The e^{st} term cancels out, and we find that

$$\frac{V_o}{U_o} = \frac{\frac{1}{m}}{s + \frac{b}{m}}. \tag{2.6}$$

For reasons that will become clear in Chapter 3, this is often written using capital letters to signify that it is the "transform" of the solution, or

$$\frac{V(s)}{U(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}}. \tag{2.7}$$

Transfer function

This expression of the differential equation (2.4) is called the **transfer function** and will be used extensively in later chapters. Note that, in essence, we have substituted s for d/dt in Eq. (2.4). This transfer function serves as a math model that relates the car's velocity to the forces propelling the car, that is, inputs from the accelerator pedal. Transfer functions of a system will be used in later chapters to design feedback controllers such as a cruise control device found in many modern cars.

2. **Time response:** The dynamics of a system can be prescribed to Matlab in terms of its transfer function as can be seen in the Matlab statements below that implements Eq. (2.7). The `step` function in Matlab calculates the time response of a linear system to a unit step input. Because the system is linear, the output for this case can be multiplied by the magnitude of the input step to derive a step response of any amplitude. Equivalently, `sys` can be multiplied by the magnitude of the input step.

Step response with Matlab

The statements

```
s=tf('s');                    % sets up the mode to define the
                                transfer function
sys = (1/1000)/(s + 50/1000); % defines the transfer function from
                                Eq. (2.7) with the numbers filled in.
step(500*sys);                % plots the step response for u = 500.
```

calculate and plot the time response of velocity for an input step with a 500-N magnitude. The step response is shown in Fig. 2.3.

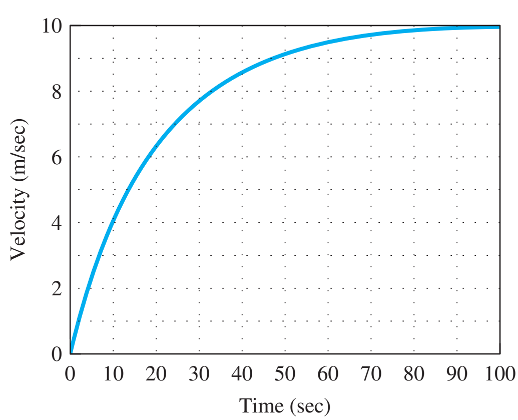


Figure 2.3
Response of the car velocity to a step in u

Newton's law also can be applied to systems with more than one mass. In this case it is particularly important to draw the free-body diagram of each mass, showing the applied external forces as well as the equal and opposite internal forces that act from each mass on the other.

EXAMPLE 2.2 *A Two-Mass System: Suspension Model*

Figure 2.4 shows an automobile suspension system. Write the equations of motion for the automobile and wheel motion assuming one-dimensional vertical motion of one quarter of the car mass above one wheel. A system consisting of one of the four wheel suspensions is usually referred to as a **quarter-car model**. The system can be approximated by the simplified system shown in Fig. 2.5 where two spring constants and a damping coefficient are defined. Assume that the model is for a car with a mass of 1580 kg, including the four wheels, which have a mass of 20 kg each. By placing a known weight (an author) directly over a wheel and measuring the car's deflection, we find that $k_s = 130{,}000$ N/m. Measuring the wheel's deflection for the same applied weight, we find that $k_w \simeq 1{,}000{,}000$ N/m. By using the step response data in Fig. 3.19(b) and qualitatively observing that the car's response to a step change matches the damping coefficient curve for $\zeta = 0.7$ in the figure, we conclude that $b = 9800$ N·sec/m.

Figure 2.4
Automobile suspension

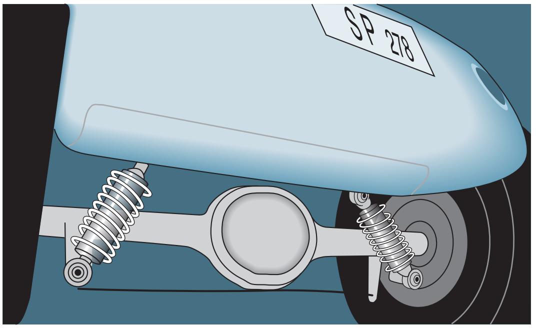
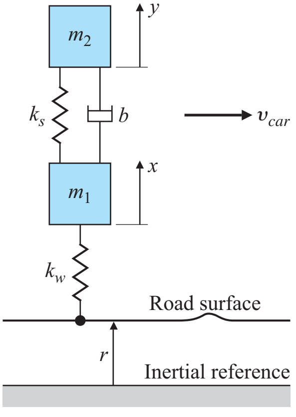


Figure 2.5
The quarter-car model



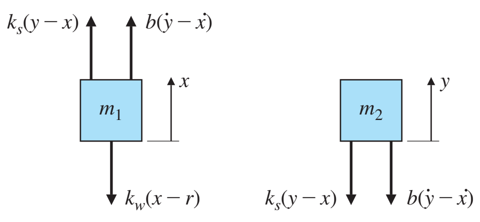
Solution. The system can be approximated by the simplified system shown in Fig. 2.5. The coordinates of the two masses, x and y, with the reference directions as shown, are the displacements of the masses from their equilibrium conditions. The equilibrium positions are offset from the springs' unstretched positions because of the force of gravity. The shock absorber is represented in the schematic diagram by a dashpot symbol with friction constant b. The magnitude of the force from the shock absorber is assumed to be proportional to the rate of change of the relative displacement of the two masses—that is, the force $= b(\dot{y}-\dot{x})$. The force of gravity could be included in the free-body diagram; however, its effect is to produce a constant offset of x and y. By defining x and y to be the distance from the equilibrium position, the need to include the gravity forces is eliminated.

