Available online at www.sciencedirect.com
SCIENCE DIRECT®

automatica

Automatica 41 (2005) 1979–1985

www.elsevier.com/locate/automatica

Brief paper

A note on the use of the Lambert W function in the stability analysis of time-delay systems☆

Chyi Hwang[a,*], Yi-Cheng Cheng[b]

[a] *Department of Chemical Engineering, I-Shou University, Ta-Hsu, Kaohsiung 840, Taiwan*
[b] *Department of Chemical Engineering, National Chung Cheng University, Chia-Yi 621, Taiwan*

Received 15 September 2004; received in revised form 23 May 2005; accepted 30 May 2005

Abstract

The Lambert W function is defined to be the multivalued inverse of the function $w \to we^w = z$. This function has been used in an extremely wide variety of applications, including the stability analysis of fractional-order as well as integer-order time-delay systems. The latter application is based on taking the *m*th power and/or *n*th root of the transcendental characteristic equation (TCE) and representing the roots of the derived TCE(s) in terms of W functions. In this note, we re-examine such an application of using the Lambert W function through actually computing the root distributions of the derived TCEs of some chosen orders. It is found that the rightmost root of the original TCE is not necessarily a principal branch Lambert W function solution, and that a derived TCE obtained by taking the *m*th power of the original TCE introduces superfluous roots to the system. With these observations, some deficiencies displayed in the literature (Chen, Y. Q., & Moore, K. L. (2002a). Analytical stability bound for delayed second-order systems with repeating poles using Lambert W function. *Automatica*, *38*(5), 891–895, Chen, Y. Q., & Moore, K. L. (2002b). Analytical stability bound for a class of delayed fractional-order dynamic systems. *Nonlinear Dynamics*, *29*(1–4), 191–200) are pointed out. Moreover, we clarify the correct use of Lambert W function to stability analysis of a class of time-delay systems. This will actually enlarge the application scope of the Lambert W function, which is becoming a standard library function for various commercial symbolic software packages, to time-delay systems.

Keywords: Lambert W function; Delay; Transcendental equation; Stability; Fractional-order systems

1. Introduction

Time-delay systems are often described by delay-differential equations (Kuang, 1993). A linear or linearized time-invariant system with a single delay has in general a transcendental characteristic equation (TCE) of the form $A(s) + B(s)e^{-\tau s} = 0$, where τ is the delay time. Due to the presence of the exponential function $e^{-\tau s}$, this equation has an infinite number of roots, which makes the analytical stability analysis of a time-delay system extremely difficult. Up to now, no simple and general algebraic criterion, like the Routh–Hurwitz criterion for delay-free systems, has been presented in the literature for testing the root distribution of a TCE with respect to the imaginary axis of the complex plane. Usually, the stability analysis of time-delay systems relies on graphical methods, e.g., Nyquist criterion (Mossaheb, 1980) or D-partition technique (Neimark, 1973; Hwang & Hwang, 2004).

Recently, an approach of using the Lambert W function (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996) has been presented by Chen and Moore(2002a,b) to obtain a stability bound for a class of time-delay systems having the TCE

$$(s+\alpha)^{n/m} + K_p e^{-\tau s} = 0, \qquad (1)$$

where n and m are positive integers and $\alpha, \tau > 0$, K_p are real numbers. Such an approach has also been applied by Asl and Ulsoy (2003) to calculate time responses of time-delay

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor George Weiss under the direction of Editor Roberto Tempo.

* Corresponding author. Tel.: +886 5 242 8306; fax: +886 7 657 8945.
E-mail address: chyi@isu.edu.tw (C. Hwang).

0005-1098/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2005.05.020

systems and by Weiss, Zhong, and Green (2004) to investigate the stability properties of a repetitive control system which contains an internal model of delayed positive feedback loop. These applications are based on casting TCE (1) in the form

$$(as+b)e^{cs}+d=0 \tag{2}$$

whose roots can be represented by the Lambert W function as

$$s=\frac{1}{c}W\left(-\frac{dc}{a}e^{bc/a}\right)-\frac{b}{a}, \tag{3}$$

where $W(z)$ satisfies the functional equation $W(z)e^{W(z)}=z$. The function $W(z)$ is multivalued and the different possible solutions are denoted by $W_k(z)$ for $k=0, \pm 1, \pm 2$, etc. Chen and Moore used (3) with only the principal branch solution $W=W_0$ and with only a real argument to evaluate the stability bound in the parameter space for the time-delay systems whose TCEs are of the form (1).

As it can be noted, the cast of the original TCE (1) into a derived TCE of the form (2) is achieved through taking the *m*th power and *n*th root on the equation involved. One may naturally ask if every W function solution to the derived TCE also satisfies the original TCE. Due to the fact that the rightmost root of the original TCE (1) play an important role in the stability analysis, one may also ask if the rightmost root in the set of W function solutions to the derived TCEs of a delay system belongs to the set of principal branch W function solutions. To answer these two questions, we reexamine in this note the application of Lambert W function in stability analysis for the class of time-delay systems having the TCE (1). Through actually computing the W function solutions of the derived TCEs for the some specific systems, using the Lambert W function routine in the symbolic software package Maple, it is found that the answers are negative. The observation from the case studies motivates us to present a correct use of the Lambert W function in solving stability analysis problems of time-delay systems.

