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The main results in theory of controllability are formulated for deterministic or stochastic control systems given in a standard form. i.e., given as a first order differential equation driven by an infinitesimal generator of strongly continuous semigroup in an abstract Hilbert or Banach space and disturbed by a deterministic function or by a white noise process. At the same time, some deterministic or stochastic linear systems can be written in a standard form if the state space is enlarged. Respectively, the ordinary controllability conditions for them are too strong since they assume extended state space. It is reasonable to introduce partial controllability concepts, which assume original state space. In this paper, we study necessary and sufficient conditions of partial controllability for deterministic and stochastic linear control systems given in a standard form and their implications to particular cases.

1. Introduction

Theory of controllability, originating from the famous work of Kalman (1960), is well developed for deterministic linear control systems (see for example, Curtain and Pritchard (1978), Curtain and Zwartz (1995), Zabczyk (1995), etc.). There is an essential progress in this theory for stochastic linear control systems as well: the concepts of C- and S-controllability, as extensions of the exact and approximate controllability, are introduced and studied in Bashirov and Kerimov (1997) and Bashirov and Mahmudov (1999) for the partially observed linear control systems with independent white noises and with minor changes these developments are extended to linear control systems with correlated white noises in Bashirov (2003).

In general, the main results in theory of controllability are formulated for deterministic or stochastic control systems given in a *standard form* i.e., given as a first order differential equation driven by an infinitesimal generator of strongly continuous semigroup in an abstract Hilbert or, more generally, Banach space and disturbed by a deterministic function or by a white noise process. At the same time, there are deterministic and stochastic systems, which can be written in a standard form only by enlarging the dimension of the state space. The most popular deterministic systems having this property are higher order differential equations, wave equations and delay equations. The stochastic linear systems driven by coloured and wide band noises can also be reduced to white noise driven linear systems and thus, written in a standard form by enlarging the dimension of the state space (see Bashirov (2003)). The ordinary deterministic and stochastic controllability conditions for them are too strong because these conditions involve the enlarged state space. Therefore, it is reasonable to introduce partial controllability concepts, which assume original state space in cases of such control systems.

Our aim in this paper is to study partial controllability concepts for deterministic and stochastic linear control systems given in a standard form. Our investigations show that the partial controllability concepts have similar properties to the ordinary controllability concepts. Even the respective proofs are similar with minor changes. Therefore, we give guidelines to the proofs, concentrating on new elements. But the area of

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applications of partial controllability concepts is essentially wider than the same for ordinary controllability concepts. Especially, the conditions for partial C- and S-controllability allow to get controllability results for stochastic control systems with coloured and wide band noises. These applications are considered in detail.

2. Notation

We assume that X, U and M are real separable Hilbert spaces $\mathbb{R}^k$ denotes the k-dimensional real Euclidean space. The closure of the set D is denoted by $\overline{D}$. The space of all bounded linear operators from X to U is denoted by $\mathcal{L}(X, U)$. We let $\mathcal{L}(X) = \mathcal{L}(X, X)$ as well. A^* denotes the adjoint of the operator A. The trace of the operator A is denoted by $\operatorname{tr} A$. If $A \in \mathcal{L}(X)$ is self-adjoint and $\langle h, Ah\rangle \geq 0$ for all $h \in X$ (respectively $\langle h, Ah\rangle > 0$ for all $h \in X$ with $h \neq 0$), then we write $A \geq 0$ (respectively, $A > 0$), where $\langle\cdot,\cdot\rangle$ is an inner product and $\|\cdot\|$ is a norm. $A \in \mathcal{L}(X)$ is said to be coercive if it is self-adjoint and $\langle h, Ah\rangle \geq c\|h\|^2$ for all $h \in X$, where $c = \text{const} > 0$. The symbol I denotes an identity operator. Zero operator, zero vector and number zero are denoted by 0 being clear which is meant from the context.

Always it is supposed that two instants are given. The initial time moment is identified with zero and its fixed. The terminal one is denoted by $T (T > 0)$ and it is considered as variable. $L_2(0, T; X)$ denotes the space of equivalence classes of all Lebesque measurable and square integrable with respect to the Lebesgue measure functions from $[0, T]$ to X. The notation $\Delta_T = \{(t, s) : 0 \leq s \leq t \leq T\}$ is used for the triangular set over $[0, T]$. $B_2(\Delta_T, \mathcal{L}(X, U))$ denotes the class of all $\mathcal{L}(X, U)$-valued functions on Δ_T that are strongly measurable and square integrable with respect to the Lebesgue measure on Δ_T (see, for example Curtain and Pritchard (1978), Curtain and Zwart (1995), Bashirov (2003).

