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IMPROVING STABILITY MARGINS VIA TIME DELAY CONTROL

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ABSTRACT

While time delays typically lead to poor control performance, and even instability, previous research has shown that introduction of time delays in controlling a dynamic system can, in some cases, be beneficial. This paper presents a new benefit of time delay control for single-input single-output linear time invariant systems: it can be used to improve robustness, as measured by increased stability margins. The proposed method utilizes time delays to approximate state-derivative feedback, which can be used, together with state feedback, to reduce sensitivity and improve robustness. Additional sensors are not required since the state-derivatives are approximated using available measurements and time delays. The method is introduced using a scalar example, then applied to a single degree-of-freedom mechanical vibration control problem in simulations to demonstrate excellent performance with improved stability margins.

1. INTRODUCTION

Linear time delay systems (TDS) exhibit an infinite eigenspectrum, which can lead to difficulties in stability analysis, in obtaining free and forced solutions, as well as controller design [1-5]. However, TDS can also present an opportunity for improved control performance. This paper presents a potential benefit of using time delay control. It is shown that in single-input single-output (SISO) systems time delays can be used to improve stability margins. In essence, the proposed method utilizes time delays to approximate state-derivative feedback, which can be used together with state feedback to reduce sensitivity and improve robustness.

Control systems are typically designed based on nominal plant models, and variations in parameters are often encountered in practice. Consequently, the last few decades have seen significant research interest in robust control design, where such model uncertainties are explicitly considered in the design process. A variety of methods have been developed for achieving robust control design [6-8]. One of those methods is the use of state-derivative feedback, in addition to state feedback, to reduce closed loop system sensitivity to plant parameter variations and to disturbance inputs [9]. However, state plus state-derivative feedback (SSD) is difficult to realize in practice, because additional sensors are needed to measure the state-derivatives in addition to the states. Even measuring all the states is too restrictive in most applications. Thus, measuring state derivatives as well as states is typically impractical. For example, in a single degree-of-freedom mechanical system one would need to measure acceleration, in addition to the states (i.e., position and velocity), to implement state plus state-derivative feedback. In this paper, the use of time delays in the control to approximate the state derivatives is considered, and shown to yield improved stability margins. Consequently, with time delay control, the benefits of improved robustness can be realized without the need for additional sensors beyond those required for state feedback.

Time delays occur in many natural and engineered systems and often lead to problems such as poor control or even instability. For example, delays can be a significant concern in networked control systems [10-11], in engine control [12], or in teleoperation of robots [13]. Such TDS are described by delay differential equations (DDEs), and have an infinite eigenspectrum, which makes their analysis and control challenging. Nevertheless, a number of researchers have reported various benefits of using time delays in controller design. For example, Yang and Mote [14-15] have demonstrated that time delays can be used for noncolocated vibration control in bandsawing, to improve accuracy and reduce raw material waste. They use time delays to predict the response of a axially-translating flexible structure at points where no sensor is located. Thus, avoiding instability and robustness problems associated with noncolocation of sensors and actuators. Udwadia and co-workers have also shown benefits of time delay control for flexible structures [16-18]. They consider large time delays in the controller, comparable to the period of the uncontrolled structure, and demonstrate effective performance for both single and multi degree-of-freedom systems. Time delay control has also been shown to improve performance for both linear and nonlinear systems with uncertainty [19-20]. The approach uses information in the recent past, through the time delay, to directly estimate the unknown dynamics at any given instant.

Time delays have also been used in a state feedback controller to approximate integral and derivative actions [21]. The use of state-derivative feedback, in addition to state feedback, has previously been shown to reduce closed-loop sensitivity and improve disturbance rejection [9]. However, it requires measurement of state-derivatives in addition to the system states, which is often not practical. Consequently, approximate derivatives were considered in [22]. Those approximations were based on low-pass filters and it was found that most of the robustness benefits of SSD were then lost. It has also been shown that state derivative feedback controllers may be fragile, in the sense that arbitrarily small modeling errors and time delays may destroy stability [23], and that small uncertain feedback delays cannot always be safely neglected if state derivatives are used in the feedback [24].

In this paper the approximation of the state-derivatives using finite differencing, rather than approximate derivatives via filtering, is considered. This leads to a retarded DDE, which is then analyzed using a recently developed approach for the analysis and control of TDS based on the Lambert W function [4]. Conditions can then be found where the benefits of SSD are achieved with a time delay controller. To avoid fragility problems associated with SSD in the presence of delays, it is important to have such analysis and design tools for appropriate selection of the controller gains and time delay.

The purpose of this paper is to present a new method for improving the robustness of control systems by the intentional use of time delays in the control. In Section 2 time delays are used to approximate the derivatives of the state, so that SSD feedback can be approximately achieved. The state-derivative feedback can then be realized without the need for additional sensors to measure state-derivatives. A scalar example is used throughout to show that the desired closed-loop performance can be achieved, and with improved stability margins. Section 3 presents an application of the proposed method to a single degree-of-freedom (DOF) vibration control problem. A summary and conclusions are presented in Section 4.

2. THEORY AND SCALAR EXAMPLE

Consider a SISO linear time invariant (LTI) plant in standard state equation form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{B}u \\ y &= \mathbf{Cx} \end{aligned} \tag{1}$$

Assuming all states are measurable a state feedback (SF) controller, $u = -\mathbf{Kx}$, yields the closed-loop system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_c\mathbf{x} = (\mathbf{A} - \mathbf{BK})\mathbf{x} \\ y &= \mathbf{Cx} \end{aligned} \tag{2}$$

It is well-known that if (**A**,**B**) is controllable, **K** can be selected (e.g., by eigenvalue assignment or optimal control methods) to achieve the desired closed-loop performance. If not all the states are measured, but (**A**,**C**) is observable, then one can also design a state estimator, or observer, to estimate the states from the output and use those estimated states in the feedback control [25]. Furthermore, the SF controllers can be designed to achieve not only the desired closed-loop performance, but can also be designed for excellent robustness properties as measured by stability margins. However, the use of a state estimator will typically reduce those stability margins [26].

