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A common Lyapunov function for a differential inclusion with matrices having small commutators

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ABSTRACT

Let A_1 and A_2 be asymptotically stable matrices. We consider switched continuous time linear systems described by the differential inclusion $\dot{x}(t) = \{y(t) : y(t) = Ax(t), A \in \{A_1, A_2\}\}$. The paper demonstrates that a common quadratic Lyapunov function exists for such systems, provided the commutator $[A_1, A_2] = A_1A_2 - A_2A_1$ has a sufficiently small norm. Our results generalize the ones of Narendra and Balakrishnan (1994).

1. Introduction and statement of the main result

Narendra and Balakrishnan have shown in Narendra and Balakrishnan (1994) that if one has a finite family of exponentially stable linear systems whose matrices commute, then the corresponding switched linear system is exponentially stable for arbitrary switching. Moreover, they gave an explicit expression for a common Lyapunov function for the individual linear considered switched system. The motivation for considering this problem arises from various areas of mathematics and control theory. The results presented in Narendra and Balakrishnan (1994) aroused great interest among specialists and found applications to solve various problems, cf. Linh and Liberzon (2005), Michaletzky and Gerencsér (2002), Mori, Mori, and Kokame (2001), Mori, Nguyen, Mori, and Kokame (2006), Shorten and Narendra (2003) and Guisheng and Xu (2011) and references given therein. In particular, in the paper (Shorten & Narendra, 2003) the authors derive the necessary and sufficient conditions for the existence of the common quadratic Lyapunov functions for continuous-time linear time-invariant switched systems whose system matrices are in companion form. In addition, in the paper (Mori, Mori, & Kuroe, 1997) the sufficient existence conditions are established, provided that there exists a non-singular matrix U such that every $U^{-1}A_iU$ is upper triangular for each system matrix A_i. Below we do not assume that the system matrices have the companion form or simultaneously triangularizable. For the background material on the continuous-time and discrete-time switched systems we refer the reader to Gao, Liberzon, Liu, and Basar (2018), Jan and Francoise (2009), Pandey and de Oliveira (2017), Sun and Ge (2005) and Sun and Ge (2011).

Below $\mathbf{C}^n$ is the complex n-dimensional Euclidean space with a scalar product $\langle ., . \rangle$, the Euclidean norm $\|.\| = \sqrt{\langle ., . \rangle}$ and unit matrix I, $\mathbf{C}^{n\times n}$ is the set of all $n \times n$ matrices. For an $A \in \mathbf{C}^{n\times n}$, A^* is the adjoint matrix; $\lambda_k(A)$, $k = 1, \dots, n$, are the eigenvalues of $A \in \mathbf{C}^{n\times n}$, counted with their multiplicities; $\alpha(A) := \max_k \Re\lambda_k(A)$; $\|A\| = \sup_{h\in\mathbf{C}^n} \|Ah\|/\|h\|$ is the spectral norm (the operator norm with respect to the Euclidean vector norm), $\nu(A) := \min_k \lambda_k(A + A^*)/2$ (the smallest eigenvalue of the matrix $(A^* + A)/2$). In addition, put

$$\gamma(A) = \int_0^\infty \|e^{At}\|^2 dt \quad (A \in \mathbf{C}^{n\times n};\ \alpha(A) < 0).$$

Furthermore, let $\mathcal{A} = \{A_1, A_2\}$ be the set of two Hurwitz $n \times n$-matrices A_1 and A_2: $\alpha(A_1) < 0, \alpha(A_2) < 0$. Consider the differential inclusion

$$\dot{x}(t) \in \{Ax(t), A \in \mathcal{A}\} \quad (t \ge 0;\ \dot{x} = dx/dt). \tag{1.1}$$

If a function $V(.)$ is a Lyapunov one for both the individual systems $\dot{x}(t) = A_jx, j = 1, 2$, then it is said to be a common Lyapunov function for (1.1).

In the present paper we show that (1.1) has a common Lyapunov function, provided the commutator $[A_1, A_2] = A_1A_2 - A_2A_1$ has a sufficiently small norm. Besides, we generalize Theorem 1 from Narendra and Balakrishnan (1994). Recall that the existence of a common Lyapunov function implies the exponential stability of (1.1), cf. Jan and Francoise (2009, Theorem 4.5).

Now we are in a position to formulate our main result.

Theorem 1. *Consider switched system (1.1) with $\alpha(A_k) < 0$ $(k = 1, 2)$ and put*

$$P_1 := \int_0^\infty e^{A_1^*t} e^{A_1t} dt \text{ and } P_2 = \int_0^\infty e^{A_2^*t} P_1 e^{A_2t} dt.$$

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Assume that

$$4\|[A_2, A_1]\||\nu(A_2)|\gamma(A_1)\gamma^2(A_2) < 1. \quad (1.2)$$

Then the function $V(x) := x^* P_2 x$ $(x \in \mathbf{C}^n)$ *is a common Lyapunov function for system* (1.1).