All integrals of vector-valued functions are considered in the Bochner sense. For probability, expectation and conditional expectation, the notations **P, E** and $\mathbf{E}(\cdot|\cdot)$ are used, respectively. $\operatorname{cov}(x, y)$ is the covariance operator of the random variables x and y. We let $\operatorname{cov} x = \operatorname{cov}(x, x)$. The integrals of operator-valued functions (except stochastic integrals) are in the strong Bochner sense.

3. Definitions

Consider a control system on $[0, \infty)$. Let x_t^u be its (random or not) state value at time t, corresponding to the control u taken from the set of the admissible controls $\mathcal{U}$. If the control system under consideration is stochastic, then by $\mathcal{F}_t^u$ we denote the σ-algebra generated by the observations on the time interval $[0, t]$ corresponding to the control u. Suppose that X is the state space and $L \in \mathcal{L}(X)$. Assume that the range H of L is a closed subspace of X. For $0 \leq \varepsilon < \infty, 0 \leq p \leq 1$, and $T > 0$, introduce the sets

$$D_T = \{Lx_T^u : u \in \mathcal{U}\}, \tag{1}$$

$$S_T(\varepsilon, p) = \Big\{h \in H : \exists u \in \mathcal{U} \text{ such that } \mathbf{P}\big(\|L\mathbf{E}(x_T^u|\mathcal{F}_T^u) - h\|^2 > \varepsilon\big) \leq 1 - p\Big\}, \tag{2}$$

$$C_T(\varepsilon, p) = \Big\{h \in H : \exists u \in \mathcal{U} \text{ such that } \mathbf{P}(\|L\mathbf{E}(x_T^u|\mathcal{F}_T^u) - h\|^2 > \varepsilon) \leq 1 - p \text{ and } h = L\mathbf{E}x_T^u\Big\}. \tag{3}$$

Definition 1: A deterministic control system will be called

(a) L-partially D^c-controllable for the time T if $D_T = H$;
(b) L-partially D^a-controllable for the time T if $\overline{D_T} = H$.

In particular, if $L = I$ (or, more generally, L is a surjection), then $H = X$, and, consequently, the concepts of L-partial D^c- and D^a-controllability coincide with the well known concepts of exact (or complete) and approximate controllability for deterministic control systems, respectively.

Following to the concepts of S-, C-, and S^0-controllability from Bashirov and Mahmudov (1999) and Bashirov (2003), we introduce their partial versions in the following way.

Definition 2: A stochastic control system will be called

(a) L-partially S-controllable for the time T if $S_T(\varepsilon, p) = H$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$;
(b) L-partially C-controllable for the time T if $C_T(\varepsilon, p) = H$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$;
(c) L-partially S^0- controllable for the time T if $0 \in S_T(\varepsilon, p)$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$.

These concepts of controllability for stochastic systems are well-discussed and their usefulness are justified in Bashirov and Kerimov (1997), Bashirov and Mahmudov (1999) and Bashirov (2003) in case $H = X$. The same are applied to the case when H is a proper subspace of X.

4. Description of the system

Assume that A is the infinitesimal generator of a strongly continuous semigroup e^{At}, $t \geq 0$, $B \in \mathcal{L}(U, X), C \in \mathcal{L}(X, \mathbb{R}^k)$ f: $[0, \infty) \to X$ is so that for all $T > 0$, its restriction on $[0, T]$ belongs to $L_2(0, T; X)$, x_0 is a Gaussian random variable with cov $x_0 = P_0$, m and n are M- and $\mathbb{R}^k$-valued correlated Wiener processes, respectively, with $n_0=0$, $m_0=0$, $\mathbf{E}n_t=0$, $\mathbf{E}m_t=0$,

$$\operatorname{cov}\begin{bmatrix} m_t \\ n_t \end{bmatrix} = \begin{bmatrix} \Phi & \Psi \\ \Psi^* & I \end{bmatrix} t,$$

where Φ is a nuclear operator on M, and x_0 and (m, n) are independent. Consider the partially observable linear stochastic control system

$$\begin{cases} \mathrm{d}x_t^u = (Ax_t^u + Bu_t + f_t)\mathrm{d}t + \mathrm{d}m_t, & t > 0,\ x_0^u = x_0, \\ \mathrm{d}\xi_t^u = Cx_t^u \mathrm{d}t + \mathrm{d}n_t, & t > 0,\ \xi_0^u = 0, \end{cases} \tag{4}$$

where x^u, u and ξ^u are state, control and observation processes. Under the set $\mathcal{U}$ of admissible controls we consider the set of all controls in the linear feedback form

$$u_t = v_t + \int_0^t K_{t,s}\,\mathrm{d}\xi_s^u,$$

where $v : [0, \infty) \to U$ and $K : \{(t, s) : 0 \le s \le t\} \to \mathcal{L}(\mathbb{R}^k, U)$ are so that for all $T > 0$, their restrictions to $[0, T]$ and Δ_T belong respectively to $L_2(0, T; U)$ and $B_2(\Delta_T, \mathcal{L}(\mathbb{R}^k, U))$.

