Ghi chú:

Thứ

Ngày / /

Buổi 3/01/04

Tiết 1: chữa BT giới hạn

Tiết 2: Học về hàm liên tục, giới hạn trái / phải

Thực hành

Tiết 3: Dạy về quy tắc L'Hospital

BTVN.

Chữa BT: 3 → 8 (NĐT).

3) ① $I = \lim\limits_{x \to 2} \dfrac{(x^2-x-2)^{20}}{(x^3-12x+16)^{10}}$ $\left(\dfrac{0}{0}\right)$ tử $2^2-2-2=0$, mẫu $2^3-12\cdot 2+16=0$.

$x^2-x-2 = (x-2)(x+1)$

$(x^3-12x+16) = (x-2)\underbrace{(x^2+2x-8)}$

$= (x-2)\,(x-2)(x+4)$

$= (x-2)^2\,(x+4)$.

Do đó $I = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)^{20}}\,(x+1)^{20}}{\cancel{(x-2)^{20}}\,(x+4)^{10}}$

$= \lim\limits_{x \to 2} \dfrac{(x+1)^{20}}{(x+4)^{10}} = \dfrac{3^{20}}{6^{10}} = \dfrac{3^{10}}{2^{10}}$

② $I = \lim\limits_{x \to 1} \dfrac{x+x^2+\dots+x^n-n}{x-1}$

$I = \lim\limits_{x \to 1} \dfrac{x-1+(x^2-1)+(x^3-1)+\dots+(x^n-1)}{x-1}$

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Thứ

Ngày / /

Chú ý: $x^k - 1 = (x-1)(x^{k-1} + x^{k-2} + \dots + 1)$.

Do đó

$$I = \lim_{x\to 1} 1 + (x+1) + (x^2+x+1) + \dots + (x^{n-1} + x^{n-2} + \dots + 1)$$

$$= 1 + 2 + 3 + \dots + n$$

$$= \frac{n(n+1)}{2}.$$

③ $\lim\limits_{x\to 1} \dfrac{x^{100} - 2x + 1}{x^{60} - 2x + 1}$ (dạng $\frac{0}{0}$)

$TS = x^{100} - 2x + 1 = x^{100} - 1 - (2x-2)$

$= (x-1)[x^{99} + x^{98} + \dots + 1] - 2(x-1)$.

$= (x-1)[x^{99} + x^{98} + \dots + 1 - 2]$

$MS = x^{60} - 2x + 1$

$\underline{\underline{tt}}$ $(x-1)[x^{59} + x^{58} + \dots + 1 - 2]$.

Ta có $I = \lim\limits_{x\to 1} \dfrac{TS}{MS} = \lim\limits_{x\to 1} \dfrac{x^{99} + x^{98} + \dots + 1 - 2}{x^{59} + x^{58} + \dots + 1 - 2}$

$$= \frac{100 - 2}{60 - 2} = \frac{98}{58}.$$

④ $I = \lim\limits_{x\to a} \dfrac{x^n - a^n - (x-a)na^{n-1}}{(x-a)^2}$ $\left(\frac{0}{0}\right)$

$TS = x^n - a^n - (x-a)na^{n-1}$

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~~$x^n - a^n - xna^{n-1} + na^n$~~

$\boxed{C1}$ Tìm cách phân tích tử số thành

$$(x-a)^2\,(\ldots\ldots) \Rightarrow \text{hơi khá dài.}$$

$\boxed{C2}$ Dùng quy tắc L'Hospital $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \left(\text{nếu dạng } \frac{0}{0}, \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)} \quad (\,' : \text{đạo hàm theo } x)$$

Áp dụng: $I = \lim_{x \to a} \dfrac{x^n - a^n - xna^{n-1} + na^n}{(x-a)^2}$

$$\overset{L'H}{=} \lim_{x \to a} \frac{nx^{n-1} - na^{n-1}}{2(x-a)} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x \to a} \frac{n(n-1)x^{n-2}}{2} = \frac{n(n-1)a^{n-2}}{2},$$

