Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

Controllability and Observability of Systems of Linear Delay Differential Equations via the Matrix Lambert W Function

Sun Yi, Patrick W. Nelson, and A. Galip Ulsoy

Abstract— **Controllability and observability of linear time delayed systems have been studied, and various definitions and criteria have been presented since the 1960s. However, the lack of an analytical solution approach has limited the applicability of the existing theories. Recently, the solution to systems of linear delay differential equations has been derived using the matrix Lambert W function, in a form similar to the transition matrix in ordinary differential equations. The criteria for controllability and observability, and their Gramians, for systems of delay differential equations using the solution in terms of the matrix Lambert W function are presented for the first time and illustrated with examples.**

I. Introduction

Controllability and observability are two fundamental attributes of a dynamical system. Such properties of Time delayed systems (TDS) have been explored since the 1960s [22]. The Gramians for controllability and observability for TDS were presented by Weiss (1967, [25]) and Delfour *et al.* (1972, [9]) based upon assumed solution forms of the delay differential equations (DDEs). However, due to the lack of an analytical solution of DDEs, application of the Gramians to verify controllability and observability of linear time delay systems has not been possible, [17].

Recently, an analytic approach to obtain the complete solution of linear systems of DDEs, based on the concept of the Lambert W function, was developed for both scalar and matrix-vector cases when the coefficient matrices commute in Asl and Ulsoy [1]. The approach was extended to non-homogeneous DDEs and to general systems of DDEs in [27] (see Table I). Consider a linear system of DDEs with a single constant delay, h

$$\begin{array}{rcll} \dot{\mathbf{x}}(t) & = & \mathbf{A}\mathbf{x}(t)+\mathbf{A_d}\mathbf{x}(t-h)+\mathbf{B}\mathbf{u}(t) & t>0 \\ \mathbf{x}(t) & = & \mathbf{g}(t) & t\in[-h,0) \\ \mathbf{x}(t) & = & \mathbf{x}_0 & t=0 \end{array} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{A_d}$ are $n\times n$ coefficient matrices, and $\mathbf{x}(t)$ is an $n\times 1$ state vector, $\mathbf{B}$ is an $n\times r$ matrix, $\mathbf{u}(t)$, an $r\times 1$ vector, is a function representing the external excitation, and $\mathbf{g}(t)$ and $\mathbf{x}_0$ are a specified preshape function and an initial state respectively. The solution to (1) in terms of the matrix Lambert W function is [27]

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I+\int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^N\mathbf{B}\mathbf{u}(\xi)d\xi \tag{2}$$

where

$$\mathbf{S}_k=\frac{1}{h}\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)+\mathbf{A} \tag{3}$$

The coefficient $\mathbf{C}_k^I$ in (2) is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and the preshape function $\mathbf{g}(t)$ and the initial condition $\mathbf{x}_0$, while $\mathbf{C}_k^N$ is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and does not depend on $\mathbf{g}(t)$ or $\mathbf{x}_0$. The methods for computing $\mathbf{C}_k^I$ and $\mathbf{C}_k^N$ were developed in [1], [28]. The matrix $\mathbf{Q}_k$ in (2) can be obtained from the following condition,

$$\mathbf{W}_k(\mathbf{A}_d h\mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)+\mathbf{A}h}=\mathbf{A_d}h \tag{4}$$

Note that $\mathbf{W}_k$ in (3) and (4) denotes the matrix Lambert W function as detailed in [1], [8]. From the Laplace transform of the system (1), the solution to (1) in the Laplace domain is obtained in [28]. Comparing the solution in the Laplace domain with the solution in the time domain, in terms of the matrix Lambert W function in (2), yields [28]:

$$\mathfrak{L}^{-1}\left\{\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)\right\}^{-1}=\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N \tag{5}$$

where s is the Laplace variable.

This approach using the Lambert W function provides a solution form for DDEs similar to that of the matrix exponential for ordinary differential equations (ODEs) (see Table I). The approach using the matrix Lambert W function provides a solution form for DDEs which enables the practical use of observability and controllability Gramians for TDS.

In this paper, the properties of controllability and observability for TDS are studied using the solution form in terms of the matrix Lambert W function. For the first time both algebraic conditions and Gramians for observability and controllability of DDEs are derived in a manner analogous to the well-known observability and controllability results for the ODE case.

II. Controllability

The concept of *point-wise controllability* of a system of DDEs, as in (1), and the related conditions were introduced in [25].

