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Stabilization of discrete-time switched singular systems with state, output and switching delays

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Abstract

This paper is concerned with state feedback stabilization of discrete-time switched singular systems with time-varying delays existing simultaneously in the state, the output and the switching signal of the switched controller. On the basis of equivalent dynamics decomposition and Lyapunov–Krasovskii method, exponential estimates for the response of slow states of the closed-loop subsystems running in asynchronous and synchronous periods are first given. Exponential estimates for the response of fast states are also provided by establishing an analytic equation to solve the fast states and using some algebraic techniques. Then, by employing the obtained exponential estimates and the piecewise Lyapunov function approach with average dwell time (ADT) switching, sufficient conditions for the existence of a class of stabilizing switching signals and state feedback gains are derived, which explicitly depend on upper bounds on the delays and a lower bound on the ADT. Finally, two numerical examples are provided to illustrate the effectiveness of the obtained theoretical results.

1. Introduction

Switched systems have attracted considerable attention since 1990s because of their great capability in modeling and control of engineering systems with abrupt parameter and/or

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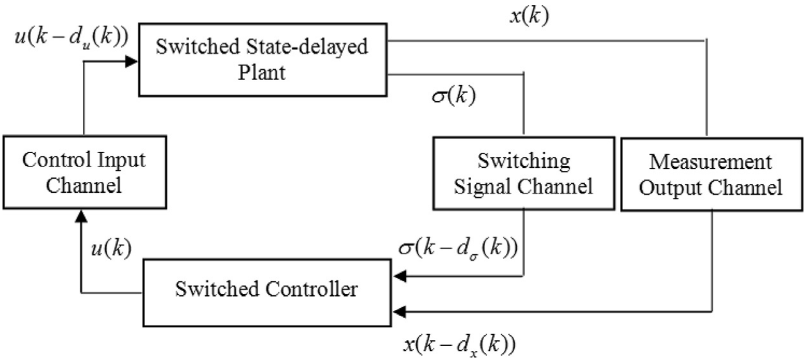

Fig. 1. Feedback switched systems with time delays.

structure variations, for example power electronics, chemical processes, networked and embedded systems [1,2]. A switched system is composed of a family of subsystems and a switching rule orchestrating the switching among them. The co-design of stabilizing switching strategies and feedback control laws (i.e. *feedback stabilization*) is one of the most challenging problems in the study of switched systems [2], which has been investigated in many studies; see, e.g. [2–8] and the references therein.

As is well known, time delay naturally appears in feedback control systems either in the state, the measurement output or in the control input, and it often incurs performance degradation and instability of the system [9]. In real switched control systems, due to intrinsic and extrinsic causes such as unknown or nondeterministic triggering of switching functions [10], disturbances [11], and signal transmission constraints [12], the change of plant's switching signal often cannot be detected instantly by the switched controller, but only after a time delay also called *switching delay* (see the switching signal channel shown in Fig. 1). Switching delay leads to asynchronous switching between the system modes and the switched controller, which makes the feedback stabilization problem more difficult. Over the past decade, many works have focused on this issue, which are divided in general into three categories: (1) stabilization with only switching delay; see, for example [10,11,13–16]. The key idea behind these studies is to utilize the extended multiple Lyapunov functions, which allow the energy of active Lyapunov or Lyapunov-like functions to be limitedly increased over asynchronous switching time interval. (2) stabilization with state and switching delays. Research on this topic is carried out mainly by combining the extended multiple Lyapunov functions approach with Lyapunov–Krasovskii (L–K) method; see [17–20] for some recent publications; and (3) stabilization with output (feedback state) and switching delays. Research on this theme is difficult because the delayed output and the delayed switching signal exist in two different types of index set [21]. Only a few results are available for continuous-time switched systems [21–24].

It is noteworthy that all the aforementioned works consider only switched *regular* (or state-space) systems. However, many real-world systems are more appropriately represented by switched *singular* (SS) system models due to additional algebraic constraints among state variables, for example electrical circuits [25], power systems [26], and chemical processes [27] (also see [28] for more applications). Singular systems are also referred to as descriptor or generalized state-space systems, differential-algebraic equations, etc. Analysis and synthesis of SS time-delay systems are more difficult than those of switched regular time-delay systems because of the additional switched delay algebraic equations. The solutions to the system may not exist and the system may have impulsive modes (for continuous system) or is not always causal (for discrete system). Recently, some results on stability and control of SS systems with state delay have been reported; see, e.g. [29–35]. So far, there are only a few works dealing with feedback stabilization of discrete-time SS systems with state and switching delays [36,37]. Moreover, it should be pointed out that the methods in [36,37] have two shortcomings. Firstly, in order to avoid dealing directly with the algebraic subsystems, the original system in [36] was transformed into a switched regular state-delayed system through model augmentation, which leads to high computation cost. While in [37], some delayed state terms were discarded to derive exponential estimates for solutions of the system (see the last two inequalities in Section Appendix therein), which brings about conservatism. Secondly, the switching conditions designed in [36,37] require preassigning the length of total asynchronous switching time. In practice, however, it is hard to know the length in advance.

To the best of authors' knowledge, the state feedback stabilization problem for discrete-time SS systems (even for switched regular systems) subject to simultaneously state, output and switching delays has not been investigated yet, which is the focus of this paper. This issue is quite complicated since the resulting closed-loop systems include not only multiple state delays but also a switching delay. To gain insights into the complexity of the issue, all the delays are admitted to time-varying. The fundamental questions to be addressed are stated as follows: [Q1] *How to derive exponential estimates for solutions of discrete singular systems with multiple state delays, especially of the algebraic subsystems?* This question is solved in two steps. On the basis of equivalent dynamics decomposition and L–K method, exponential estimates for the solutions of slow subsystems in asynchronous and synchronous cases are first given. Then, by establishing an analytic equation to solve fast state variables and utilizing some algebraic techniques, exponential estimates for fast subsystems are presented. [Q2] *How to design stabilizing switching laws which can synthetically reveal the effects of the state, output and switching delays on stability of the switched systems?* By using the average dwell time (ADT) switching scheme, we identify a class of exponentially stabilizing switching signals and give the existence conditions, which depend on the upper bounds of the delays, a lower bound on the ADT and the decay rate of the closed-loop system. [Q3] *How to give simple state feedback stabilization conditions?* By introducing an exponential finite sum inequality, this paper provides LMI-based stabilization conditions which do not include any free-weighting matrices. We would like to emphasize that the results in this paper are of importance in studying remote control strategies for SS systems and designing SS control systems under networked environments.

This paper is organized as follows. The problem description, necessary definitions and lemmas are given in Section 2. Section 3 gives exponential estimation for the closed-loop control system in asynchronous and synchronous cases. In Section 4, the switching signal and the state feedback gains are designed. Numerical examples are provided in Section 5 and conclusions are stated in Section 6.

Notations. For a symmetric matrix P, $P>0$ (≥ 0) means that P is positive definite (semi-positive definite). $\lambda_{\max}(P)(\lambda_{\min}(P))$ denotes the largest (smallest) eigenvalue of P. I is an identity matrix with appropriate dimension. $\mathbb{Z}$ denotes the set of all integer numbers and $\mathbb{Z}^{+}$ denotes the set of all non-negative integers. The superscript $'\top'$ represents the transpose and $'*'$ denotes the symmetric terms in a symmetric matrix. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. For a real matrix A, $\mathrm{Sym}(A)$ is defined as $A + A^\top$. $\|\cdot\|$ denotes the Euclidean norm of a vector. For a function sequence $x = \{x(t)\}$, we define $\|x(t)\|_{\bar{d}} = \sup_{t-\bar{d}\le\theta\le t}\|x(\theta)\|$.

2. Problem formulation and preliminaries

Consider the following discrete-time SS systems with a state delay

$$\begin{cases} Ex(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d_1(k)) + B_{\sigma(k)}u(k), \\ x(k) = \phi_0(k), \;\; k = -\bar{d}_1, -\bar{d}_1+1, \ldots, 0 \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, and $u(k) \in \mathbb{R}^m$ is the control input. $\sigma : \mathbb{Z}^+ \to \mathcal{I} = \{1, 2, \ldots, N\}$ is the switching signal which is a piecewise constant right continuous function of time; N is the number of subsystems or modes. $d_1(k)$ is a positive integer function representing the time-varying state delay and satisfies $0 < \underline{d}_1 \le d_1(k) \le \bar{d}_1$, where $\underline{d}_1$ and $\bar{d}_1$ are known positive integers. $\phi_0(k)$ is a compatible vector valued initial function. $E \in \mathbb{R}^{n\times n}$ is a constant matrix satisfying $\mathrm{rank}E = r \le n$. A_j, A_{dj} and B_j, $\forall j \in \mathcal{I}$, are known constant matrices with appropriate dimensions. For a σ, the corresponding switching instants $0 \triangleq k_0 < k_1 < \cdots k_i < k_{i+1} < \cdots$ are defined recursively as $k_{i+1} = \inf\{k > k_i : \sigma(k) \neq \sigma(k_i)\}$. When $k \in [k_i, k_{i+1})$, $\forall i \in \mathbb{Z}^+$, we let $\sigma(k) = \sigma(k_i) = l_i \in \mathcal{I}$, and quadruple-matrix $(E, A_{l_i}, A_{dl_i}, B_{l_i})$, denoting the l_ith subsystem of Eq. (1), are activated.

This paper is concerned with state feedback stabilization of system (1). Ideally, a switched state feedback control pattern $u(k) = K_{\sigma(k)}x(k)$ is used, where $K_{\sigma(k)}$ are the controller gains to be determined. However, in practice the controller usually receives past measurements of both the state and the switching signal of the plant because of the existence of switching delay. Thus, we consider the following state feedback control law:

$$u(k) = K_{\sigma(k-\tau_\mathrm{s}(k))}x(k - d_2(k)), \tag{2}$$

where $\tau_\mathrm{s}(k)$ is the switching delay satisfying $0 < \tau_\mathrm{s}(k) \le \bar{\tau}_\mathrm{s}$. Here, without loss of generality [13,23], it is assumed that the maximal switching delay $\bar{\tau}_\mathrm{s}$ is known a *priori* and $\bar{\tau}_\mathrm{s} < k_{i+1} - k_i$ for all $i \in \mathbb{Z}^+$. $d_2(k)$ is the output delay satisfying $0 < \underline{d}_2 \le d_2(k) \le \bar{d}_2$, where $\underline{d}_2$ and $\bar{d}_2$ are positive integers.

Remark 1. If a control input delay $d_\mathrm{u}(k)$ also exists in the feedback loop, then the actual control law becomes $u(k) = K_{\sigma(k-\tau_\mathrm{s}(k)-d_\mathrm{u}(k))}x(k - d_2(k) - d_\mathrm{u}(k))$, which is also of the form (2).

Substituting Eq. (2) into Eq. (1) yields the following closed-loop system:

$$Ex(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d_1(k)) + B_{\sigma(k)}K_{\sigma(k-\tau_\mathrm{s}(k))}x(k-d_2(k)) \tag{3}$$

with initial condition $x(k) = \phi(k), \;\; k = -\bar{d}, -\bar{d}+1, \ldots, 0, \; \bar{d} = \max\{\bar{d}_1, \bar{d}_2\}$.

Definition 1 [38]**.** The discrete singular system $Ex(k+1) = Ax(k)$ (or the pair (E, A)) is said to be regular if $\det(zE - A)$ is not identically zero and causal if it is regular and $\deg(\det(zE - A)) = \mathrm{rank}E$.

Definition 2. Consider a singular delay system $\Sigma: \;\; Ex(k+1) = Ax(k) + \sum_{s=1}^{p} A_{ds}x(k - d_s(k))$ where $0 < \underline{d}_s \le d_s(k) \le \bar{d}_s$, $\underline{d}_s$ and $\bar{d}_s$ are positive scalars, and $p \ge 1$. The system Σ is said to be regular and causal if the pair (E, A) is regular and causal.

Remark 2. The regularity and causality of the system Σ guarantee that the solution to it exists and is unique. This can be easily proved by using the regularity and causality of the pair (E, A) and the equivalent decomposition in [39].

Definition 3. Under switching signal σ, the closed-loop system (3) is said to be

(1) regular and causal if the pair (E, A_{l_i}) is regular and causal, $\forall l_i \in \mathcal{I}$;
(2) exponentially stable with a decay rate λ $(0<\lambda<1)$ if its solutions satisfy $\|x(k)\| \leq c\lambda^k\|x(0)\|_{\bar{d}}$ for all $k\geq 0$ and any compatible initial condition $\phi(k)$, $k=-\bar{d}, -\bar{d}+1, \dots, 0$, where $c>0$ is the decay coefficient;
(3) λ-exponentially admissible if it is regular, causal and exponentially stable with a decay rate λ.

Definition 4. Consider two singular systems with p $(p\geq 1)$ state delays $\Sigma : Ex(k+1) = Ax(k) + \sum_{s=1}^{p} A_{ds}x(k-d_s(k))$ and $\tilde{\Sigma} : \tilde{E}\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \sum_{s=1}^{p} \tilde{A}_{ds}\tilde{x}(k-d_s(k))$. If there exist two invertible matrices Q and P such that $QEP = \tilde{E}$, $QAP = \tilde{A}$, $QA_{ds}P = \tilde{A}_{ds}$, $s = 1, \dots, p$, and $P^{-1}x(k) = \tilde{x}(k)$, then systems Σ and $\tilde{\Sigma}$ are called restricted system equivalent (r.s.e.) under transformation (Q, P). Such an equivalent relation is denoted by $\Sigma \overset{Q,P}{\Longleftrightarrow} \tilde{\Sigma}$ or $(E, A, A_{d1}, \dots, A_{dp}) \overset{Q,P}{\Longleftrightarrow} (\tilde{E}, \tilde{A}, \tilde{A}_{d1}, \dots, \tilde{A}_{dp})$ throughout the paper.

Remark 3. By the proof of [38, Lemma 1], it can be easily verified that the regularity, causality and exponential stability of a singular system with state delays are preserved under r.s.e. transformation. When $\tilde{E} = \mathrm{diag}\{I_{r_E}, 0\}$, where $r_E = \mathrm{rank}(E)$, system $\tilde{\Sigma}$ is of the so-called dynamics decomposition form [39]. In this case, let $\tilde{x}(k) = [\tilde{x}_1^\top(k)\ \tilde{x}_2^\top(k)]^\top$ with $\tilde{x}_1(k) \in \mathbb{R}^{r_E}$ and $\tilde{x}_2(k) \in \mathbb{R}^{n-r_E}$. We denote $\tilde{x}_1(k)$ and $\tilde{x}_2(k)$ the slow and fast state variables, respectively.

Definition 5 [13]**.** For switching signal σ and any $k_e > k_b \geq k_0$, let $N_\sigma(k_b, k_e)$ be the switching numbers of σ over the interval $[k_b, k_e)$. If for given $N_0 \geq 1$ and $\tau_a > 0$, $N_\sigma(k_b, k_e) \leq N_0 + (k_e - k_b)/\tau_a$ holds, then τ_a and N_0 are called the ADT and the chatter bound, respectively. Note here that we denote $S_{\mathrm{ave}}[\tau_a, N_0]$ the class of switching signals with ADT τ_a and chatter bound N_0.

The objective of this paper is to determine a set of state feedback gains K_{l_i}, $\forall l_i \in \mathcal{I}$ and find a class of corresponding switching signals specified by ADT such that the resulting closed-loop system (3) is exponentially admissible in the presence of state, output and switching delays.

For simplicity of notation, in what follows, the matrices $B_{l_i}K_{l_{i-1}}$ and $B_{l_i}K_{l_i}$ in the closed-loop system (3) will be denoted by $\mathrm{B}_{l_il_{i-1}}$ and B_{l_i}, respectively. Moreover, for the closed-loop system (3), the subsystem acting during the time interval $[k_i, k_i+\tau_s(k_i))$ will be denoted by $\mathcal{S}_{l_il_{i-1}}$ or quadruple-matrix $(E, A_{l_i}, A_{dl_i}, \mathrm{B}_{l_il_{i-1}})$, the subsystem running during the time interval $[k_i+\tau_s(k_i), k_{i+1})$ will be denoted by $\mathcal{S}_{l_i}$ or quadruple-matrix $(E, A_{l_i}, A_{dl_i}, \mathrm{B}_{l_i})$, and so on.

The following lemmas are given for later development.

Lemma 1 *[40]. Given a matrix D, let a matrix $S>0$ and a scalar $\eta \in (0, 1)$ exist such that $D^\top SD - \eta^2 S < 0$. Then, the matrix D satisfies the bound $\|D^\ell\| \leq \chi e^{-\delta\ell}$, $\forall \ell \in \mathbb{Z}^+$, where $\chi = \sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}}$ and $\delta = -\ln\eta$.*

Lemma 2. *For a constant matrix $S \in \mathbb{R}^{n\times n}$, $S>0$, integers $d(k_1)$ and $d(k_2)$ satisfying $0<d(k_1)\leq d(k_2)$, a scalar $v>0$ and a vector function $y(k) : \mathbb{Z} \to \mathbb{R}^n$, the following* *exponential finite sum inequality holds:*

$$\sum_{j=k-d(k_2)}^{k-d(k_1)} v^{k-j} y^\top(j) E^\top S E y(j) \geq w \left(\sum_{j=k-d(k_2)}^{k-d(k_1)} E y(j) \right)^\top S \left(\sum_{j=k-d(k_2)}^{k-d(k_1)} E y(j) \right), \tag{4}$$

where $w = \begin{cases} v^{d(k_2)}(1-v)/(1-v^{d(k_2)-d(k_1)+1}), & v \neq 1 \\ d(k_2) - d(k_1) + 1, & v = 1 \end{cases}$.