4) ① $I = \lim_{x \to +\infty} \dfrac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{\sqrt{x+1}}$ $\left(\text{dạng } \frac{\infty}{\infty}\right)$

$$= \lim_{x \to +\infty} \frac{\sqrt{1 + \sqrt{\frac{x+\sqrt{x}}{x^2}}}}{\sqrt{1 + \frac{1}{x}}} = 1$$

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Thứ

Ngày / /

(2) $I = \lim\limits_{x \to +\infty} \dfrac{\sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}}{\sqrt{2x+1}} = \lim\limits_{x \to +\infty} \dfrac{x^{1/2} + x^{1/3} + x^{1/4}}{\sqrt{2x+1}}$

$= \lim\limits_{x \to +\infty} \dfrac{1 + x^{-1/6} + x^{-1/4}}{\sqrt{2 + \frac{1}{x}}}$ (chia cả tử & mẫu cho $\sqrt{x}$)

$= \dfrac{1}{2}$.

6) (1) $I = \lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a} = \lim\limits_{x \to a} \dfrac{2 \cos \frac{x+a}{2} \sin \frac{x-a}{2}}{2 \cdot \frac{x-a}{2}}$

$= \lim\limits_{x \to a} \cos \dfrac{x+a}{2} \cdot \dfrac{\sin \frac{x-a}{2}}{\left(\frac{x-a}{2}\right)} = \cos a \cdot 1 = \cos a$

(2) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} = \lim\limits_{x \to 0} \dfrac{1 + \tan x - (1 + \sin x)}{x^3 (\sqrt{\cdots} + \sqrt{\cdots})}$

$= \lim\limits_{x \to 0} \dfrac{\frac{\sin x}{\cos x} - \sin x}{x^3 (\sqrt{\cdots} + \sqrt{\cdots})} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{\frac{1 - \cos x}{\cos x}}{x^2 \cdot (\sqrt{\cdots} + \sqrt{\cdots})}$

$= 1 \cdot \lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} \cdot \dfrac{1}{x^2 (\sqrt{1+\tan x} + \sqrt{1+\sin x})}$

$= 1 \cdot \dfrac{1}{2}$

(3) $I = \lim\limits_{x \to 0} \dfrac{1 - \cos x + \cos x - \cos x \cos 2x + \cos x \cos 2x (1 - \cos 3x)}{1 - \cos x}$

$= \lim\limits_{x \to 0} 1 + \cos x \cdot \dfrac{1 - \cos 2x}{1 - \cos x} + \cos x \cos 2x \cdot \dfrac{1 - \cos 3x}{1 - \cos x}$

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Chú ý: $\dfrac{1-\cos(kx)}{1-\cos x} = \dfrac{2\sin^2\frac{kx}{2}}{2\sin^2\frac{x}{2}}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{1-\cos(kx)}{1-\cos x} = \lim\limits_{x\to 0} \ldots = k^2$

Do đó $I = 1 + 1.2^2 + 1.1.3^2 = 14$.

(4) $I = \lim\limits_{x\to 0} \dfrac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x} = \lim\limits_{x\to 0} \dfrac{\sqrt{\cos x}-1}{\sin^2 x} + \dfrac{1-\sqrt[3]{\cos x}}{\sin^2 x}$

x l/hợp $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{\sin^2 x \cdot (\sqrt{\cos x}+1)}$

$+ \dfrac{1-\cos x}{\sin^2 x\left((\sqrt[3]{\cos x})^2 + \sqrt[3]{\cos x} + 1\right)}$

$= \dfrac{1}{(1+\sqrt{1})} \cdot \dfrac{-1}{2} + \dfrac{+1}{2} \cdot \dfrac{1}{3} = \dfrac{-1}{12}$

7) Có 2 câu đến x liên hợp, dễ quá nên các em tự xem giải

8) (1) $I = \lim\limits_{x\to +\infty} \left(\dfrac{3x^2-x+1}{2x^2+x+1}\right)^{\frac{x^3}{1-x}}$

Vì $\lim\limits_{x\to +\infty} \dfrac{3x^2-x+1}{2x^2+x+1} = \dfrac{3}{2}$.