To the system (4) one can associate two control systems. The first one is the deterministic control system

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t^v = Ay_t^v + Bv_t + f_t, \quad t > 0, \quad y_0^v = y_0 = \mathbf{E}x_0, \tag{5}$$

where v is a control on the set $\mathcal{V}$ of all functions v as described above. The second one is the partially observable stochastic control system

$$\begin{cases} \mathrm{d}z_t^w = (Az_t^w + Bw_t)\mathrm{d}t + \mathrm{d}m_t, & t > 0, \quad z_0^w = z_0 = x_0 - \mathbf{E}x_0, \\ \mathrm{d}\eta_t^w = Cz_t^w \mathrm{d}t + \mathrm{d}n_t, & t > 0, \quad \eta_0^w = 0, \end{cases} \tag{6}$$

where w is a control in the set $\mathcal{W}$ of all controls in the linear feedback form

$$w_t = \int_0^t K_{t,s}\,\mathrm{d}\eta_s^w$$

with K as described above. The solutions of the equations in (4), (5) and (6) are meant in the mild sense (see Curtain and Prichard (1978) and Bashirov (2003)).

5. Partial D^c- and D^a-controllability

Fix $T > 0$. To present necessary and sufficient conditions for L-partial D^c- and D^a-controllability for the time T of the system (5) on $\mathcal{V}$, introduce the operator $\Lambda_{T-t} : L_2(t, T; U) \to X$ by

$$\Lambda_{T-t} v = \int_t^T e^{A(T-s)} Bv(s)\,\mathrm{d}s$$

and let $\Gamma_{T-t} = \Lambda_{T-t}\Lambda_{T-t}^*$. Γ_{T-t} is called the controllability operator. We will consider the operators $\tilde{\Lambda}_{T-t} = L\Lambda_{T-t}$ and $\tilde{\Gamma}_{T-t} = L\Gamma_{T-t}L^*$. One can see that $\tilde{\Gamma}_{T-t} \ge 0$. Hence, the resolvent $R(\lambda, -\tilde{\Gamma}_{T-t}) = (\lambda I + \Gamma_{T-t})^{-1}$ is well defined for all $\lambda > 0$ and for all $0 \le t \le T$. If $\tilde{\Gamma}_{T-t}$ is coercive, then $R(\lambda, -\tilde{\Gamma}_{T-t})$ is defined for $\lambda = 0$ as well. The following demonstrates that the role of $\tilde{\Gamma}_{T-t}$ for the concepts of partial controllability is similar to the role of Γ_{T-t} for the ordinary concepts of controllability.

Theorem 1: *Under the above conditions and notation, the following statements are equivalent*:

(a) *The system* (5) *on* $\mathcal{V}$ *is L-partially* D^c*-controllable for the time T*;
(b) $\tilde{\Gamma}_T$ *is coercive*;
(c) $R(\lambda, -\tilde{\Gamma}_T)$ *converges as* $\lambda \to 0$ *in the uniform operator topology*;
(d) $R(\lambda, -\tilde{\Gamma}_T)$ *converges as* $\lambda \to 0$ *in the strong operator topology*;
(e) $R(\lambda, -\tilde{\Gamma}_T)$ *converges as* $\lambda \to 0$ *in the weak operator topology*;
(f) $\lambda R(\lambda, -\tilde{\Gamma}_T) \to 0$ *as* $\lambda \to 0$ *in the uniform operator topology*.

Theorem 2: *Under the above conditions and notation, the following statements are equivalent*:

(a) *The system* (5) *on* $\mathcal{V}$ *is L-partially* D^a*-controllable for the time T*;
(b) $\tilde{B}^* e^{\tilde{A}^* t} L^* h = 0$ *for all* $0 < t \le T$ *implies* $h = 0$;
(c) $\lambda R(\lambda, -\tilde{\Gamma}_T) \to 0$ *as* $\lambda \to 0$ *in the strong operator topology*;
(d) $\lambda R(\lambda, -\tilde{\Gamma}_T) \to 0$ *as* $\lambda \to 0$ *in the weak operator topology*.