Proof. By using Schur complement for non-strict inequalities (Eq. (2.41) in [41]), it is easy to obtain $\begin{bmatrix} v^{k-j} y^\top(j) E^\top S E y(j) & y^\top(j) E^\top \\ * & v^{j-k} S^{-1} \end{bmatrix} \geq 0$. Taking the sum in j in this inequality from $k-d(k_2)$ to $k-d(k_1)$ yields

$$\begin{bmatrix} \sum_{j=k-d(k_2)}^{k-d(k_1)} v^{k-j} y^\top(j) E^\top S E y(j) & \sum_{j=k-d(k_2)}^{k-d(k_1)} y^\top(j) E^\top \\ * & \sum_{j=k-d(k_2)}^{k-d(k_1)} v^{j-k} S^{-1} \end{bmatrix} \geq 0.$$

Applying Schur complement again on the above inequality results (4). □

Lemma 3 *[42]. Given matrices X, $Y = Y^\top$ and Z with appropriate dimensions. Then there exists a scalar $\rho > 0$ such that $\rho I + Y > 0$ and $-X^\top Z - Z^\top X - Z^\top Y Z \leq X^\top(\rho I + Y)^{-1} X + \rho Z^\top Z$.*

3. Exponential estimation for the closed-loop subsystems

In this section, for the closed-loop subsystems $\mathcal{S}_{l_i l_{i-1}}$ and $\mathcal{S}_{l_i}$, we give exponential estimates for solutions of restricted equivalent subsystems in the dynamics decomposition form, which will constitute a basis for proving the main result Theorem 1 in the next section. First, sufficient conditions for the existence of the equivalent subsystems and exponential estimates for the solutions of slow state variables via the L–K method are presented in Proposition 1. Then, exponential estimates for the solutions of fast state variables are derived in Proposition 2.

For subsystems $\mathcal{S}_{l_i l_{i-1}}$ and $\mathcal{S}_{l_i}$, the following L–K functions are constructed:

$$\begin{aligned} V_{\tilde{\sigma}}(k) = {} & x^\top(k) E^T P_{\tilde{\sigma}} E x(k) + \sum_{s=1}^{2} \Bigg\{ \sum_{j=k-d_s(k)}^{k-1} x^\top(j)(1+\hbar_{\tilde{\sigma}})^{k-1-j} Q_{\tilde{\sigma}1_s} x(j) \\ & + \sum_{i=-\bar{d}_s+1}^{-\underline{d}_s} \sum_{j=k+i}^{k-1} x^\top(j)(1+\hbar_{\tilde{\sigma}})^{k-1-j} Q_{\tilde{\sigma}1_s} x(j) \\ & + \sum_{j=k-\underline{d}_s}^{k-1} x^\top(j)(1+\hbar_{\tilde{\sigma}})^{k-1-j} Q_{\tilde{\sigma}2_s} x(j) + \sum_{j=k-\bar{d}_s}^{k-1} x^\top(j)(1+\hbar_{\tilde{\sigma}})^{k-1-j} Q_{\tilde{\sigma}3_s} x(j) \\ & + \sum_{i=-\bar{d}_s}^{-1} \sum_{j=k+i}^{k-1} y^\top(j) E^\top (1+\hbar_{\tilde{\sigma}})^{k-1-j} Z_{\tilde{\sigma}1_s} E y(j) \\ & + \sum_{i=-\bar{d}_s}^{-\underline{d}_s-1} \sum_{j=k+i}^{k-1} y^\top(j) E^\top (1+\hbar_{\tilde{\sigma}})^{k-1-j} Z_{\tilde{\sigma}2_s} E y(j) \Bigg\}, \quad \tilde{\sigma} = l_i l_{i-1}, l_i, \end{aligned} \tag{5}$$

where $P_{\tilde{\sigma}} > 0$, $Q_{\tilde{\sigma}1_s} > 0$, $Q_{\tilde{\sigma}2_s} > 0$, $Q_{\tilde{\sigma}3_s} > 0$, $Z_{\tilde{\sigma}1_s} > 0$, $Z_{\tilde{\sigma}2_s} > 0$, $s = 1, 2$, $y(k) = x(k+1) - x(k)$, and $\hbar_{\tilde{\sigma}} = \begin{cases} \beta, & \tilde{\sigma} = l_i l_{i-1} \\ -\alpha, & \tilde{\sigma} = l_i \end{cases}$ with $0 < \alpha < 1$ and $\beta > 0$.

Remark 4. Note that for the L–K functionals in Eq. (5), the terms $\sum_{j=k-d_s(k)}^{k-1} \cdot$ and $\sum_{i=-\bar{d}_s+1}^{-\underline{d}_s} \sum_{j=k+i}^{k-1} \cdot$ correspond to delay-dependent stability condition while the terms $\sum_{j=k-\underline{d}_s}^{k-1} \cdot$ and $\sum_{j=k-\bar{d}_s}^{k-1} \cdot$ relate to delay-range-dependent stability condition. The double summation terms of the difference of state are used to establish relations between $x(k)$ and $x(k-d_s(k))$, $x(k-d_s(k))$ and $x(k-\underline{d}_s)$, as well as $x(k-d_s(k))$ and $x(k-\bar{d}_s)$, which are helpful in reducing conservatism of the derived stability condition. The introduction of $(1+\hbar_{\tilde{\sigma}})^{k-1-j}$ aims to derive exponentially increasing and decreasing bounds on the functionals (see Eqs. (9) and (10) later). The proposed $V_{\tilde{\sigma}}(k)$ is actually a very general form of discrete L–K functional. For example, setting $s = 1$, $\hbar_{\tilde{\sigma}} = 0$, and $Q_{\tilde{\sigma}2_s} = Q_{\tilde{\sigma}3_s} = Z_{\tilde{\sigma}1_s} = 0$ yields the L–K functional in [43]. If choosing $E = I$ and $s = 1$, $V_{l_i}(k)$ reduces to the L–K functional in [44]. It is worth pointing out that by employing the delay-partitioning technique [45] or introducing more delay terms [46], some improved L–K functionals can be easily constructed for the subsystems $\mathcal{S}_{l_i l_{i-1}}$ and $\mathcal{S}_{l_i}$, which will yield more less conservative stability conditions but give rise to higher computation burden.

Proposition 1. *Consider the subsystems $\mathcal{S}_{l_i l_{i-1}}$ and $\mathcal{S}_{l_i}$ in Eq. (3), and let $0 < \alpha < 1$, $\beta > 0$ and $0 < \underline{d}_s < \bar{d}_s$, $s = 1, 2$, be given constants satisfying $(1+\beta)^{\underline{d}_s}/(1+\tilde{d}_s) < 1$, where $\tilde{d}_s = \bar{d}_s - \underline{d}_s$. If there exist matrices $P_{\tilde{\sigma}} > 0$, $Q_{\tilde{\sigma}w_s} > 0$, $Z_{\tilde{\sigma}1_s} > 0$, $Z_{\tilde{\sigma}2_s} > 0$ and $Y_{\tilde{\sigma}} = Y_{\tilde{\sigma}}^\top$, $\tilde{\sigma} = l_i l_{i-1}, l_i$, $w = 1, 2, 3$, $s = 1, 2$, such that*

$$\Psi_{\tilde{\sigma}} = \begin{bmatrix} \Sigma_{\tilde{\sigma}} & (\widehat{A}_{\tilde{\sigma}} - \mathrm{I}_1 E)^\top \widehat{Z}_{\tilde{\sigma}} \\ * & -\widehat{Z}_{\tilde{\sigma}} \end{bmatrix} < 0, \ \tilde{\sigma} = l_i l_{i-1}, l_i, \tag{6}$$

where $\widehat{Z}_{\tilde{\sigma}} = \sum_{s=1}^{2} (\bar{d}_s Z_{\tilde{\sigma}1_s} + \tilde{d}_s Z_{\tilde{\sigma}2_s})$,

$$\begin{aligned} \Sigma_{\tilde{\sigma}} = {} & \Phi_{\tilde{\sigma}} + \widehat{A}_{\tilde{\sigma}}^\top P_{\tilde{\sigma}} \widehat{A}_{\tilde{\sigma}} - \widehat{A}_{\tilde{\sigma}}^\top R^\top Y_{\tilde{\sigma}} R \widehat{A}_{\tilde{\sigma}} - \rho_{\tilde{\sigma}1_1} (\mathrm{I}_1 - \mathrm{I}_2)^\top E^\top Z_{\tilde{\sigma}1_1} E (\mathrm{I}_1 - \mathrm{I}_2) \\ & - \rho_{\tilde{\sigma}1_2} (\mathrm{I}_1 - \mathrm{I}_5)^\top E^\top Z_{\tilde{\sigma}1_2} E (\mathrm{I}_1 - \mathrm{I}_5) - \rho_{\tilde{\sigma}2_1} (\mathrm{I}_2 - \mathrm{I}_4)^\top E^\top (Z_{\tilde{\sigma}1_1} + Z_{\tilde{\sigma}2_1}) E \\ & \times (\mathrm{I}_2 - \mathrm{I}_4) - \rho_{\tilde{\sigma}2_2} (\mathrm{I}_5 - \mathrm{I}_7)^\top E^\top (Z_{\tilde{\sigma}1_2} + Z_{\tilde{\sigma}2_2}) E (\mathrm{I}_5 - \mathrm{I}_7) - \rho_{\tilde{\sigma}2_1} (\mathrm{I}_3 - \mathrm{I}_2)^\top \\ & \times E^\top Z_{\tilde{\sigma}2_1} E (\mathrm{I}_3 - \mathrm{I}_2) - \rho_{\tilde{\sigma}2_2} (\mathrm{I}_6 - \mathrm{I}_5)^\top E^\top Z_{\tilde{\sigma}2_2} E (\mathrm{I}_6 - \mathrm{I}_5), \ \tilde{\sigma} = l_i l_{i-1}, l_i, \end{aligned}$$

$$\begin{aligned} \Phi_{\tilde{\sigma}} = \mathrm{diag}\Big\{ & -(1+\hbar_{\tilde{\sigma}}) E^\top P_{\tilde{\sigma}} E + \sum_{s=1}^{2} \big[(1+\tilde{d}_s) Q_{\tilde{\sigma}1_s} + Q_{\tilde{\sigma}2_s} + Q_{\tilde{\sigma}3_s} \big], \\ & -(1+\hbar_{\tilde{\sigma}})^{\mathrm{d}_{\tilde{\sigma}1}} Q_{\tilde{\sigma}1_1}, -(1+\hbar_{\tilde{\sigma}})^{\underline{d}_1} Q_{\tilde{\sigma}2_1}, -(1+\hbar_{\tilde{\sigma}})^{\bar{d}_1} Q_{\tilde{\sigma}3_1}, \\ & -(1+\hbar_{\tilde{\sigma}})^{\mathrm{d}_{\tilde{\sigma}2}} Q_{\tilde{\sigma}1_2}, -(1+\hbar_{\tilde{\sigma}})^{\underline{d}_2} Q_{\tilde{\sigma}2_2}, -(1+\hbar_{\tilde{\sigma}})^{\bar{d}_2} Q_{\tilde{\sigma}3_2} \Big\}, \ \tilde{\sigma} = l_i l_{i-1}, l_i, \end{aligned}$$

$$\begin{aligned} \widehat{A}_{\tilde{\sigma}} &= [A_{l_i} \quad A_{dl_i} \quad 0 \quad 0 \quad \mathrm{B}_{\tilde{\sigma}} \quad 0 \quad 0], \ \tilde{\sigma} = l_i l_{i-1}, l_i, \\ \mathrm{I}_t &= [0_{n\times(t-1)n} \quad I_n \quad 0_{n\times(7-t)n}], \quad t = 1, \ldots, 7, \end{aligned}$$

with $\mathrm{d}_{l_i l_{i-1} s} = \underline{d}_s$, $\mathrm{d}_{l_i s} = \bar{d}_s$, $\rho_{l_i 1_s} = \alpha(1-\alpha)^{\bar{d}_s}/(1-(1-\alpha)^{\bar{d}_s})$, $\rho_{l_i 2_s} = \alpha(1-\alpha)^{\bar{d}_s}/(1-(1-\alpha)^{\tilde{d}_s})$, $\rho_{l_i l_{i-1} 1_s} = \beta(1+\beta)^{\underline{d}_s}/((1+\beta)^{\underline{d}_s} - 1)$, $\rho_{l_i l_{i-1} 2_s} = \beta(1+\beta)^{\underline{d}_s}/((1+\beta)^{\tilde{d}_s} - 1)$, $s = 1, 2$, *$R \in \mathbb{R}^{n\times n}$ is any matrix satisfying $RE = 0$ and $\mathrm{rank}(R) = n - r$, then the following results hold:*

(1) *The subsystems* $\mathcal{S}_{l_il_{i-1}}$ *and* $\mathcal{S}_{l_i}$ *are regular and causal;*

(2) *There exist invertible matrices* G_{l_i} *and* H *such that* $\mathcal{S}_{l_il_{i-1}} \overset{G_{l_i},H}{\Longleftrightarrow} \tilde{\mathcal{S}}_{l_il_{i-1}}$ *(or* $(\tilde{E}, \tilde{A}_{l_i}, \tilde{A}_{dl_i}, \tilde{\mathrm{B}}_{l_il_{i-1}})$*) and* $\mathcal{S}_{l_i} \overset{G_{l_i},H}{\Longleftrightarrow} \tilde{\mathcal{S}}_{l_i}$ *(or* $(\tilde{E}, \tilde{A}_{l_i}, \tilde{A}_{dl_i}, \tilde{\mathrm{B}}_{l_i})$*), where* $\tilde{E} = \mathrm{diag}\{I_r, 0\}$ *and* $\tilde{A}_{l_i}$ *has the structure of* $\begin{bmatrix} \tilde{A}_{l_i}^{11} & 0 \\ \tilde{A}_{l_i}^{21} & I_{n-r} \end{bmatrix}$*.Furthermore, letting* $\tilde{A}_{dl_i} = \begin{bmatrix} \tilde{A}_{dl_i}^{11} & \tilde{A}_{dl_i}^{12} \\ \tilde{A}_{dl_i}^{21} & \tilde{A}_{dl_i}^{22} \end{bmatrix}$ *and* $\tilde{\mathrm{B}}_{\tilde{\sigma}} = \begin{bmatrix} \tilde{\mathrm{B}}_{\tilde{\sigma}}^{11} & \tilde{\mathrm{B}}_{\tilde{\sigma}}^{12} \\ \tilde{\mathrm{B}}_{\tilde{\sigma}}^{21} & \tilde{\mathrm{B}}_{\tilde{\sigma}}^{22} \end{bmatrix}$, $\tilde{\sigma} = l_il_{i-1}, l_i$, *there exist scalars* $\chi_{l_il_{i-1}s} > 1$ *and* $\chi_{l_is} > 1$, $s = 1, 2$ *such that* $\forall \ell \in \mathbb{Z}^+$,

$$\begin{aligned} &\|(\tilde{A}_{dl_i}^{22})^{\ell}\| \le \chi_{l_il_{i-1}1}\big[(1+\beta)^{\frac{d_1}{2}}/(1+\tilde{d}_1)^{\frac{1}{2}}\big]^{\ell}, \\ &\|(\tilde{\mathrm{B}}_{l_il_{i-1}}^{22})^{\ell}\| \le \chi_{l_il_{i-1}2}\big[(1+\beta)^{\frac{d_2}{2}}/(1+\tilde{d}_2)^{\frac{1}{2}}\big]^{\ell}, \end{aligned} \tag{7}$$

$$\begin{aligned} &\|(\tilde{A}_{dl_i}^{22})^{\ell}\| \le \chi_{l_i1}\big[(1-\alpha)^{\frac{\bar{d}_1}{2}}/(1+\tilde{d}_1)^{\frac{1}{2}}\big]^{\ell}, \\ &\|(\tilde{\mathrm{B}}_{l_i}^{22})^{\ell}\| \le \chi_{l_i2}\big[(1-\alpha)^{\frac{\bar{d}_2}{2}}/(1+\tilde{d}_2)^{\frac{1}{2}}\big]^{\ell}. \end{aligned} \tag{8}$$

(3) *Letting* $\tilde{x}(k) = H^{-1}x(k) = [\tilde{x}_1^{\top}(k)\ \tilde{x}_2^{\top}(k)]$, *where* $\tilde{x}_1(k) \in \mathbb{R}^r$ *and* $\tilde{x}_2(k) \in \mathbb{R}^{n-r}$, $V_{l_il_{i-1}}(k)$ *and* $V_{l_i}(k)$ *defined in Eq. (5) satisfy the following estimations:*

$$a_{l_il_{i-1}}\|\tilde{x}_1(k)\|^2 \le V_{l_il_{i-1}}(k) \le (1+\beta)^{k-k_i}V_{l_il_{i-1}}(k_i), \ \forall k \in [k_i, k_i+\tau_{\mathrm{s}}(k_i)), \tag{9}$$

$$a_{l_i}\|\tilde{x}_1(k)\|^2 \le V_{l_i}(k) \le (1-\alpha)^{k-k_i-\tau_{\mathrm{s}}(k_i)}V_{l_i}(k_i+\tau_{\mathrm{s}}(k_i)), \quad \forall k \in [k_i+\tau_{\mathrm{s}}(k_i), k_{i+1}), \tag{10}$$

where $a_{l_il_{i-1}}$ *and* a_{l_i} *are positive scalars*

Proof. For convenience of presentation, we first consider subsystem $\mathcal{S}_{l_i}$. For this subsystem, the parameters $\tilde{\sigma}$ and $\hbar_{\tilde{\sigma}}$ in Eq. (5) equal l_i and $-\alpha$, respectively.