$\Rightarrow$ I k° có dạng vô định

Vì $\lim\limits_{x\to +\infty} \dfrac{x^3}{1-x} = -\infty$.

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$$\Rightarrow I = \left(\frac{3}{2}\right)^{-\infty} = 0.$$

② Cũng chỉ phải dạng vô định.

$$I = 1^{1} = 1.$$

③ $I = \lim\limits_{x \to 0} \sqrt[x]{1-2x}$ (Dạng 1^∞)

$$= \lim_{x \to 0} (1-2x)^{1/x}$$

Ta có $I = \lim\limits_{x \to 0} \left(1 + (-2x)\right)^{\frac{1}{-2x} \cdot \frac{-2x}{x}}$

$$= \lim_{u \to 0} (1+u)^{\frac{1}{u} \cdot 2} = e^{2}$$

$(u = -2x)$

④ $I = \lim\limits_{x \to 0} \sqrt[x]{\cos\sqrt{x}} = \lim\limits_{x \to 0} (\cos\sqrt{x})^{1/x}$ (Dạng 1^∞)

Đặt $\sqrt{x} = y \Rightarrow x = y^2 \Rightarrow \frac{1}{x} = \frac{1}{y^2}$

$$I = \lim_{y \to 0} (\cos y)^{1/y^2} = \lim_{y \to 0} \left(1 - (1-\cos y)\right)^{\frac{1}{y^2}}$$

$$= \lim_{y \to 0} \left(1 - 2\sin^2\frac{y}{2}\right)^{1/y^2} = \lim_{y \to 0} e^{\frac{-2\sin^2\frac{y}{2}}{y^2}}$$

$$= e^{-\frac{1}{2}}$$

⑤ $I = \lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ (Dạng 1^∞)

5 10 15 20

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Ngày / /

Đổi biến $y = \frac{\pi}{2} - x \Rightarrow x \to \frac{\pi}{2}$ thì $y \to 0$.

$\sin x = \sin\left(\frac{\pi}{2} - y\right) = \cos y = 1 - 2\left(\sin\frac{y}{2}\right)^2$

$\tan x = \tan\left(\frac{\pi}{2} - y\right) = \cot y = \frac{\cos y}{\sin y}$.

Do đó $I = \lim\limits_{y \to 0}\left[1 - 2\left(\sin\frac{y}{2}\right)^2\right]^{\frac{\cos y}{\sin y}}$

$$= e^{\lim\limits_{y\to 0} \frac{-2\left(\sin\frac{y}{2}\right)^2 \cdot \cos y}{\sin y}}$$

Từ CThức $\sin y = 2\sin\frac{y}{2}\cos\frac{y}{2}$

$$I = e^{\lim\limits_{y\to 0}\frac{-\sin\frac{y}{2}\cos y}{\cos\frac{y}{2}}} = e^{\frac{0 \cdot 1}{1}} = 1.$$

(6) $I = \lim\limits_{x\to+\infty} \ldots \Rightarrow$ Tôi cũng làm như sách giải của thầy Trí thôi.

(7) $I = \lim\limits_{x\to 0}\frac{e^{\alpha x} - e^{\beta x}}{\sin(\alpha x) - \sin(\beta x)}$ Dùng quy tắc L'Hospital là nhanh nhất

$\overset{L'H}{=} \lim\limits_{x\to 0}\frac{\alpha e^{\alpha x} - \beta e^{\beta x}}{\alpha\cos(\alpha x) - \beta\cos(\beta x)}$

thay $x=0$ $\frac{\alpha - \beta}{\alpha - \beta} = 1.$

(8) Thầy cũng làm như sách giải thầy Trí.