The proofs of (1a)⇔(1b) and (2a)⇔(2b) are based on the well-known facts about linear operators and identical to the respective proofs from Curtain and Pritchard (1978); pp. 56–57 and 60, given in the case $H = X$. The proofs of other parts of Theorems 1 and 2 are identical to the respective proofs from Bashirov and Mahmudov (1999) (see also Bashirov (2003)), also given in the case $H = X$. A minor change is needed in the proof of (2a)⇔(2c): instead of Lemma 2 from Bashirov and Mahmudov (1999) (or Lemma 12.10 from Bashirov (2003)), its following modification must be used.

Lemma 1: *Under the above conditions and notation, for given* $h \in H$ *and* $\lambda > 0$, *there exists an optimal* *control $v^\lambda \in \mathcal{V}$ at which the functional*

$$J^\lambda(v) = \|Ly_T^v - h\|^2 + \lambda \int_0^T \|v_t\|^2 \mathrm{d}t \tag{7}$$

along the system (5) *takes on its minimal value on $\mathcal{V}$. Moreover,*

$$v_t^\lambda = -\lambda^{-1} B^* e^{A^*(T-t)} L^*(Ly_T^{v^\lambda} - h), \quad \textit{a.e. on } [0,T], \tag{8}$$

$$Ly_T^{v^\lambda} - h = \lambda R(\lambda, -\tilde{\Gamma}_T)(Le^{AT}y_0 + Lg - h), \tag{9}$$

where $g = \int_0^T e^{A(T-t)} f_t \mathrm{d}t$

Proof: Calculating the variation of the functional J^λ, we obtain the formula (8). Hence,

$$y_T^{v^\lambda} = e^{AT}y_0 + \int_0^T e^{A(T-t)} f_t \mathrm{d}t - \lambda^{-1} \int_0^T e^{A(T-t)} BB^* e^{A^*(T-t)} L^*(Ly_T^{v^\lambda} - h)\mathrm{d}t.$$

This yields

$$Ly_T^{v^\lambda} - h = -\lambda^{-1}\tilde{\Gamma}_T(Ly_T^{v^\lambda} - h) + Le^{AT}y_0 + Lg - h.$$

Consequently,

$$(\lambda I + \tilde{\Gamma}_T)(Ly_T^{v^\lambda} - h) = \lambda(Le^{AT}y_0 + Lg - h).$$

This implies (9). □

Thus we see that the operator $\tilde{\Gamma}_{T-t}$ well-substitutes the controllability operator Γ_{T-t} for partial versions of the concepts of controllability. Therefore, it is reasonable to call $\tilde{\Gamma}_T$ as a L-partial controllability operator. Application of Theorems 1 and 2 to specific systems gives the well-known results for higher order differential equations (see Zabczyk (1995), p. 22) and wave equations (see Zabczyk (1995), Theorem 2.10, and Curtain and Pritchard (1978), Example 3.8). For delay equations, we obtain the following.

Example 1: Consider the following controlled differential delay equation in the Hilbert space H:

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t^v = Ay_t^v + Ny_{t-\varepsilon}^v + \int_{-\varepsilon}^0 M_\theta y_{t+\theta}^v \mathrm{d}\theta + Bv_t, \quad t > 0, \tag{10}$$

with the initial conditions

$$y_0^v = \xi, \quad y_\theta^v = \eta_\theta, \quad -\varepsilon \le \theta < 0, \tag{11}$$

where $\varepsilon > 0$ is a fixed value, A is the generator of the strongly continuous semigroup e^{At}, $t \ge 0$, $N \in \mathcal{L}(H)$, M is a Lebesgue measurable and essentially bounded $\mathcal{L}(H)$-valued function on $[-\varepsilon, 0]$, $B \in \mathcal{L}(U,H)$, $\xi \in H$, $\eta \in L_2(-\varepsilon, 0; H)$ with $\mathrm{d}\eta/\mathrm{d}\theta \in L_2(-\varepsilon, 0; H)$ and $\eta_0 = \xi$. Here, the original state space is H. Enlarge it to $X = M^2(-\varepsilon, 0; H) = H \times L_2(-\varepsilon, 0; H)$ and let $\tilde{A}$ be a linear operator from $D(\tilde{A}) \subseteq X$ to X, defined by

$$\tilde{A}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} A\xi + N\eta_{-\varepsilon} + \int_{-\varepsilon}^0 M_\theta \eta_\theta \mathrm{d}\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}(\eta - \xi) \end{bmatrix},$$

where

$$D(\tilde{A}) = \left\{ \begin{bmatrix} \xi \\ \eta \end{bmatrix} : \xi \in H, \eta \in L_2(-\varepsilon, 0; H), \frac{\mathrm{d}\eta}{\mathrm{d}\theta} \in L_2(-\varepsilon, 0; H), \eta_0 = \xi \right\}.$$