(1). Since $\mathrm{rank}E = r \le n$, there exist invertible matrices G and H such that $\bar{E} = GEH = \mathrm{diag}\{I_r, 0\}$. Let $\bar{A}_{l_i} = GA_{l_i}H$, $\bar{A}_{dl_i} = GA_{dl_i}H$, $\bar{\mathrm{B}}_{l_i} = G\mathrm{B}_{l_i}H$, $\bar{P}_{l_i} = G^{-\top}P_{l_i}G^{-1}$, $\bar{Q}_{l_iw_s} = H^{\top}Q_{l_iw_s}H$, $w = 1,2,3$, $s = 1,2$, $\bar{Z}_{l_i1_s} = G^{-\top}Z_{l_i1_s}G^{-1}$, $\bar{Z}_{l_i2_s} = G^{-\top}Z_{l_i2_s}G^{-1}$, $RG^{-1} = [R_1\ R_2]$, where

$$\bar{A}_{l_i} = \begin{bmatrix} \bar{A}_{l_i}^{11} & \bar{A}_{l_i}^{12} \\ \bar{A}_{l_i}^{21} & \bar{A}_{l_i}^{22} \end{bmatrix}, \ \bar{A}_{dl_i} = \begin{bmatrix} \bar{A}_{dl_i}^{11} & \bar{A}_{dl_i}^{12} \\ \bar{A}_{dl_i}^{21} & \bar{A}_{dl_i}^{22} \end{bmatrix}, \ \bar{\mathrm{B}}_{l_i} = \begin{bmatrix} \bar{\mathrm{B}}_{l_i}^{11} & \bar{\mathrm{B}}_{l_i}^{12} \\ \bar{\mathrm{B}}_{l_i}^{21} & \bar{\mathrm{B}}_{l_i}^{22} \end{bmatrix},$$

$$\bar{P}_{l_i} = \begin{bmatrix} \bar{P}_{l_i}^{11} & \bar{P}_{l_i}^{12} \\ * & \bar{P}_{l_i}^{22} \end{bmatrix}, \ \bar{Q}_{l_iw_s} = \begin{bmatrix} \bar{Q}_{l_iw_s}^{11} & \bar{Q}_{l_iw_s}^{12} \\ * & \bar{Q}_{l_iw_s}^{22} \end{bmatrix},$$

$$\bar{Z}_{l_i1_s} = \begin{bmatrix} \bar{Z}_{l_i1_s}^{11} & \bar{Z}_{l_i1_s}^{12} \\ * & \bar{Z}_{l_i1_s}^{22} \end{bmatrix}, \ \bar{Z}_{l_i2_s} = \begin{bmatrix} \bar{Z}_{l_i2_s}^{11} & \bar{Z}_{l_i2_s}^{12} \\ * & \bar{Z}_{l_i2_s}^{22} \end{bmatrix}.$$

Using the expressions of $\bar{E}$ and RG^{-1}, it follows from $RE = 0$ and $\mathrm{rank}(R) = n - r$ that $R_1 = 0$ and $\mathrm{rank}(R_2) = n - r$. Pre- and post-multiplying $\Sigma_{l_i} < 0$ in (6) by I_1 and $\mathrm{I}_1^{\top}$, respectively, noting $P_{l_i} > 0$, $Q_{l_iw_s} > 0$, and using the expressions of $\bar{E}$, $\bar{P}_{l_i}$, $\bar{A}_{dl_i}$, RG^{-1}, $\bar{Z}_{l_i1_s}$ and $\bar{Z}_{l_i2_s}$, it is easy to obtain $-(\bar{A}_{l_i}^{22})^{\top}R_2^{\top}Y_{l_i}R_2\bar{A}_{l_i}^{22} < 0$, which implies that $\bar{A}_{l_i}^{22}$ is invertible and then the pair (E, A_{l_i}) is regular and causal [38]. So, by Definition 2, subsystem $\mathcal{S}_{l_i}$ is regular and causal.

(2). Set $G_{l_i} = \begin{bmatrix} I_r & -\bar{A}_{l_i}^{12}(\bar{A}_{l_i}^{22})^{-1} \\ 0 & (\bar{A}_{l_i}^{22})^{-1} \end{bmatrix} G$. It can be verified that $G_{l_i}EH = \text{diag}\{I_r, 0\} \triangleq \tilde{E}$ and $G_{l_i}A_{l_i}H = \begin{bmatrix} \tilde{A}_{l_i}^{11} & 0 \\ \tilde{A}_{l_i}^{21} & I_{n-r} \end{bmatrix} \triangleq \tilde{A}_{l_i}$, where $\tilde{A}_{l_i}^{11} = \bar{A}_{l_i}^{11} - \bar{A}_{l_i}^{12}(\bar{A}_{l_i}^{22})^{-1}\bar{A}_{l_i}^{21}$ and $\tilde{A}_{l_i}^{21} = (\bar{A}_{l_i}^{22})^{-1}\bar{A}_{l_i}^{21}$. Let $G_{l_i}A_{dl_i}H \triangleq \tilde{A}_{dl_i}$ and $G_{l_i}\mathrm{B}_{l_i}H \triangleq \tilde{\mathrm{B}}_{l_i}$. Then, by Definition 4, we have $\mathcal{S}_{l_i} \overset{G_{l_i},H}{\Longleftrightarrow} \tilde{\mathcal{S}}_{l_i}$ (or $(\tilde{E}, \tilde{A}_{l_i}, \tilde{A}_{dl_i}, \tilde{\mathrm{B}}_{l_i})$).

In order to prove Eq. (8), denote $RG_{l_i}^{-1} = [0\ \tilde{R}_2] \triangleq \tilde{R}$, $G_{l_i}^{-\top}P_{l_i}G_{l_i}^{-1} = \begin{bmatrix} \tilde{P}_{l_i}^{11} & \tilde{P}_{l_i}^{12} \\ * & \tilde{P}_{l_i}^{22} \end{bmatrix} \triangleq \tilde{P}_{l_i}$, $G_{l_i}^{-\top}Z_{l_i1_s}G_{l_i}^{-1} = \begin{bmatrix} \tilde{Z}_{l_i1_s}^{11} & \tilde{Z}_{l_i1_s}^{12} \\ * & \tilde{Z}_{l_i1_s}^{22} \end{bmatrix} \triangleq \tilde{Z}_{l_i1_s}$ and $G_{l_i}^{-\top}Z_{l_i2_s}G_{l_i}^{-1} = \begin{bmatrix} \tilde{Z}_{l_i2_s}^{11} & \tilde{Z}_{l_i2_s}^{12} \\ * & \tilde{Z}_{l_i2_s}^{22} \end{bmatrix} \triangleq \tilde{Z}_{l_i2_s}$, $s = 1, 2$. For $\Psi_{l_i} < 0$, using matrix theory and noting $P_{l_i} > 0$, $Q_{l_iw_s} > 0$, $Z_{l_i1_s} > 0$, $Z_{l_i2_s} > 0$, $w = 1, 2, 3$, $s = 1, 2$, we can obtain

$$\Pi_{l_if} = \begin{bmatrix} \Pi_{l_if}^{11} & -A_{l_i}^\top R^\top Y_{l_i} R\mathcal{A}_{l_if} + \rho_{l_i1_f}E^\top Z_{l_i1_f}E \\ * & \Pi_{l_if}^{22} \end{bmatrix} < 0, \quad f = 1, 2, \tag{11}$$

where $\mathcal{A}_{l_i1} = A_{dl_i}$, $\mathcal{A}_{l_i2} = \mathrm{B}_{l_i}$, $\Pi_{l_if}^{11} = -(1-\alpha)E^\top P_{l_i}E - A_{l_i}^\top R^\top Y_{l_i}RA_{l_i} + (1+\tilde{d}_f)Q_{l_i1_f} - \sum_{s=1}^{2}\rho_{l_i1_s}E^\top Z_{l_i1_s}E$ and $\Pi_{l_if}^{22} = -\mathcal{A}_{l_if}^\top R^\top Y_{l_i}R\mathcal{A}_{l_if} - (1-\alpha)^{\bar{d}_f}Q_{l_i1_f} - \rho_{l_i1_f}E^\top Z_{l_i1_f}E - \rho_{l_i2_f}E^\top(Z_{l_i1_f} + 2Z_{l_i2_f})E$. Pre- and post-multiplying $\Pi_{l_i1} < 0$ and $\Pi_{l_i2} < 0$ by diag$\{H^\top, H^\top\}$ and its transpose, respectively, and using the expressions of $\tilde{E}$, $\tilde{R}$, $\tilde{P}_{l_i}$, $\tilde{A}_{l_i}$, $\tilde{A}_{dl_i}$, $\tilde{\mathrm{B}}_{l_i}$, $\bar{Q}_{l_i1_s}$, $\tilde{Z}_{l_i1_s}$ and $\tilde{Z}_{l_i2_s}$, it follows

$$\mathrm{O}_{l_if} = \begin{bmatrix} -\tilde{R}_2^\top Y_{l_i}\tilde{R}_2 + (1+\tilde{d}_f)\bar{Q}_{l_i1_f}^{22} & -\tilde{R}_2^\top Y_{l_i}\tilde{R}_2\tilde{\mathcal{A}}_{l_if}^{22} \\ * & -(\tilde{\mathcal{A}}_{l_if}^{22})^\top\tilde{R}_2^\top Y_{l_i}\tilde{R}_2\tilde{\mathcal{A}}_{l_if}^{22} - (1-\alpha)^{\bar{d}_f}\bar{Q}_{l_i1_f}^{22} \end{bmatrix} < 0,$$

where $f = 1, 2$, $\tilde{\mathcal{A}}_{l_i1}^{22} = \tilde{A}_{dl_i}^{22}$ and $\tilde{\mathcal{A}}_{l_i2}^{22} = \tilde{\mathrm{B}}_{l_i}^{22}$. Pre- and post-multiplying $\mathrm{O}_{l_i1} < 0$ by $[-(\tilde{\mathcal{A}}_{l_i1}^{22})^\top\ I]$ and its transpose, and $\mathrm{O}_{l_i2} < 0$ by $[-(\tilde{\mathcal{A}}_{l_i2}^{22})^\top\ I]$ and its transpose, respectively, lead to

$$(\tilde{\mathcal{A}}_{l_if}^{22})^\top\bar{Q}_{l_i1_f}^{22}\tilde{\mathcal{A}}_{l_if}^{22} - \eta_f\bar{Q}_{l_i1_f}^{22} < 0, \ f = 1, 2, \tag{12}$$

where $\eta_f = (1-\alpha)^{\bar{d}_f}/(1+\tilde{d}_f)$. Thus, by Lemma 1, there exist constants $\chi_{l_if} = \sqrt{\frac{\lambda_{\max}(\bar{Q}_{l_i1_f}^{22})}{\lambda_{\min}(\bar{Q}_{l_i1_f}^{22})}}$ and $\delta_f = -\ln\sqrt{\eta_f}$, $f = 1, 2$, such that $\|(\tilde{\mathcal{A}}_{l_if}^{22})^\ell\| \le \chi_{l_if}e^{-\delta_f\ell}$, $\forall\ell \in \mathbb{Z}^+$, i.e. Eq. (8) holds.

(3). Let $x_s^\top(k) = \begin{bmatrix} x^\top(k - d_s(k)) & x^\top(k - \underline{d}_s) & x^\top(k - \bar{d}_s) \end{bmatrix}$, $s = 1, 2$, and $x^\top(k) = \begin{bmatrix} x^\top(k) & x_1^\top(k) & x_2^\top(k) \end{bmatrix}$. Define $\triangle V_{l_i}(k) = V_{l_i}(k+1) - (1-\alpha)V_{l_i}(k)$. Then, along the system trajectory of $\mathcal{S}_{l_i}$, $\triangle V_{l_i}(k)$ is bounded by

$$\begin{aligned} &\triangle V_{l_i}(k) \\ &\le x^\top(k+1)E^\top P_{l_i}Ex(k+1) - (1-\alpha)x^\top(k)E^\top P_{l_i}Ex(k) + \sum_{s=1}^{2}\Big\{x^\top(k) \\ &\quad \times (1+\tilde{d}_s)Q_{l_i1_s}x(k) - x^\top(k-d_s(k))(1-\alpha)^{d_s(k)}Q_{l_i1_s}x(k-d_s(k)) \\ &\quad + x^\top(k)(Q_{l_i2_s} + Q_{l_i3_s})x(k) - x^\top(k-\underline{d}_s)(1-\alpha)^{\underline{d}_s}Q_{l_i2_s}x(k-\underline{d}_s) \\ &\quad - x^\top(k-\bar{d}_s)(1-\alpha)^{\bar{d}_s}Q_{l_i3_s}x(k-\bar{d}_s) + y^\top(k)E^\top(\bar{d}_sZ_{l_i1_s} + \tilde{d}_sZ_{l_i2_s})Ey(k) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=k-d_s(k)}^{k-1} y^\top(j)E^\top(1-\alpha)^{k-j} Z_{l_i 1_s} E y(j) - \sum_{j=k-\bar{d}_s}^{k-d_s(k)-1} y^\top(j)E^\top(1-\alpha)^{k-j} \\
&\times (Z_{l_i 1_s} + Z_{l_i 2_s}) E y(j) - \sum_{j=k-d_s(k)}^{k-\underline{d}_s-1} y^\top(j)E^\top(1-\alpha)^{k-j} Z_{l_i 2_s} E y(j) \Bigg\}. \qquad (13)
\end{aligned}
$$

By Lemma 2, the following inequalities hold:

$$
\begin{aligned}
&- \sum_{j=k-d_s(k)}^{k-1} y^\top(j)E^\top(1-\alpha)^{k-j} Z_{l_i 1_s} E y(j) \\
&\le -\frac{\alpha(1-\alpha)^{d_s(k)}}{1-(1-\alpha)^{d_s(k)}} \left(\sum_{j=k-d_s(k)}^{k-1} E y(j) \right)^\top Z_{l_i 1_s} \left(\sum_{j=k-d_s(k)}^{k-h} E y(j) \right) \\
&\le -\rho_{l_i 1_s} [x(k) - x(k-d_s(k))]^\top E^\top Z_{l_i 1_s} E [x(k) - x(k-d_s(k))], \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=k-\bar{d}_s}^{k-d_s(k)-1} y^\top(j)E^\top(1-\alpha)^{k-j} (Z_{l_i 1_s} + Z_{l_i 2_s}) E y(j) \\
&\le -\rho_{l_i 2_s} [x(k-d_s(k)) - x(k-\bar{d}_s)]^\top E^\top (Z_{l_i 1_s} + Z_{l_i 2_s}) E \\
&\quad \times [x(k-d_s(k)) - x(k-\bar{d}_s)], \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=k-d_s(k)}^{k-\underline{d}_s-1} y^\top(j)E^\top(1-\alpha)^{k-j} Z_{l_i 2_s} E y(j) \\
&\le -\rho_{l_i 2_s} [x(k-\underline{d}_s) - x(k-d_s(k))]^\top E^\top Z_{l_i 2_s} E [x(k-\underline{d}_s) - x(k-d_s(k))]. \qquad (16)
\end{aligned}
$$

From $RE = 0$, it follows that for any symmetric matrix Y_{l_i} with appropriate dimension

$$
0 = -x^\top(k+1)E^\top R^\top Y_{l_i} R E x(k+1). \qquad (17)
$$

Substituting Eqs. (14)–(16) into Eq. (13) and adding the right side in Eq. (17) yield $\triangle V_{l_i}(k) \le x^\top(k)(\Sigma_{l_i} + (\widehat{A}_{l_i} - \mathrm{I}_1 E)^\top \widehat{Z}_{l_i} (\widehat{A}_{l_i} - \mathrm{I}_1 E))x(k)$. By using the Schur complement to $\Psi_{l_i} < 0$ in Eq. (6), it follows that $\Sigma_{l_i} + (\widehat{A}_{l_i} - \mathrm{I}_1 E)^\top \widehat{Z}_{l_i} (\widehat{A}_{l_i} - \mathrm{I}_1 E) < 0$, which means $\triangle V_{l_i}(k) < 0$, i.e. $V_{l_i}(k+1) \le (1-\alpha) V_{l_i}(k)$. Then, we have $V_{l_i}(k) \le (1-\alpha)^{k-k_i-\tau_s(k_i)} V_{l_i}(k_i + \tau_s(k_i))$, $\forall k \in [k_i + \tau_s(k_i), k_{i+1})$. In addition, noting $x^\top(k)E^\top P_{l_i} E x(k) = \tilde{x}_1^\top(k) \tilde{P}_{l_i}^{11} \tilde{x}_1(k)$, we have from Eq. (5) that $V_{l_i}(k) \ge x^\top(k)E^\top P_{l_i} E x(k) \ge a_{l_i} \|\tilde{x}_1(k)\|^2$, where $a_{l_i} = \lambda_{\min}(\tilde{P}_{l_i}^{11})$. Therefore, Eq. (10) holds.

For subsystem $\mathcal{S}_{l_i l_{i-1}}$, the regularity and causality, Eqs. (7) and (9) can be proved by using similar proof procedure of subsystem $\mathcal{S}_{l_i}$, and thus they are omitted. This completes the proof. □

Remark 5. Note that the restricted equivalent subsystems $\tilde{\mathcal{S}}_{l_i l_{i-1}}$ and $\tilde{\mathcal{S}}_{l_i}$ in Proposition 1 are not unique since they depend on the matrices G and H. A systematic way to find G and H is to use the singular value decomposition on the matrix E [25].

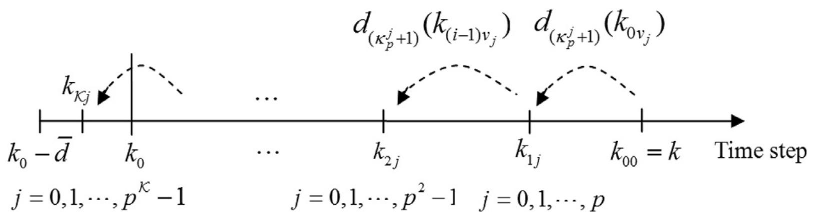

Fig. 2. An illustrative diagram of time instant k with respect to the past instants.