Definition 1: The system (1) is *point-wise controllable* if, for any given initial conditions $\mathbf{g}$ and $\mathbf{x}_0$, there exist a time

This work was supported by NSF Grant No. 0555765

Sun Yi (corresponding author), A. Galip Ulsoy are with the Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109. syjo@umich.edu

Patrick W. Nelson is with the Department of Mathematics, University of Michigan, Ann Arbor, MI 48109.

TABLE I

COMPARISON OF THE SOLUTIONS TO ODES AND DDES. THE SOLUTION TO DDES IN TERMS OF THE LAMBERT W FUNCTION SHOWS A FORMAL SEMBLANCE TO THAT OF ODES [29]

ODEs	DDEs
Scalar Case	
$\dot{x}(t) = ax(t) + bu(t), \quad t > 0$	$\dot{x}(t) = ax(t) + a_d x(t-h) + bu(t), \quad t > 0$
$x(t) = x_0, \quad t = 0$	$x(t) = g(t), \quad t \in [-h, 0)$; $x(t) = x_0, \quad t = 0$
$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\xi)}bu(\xi)d\xi$	$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\xi)} C_k^N bu(\xi)d\xi$ where, $S_k = \frac{1}{h}W_k(a_d h e^{-ah}) + a$
Matrix-Vector Case	
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad t > 0$	$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t), \quad t > 0$
$\mathbf{x}(t) = \mathbf{x}_0, \quad t = 0$	$\mathbf{x}(t) = \mathbf{g}(t), \quad t \in [-h, 0)$; $\mathbf{x}(t) = \mathbf{x}_0, \quad t = 0$
$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\xi)}\mathbf{B}\mathbf{u}(\xi)d\xi$	$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t} \mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)} \mathbf{C}_k^N \mathbf{B}\mathbf{u}(\xi)d\xi$ where, $\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) + \mathbf{A}$

t_1, $0 < t_1 < \infty$, and an admissible (i.e., measurable and bounded on a finite time interval) control segment $\mathbf{u}_{[0,t_1+h]}$ such that $\mathbf{x}(t; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}) = \mathbf{x}_1$ at $t = t_1$ for all $\mathbf{x}_1 \in \Re^n$ with initial conditions $\mathbf{g}(t)$, $\mathbf{x}_0$ and control $\mathbf{u}(t)$.

Based on an assumed solution form to (1) and using the kernel function, the condition for *point-wise controllability* is derived in [25]. Even though the equations to obtain the kernel function were presented in [25] based on the results of [2], the lack of an analytical solution of the systems of DDEs has prevented its evaluation and application [17]. This has prompted many authors to develop algebraic controllability criteria in terms of systems matrices [4], [6], [10], [13], [20], [24]. Other definitions of controllability which belong in different classifications, such as spectral controllability, have alternatively been provided [18], [21], and the corresponding criterion was characterized in terms of the rank of the associated matrices in [11]. For a detailed study, refer to [17], [22]. Using the matrix Lambert W function, however, the linear system with single time-invariant delay can be solved as in (2), and so the kernel function used in the condition for point-wise controllability can be derived as in (5).

Theorem 1: If a system (1) is point-wise complete [7], there exist a control which results in *point-wise controllability* in finite time of the solution of (1) for any initial conditions $\mathbf{g}$ and $\mathbf{x}_0$, if and only if the controllability Gramian, $\mathcal{C}$, computed with the kernel defined in (5) satisfies the following rank condition,

$$\mathrm{rank}[\mathcal{C}(0,t_1)] \equiv \mathrm{rank}\left[\int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \times \mathbf{B}\mathbf{B}^T \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi\right] = n \tag{6}$$

where ^{T} indicates transpose.

Proof: *Sufficiency* In (2), in order to transfer $\mathbf{x}(t)$ to $\mathbf{x}_1$ at t_1, substitute an input obtained with the inverse of the controllability Gramian in (6)

$$\mathbf{u}(t) = -\mathbf{B}^T \left\{\mathbf{K}(t,t_1)\right\}^T \mathcal{C}^{-1}(0,t_1) \times \left\{\mathbf{M}(t_1; 0, \mathbf{g}, \mathbf{x}_0) - \mathbf{x}_1\right\} \tag{7}$$

where $\mathbf{M}$ is the free solution to (1), that is

$$\mathbf{M}(t_1; 0, \mathbf{g}, \mathbf{x}_0) \equiv \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-0)}\mathbf{C}_k^I \tag{8}$$

The kernel function $\mathbf{K}(\xi, t_1)$ is obtained as

$$\mathbf{K}(\xi, t_1) \equiv \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \tag{9}$$

Then $\mathbf{x}(t_1) = \mathbf{x}_1$.