Consider a state plus state derivative (SSD) feedback controller, $u = -\mathbf{Fx}(t) - \mathbf{G}\dot{\mathbf{x}}(t)$, for the system in Eq. (1). The closed-loop system becomes:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x} = (\mathbf{I} + \mathbf{BG})^{-1}(\mathbf{A} - \mathbf{BF})\mathbf{x}(t) \tag{3}$$

As described in detail in [9], one can first select **K** in Eq. (2) to obtain a desired $\mathbf{A_c}$, then select **G** in Eq. (3) based on sensitivity considerations, and finally determine **F** from Eq. (3), given $\mathbf{A_c}$, **G**, **A** and **B**.

Example 1. Sensitivity reduction by SSD feedback

Let a = -0.05 and b = 1.0 for the system $\dot{x}(t) = ax(t) + bu(t)$ and consider the SF controller $u(t) = -kx(t)$ to obtain $\dot{x}(t) = a_c x(t) = (a - bk)x(t)$. If one selects, a_c = -2, then $k = (a - a_c)/b = 1.95$. Thus, the closed-loop system will have a 2% settling time of t_s = 4(1/2) = 2 s, improving on the open-loop settling time of t_s = 4(20) = 80 s.

Next consider the SSD controller $u(t) = -fx(t) - g\dot{x}(t)$ to obtain the closed-loop system $\dot{x}(t) = (1 + bg)^{-1}(a - bf)x(t) = a_c x(t)$. Since a_c = -2, for a given g one can then obtain the required value of f, i.e., $f = (a - a_c(1 + bg))/b$, as summarized in Table 1. Note that, to avoid division by zero, one cannot select $g = -(1/b) = -1$ in this example.

Table 1. Values of the gain *f*, and sensitivities in Eq. (4), for a given value of the gain *g*, to achieve a_c = -2.

g	f	S^λ_a	S^λ_b
0	1.95	1	-1.95
1	3.95	0.5	-0.9750
5	11.95	0.1667	-0.3250
10	21.95	0.0909	-0.1773

For SSD feedback control the sensitivity of the closed-loop system eigenvalue (i.e., $\lambda = a_c = -2$) with respect to variations in the open-loop parameters a and b can be obtained as:

$$
\begin{aligned}
S_a^{\lambda} &= \frac{\partial}{\partial a}[(1+bg)^{-1}(a-bf)] = (1+bg)^{-1} \\
S_b^{\lambda} &= \frac{\partial}{\partial b}[(1+bg)^{-1}(a-bf)] = \frac{-(f+ag)}{(1+bg)^{2}}
\end{aligned}
\tag{4}
$$

For the state feedback controller the corresponding sensitivity expressions can be obtained from Eq. (4) when $g = 0$. As shown in Table 1, for $g > 0$, the sensitivities for SSD feedback control are reduced compared to SF control (i.e., $g = 0$). The use of SSD feedback also improves the stability margins, as discussed later in this section. Although not shown here, values of $g > 0$ in SSD feedback also reduce the effects of any external disturbances acting on the system [9]. Thus, the benefits of SSD feedback can be significant.

Despite its sensitivity reduction benefits, in practice, it is often difficult to obtain measurements of the state derivatives to implement SSD feedback control. Consequently, one can instead implement the following time delay controller (TDC):

$$
u(t) = -\mathbf{K}_p\mathbf{x}(t) - \mathbf{K}_d\mathbf{x}(t-h) \tag{5}
$$

where h is a time delay. This yields the closed-loop system:

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{B}\mathbf{K}_p)\mathbf{x}(t) - \mathbf{B}\mathbf{K}_d\mathbf{x}(t-h) \\
&= \mathbf{A}_c\mathbf{x}(t) + \mathbf{A}_d\mathbf{x}(t-h)
\end{aligned}
\tag{6}
$$

The system in Eq. (6) is a retarded DDE and has an infinite eigenspectrum. However, in the limit as h approaches zero, Eq. (6) can also be viewed as an approximation to the closed-loop system in Eq. (3) where SSD feedback was utilized. Consider the approximation, for small h:

$$
\dot{\mathbf{x}}(t) \approx \frac{\mathbf{x}(t) - \mathbf{x}(t-h)}{h} \tag{7}
$$

Then the SSD feedback control can be approximated as:

$$
\begin{aligned}
u(t) &= -\mathbf{F}\mathbf{x}(t) - \mathbf{G}\dot{\mathbf{x}}(t) \approx -\mathbf{F}\mathbf{x}(t) - (1/h)\mathbf{G}(\mathbf{x}(t) - \mathbf{x}(t-h)) \\
&= -(\mathbf{F} + (1/h)\mathbf{G})\mathbf{x}(t) + (1/h)\mathbf{G}\mathbf{x}(t-h)
\end{aligned}
\tag{8}
$$

Thus, comparing Eq. (5) and Eq. (8) one obtains:

$$
\begin{aligned}
\mathbf{K}_p &= \mathbf{F} + (1/h)\mathbf{G} \\
\mathbf{K}_d &= -(1/h)\mathbf{G}
\end{aligned}
\tag{9}
$$

Consequently, if one selects h to be sufficiently small, and selects $\mathbf{G}$ to reduce sensitivity, the resulting closed-loop system in Eq. (6) should have the specified closed-loop performance (as with SF control only) plus reduced sensitivity, as with SSD feedback control. However, Eq. (6) is a DDE, thus, its analysis, to confirm closed-loop performance and robustness for particular values of h, $\mathbf{F}$ and $\mathbf{G}$, can be challenging.

Example 2. Approximation of state derivative

Consider the scalar system in Ex. 1, and approximate the SSD feedback controller, $u(t) = -fx(t) - g\dot{x}(t)$, by using time delay h in the control, as $u = -k_p x(t) - k_d x(t-h)$. Given f and g, and the time delay h, one can determine the gains k_p and k_d for this controller using Eq. (9). Table 2 summarizes these for $h = 0.1$ s (small relative to the specified closed-loop settling time of 2 s) and selected values of g. Note in Eq. (9) that all elements of $\mathbf{G}$ are divided by h. Thus, the same gains can be obtained by scaling either the time delay or the elements of the gain $\mathbf{G}$. For example, the same gains $k_p = 61.95$ and $k_d = -50$ are obtained for $g = 0.5$ and $h = 0.01$ as for $g = 5$ and $h = 0.1$.