2. The Lambert W function

The Lambert W function is the complex function which solves for w the following equation:

$$we^{w}=z, \quad w, z \in \mathbf{C}, \tag{4}$$

where $\mathbf{C}$ is the set of complex numbers. The complex function $W(z)$ satisfies the functional equation (4) for $z \in \mathbf{C}$. This function has acquired popularity only recently, due to advances in computational mathematics and its implementation in the mathematical library of the computer algebra program Maple (Char et al., 1991). Actually, Lambert W function received its name during the implementation of this function in Maple.

The Lambert function $W(z)$ is multivalued and as such it has many branches (Corless et al., 1996; Jeffrey, Hare, & Corless, 1996). The different possible branches are denoted by $W_k(z)$ for any $k=0, \pm 1, \pm 2$, etc. The ranges of the branches $W_k(z)$ are shown in Fig. 1. According to the implementation in Maple (Corless et al., 1996), the curve which separates the principal branch $W_0(z)$ from the branches $W_1(z)$ and $W_{-1}(z)$ is

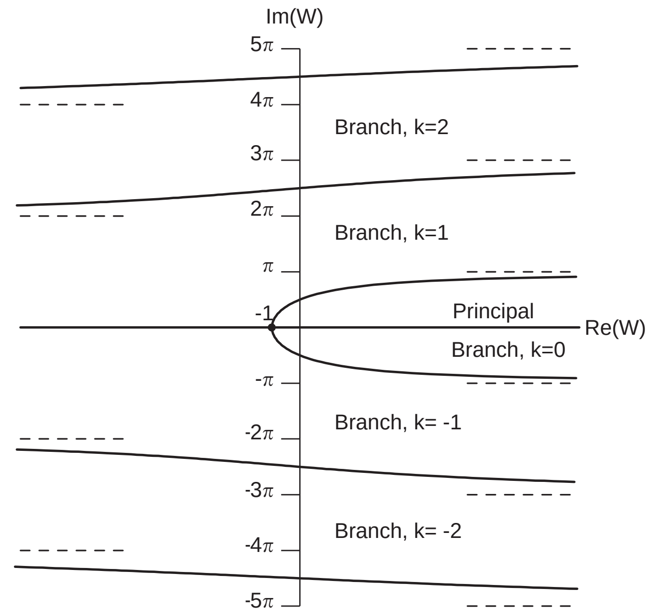

Fig. 1. Ranges of the various branches of Lambert W functions.

$$\{-\eta \cot \eta + \mathbf{i}\eta : -\pi < \eta < \pi\}, \quad \mathbf{i}=\sqrt{-1}. \tag{5}$$

The curve separating $W_1(z)$ and $W_{-1}(z)$ is simply $(-\infty, -1]$. Finally, the curves separating the remaining branches are

$$\{-\eta \cot \eta + \mathbf{i}\eta : 2k\pi < \pm\eta < (2k+1)\pi\}, \quad k=1, 2, \ldots. \tag{6}$$

Since the images of boundary curves under the mapping $z=we^{w}$ with $w=W(z)$ are the branch cuts in the z-plane, each boundary curve in Fig. 1 belongs to the region below it.

The branch $W_0(z)$ is called the principal branch of W. It contains the real line $[-1, +\infty)$ in its range and has a double branch point at $z=-e^{-1}$ corresponding to $w=-1$, which it shares with both $W_1(z)$ and $W_{-1}(z)$. It is noted that $W_0(z)$ and $W_{-1}(z)$ are real for real $z \in [-e^{-1}, \infty)$ and $z \in [-e^{-1}, 0)$, respectively. Thus, $W_0(z)$ and $W_{-1}(z)$ are the only branches of W that take on real values.

3. Integer-order time-delay systems

We let in Fig. 2, $G_c(s)=K_p$ and

$$G_p(s)=\frac{e^{-\tau s}}{(s+\alpha)^n}, \tag{7}$$

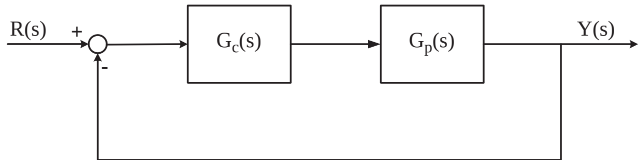

Fig. 2. A typical feedback control system.

where n is a positive integer and τ is the delay time. The closed-loop TCE of this feedback system is given by

$$(s+\alpha)^n \mathrm{e}^{\tau s} = -K_p. \tag{8}$$

For $n=1$, Asl and Ulsoy (2003) obtained roots s_k of the above characteristic equation as follows:

$$s_k = \frac{1}{\tau} W_k(-K_p \tau \mathrm{e}^{\tau\alpha}) - \alpha, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{9}$$

