T) W_I < 0, \quad Q > 0 \end{cases} \tag{4.71}$$

However, as W_I is a matrix whose columns span the null space of the identity matrix which is $\mathscr{N}(I) = \{0\}$ the above equation simply reduces to

$$W_B^T (AQ + QA^T) W_B < 0, \quad Q > 0 \tag{4.72}$$

which is an LMI problem. Alternatively, using Finsler's Lemma we get

$$\begin{cases} AQ + QA^T - \sigma BB^T < 0, \quad Q > 0 \\ AQ + QA^T - \sigma I < 0, \quad Q > 0 \end{cases} \tag{4.73}$$

However, we can neglect the second inequality because if we can find a σ satisfying the first inequality, we can always find one which satisfies the second.

Notice that the use of both the projection lemma and Finsler's Lemma effectively reduces our original LMI problem into two separate ones: the first LMI problem involves the calculation of $Q > 0$; the second ivolves the back-substitution of Q into the original problem in order for us to find L (and then F). The reader is, however, cautioned against the possibility of ill-conditioning in this two-step approach. For some problems, normally those with large numbers of variables, X can be poorly conditioned, which can hinder the numerical determination of Λ from equation (4.65).

4.3 Examples

4.3.1 Lyapunov Stability for Continuous-Time Systems

The stability of a nonlinear systems is generally more difficult to ascertain than that of a linear systems. A sufficient (but not necessary) condition was given by Lyapunov; see, for example, [7].

Theorem 4.1 (Lyapunov's Theorem for continuous-time systems). *Given an unbounded positive definite function $V(x) > 0 \quad \forall x \neq 0$ and an autonomous system $\dot{x} = f(x)$, then the system $\dot{x} = f(x)$ is asymptotically stable if*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \quad \forall x \neq 0 \tag{4.74}$$

For linear systems described by

$$\dot{x} = Ax$$

this condition (4.74) is necessary and sufficient. A suitable Lyapunov function is

$$V(x) = x^T Px, \quad P > 0.$$

This implies that the linear system is asymptotically stable if and only if

$$x^T PAx + x^T A^T Px < 0 \quad \forall x \neq 0$$

This is equivalent to what has was previously said in Example 1.

4.3.2 $\mathscr{L}_2$ Gain

In linear systems, the $\mathscr{H}_\infty$ norm is equivalent to the maximum RMS energy gain of the system and is also called the $\mathscr{H}_\infty$ -gain of the linear system. The equivalent measure for nonlinear systems is the so-called $\mathscr{L}_2$ gain, which is a bound on the RMS energy gain. Specifically a *nonlinear* system with input $w(t)$ and output $z(t)$ (see Figure 4.1) is said to have an $\mathscr{L}_2$ gain of γ if

$$\|z\|_2 < \gamma \|w\|_2 + \beta \tag{4.75}$$

where β is a positive constant and $\|(.)\|_2$ denotes the standard 2-norm-in-time ($\mathscr{L}_2$ norm) of a vector, i.e. $\|x\|_2 = \sqrt{\int_{t=0}^{\infty} x^T(t)x(t)dt}$. Thus the $\mathscr{L}_2$ gain of a system can be taken as a measure of the size of its output relative to the size of its input. For a linear system

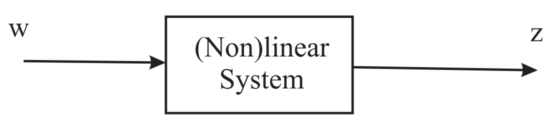

Fig. 4.1. Nonlinear System

$$\dot{x} = Ax + Bw \tag{4.76}$$

$$z = Cx + Dw \tag{4.77}$$

this fact can be derived from the LMIs (4.26)-(4.27) as this is equivalent to:

$$\begin{bmatrix} A^T P + PA + \frac{1}{\gamma} C^T C & PB + \frac{1}{\gamma} C^T D \\ B^T P + \frac{1}{\gamma} D^T C & -\gamma I + \frac{1}{\gamma} D^T D \end{bmatrix} < 0 \tag{4.78}$$

for $P > 0$. Writing this matrix inequality in terms of the vector $\begin{bmatrix} x^T & w^T \end{bmatrix}^T$, it follows that we need to find the minimum of γ so that