Table 2. Values of the gains k_p and k_d in Eq. (5) for given values of g, and for time delay $h = 0.1$ s, to achieve $a_c = -2$.

g	f	k_p	k_d
0	1.95	1.95	0
1	3.95	13.95	-10
5	11.95	61.95	-50
10	21.95	121.95	-100

One is interested in the performance, and sensitivity, of the approximate system in Eq. (6). Ideally, one would like to see the performance of this closed-loop system be close to that specified by $\mathbf{A_c}$ in Eq. (2) or (3). The system of DDEs in Eq. (6) possesses an infinite number of eigenvalues due to the presence of the delay h. However, the overall system response, as in higher-order linear systems without delay, will be dominated by the eigenvalues of the system which are closest to the imaginary axis; that is the rightmost eigenvalues. These rightmost eigenvalues will determine stability as well as transient response characteristics, such as settling time and overshoot [27]. A method, based on the Lambert W function, has been developed that enables the analysis of LTI TDS as in Eq. (6) to obtain rightmost eigenvalues and the time response [4]. One can utilize this method to establish the performance of the closed-loop system in Eq. (6).

Example 3. Eigenvalues and response of the system in Eq. (6)

Consider the same system as in Exs. 1-2. The closed-loop system is represented by the DDE:

$$
\begin{aligned}
\dot{x}(t) &= (a - bk_p)x(t) - (bk_d)x(t-h) \\
&= -(0.05 + k_p)x(t) - bk_d x(t-h)
\end{aligned}
\tag{10}
$$

For different values of g and h one can obtain the gains k_p and k_d as in Ex. 2 (see Table 2). Then, for each set of gains, one can determine the rightmost eigenvalue from the analytical expression [4]:

$$
s_0 = \frac{1}{h}W_0(-bk_d h e^{-(a-bk_p)h}) + (a - bk_p) \tag{11}
$$

where W_0 is the principal ($k = 0$) branch of the Lambert W function, W_k. This can be readily evaluated, for example using the *lambertw* command in MATLAB, or using functions available in the open-source *LambertW_DDE Toolbox* [28].

These rightmost (i.e., dominant) eigenvalues are given in Table 3 for selected values of g and h. Furthermore, either using the series solution in [4], or a numerical integration routine (e.g., *dde23* in MATLAB) one can obtain the time response for the system in Eq. (10).

The results in Table 3 show that the rightmost eigenvalue is close to the desired value of -2. Given g, as we decrease h, the approximation in Eq. (7) becomes more accurate and the closed-loop eigenvalues approach the desired value of -2. As shown in Fig. 1, for g = 1 and h = 0.1s, the time response is very close to the g = 0 (i.e., state feedback only) case with a settling time of approximately 2 s.

Table 3. Rightmost eigenvalue of Eq. (10) for selected values of *g* and *h*.

	***h*=0.01**	***h*=0.05**	***h*=0.1**
***g*=0**	-2.0	-2.0	-2.0
***g*=1**	-1.9900	-1.9508	-1.9034
***g*=5**	-1.9835	-1.9206	-1.8484
***g*=10**	-1.9820	-1.9140	-1.8368

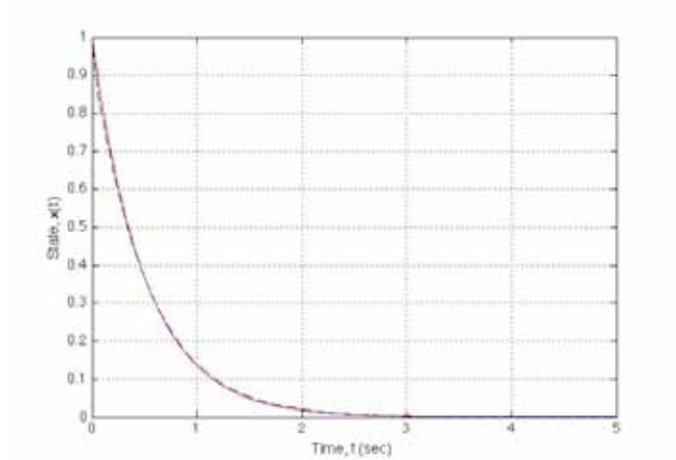

Figure 1. Closed-loop response to initial condition $x(t)$ = 1.0 for $-h < t \leq 0$ for the scalar system in Ex. 3 with SF control (solid red line) and with TDC (dashed blue line) when g = 1 and h = 0.1 s.

In addition to achieving the specified performance, one would like to see improved sensitivity (or robustness) for $g > 0$, due to the addition of the time delay terms in the control. As we see in Table 1, sensitivities will be reduced for $g > 0$. Standard measures of the robustness of SISO control systems are their stability margins, i.e., the gain margin, *GM*, and the phase margin, *PM*. For a typical closed-loop system, as in Fig. 2, the stability margins are determined from a Bode (or Nyquist) plot of the loop transfer function *C(s)G(s)*. A frequency domain control design rule-of-thumb is to specify *GM* > 6 dB and *PM* > 30 to 40 degrees. The stability margins can be readily obtained, for example using the *margin* command in MATLAB, or from a Bode or Nyquist plot of the open loop transfer function *C(s)G(s)*. A pure time delay of h seconds contributes, in the frequency response, a gain of one (i.e., 0 dB) and a phase lag of $-\omega h$. For example, when $\omega h = \pi$ the phase lag is 180 degrees. Consequently, if the desired closed-loop bandwidth is ω_{bw}, then delays of $\pi/\omega_{bw} \gg h$ are desired [29].

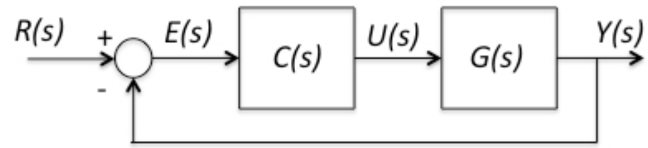

Figure 2. Block diagram of unity feedback control system with controller *C(s)* and plant *G(s)*.