Chen and Moore (2002a) applied the above approach to evaluate the stability bound on the negative controller gain K_p for $n=2$. With $n=2$ and K_p being replaced by $-K_p$, they first took the square root for both sides of (8) to obtain

$$(s+\alpha)\mathrm{e}^{(\tau/2)s} = \pm\sqrt{K_p}, \tag{10}$$

then they represented the roots of the above-derived TCE in terms of the Lambert W function as

$$s = \frac{2}{\tau} W\left(\frac{\tau}{2}\mathrm{e}^{(\tau/2)\alpha}\left(\pm\sqrt{K_p}\right)\right) - \alpha \tag{11}$$

and finally they concluded that the stability condition for all possible τ, α and K_p is

$$\frac{2}{\tau} W\left(\frac{\tau}{2}\mathrm{e}^{(\tau/2)\alpha}\left(\pm\sqrt{K_p}\right)\right) - \alpha \leqslant 0. \tag{12}$$

Moreover, they remarked that the stability bound of the closed-loop system with a positive integer order n is given by

$$\frac{n}{\tau} W\left(\frac{\tau}{n}\mathrm{e}^{(\tau/n)\alpha}\sqrt[n]{K_p}\right) - \alpha \leqslant 0. \tag{13}$$

Before making comments on the result of Chen and Moore (2002a), we first consider the Lambert W function solutions to (8) for an arbitrary positive integer n. Taking the nth root for both sides of (8) gives the following n derived TCEs:

$$(s_l+\alpha)\mathrm{e}^{(\tau/n)s_l} = \begin{cases} \sqrt[n]{|K_p|}\exp\left(\mathbf{i}\,\frac{(2l+1)\pi}{n}\right) \\ \quad \text{if } K_p \geqslant 0, \\ \sqrt[n]{|K_p|}\exp\left(\mathbf{i}\,\frac{2l\pi}{n}\right) \\ \quad \text{if } K_p < 0. \end{cases} \tag{14}$$

The roots of these TCEs can be written as

$$s_{l,k} = \begin{cases} \frac{n}{\tau} W_k\left(\frac{\tau}{n}\mathrm{e}^{(\tau/n)\alpha}\sqrt[n]{|K_p|}\exp\left(\mathbf{i}\,\frac{(2l+1)\pi}{n}\right)\right) - \alpha \\ \quad \text{if } K_p \geqslant 0, \\ \frac{n}{\tau} W_k\left(\frac{\tau}{n}\mathrm{e}^{(\tau/n)\alpha}\sqrt[n]{|K_p|}\exp\left(\mathbf{i}\,\frac{2l\pi}{n}\right)\right) - \alpha \\ \quad \text{if } K_p < 0, \end{cases} \tag{15}$$

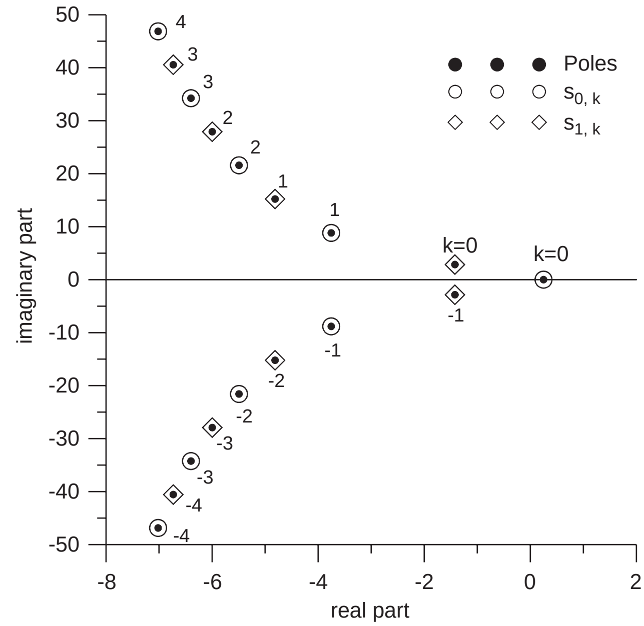

Fig. 3. Root distribution of the derived TCEs for $(n, K_p, \alpha, \tau) = (2, -2, 1, 1)$.