The proof of this theorem is presented in the next section. Theorem 1 is sharp: if $[A_2, A_1] = 0$, then (1.2) is automatically holds and we obtain the result of Narendra and Balakrishnan.

If A is a normal matrix: $AA^* = A^*A$, then as is well-known $\|e^{At}\| = e^{\alpha(A)t}$, for instance see Gil (2018, Theorem 3.5) and references given therein. So

$$\gamma(A) \le \frac{1}{2|\alpha(A)|}. \quad (1.3)$$

Below we suggest estimates for $\gamma(A)$ of a non-normal matrix A.

2. Proof of Theorem 1

As is well known, P_1 and P_2 are solutions of the Lyapunov equations

$$A_1^* P_1 + P_1 A_1 = -I \quad (2.1)$$

and

$$A_2^* P_2 + P_2 A_2 = -P_1, \quad (2.2)$$

cf. Vidyasagar (1993). Put

$$M_2 := \int_0^\infty e^{A_2^* t} e^{A_2 t} dt$$

and

$$K_0 := \int_0^\infty \int_0^\infty e^{A_2^*(t+s)} P_1 e^{sA_2} [A_2, A_1] e^{tA_2} ds\, dt.$$

For Hermitian matrices S and S_1 we write $S < 0$, if S is negative definite, i.e. $\langle Sx, x\rangle \le -b_0 \langle x, x\rangle$ $(x \in \mathbf{C}^n)$ for a constant $b_0 > 0$, and $S < S_1$ means that $S - S_1 < 0$.

Lemma 2. *Let*

$$K_0 + K_0^* < M_2. \quad (2.3)$$

Then

$$A_1^* P_2 + P_2 A_1 < 0, \quad (2.4)$$

and therefore (according to (2.2)*)* $x^* P_2 x$ *is a Lyapunov function of* (1.1).

Proof. Note that $e^{A_2 t} A_1 = A_1 e^{A_2 t} + [e^{A_2 t}, A_1]$, where $[e^{A_2 t}, A_1] := e^{A_2 t} A_1 - A_1 e^{A_2 t}$. Similarly,

$$A_1^* e^{A_2^* t} = e^{A_2^* t} A_1^* + [A_1^*, e^{A_2^* t}] = e^{A_2^* t} A_1^* + [e^{tA_2}, A_1]^*.$$

We thus have

$$\begin{aligned} A_1^* P_2 + P_2 A_1 &= \int_0^\infty A_1^* e^{A_2^* t} P_1 e^{A_2 t} dt + \int_0^\infty e^{A_2^* t} P_1 e^{A_2 t} A_1 dt \\ &= \int_0^\infty e^{A_2^* t} A_1^* P_1 e^{A_2 t} dt + \int_0^\infty e^{A_2^* t} P_1 A_1 dt e^{A_2 t} + K + K^* \\ &= \int_0^\infty e^{A_2^* t} (A_1^* P_1 + P_1 A_1) e^{A_2 t} dt + K + K^* \\ &= -\int_0^\infty e^{A_2^* t} e^{A_2 t} dt + K + K^* = -M_2 + K + K^*, \end{aligned}$$

where

$$K = \int_0^\infty e^{A_2^* t} P_1 [e^{A_2 t}, A_1] dt$$

and thus

$$K^* = \int_0^\infty [A_1^*, e^{A_2^* t}] P_1 e^{A_2 t} dt.$$

So

$$A_1^* P_2 + P_2 A_1 = -M_2 + K + K^*. \quad (2.5)$$

But for arbitrary $n \times n$ matrices A_1 and A_2, due to Gil (2018, Lemma 13.3),

$$[e^{A_2 t}, A_1] = \int_0^t e^{sA_2} [A_2, A_1] e^{(t-s)A_2} ds \ \ (t \ge 0). \quad (2.6)$$

So

$$K = \int_0^\infty e^{A_2^* t} P_1 \int_0^t e^{sA_2} [A_2, A_1] e^{(t-s)A_2} ds\, dt,$$

and consequently

$$\begin{aligned} K &= \int_0^\infty \int_s^\infty e^{A_2^* t} P_1 e^{sA_2} [A_2, A_1] e^{(t-s)A_2}\, dt\, ds \\ &= \int_0^\infty \int_0^\infty e^{A_2^*(t+s)} P_1 e^{sA_2} [A_2, A_1] e^{tA_2} ds\ \ dt = K_0. \end{aligned}$$

Therefore, by (2.5) $A_1^* P_2 + P_2 A_1 = -M_2 + K_0 + K_0^*$. If (2.3) holds, then (2.4) is valid, as claimed. □

Proof of Theorem 1. Put $y(t) = e^{A_2 t} x$ $(x \in \mathbf{C}^n, t \ge 0)$. Then

$$\begin{aligned} \frac{d}{dt}\langle y(t), y(t)\rangle &= \langle \dot{y}(t), y(t)\rangle + \langle y(t), \dot{y}(t)\rangle \\ &= \langle (A_2 + A_2^*) y(t), y(t)\rangle \ge 2\nu(A_2)\langle y(t), y(t)\rangle. \end{aligned}$$