Necessity Given any $\mathbf{g}$, suppose there exist $t_1 > 0$ and a control $\mathbf{u}_{[0,t_1]}$ such that $\mathbf{x}(t_1) = \mathbf{0}$, but (6) does not hold. The latter implies that there exists a non-zero vector $\mathbf{x}_1 \in \Re^n$ such that $\mathbf{x}_1^T\mathbf{K}(t,t_1)\mathbf{B} = \mathbf{0}, \ 0 \le t \le t_1$ due to the following fact. Let $\mathbf{F}$ be an $n \times p$ matrix. Define

$$\mathcal{P}_{(t_1,t_2)} \equiv \int_{t_1}^{t_2} \mathbf{F}(t)\mathbf{F}^T(t)dt \tag{10}$$

Then the rows of $\mathbf{F}$ are linearly independent on $[t_1, t_2]$ if and only if the $n \times n$ constant matrix $\mathcal{P}_{(t_1,t_2)}$ is nonsingular [5]. Then, from (2),

$$\mathbf{x}_1^T\mathbf{x}(t_1) = \mathbf{x}_1^T\mathbf{M}(t_1, 0, \mathbf{g}, \mathbf{x}_0) + \int_0^{t_1} \mathbf{x}_1^T\mathbf{K}(\xi, t_1)\mathbf{B}\mathbf{u}(\xi)d\xi \tag{11}$$

and $\mathbf{0} = \mathbf{x}_1^T\mathbf{M}(t_1, 0, \mathbf{g}, \mathbf{x}_0)$. By hypothesis, however, $\mathbf{g}$ and $\mathbf{x}_0$ can be chosen such that $\mathbf{M}(t_1, 0, \mathbf{g}, \mathbf{x}_0) = \mathbf{x}_1$. Then $\mathbf{x}_1^T\mathbf{x}_1 = \mathbf{0}$ which contradicts the assumption that $\mathbf{x}_1 \neq \mathbf{0}$ ■

In the ODE case, the input computed using the controllability Gramian will use the minimal energy in transferring $(\mathbf{x}_0, 0)$ to $(\mathbf{x}_1, t_1)$ [5]. Using (6), one can prove that such result is also available for DDE's.

Theorem 2: The input defined in (7) consumes the *minimal* amount of energy, among all the $\mathbf{u}'s$ that can transfer $(\mathbf{x}_0, 0)$ to $(\mathbf{x}_1, t_1)$.

Proof: Omitted due to space limitation: the proof is similar to that for systems of ODEs [5]. ■

With Theorem 1 and (10), we can conclude

Corollary 1: If and only if all rows of

$$\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{B} \tag{12}$$

are linearly independent on $[0, \infty)$, then, the system in (1) is *point-wise controllable*.

Since the Laplace transform is a one-to-one linear operator, using (5)

Corollary 2: If and only if all rows of

$$\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1}\mathbf{B} \tag{13}$$

are linearly independent, over the field of complex numbers, then, the system in (1) is *point-wise controllable*.

In systems of ODEs, if the state variable $\mathbf{x}(t)$ is forced to zero at $t = t_1$, it stays at zero on $[t_1, \infty)$. However, because the system of DDEs in (1) has a *delayed term*, $\mathbf{x}(t-h)$, in its equation, even though all the individual state variables are zero at $t = t_1$ they can become non-zero again after t_1. Therefore, we need an additional definition of controllability for systems of DDEs [25].

Definition 2: The system (1) is *absolutely controllable* if, for any given preshape function $\mathbf{g}$, there exist a time t_1, $0 < t_1 < \infty$, and an admissible control segment $\mathbf{u}_{[0,t_1+h]}$ such that $\mathbf{x}(t; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}) = \mathbf{0}$, $t_1 < t < t_1 + h$.

Theorem 3: A point-wise complete system (1) is *absolutely controllable* if and only if

(1) there exists $t_1 > 0$ such that (6) holds;

(2) given $\mathbf{g}$ and $\mathbf{x}_0$, then with t_1 as in (6) and for some admissible $\mathbf{u}_{[0,t_1+h]}$ such that $\mathbf{x}(t_1; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}) = \mathbf{0}$, the equation

$$\mathbf{Bu}(t) = -\mathbf{A_d}\mathbf{x}(t-h; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}) \tag{14}$$

has an admissible solution, $\mathbf{u}(\cdot)$ on the interval $(t_1, t_1 + h)$. $\mathbf{x}(t-h; 0, \mathbf{g}, \mathbf{u})$ in (14) is obtained from (2)

$$\begin{aligned}\mathbf{x}(t-h) &= \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-h)}\mathbf{C}_k^I \\ &\quad + \int_0^{t-h} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-h-\xi)}\mathbf{C}_k^N\mathbf{Bu}(\xi)d\xi\end{aligned} \tag{15}$$