Example 4. Stability margins of example system

Consider the same system as in Exs. 1-3. The plant transfer function $G(s) = b/(s-a) = 1/(s+0.05)$ and the controller transfer function $C(s) = k_p + k_d e^{-sh}$. When h = 0 (no delay), then k_d = 0 and k_p = k, for SF control only. The results are summarized in Table 4 for selected values of the SSD feedback gain, g, and the delay time, h, ranging from 5 to 100 ms. As a basis for comparison, when g = 0 (SF only) the stability margins are *GM* = ∞ and *PM* = 91.5 degrees (see Fig. 3). For exact SSD, without the use of delay to approximate the state derivative, $C(s) = f + gs$ and one obtains *GM* = ∞ and *PM* = 180 degrees (see Fig. 4). Consequently, exact SSD, if it were implementable, would nearly double the *PM* compared to SF control.

Table 4. Phase margins, *PM*, in degrees, for the closed-loop system in Eq. (10) for combinations of *g* and *h*. Note that *g* = 0 is SF only, while *h* = 0 is the exact SSD case.

	***h*=0**	***h*=0.005**	***h*=0.01**	***h*=0.05**	***h*=0.1**
***g*=0**	91.5	NA	NA	NA	NA
g=0.5	180	120.8	120.8	120.6	120.0
g=0.75	180	139.1	139.1	138.0	133.5
***g*=1**	180	163.5	156.8	131.1	115.3
g=1.25	180	113.8	112.4	103.3	95.3
***g*=5**	180	31.7	32.1	35.3	39.3
***g*=10**	180	17.6	18.5	26.2	35.7

Note that when $g > 0$, it is possible to achieve improved phase margins (i.e., greater than 91.5 degrees) with the approximate SSD using delays. The *GM* is no longer infinite, as it was for SF control, due to the time delay. However, the *GM* ≈ 40 dB for h = 0.1 s, and is still well above the typical 6 dB design guideline. In this example, Table 4 shows that phase margins are improved for all values of $g \leq 1.25$ and $h \leq 0.1$. For example, when g = 1 and h = 0.1 it is possible to obtain the desired response (see Fig. 1), while improving the phase margin by 23.8 degrees (see Fig. 5). When h = 0.005 and g = 1, the response is indistinguishable from state feedback control and the phase margin is improved by 72 degrees. Figures 3, 4 and 5 show, respectively, the Bode plots for *C(s)G(s)* for SF control, for SSD feedback with g = 1, and approximate SSD feedback with g = 1 and h = 0.1. Note that with SSD feedback, the phase characteristics are improved compared to state feedback only. Furthermore, the TDC (i.e., approximate SSD) also exhibits improved phase characteristics compared to SF control.

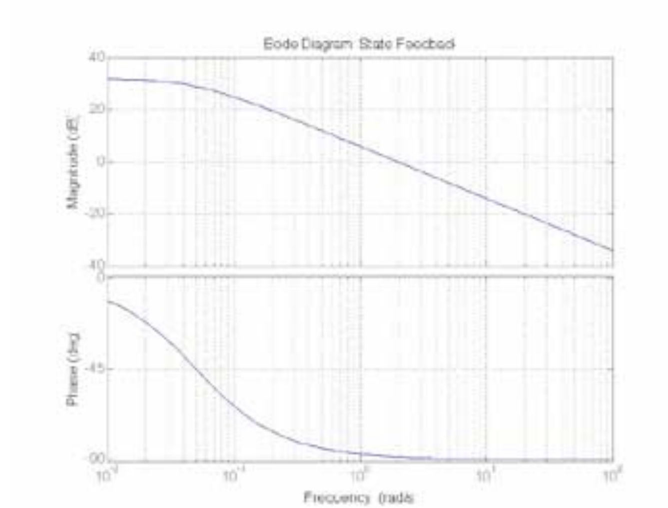

Figure 3. Bode plot of *C(s)G(s)* for SF control. $GM = \infty$ and PM = 91.5 deg.

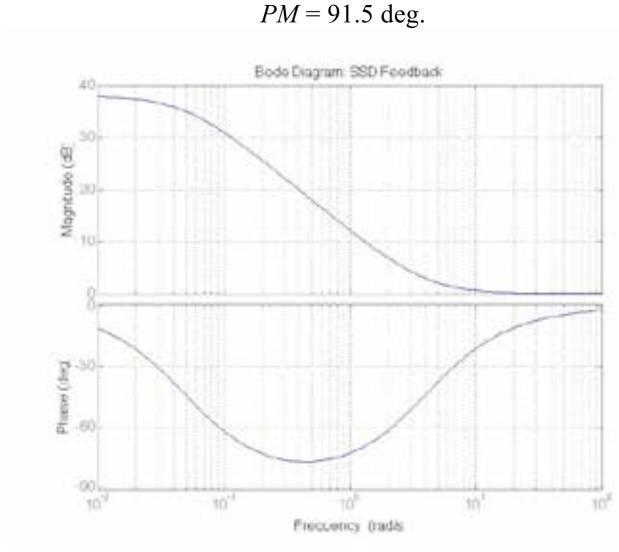

Figure 4. Bode plot of *C(s)G(s)* for SSD control with g=1. $GM = \infty$ and PM = 180 deg.

Consider a specific numerical comparison by designing the controllers using the given nominal parameter values a = -0.05 and b = 1.0, but applying these controllers to a system with the actual parameter values a = -0.01 and b = 0.5. Then the state feedback controller yields a closed-loop settling time of t_s = 4.1 s, the exact SSD controller with g = 1 yields t_s = 3 s and the approximate SSD controller with g = 1 and h = 0.005 also yields t_s = 3 s. The SSD controllers are less sensitive to the parameter perturbations and lead to settling times closer to the desired value of t_s = 2 s. Furthermore, the delay approximation is excellent for this small 5 ms delay time. When the delay time is increased to 100 ms (i.e., h = 0.1 s), with g = 1, one obtains t_s = 3.1 s. Thus, while still a major improvement over the state feedback only case, this approximation with larger delay is slightly less effective compared to the exact SSD controller.