Solving this inequality, we have

$$\|e^{A_2 t} x\| \ge \|x\| e^{\nu(A_2)t} \ \ (x \in \mathbf{C}^n, t \ge 0).$$

Besides, $\nu(A_2) < 0$, since A_2 is a Hurwitz matrix. Consequently,

$$\begin{aligned} \langle M_2 x, x\rangle &= \int_0^\infty \langle e^{A_2 t} x, e^{A_2 t} x\rangle dt \ge \int_0^\infty e^{2\nu(A_2)t} dt \\ &= \frac{1}{2|\nu(A_2)|}\langle x, x\rangle. \end{aligned} \quad (2.7)$$

In addition,

$$\|K_0\| \le$$

$$\begin{aligned} &\|P_1\| \|[A_2, A_1]\| \int_0^\infty \int_0^\infty \|e^{A_2 s}\| \|e^{A_2 t}\| \|e^{sA_2}\| \|e^{tA_2}\| ds\ \ dt \\ &= \|[A_2, A_1]\| \|P_1\| (\int_0^\infty \|e^{sA_2}\|^2 ds)^2. \end{aligned}$$

Obviously $\|P_1\| \le \gamma(A_1)$. Thus,

$$\|K_0\| \le \|[A_2, A_1]\| \gamma(A_1)\gamma^2(A_2).$$

Furthermore, inequality (1.2) implies

$$4\|K_0\| \le 4\|[A_2, A_1]\| \gamma(A_1)\gamma^2(A_2) < \frac{1}{\nu(A_2)}.$$

Thus according to (2.7),

$$M_2 \ge \frac{1}{2\nu(A_2)} I > 2\|K_0\| I \ge K_0 + K_0^*.$$

This and Lemma 2 prove the theorem. □

3. An estimate for $\gamma(A)$ and example

Let $|A|_F = (\text{trace}\ \ AA^*)^{1/2}$ be the Frobenius (Hilbert–Schmidt) norm of A. Introduce the quantity (the departure from normality of A)

$$g(A) = [|A|_F^2 - \sum_{k=1}^{n} |\lambda_k(A)|^2 \ ]^{1/2}.$$

The following relations are checked in Gil (2018, Section 3.1):

$$g^2(A) \le |A|_F^2 - |\text{trace } A^2| \text{ and } g^2(A) \le \frac{|A - A^*|_F^2}{2}.$$

As it is proved in Gil (2018, Example 3.2), for any $n \times n$ matrix A, the following inequality

$$\|e^{At}\| \le e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}} \ \ (t \ge 0)$$

is valid. Denote

$$\gamma_0(A) := \sum_{j,k=0}^{n-1} \frac{(k+j)! g^{k+j}(A)}{|2\alpha(A)|^{k+j+1}(k!j!)^{1/2}}$$

provided $\alpha(A) < 0$. Then we have

$$\gamma(A) = \int_0^\infty \|e^{At}\|^2 dt \le \int_0^\infty e^{2\alpha(A)t} \sum_{j,k=0}^{n-1} \frac{t^{k+j} g^{k+j}(A)}{(j!k!)^{3/2}} dt$$
$$= \gamma_0(A). \tag{3.1}$$

This inequality is sharp: if A is normal, then $g(A) = 0$ and with $0^0 = 1$ we obtain (1.3).

Now Theorem 1.1 implies.

Corollary 3. *The function $V(x) = x^* P_2 x$ ($x \in \mathbf{C}^n$) is a Lyapunov function of (1.1), provided*

$$4\|[A_2, A_1]\| |\nu(A_2)| \gamma_0(A_1)\gamma_0^2(A_2) < 1. \tag{3.2}$$

Example 4. Let

$$A_1 = \begin{pmatrix} -2 & 0.1 \\ 0.1 & -3 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} -2 & 0.1 \\ 0 & -3 \end{pmatrix}.$$

We have $\|[A_1, A_2]\| \le 1.12$, $A_1^* = A_1$, $g(A_1) = 0$,

$$g(A_2) = 0.1, \alpha(A_1) < -1.9, \alpha(A_2) = -2, |\nu(A_2)| < 3.05.$$

So

$$\gamma_0(A_1) < \frac{1}{2 \cdot 1.9} < 0.3, \ \gamma_0(A_2) = \frac{1}{4} + \frac{2 \cdot 0.1}{4^2}$$
$$+ \frac{2 \cdot 0.1^2}{4^3} < 0.3.$$

Since

$$4\|[A_2, A_1]| \nu(A_2)| \| \gamma_0(A_1)\gamma_0^2(A_2)$$
$$< 4 \cdot 3.05 \cdot 1.12 \cdot 0.3 \cdot (0.3)^2 < 1,$$

Corollary 3 implies that the considered system has a common quadratic Lyapunov function.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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