Proof: By *Theorem* 1 we have that for any $\mathbf{g}$ there exists $\mathbf{u}_{[0,t_1]}$ such that $\mathbf{x}(t_1; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}_{[0,t_1]}) = \mathbf{0}$. If (14) holds, then over the interval $(t_1, t_1 + h)$, (1) becomes

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t), \quad \mathbf{x}(t_1) = \mathbf{0} \tag{16}$$

It follows by the uniqueness theorem for ordinary differential equations that $\mathbf{x}(t) = \mathbf{0}$ for all $t \in [t_1, \infty)$. ■

III. Observability

Consider the system in (1) with output equation

$$\mathbf{y}(t) = \mathbf{Cx}(t) \tag{17}$$

where $\mathbf{C}$ is a coefficient $p \times n$ matrix, and $\mathbf{y}(t)$ is a $p \times 1$ measured output vector. If one knows the initial state of a TDS, then one can know all state variables for any time using the solution in (2) of the system. As seen in (2), however, the main obstacle is the fact that the free solution doesn't have the form of just the product of an initial condition matrix and a transition matrix in contrast to the ODE case (see Table I). Then, we can define

Definition 3: The system of (1) with (17) is *point-wise observable*, (or *observable*) in $[0, t_1]$ if the point $\mathbf{x}_0$ can be uniquely determined from a knowledge of $\mathbf{u}(t)$, $\mathbf{g}(t)$, and $\mathbf{y}(t)$. [9]

Just as the case of controllability, the lack of an analytical solution of the systems of DDEs has prevented the evaluation and application of the above condition. Bhat and Koivo [3] used spectral decomposition to decompose the state space into a finite-dimensional and a complementary part. In [14], various types of observability of TDS and corresponding algebraic conditions were presented. For a detailed study, refer to [14], [17].

Applying the kernel function in (9) to the observability Gramian defined in [9], one can present the following condition for observability for systems of DDEs.

Theorem 4: If and only if the observability Gramian $\mathcal{O}(0, t_1)$ computed with the kernel defined in (9) satisfies the following condition, the system of (1) and (17) is *point-wise observable*, i.e.,

$$\begin{aligned}\mathrm{rank}[\mathcal{O}(0, t_1)] \equiv \mathrm{rank}\Bigg[\int_0^{t_1} \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T \\ \times\mathbf{C}^T\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi\Bigg] = n\end{aligned} \tag{18}$$

has rank n, the system of (1) and (17) is *point-wise observable*.

Proof: This theorem is equivalent to Theorem 5 and the proof is in the proof of Theorem 5 below. ■

With Theorem 4 and (10), we can conclude

Corollary 3: If and only if all columns of the matrix

$$\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N \tag{19}$$

is are linearly independent, the system of (1) and (17) is *point-wise observable*.

Since the Laplace transform is a one-to-one linear operator, using (5)

Theorem 5: The system of (1) and (17) is *point-wise observable* if and only if all columns of the matrix

$$\mathbf{C}\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1} \tag{20}$$

are linearly independent.

Proof: The solution to (1) in the Laplace domain has a form of the product of a transition matrix and initial conditions and multiplying the coefficients $\mathbf{C}$ on both sides yields

$$\begin{aligned}\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) = \\ \mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\left\{\mathbf{x}_0+\mathbf{A_d}\mathbf{G}(s)e^{-sh}\right\} \\ +\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\left\{\mathbf{BU}(s)\right\}\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\underbrace{\begin{aligned}\mathbf{Y}(s)-\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\left\{\mathbf{BU}(s)\right\} \\ -\,\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\left\{\mathbf{A_d}\mathbf{G}(s)e^{-sh}\right\}\end{aligned}}_{\equiv\overline{\mathbf{Y}}(s),\ \text{known}[p\times 1]} \\ &= \underbrace{\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}}_{[p\times n]}\ \underbrace{\mathbf{x}_0}_{\text{unknown}[n\times 1]}\end{aligned} \tag{22}$$

Then the inverse Laplace transform of (22) is

$$\overline{\mathbf{y}}(t) = \mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{x}_0 \tag{23}$$