Then for

$$\tilde{y}_t^v = \begin{bmatrix} y_t^v \\ \bar{y}_t^v \end{bmatrix}, \quad \tilde{y}_0 = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

where

$$[\bar{y}_t^v]_t(\theta) = \begin{cases} y_{t+\theta}, & t+\theta > 0, \\ \eta_{t+\theta}, & t+\theta \le 0, \end{cases}$$

the system (10)–(11) can be written in the standard form

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\tilde{y}_t^v = \tilde{A}\tilde{y}_t^v + \tilde{B}v_t, \ t > 0, \ \tilde{y}_0^v = \tilde{y}_0,$$

where the first component of $\tilde{y}_t$ coincides with the original state process y_t described by (10)–(11), but the second one is supplied to bring (10)–(11) into a standard form. Introduce the sets

$$D_T = \{y_T^v : v \in L_2(0,T;U)\} \quad \text{and}$$
$$\tilde{D}_T = \{\tilde{y}_T^v : v \in L_2(0,T;U)\}$$

Following Delfour and Mitter (1972) (see Curtain and Zwart (1995) as well), four basic controllability concepts can be defined for the system (10)–(11):

(a) exact controllability for the time T if $D_T = H$;
(b) approximation controllability for the time T if $\overline{D}_T = H$;
(c) exact M^2-controllability for the time T if $\tilde{D}_T = M^2(-\varepsilon, 0; H) = H \times L_2(-\varepsilon, 0; H)$,
(d) approximate M^2-controllability for the time T if $\overline{\tilde{D}}_T = M^2(-\varepsilon, 0; H) = H \times L_2(-\varepsilon, 0; H)$.

Exact and approximate M^2-controllability concepts for the system (10)–(11) are too hard because they use the extended state space $M^2(-\varepsilon, 0; H)$. In Delfour and Mitter (1972) it is shown that the system (10)–(11) is never exactly M^2-controllable if H is a finite dimensional Euclidean space. Regarding approximate M^2-controllability one can apply non-partial version of Theorem 2 in part (2a)⇔(2b) and get rather strong condition. To get the conditions for exact and approximate controllability of the system (10)–(11) via Theorems 1 and 2 let L be the projection operator

$$L = [I \quad 0] : H \times L_2(-\varepsilon, 0; H) \to H.$$

The semigroup generated by the operator $\tilde{A}$ has the form

$$e^{\tilde{A}t} = \begin{bmatrix} \mathcal{Y}_t & \mathcal{Y}_{1,t} \\ \mathcal{Y}_{2,t} & \mathcal{Y}_{3,t} \end{bmatrix},$$

where $\mathcal{Y}_t$ is a solution of the equation

$$\mathcal{Y}_t = e^{At} + \int_0^{\max(t-\varepsilon,0)} e^{Ar}\left(N\mathcal{Y}_{t-r-\varepsilon} + \int_{-\varepsilon}^{0} M_\theta \mathcal{Y}_{t-r+\theta}\,\mathrm{d}\theta\right)\mathrm{d}r.$$

Then the respective L-partial controllability operator is

$$\tilde{\Gamma}_T = L\Gamma_T L^* = \int_0^T \mathcal{Y}_{T-s}BB^*\mathcal{Y}^*_{T-s}\,\mathrm{d}s.$$

Hence, by (1a)⇔(1b) in Theorem 1, the system (10)–(11) is exactly controllable for the time T if and only if $\int_0^T \mathcal{Y}_{T-s}BB^*\mathcal{Y}^*_{T-s}\,\mathrm{d}s$ is coercive. As far as we know this condition for the exact controllability of the system (10)–(11) is new. Furthermore,

$$\tilde{B}^* e^{\tilde{A}^* t} L^* = B^* \mathcal{Y}^*_t.$$

Respectively, by (2a) ⇔ (2b) in Theorem 2, the system (10)–(11) is approximately controllable for the time T if and only if $B^*\mathcal{Y}^*_t h = 0$ for every $t \in (0, T]$ implies $h = 0$. This condition in the form $\int_0^T \mathcal{Y}_{T-s}BB^*\mathcal{Y}^*_{T-s}\,\mathrm{d}s > 0$ was obtained previously in Delfour and Mitter (1972).