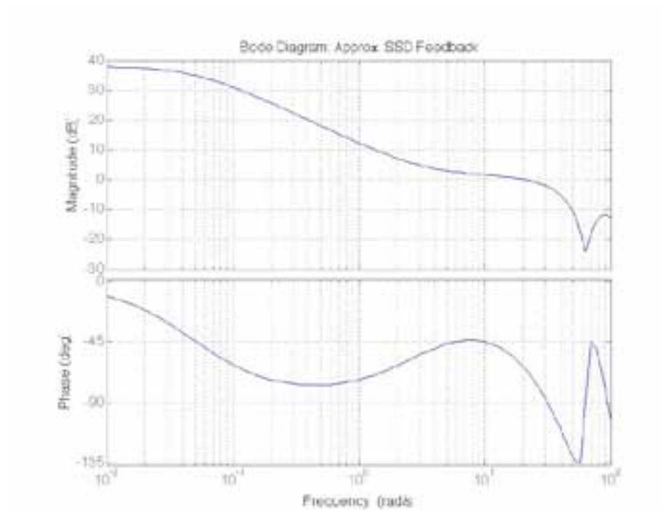

Figure 5. Bode plot of *C(s)G(s)* for TDC (i.e., approximate SSD control) with g = 1 and h = 0.1s. Note: wrapping occurs for frequencies above 20π rad/s (i.e., $1/h$ = 10 Hz) and is a computational artifact of the *bode* or *margin* commands in MATLAB. GM = 40 dB and PM = 115.3 deg.

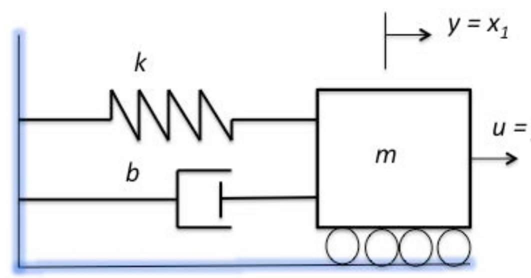

Figure 6. Schematic of a single degree-of-freedom mass-spring-damper system

3. SINGLE DOF MECHANICAL VIBRATION CONTROL

Consider a single DOF mechanical system, as shown in Fig. 6. The system dynamics can be described by the following state equations:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{m} \end{Bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{12}$$

and the transfer function of the system is:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{(1/m)}{s^2+(b/m)s+(k/m)} \tag{13}$$

where $x_1 = y$, $x_2 = dy/dt$, and $u = f$. With parameter values $m = k = 1$ and $b = 0.1$, the open-loop system is lightly damped (i.e., open-loop damping ratio $\zeta \approx 0.05$), with a 2% settling time of $t_s \approx 80$ seconds. The goal is to design a controller to achieve a closed-loop settling time of $t_s \approx 3$ seconds with little or no overshoot (i.e., closed-loop damping ratio $\zeta \approx 1$).

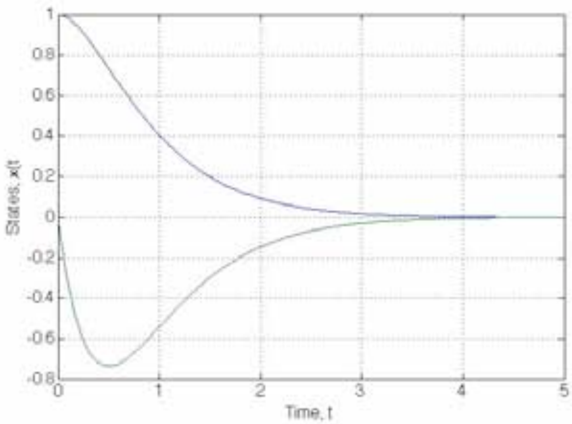

Figure 7. Response **x**(t) of single DOF system with SF control ($K_1 = 3$, $K_2 = 3.9$) to initial condition $\mathbf{x}(0) = [1\ 0]^T$.

An SF controller, $u=-\mathbf{K}\mathbf{x}=-K_1x_1-K_2x_2$, leads to the closed-loop system:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{bmatrix} 0 & 1 \\ -\frac{(k+K_1)}{m} & -\frac{(b+K_2)}{m} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$
$$y=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{14}$$

The controller can also be represented in transfer function form (see Fig. 2) as:

$$C(s)=(K_1+K_2s) \tag{15}$$

Selecting desired closed-loop eigenvalues at $s_1 = s_2 = -2$, gives the gains $K_1 = 3$ and $K_2 = 3.9$. The closed-loop response, with $t_s \approx 3$ seconds and no overshoot as desired, is shown in Fig. 7. The stability margins can be determined from a Bode plot of *C(s)G(s)* (see Fig. 8), or using the *margin* command in MATLAB, to be $GM = \infty$ and $PM = 81.1$ degrees, so this system also has excellent robustness. The closed loop bandwidth will be $\omega_{bw} \approx 1$ rad/s. However, the SF controller requires the measurement of velocity, x_2, as well as position, x_1, to implement.

Next lets illustrate how the approximation in Eq. (7) can be used to eliminate the need for velocity measurement, while maintaining the desired closed-loop response and good robustness. As shown in Eq. (15), the SF controller can be viewed as a Proportional plus Derivative (PD) controller, where the derivative of the position (i.e., velocity) is directly measured. As discussed in [21] a time delay controller can be used to approximate the derivative term, as in Eq. (7). Thus, the TDC becomes:

$$u(t)=-K_px_1(t)-K_dx_1(t-h) \tag{16}$$

with the controller transfer function:

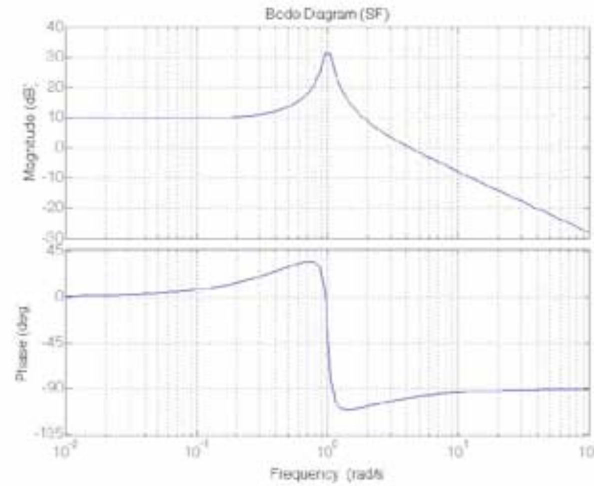

Figure 8. Loop transfer function Bode plot for SF control ($K_1 = 3$, $K_2 = 3.9$) of single DOF system. $GM = \infty$ and $PM = 81.1$ deg.