Multiply $\{\mathbf{K}(0,\xi)\}^T\,\mathbf{C}^T$ on both sides and integrate, then

$$\begin{aligned}&\int_0^{t_1}\{\mathbf{K}(0,\xi)\}^T\,\mathbf{C}^T\overline{\mathbf{y}}(t)d\xi \\ &= \underbrace{\int_0^{t_1}\{\mathbf{K}(0,\xi)\}^T\,\mathbf{C}^T\mathbf{C}\mathbf{K}(0,\xi)d\xi}_{\mathcal{O}(0,t_1)}\,\mathbf{x}_0\end{aligned} \tag{24}$$

Because $\mathcal{O}_{(0,t_1)}^{-1}$ always exists by assumption,

$$\mathcal{O}_{(0,t_1)}^{-1}\int_0^{t_1}\{\mathbf{K}(0,\xi)\}^T\,\mathbf{C}^T\overline{\mathbf{y}}(t)d\xi = \mathbf{x}_0 \tag{25}$$

and $\mathbf{x}_0$ is determined uniquely. This condition is not only sufficient but also a necessary condition and can be proven by contradiction as in the ODE case by supposing (1) and (17) is *point-wise observable* but the columns of $\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}$ are linearly dependent. Then there exists an non-zero constant vector $\boldsymbol{\alpha}$ such that

$$\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\boldsymbol{\alpha} = \mathbf{0} \tag{26}$$

Let us choose $\mathbf{x}_0 = \boldsymbol{\alpha}$, then from (22),

$$\begin{aligned}\overline{\mathbf{Y}}(s) &= \mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\mathbf{x}_0 \\ &= \mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\boldsymbol{\alpha} \\ &= \mathbf{0}\end{aligned} \tag{27}$$

Hence $\mathbf{x}_0$ cannot be detected. This contradicts the assumption that the system of (1) and (17) is *point-wise observable*. ■

For the case that $\mathbf{g}(t)$ is unknown, if $\mathbf{g}(t)$, as well as $\mathbf{x}_0$, can be determined uniquely from a knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$, the system of (1) and (17) is *absolutely observable* (or *strongly observable*) [9]. For example, if the matrix $\mathbf{C}$ is in (17) is a nonsingular $n\times n$ matrix, the system is *absolutely observable*. Even though we omit the proof here, it follows an argument similar to the proof of Theorem 4. For a detailed explanation of the definition of *absolute observability* and the corresponding conditions, the reader is referred to [9].

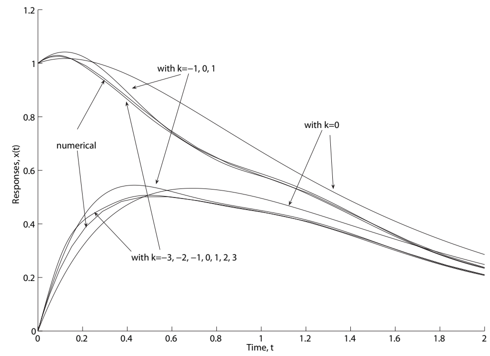

Fig. 1. Comparison for the example in (28) of results obtained by the numerical integration method, *dde23* of Matlab vs. the matrix Lambert W function approach with one, three and seven terms. With more branches the results show better agreement [29].

IV. EXAMPLE

Consider a system of DDEs, from [15], with external input:

$$\begin{aligned}\begin{Bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{Bmatrix} &= \begin{bmatrix}1 & -3\\ 2 & -5\end{bmatrix}\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix} \\ &\quad+\begin{bmatrix}1.66 & -0.697\\ 0.93 & -0.330\end{bmatrix}\begin{Bmatrix}x_1(t-1)\\ x_2(t-1)\end{Bmatrix} \\ &\quad+\mathbf{Bu}(t),\quad t>0 \\ \begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix} &= \begin{Bmatrix}1\\ 0\end{Bmatrix},\quad t\in[-1,0] \\ \mathbf{y}(t) &= \mathbf{C}\begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix}\end{aligned} \tag{28}$$

The response by the solution form in (2) is depicted in Fig. 1 and compared with the numerically obtained one. They show good agreement as the number of branches used increases.