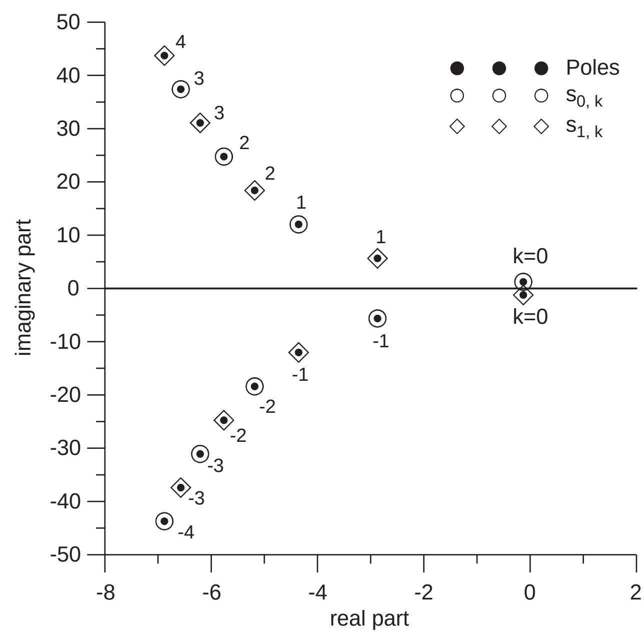

Fig. 4. Root distribution of the derived TCEs for $(n, K_p, \alpha, \tau)=(2, 2, 1, 1)$.

where $s_{l,k}$ denotes the kth $(k = 0, \pm 1, \pm 2, \ldots)$ root of the lth derived TCE in (14).

Invoking the Lambert W function LambertW(k, z) of Maple to compute the value $W_k(z)$, we plot in Figs. 3 and 4 the root distributions of $s_{l,k}$ for $(n, K_p) = (2, -2)$ and $(2, 2)$, respectively, where the parameters (α, τ) are set to be $(1, 1)$. It is noted that in these figures the roots marked with a solid circle satisfy the original TCE (8) and they are the poles of the closed-loop system. From Figs. 3 to 4, we have the following observations: (i) all the roots $s_{l,k}$ of the derived TCEs in (14) are the poles of the closed-loop system; (ii) the rightmost root(s) of the system can be real root or complex conjugate and the principal branch solution $s_{0,0}$ of the derived TCE in (14) with $l=0$ is a rightmost root; (iii) the roots of the n derived TCE locate in an interlaced manner in either upper or lower root branch.

Now, we make some comments on the use of (10)–(13) for estimating stability bounds. First, according to Eqs. (12) and (13) and Fig. 1 shown in Chen and Moore (2002a), it seems that they have made the following two assumptions: (i) the arguments of the W functions in (12) and (13) are real; (ii) the principal branch Lambert W function of the first derived TCE, i.e., the equation with $l=0$ in (14), also ensures real value. The disadvantage of assumption (i) is that it restricts the gain K_p to be negative because, according to (15), a positive K_p leads to a complex argument for the W function. Negative gain is seldom used in a practical feedback control system since a positive feedback often tends to destabilize the system. Next, we note from Fig. 4 that the principal branch solutions $s_{0,0}$ and $s_{1,0}$ are complex conjugate. Moreover, the value of the principal branch W function is complex if its argument is real and less than $-\mathrm{e}^{-1}$. Hence, inequality (13), which is a consequence of assumption (ii), becomes sometimes meaningless when the principal branch solution $s_{0,0}$ is not a real value. Based on computed results, we can deduce that the generally correct equation for describing the boundary of the stability region in the parameter space should read as

$$\Re\{s_{0,0}\} = 0, \tag{16}$$

where $\Re\{\cdot\}$ denotes the real part of the indicated quantity. Finally, it should be noted that in Chen and Moore (2002a) the stability region in $\tau - K_p$ plane is constructed through first making grids on the parameter plane, then evaluating the rightmost root $s_{0,0}$ with the Lambert W library function of Maple for each grid point, and finally obtaining the stable region by identifying the area of those grid points at each of which $s_{0,0}$ has a negative real part. Obviously, this brute-force approach to constructing stability region is by no means efficient and it cannot produce an analytic stability bound.

Finally, we note that the stability boundary consists of those points in the parameter space at which the principal branch Lambert function W_0 in (16) has a real part α. In a two-dimensional parameter plane, say, $K_p - \tau$ plane, (16) defines one-dimensional manifolds or curves. Using a path-following algorithm, we have constructed in Fig. 5 the stability domain in the $K_p - \tau$ parameter plane for the system with $n=2$ and $\alpha=1$. It is seen from this figure that the stability domain covers both positive and negative regions of K_p. However, the stability domain shown in Fig. 1 of Chen and Moore (2002a) only lies in negative K_p half plane, which

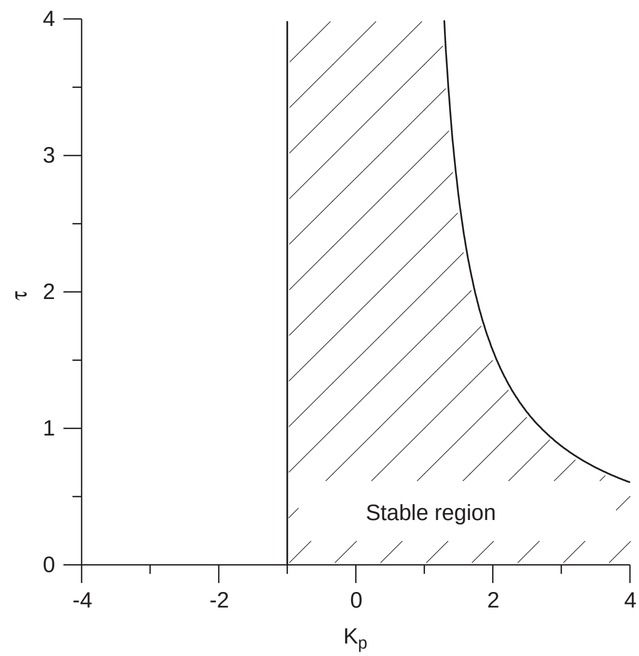

Fig. 5. Stable region in the $K_p - \tau$ plane for the system with $n=2$ and $\alpha=1$.

suggests a positive feedback has to be used in the control system of Fig. 2.

