# On the Direct Computation of the Byrnes-Isidori Normal Form

**Klaus Röbenack**[1,*]

[1] Institut für Regelungs- und Steuerungstheorie, Fakultät Elektrotechnik und Informationstechnik, Technische Universität Dresden, 01062 Dresden, Germany

If an input-affine single-input single-output control system has a well-defined relative degree, there exists a (local) change of coordinates transforming the system into the Byrnes-Isidori normal form. By this transformation, the system is decomposed into two subsystems. The first subsystem can be linearized by feedback, and the second subsystem does not directly depend on the input. To achieve this structure, the last components of the transformation have to fulfill a partial differential equation. Practically, the appropriate choice of these coordinates is usually left to the (experienced) user. In this note, the author suggests a direct calculation based on the computation of flows of certain vector fields.



## 1 Preliminaries

Consider a nonlinear input-affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u, \quad y = h(\mathbf{x}) \tag{1}$$

with smooth vector fields $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ and a smooth scalar field $h : \mathbb{R}^n \to \mathbb{R}$. The *Lie derivative* of the scalar field $h$ along the vector field $f$ is defined by $L_{\mathbf{f}}h(\mathbf{x}) := h'(\mathbf{x})\mathbf{f}(\mathbf{x})$, the *Lie bracket* of the vector fields $\mathbf{f}$ and $\mathbf{g}$ is given by $[\mathbf{f},\mathbf{g}](\mathbf{x}) := \mathbf{g}'(\mathbf{x})\mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{g}(\mathbf{x})$, see [1,2]. We recall the following results [3,4]:

**Theorem 1.1** (Rectification of Vector Fields) *Consider $k$ linearly independent vector fields $\mathbf{f}_1, \ldots, \mathbf{f}_k : \mathbb{R}^n \to \mathbb{R}^n$ with $[\mathbf{f}_i, \mathbf{f}_j] \equiv \mathbf{0}$ for $1 \le i, j \le k$ in a neighborhood of $\mathbf{p} \in \mathbb{R}^n$. Then, there exists a local diffeomorphism $\mathbf{z} = T(\mathbf{x})$, $\mathbf{x} = S(\mathbf{z})$ with $T'(\mathbf{x})\mathbf{f}_i(\mathbf{x}) = \frac{\partial}{\partial z_i}$ for $1 \le i \le k$.*

After adding $n-k$ further linearly independent vector fields $\mathbf{f}_{k+1}, \ldots, \mathbf{f}_n : \mathbb{R}^n \to \mathbb{R}^n$, the change of coordinates can be constructed by the following concatenation of flows [3, Theorem 2.36]: $\mathbf{x} = S(\mathbf{z}) = \varphi_{z_1}^{\mathbf{f}_1} \circ \cdots \circ \varphi_{z_n}^{\mathbf{f}_n}(\mathbf{p})$.

**Lemma 1.2** *Let $\Delta$ be an involutive distribution with $k = \dim \Delta$. Then, there exists a basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ of $\Delta$ with $[\mathbf{f}_i, \mathbf{f}_j] \equiv \mathbf{0}$ for $1 \le i, j \le k$ in a neighborhood of $\mathbf{p} \in \mathbb{R}^n$.*

**Theorem 1.3** (Frobenuis) *Let $\Delta$ be an involutive distribution with $k = \dim \Delta$ in a neighborhood of $\mathbf{p} \in \mathbb{R}^n$. Then, there exist $n-k$ scalar fields $\lambda_1, \ldots, \lambda_{n-k}$ with $\Delta^\perp = \mathrm{span}\{\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_{n-k}\}$.*

Proof. If $\Delta$ is involutive, then there are vector fields $\mathbf{f}_1, \ldots, \mathbf{f}_k$ with $\Delta = \mathrm{span}\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ and $[\mathbf{f}_i, \mathbf{f}_j] = \mathbf{0}$ for $1 \le i, j \le k$ (Lemma 1.2). According to Theorem 1.1 there exists a change of coordinates $\mathbf{z} = T(\mathbf{x})$ such that $\Delta = \mathrm{span}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k}\}$. The annihilator is given by $\Delta^\perp = \mathrm{span}\{\mathrm{d}z_{r+1}, \ldots, \mathrm{d}z_n\}$. In original coordinates, the annihilator is described by the gradients of the last $n-r$ components of the transformation $T$, i.e., $\lambda_1 := t_{r+1}, \ldots, \lambda_{n-r} := t_n$. □

## 2 Relative Degree and Byrnes-Isidori Normal Form

**Definition 2.1** The system (1) has a *relative degree $r$* at a point $\mathbf{p} \in \mathbb{R}^n$ if $L_{\mathbf{g}}h(\mathbf{x}) = L_{\mathbf{g}}L_{\mathbf{f}}h(\mathbf{x}) = \cdots = L_{\mathbf{g}}L_{\mathbf{f}}^{r-2}h(\mathbf{x}) = 0$ for all $\mathbf{x}$ in a neighborhood of $\mathbf{p}$ and $L_{\mathbf{g}}L_{\mathbf{f}}^{r-1}h(\mathbf{p}) \neq 0$.