By the criterion in [7], the system in (28) is point-wise complete. Setting $\mathbf{B}=[\ 1\quad 0\ ]^T$ and applying the numerical values, $\mathbf{S}_k$, $\mathbf{C}_k^N$, introduced in Section I, we can compute the controllability Gramian $\mathcal{C}(0,t_1)$ in (6). Then in order for the system (28) to be *point-wise controllable* as defined in section II, $\mathcal{C}(0,t_1)$ should have full rank. This means that the determinant of the matrix is non-zero. That is,

$$\det|\mathcal{C}(0,t_1)| \neq 0 \tag{29}$$

Computing the determinant of the matrix for an increasing number of branches yields the result in Fig. 2. As more branches are included, the value of determinant converges to a non-zero value. Therefore, this system can be said to be *point-wise controllable*. Using the inverse of the controllability Gramian, it is possible to derive the input to drive the state variable $\mathbf{x}(t)$ to zero at $(t = t_1)$ with (7). Fig. 3 is the response when the input is applied. One can confirm that $x_1 = x_2 = 0$ at $t_1 = 4$. However, because the system of DDEs in (28) has a *delayed term* in its equation, even though all the individual state variables are zero at $t = t_1$,

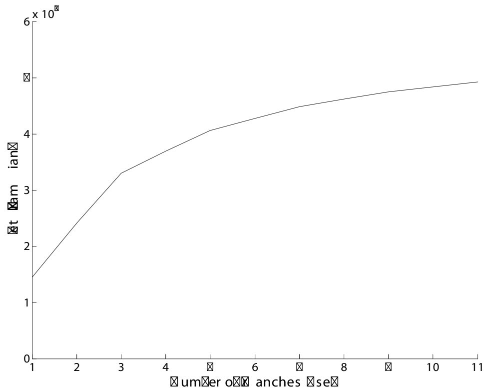

Fig. 2. Determinant of the controllability Gramian with increasing number of branches. As more branches are included, the value of determinant converges to a non-zero value.

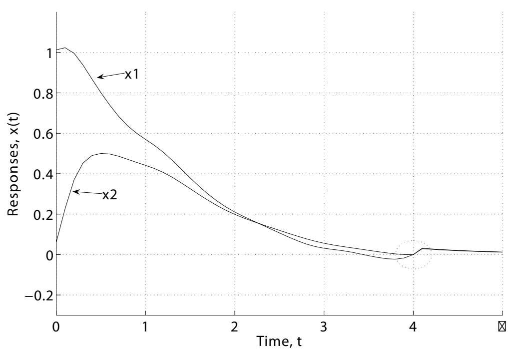

Fig. 3. Point-wise controllable; at $t_1 = 4$, all the states are at zero, but after $t_1 = 4$, start oscillating

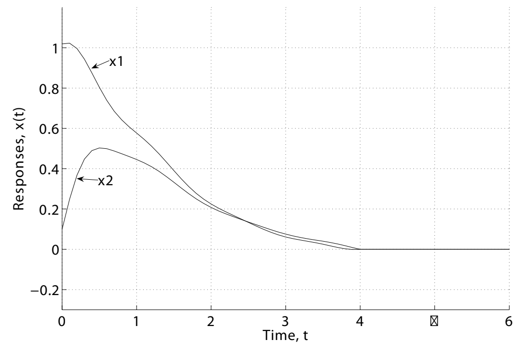

Fig. 4. Absolutely controllable: after $t_1 = 4$, all the states are at zero

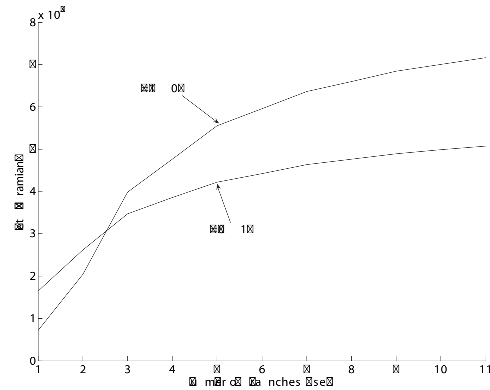

Fig. 5. Determinant of Observability Gramian when $\mathbf{C} = [0 \;\; 1]$ and $\mathbf{C} = [1 \;\; 0]$. As the number of branches used increases, the value of the determinant in case of $\mathbf{C} = [1 \;\; 0]$ tends to converge on higher value than the case of $\mathbf{C} = [0 \;\; 1]$

they become non-zero again after t_1. For a system to be *absolutely controllable* as defined in section II, (14) should have admissible solution $\mathbf{u}(\cdot)$on the interval $(t_1, t_1{+}h)$. With $\mathbf{B} = [\; 1 \quad 0 \;]^T$, there does not exist any such solution $\mathbf{u}(t)$. However, if the matrix $\mathbf{B}$ is a nonsingular $n \times n$ matrix, we can ensure that there exists a solution as (14). For example, if $\mathbf{B}$= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then the system is *absolutely controllable*, and we can confirm that the state variable stays at zero after $t_1 = 4$ (see Fig. 4). Even in case that $\mathbf{B}$ is singular as well as nonsingular, the solution to (14) can exist depending on the form of $\mathbf{A_d}$.