4. Fractional-order time-delay systems

Many real-world physical systems are well characterized by fractional-order differential equations (Ortigueira, 2000), i.e., equations involving noninteger-order derivatives. Moreover, fractional-order controllers (Podlubny, 1999; Raynaud & Zergainoh, 2000), have been designed and applied to control a variety of dynamical processes of noninteger orders. The latter development has motivated the study of stability analysis for fractional-order control systems with or without time delays (Bonnet & Partington, 2002). In this line of research, Chen and Moore (2002b) have recently applied the Lambert W function to evaluate the stability bound for a class of fractional-order systems. Here, we shall revisit the problem of stability analysis for fractional-order systems using the Lambert W function and present our findings.

Let $G_p(s)$ in Fig. 2 be a fractional-order plant

$$G_p(s) = \frac{\mathrm{e}^{-\tau s}}{(s+\alpha)^{n/m}}. \tag{17}$$

It is readily shown that the closed-loop system has the TCE in (1), which can be cast as

$$(s+\alpha)^{n/m}\mathrm{e}^{\tau s} = -K_p = \begin{cases} |K_p|\mathrm{e}^{\mathbf{i}\pi}, & K_p \geqslant 0, \\ |K_p|, & K_p < 0. \end{cases} \tag{18}$$

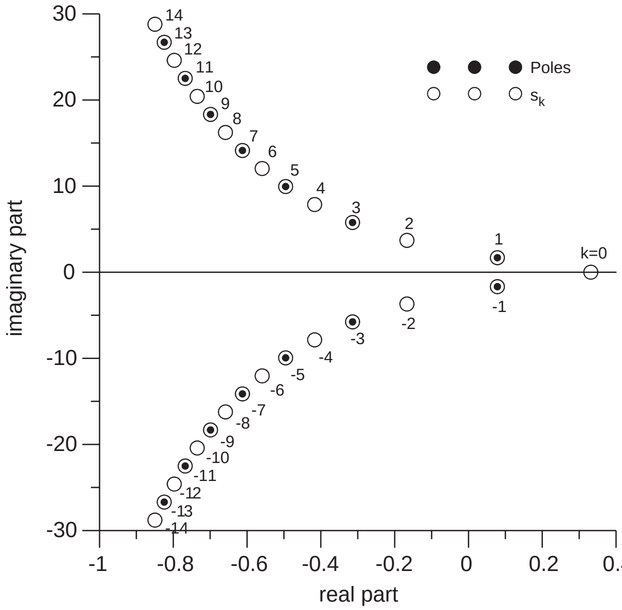

Fig. 6. Root distribution of the derived TCE for $(n/m, K_p, \alpha, \tau) = (\frac{1}{2}, 1.5, 0.5, 1.5)$.

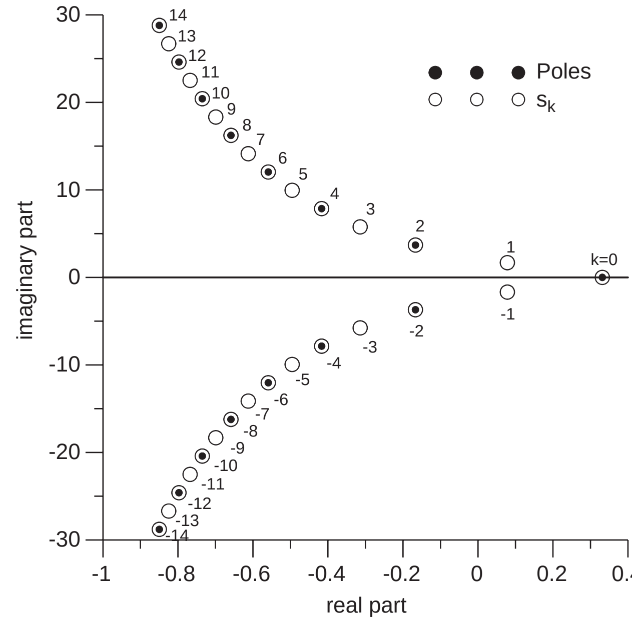

Fig. 7. Root distribution of the derived TCE for $(n/m, K_p, \alpha, \tau) = (\frac{1}{2}, -1.5, 0.5, 1.5)$.