6. Partial S^0-controllability

To prove a sufficient condition for the L-partial S^0-controllability of the system (6), we will use the following:

Lemma 2: *Under the above conditions and notation, for given $\lambda > 0$, there exists an optimal control $w^\lambda \in \mathcal{W}$ at which the functional*

$$J^\lambda(w) = \mathbf{E}\left(\|L\mathbf{E}(z^w_T \mid \mathcal{F}^w_T)\|^2 + \lambda\int_0^T \|w_t\|^2 \mathrm{d}t\right) \tag{12}$$

along the system (6) *takes its minimal value on $\mathcal{W}$. Moreover,*

$$w^\lambda_t = -\lambda^{-1}B^*e^{A^*(T-t)}L^*L\mathbf{E}\left(z^{w^\lambda}_T \mid \mathcal{F}^{w^\lambda}_T\right), \quad \textit{a.e. on } [0, T], \tag{13}$$

$$L\mathbf{E}\left(z^{w^\lambda}_T \mid \mathcal{F}^{w^\lambda}_T\right) = \lambda R(\lambda, -\tilde{\Gamma}_T)L\int_0^T e^{A(T-s)}(P_sC^* + \Psi)\mathrm{d}\overline{\eta}_s, \tag{14}$$

where P is a solution of the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t - AP_t - P_tA^* - \Phi - (P_tC^* + \Psi)(CP_t + \Psi^*) = 0,$$
$$t > 0, \quad P_0 = \operatorname{cov} z_0, \tag{15}$$

satisfying $P_t \geq 0$ for all $t \geq 0$, and $\overline{\eta}$ is an innovation process defined by

$$\mathrm{d}\overline{\eta}_t = \mathrm{d}\,\eta^w_t - C\overline{z}^w_t\, d_t, \quad t > 0, \quad \overline{\eta}_0 = 0. \tag{16}$$

Furthermore,

$$J^\lambda(w^\lambda) = \int_0^T \operatorname{tr}((CP_s + \Psi^*)\, Q^\lambda_t (P_sC^* + \Psi))\mathrm{d}s, \tag{17}$$

where

$$Q^\lambda_t = \lambda e^{A^*(T-s)}L^*R(\lambda, -\tilde{\Gamma}_{T-t})Le^{A(T-s)}, \quad t \geq 0. \tag{18}$$

Proof: Let $\hat{z}^w_t = \mathbf{E}(z^w_t \mid \mathcal{F}^w_t)$. By Kalman filtering result, $\hat{z}^w$ satisfies

$$\mathrm{d}\hat{z}^w_t = (A\hat{z}^w_t + Bw_t)\mathrm{d}t + (P_tC^* + \Psi)\mathrm{d}\overline{\eta}_t, \quad t > 0, \quad \hat{z}^w_0 = 0, \tag{19}$$

where P and $\overline{\eta}$ are defined by (15) and (16), respectively. Note that the solution P of the equation (15) is understood in the scalar product sense and it satisfies $P_t \geq 0$ for all $t \geq 0$, and $\overline{\eta}$ is a Wiener process with $\operatorname{cov}\overline{\eta}_t = It$ and it is independent on w (see Curtain and Pritchard (1978) and Bashirov (2003)). Replacing the system (6) by (19), one can derive the formulae (13) and (14) in a similar way as in the proof of Lemma 1. Substituting them in (12), one can obtain the formula (17) with (18). □

Theorem 3: *Under the above conditions and notation, the system* (6) *is L-partially S^0-controllable for the time T if the system* (5) *is L-partially D^a-controllable for each time $t \in (0, T]$.*

Proof: The arguments given in the proofs of Lemmas 12.15 and 12.16 from Bashirov (2003) together with the Lemma 2 allow to prove that the finite non-negative limit

$$a_T = \lim_{\lambda\to 0}\int_0^T \operatorname{tr}((CP_s + \Psi^*)\, Q^\lambda_s (P_sC^* + \Psi)\mathrm{d}s$$

exists and

$$\inf_{\mathcal{W}} \mathbf{E}\|L\mathbf{E}(z^w_T \mid \mathcal{F}^w_T)\|^2 = a_T.$$

Using this, in a similar way as in the proof of Theorem 12.17 from Bashirov (2003), one can show that if $a_T < \varepsilon(1-p)$, then $0 \in S_T(\varepsilon, p)$ for every $\varepsilon > 0$ and $0 \leq p < 1$, where $S_T(\varepsilon, p)$ is the set defined by (2) for the system (6) on $\mathcal{W}$. This immediately implies that $a_T = 0$ is sufficient for the system (6) on $\mathcal{W}$ be

L-partially S^0-controllable. Finally, the arguments used in the proof of Theorem 12.19 from Bashirov (2003) together with the implication (2a)$\Rightarrow$(2c) from Theorem 2, yield that if the system (5) on $\mathcal{V}$ is L-partially D^a-controllable for the each time $t \in (0, T]$, then $a_T = 0$.