Even though a system satisfies the algebraic criteria already provided in previous works, such as [14], [17], in case that the determinant of the observability Gramian is smaller than a specific value, then it is not possible to design an observer as the gains in the observer can become unrealistically high. Consider the determinant of the observability Gramian corresponding the coefficient $\mathbf{C}$ in (28). For example, the determinants of the observability Gramian when $\mathbf{C} = [\; 1 \quad 0 \;]$ and $\mathbf{C} = [\; 0 \quad 1 \;]$ are compared in Fig. 5 with $t_1 = 4$. As the number of branches used increases, the value of the determinant in case of $\mathbf{C} = [\; 1 \quad 0 \;]$ tends to converge to a higher value than the case of $\mathbf{C} = [\; 0 \quad 1 \;]$.

The presented results agree with those obtained using the algebraic methods. However, using the method of Gramians developed in this paper, we can acquire more information. The controllability and observability Gramians indicate how controllable and observable the corresponding states are [12]. The algebraic conditions for controllability/observability tell only whether a system is controllable/observable or not. On the other hand, with the condition using Gramian concepts, we can determine how the change in some specific parameters of the system or the delay time, h, affect the controllability and observability of the system via the changes in the Gramians. In this way, one can track how parameters and/or the delay time, h, of the system can have influence on the input property (controllability) and the output property (observability) [23].

For systems of ODEs, a balanced realization in which the controllability Gramian and observability Gramian of a system are equal and diagonal was introduced in [19] and its existence was investigated in [23]. By 'balancing' a realization we mean that we 'symmetrize' a certain input property (controllability) with a certain output property (observability) through a suitable choice of basis [23]. The significance of the method has been established because of its desirable properties such as good error bounds, computational simplicity, stability, and its close connection to robust multivariable control [16]. However, for systems of DDEs, results on balanced realizations have been lacking. It is shown below that, using the Gramians defined in Section II and III, the concept of the balanced realization can be extended to systems of DDEs. Let $\mathbf{T}$ be a nonsingular state transformation

$$\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t) \tag{30}$$

Then, $\hat{\mathcal{C}}(0, t_1)$ and $\hat{\mathcal{O}}(0, t_1)$ can be made equal and diagonal with the aid of a suitably chosen matrix $\mathbf{T}$. In the numerical example in (28), when $\mathbf{B} = [\ 1 \quad 0\ ]^T$ and $\mathbf{C} = [\ 1 \quad 0\ ]$, the transformation

$$\mathbf{T} \;=\; \begin{bmatrix} -0.3929 & 1.1910 \\ 1.0880 & -0.5054 \end{bmatrix} \tag{31}$$

makes the the Gramians 'balanced', i.e., equal to each other and diagonalized,

$$\hat{\mathcal{C}}(0, t_1) \;=\; \hat{\mathcal{O}}(0, t_1) \;=\begin{bmatrix} 0.0238 & 0.0000 \\ 0.0000 & 0.2497 \end{bmatrix} \tag{32}$$

when computed using 11 branches of the matrix Lambert W function.

V. Conclusions and Future work

The controllability and observability of linear systems of DDEs have been studied using the solution form based on the matrix Lambert W function. It is possible, for the first time, to derive not only necessary and sufficient conditions for point-wise controllability and observability based on the solution of DDEs, but also to determine the input which takes the system to a desired state. Also for the first time for systems of DDEs, the balanced realization, in which the Gramians are equal and diagonal [19], is investigated in the time domain as in the case of ODEs. An example is presented to demonstrate the utility of the theoretical results.

Based upon the results presented, extension of well-established control design concepts for systems of ODEs to systems of DDEs appears feasible. For example, the design of feedback controllers and observers for DDEs can be developed in a manner analogous to ODEs and is already being investigated by the authors.

VI. Acknowledgments

The authors gratefully acknowledge the support of National Science Foundation (Grant No. 0555765).

References

[1] F. M. Asl and A. G. Ulsoy, "Analysis of a system of linear delay differential equations," *J. Dyn. Syst. Meas. Control*, vol. 125, pp. 215-223, 2003.

[2] R. E. Bellman and K. L. Cooke, *Differential-Difference Equations*, New York: Academic Press, 1963.

[3] K. P. Bhat and H. N. Koivo, "Modal Characterizations of Controllability and Observability in Time Delay Systems," *IEEE Trans. Aut. Cont.*, vol. 21, pp. 292-293, 1976.

[4] A. F. Buchalo, "Explicit conditions for controllability of linear systems with time lag," *IEEE Trans. Aut. Cont.*, vol. 13, pp. 193-195, 1968.