Taking first the mth power and then the nth root on both sides of the above equation, we have

$$(s_l+\alpha)\mathrm{e}^{(m\tau/n)s_l} = \begin{cases} \sqrt[n]{|K_p|^m}\exp\left(\mathbf{i}\,\frac{(2l+m)\pi}{n}\right) \\ \quad \text{if } K_p \geqslant 0, \\ \sqrt[n]{|K_p|^m}\exp\left(\mathbf{i}\,\frac{2l\pi}{n}\right) \\ \quad \text{if } K_p < 0, \end{cases} \tag{19}$$

where $l=0, 1, \ldots, n-1$. According to (2) and (3), the roots of the derived TCEs in (19) are given by

$$s_{l,k} = \begin{cases} \frac{n}{m\tau}\,W_k\left(\frac{m\tau}{n}\,\mathrm{e}^{(m\tau/n)\alpha}\sqrt[n]{|K_p|^m}\exp\left(\mathbf{i}\,\frac{(2l+m)\pi}{n}\right)\right) - \alpha \\ \quad \text{if } K_p \geqslant 0, \\ \frac{n}{m\tau}\,W_k\left(\frac{m\tau}{n}\,\mathrm{e}^{(m\tau/n)\alpha}\sqrt[n]{|K_p|^m}\exp\left(\mathbf{i}\,\frac{2l\pi}{n}\right)\right) - \alpha \\ \quad \text{if } K_p < 0. \end{cases} \tag{20}$$

To have an insight into the relationship between the roots of the derived TCEs and the poles of the system, we actually compute the roots $s_{l,k}$ for systems of orders $n/m = \frac{1}{2}$ and $\frac{1}{3}$ using the Lambert W function implemented in Maple. First, consider the system of order $n/m=\frac{1}{2}$. We show in Figs. 6 and 7, where $\alpha=0.5$, $\tau=1.5$, the root distributions for $K_p=1.5$ and -1.5, respectively. For $n=1$, there is only one derived TCE and its roots are denoted by s_k in these two figures. The roots s_k that satisfy the original TCE (1) are marked with a solid circle. It is observed from Figs. 6 and 7 that the roots $s_{\pm 1}, s_{\pm 3}, \ldots$, for $K_p=-1.5$ and the roots $s_0, s_{\pm 2}, s_{\pm 4}, \ldots$, for $K_p=1.5$ do not satisfy the original TCE. Also observed is that the true roots and superfluous roots interlace. These observations imply that taking power of 2 on both sides of the original TCE (1) gives rise to a derived TCE which introduces superfluous roots to the original one. To verify this implication, we then consider the system of order $\frac{1}{3}$. In this case, the derived TCE is obtained by taking the power of 3 on both sides of the original TCE (1). Figs. 8 and 9 show the root distributions for $K_p=1$ and -1, respectively, whereas the parameters (α, τ) are set as $(0.5, 1)$. It is seen from these two figures that on each root branch, there is only one pole of the system among every three adjacent roots of the derived TCEs.

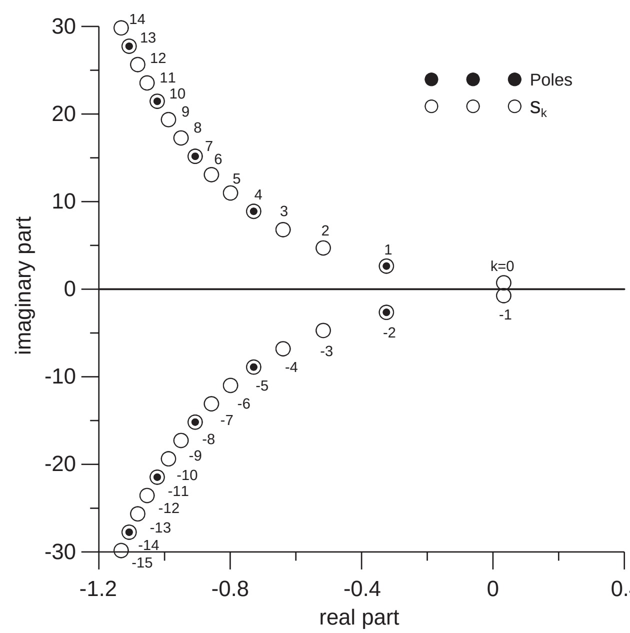

Fig. 8. Root distribution of the derived TCE for $(n/m, K_p, \alpha, \tau) = (\frac{1}{3}, 1, 0.5, 1)$.