HONG HA

Ghi chú:

Thứ

Ngày / /

(8) $$I = \lim_{n\to\infty} n^2 \, x^{\frac{1}{n+1}} \left(x^{\frac{1}{n(n+1)}} - 1 \right)$$

$$= \lim_{n\to\infty} x^{\frac{1}{n+1}} \cdot \frac{n^2}{n(n+1)} \cdot \lim_{n\to\infty} \frac{x^{\frac{1}{n(n+1)}} - 1}{\left(\frac{1}{n(n+1)}\right)}$$

$\downarrow$ $x^0 = 1$ $\quad$ $\downarrow$ 1 $\quad$ $\downarrow$ dạng $\lim_{y\to 0} \frac{x^y - 1}{y}$

Chú ý: các đẳng thức giới hạn đáng nhớ thấy cho

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1} , \boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a}$$

Do đó ta có $I = 1 \cdot 1 \cdot \ln(x) = \ln(x)$.

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Bài tập bổ sung:

5) $$I = \lim_{x\to 0}\left(1-\tan^2 x\right)^{\frac{1}{\sin^2(2x)}}$$

$$= e^{\lim_{x\to 0}\frac{-\tan^2 x}{\sin^2(2x)}}$$

(vì $\lim_{x\to 0}\tan x = 0$

và ta s.dụng $\lim_{u\to 0}(1+u)^{\frac{1}{u}} = e$)

Do đó ta cần tìm $I_2 = \lim_{x\to 0}\frac{-\tan^2 x}{\sin^2(2x)}$

$$= \lim_{x\to 0}\frac{-\frac{\sin^2(x)}{\cos^2(x)}}{2^2\sin^2(x)\cos^2(x)} = \lim_{x\to 0}\frac{-1}{2^2\cos^4(x)}$$

$$= \frac{-1}{4}$$

Vì vậy $I = e^{I_2} = e^{-1/4}$

6) $$I = \lim_{x\to 0}\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{x^2}}$$

$$= e^{\lim_{x\to 0}\frac{-2\sin^2\frac{x}{2}}{x^2}} = e^{-\frac{1}{2}}$$

~~Các câu 8, 10 làm~~ Câu 7 bỏ vì chứa cosh.

8) $$I = \lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2-1}\right)^{x^2}$$

$$= \lim_{x\to\infty}\left(1+\frac{4}{2x^2-1}\right)^{x^2}$$

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$$= \lim_{x\to\infty}\left(1+\frac{1}{\left(\frac{2x^2-1}{4}\right)}\right)^{x^2}$$

$$= e^{\lim_{x\to\infty}\frac{x^2}{(2x^2-1)/4}}$$

$$= e^{\lim_{x\to\infty}\frac{4x^2}{2x^2-1}} = e^2.$$

9) $$\lim_{x\to 2}\frac{2^x-x^2}{x-2} \overset{\text{L'Hospital}}{=} \lim_{x\to 2}\frac{2^x\ln 2-2x}{1}$$

$$= 4\ln 2-4.$$

10) $$I=\lim_{x\to\infty}\left(e^{\frac{1}{x}}+\frac{1}{x}\right)^x$$

$$\Rightarrow \ln I = \lim_{x\to\infty} x\ln\left(e^{\frac{1}{x}}+\frac{1}{x}\right)$$

Đổi biến $y=\frac{1}{x} \Rightarrow \ln I = \lim_{u\to 0}\frac{\ln(e^u+u)}{u}$

$$\overset{L'H}{=} \lim_{u\to 0}\frac{\frac{e^u+1}{e^u+u}}{1} = \frac{e^0+1}{e^0+0}=2$$

$$\Rightarrow I = e^2.$$

Bài 5 (sgk - 113).

1) $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x}}{x}$

Ta có: $\dfrac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x}}{x} = \dfrac{(\sqrt[m]{1+\alpha x} - 1) + (1 - \sqrt[n]{1+\beta x})}{x}$

$= \dfrac{\sqrt[m]{1+\alpha x} - 1}{x} - \dfrac{\sqrt[n]{1+\beta x} - 1}{x}$.