$$C(s)=(K_p+K_de^{-sh}) \tag{17}$$

and the closed-loop system is:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{bmatrix} 0 & 1 \\ -\frac{(k+K_p)}{m} & -\frac{b}{m} \end{bmatrix}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$
$$+\begin{bmatrix} 0 & 0 \\ -\frac{K_d}{m} & 0 \end{bmatrix}\begin{Bmatrix} x_1(t-h) \\ x_2(t-h) \end{Bmatrix} \tag{18}$$
$$y=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Using Eq. (7) to approximate the output derivative term in the PD control gives $K_p=(K_1+(K_2/h))$ and $K_d=(-K_2/h)$. For example, if one selects $h = 0.1$ then $K_p = 42$ and $K_d = -39$. The response for the system in Eq. (18) is shown in Fig. 9, and can be obtained using routines in the *LambertW_DDE Toolbox* [28], or the numerical simulation routine *dde23* in MATLAB, and is almost identical to the response of the system with SF control (see Fig. 7). However, this TDC, which approximates PD control, uses only output (position) measurement.

However, the system with TDC has GM = 21.7 dB and PM = 69.4 degrees when h = 0.1 sec as shown in the Bode plot in Fig. 10. While these are still excellent stability margins, they do represent a loss of robustness compared to the system with SF. Note that the frequency response plots in Fig. 8 (for PD control) and in Fig. 10 (for TDC, approximate PD control) match well for frequencies up to one decade below $1/h$ = 10 Hz = 20π rad/s. For smaller values of h the range extends to even higher frequencies. Since the closed-loop system bandwidth is approximately 1 rad/s, this approximation with h = 0.1 s is quite effective.

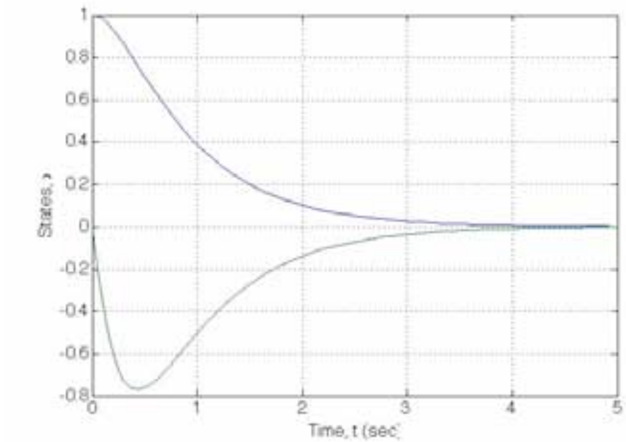

Figure 9. Response of the single DOF system with approximate PD control (TDC, h = 0.1, K_p = 42, K_d = -39) to initial condition $\mathbf{x}(0) = [1\ 0]^T$.

Table 5. The gains, rightmost poles, and stability margins for the closed-loop system in Eq. (20) with approximate PD control for selected values of the delay time, *h*.

	***h* = 1e-4**	***h* = 5e-4**	***h* = 1e-3**	***h* = 5e-3**	***h* = 1e-2**	***h* = 5e-2**	***h* = 1e-1**
K_p	39003	7803	3903	783	393	81	42
K_d	-39000	-7800	-3900	-780	-390	-78	-39
Right-most poles	-2.0 ± 0.03*i*	-1.95 -2.06	-2.0 ± 0.06*i*	-2.01 ± 0.05*i*	-2.01 ± 0.17*i*	-2.1 ± 0.30*i*	-2.05 ± 0.71*i*
***GM* (dB)**	82	68.1	62	48.1	42	27.9	21.7
***PM* (deg)**	81	81	80.9	80.5	79.9	75.3	69.4

The effect of the time delay, h, on the gains, rightmost poles, and stability margins is summarized in Table 5. Note that as the delay time, h, increases, the rightmost (i.e., dominant) eigenvalue is farther from the desired value of -2, and the stability margins are also reduced. Since typical frequency domain control design guidelines are GM > 6 dB and PM > 30 to 40 degrees, the delays considered here all yield good designs with acceptable closed-loop performance and stability margins. As noted before, when $\omega h = \pi$ radians the delay introduces a phase lag of 180 degrees. Consequently, if the desired closed-loop bandwidth is ω_{bw}, then delay times of $h \ll \pi / \omega_{bw}$ are desired. Thus, in this particular example the closed loop bandwidth is $\omega_{bw} \approx 1$ rad/s, and one requires for good performance that $h \ll \pi$. Selecting h = 0.1 sec, or smaller, is seen to be a good choice from the results in Table 1 for selected values of h.

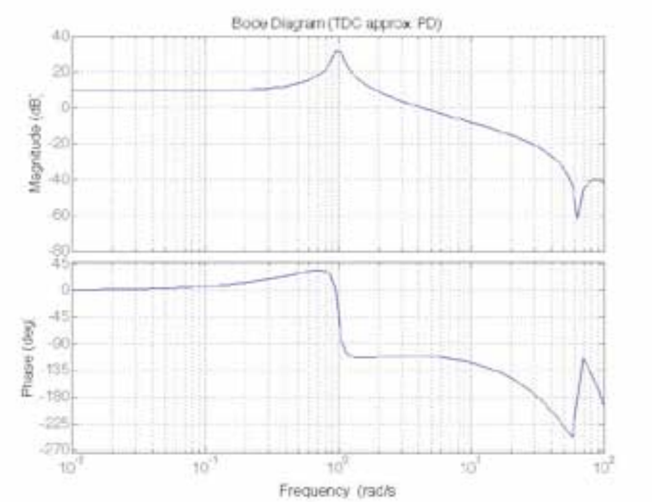

Figure 10. Loop transfer function Bode plot for approximate PD control (TDC, h = 0.1, K_p = 42, K_d = -39) of a single DOF system. GM = 21.7 dB and PM = 69.4 deg. Note: wrapping which occurs here for frequencies above 20π rad/s (i.e., $1/h$ = 10 Hz) is a computational artifact of the *bode* or *margin* commands in MATLAB.