Remark 6. Since $V_{l_i l_{i-1}}(k)$ and $V_{l_i}(k)$ are bounded functionals, there exist sufficiently large scalars $b_{l_i l_{i-1}}$ and b_{l_i} such that $V_{l_i l_{i-1}}(k_i) \le b_{l_i l_{i-1}} \|x(k_i)\|_{\bar{d}}^2$ and $V_{l_i}(k_i + \tau_s(k_i)) \le b_{l_i} \|x(k_i + \tau_s(k_i))\|_{\bar{d}}^2$. Thus, by Eqs. (9) and (10), we have $\|\tilde{x}_1(k)\| \le \sqrt{b_{l_i l_{i-1}}/a_{l_i l_{i-1}}}(1+\beta)^{(k-k_i)/2}\|x(k_i)\|_{\bar{d}}$, $\forall k \in [k_i, k_i + \tau_s(k_i))$, and $\|\tilde{x}_1(k)\| \le \sqrt{b_{l_i}/a_{l_i}}(1-\alpha)^{(k-k_i-\tau_s(k_i))/2}\|x(k_i+\tau_s(k_i))\|_{\bar{d}}$, $\forall k \in [k_i + \tau_s(k_i), k_{i+1})$, which imply that the responses of slow state variables of $\tilde{\mathcal{S}}_{l_i l_{i-1}}$ and $\tilde{\mathcal{S}}_{l_i}$ are exponentially divergent and exponentially convergent, respectively.

In what follows, we will give exponential estimates for solutions of fast state variables of the equivalent subsystems $\tilde{\mathcal{S}}_{l_i l_{i-1}}$ and $\tilde{\mathcal{S}}_{l_i}$. For the sake of discussion, we consider a general discrete-time singular system with p ($p \ge 1$) state delays described by the following dynamics decomposition form:

$$\begin{cases} x_1(k+1) = A^{11}x_1(k) + \sum_{s=1}^{p}\begin{bmatrix} A_{ds}^{11} & A_{ds}^{12}\end{bmatrix}x(k-d_s(k)), \\ 0 = A^{21}x_1(k) + x_2(k) + \sum_{s=1}^{p}\begin{bmatrix} A_{ds}^{21} & A_{ds}^{22}\end{bmatrix}x(k-d_s(k)), \\ x_1(k) = \psi_1(k),\ x_2(k) = \psi_2(k),\ k = k_0 - \bar{d}, \ldots, k_0, \end{cases} \tag{18}$$

where $x(k) = \begin{bmatrix} x_1^\top(k) & x_2^\top(k)\end{bmatrix}^\top \in \mathbb{R}^n$ is the state vector with $x_1(k) \in \mathbb{R}^r$ and $x_2(k) \in \mathbb{R}^{n-r}$, $d_s(k)$, $s = 1, \ldots, p$, are state delays satisfying $0 < \underline{d}_s \le d_s(k) \le \bar{d}_s$, where $\underline{d}_s$ and $\bar{d}_s$ are known positive integers. $\psi(k) = \begin{bmatrix}\psi_1^\top(k) & \psi_2^\top(k)\end{bmatrix}^\top \in \mathbb{R}^n$ is the compatible initial condition function. k_0 is the initial time step. $\bar{d}$ is defined as $\bar{d} = \max\{\bar{d}_1, \ldots, \bar{d}_p\}$. Also, let $\underline{d} = \min\{\underline{d}_1, \ldots, \underline{d}_p\}$ and $\tilde{d}_s = \bar{d}_s - \underline{d}_s$, $s = 1, \ldots, p$.

In order to describe the dependency of fast variables $x_2(k)$ on the past values of $x(k)$, inspired by [47], we define

$$k_{00} = k,\ \ k_{ij} = k_{(i-1)v_j} - d_{(\kappa_p^j+1)}(k_{(i-1)v_j}), \tag{19}$$

$$\hat{A}_{00} = I,\ \ \hat{A}_{ij} = \hat{A}_{(i-1)v_j} \times \left(-A^{22}_{d(\kappa_p^j+1)}\right), \tag{20}$$

$$\mathcal{O}_{k_0} = \left\{k_{ij} | k_{ij} \in (k_0 - \bar{d}, k_0], k_{(i-1)j} \notin (k_0 - \bar{d}, k_0]\right\}, \tag{21}$$

where v_j is the greatest integer less than or equal to $\frac{j}{p}$, and κ_p^j is the remainder of the integer division $\frac{j}{p}$ ($\kappa_p^j = j$ if $j < p$). v_j and κ_p^j are undefined if $k_{(i-1)v_j} \in \mathcal{O}_{k_0}$.

Proposition 2. *For the singular multiple delays system (18),*

(1) *there exists a limited positive integer* $\mathcal{K}$ *such that* $k_{\mathcal{K}j} \in \mathcal{O}_{k_0}$ *(see Fig. 2), and* $x_2(k)$ *can be computed as follows:*

$$x_2(k) = \sum_{i=1}^{\mathcal{K}} \sum_{\substack{j=0 \\ k_{ij}\in\mathcal{O}_{k_0}}}^{p^i-1} \left\{\hat{A}_{ij}x_2(k_{ij})\right\} - A^{21} \sum_{i=0}^{\mathcal{K}-1} \sum_{\substack{j=0 \\ k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1} \hat{A}_{ij}x_1(k_{ij})$$
$$- \sum_{i=0}^{\mathcal{K}-1} \sum_{\substack{j=0 \\ k_{iv_j}\notin\mathcal{O}_{k_0}}}^{p^{i+1}-1} \hat{A}_{iv_j}A^{21}_{d(\kappa_p^j+1)}x_1(k_{(i+1)j}). \tag{22}$$

(2) *under the following exponential convergence condition of* $x_1(k)$*, i.e.*

$$\|x_1(k)\| \le \epsilon\gamma^{k-k_0}\|\psi(k_0)\|_{\bar{d}}, \ \ \forall k \ge k_0, \tag{23}$$

where $\epsilon > 0$ *and* $0 < \gamma < 1$,

(2.1) *if for every* $s \in \{1, \dots, p\}$, $\|A^{22}_{ds}\| \le \chi_s\underline{\vartheta}^{\frac{\tilde{d}_s}{2}}/(1+\tilde{d}_s)^{\frac{1}{2}}$ *holds, where* $\chi_s > 1$ *and* $0 < \underline{\vartheta} < \gamma^2$, *then an upper bound for* $\|x_2(k)\|$ *can be obtained by*

$$\|x_2(k)\| \le \underline{\nu}_1\underline{\vartheta}^{\frac{k-k_0}{2}}\|\psi_2(k_0)\|_{\bar{d}} + \underline{\nu}_2\gamma^{k-k_0}\|\psi(k_0)\|_{\bar{d}}, \tag{24}$$

where $\underline{\nu}_1 = \Sigma_{\underline{\vartheta}}(1-\Sigma^{\mathcal{K}}_{\underline{\vartheta}})/(1-\Sigma_{\underline{\vartheta}})$, $\underline{\nu}_2 = \epsilon(\|A^{21}\| + p\gamma^{-\bar{d}}\|A^{21}_d\|)(1-\Sigma^{\mathcal{K}}_{\underline{\vartheta}})/(1-\Sigma_{\underline{\vartheta}})$, $\Sigma_{\underline{\vartheta}} = \sum_{s=1}^{p}[\chi_s/(1+\tilde{d}_s)^{\frac{1}{2}}]$ *and* $\|A^{21}_d\| = \max_{s=1,\dots,p}\left\{\|A^{21}_{ds}\|\right\}$.

(2.2) *if for every* $s \in \{1, \dots, p\}$, $\|A^{22}_{ds}\| \le \chi_s\bar{\vartheta}^{\frac{d_s}{2}}/(1+\tilde{d}_s)^{\frac{1}{2}}$ *holds, where* $\chi_s > 1$ *and* $\bar{\vartheta} > 1$, *then an upper bound for* $\|x_2(k)\|$ *can be estimated as*

$$\|x_2(k)\| \le \bar{\nu}_1\bar{\vartheta}^{\frac{k-k_0}{2}}\|\psi_2(k_0)\|_{\bar{d}} + \bar{\nu}_2\gamma^{k-k_0}\|\psi(k_0)\|_{\bar{d}}, \tag{25}$$

where $\bar{\nu}_1 = \Sigma_{\bar{\vartheta}}(1-\Sigma^{\mathcal{K}}_{\bar{\vartheta}})/(1-\Sigma_{\bar{\vartheta}})$, $\bar{\nu}_2 = \epsilon(\|A^{21}\| + p\gamma^{-\bar{d}}\|A^{21}_d\|)(1-\Sigma^{\mathcal{K}}_{\bar{\vartheta}})/(1-\Sigma_{\bar{\vartheta}})$, $\Sigma_{\bar{\vartheta}} = \sum_{s=1}^{p}[\chi_s\bar{\vartheta}^{\frac{\bar{d}_s}{2}}/\gamma^{\bar{d}_s}(1+\tilde{d}_s)^{\frac{1}{2}}]$ *and* $\|A^{21}_d\|$ *is as in (2.1).*

Proof. The proof is given in Appendix. □

Remark 7. Note that discrete singular delay systems are delay difference equations coupled with delay algebraic equations. It is difficult to obtain exponential estimates for solutions of such systems, especially for fast state variables. For stable discrete singular systems with single state delay, a function inequality was proposed in [29] (see Lemma 3 therein) to establish exponential estimates for fast state variables of the systems. However, it can neither be applied to the system with multiple state delays nor be used for the system with system matrices satisfying the norm bound conditions as stated in (2.2) in Proposition 2. For discrete-time singular systems with $p(p \ge 1)$ state delays, different from [29], Proposition 2 gives an analytic equation to solve fast state variables. Moreover, when the slow state variables are upper-bounded by an exponentially convergent function, it presents exponential estimates for the fast state variables with respect to two kinds of norm bound conditions of system matrices A^{22}_{ds}, $s = 1, \dots, p$. It should be pointed out that the results in Proposition 2, even for the case of systems with single state delay, have not been reported in the literature.

4. Design of the exponentially stabilizing control law

In this section, by using a combination of the piecewise Lyapunov function approach and the ADT scheme, together with the analysis in the previous section, we will give a sufficient condition for the existence of an exponentially stabilizing state feedback control law (2) for the system (1).

Theorem 1. *Consider the switched singular system (1) and let* $\varepsilon_{l_i1}, \varepsilon_{l_i2}, \dots, \varepsilon_{l_i9}$, $\forall l_i \in \mathcal{I}$, $\mu \geq 1$, $0<\alpha<1$, $\beta>0$ *and* $0<\underline{d}_s \leq \bar{d}_s$, $s=1,2$, *be given constants with* $(1+\beta)^{\underline{d}_s}/(1+\tilde{d}_s)<1$. *Suppose that there exist matrices* $X_{l_i}>0$, $Q_{l_iw_s}>0$, $Z_{l_i1_s}>0$, $Z_{l_i2_s}>0$, $Y_{l_i}=Y_{l_i}^\top$, K_{l_i}, $\forall l_i \in \mathcal{I}$, $w=1,2,3$, $s=1,2$, $X_{l_il_{i-1}}>0$, $Q_{l_il_{i-1}w_s}>0$, $Z_{l_il_{i-1}1_s}>0$, $Z_{l_il_{i-1}2_s}>0$, $Y_{l_il_{i-1}}=Y_{l_il_{i-1}}^\top$, $\forall (l_i,l_{i-1}) \in \mathcal{I}\times\mathcal{I}$, $l_i \neq l_{i-1}$, $w=1,2,3$, $s=1,2$, *and scalars* ϱ_{l_i}, $\forall l_i \in \mathcal{I}$, $\varrho_{l_il_{i-1}}$, $\forall (l_i,l_{i-1}) \in \mathcal{I}\times\mathcal{I}$, $l_i \neq l_{i-1}$, *such that*

$$\Gamma_{l_il_{i-1}} = \begin{bmatrix} \Lambda_{l_il_{i-1}} & \Xi_{l_il_{i-1}} \\ * & \Theta_{l_il_{i-1}} \end{bmatrix} < 0, \quad \forall (l_i,l_{i-1}) \in \mathcal{I}\times\mathcal{I},\ l_i \neq l_{i-1}, \tag{26}$$

$$\Gamma_{l_i} = \begin{bmatrix} \Lambda_{l_i} & \Xi_{l_i} \\ * & \Theta_{l_i} \end{bmatrix} < 0, \quad \forall l_i \in \mathcal{I}, \tag{27}$$

where

$$\Lambda_{\tilde{\sigma}} = \begin{bmatrix} \Lambda_{\tilde{\sigma}11} & \Lambda_{\tilde{\sigma}12} & 0 & 0 & \Lambda_{\tilde{\sigma}15} & 0 & 0 \\ * & \Lambda_{\tilde{\sigma}22} & \rho_{\tilde{\sigma}2_1}E^\top Z_{\tilde{\sigma}2_1}E & \Lambda_{\tilde{\sigma}24} & \Lambda_{\tilde{\sigma}25} & 0 & 0 \\ * & * & \Lambda_{\tilde{\sigma}33} & 0 & 0 & 0 & 0 \\ * & * & * & \Lambda_{\tilde{\sigma}44} & 0 & 0 & 0 \\ * & * & * & * & \Lambda_{\tilde{\sigma}55} & \rho_{\tilde{\sigma}2_2}E^\top Z_{\tilde{\sigma}2_2}E & \Lambda_{\tilde{\sigma}57} \\ * & * & * & * & * & \Lambda_{\tilde{\sigma}66} & 0 \\ * & * & * & * & * & * & \Lambda_{\tilde{\sigma}77} \end{bmatrix},$$

$$\begin{aligned} \Xi_{\tilde{\sigma}} = \big[& \widehat{A}_{\tilde{\sigma}}^\top R^\top \quad \mathrm{s}_{l_i} \quad \widehat{A}_{\tilde{\sigma}}^\top \quad \bar{d}_1(\widehat{A}_{\tilde{\sigma}} - \mathrm{I}_1E)^\top \quad \tilde{d}_1(\widehat{A}_{\tilde{\sigma}} - \mathrm{I}_1E)^\top \quad \bar{d}_2(\widehat{A}_{\tilde{\sigma}} - \mathrm{I}_1E)^\top \\ & \tilde{d}_2(\widehat{A}_{\tilde{\sigma}} - \mathrm{I}_1E)^\top \ \big], \\ \Theta_{\tilde{\sigma}} = \mathrm{diag}\big\{ & \varepsilon_{l_i5}^2\varrho_{\tilde{\sigma}}I - 2\varepsilon_{l_i5}I,\ -\varrho_{\tilde{\sigma}}I - Y_{\tilde{\sigma}},\ -X_{\tilde{\sigma}},\ \bar{d}_1\varepsilon_{l_i6}(\varepsilon_{l_i6}Z_{\tilde{\sigma}1_1} - 2I), \\ & \tilde{d}_1\varepsilon_{l_i7}(\varepsilon_{l_i7}Z_{\tilde{\sigma}2_1} - 2I),\ \bar{d}_2\varepsilon_{l_i8}(\varepsilon_{l_i8}Z_{\tilde{\sigma}1_2} - 2I),\ \tilde{d}_2\varepsilon_{l_i9}(\varepsilon_{l_i9}Z_{\tilde{\sigma}2_2} - 2I)\big\}, \\ \tilde{\sigma} = {} & l_{i-1}l_i, l_i, \end{aligned}$$

with

$$\begin{aligned} \Lambda_{\tilde{\sigma}11} &= \varepsilon_{l_i4}(1+\hbar_{\tilde{\sigma}})(E+E^\top+\varepsilon_{l_i4}X_{\tilde{\sigma}}) + \varepsilon_{l_i1}\mathrm{Sym}(RA_{l_i}), \\ &\quad + \sum_{s=1}^{2}[(1+\tilde{d}_s)Q_{\tilde{\sigma}1_s} + Q_{\tilde{\sigma}2_s} + Q_{\tilde{\sigma}3_s} - \rho_{\tilde{\sigma}1_s}E^\top Z_{\tilde{\sigma}1_s}E], \\ \Lambda_{\tilde{\sigma}12} &= \varepsilon_{l_i2}A_{l_i}^\top R^\top + \varepsilon_{l_i1}RA_{dl_i} + \rho_{\tilde{\sigma}1_1}E^\top Z_{\tilde{\sigma}1_1}E, \\ \Lambda_{\tilde{\sigma}15} &= \varepsilon_{l_i3}A_{l_i}^\top R^\top + \varepsilon_{l_i1}R\mathrm{B}_{\tilde{\sigma}} + \rho_{\tilde{\sigma}1_2}E^\top Z_{\tilde{\sigma}1_2}E, \\ \Lambda_{\tilde{\sigma}22} &= \varepsilon_{l_i2}\mathrm{Sym}(RA_{dl_i}) - (1+\hbar_{\tilde{\sigma}})^{\mathrm{d}_{\tilde{\sigma}1}}Q_{\tilde{\sigma}1_1} - \rho_{\tilde{\sigma}1_1}E^\top Z_{\tilde{\sigma}1_1}E \\ &\quad - \rho_{\tilde{\sigma}2_1}E^\top(Z_{\tilde{\sigma}1_1} + 2Z_{\tilde{\sigma}2_1})E, \\ \Lambda_{\tilde{\sigma}24} &= \rho_{\tilde{\sigma}2_1}E^\top(Z_{\tilde{\sigma}1_1} + Z_{\tilde{\sigma}2_1})E, \quad \Lambda_{\tilde{\sigma}25} = \varepsilon_{l_i3}A_{dl_i}^\top R^\top + \varepsilon_{l_i2}R\mathrm{B}_{\tilde{\sigma}}, \end{aligned}$$

$$\Lambda_{\tilde{\sigma}33} = -(1+\hbar_{\tilde{\sigma}})^{\underline{d}_1} Q_{\tilde{\sigma}2_1} - \rho_{\tilde{\sigma}2_1} E^\top Z_{\tilde{\sigma}2_1} E,$$
$$\Lambda_{\tilde{\sigma}44} = -(1+\hbar_{\tilde{\sigma}})^{\bar{d}_1} Q_{\tilde{\sigma}3_1} - \rho_{\tilde{\sigma}2_1} E^\top (Z_{\tilde{\sigma}1_1} + Z_{\tilde{\sigma}2_1}) E,$$
$$\Lambda_{\tilde{\sigma}55} = \varepsilon_{l_i3}\mathrm{Sym}(R\mathrm{B}_{\tilde{\sigma}}) - (1-\alpha)^{\mathrm{d}_{\tilde{\sigma}2}} Q_{\tilde{\sigma}1_2} - \rho_{\tilde{\sigma}1_2} E^\top Z_{\tilde{\sigma}1_2} E$$
$$- \rho_{\tilde{\sigma}2_2} E^\top (Z_{\tilde{\sigma}1_2} + 2Z_{\tilde{\sigma}2_2}) E,$$
$$\Lambda_{\tilde{\sigma}57} = \rho_{\tilde{\sigma}2_2} E^\top (Z_{\tilde{\sigma}1_2} + Z_{\tilde{\sigma}2_2}) E, \quad \Lambda_{\tilde{\sigma}66} = -(1+\hbar_{\tilde{\sigma}})^{\underline{d}_2} Q_{\tilde{\sigma}2_2} - \rho_{\tilde{\sigma}2_2} E^\top Z_{\tilde{\sigma}2_2} E,$$
$$\Lambda_{\tilde{\sigma}77} = -(1+\hbar_{\tilde{\sigma}})^{\bar{d}_2} Q_{\tilde{\sigma}3_2} - \rho_{\tilde{\sigma}2_2} E^\top (Z_{\tilde{\sigma}1_2} + Z_{\tilde{\sigma}2_2}) E,$$
$$\mathrm{S}_{l_i} = [\varepsilon_{l_i1} I \quad \varepsilon_{l_i2} I \quad 0 \quad 0 \quad \varepsilon_{l_i3} I \quad 0 \quad 0]^\top,$$