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \frac{1}{\gamma} C^T C & PB + \frac{1}{\gamma} C^T D \\ B^T P + \frac{1}{\gamma} D^T C & -\gamma I + \frac{1}{\gamma} D^T D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0, \quad \forall \begin{bmatrix} x^T & w^T \end{bmatrix}^T \neq 0 \tag{4.79}$$

or

$$\begin{aligned} & x^T A^T Px + x^T PAx + \frac{1}{\gamma} x^T C^T Cx + x^T (PB + \frac{1}{\gamma} C^T D) w \\ & + w^T (B^T P + \frac{1}{\gamma} D^T C) x + w^T \frac{1}{\gamma} D^T Dw - \gamma w^T w \\ & \qquad = x^T A^T Px + x^T PA^T x + 2x^T PBw + \frac{1}{\gamma} z^T z - \gamma w^T w < 0 \end{aligned} \tag{4.80}$$

Defining now $V = x^T Px$, it is easily derived from

$$\dot{V} = x^T A^T Px + x^T PAx + 2x^T PBw$$

and (4.80) that:

$$\dot{V} + \frac{1}{\gamma} z^T z - \gamma w^T w < 0$$

Integration in the interval $[0, \infty)$ implies

$$V(t = \infty) - V(t = 0) + \int_{t=0}^{\infty} \frac{1}{\gamma} z^T(s) z(s) ds - \int_{t=0}^{\infty} \gamma w^T(s) w(s) ds < 0$$

Shifting terms, taking the square root and using the triangle inequality, it is easily shown that γ is indeed the $\mathscr{L}_2$-gain of the linear system (4.76) as (4.75) follows.

4.3.3 Lyapunov Stability for Discrete-Time Systems

Discrete-time systems are often treated in less detail than continuous-time systems. However, digital technology for controller implementation makes discrete controller design and analysis a pertinent subject of interest. As for Lyapunov stability of continuous-time systems a similar stability theorem can be stated.

Theorem 4.2 (Lyapunov's Theorem for discrete systems). *Given an unbounded positive definite function* $V(x) > 0 \quad \forall x \neq 0$ *and an autonomous system* $x(k+1) = f(x(k))$*, then the system* $x(k+1) = f(x(k))$ *is asymptotically stable if*

$$\Delta V(x(k+1)) = V(x(k+1)) - V(x(k)) < 0 \quad \forall x \neq 0 \tag{4.81}$$

As before, for linear systems described by

$$x(k+1) = Ax(k)$$

this condition (4.81) is necessary and sufficient and a Lyapunov function is given by

$$V(x) = x^T Px, \quad P > 0.$$

Thus, the linear system is asymptotically stable if and only if

$$x^T A^T PAx - x^T Px < 0 \quad \forall x \neq 0$$

or

$$A^T PA - P < 0.$$

4.3.4 l_2 Gain

Similar to a continuous-time system, the computation of the l_2-gain can be very helpful to the design and analysis of discrete-time control systems. As for continuous time (4.75), a discrete-time system with input w and output z has a finite l_2-gain, γ, if

$$\|z\|_2 < \gamma\|w\|_2 + \beta \tag{4.82}$$

The significant difference is the definition of the l_2-norm $\|(.)\|_2$ which is now given by an infinite sum $\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} x^T(k)x(k)}$. For linear systems

$$x(k+1) = Ax(k) + Bw(k) \tag{4.83}$$
$$y(k) = Cx(k) + Dw(k) \tag{4.84}$$

the value of the finite l_2-gain, γ, is the same as the $\mathscr{H}_\infty$ norm or the maximum RMS energy gain of the system. The computation of the l_2-gain is easily achieved by considering the following matrix inequality problem:

$$\min \gamma \quad \text{s.t.} \tag{4.85}$$
$$\begin{bmatrix} A^T PA - P + \frac{1}{\gamma}C^T C & A^T PB + \frac{1}{\gamma}C^T D \\ B^T PA + \frac{1}{\gamma}D^T C & -\gamma I + B^T PB + \frac{1}{\gamma}D^T D \end{bmatrix} < 0 \tag{4.86}$$
$$-P < 0 \tag{4.87}$$

for $P > 0$. This readily follows from the analysis of

$$\Delta V(x(k+1)) + \frac{1}{\gamma}y^T(k)y(k) - \gamma w^T(k)w(k) < 0 \tag{4.88}$$

for a Lyapunov function $V = x^T Px$. To complete the analysis for the l_2-gain, the sum of the inequality of (4.88) over k in the interval $[0, \infty)$ is considered (rather than the integral as in Section 4.3.2). The problem with matrix inequality (4.86) is that it is not linear in γ. This is easily amended by using the Schur complement so that the l_2-gain computation can be equivalently expressed in a convex formulation:

$$\min \gamma \quad \text{s.t.} \tag{4.89}$$

$$\begin{bmatrix} A^T PA - P & A^T PB & C^T \\ B^T PA & -\gamma I + B^T PB & D^T \\ C & D & -\gamma I \end{bmatrix} < 0 \tag{4.90}$$

$$-P < 0 \tag{4.91}$$

The matrix $P > 0$ and the scalar $\gamma > 0$ are variables of this LMI-problem.

Another approach of creating a linear matrix inequality from (4.86) is using the change of variable approach from Section 4.2.1. The first step is to multiply all elements of (4.86) by γ so that

$$\min \gamma \quad \text{s.t.} \tag{4.92}$$

$$\begin{bmatrix} \gamma A^T PA - \gamma P + C^T C & \gamma A^T PB + C^T D \\ \gamma B^T PA + D^T C & -\gamma^2 I + \gamma B^T PB + D^T D \end{bmatrix} < 0 \tag{4.93}$$

$$-P < 0 \tag{4.94}$$

Defining now the new variables $Q = P\gamma$ and $\mu = \gamma^2$ allows us to formulate the equivalent optimization problem:

$$\min \mu \quad \text{s.t.} \tag{4.95}$$

$$\begin{bmatrix} A^T QA - Q + C^T C & A^T QB + C^T D \\ B^T QA + D^T C & -\mu I + B^T QB + D^T D \end{bmatrix} < 0 \tag{4.96}$$

$$-Q < 0 \tag{4.97}$$

which is clearly convex. Now the matrix $Q > 0$ and the scalar $\gamma > 0$ are variables of the LMI-problem and the l_2-gain is readily computed with $\gamma = \sqrt{\mu}$.

4.3.5 Sector Boundedness

The saturation function is defined as

$$sat(u) = [sat_1(u_1), \ldots, sat_2(u_m)]^T \tag{4.98}$$

and $sat_i(u_i) = \text{sign}(u_i) \times \min\{|u_i|, \bar{u}_i\}, \quad \bar{u}_i > 0 \quad \forall i \in \{1, \ldots, m\}$, where $\bar{u}_i$ is the i'th saturation limit. From this, the deadzone function can be defined as

$$Dz(u) = u - sat(u) \tag{4.99}$$

It is easy to verify that the saturation function, $\text{sat}_i(u_i)$ satisfies the following inequality

$$u_i \text{sat}_i(u_i) \geq \text{sat}_i^2(u_i) \tag{4.100}$$

or

$$\text{sat}_i(u_i)[u_i - \text{sat}_i(u_i)]w_i \geq 0 \tag{4.101}$$

for some $w_i > 0$. Collecting this inequality for all i we can write

$$\text{sat}(u)^T W[u - \text{sat}(u)] \geq 0 \tag{4.102}$$

for some diagonal $W > 0$. Similarly, it follows that

$$\text{Dz}(u)^T W[u - \text{Dz}(u)] \geq 0 \tag{4.103}$$

for some diagnal $W > 0$. We will make use of this inequality in the next section when computing the $\mathscr{L}_2$-gain of a linear system with input and output constraints.