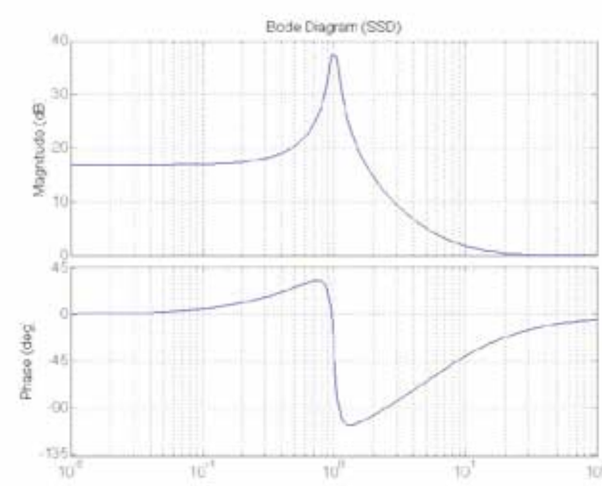

Figure 11. Loop transfer function Bode plot for SSD control (F_1 = 7, F_2 = 7.9, G_1 = 0, G_2 = 1) of a single DOF system. $GM = \infty$ and PM = 180 deg.

The rightmost eigenvalues in Table 5 are obtained using the Lambert W function method for TDS and the software available in the *LambertW_DDE Toolbox* [4, 28]. These results illustrate that by using a properly designed TDC, one can measure only position and achieve performance and stability margins very similar to SF control. Note that position feedback alone, without delay, cannot provide the desired performance (i.e., settling time of 3 sec, with no overshoot)

Next consider further robustness improvement of the closed-loop system in Eq. (18) by using SSD feedback instead of SF only. Let $u = -\mathbf{F}\mathbf{x} - \mathbf{G}\dot{\mathbf{x}} = -F_1 x_1 - F_2 x_2 - G_1 \dot{x}_1 - G_2 \dot{x}_2$, which is SSD feedback control of the single DOF system in Eq. (12). The controller transfer function is:

$$C(s) = (F_1 + (F_2 + G_1)s + G_2 s^2) \quad (19)$$

The closed-loop system can be written as:

$$\frac{d}{dt}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{(k+F_1)}{(m+G_2)} & -\dfrac{(b+F_2+G_1)}{(m+G_2)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad (20)$$

to achieve the same closed-loop eigenvalues (i.e., -2 and -2) as with state feedback, one can select G_1 and G_2, then determine F_1 and F_2 from the equations:

$$F_1 = 4(m+G_2) - k; \quad F_2 = 4(m+G_2) - b - G_1 \quad (21)$$

For example, if one selects $G_1 = 0$, $G_2 = 1.0$, $F_1 = 7$ and $F_2 = 7.9$, the response is exactly the same as in Fig. 7, but with stability margins of $GM = \infty$ and $PM = 180$ degrees (see Fig. 11). So the system with SSD control has a 98.9 degree increase in the PM compared to SF control only. The disadvantage, of course, is that now one needs to measure not only position and velocity of the mass, but also its acceleration.

Next, use of the approximation in Eq. (7) is considered to achieve improved stability margins compared to SF only, but by using only measurement of position and velocity. For the approximate SSD controller:

$$\begin{aligned} u(t) &= -\mathbf{K_p}\mathbf{x}(t) - \mathbf{K_d}\mathbf{x}(t-h) = \\ &-K_{p1}x_1(t) - K_{p2}x_2(t) - K_{d1}x_1(t-h) - K_{d2}x_2(t-h) \end{aligned} \quad (22)$$

and it has the controller transfer function:

$$C(s) = (K_{p1} + K_{p2}s + K_{d1}e^{-sh} + K_{d2}se^{-sh}) \quad (23)$$

From Eq. (10), the controller has the gains:

$$\begin{aligned} K_{d1} &= -G_1/h; \quad K_{d2} = -G_2/h \\ K_{p1} &= F_1 + G_1/h; \quad K_{p2} = F_2 + G_2/h \end{aligned} \quad (24)$$

The closed-loop system is given by:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{(k+F_1+G_1/h)}{m} & -\dfrac{(b+F_2+G_2/h)}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 \\ \dfrac{G_1}{mh} & \dfrac{G_2}{mh} \end{bmatrix} \begin{Bmatrix} x_1(t-h) \\ x_2(t-h) \end{Bmatrix} \quad (25)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

For example, with $G_1 = 0$ and $G_2 = 1.0$ as before, one obtains the gains $K_{d1} = 0$, $K_{d2} = -1/h$, $K_{p1} = 7$, and $K_{p2} = 7.9 + 1/h$. Table 6 presents the results for selected values of the delay h.

Table 6. Gains, rightmost eigenvalues, and phase margins for the system in Eq. (26) with approximate SSD control for $G_1 = 0$ and $G_2 = 1$ for selected values of h.

	h = 0.005	h = 0.01	h = 0.05	h = 0.1
Gains	K_{d1}=0; K_{d2}=-200; K_{p1}=7; K_{p2}=207.9	K_{d1}=0; K_{d2}=-100; K_{p1}=7; K_{p2}=107.9	K_{d1}=0; K_{d2}=-20; K_{p1}=7; K_{p2}=27.9	K_{d1}=0; K_{d2}=-10; K_{p1}=7; K_{p2}=17.9
Rightmost eigenvalues	-1.99 ±0.10i	-1.99 ±0.14i	-1.93 ±0.30i	-1.86 ±0.41i
***PM* (deg)**	157	148	115	98