$\widehat{A}_{\tilde{\sigma}}$, I_1, $\mathrm{d}_{\tilde{\sigma}1}$, $\mathrm{d}_{\tilde{\sigma}2}$, $\rho_{\tilde{\sigma}1_s}$, $\rho_{\tilde{\sigma}2_s}$, $\tilde{\sigma} = l_i l_{i-1}, l_i$, $s = 1, 2$, *and* R *are as in Proposition 1.*

Then, under the state feedback control (2) with a maximal switching delay $\bar{\tau}_s$, the resulting closed-loop system (3) is λ-exponentially admissible with $\sqrt{1-\alpha} < \lambda < 1$ for any switching signal $\sigma \in S_{\mathrm{ave}}[\tau_a, N_0]$ satisfying the following conditions:

$$Cond_1 : (\bar{\tau}_s + \bar{d} - 1)\ln c + 2\ln\mu \le \tau_a \ln\frac{\lambda^2}{1-\alpha}, \tag{28}$$

$$Cond_2 : N_0 \le \frac{\ln\varsigma}{(\bar{\tau}_s + \bar{d} - 1)\ln c + 2\ln\mu}, \tag{29}$$

where $\varsigma > 0$, $c = (1+\beta)/(1-\alpha)$, and μ satisfies

$$X_{l_i l_{i-1}} \le \mu X_{l_i},\ Q_{l_i w_s} \le \mu Q_{l_i l_{i-1} w_s},\ Z_{l_i 1_s} \le \mu Z_{l_i l_{i-1} 1_s},\ Z_{l_i 2_s} \le \mu Z_{l_i l_{i-1} 2_s},$$
$$w = 1,2,3,\ s = 1,2,\quad \forall (l_i, l_{i-1}) \in \mathcal{I}\times\mathcal{I},\ l_i \ne l_{i-1}, \tag{30}$$

$$X_{l_{i-1}} \le \mu X_{l_i l_{i-1}},\ Q_{l_i l_{i-1} w_s} \le \mu Q_{l_{i-1} w_s},\ Z_{l_i l_{i-1} 1_s} \le \mu Z_{l_{i-1} 1_s},\ Z_{l_i l_{i-1} 2_s} \le \mu Z_{l_{i-1} 2_s},$$
$$w = 1,2,3,\ s = 1,2,\quad \forall (l_i, l_{i-1}) \in \mathcal{I}\times\mathcal{I},\ l_i \ne l_{i-1}. \tag{31}$$

Proof. The proof is divided into four steps. In Step 1, on the basis of Proposition 1, we first prove that the closed-loop system (3) is regular and causal under conditions (26) and (27) and then give corresponding restricted equivalent closed-loop system. In Step 2, slow variables of the equivalent closed-loop system is proved to be exponentially convergent with ADT switching signals satisfying Eqs. (28) and (29). In Step 3, by Proposition 2, we prove that fast variables poss the same decay rate as slow ones. Exponential stability of the closed-loop system (3) is finally shown in Step 4.

Step 1. (regularity and causality of the closed-loop system (3)). According to Lemma 3, for every $l_i \in \mathcal{I}$ and any matrix $Y_{l_i} = Y_{l_i}^\top$, there exists a scalar $\varrho_{l_i} > 0$ such that $\varrho_{l_i} I + Y_{l_i} > 0$ and $-\widehat{A}_{l_i}^\top R^\top Y_{l_i} R \widehat{A}_{l_i} \le \mathrm{Sym}(\widehat{A}_{l_i}^\top R^\top \mathrm{S}_{l_i}) + \mathrm{S}_{l_i}^\top (\varrho_{l_i} I + Y_{l_i})^{-1} \mathrm{S}_{l_i} + \varrho_{l_i} \widehat{A}_{l_i}^\top R^\top R \widehat{A}_{l_i}$. By Lemma 3 with $\rho = 0$, $-E^\top P_{l_i} E \le \varepsilon_{l_i4}\mathrm{Sym}(E) + \varepsilon_{l_i4}^2 P_{l_i}^{-1}$ holds for an arbitrary scalar ε_{l_i4}. Using these relations and writing $P_{l_i}^{-1} \triangleq X_{l_i}$, Ψ_{l_i} in (6) satisfies

$$\Psi_{l_i} \le \begin{bmatrix} \mathrm{M}_{l_i} & (\widehat{A}_{l_i} - \mathrm{I}_1 E)^\top \widehat{Z}_{l_i} \\ * & -\widehat{Z}_{l_i} \end{bmatrix} \triangleq \tilde{\Psi}_{l_i}, \tag{32}$$

where $\mathrm{M}_{l_i} = \Lambda_{l_i} + \varrho_{l_i}\widehat{A}_{l_i}^\top R^\top R\widehat{A}_{l_i} + \mathrm{S}_{l_i}^\top(\varrho_{l_i}I + Y_{l_i})^{-1}\mathrm{S}_{l_i} + \widehat{A}_{l_i}^\top X_{l_i}^{-1}\widehat{A}_{l_i}$, and Λ_{l_i} is defined in Eq. (27). Applying the Schur complement, $\tilde{\Psi}_{l_i} < 0$ is equivalent to

$$\begin{bmatrix} \Lambda_{l_i} & \Xi_{l_i} \\ * & \begin{array}{c}\mathrm{diag}\{-\varrho_{l_i}^{-1}I, -\varrho_{l_i}I - Y_{l_i}, -X_{l_i}, -\bar{d}_1 Z_{l_i 1_1}^{-1}, \\ -\tilde{d}_1 Z_{l_i 2_1}^{-1}, -\bar{d}_2 Z_{l_i 1_2}^{-1}, -\tilde{d}_2 Z_{l_i 2_2}^{-1}\}\end{array} \end{bmatrix}. \tag{33}$$

By Lemma 3 with $\rho = 0$, $-\varrho_{l_i}^{-1}I \le \varepsilon_{l_i5}^2\varrho_{l_i}I - 2\varepsilon_{l_i5}I$, $-Z_{l_i1_1}^{-1} \le \varepsilon_{l_i6}^2 Z_{l_i1_1} - 2\varepsilon_{l_i6}I$, $-Z_{l_i2_1}^{-1} \le \varepsilon_{l_i7}^2 Z_{l_i2_1} - 2\varepsilon_{l_i7}I$, $-Z_{l_i1_2}^{-1} \le \varepsilon_{l_i8}^2 Z_{l_i1_2} - 2\varepsilon_{l_i8}I$ and $-Z_{l_i2_2}^{-1} \le \varepsilon_{l_i9}^2 Z_{l_i2_2} - 2\varepsilon_{l_i9}I$ hold for arbitrary scalars $\varepsilon_{l_i5}, \varepsilon_{l_i6}, \ldots, \varepsilon_{l_i9}$. Using these inequalities, it can be obtained that if (27) is satisfied, then (33) holds, $\forall l_i \in \mathcal{I}$, and thus $\tilde{\Psi}_{l_i} < 0$, $\forall l_i \in \mathcal{I}$. By (32), we then have $\Psi_{l_i} < 0$, $\forall l_i \in \mathcal{I}$. Similarly, it can be proved that Eq. (26) guarantees that $\Psi_{l_i l_{i-1}} < 0$ holds, $\forall (l_i, l_{i-1}) \in \mathcal{I} \times \mathcal{I}$ and $l_i \neq l_{i-1}$. Thus, by Proposition 1 (1), every subsystem of the closed-loop system (3) is regular and causal, and then system (3) is regular and causal from Definition 3. Moreover, according to Proposition 1 (2), for every $l_i \in \mathcal{I}$ there exist invertible matrices G_{l_i} and H such that $\mathcal{S}_{l_i l_{i-1}}$ and $\mathcal{S}_{l_i}$ are r.s.e. to $\tilde{\mathcal{S}}_{l_i l_{i-1}}$ and $\tilde{\mathcal{S}}_{l_i}$, respectively, i.e. the closed-loop system (3) is r.s.e. to

$$\tilde{E}\tilde{x}(k+1) = \begin{cases} \tilde{A}_{l_i}\tilde{x}(k) + \tilde{A}_{dl_i}\tilde{x}(k - d_1(k)) + \tilde{\mathrm{B}}_{l_i l_{i-1}}\tilde{x}(k - d_2(k)), \\ \forall k \in [k_i, k_i + \tau_{\mathrm{s}}(k_i)), \ i \in \mathbb{Z}^+, \\ \tilde{A}_{l_i}\tilde{x}(k) + \tilde{A}_{dl_i}\tilde{x}(k - d_1(k)) + \tilde{\mathrm{B}}_{l_i}\tilde{x}(k - d_2(k)), \\ \forall k \in [k_i + \tau_{\mathrm{s}}(k_i), k_{i+1}), \ i \in \mathbb{Z}^+. \end{cases} \tag{34}$$

Step 2. (exponential convergence of slow variables of system (34)). In view of Eq. (5), choose the following piecewise L–K functional for the closed-loop system (3):

$$V_{\tilde{\sigma}(k)}(k) = \begin{cases} V_{l_i l_{i-1}}(k), \ \forall k \in [k_i, k_i + \tau_{\mathrm{s}}(k_i)), \ i \in \mathbb{Z}^+, \\ V_{l_i}(k), \ \forall k \in [k_i + \tau_{\mathrm{s}}(k_i), k_{i+1}), \ i \in \mathbb{Z}^+. \end{cases} \tag{35}$$

From Eqs. (9), (10) and (35), it follows that

$$V_{\tilde{\sigma}(k)}(k) \le \begin{cases} (1+\beta)^{k-k_i} V_{l_i l_{i-1}}(k_i), \ \forall k \in [k_i, k_i + \tau_{\mathrm{s}}(k_i)), \ i \in \mathbb{Z}^+, \\ (1-\alpha)^{k-k_i-\tau_{\mathrm{s}}(k_i)} V_{l_i}(k_i + \tau_{\mathrm{s}}(k_i)), \ \forall k \in [k_i + \tau_{\mathrm{s}}(k_i), k_{i+1}), \ i \in \mathbb{Z}^+. \end{cases} \tag{36}$$

In addition, by Eqs. (5), (30) and (31), it is easy to be verified that

$$\begin{aligned} &V_{l_i}(k_i + \tau_{\mathrm{s}}(k_i)) \le c^{\bar{d}-1}\mu V_{l_i l_{i-1}}(k_i + \tau_{\mathrm{s}}(k_i)), \\ &V_{l_i l_{i-1}}(k_i) \le \mu V_{l_{i-1}}(k_i), \ \forall (l_{i-1}, l_i) \in \mathcal{I} \times \mathcal{I}, \ l_{i-1} \neq l_i, \end{aligned} \tag{37}$$

where $c = (1+\beta)/(1-\alpha)$.

For any $k \in [k_i + \tau_{\mathrm{s}}(k_i), k_{i+1})$, $i \in \mathbb{Z}^+$, it follows from Eqs. (36) and (37) that

$$\begin{aligned} &V_{\tilde{\sigma}(k)}(k) \\ &\le (1-\alpha)^{k-(k_i+\tau_{\mathrm{s}}(k_i))} V_{l_i}(k_i + \tau_{\mathrm{s}}(k_i)) \\ &\le (1-\alpha)^{k-(k_i+\tau_{\mathrm{s}}(k_i))} c^{\bar{d}-1}\mu V_{l_i l_{i-1}}(k_i + \tau_{\mathrm{s}}(k_i)) \\ &\le c^{\bar{d}-1}\mu(1-\alpha)^{k-(k_i+\tau_{\mathrm{s}}(k_i))}(1+\beta)^{\tau_{\mathrm{s}}(k_i)} V_{l_i l_{i-1}}(k_i) \\ &\le c^{\bar{d}-1}\mu(1-\alpha)^{k-(k_i+\tau_{\mathrm{s}}(k_i))}(1+\beta)^{\tau_{\mathrm{s}}(k_i)}\mu V_{l_{i-1}}(k_i) \\ &\le c^{\bar{d}-1}\mu^2(1-\alpha)^{k-k_{i-1}-\tau_{\mathrm{s}}(k_i)-\tau_{\mathrm{s}}(k_{i-1})}(1+\beta)^{\tau_{\mathrm{s}}(k_i)} V_{l_{i-1}}(k_{i-1} + \tau_{\mathrm{s}}(k_{i-1})) \\ &\le c^{2(\bar{d}-1)}\mu^3(1-\alpha)^{k-k_{i-1}-\tau_{\mathrm{s}}(k_i)-\tau_{\mathrm{s}}(k_{i-1})}(1+\beta)^{\tau_{\mathrm{s}}(k_i)+\tau_{\mathrm{s}}(k_{i-1})} V_{l_{i-1}l_{i-2}}(\tau_{\mathrm{s}}(k_{i-1})) \end{aligned}$$

$$
\begin{aligned}
&\leq \cdots \\
&\leq c^{(i+1)(\bar{d}-1)}\mu^{2i+1}(1-\alpha)^{k-k_0-(\tau_s(k_i)+\cdots+\tau_s(k_0))}(1+\beta)^{\tau_s(k_i)+\cdots+\tau_s(k_0)}V_{l_0 l_{-1}}(k_0) \\
&\leq c^{(N_\sigma(k_0,k)+1)(\bar{d}-1)}\mu^{2N_\sigma(k_0,k)+1}(1-\alpha)^{k-k_0-\tau_s[k_0,k)}(1+\beta)^{\tau_s[k_0,k)}V_{l_0 l_{-1}}(k_0) \\
&= c^{(N_\sigma(k_0,k)+1)(\bar{d}-1)+\tau_s[k_0,k)}\mu^{2N_\sigma(k_0,k)+1}(1-\alpha)^{k-k_0}V_{l_0 l_{-1}}(k_0),
\end{aligned}
\tag{38}
$$

where $\tau_s[k_0,k)=\sum_{\iota=0}^{i}\tau_s(k_\iota)$. Similarly, for any $k\in[k_i,k_i+\tau_s(k_i))$, $i\in\mathbb{Z}^+$, it can be obtained that

$$
\begin{aligned}
V_{\tilde{\sigma}(k)}(k) &\leq c^{N_\sigma(k_0,k)(\bar{d}-1)}\mu^{2N_\sigma(k_0,k)}(1-\alpha)^{k-k_0-\tau_s[k_0,k)}(1+\beta)^{\tau_s[k_0,k)}V_{l_0 l_{-1}}(k_0) \\
&= c^{N_\sigma(k_0,k)(\bar{d}-1)+\tau_s[k_0,k)}\mu^{2N_\sigma(k_0,k)}(1-\alpha)^{k-k_0}V_{l_0 l_{-1}}(k_0).
\end{aligned}
\tag{39}
$$

Because $\tau_s(k)\leq\bar{\tau}_s$, we conclude that the total time in $(k_0,\,k)$ for which $\sigma(k)\neq\sigma(k-\tau_s(k))$ will be at most $(N_\sigma(k_0,k)+1)\bar{\tau}_s$, which implies that $\tau_s[k_0,k)\leq(N_\sigma(k_0,k)+1)\bar{\tau}_s$. In addition, since $\sigma\subset\mathcal{S}_{\text{ave}}[\tau_a,N_0]$, it holds that

$$
\tau_s[k_0,k)\leq(N_0+1)\bar{\tau}_s+\bar{\tau}_s(k-k_0)/\tau_a. \tag{40}
$$

Hence,

$$
c^{\tau_s[k_0,k)}\leq c^{(N_0+1)\bar{\tau}_s+\bar{\tau}_s(k-k_0)/\tau_a}. \tag{41}
$$

If conditions (28) and (29) are satisfied, then we can obtain

$$
c^{(\bar{\tau}_s+\bar{d}-1)(k-k_0)/\tau_a}\mu^{2(k-k_0)/\tau_a}(1-\alpha)^{k-k_0}\leq\lambda^{2(k-k_0)},
$$

$$
c^{(\bar{\tau}_s+\bar{d}-1)(N_0+1)}\mu^{2N_0+1}\leq\varsigma\mu c^{\bar{\tau}_s+\bar{d}-1},
$$

which, combined with Eq. (41), yield

$$
c^{(N_\sigma(k_0,k)+1)(\bar{d}-1)+\tau_s[k_0,k)}\mu^{2N_\sigma(k_0,k)+1}(1-\alpha)^{k-k_0}\leq\varsigma\mu c^{\bar{\tau}_s+\bar{d}-1}\lambda^{2(k-k_0)}, \tag{42}
$$

$$
c^{N_\sigma(k_0,k)(\bar{d}-1)+\tau_s[k_0,k)}\mu^{2N_\sigma(k_0,k)}(1-\alpha)^{k-k_0}\leq\varsigma c^{\bar{\tau}_s}\lambda^{2(k-k_0)}. \tag{43}
$$

Now, substituting Eqs. (42) and (43) into Eqs. (38) and (39), respectively, and in view of $c^{\bar{d}-1}\mu>1$, we have that for any $k\in[k_i,k_{i+1})$, $i\in\mathbb{Z}^+$,

$$
V_{\tilde{\sigma}(k)}(k)\leq\varsigma\mu c^{\bar{\tau}_s+\bar{d}-1}\lambda^{2(k-k_0)}V_{l_0 l_{-1}}(k_0). \tag{44}
$$

From Eqs. (35), (9) and (10), it follows that $a\|\tilde{x}_1(k)\|^2\leq V_{\tilde{\sigma}(k)}(k)$, where $a=\min_{\forall(l_i\times l_{i-1})\in\mathcal{I}\times\mathcal{I},l_{i-1}\neq l_i}\{a_{l_i l_{i-1}},a_{l_i}\}$. In addition, since $V_{\tilde{\sigma}(k)}(k)$ is bounded, there exists a sufficiently large scalar b such that $V_{\tilde{\sigma}(k)}(k_0)\leq b\|\phi(k_0)\|_{\bar{d}}^2$. Then,

$$
\|\tilde{x}_1(k)\|\leq\sqrt{\frac{b}{a}\varsigma\mu c^{\bar{\tau}_s+\bar{d}-1}}\lambda^{k-k_0}\|\phi(k_0)\|_{\bar{d}}\triangleq r_1\lambda^{k-k_0}\|\phi(k_0)\|_{\bar{d}}, \tag{45}
$$

which means that slow variables of system (34) are exponentially convergent with decay rate λ.