4.3.6 A Slightly More Detailed Example

Suppose we consider the computation of the $\mathscr{L}_2$-gain for the linear SISO system with saturated input signal u:

$$\dot{x} = Ax + b\mathrm{sat}(u),\ x \in \mathbb{R}^n \tag{4.104}$$

which is subject to a limitation in the measurement range of the output y (see Figure 4.2):

$$y = \mathrm{sat}(cx + d\mathrm{sat}(u)). \tag{4.105}$$

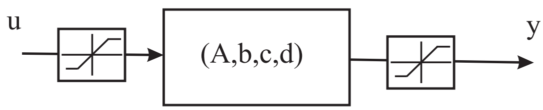

Fig. 4.2. Linear SISO System with input/output saturation

This is usually the case for many practically controlled systems. For instance actuator inputs u can be constrained due to mechanical limits (e.g., valves) or due to digital-to-analogue converter voltage signal limits, while sensor signals are possibly constrained due to sensor voltage range limits or simply by analogue-to-digital converter limits. Hence, the analysis of such systems is vital to practical control systems and will be pursued in greater detail in a later chapter.

An upper bound of the $\mathscr{L}_2$-gain of this system can be computed by considering sector constraints for the saturation functions. Furthermore, the Projection Lemma will be used to show that the $\mathscr{L}_2$-gain of (4.104)–(4.105) is in actual fact not larger than the $\mathscr{L}_2$-gain of the system without any of the saturation limits.

For analysis, we may define

$$s = \mathrm{sat}(u).$$

Hence, similar to (4.103), it follows that

$$sw_1(u - s) \geq 0, \tag{4.106}$$

where $w_1 > 0$. In the same way, it can be shown that for the output signal y we have:

$$yw_2(cx + ds - y) \geq 0 \tag{4.107}$$

for an arbitrary diagonal matrix $w_2 > 0$. We know from Section 4.3.2 that from

$$\dot{V} + \frac{1}{\gamma}y^2 - \gamma u^2 \leq 0$$

it follows that our system of interest (4.104-4.105) has the $\mathscr{L}_2$-gain γ. However, in our case, this analysis is conducted under the condition (4.106–4.107). Hence, from the S-procedure of Section 4.2.4, it follows that

$$\dot{V} + \frac{1}{\gamma}y^2 - \gamma u^2 + 2sw_1(u - s) + 2yw_2(cx + ds - y) < 0 \tag{4.108}$$

for $\begin{bmatrix} x^T & s & u & y \end{bmatrix} \neq 0$ implies similarly an $\mathscr{L}_2$-gain of γ. This is also easily derived when integrating the inequality over the interval $[0,\infty]$ above and considering a similar procedure as in Section 4.3.2 under the assumptions of (4.106–4.107). By computing the expression for $\dot{V}$, it follows that:

$$\begin{aligned} & x^T A^T Px + xPAx + x^T Pbs + sb^T Px \\ & + \frac{1}{\gamma} y^2 - \gamma u^2 + 2sw_1(u-s) + 2yw_2(cx+ds-y) < 0 \end{aligned} \tag{4.109}$$

Rewriting this inequality, it is shown that:

$$\begin{bmatrix} x \\ s \\ u \\ y \end{bmatrix}^T \begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & -2w_1 & w_1 & dw_2 \\ 0 & w_1 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} x \\ s \\ u \\ y \end{bmatrix} < 0$$

for

$$\begin{bmatrix} x^T & s & u & y \end{bmatrix} \neq 0.$$

This is equivalent to the matrix inequality of

$$\begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & -2w_1 & w_1 & dw_2 \\ 0 & w_1 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix} < 0. \tag{4.110}$$

The target is now to minimize the value of $\gamma > 0$ to find the $\mathscr{L}_2$-gain of our system of interest (4.104–4.105). However, the variable γ appears twice in the matrix inequality, once in the denominator of $\frac{1}{\gamma}$. This results in a non-linear and non-convex matrix inequality and it is not easily possible to avoid this non-linearity. Furthermore, this matrix inequality looks rather complicated as the positive definite matrix P, γ, w_1 and w_2 are variables.