Assume the system has a well-defined relative degree $r$ at $p$. Then, the covectors $\mathrm{d}h(\mathbf{p}), \ldots, \mathrm{d}L_{\mathbf{f}}^{r-1}h(\mathbf{p})$ are linearly independent [1, Lemma 4.1.1]. This means that the associated Lie derivatives define a partial change of coordinates, i.e., $z_1 = \phi_1(\mathbf{x}) = h(x), \ldots, z_r = \phi_r(\mathbf{x}) = L_{\mathbf{f}}^{r-1}h(x)$. The next theorem describes a special choice of the additional coordinates:

**Theorem 2.2** *Assume the system has a well-defined relative degree $r$ at $p$. Then there exists a diffeomorphism $\mathbf{z} = \Phi(\mathbf{x})$ transforming the system into the* Byrnes-Isidori normal form

$$\begin{array}{rclcrcl} \dot{z}_1 & = & z_2, & \qquad & \dot{z}_{r+1} & = & q_1(\mathbf{z}), \\ & \vdots & & & & \vdots & \\ \dot{z}_{r-1} & = & z_r, & & \dot{z}_n & = & q_{n-r}(\mathbf{z}), \\ \dot{z}_r & = & \alpha(\mathbf{z}) + \beta(\mathbf{z})u, & & y & = & z_1. \end{array} \tag{2}$$

* Corresponding author: e-mail klaus.roebenack@tu-dresden.de, phone +49 351 463 33940, fax +49 351 463 37281

P r o o f. If system (1) has a well-defined relative degree $r$ at $p$, the distribution $\Delta := \mathrm{span}\{\mathbf{g}\}$ is regular with $\dim \Delta = 1$ in a neighborhood of $\mathbf{p}$. In addition, $\Delta$ is also involutive due to $[\mathbf{g}, \mathbf{g}] \equiv \mathbf{0}$. Theorem 1.3 implies the existence of $n-1$ scalar fields $\lambda_1, \ldots, \lambda_{n-1}$ with $\Delta^\perp = \mathrm{span}\{\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_{n-1}\}$. With the well-defined relative degree we have $L_{\mathbf{g}} L_{\mathbf{f}}^i h = \langle \mathrm{d}L_{\mathbf{f}}^i h, \mathbf{g} \rangle = 0$ for $0 \le i \le r-2$. This imples $\mathrm{d}h, \mathrm{d}L_{\mathbf{f}}h \ldots, \mathrm{d}L_{\mathbf{f}}^{r-2}h \in \Delta^\perp$. The Basis Exchange Lemma allows the relacement of $r-1$ elements in the annihilator. Without loss of generality we have $\Delta^\perp = \mathrm{span}\{\mathrm{d}h, \mathrm{d}L_{\mathbf{f}}h \ldots, \mathrm{d}L_{\mathbf{f}}^{r-2}h, \mathrm{d}\lambda_r, \ldots, \mathrm{d}\lambda_{n-1}\}$. The new coordinates are given by $\phi_{r+1} := \lambda_r, \ldots, \phi_n := \lambda_{n-1}$. □