7. Partial *C*- and *S*-controllability

In this section we present necessary and sufficient conditions for L-partial C- and S-controllability of the system (4) on $\mathcal{U}$.

Theorem 4: *Under the above conditions and notation, the system* (4) *on* $\mathcal{U}$ *is* L*-partially* C*-controllable for every time* $T > 0$ *if and only if the system* (5) *on* $\mathcal{V}$ *is* L*-partially* D^c*-controllable for every time* $T > 0$.

Proof: At first, similarly to the proof of Theorem 12.24 from Bashirov (2003), one can show that for fixed $T > 0$, the system (4) on $\mathcal{U}$ is L-partially C-controllable for the time T if and only if the system (5) on $\mathcal{V}$ is L-partially D^c-controllable for the time T and the system (6) on $\mathcal{W}$ is L-partially S^0-controllable for the time T. This implies the necessity part of the theorem. For the sufficiency part, additionally, Theorem 3 must be used. □

Theorem 5: *Under the above conditions and notation, the system* (4) *on* $\mathcal{U}$ *is* L*-partially* S*-controllable for every time* $T > 0$ *if and only if the system* (5) *on* $\mathcal{V}$ *is* L*-partially* D^a*-controllable for every time* $T > 0$.

Proof: Using the fact that a linear transformation of a Gaussian random variable is again Gaussain, in a similar way as in the proof of Theorem 12.31 from Bashirov (2003), one can prove that for fixed $T > 0$, the system (4) on $\mathcal{U}$ is L-partially S-controllable for the time T if and only if the system (5) on $\mathcal{V}$ is L-partially D^a-controllable for the time T and the system (6) on $\mathcal{W}$ is L-partially S^0-controllable for the time T. This implies the necessity part of the theorem. For the sufficiency part additionally Theorem 3 must be used. □

8. Examples

Example 2 (Colored noise driven systems): Consider the system

$$\begin{cases} dx_t^u = (Ax_t^u + Bu_t + f_t + \varphi_t)dt + dm_t, & t > 0,\ x_0^u = x_0, \\ d\xi_t^u = (Cx_t^u + \psi_t)dt + dn_t, & t > 0,\ \xi_0^u = 0, \end{cases} \tag{20}$$

on $\mathcal{U}$ as defined in Section 4, which differs from the system (4) by additional noise processes φ and ψ. Assume that φ and ψ are colored noises, i.e., they are solutions of the linear stochastic equations

$$d\varphi_t = A_1\varphi_t dt + dm'_t, \quad t > 0,\ \varphi_0 = 0,$$
$$d\psi_t = A_2\psi_t dt + dm''_t, \quad t > 0,\ \psi_0 = 0,$$

where A_1 and A_2 are the infinitesimal generators of strongly continuous semigroups, and m' and m'' are two Wiener processes correlated with (m, n). Denote

$$\tilde{A} = \begin{bmatrix} A & I & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 & I \end{bmatrix}$$

and

$$\tilde{x}_t^u = \begin{bmatrix} x_t^u \\ \varphi_t \\ \psi_t \end{bmatrix}, \quad \tilde{x}_0 = \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{m}_t = \begin{bmatrix} m_t \\ m'_t \\ m''_t \end{bmatrix} \quad \tilde{f}_t = \begin{bmatrix} f_t \\ 0 \\ 0 \end{bmatrix}.$$

Then the system (20) can be enlarged to the standard form

$$\begin{cases} d\tilde{x}_t^u = (\tilde{A}\tilde{x}_t^u + \tilde{B}u_t + \tilde{f}_t)dt + d\tilde{m}_t, & t > 0, \tilde{x}_0^u = \tilde{x}_0, \\ d\xi_t^u = \tilde{C}\tilde{x}_t^u dt + dn_t, & t > 0, \xi_0^u = 0. \end{cases} \tag{21}$$

Hence, letting the operator L be the projection operator from the state space of the system (21) to the state space of (20) and applying Theorems 4 and 5 to the system (21), we conclude that the system (20) on $\mathcal{U}$ is C-controllable (respectively, S-controllable) for every $T > 0$ if and only if the system (5) on $\mathcal{V}$ is exactly (respectively, approximately) controllable for every $T > 0$.