The Projection Lemma allows us to derive a significantly simpler matrix inequality which delivers the $\mathscr{L}_2$-gain of our system. The first step is to rewrite the matrix inequality of (4.110) in the same format as for (4.65). Hence,

$$\begin{aligned} & \begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & 0 & 0 & dw_2 \\ 0 & 0 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix} \\ & + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w_1 \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} w_1 \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} < 0. \end{aligned} \tag{4.111}$$

Defining the matrices

$$g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; h_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \; \Psi_1 = \begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & 0 & 0 & dw_2 \\ 0 & 0 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix},$$

allows us to write

$$\Psi_1 + g_1 w_1 h_1^T + h_1 w_1^T g_1^T < 0. \tag{4.112}$$

Thus, the null space matrices W_{g_1} and W_{h_1} for g_1 and h_1 need to be found so that $\left[W_{g_1}^T \; g_1\right]$ and $\left[W_{h_1}^T \; h_1\right]$ are full rank and $W_{g_1} g_1 = 0$, $W_{h_1} h_1 = 0$. This implies:

$$W_{g_1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; W_{h_1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, it follows from (4.66) that both

$$W_{g_1} \Psi_1 W_{g_1}^T = \begin{bmatrix} A^T P + PA & 0 & c^T w_2 \\ 0 & -\gamma & 0 \\ w_2 c & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix} < 0$$

and

$$W_{h_1} \Psi_1 W_{h_1}^T = \begin{bmatrix} A^T P + PA & Pb & c^T w_2 \\ b^T P & -\gamma & d w_2 \\ w_2 c & w_2 d & -2w_2 + \frac{1}{\gamma} \end{bmatrix} < 0$$

have to be satisfied for (4.110) to be true. An extra iteration using the Projection Lemma for w_2 as the variable implies that if $W_{h_1} \Psi_1 W_{h_1}^T < 0$ then we also have $W_{g_1} \Psi_1 W_{g_1}^T < 0$. Hence, we can focus our investigation on $W_{h_1} \Psi_1 W_{h_1}^T$ only:

$$W_{h_1} \Psi_1 W_{h_1}^T = \Psi_2 + g_2 w_2 h_2^T + h_2 w_2 g_2^T \tag{4.113}$$

where

$$g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; h_2 = \begin{bmatrix} c^T \\ d \\ -1 \end{bmatrix}, \; \Psi_2 = \begin{bmatrix} A^T P + PA & Pb & 0 \\ b^T P & -\gamma & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

This allows us to derive the null space matrices W_{g_2} and W_{h_2} for g_2 and h_2

$$W_{g_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; W_{h_2} = \begin{bmatrix} I & 0 & c^T \\ 0 & 1 & d \end{bmatrix}$$

so that

$$W_{g_2} \Psi_2 W_{g_2}^T = \begin{bmatrix} A^T P + PA & Pb \\ b^T P & -\gamma \end{bmatrix}, \; W_{h_2} \Psi_2 W_{h_2}^T = \begin{bmatrix} A^T P + PA + \frac{c^T c}{\gamma} & Pb + \frac{c^T d}{\gamma} \\ b^T P + \frac{dc}{\gamma} & -\gamma + \frac{d^2}{\gamma} \end{bmatrix}.$$

As before we can easily see that $W_{g_2} \Psi_2 W_{g_2}^T < 0$ is always satisfied if $W_{h_2} \Psi_2 W_{h_2}^T < 0$. Hence, the matrix inequality

$$\begin{bmatrix} A^T P + PA + \frac{c^T c}{\gamma} & Pb + \frac{c^T d}{\gamma} \\ b^T P + \frac{dc}{\gamma} & -\gamma + \frac{d^2}{\gamma} \end{bmatrix} < 0 \tag{4.114}$$

is satisfied for $P > 0$ if and only if (4.110) holds. From (4.78), the $\mathscr{L}_2$-gain of the non-linear system (4.104)-(4.105) is exactly the same as for the linear system defined by the system quadruple (A, b, c, d). The computation of the $\mathscr{L}_2$-gain follows the approach shown in Example 2.