Step 3. (exponential convergence of fast variables of system (34)). Without loss of generality, we assume $k\in[k_i+\tau_s(k_i),k_{i+1})$, $i\in\mathbb{Z}^+$. In what follows, we first give the estimation of $\|\tilde{x}_2(k)\|$. When $k\in[k_i+\tau_s(k_i),k_{i+1})$, subsystem $\mathcal{S}_{l_i}$ is activated. As proved in Step 1, $\mathcal{S}_{l_i}$ can be equivalently transformed into $\tilde{\mathcal{S}}_{l_i}$ (or $(\tilde{E},\tilde{A}_{l_i},\tilde{A}_{dl_i},\tilde{\mathrm{B}}_{l_i})$) under condition (27). And, sub-matrices $\tilde{A}^{22}_{dl_i}$ and $\tilde{\mathrm{B}}^{22}_{l_i}$ in $\tilde{A}_{dl_i}$ and $\tilde{\mathrm{B}}_{l_i}$ satisfy (8). Then, by Proposition 2 (1), there exists a limited integer $\mathcal{K}_{l_i} > 0$ such that $\tilde{x}_2(k)$ depends on $\mathcal{K}_{l_i}$ times $\tilde{x}(k)$ before $k_i + \tau_s(k_i)$, and $\tilde{x}_2(k)$ can be computed by Eq. (22) with $\mathcal{K} \to \mathcal{K}_{l_i}$, $p \to 2$, $\mathcal{O}_{k_0} \to \mathcal{O}_{k_i+\tau_s(k_i)}$, $A^{21} \to \tilde{A}^{21}_{l_i}$, $A^{21}_{d1} \to \tilde{A}^{21}_{dl_i}$, $A^{21}_{d2} \to \tilde{\mathrm{B}}^{21}_{l_i}$, $A^{22}_{d1} \to \tilde{A}^{22}_{dl_i}$ and $A^{22}_{d2} \to \tilde{\mathrm{B}}^{22}_{l_i}$. Noting that $0 < 1-\alpha < \lambda^2 < 1$ and by Proposition 2 (2.1), $\|\tilde{x}_2(k)\|$ can then be bounded by

$$\begin{aligned}\|\tilde{x}_2(k)\| \leq& \underline{\nu}_{l_i1}\|\tilde{x}_2(k_i+\tau_s(k_i))\|_{\bar{d}}(1-\alpha)^{(k-k_i-\tau_s(k_i))/2} \\ &+ \underline{\nu}_{l_i2}\|\tilde{x}(k_i+\tau_s(k_i))\|_{\bar{d}}\lambda^{k-k_i-\tau_s(k_i)} \\ \leq& \underline{\nu}_{l_i1}\|\tilde{x}_2(k_i+\tau_s(k_i))\|_{\bar{d}}(1-\alpha)^{(k-k_i-\tau_s(k_i))/2} \\ &+ \underline{\nu}_{l_i2}(\|\tilde{x}_1(k_i+\tau_s(k_i))\|_{\bar{d}} + \|\tilde{x}_2(k_i+\tau_s(k_i))\|_{\bar{d}})\lambda^{k-k_i-\tau_s(k_i)},\end{aligned} \tag{46}$$

where $\underline{\nu}_{l_i1} > 0$ and $\underline{\nu}_{l_i2} > 0$.

By Eq. (45), we have $\|\tilde{x}_1(k_i+\tau_s(k_i))\|_{\bar{d}} \leq r_1\lambda^{-\bar{d}}\lambda^{k_i+\tau_s(k_i)-k_0}\|\phi(k_0)\|_{\bar{d}}$. Thus,

$$\|\tilde{x}_1(k_i+\tau_s(k_i))\|_{\bar{d}}\lambda^{k-k_i-\tau_s(k_i)} \leq r_1\lambda^{-\bar{d}}\lambda^{k-k_0}\|\phi(k_0)\|_{\bar{d}}. \tag{47}$$

In order to obtain the estimation of $\|\tilde{x}_2(k_i+\tau_s(k_i))\|_{\bar{d}}$, consider the time interval $[k_i, k_i + \tau_s(k_i))$ on which subsystem $\mathcal{S}_{l_il_{i-1}}$ is activated. According to the analysis in Step 1, $\mathcal{S}_{l_il_{i-1}}$ is r.s.e. to $\tilde{\mathcal{S}}_{l_il_{i-1}}$ (or $(\tilde{E}, \tilde{A}_{l_i}, \tilde{A}_{dl_i}, \tilde{\mathrm{B}}_{l_il_{i-1}})$). The sub-matrices $\tilde{A}^{22}_{dl_i}$ and $\tilde{\mathrm{B}}^{22}_{l_il_{i-1}}$ in $\tilde{A}_{dl_i}$ and $\tilde{\mathrm{B}}_{l_il_{i-1}}$ satisfy Eq. (7). In view of $\beta > 0$, using Proposition 2 (1) and (2.2) and the relation $\|\tilde{x}(k_i)\|_{\bar{d}} \leq \|\tilde{x}_1(k_i)\|_{\bar{d}} + \|\tilde{x}_2(k_i)\|_{\bar{d}}$, $\|\tilde{x}_2(k_i+\tau_s(k_i))\|_{\bar{d}}$ can be bounded by

$$\begin{aligned}\|\tilde{x}_2(k_i+\tau_s(k_i))\|_{\bar{d}} \leq& \bar{\nu}_{l_il_{i-1}1}\|\tilde{x}_2(k_i)\|_{\bar{d}}(1+\beta)^{\tau_s(k_i)/2} + \bar{\nu}_{l_il_{i-1}2}(\|\tilde{x}_1(k_i)\|_{\bar{d}} \\ &+ \|\tilde{x}_2(k_i)\|_{\bar{d}})\lambda^{\tau_s(k_i)},\end{aligned} \tag{48}$$

where $\bar{\nu}_{l_il_{i-1}1} > 0$ and $\bar{\nu}_{l_il_{i-1}2} > 0$. Substituting Eqs. (47) and (48) into Eq. (46) yields

$$\begin{aligned}\|\tilde{x}_2(k)\| \leq& \underline{\nu}_{l_i1}\bar{\nu}_{l_il_{i-1}1}\|\tilde{x}_2(k_i)\|_{\bar{d}}(1+\beta)^{\tau_s(k_i)/2}(1-\alpha)^{(k-k_i-\tau_s(k_i))/2} \\ &+ \underline{\nu}_{l_i1}\bar{\nu}_{l_il_{i-1}2}(\|\tilde{x}_2(k_i)\|_{\bar{d}} + \|\tilde{x}_1(k_i)\|_{\bar{d}})\lambda^{\tau_s(k_i)}(1-\alpha)^{(k-k_i-\tau_s(k_i))/2} \\ &+ \underline{\nu}_{l_i2}\bar{\nu}_{l_il_{i-1}1}\|\tilde{x}_2(k_i)\|_{\bar{d}}(1+\beta)^{\tau_s(k_i)/2}\lambda^{k-k_i-\tau_s(k_i)} \\ &+ \underline{\nu}_{l_i2}\bar{\nu}_{l_il_{i-1}2}(\|\tilde{x}_2(k_i)\|_{\bar{d}} + \|\tilde{x}_1(k_i)\|_{\bar{d}})\lambda^{k-k_i} + \underline{\nu}_{l_i2}r_1\lambda^{-\bar{d}}\|\phi(k_0)\|_{\bar{d}}\lambda^{k-k_0}.\end{aligned} \tag{49}$$

Noting that $0 < 1-\alpha < \lambda^2 < 1$ and $\|\tilde{x}_1(k_i)\|_{\bar{d}} \leq r_1\lambda^{-\bar{d}}\|\phi(k_0)\|_{\bar{d}}\lambda^{k_i-k_0}$, it follows from Eq. (49) that

$$\begin{aligned}\|\tilde{x}_2(k)\| \leq& (\underline{\nu}_{l_i1} + \underline{\nu}_{l_i2})\bar{\nu}_{l_il_{i-1}1}\|\tilde{x}_2(k_i)\|_{\bar{d}}(1+\beta)^{\tau_s(k_i)/2}\lambda^{k-k_i-\tau_s(k_i)} \\ &+ (\underline{\nu}_{l_i1} + \underline{\nu}_{l_i2})\bar{\nu}_{l_il_{i-1}2}\|\tilde{x}_2(k_i)\|_{\bar{d}}\lambda^{k-k_i} \\ &+ ((\underline{\nu}_{l_i1} + \underline{\nu}_{l_i2})\bar{\nu}_{l_il_{i-1}2} + \underline{\nu}_{l_i2})\lambda^{-\bar{d}}r_1\|\phi(k_0)\|_{\bar{d}}\lambda^{k-k_0} \\ \triangleq& \nu_{l_i1}\|\tilde{x}_2(k_i)\|_{\bar{d}}(1+\beta)^{\tau_s(k_i)/2}\lambda^{k-k_i-\tau_s(k_i)} + \nu_{l_i2}\|\tilde{x}_2(k_i)\|_{\bar{d}}\lambda^{k-k_i} \\ &+ \nu_{l_i3}\|\phi(k_0)\|_{\bar{d}}\lambda^{k-k_0}.\end{aligned} \tag{50}$$

To bound the term $\|\tilde{x}_2(k_i)\|_{\bar{d}}$, consider the time interval $k \in [k_{i-1} + \tau_s(k_{i-1}), k_i)$ on which subsystem $\tilde{\mathcal{S}}_{l_{i-1}}$ works. Similar to the derivation of (46), $\|\tilde{x}_2(k_i)\|_{\bar{d}}$ can be bounded by

$$\begin{aligned}\|\tilde{x}_2(k_i)\|_{\bar{d}} \leq& \underline{\nu}_{l_{i-1}1}\|\tilde{x}_2(k_{i-1}+\tau_s(k_{i-1}))\|_{\bar{d}}(1-\alpha)^{(k_i-k_{i-1}-\tau_s(k_{i-1}))/2} \\ &+ \underline{\nu}_{l_{i-1}2}(\|\tilde{x}_1(k_{i-1}+\tau_s(k_{i-1}))\|_{\bar{d}}\end{aligned}$$

$$+\|\tilde{x}_2(k_{i-1}+\tau_s(k_{i-1}))\|_{\bar{d}})\lambda^{k_i-k_{i-1}-\tau_s(k_{i-1})}, \tag{51}$$

where $\underline{\nu}_{l_{i-1}1}>0$ and $\underline{\nu}_{l_{i-1}2}>0$. Substituting Eq. (51) into Eq. (50) and using $0<1-\alpha<\lambda^2<1$ yields

$$\begin{aligned}\|\tilde{x}_2(k)\| \le & \nu_{l_i1}(\underline{\nu}_{l_{i-1}1}+\underline{\nu}_{l_{i-1}2})\|\tilde{x}_2(k_{i-1}+\tau_s(k_{i-1}))\|_{\bar{d}}(1+\beta)^{\tau_s(k_i)/2}\\ & \times\lambda^{k-k_{i-1}-\tau_s(k_i)-\tau_s(k_{i-1})}\\ & +\nu_{l_i2}(\underline{\nu}_{l_{i-1}1}+\underline{\nu}_{l_{i-1}2})\|\tilde{x}_2(k_{i-1}+\tau_s(k_{i-1}))\|_{\bar{d}}\lambda^{k_i-k_{i-1}-\tau_s(k_{i-1})}\\ & +\{r_1\lambda^{-\bar{d}}(\nu_{l_i1}(1+\beta)^{\tau_s(k_i)/2}/\lambda^{\tau_s(k_i)}+\nu_{l_i2})\underline{\nu}_{l_{i-1}2}+\nu_{l_i3}\}\|\phi(k_0)\|_{\bar{d}}\lambda^{k-k_0}.\end{aligned}$$

Next, by successively bounding the terms $\|\tilde{x}_2(k_i)\|_{\bar{d}}$, $\|\tilde{x}_2(k_{i-1}+\tau_s(k_{i-1}))\|_{\bar{d}},\ldots,$ $\|\tilde{x}_2(k_0+\tau_s(k_0))\|_{\bar{d}}$ as above and using some arithmetical calculations, $\|\tilde{x}_2(k)\|$ can be finally bounded by

$$\begin{aligned}\|\tilde{x}_2(k)\| \le & \nu_{l_01}\|\tilde{x}_2(k_0)\|_{\bar{d}}(1+\beta)^{\tau_s[k_0,k)/2}\lambda^{k-k_0-\tau_s[k_0,k)}+\nu_{l_02}\|\tilde{x}_2(k_0)\|_{\bar{d}}\lambda^{k-k_0}\\ & +\nu_{l_03}\|\phi(k_0)\|_{\bar{d}}\lambda^{k-k_0},\end{aligned} \tag{52}$$

where ν_{l_01}, ν_{l_02} and ν_{l_03} are positive scalars.

Now, we prove exponential convergence of $\|\tilde{x}_2(k)\|$. Note that condition (28) implies that $\bar{\tau}_s\ln((1+\beta)/(1-\alpha))\le\tau_a\ln(\lambda^2/(1-\alpha))$, that is,

$$\frac{\bar{\tau}_s}{2\tau_a}\ln(1+\beta)\le\left(\frac{\bar{\tau}_s}{\tau_a}-1\right)\ln(1-\alpha)^{\frac{1}{2}}+\ln\lambda. \tag{53}$$

In view of $0<1-\alpha<\lambda^2$, it follows from Eq. (53) that $(\bar{\tau}_s/2\tau_a)\ln(1+\beta)\le(\bar{\tau}_s/\tau_a-1)\ln\lambda+\ln\lambda$. Then, it is easy to obtain

$$(1+\beta)^{\bar{\tau}_s(k-k_0)/2\tau_a}\lambda^{(1-\bar{\tau}_s/\tau_a)(k-k_0)}\le\lambda^{k-k_0}. \tag{54}$$

Therefore, it follows from (40), (52) and (54) that

$$\begin{aligned}\|\tilde{x}_2(k)\| & \le \nu_{l_01}(\sqrt{1+\beta}/\lambda)^{(N_0+1)\bar{\tau}_s}\|\tilde{x}_2(k_0)\|_{\bar{d}}\lambda^{k-k_0}+\nu_{l_02}\|\tilde{x}_2(k_0)\|_{\bar{d}}\lambda^{k-k_0}\\ & \quad +\nu_{l_03}\|\phi(k_0)\|_{\bar{d}}\lambda^{k-k_0}\\ & \le\left(\nu_{l_01}(\sqrt{1+\beta}/\lambda)^{(N_0+1)\bar{\tau}_s}+\nu_{l_02}+\nu_{l_03}\right)\lambda^{k-k_0}\|\phi(k_0)\|_{\bar{d}}\\ & \triangleq r_2\lambda^{k-k_0}\|\phi(k_0)\|_{\bar{d}},\end{aligned} \tag{55}$$

which means that fast variables of the equivalent system (34) are exponentially convergent with decay rate λ.