Đặt $\sqrt[m]{1+\alpha x} = y$. Khi $x \to 0$ thì $y \to 1$. do đó:

$\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+\alpha x} - 1}{x} = \cancel{\lim\limits_{y\to 1} \dfrac{y-1}{y^2-1}}\ \lim \dfrac{\alpha(y-1)}{y^m - 1} = \dfrac{\alpha}{m}$.

Tương tự ta có $\lim\limits_{x\to 0} \dfrac{\sqrt[n]{1+\beta x} - 1}{x} = \dfrac{\beta}{n}$.

Vậy $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x}}{x} = \dfrac{\alpha}{m} - \dfrac{\beta}{n}$

2) $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+\alpha x} \cdot \sqrt[n]{1+\beta x} - 1}{x}$.

Ta có: $\sqrt[m]{1+\alpha x} \cdot \sqrt[n]{1+\beta x} - 1$

$= (\sqrt[m]{1+\alpha x} - 1) \cdot \sqrt[n]{1+\beta x} + (\sqrt[n]{1+\beta x} - 1)$.

Do đó theo ý (1) ta suy ra:

$\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+\alpha x} \cdot \sqrt[n]{1+\beta x} - 1}{x} = \dfrac{\alpha}{m} + \dfrac{\beta}{n}$.

BÀI TẬP BỔ SUNG

1. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(x+1)} = \lim\limits_{x\to 2} \dfrac{x+2}{x+1} = \dfrac{4}{3}$

2. $\lim\limits_{x\to 0} \dfrac{\sqrt[5]{32+x}-2}{x} = \lim\limits_{x\to 0} \dfrac{32+x-32}{x\left(\sqrt[5]{(32+x)^4}+2\sqrt[5]{(32+x)^3}+4\sqrt[5]{(32+x)^2}+8\sqrt[5]{32+x}+16\right)}$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt[5]{(32+x)^4}+2\sqrt[5]{(32+x)^3}+4\sqrt[5]{(32+x)^2}+8\sqrt[5]{32+x}+16}$

$= \dfrac{1}{80}$

3. $\lim\limits_{x\to 0} \dfrac{\cos 3x-\cos 7x}{x^2} = \lim\limits_{x\to 0} \dfrac{-(\cos 7x-\cos 3x)}{x^2} = \lim\limits_{x\to 0} \dfrac{2\sin 5x \cdot \sin 2x}{x^2}$

$= \lim\limits_{x\to 0} 2 \cdot \dfrac{\sin 5x}{5x} \cdot \dfrac{\sin 2x}{2x} \cdot 5 \cdot 2 = 2.5.2 = 20$

4. $\lim\limits_{x\to \frac{\pi}{4}} \cot 2x \cdot \cot\left(\frac{\pi}{4}-x\right)$

Ta có: $\cot 2x \cdot \cot\left(\dfrac{\pi}{4}-x\right) = \dfrac{\cos 2x \cdot \cos\left(\frac{\pi}{4}-x\right)}{\sin 2x \cdot \sin\left(\frac{\pi}{4}-x\right)} = \dfrac{(\cos^2 x-\sin^2 x)\ \frac{\sin x+\cos x}{\sqrt{2}}}{-2\sin x\cos x\ \frac{\sin x-\cos x}{\sqrt{2}}}$

$= \dfrac{(\sin x+\cos x)^2 \cdot (\cos x-\sin x)}{2\sin x \cdot \cos x \cdot (\cos x-\sin x)} = \dfrac{(\sin x+\cos x)^2}{2\sin x \cdot \cos x}$

$\Rightarrow \lim\limits_{x\to \frac{\pi}{4}} \cot 2x \cdot \cot\left(\dfrac{\pi}{4}-x\right) = \lim\limits_{x\to \frac{\pi}{4}} \dfrac{(\sin x+\cos x)^2}{2\sin x\cos x} = 2$