[5] C. T. Chen, *Linear System Theory and Design*, Harcourt Brace Jovanovich, 1984.

[6] A. K. Choudhury, "Algebraic and transfer-function criteria of fixed-time controllability of delay-differential systems," *Int. J. Control*, vol. 16, pp. 1073-1082, 1972.

[7] A. K. Choudhury, "Necessary and Sufficient Conditions of Pointwise Completeness of Linear Time-Invariant Delay-Differential Systems," *Int. J. Control*, Vol. 16, pp. 1083-1100, 1972.

[8] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, "On the Lambert W function," *Adv. Comput. Math.*, vol. 5, pp. 329-359, 1996.

[9] M. C. Delfour and S. K. Mitter, "Controllability, observability and optimal feedback control of affine hereditary differential systems," *SIAM J. Control*, vol. 10, pp. 298-328, 1972.

[10] M. Fliess and H. Mounier, "Interpretation and comparison of various types of delay system controllabilities," in *IFAC Conference, System Structure and Control*, 1995, pp. 330-335.

[11] H. Glusing-Luerssen, "A behavioral approach to delay-differential systems," *SIAM J. Cont. Opt.*, vol. 35, pp. 480-499, 1997.

[12] S. Holford and P. Agathoklis, "Use of model reduction techniques for designing IIR filters with linear phase in the passband," *IEEE Trans. Signal Process.*, vol. 44, pp. 2396-2404, 1996.

[13] F. M. Kirillova and S. V. Churakova, "The controllability problem for linear systems with aftereffect," *Differ. Equ.*, vol. 3, pp. 221-225, 1967.

[14] E. B. Lee and A. Olbrot, "Observability and related structural results for linear hereditary systems," *Int. J. Control*, vol. 34, pp. 1061-1078, 1981.

[15] T. N. Lee and S. Dianat, "Stability of time-delay systems," *IEEE Trans. Aut. Cont.*, Vol. 26, pp. 951-953, 1981

[16] W. S. Lu, E. B. Lee and Q. T. Zhang, "Balanced approximation of two-dimensional and delay-differential systems," *Int. J. Control*, vol. 46, 1987.

[17] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems : Analysis, Optimization, and Applications.* New York, U.S.A.: Elsevier Science Pub., 1987, pp. 504.

[18] A. Manitius and A. W. Olbrot, "Finite spectrum assignment problem for systems with delays," *IEEE Trans. Auto. Cont.*, vol. 24, pp. 541-553, 1979.

[19] B. C. Moore, "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Trans. Aut. Cont.*, vol. 26, pp. 17-32, 1981.

[20] A. W. Olbrot, "On controllability of linear systems with time delays in control," *IEEE Trans. Aut. Cont.*, vol. 17, pp. 664-666, 1972.

[21] P. Picard, O. Sename and J. F. Lafay, "Weak controllability and controllability indices for linear neutral systems," *Math. Comput. Simul.*, vol. 45, pp. 223-233, 1998.

[22] J. P. Richard, "Time-delay systems: an overview of some recent advaces and open problems," *Automatica*, vol. 39, pp. 1667-1964, 2003.

[23] E. I. Verriest and T. Kailath, "On generalized balanced realizations," *IEEE Trans. Aut. Cont.*, vol. 28, pp. 833-844, 1983.

[24] L. Weiss, "An algebraic criterion for controllability of linear systems with time delay," *IEEE Trans. Aut. Cont.*, vol. 15, pp. 443-444, 1970.

[25] L. Weiss, "On the controllability of delay-differential equations,;" *SIAM J. Cont. Opt.*, vol. 5, pp. 575-587, 1967.

[26] S. Yi, P. W. Nelson and A. G. Ulsoy, "Chatter Stability Analysis Using the Matrix Lambert Function and Bifurcation Analysis," *Math. Biosci. Eng.*, vol. 4, pp. 355-368, 2007.

[27] S. Yi and A. G. Ulsoy, "Solution of a system of linear delay differential equations using the matrix Lambert function," *Proc. 25th American Control Conference*, Minnearpolis, MN, Jun. 2006, pp. 2433-2438.

[28] S. Yi and A. G. Ulsoy, and P. W. Nelson, "Solution of systems of linear delay differential equations via Laplace transformation," *Proc. 45th IEEE Conf. on Decision and Control*, San Diego, CA, Dec. 2006, pp. 2535-2540.

[29] S. Yi and A. G. Ulsoy and P. W. Nelson, "Analysis of Systems of Linear Delay Differential Equations Using the Matrix Lambert Function and the Laplace Transformation," *Automatica*, (in press).