## 3 Computation

The proof of Theorem 2.2 based on Theorem 1.3 is constructive. The following procedure to compute the Byrnes-Isidori normal form can be derived [4]: We start with the input vector field $\mathbf{g}_1 := \mathbf{g}$ and add $n-1$ further linearly independent vector fields $\mathbf{g}_2, \ldots, \mathbf{g}_n$, e.g. unit vector fields $\frac{\partial}{\partial x_i}$. Then, we compute the associated flows $\varphi_t^{\mathbf{g}_1}, \ldots, \varphi_t^{\mathbf{g}_n}$. The concatenation of these flows yields a transformation $\mathbf{x} = S(\tilde{\mathbf{z}}) = \varphi_{\tilde{z}_1}^{\mathbf{g}_1} \circ \cdots \circ \varphi_{\tilde{z}_n}^{\mathbf{g}_n}(\mathbf{p})$ according to Theorem 1.1. Next, we compute the inverse map $\tilde{\mathbf{z}} = T(\mathbf{x})$ with components $t_1, \ldots, t_n$. Due to Theorem 1.3 we have $\mathrm{span}\{\mathbf{g}_1\}^\perp = \mathrm{span}\{\mathrm{d}t_2, \ldots, \mathrm{d}t_n\}$. From this basis we select elements $\mathrm{d}t_{i_j}$ to augment the gradients resulting from the first subsystem such that $\mathrm{span}\{\mathbf{g}_1\}^\perp = \mathrm{span}\{\mathrm{d}h, \mathrm{d}L_{\mathbf{f}}h, \ldots, \mathrm{d}L_{\mathbf{f}}^{r-2}h, \mathrm{d}t_{i_1}, \ldots, \mathrm{d}t_{i_{n-r}}\}$. The corresponding coordinates of the second subsystem are given by $\phi_{r+1} := t_{i_1}, \ldots, \phi_n := t_{i_{n-r}}$.

## 4 Example

We consider the underactuated two-degree-of-freedom horizontal manipulator given in [5]. A partial feedback linearization with relative degree $r = 2$ yields the model

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad \dot{x}_3 = x_4, \quad \dot{x}_4 = \kappa x_2^2 \sin x_3 - (1 + \kappa \cos x_3)u, \quad y = x_1 \tag{3}$$

with the dimensionless parameter $\kappa > 0$ and the input $u$, see [6]. The $r$-dimensional first subsystem having the coordinates $\{x_1, x_2\}$ is already in the form given by Eq. (2). The associated coordinates are given by $z_1 = \phi_1(\mathbf{x}) = h(\mathbf{x}) = x_1$ and $z_2 = \phi_2(\mathbf{x}) = L_{\mathbf{f}}h(\mathbf{x}) = x_2$. However, the second subsystem consisting of $\{x_3, x_4\}$ does still depend on the input $u$. To compute the $n-r$ coordinates for the second subsystem we consider the input vector field $\mathbf{g}(\mathbf{x}) = \frac{\partial}{\partial x_2} - (1 + \kappa \cos x_3)\frac{\partial}{\partial x_4}$ obtained from (3). We augment this vector field with $\mathbf{g}_2 = \frac{\partial}{\partial x_1}$, $\mathbf{g}_3 = \frac{\partial}{\partial x_3}$, $\mathbf{g}_4 = \frac{\partial}{\partial x_4}$. The concatenation of the associated flows yields the transformation $S$ with the components $x_1 = \tilde{z}_2$, $x_2 = \tilde{z}_1$, $x_3 = \tilde{z}_3$, $x_4 = \tilde{z}_4 - (1 + \kappa \cos \tilde{z}_3)\tilde{z}_1$ and the inverse $T$ with $\tilde{z}_1 = t_1(\mathbf{x}) = x_2$, $\tilde{z}_2 = t_2(\mathbf{x}) = x_1$, $\tilde{z}_3 = t_3(\mathbf{x}) = x_3$ and $\tilde{z}_4 = t_4(\mathbf{x}) = x_4 + (1 + \kappa \cos x_3)x_2$. The coordinates for the second subsystem can be chosen from $t_2, \ldots, t_3$. Since $t_2(\mathbf{x}) = x_1$ already occurs as a coordinate of the first subsystem, we select $z_3 = \phi_3(\mathbf{x}) = t_3(\mathbf{x})$ and $z_4 = \phi_4(\mathbf{x}) = t_4(\mathbf{x})$ as additional coordinates. With $\mathbf{z} = \Phi(\mathbf{x})$ system (3) is transformed into

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = u, \quad \dot{z}_3 = z_4 - (1 + \kappa \cos z_3)z_2, \quad \dot{z}_4 = -\kappa z_2(z_4 - \kappa z_2 \cos z_3)\sin z_3, \quad y = z_1.$$

This system has the structure (2) and coincides with the normal form computed in [7].

## 5 Conclusions

A method for the systematic computation of the Byrnes-Isidori normal form for input-affine single-input single-output control systems has been derived. The calculation can still be difficult since the flow of the input vector field needs to be computed. This part can be simplified performing a partial feedback linearization in advance. The computation of the flows of the other vector fields is straightforward using constant vector fields. The second obstracle is the symbolic inversion of a nonlinaer map.

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