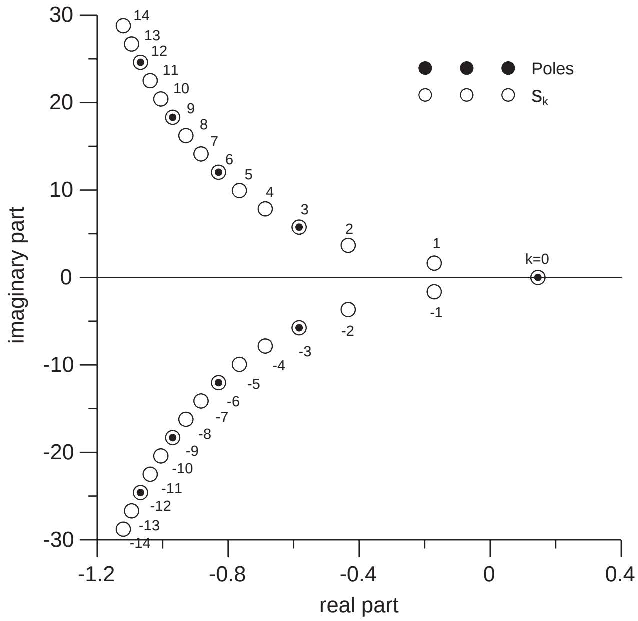

Fig. 9. Root distribution of the derived TCE for $(n/m, K_p, \alpha, \tau) = (\frac{1}{3}, -1, 0.5, 1)$.

With the observations on root distributions shown in Figs. 6–9, we may reasonably infer that for a TCE derived from the original TCE by taking an integer power of $m > 1$, there is only one true root for the original TCE in the m-element root set $\{s_{l,k\pm j}\}_{j=0}^{m-1}$, $k \geqslant 0$. Without recognizing the generation of false poles using the Lambert W function representation, direct use of the roots (20) will result in an incorrect result. Moreover, as it can be seen from Figs. 6 and 8, the rightmost root(s) of the derived TCEs are not necessarily the roots of the original TCE. Hence, the stability bound for the fractional delay system cannot be evaluated directly using (16). To be safe, the stability boundary in parameters space, can be evaluated with the formula:

$$\max_{\substack{l=0,n-1,k=0,m-1\\ \Delta(s_{l,k})=0}} \Re\{s_{l,k}\} = 0, \tag{21}$$

where $\Delta(s) = 0$ denotes the original TCE (1).

Finally, it is noted that Chen and Moore (2002b) have applied the Lambert W function to evaluate the stability bound for TCE (1) with $n/m = r$ and K_p replaced by $-K_p$. They arrived at the stability condition

$$\frac{r}{\tau} W\left(\frac{\tau}{r} \mathrm{e}^{(\tau/r)\alpha}(K_p)^{1/r}\right) - \alpha \leqslant 0. \tag{22}$$

Due to the facts that the same assumptions as those stated for integer-order delayed systems were made, and that the unawareness of superfluous roots generated from derived TCEs, their results are by no means generally correct for evaluating stability bounds for fractional-order systems. Indeed, the stability bound shown in Fig. 2 of Chen and Moore (2002b) for $\alpha = 0.5$ and $r = \frac{1}{3}$ is incomplete since the surface corresponding to the Lambert W function solutions

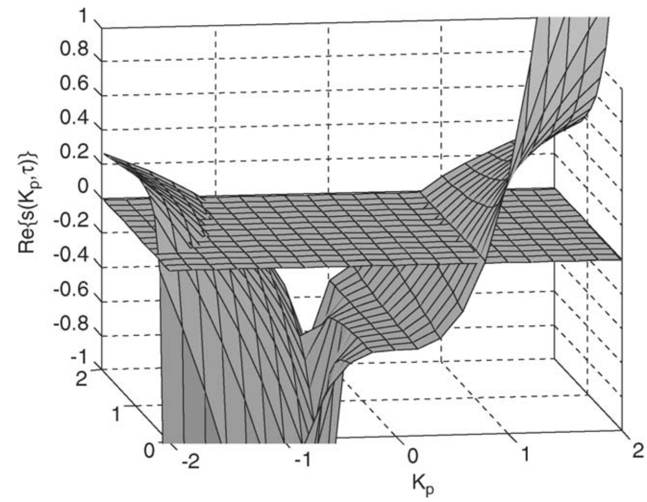

Fig. 10. The stability bound for system with $n/m = \frac{1}{3}$ and $\alpha = 0.5$.

with nonzero imaginary parts is missed. We present the correct one in Fig. 10, in which the vertical coordinate $\Re\{s(k_p, \tau)\}$ represents the real part of the rightmost root of the original TCE. Note that the missing surface in Fig. 2 of Chen and Moore (2002b) might be due to an inappropriate use of Maple's plot function.[1] However, even if the Maple's plot function is invoked correctly, the use of principal branch W function solutions as the rightmost roots of the original TCE also gives rise to an erroneous plot since the missing solution surface locates in an area of the $\tau - K_p$ plane within which the rightmost root of the original is the W_1 branch rather than the principal branch W_0.