The force from the car suspension acts on both masses in proportion to their relative displacement with spring constant k_s. Figure 2.6 shows the free-body diagram of each mass. Note that the forces from the spring on the two masses are equal in magnitude but act in opposite directions, which is also the case for the damper. A positive displacement y of mass m_2 will result in a force from the spring on m_2 in the direction shown and a force from the spring on m_1 in the direction shown. However, a positive displacement x of mass m_1 will result in a force from the spring k_s on m_1 in the opposite direction to that drawn in Fig. 2.6, as indicated by the *minus* x term for the spring force.

The lower spring k_w represents the tire compressibility, for which there is insufficient damping (velocity-dependent force) to warrant including a dashpot in the model. The force from this spring is proportional to the distance the tire is compressed and the nominal equilibrium force would be that required to support m_1 and m_2 against gravity. By defining x to be the distance from equilibrium, a force will result if either the road surface has a bump (r changes from its equilibrium value of zero) or the wheel bounces (x changes). The motion of the simplified car over a bumpy road will result in a value of $r(t)$ that is not constant.

As previously noted, there is a constant force of gravity acting on each mass; however, this force has been omitted, as have been the equal and opposite forces from the springs. Gravitational forces can always be omitted from vertical-spring mass systems (1) if the position coordinates are defined from the equilibrium position that results when gravity is acting, and (2) if the spring forces used in the analysis are actually the perturbation in spring forces from those forces acting at equilibrium.

Figure 2.6
Free-body diagrams for suspension system



Applying Eq. (2.1) to each mass and noting that some forces on each mass are in the negative (down) direction yields the system of equations

$$b(\dot{y}-\dot{x})+k_s(y-x)-k_w(x-r)=m_1\ddot{x}, \tag{2.8}$$

$$-k_s(y-x)-b(\dot{y}-\dot{x})=m_2\ddot{y}. \tag{2.9}$$

Some rearranging results in

$$\ddot{x}+\frac{b}{m_1}(\dot{x}-\dot{y})+\frac{k_s}{m_1}(x-y)+\frac{k_w}{m_1}x=\frac{k_w}{m_1}r, \tag{2.10}$$

$$\ddot{y}+\frac{b}{m_2}(\dot{y}-\dot{x})+\frac{k_s}{m_2}(y-x)=0. \tag{2.11}$$

Check for sign errors

The most common source of error in writing equations for systems like these are sign errors. The method for keeping the signs straight in the preceding development entailed mentally picturing the displacement of the masses and drawing the resulting force in the direction that the displacement would produce. Once you have obtained the equations for a system, a check on the signs for systems that are obviously stable from physical reasoning can be quickly carried out. As we will see when we study stability in Section 6 of Chapter 3, a stable system always has the same signs on similar variables. For this system, Eq. (2.10) shows that the signs on the $\ddot{x}$, $\dot{x}$, and x terms are all positive, as they must be for stability. Likewise, the signs on the $\ddot{y}$, $\dot{y}$, and y terms are all positive in Eq. (2.11).

The transfer function is obtained in a similar manner as before for zero initial conditions. Substituting s for d/dt in the differential equations yields

$$s^2X(s)+s\frac{b}{m_1}(X(s)-Y(s))+\frac{k_s}{m_1}(X(s)-Y(s))+\frac{k_w}{m_1}X(s)=\frac{k_w}{m_1}R(s),$$

$$s^2Y(s)+s\frac{b}{m_2}(Y(s)-X(s))+\frac{k_s}{m_2}(Y(s)-X(s))=0,$$

which, after some algebra and rearranging to eliminate $X(s)$, yields the transfer function

$$\frac{Y(s)}{R(s)}=\frac{\frac{k_w b}{m_1m_2}\left(s+\frac{k_s}{b}\right)}{s^4+\left(\frac{b}{m_1}+\frac{b}{m_2}\right)s^3+\left(\frac{k_s}{m_1}+\frac{k_s}{m_2}+\frac{k_w}{m_1}\right)s^2+\left(\frac{k_w b}{m_1m_2}\right)s+\frac{k_wk_s}{m_1m_2}}. \tag{2.12}$$

To determine numerical values, we subtract the mass of the four wheels from the total car mass of 1580 kg and divide by 4 to find that $m_2=375$ kg. The wheel mass was measured directly to be $m_1=20$ kg. Therefore, the transfer function with the numerical values is

$$\frac{Y(s)}{R(s)}=\frac{1.31e06(s+13.3)}{s^4+(516.1)s^3+(5.685e04)s^2+(1.307e06)s+1.733e07}. \tag{2.13}$$