Example 3 (Wide band noise driven systems): Consider the system (20) from Example 2 and assume that φ and ψ are stationary wide band noise processes, i.e., their autocovariance functions are nonzero within some small time intervals and vanish outside. Indeed, wide band noise processes more adequately describe the noises met in applied engineering problems (see Fleming and Rishel (1975, p. 126). In Bashirov (2003) it is shown that under sufficiently general conditions, the wide band noise processes φ and ψ can be represented in the form

$$\varphi_t = \int_{\max(0,\, t-\varepsilon)}^{t} \Phi_{\theta - t}\, dm'_\theta \quad \text{and} \quad \psi_t = \int_{\max(0,\, t-\delta)}^{t} \Psi_{\alpha - t}\, dm''_\alpha,$$

where $\varepsilon > 0$, $\delta > 0$, Φ and Ψ are operator-valued functions on $[-\varepsilon, 0]$ and $[-\delta, 0]$, respectively, and m' and m'' are Weiner processes. Let (m_t, n_t) and (m'_t, m''_t) be independent or correlated. Assuming that Φ and Ψ are differentiable with the square integrable derivatives and with $\Phi_{-\varepsilon} = 0$ and $\Psi_{-\delta} = 0$ (these conditions are natural from applied point of view; see the discussion in

Bashirov (2003)), introduce the functions $\tilde{\varphi} : [0, \infty) \to L_2(-\varepsilon, 0; X)$ and $\tilde{\psi} : [0, \infty) \to L_2(-\delta, 0; \mathbb{R}^k)$ by

$$\mathrm{d}\tilde{\varphi}_t = (\mathrm{d}/\mathrm{d}\theta)\tilde{\varphi}_t + (\mathrm{d}\Phi/\mathrm{d}\theta)\mathrm{d}m'_t, \quad t > 0,\ \tilde{\varphi}_0 = 0,$$
$$\mathrm{d}\tilde{\psi}_t = (\mathrm{d}/\mathrm{d}\alpha)\tilde{\psi}_t + (\mathrm{d}\Psi/\mathrm{d}\alpha)\mathrm{d}m''_t, \quad t > 0,\ \tilde{\psi}_0 = 0.$$

Then

$$\varphi_t = \int_{-\varepsilon}^{0} \tilde{\varphi}_t \,\mathrm{d}\theta, \quad \text{and} \quad \psi_t = \int_{-\delta}^{0} \tilde{\psi}_t \,\mathrm{d}\alpha.$$

Therefore, for

$$\tilde{A} = \begin{bmatrix} A & I_\varepsilon & 0 \\ 0 & \mathrm{d}/\mathrm{d}\theta & 0 \\ 0 & 0 & \mathrm{d}/\mathrm{d}\alpha \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 & I_\delta \end{bmatrix},$$

where I_ε and I_δ are operators of integration on the intervals $[-\varepsilon, 0]$ and $[-\delta, 0]$, respectively, and for

$$\tilde{x}^u_t = \begin{bmatrix} x^u_t \\ \tilde{\varphi}_t \\ \tilde{\psi}_t \end{bmatrix}, \quad \tilde{x}_0 = \begin{bmatrix} x_0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{m}_t = \begin{bmatrix} m_t \\ (\mathrm{d}\Phi/\mathrm{d}\theta)m'_t \\ (\mathrm{d}\Psi/\mathrm{d}\alpha)m''_t \end{bmatrix},$$
$$\tilde{f}_t = \begin{bmatrix} f_t \\ 0 \\ 0 \end{bmatrix},$$

the system (20) with the wide band noises φ and ψ can be enlarged to the standard form (21), Again, letting the operator L be the projection operator from the state space of the system (21) to the state space of (20) and applying Theorems 4 and 5 to the system (21), we conclude that the system (20) on $\mathcal{U}$ is C-controllable (respectively, S-controllable) for every $T > 0$ if and only if the system (5) on $\mathcal{V}$ is exactly (respectively, aproximately) controllable for every $T > 0$.

9. Conclusion

In the paper the concepts of partial cotrollability for deterministic and strochastic control systems are introduced and necessary and suffucuent conditions for them are derived. The usefulness of these concepts are demonstrated on the examples of linear differential delay systems and partially observable linear systems driven by colored and wide band noises.

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