5. Conclusions

In this note, the application of Lambert W function to the stability analysis of time-delay systems is re-examined. Through actually computing the root distributions for both integer- and fractional-order delayed control systems of some specified orders using Lambert W function solutions of derived transcendental characteristic equations, we have pointed out some pitfalls in the use of the Lambert W function to analyze time-delay systems. In particular, we have the following key observations obtained from the case study of fractional-order systems: (i) the operation of taking an integer power on a TCE gives rise to a derived TCE which introduces superfluous roots to the original TCE; (ii) the rightmost root of the derived TCEs is not necessarily the principal branch Lambert W function solution of a derived TCE. With these observations, we have clarified the correct use of Lambert W function to stability analysis of a class of time-delay systems. Moreover, some unclear points or deficiencies displayed in the literature are pointed out. The

[1] It is a trick of Maple's plot function that, when assigning a complex value to a coordinate, the corresponding point will not be displayed.

critical points clarified in the paper will actually enlarge the application scope of the Lambert W function, which is becoming a standard library function for various commercial symbolic software packages, to time-delay systems.

Acknowledgements

This work was supported by the National Science Councils of the Republic of China under Grants NSC93-2214-E-214-006 and NSC93-2214-E-214-007.

References

Asl, F. M., & Ulsoy, A. G. (2003). Analysis of a system of linear differential equations. *J. Dynamic Systems, Measurement, and Control, Transactions of the ASME*, *125*(3), 215–223.

Bonnet, C., & Partington, J. R. (2002). Analysis of fractional delay systems of retarded and neutral type. *Automatica*, *38*(7), 1133–1138.

Char, B. W., Geddes, K. O., Gonnet, G. H., Leong, M. B., Monagan, M. B., & Watt, S. M. (1991). *The Maple V language reference manual*. Berlin: Springer.

Chen, Y. Q., & Moore, K. L. (2002a). Analytical stability bound for delayed second-order systems with repeating poles using Lambert W function. *Automatica*, *38*(5), 891–895.

Chen, Y. Q., & Moore, K. L. (2002b). Analytical stability bound for a class of delayed fractional-order dynamic systems. *Nonlinear Dynamics*, *29*(1–4), 191–200.

Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J., & Knuth, D. E. (1996). On the Lambert W function. *Advances in Computational Mathematics*, *5*(4), 329–359.

Hwang, C., & Hwang, J. H. (2004). Stabilization of first-order plus dead-time unstable processes using PID controllers. *IEE Proceedings—Control Theory and Applications*, *151*(1), 89–94.

Jeffrey, D. J., Hare, D. E. G., & Corless, R. M. (1996). Unwinding the branches of the Lambert W function. *The Mathematical Scientist*, *21*(1), 1–7.

Kuang, Y. (1993). *Delay differential equations with applications in population dynamics*. Boston, MA, USA: Academic Press.

Mossaheb, S. (1980). A Nyquist type stability criterion for linear multivariable delayed systems. *International Journal of Control*, *32*(5), 821–847.

Neimark, Ju. I. (1973). *D*-decomposition of the space of quasi-polynomials (on the stability of linearized distributive systems). *American mathematical society translations*, *Series 2* (Vol. 102, pp. 95–131). Providence, RI: American Mathematical Society.

Ortigueira, M. D. (2000). Introduction to fractional linear systems. Part 1. Continuous-time case. *IEE Proceedings—Vision, Image and Signal Processing*, *147*(1), 62–70.

Podlubny, I. (1999). Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers. *IEEE Transactions on Automatic Control*, *44*(1), 208–214.

Raynaud, H. F., & Zergainoh, A. (2000). State-space representation for fractional order controllers. *Automatica*, *36*(7), 1017–1021.

Weiss, G., Zhong, Q.-C., & Green, T. C. (2004). H^{∞} repetitive control of DC-AC converters in microgrids. *IEEE Transactions on Power Electronics*, *19*(1), 219–230.

Chyi Hwang was born in Jinmen Island, Fukien Province, China, in May 18, 1952. He received the B.S., M.S., and Ph.D. degrees in chemical engineering from National Cheng Kung University, Taiwan, in 1976, 1978, and 1981, respectively.
From December 1981 to July 1983, he served in the Department of Chemical Engineering, National Cheng Kung University, where he was promoted to the professor rank in 1987. He founded the Department of Chemical Engineering of National Chung Cheng University in August 1993, and served as the Department Head from August 1993 to July 1999. Dr. Hwang was elected chair professor of National Chung Cheng University in January 2002. In February 2004, he joined the Department of Chemical Engineering, I-Shou University, and was appointed a chair professor of the University.
Dr. Hwang has great interests in linear systems theory, robust control, real-time control systems, nonlinear systems dynamics, and optimization. He has published about two hundred technical papers.
Dr. Hwang is a member of the Chinese Institute of Engineers (CIE). He received the Outstanding Young Engineer Award in 1987 from the CIE. In 1985–1995, he received the Outstanding Research Award from the National Science Council (NSC) of the Republic of China. In 1995–2001, he was appointed as a Research Specialist by the NSC.

Yi-Cheng Cheng was born in 1976. He received the B.S. and M.S. degrees from Feng Chia University, Taiwan, in 1999 and 2001, respectively. He is now pursuing the Ph.D. degree in the Department of Chemical Engineering, National Chung Cheng University, Taiwan. His current research focuses include fractional-order systems, time-delay systems, and controller design.