Step 4. (exponential stability of the closed-loop system (3)). Noting that $x(k)=H\tilde{x}(k)$ and using (45) and (55), we have

$$\|x(k)\|\le\|H\|(\|\tilde{x}_1(k)\|+\|\tilde{x}_2(k)\|)\le\|H\|\sqrt{r_1^2+r_2^2}\lambda^{k-k_0}\|\phi(k_0)\|_{\bar{d}}.$$

This completes the proof. □

Remark 8. The switching condition (28) provides an explicit relationship among the maximal switching delay $\bar{\tau}_s$, upper bound on state delay $\bar{d}$ and ADT τ_a of the closed-loop system (3). For a fixed τ_a, a larger $\bar{d}$ needs a smaller $\bar{\tau}_s$ and vice versa. When $\bar{d}=0$, $\bar{\tau}_s$ reaches a maximum and equals $(\ln(\lambda^2/(1-\alpha))-\ln(c\mu^2))/\ln c$. For a fixed $\bar{\tau}_s$, a larger τ_a permits a larger $\bar{d}$ and vice versa. For a fixed $\bar{d}$, a larger τ_a also permits a larger $\bar{\tau}_s$ and vice versa. One can sacrifice the ADT to obtain relatively large switching delay and state delay. Moreover, compared with the asynchronous switching conditions designed in [10,11,13–20], the switching condition (28) explicitly contains exponential decay rate factor λ of the closed-loop system. Therefore, the switching conditions proposed in Theorem 1 are more desirable for system analysis and control synthesis.

When the switching delay $\tau_s(k) = 0$, we have the following corollary.

Corollary 1. *Consider the switched singular system (1), and let* $0<\alpha<1$, $0 < \underline{d}_s < \bar{d}_s$, $s = 1, 2$, $\varepsilon_{l_i1}, \varepsilon_{l_i2}, \dots, \varepsilon_{l_i9}$, $\forall l_i \in \mathcal{I}$, *and* $\mu \geq 1$ *be given constants. Suppose that there exist matrices* $X_{l_i} > 0$, $Q_{l_iw_s} > 0$, $Z_{l_i1_s} > 0$, $Z_{l_i2_s} > 0$, $w = 1, 2, 3$, $s = 1, 2$, $Y_{l_i} = Y_{l_i}^{\top}$, K_{l_i}, $\forall l_i \in \mathcal{I}$, *and scalar* ρ_{l_i}, $\forall l_i \in \mathcal{I}$, *such that (27) holds with* $X_{l_{i-1}} \leq \mu X_{l_i}$, $Q_{l_iw_s} \leq \mu Q_{l_{i-1}w_s}$, $Z_{l_i1_s} \leq \mu Z_{l_{i-1}1_s}$, $Z_{l_i2_s} \leq \mu Z_{l_{i-1}2_s}$, $w = 1, 2, 3$, $s = 1, 2$, $\forall (l_i, l_{i-1}) \in \mathcal{I} \times \mathcal{I}$, $l_i \neq l_{i-1}$. *Then, under the state feedback control (2) with* $\tau_s(k) = 0$, *the resulting closed-loop system (3) is exponentially admissible for any switching signal* $\sigma \in [\tau_a, N_0]$ *satisfying* $\tau_a \geq -\frac{\ln \mu}{\ln(1-\alpha)}$. *Moreover, the decay rate is* $\lambda = \sqrt{(1-\alpha)\mu^{1/\tau_a}}$.

Remark 9. Different from our previous works [29,36,37], benefiting from the exponential finite sum inequality in Lemma 2, the exponential admissibility and stabilization conditions obtained in this paper do not contain free-weighting matrices. Thus, they are more concise and easily tested.

5. Numerical examples

The following examples are provided to illustrate the validity of the obtained results.

Example 1. Consider the switched singular system (1) with two subsystems (i.e., $\mathcal{I} = \{1, 2\}$) as follows:

$$E = \begin{bmatrix} 6 & 3 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 1.5 \\ -2 & 1.1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.25 & -0.2 \\ 0.1 & -0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.2 & 1.5 \\ -1.1 & 1.5 \end{bmatrix},\ A_{d2} = \begin{bmatrix} -0.5 & 0.3 \\ -0.1 & -0.1 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 1.1 \\ -1.15 \end{bmatrix},$$

$$1 \leq d_1(k) \leq 3 \ \text{(i.e., } \underline{d}_1 = 1,\ \bar{d}_1 = 3\text{)}.$$

For convenience, denote the above two subsystems by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. The objective is to design a switched state feedback control law in the form of Eq. (2) with $\tau_s(k)$ satisfying $0 < \tau_s(k) \leq 1$ and $d_2(k)$ satisfying $1 \leq d_2(k) \leq 3$, i.e., $\bar{\tau}_s = 1$, $\underline{d}_2 = 1$ and $\bar{d}_2 = 3$, such that the resulting closed-loop system is exponentially admissible.

Firstly, we assume no switching delay exists in the switching signal of the controller, i.e., $\tau_s(k) = 0$. For $\alpha = 0.28$ and $\mu = 1.05$, choosing $R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\varepsilon_{11} = \varepsilon_{21} = -20$, $\varepsilon_{12} = \varepsilon_{22} = -0.01$, $\varepsilon_{13} = \varepsilon_{23} = -0.01$, $\varepsilon_{14} = \varepsilon_{24} = -13$, $\varepsilon_{15} = \varepsilon_{25} = \cdots = \varepsilon_{19} = \varepsilon_{29} = 12$, and by solving Corollary 1 with MATLAB LMI toolbox, we get minimal ADT (denoted by τ_a^*) $\tau_a^* = 0.1485$ and corresponding feedback gains

$$K_1 = \begin{bmatrix} -0.0074 & -0.0039 \end{bmatrix},\ K_2 = \begin{bmatrix} -0.0031 & -0.0003 \end{bmatrix}. \tag{56}$$

Suppose that the subsystems are activated in the following sequence:

$\mathcal{S}_1\mathcal{S}_1\mathcal{S}_2\mathcal{S}_2\mathcal{S}_1\mathcal{S}_1\mathcal{S}_2\mathcal{S}_2\mathcal{S}_1\mathcal{S}_1\mathcal{S}_2\mathcal{S}_2\mathcal{S}_1\mathcal{S}_1\mathcal{S}_2\mathcal{S}_2\cdots$ (i.e., $\tau_a = 2$).

Choosing initial condition function $\phi(\kappa)=\begin{bmatrix}0.29 & 0.35\end{bmatrix}^\top$, $\kappa=-3,\dots,0$, and selecting $d_1(k)=2+\sin(0.5\pi k)$ and $d_2(k)=2+\cos(0.5\pi k)$, the state responses of the closed-loop system with feedback gains (56) are depicted in Fig. 3(a). Now, if there exists a switching delay $\tau_s(k)=1$ in the switching signal of the controller, by applying the obtained controller, the corresponding state responses of the closed-loop system are given in Fig. 3(b). From Fig. 3(b), it is seen that the responses have larger overshoots and longer setting time.

Next, we consider Theorem 1. For the same α, μ, R, ε_{sf}, $s=1,2$, $f=1,\dots,9$ as above, choosing $\beta=0.05$, the LMIs (26) and (27) are feasible and the corresponding feedback gains are solved as

$$K_1=\begin{bmatrix}-0.0096 & -0.0048\end{bmatrix},\ K_2=\begin{bmatrix}-0.0329 & -0.0165\end{bmatrix}. \tag{57}$$

Setting $\lambda=0.95$ and $\varsigma=1$, it follows from Eq. (28) that the admissible ADT of the switching signal σ should satisfy $\tau_a\geq 5.2261$. Suppose that the two subsystems are activated in the following sequence:

$\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\dots$ (i.e., $\tau_a=6$).

Choosing the same initial condition $\phi(\kappa)$ and state delays $d_1(k)$, $d_2(k)$ as above, the corresponding closed-loop state responses with $\tau_s(k)=1$ are depicted in Fig. 3(c). It can be seen from the curves that the proposed feedback stabilization method is valid despite the existing of switching delay.

Example 2. Consider a direct current (DC) motor driving a load via a gearbox in [25] (pp. 4 in Section 1) as shown in Fig. 4. Letting $v(t)$, $i(t)$ and $\omega(t)$ denote the armature voltage, the armature current and the angular velocity of motor shaft, and neglecting the armature inductance L_m, the singular model of the system is given by [25]

$$\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\dot{x}(t)=\begin{bmatrix}R & K_E\\ \frac{K_T}{J} & -\frac{b}{J}\end{bmatrix}x(t)-\begin{bmatrix}1\\ 0\end{bmatrix}u(t). \tag{58}$$

where $x(t)=[x_1(t)\ x_2(t)]^\top$ is the state with $x_1(t)=i(t)$ and $x_2(t)=\omega(t)$, and $u(t)=v(t)$ is the control input. R is the armature resistor, K_E is the electromotive force constant, K_T is the torque constant, and J and b are the converted moment of inertia and damping ratio, respectively, which are defined by $J=J_m+J_c/\eta^2$ and $b=b_m+b_c/\eta^2$, where J_m and J_c are the moments of inertia of the rotor and the load, b_m and b_c are the damping ratios of the motor and the load, and η is the gear ratio.

Assume that the mass and/or radius of the load change abruptly. Then, the changes can be represented by the jumping of the inertia J. In this paper, J is assumed to belong to a set: $J\in\{0.9\,\mathrm{Kg\,m^2}, 1.2\,\mathrm{Kg\,m^2}\}$. Let $R=1.5\,\Omega$, $K_E=K_T=1.8\,\mathrm{V\,s/rad}$, and $b=0.3\,\mathrm{N\,m/rad\,s}$. The discretization of system (58) with the above parameters and a sampling time 0.1s results in the following discrete SS system:

$$(\mathcal{S}_{\sigma(k)}):Ex(k+1)=A_{\sigma(k)}x(k)+B_{\sigma(k)}u(k),\ \ k\geq 0$$

where $\sigma(k)\in\mathcal{I}=\{1,2\}$, $E=\mathrm{diag}\{0,1\}$,

$$A_1=\begin{bmatrix}1.5 & 1.8\\ 0 & 0.7608\end{bmatrix}, B_1=\begin{bmatrix}-1\\ 0.1167\end{bmatrix}, A_2=\begin{bmatrix}1.5 & 1.8\\ 0 & 0.8146\end{bmatrix}, B_2=\begin{bmatrix}-1\\ 0.0904\end{bmatrix}.$$

For this system, our objective is to design a switched state feedback control law in form (2) with $\tau_s(k)$ satisfying $0<\tau_s(k)\leq 1$ and $d_2(k)$ satisfying $1\leq d_2(k)\leq 2$, i.e. $\bar{\tau}_s=1$, $\underline{d}_2=1$ and $\bar{d}_2=2$, such that the resulting closed-loop system is exponentially admissible. Set $\alpha=$

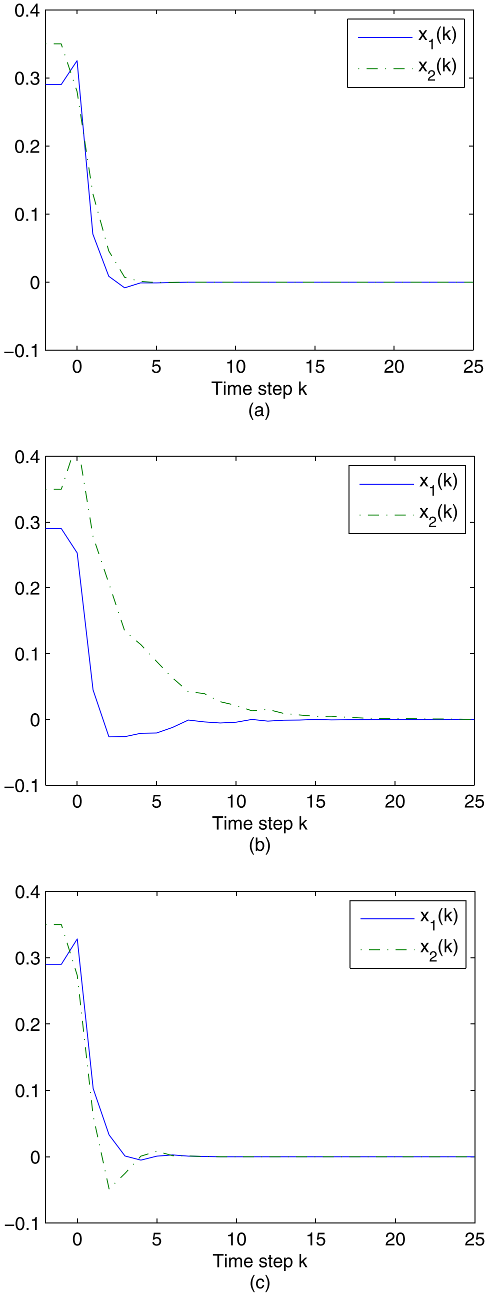

Fig. 3. State trajectories of the closed-loop system. (a) State trajectories of the system with $\tau_s(k) = 0$, $\tau_a = 2$ and feedback gains (56). (b) State trajectories of the system with $\tau_s(k) = 1$, $\tau_a = 2$ and feedback gains (56). (c) State trajectories of the system with $\tau_s(k) = 1$, $\tau_a = 6$ and feedback gains (57).

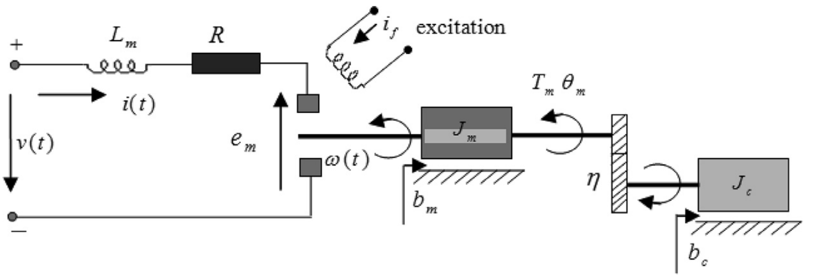

Fig. 4. Block diagram of a DC motor driving a load.

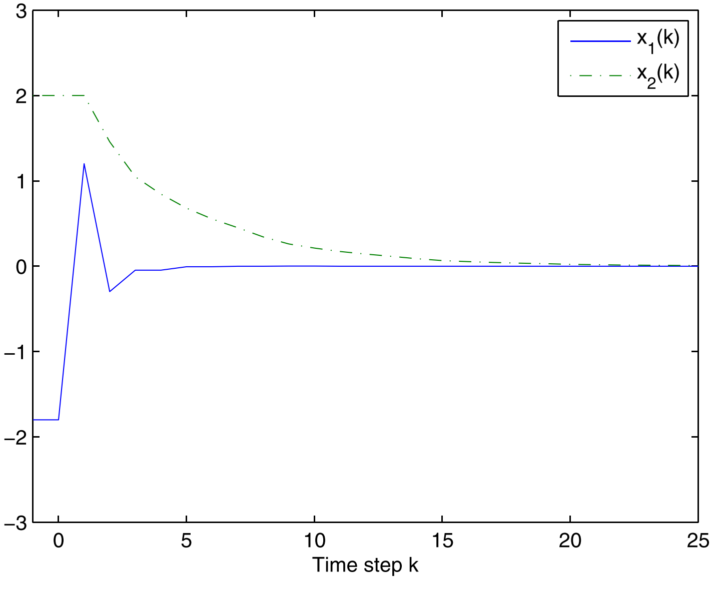

Fig. 5. State trajectories of the closed-loop system in Example 2.

0.25, $\beta = 0.1$ and $\mu = 1.1$, and let $R = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, $\varepsilon_{11} = \varepsilon_{21} = -0.1$, $\varepsilon_{13} = \varepsilon_{23} = -10^4$, $\varepsilon_{14} = \varepsilon_{24} = -10^5$, $\varepsilon_{15} = \varepsilon_{25} = 0.1$, $\varepsilon_{18} = \varepsilon_{28} = 0.1$, $\varepsilon_{19} = \varepsilon_{29} = 0.1$, $\varepsilon_{12} = \varepsilon_{22} = 0$, $\varepsilon_{16} = \varepsilon_{26} = 0$, $\varepsilon_{17} = \varepsilon_{27} = 0$. By solving Theorem 1 with $Q_{s1_1} = Q_{s2_1} = Q_{s3_1} = 0$, $Z_{s1_1} = Z_{s2_1} = 0$, $s = 1, 2$, $Q_{ss'1_1} = Q_{ss'2_1} = Q_{ss'3_1} = 0$, $Z_{ss'1_1} = Z_{ss'2_1} = 0$, $s = 1, 2$, $s' = 1, 2$, we can obtain

$$K_1 = \begin{bmatrix} 0.2677 & -0.0439 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.2681 & -0.0430 \end{bmatrix}.$$

Now, letting $\lambda = 0.96$ and $\varsigma = 1$, by Eq. (28), the admissible ADT of σ should satisfy $\tau_a \geq 6.0391$. Suppose the two subsystems are activated in the following sequence: $\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_2\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1\mathcal{S}_1 \ldots$ (i.e., $\tau_a = 7$). Choosing initial condition function $\phi(\kappa) = [-1.8 \quad 2]^\top$, $\kappa = -2, -1, 0$, state delay function $d_2(k) = 1 + |\cos(0.5\pi k)|$ and $\tau_s(k) = 1$, the state responses of the closed-loop with are depicted in Fig. 5. From Fig. 5, we can see that the closed-loop system is stable.

6. Conclusions

The problem of state feedback stabilization for discrete-time switched singular systems with a state delay, an output delay and a switching delay under ADT switching constraints is studied in the paper. An LMI-based condition in terms of upper bounds on the delays and a lower bound on the ADT is derived to guarantee the regularity, the causality and exponential stability of the closed-loop system. Future research will be focused on the extension of the controlled plant to more complex settings, for example systems with switched singular matrix $E_{\sigma(k)}$ and uncertain system parameters (as in [48]), and the incorporation of more performance requirements such as input-to-state stability (as in [21,23]) and dissipativity (as in [49]) into the stabilization design.