This result is easily understood by the following argument. For small input signals of u, the non-linearities in the system of (4.104)–(4.105) are not active. The system acts as a linear system for which the $\mathscr{L}_2$-gain has been found in Section 4.3.2. When the signal u reaches the saturation limits, the linear system is non-linearly constrained and the output may also be non-linearly limited. Hence, the level of the $\mathscr{L}_2$-gain for large amplitude signals must decrease. Thus, the maximum achievable $\mathscr{L}_2$-gain is indeed given only by the linear component of (4.104)-(4.105).

4.4 Summary

This chapter has introduced some important tools for the analysis of closed-loop control systems via linear matrix inequalities. It has been shown that quadratic performance constraints, such as $\mathscr{L}_2$-gain computation, can be equivalently encoded via linear matrix inequalities. Furthermore, we have shown that non-linearities can be considered using sector bounds. By posing control problems in terms of linear matrix inequalities a highly flexible design and anaiysis tool is obtained which allows the combination of different control requirements. wellestablished numerical solvers for semi-definite programming allow the efficient solution of these convex linear matrix inequality problems.

References

1. M. Athans. *Optimal control : an introduction to the theory and its applications*. McGraw-Hill, New York, London, 1966.
2. S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics, 1994.
3. J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Transactions on Automatic Control*, 34(8):831–847, 1989.
4. P. Finsler. Uber das Vorkommen definiter und semi-definiter Formen in Scharen quadratischer Formen. *Comentarii Mathematica Helvetici*, 9:192–199, 1937.
5. P. Gahinet and P. Apkarian. A linear matrix inequality approach to $\mathscr{H}_\infty$ control. *International Journal of Robust and Nonlinear Control*, 4:421–448, 1994.
6. P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali. *LMI Control Toolbox*. The MathWorks Inc., 1995.
7. H.K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, 1996.
8. H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. Wiley-Interscience, New York, 1972.
9. F. L. Lewis. *Optimal control*. John Wiley and Sons, New York, 1986.
10. P. Moylan and D. Hill. Stability criteria for large-scale systems. *IEEE Transactions on Automatic Control*, 23:143–149, 1978.
11. Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. Studies in Applied Mathematics. SIAM, Philadelphia, 1993.
12. S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: Analysis and Design*. Wiley, Chichester, UK, 2nd edition, 2005.
13. C. W. Scherer. Mixed H_2/H_∞ control for time-varying and linear parametrically-varying systems. *Intenational Journal of Robust and Nonlinear Control*, 6:929–952, 1996.

14. C. W. Scherer, P. Gahinet, and M. Chilali. Multi-objective output-feedback control via lmi optimization. *IEEE Transactions on Automatic Control*, 42:896–911, 1997.
15. M.C. Turner, G. Herrmann, and I. Postlethwaite. Accounting for uncertainty in anti-windup synthesis. *submitted*, 2003.
16. M.C. Turner, G. Herrmann, and I. Postlethwaite. An introduction to linear matrix inequalities in control. *University of Leicester Department of Engineering Techincal Report no 02-04*, 2004.
17. A. van der Schaft. *L2-Gain and Passivity Techniques in Nonlinear Control*. Communications and Control Engineering Series. Springer-Verlag, Berlin, 2nd edition, 2000.
18. L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38:49–95, 1996.
19. J. C. Willems. Least stationary optimal control and the algebraic riccati equation. *IEEE Transactions on Automatic Control*, AC-16(6):621–634, 1971.
20. K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, New Jersey, 1996.