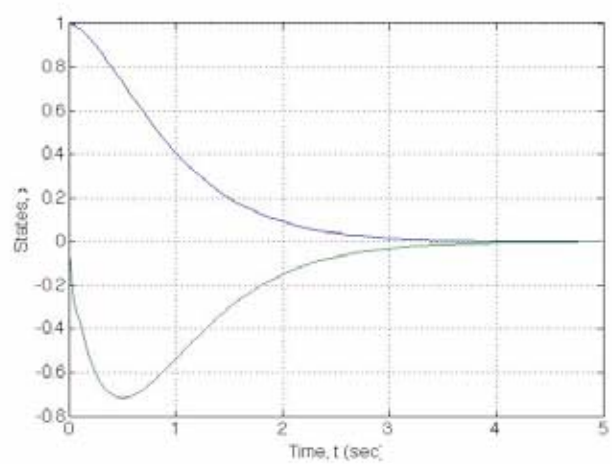

Figure 12. Response of the single DOF system with TDC (i.e., approximate SSD control) and $h = 0.1$ to initial condition $\mathbf{x}(0) = [1\ 0]^T$.

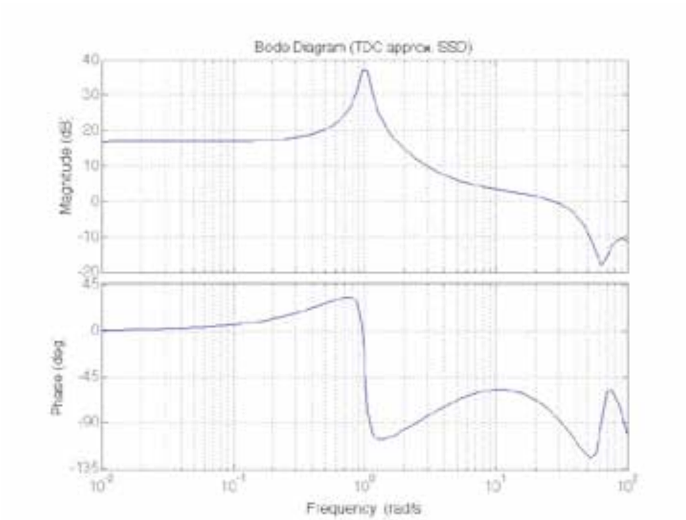

Figure 13. Loop transfer function Bode plot for TDC (i.e., approximate SSD control) and h = 0.1 of a single DOF system. GM = 30 dB and PM = 98 deg. Note: wrapping which occurs here for frequencies above 20π rad/s (i.e., $1/h$ = 10 Hz) is a computational artifact of the *bode* or *margin* commands in MATLAB.

The response of the closed-loop system with approximate SSD feedback (h = 0.1) is shown in Fig. 12 and is very similar to the response shown in Fig. 7 for the system with SF control. The loop transfer function Bode plot for approximate SSD control (h = 0.1) is shown in Fig. 13. For h = 0.1 s the PM = 98 degrees, which is a 16.9 degree improvement over the SF control. As shown in Table 6, smaller values of h can lead to even larger improvements in the PM. Note that the closed-loop system bandwidth is approximately 1 rad/s, and for frequencies one decade below $1/h$ = 10 Hz = 20π rad/s, the frequency response curves in Fig. 7 (for SSD control) and in Fig. 13 (for approximate SSD control) match quite well. Consequently, values of h = 0.1 s, or smaller, yield good results as shown in Table 6. The gain margins are no longer infinite, as for SF control, since the phase does cross 180 degrees at high frequencies due to the delay. However, for small values of h as considered in Table 6, the gain margins are well above the recommended design guideline of 6 dB. For example, when h = 0.1 s, the GM $\approx$ 30 dB, and is larger for smaller values of h.

The use of time delay for vibration control of the single DOF mechanical system in Fig. 6, can be beneficial in each of the following ways:

1. By measuring position only, and using time delay to approximate velocity, one can achieve essentially the same performance as with state derivative feedback, but with some loss of robustness (i.e., reduction in stability margins) depending on the selected delay time, h.
2. If both position and velocity are measured to implement state feedback control, the delay can be used to approximate acceleration and implement approximate SSD feedback, thus, improving stability margins beyond what can be achieved with state feedback only.

To achieve these robustness benefits of TDC, while maintaining the desired transient response, it is necessary to first design an SSD feedback controller [9], and then select a suitable delay time, h, in approximating the SD controller using Eq. (7). The recently developed Lambert W function based methods for time delay systems, and the open source software *LambertW_DDE Toolbox*, can be used to facilitate these steps by obtaining rightmost eigenvalues and the time response of the system of DDE's in Eq. (25) [4, 28].

4. SUMMARY AND CONCLUSIONS

Time delays can be used to reduce sensitivity and improve robustness by approximating the state derivatives in a state plus state derivative feedback controller. First an ideal SSD controller, assuming measurement of state derivatives, is designed. Then the state derivatives are approximated using a time delay. Methods based on the Lambert W function, for analysis of time delay systems, are used to select a suitable delay time, h, and SSD gain, $\mathbf{G}$. The method is first illustrated for a scalar system and shown to yield the specified performance with improved stability margins when compared to state feedback control. The method is then applied to a SISO vibration control problem for a single DOF system, and shown to yield the specified performance with improved stability margins when compared to state feedback control only.

In conclusion, TDC can approximate SSD control, and be utilized without requiring additional measurements, to improve stability margins compared to SF control only. Thus, the proper introduction of time delays in the control can improve robustness without sacrificing performance or requiring additional measurements. This paper has presented a systematic design approach for such controllers. The approach is based on fist designing an SSD feedback controller, and then using a finite difference approximation of the state derivative term using time delays. The appropriate time delay can be selected by solving a system of DDEs, for rightmost eigenvalues and/or time response, using a Lambert W function based method and related open source software.

In this paper we have considered only SISO LTI systems, and assumed measurement of state variables. Specifically, a scalar example and an application to a single DOF mechanical vibration control problem were considered. The method has also been successfully applied to SISO two DOF mechanical vibration problems, but those results are not included here due to space limitations. The extension of these results to multi-input multi-output LTI systems, and to observer-based controllers, appears to be relatively straightforward but is left to future research. An important topic for future research is, of course, to consider the effects of output measurement noise.

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