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Proof of Proposition 2. (1). (Proof of Eq. (22)). From (18), $x_2(k)$ equals $-A^{21}x_1(k) + \sum_{s=1}^{p}\{-A^{21}_{ds}x_1(k-d_s(k)) - A^{22}_{ds}x_2(k-d_s(k))\}$, which can be rewritten as $-A^{21}x_1(k) + \sum_{j=0}^{p-1}\{-A^{21}_{d(j+1)}x_1(k_{1j}) + \hat{A}_{1j}x_2(k_{1j})\}$ by using Eqs. (19) and (20). Considering Eq. (21), $x_2(k)$ can be further rewritten as

$$-A^{21}x_1(k) - \sum_{j=0}^{p-1}\{A^{21}_{d(j+1)}x_1(k_{1j})\} + \sum_{\substack{j=0\\ k_{1j}\in\mathcal{O}_{k_0}}}^{p-1}\left\{\hat{A}_{1j}x_2(k_{1j})\right\} + \sum_{\substack{j=0\\ k_{1j}\notin\mathcal{O}_{k_0}}}^{p-1}\left\{\hat{A}_{1j}x_2(k_{1j})\right\}.$$

If $k_{1j}\notin\mathcal{O}_{k_0}$, it holds from Eqs. (18)–(21) that $\hat{A}_{1j}x_2(k_{1j}) = -\hat{A}_{1j}A^{21}x_1(k_{1j}) + \sum_{f=jp}^{(j+1)p-1}\{-\hat{A}_{1j}A^{21}_{d(\kappa_p^f+1)}x_1(k_{2f}) + \hat{A}_{2f}x_2(k_{2f})\}$. Then, $x_2(k)$ can be computed as

$$\begin{aligned}
&-A^{21}x_1(k) + \sum_{j=0}^{p-1}\{-A^{21}_{d(j+1)}x_1(k_{1j})\} + \sum_{\substack{j=0\\ k_{1j}\in\mathcal{O}_{k_0}}}^{p-1}\left\{\hat{A}_{1j}x_2(k_{1j})\right\}\\
&\quad - \sum_{\substack{j=0\\ k_{1j}\notin\mathcal{O}_{k_0}}}^{p-1}\left\{\hat{A}_{1j}A^{21}x_1(k_{1j})\right\}\\
&\quad + \sum_{\substack{j=0\\ k_{1j}\notin\mathcal{O}_{k_0}}}^{p-1}\sum_{f=jp}^{(j+1)p-1}\left\{-\hat{A}_{1j}A^{21}_{d(k_p^f+1)}x_1(k_{2f}) + \hat{A}_{2f}x_2(k_{2f})\right\}\\
&= -A^{21}\sum_{i=0}^{1}\sum_{\substack{j=0\\ k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\left\{\hat{A}_{ij}x_1(k_{ij})\right\} + \sum_{i=0}^{1}\sum_{\substack{j=0\\ k_{iv_j}\notin\mathcal{O}_{k_0}}}^{p^{i+1}-1}\left\{-\hat{A}_{iv_j}A^{21}_{d(\kappa_p^j+1)}x_1(k_{(i+1)j})\right\}
\end{aligned}$$

$$+\sum_{\substack{j=0\\k_{1j}\in\mathcal{O}_{k_0}}}^{p-1}\left\{\hat{A}_{1j}x_2(k_{1j})\right\}+\sum_{\substack{j=0\\k_{1v_j}\notin\mathcal{O}_{k_0}}}^{p^2-1}\left\{\hat{A}_{2j}x_2(k_{2j})\right\}$$

$$=-A^{21}\sum_{i=0}^{1}\sum_{\substack{j=0\\k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\left\{\hat{A}_{ij}x_1(k_{ij})\right\}-\sum_{i=0}^{1}\sum_{\substack{j=0\\k_{iv_j}\notin\mathcal{O}_{k_0}}}^{p^{i+1}-1}\left\{\hat{A}_{iv_j}A^{21}_{d(\kappa_p^j+1)}x_1(k_{(i+1)j})\right\}$$

$$+\sum_{i=1}^{2}\sum_{\substack{j=0\\k_{ij}\in\mathcal{O}_{k_0}}}^{p^i-1}\left\{\hat{A}_{ij}x_2(k_{ij})\right\}+\sum_{\substack{j=0\\k_{2j}\notin\mathcal{O}_{k_0}}}^{p^2-1}\left\{\hat{A}_{2j}x_2(k_{2j})\right\}.$$

Similarly, if $k_{2j}\notin\mathcal{O}_{k_0}$, from Eqs. (18)–(21), we have $\hat{A}_{2j}x_2(k_{2j})=-\hat{A}_{2j}A^{21}x_1(k_{2j})+\sum_{f=jp}^{(j+1)p-1}\{-\hat{A}_{2j}A^{21}_{d(\kappa_p^f+1)}x_1(k_{3f})+\hat{A}_{3f}x_2(k_{3f})\}$ and

$$x_2(k)=-A^{21}\sum_{i=0}^{2}\sum_{\substack{j=0\\k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\left\{\hat{A}_{ij}x_1(k_{ij})\right\}-\sum_{i=0}^{2}\sum_{\substack{j=0\\k_{iv_j}\notin\mathcal{O}_{k_0}}}^{p^{i+1}-1}\left\{\hat{A}_{iv_j}A^{21}_{d(\kappa_p^j+1)}x_1(k_{(i+1)j})\right\}$$

$$+\sum_{i=1}^{3}\sum_{\substack{j=0\\k_{ij}\in\mathcal{O}_{k_0}}}^{p^i-1}\left\{\hat{A}_{ij}x_2(k_{ij})\right\}+\sum_{\substack{j=0\\k_{3j}\notin\mathcal{O}_{k_0}}}^{p^3-1}\left\{\hat{A}_{3j}x_2(k_{3j})\right\}.$$

Note that $k_{ij}=k_{(i-1)v_j}-d_{(\kappa_p^j+1)}(k_{(i-1)v_j})\le k_{(i-1)v_j}-\underline{d}_{(\kappa_p^j+1)}(k_{(i-1)v_j})<k_{(i-1)v_j}$. Then, by continuously decomposing $\hat{A}_{ij}x_2(k_{ij})$ for $i=3,4,\ldots,$ it can be concluded that there exists a limited positive integer $\mathcal{K}$ such that $k_{\mathcal{K}j}\le k_0$ and Eq. (22) holds.

(2). (Proof of Eq. (24)). From Eq. (22), we have

$$\|x_2(k)\|\le\underbrace{\sum_{i=1}^{\mathcal{K}}\sum_{\substack{j=0\\k_{ij}\in\mathcal{O}_{k_0}}}^{p^i-1}\left\{\|\hat{A}_{ij}\|\right\}\|\psi_2(k_0)\|_{\bar{d}}}_{1}+\underbrace{\|A^{21}\|\sum_{i=0}^{\mathcal{K}-1}\sum_{\substack{j=0\\k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\left\{\|\hat{A}_{ij}\|x_1(k_{ij})\right\}}_{2}$$

$$+\underbrace{\sum_{i=0}^{\mathcal{K}-1}\sum_{\substack{j=0\\k_{iv_j}\notin\mathcal{O}_{k_0}}}^{p^{i+1}-1}\|\hat{A}_{iv_j}A^{21}_{d(\kappa_p^j+1)}x_1(k_{(i+1)j})\|}_{3}\,. \tag{A.1}$$

Then the proof of Eq. (24) boils down to finding the upper bounds for the terms $\underbrace{\cdot}_{1}$, $\underbrace{\cdot}_{2}$ and $\underbrace{\cdot}_{3}$ in Eq. (A.1), which is decomposed in the following three parts:

Part 1. (Upper bound for the term $\underbrace{\cdot}_{1}$ in Eq. (A.1)). By Eqs. (19) and (20), k_{ij} and $\hat{A}_{ij}$ can be re-written as follows:

$$k_{ij}=k_{(i-1)v_j}-d_{(\kappa_p^j+1)}(k_{(i-1)v_j})\triangleq k_{(i-1)v_j}-d_{k_1},$$

$$\begin{aligned}\hat{A}_{ij} &= \hat{A}_{(i-1)v_j}\left(-A^{22}_{d(\kappa^j_p+1)}\right)\underline{\vartheta}^{-\bar{d}_{(\kappa^j_p+1)}(k_{(i-1)v_j})/2}\underline{\vartheta}^{\bar{d}_{(\kappa^j_p+1)}(k_{(i-1)v_j})/2}\\ &\triangleq \hat{A}_{(i-1)v_j}\left(-A^{22}_{dk_1}\right)\underline{\vartheta}^{-\bar{d}_{k_1}/2}\underline{\vartheta}^{\bar{d}_{k_1}/2},\end{aligned}$$

where k_1 is a positive integers between 1 and p. Similarly, by successively factorizing $k_{(i-1)v_j}, k_{(i-2)v_{v_j}}, \cdots$, and $\hat{A}_{(i-1)v_j}, \hat{A}_{(i-2)v_{v_j}}, \cdots$, we have

$$\begin{aligned}k_{ij} &= k - d_{k_i} - \cdots - d_{k_1},\\ \hat{A}_{ij} &= \left(-A^{22}_{dk_i}\right)\underline{\vartheta}^{-\bar{d}_{k_i}/2}\cdots\left(-A^{22}_{dk_1}\right)\underline{\vartheta}^{-\bar{d}_{k_1}/2}\underline{\vartheta}^{\hat{d}_{ij}/2},\end{aligned} \tag{A.2}$$

where $k_1, \ldots, k_i$ are positive integers between 1 and p, and $\hat{d}_{ij} \triangleq \sum_{\varepsilon=1}^{i} \bar{d}_{k_\varepsilon}$. When $k_{ij} \in \mathcal{O}_{k_0}$, it follows from Eq. (21) that $k_0 - \bar{d} \leq k - (d_{k_1} + \cdots + d_{k_i}) \leq k_0$, which implies that $k - \hat{d}_{ij} \leq k_0$. Thus, using this relation, Eq. (A.2) and noting $0 < \theta < 1$, the term $\underbrace{\cdot}_{1}$ in Eq. (A.1) can be bounded by

$$\sum_{i=1}^{\mathcal{K}} \sum_{\substack{j=0\\ k_{ij}\in\mathcal{O}_{k_0}}}^{p^i-1} \left\{\|\hat{A}_{ij}\|\right\} \|\psi_2(k_0)\|_{\bar{d}} \leq \sum_{i=1}^{\mathcal{K}} \sum_{\substack{j=0\\ k_{ij}\in\mathcal{O}_{k_0}}}^{p^i-1} \left\{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\right\} \|\psi_2(k_0)\|_{\bar{d}}\underline{\vartheta}^{(k-k_0)/2}. \tag{A.3}$$

For every i, the term $\sum_{k_{ij}\in\mathcal{O}_{k_0}} \{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\}$ sums up all $\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|$ satisfying $k_{ij} \in \mathcal{O}_{k_0}$, and its maximum value occurs in the case that $k_{ij} \in \mathcal{O}_{k_0}$ for all j between 0 and $p^i - 1$. Thus, $\sum_{j=0\ k_{ij}\in\mathcal{O}_{k_0}}^{p^i-1} \{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\}$ can be bounded by

$$\sum_{j=0}^{p^i-1} \left\{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\right\} = \sum_{j=0}^{p-1} \|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\| + \cdots + \sum_{j=p^i-p}^{p^i-1} \|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|. \tag{A.4}$$

Using Eqs. (19), (20) and the definition of $\hat{d}_{ij}$, $\sum_{j=0}^{p^i-1} \{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\}$ equals

$$\begin{aligned}&\sum_{j=0}^{p-1} \|\hat{A}_{(i-1)v_j}\underline{\vartheta}^{-\hat{d}_{(i-1)v_j}/2}A^{22}_{d(\kappa^j_p+1)}\underline{\vartheta}^{-\bar{d}_{(\kappa^j_p+1)}/2}\| + \cdots\\ &\quad + \sum_{j=p^i-p}^{p^i-1} \|\hat{A}_{(i-1)v_j}\underline{\vartheta}^{-\hat{d}_{(i-1)v_j}/2}A^{22}_{d(\kappa^j_p+1)}\underline{\vartheta}^{-\bar{d}_{(\kappa^j_p+1)}/2}\|\\ &\leq \|\hat{A}_{(i-1)v_0}\underline{\vartheta}^{-\hat{d}_{(i-1)v_0}/2}\| \sum_{j=0}^{p-1} \|A^{22}_{d(\kappa^j_p+1)}\underline{\vartheta}^{-\bar{d}_{(\kappa^j_p+1)}/2}\|\\ &\quad + \cdots + \|\hat{A}_{(i-1)v_{(p^i-p)}}\underline{\vartheta}^{-\hat{d}_{(i-1)v_{(p^i-p)}}/2}\| \sum_{j=p^i-p}^{p^i-1} \|A^{22}_{d(\kappa^j_p+1)}\underline{\vartheta}^{-\bar{d}_{(\kappa^j_p+1)}/2}\|.\end{aligned} \tag{A.5}$$

According to the definition of κ^j_p, all the summation terms on the right of $'\leq'$ in Eq. (A.5) can be bounded by $\sum_{s=1}^{p} \|A^{22}_{ds}\underline{\vartheta}^{-\bar{d}_s/2}\|$. Therefore,

$$\sum_{j=0}^{p^i-1} \{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\} \leq \sum_{j=0}^{p^{i-1}-1} \|\hat{A}_{(i-1)j}\underline{\vartheta}^{-\hat{d}_{(i-1)j}/2}\| \sum_{s=1}^{p} \|A^{22}_{ds}\underline{\vartheta}^{-\bar{d}_s/2}\|.$$

Factorizing $\sum_{j=0}^{p^{i-1}-1} \|\hat{A}_{(i-1)j}\underline{\vartheta}^{-\hat{d}_{(i-1)j}/2}\|$ with the same procedures in Eqs. (A.4) and (A.5) and iterating until $i=0$ yield

$$\sum_{j=0}^{p^i-1}\left\{\|\hat{A}_{ij}\underline{\vartheta}^{-\hat{d}_{ij}/2}\|\right\} \le \left[\sum_{s=1}^{p}\|A_{ds}^{22}\underline{\vartheta}^{-\bar{d}_s/2}\|\right]^i. \tag{A.6}$$

From $\|A_{ds}^{22}\| \le \chi_s\underline{\vartheta}^{\bar{d}_s/2}/(1+\tilde{d}_s)^{1/2}$, $\forall 1\le s\le p$, it follows that

$$\sum_{s=1}^{p}\|A_{ds}^{22}\underline{\vartheta}^{-\bar{d}_s/2}\| \le \sum_{s=1}^{p}\left[\chi_s/(1+\tilde{d}_s)^{1/2}\right]. \tag{A.7}$$

Then, using Eqs. (A.3), (A.6) and (A.7), $\underbrace{\cdot}_{1}$ in Eq. (A.1) can be bounded by

$$\sum_{i=1}^{\mathcal{K}}\left[\sum_{s=1}^{p}\|A_{ds}^{22}\underline{\vartheta}^{-\bar{d}_s/2}\|\right]^i\|\psi_2(k_0)\|_{\bar{d}}\underline{\vartheta}^{(k-k_0)/2} \le \underline{\upsilon}_1\underline{\vartheta}^{(k-k_0)/2}\|\psi_2(k_0)\|_{\bar{d}}. \tag{A.8}$$

Part 2. (upper bound for the term $\underbrace{\cdot}_{2}$ in Eq. (A.1)). From Eqs. (19) and (20), it holds that

$$\begin{aligned}
\|\hat{A}_{ij}\|\gamma^{k_{ij}-k_0} &\le \|\hat{A}_{(i-1)v_j}A^{22}_{d(\kappa_p^j+1)}\gamma^{k_{(i-1)v_j}-k_0}\gamma^{-d_{(\kappa_p^j+1)}(k_{(i-1)v_j})}\| \\
&\le \|\hat{A}_{(i-1)v_j}\gamma^{k_{(i-1)v_j}-k_0}A^{22}_{d(\kappa_p^j+1)}\gamma^{-\bar{d}_{(\kappa_p^j+1)}}\| \\
&\triangleq \|\hat{A}_{(i-1)v_j}\gamma^{k_{(i-1)v_j}-k_0}A^{22}_{dk_1}\gamma^{-\bar{d}_{k_1}}\|.
\end{aligned}$$

By continuously factorizing $\hat{A}_{(i-1)v_j}$ and $k_{(i-1)v_j}$, $\hat{A}_{(i-2)v_{v_j}}$ and $k_{(i-2)v_{v_j}}$, ..., in the same manner, we can obtain

$$\|\hat{A}_{ij}\|\gamma^{k_{ij}-k_0} \le \|I\gamma^{k-k_0}A^{22}_{dk_i}\gamma^{-\bar{d}_{k_i}}\cdots A^{22}_{dk_1}\gamma^{-\bar{d}_{k_1}}\| \le \|\hat{A}_{ij}\gamma^{-\hat{d}_{ij}}\|\gamma^{k-k_0}. \tag{A.9}$$

Using Eqs. (23) and (A.9), the term $\underbrace{\cdot}_{2}$ in Eq. (A.1) can be bounded by

$$\begin{aligned}
\|A^{21}\|\sum_{i=0}^{\mathcal{K}-1}\sum_{\substack{j=0\\k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\|\hat{A}_{ij}\|\|x_1(k_{ij})\| &\le \epsilon\|\psi(k_0)\|_{\bar{d}}\|A^{21}\|\sum_{i=0}^{\mathcal{K}-1}\sum_{\substack{j=0\\k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\|\hat{A}_{ij}\|\gamma^{k_{ij}-k_0} \\
&\le \epsilon\|\psi(k_0)\|_{\bar{d}}\|A^{21}\|\sum_{i=0}^{\mathcal{K}-1}\sum_{\substack{j=0\\k_{ij}\notin\mathcal{O}_{k_0}}}^{p^i-1}\left\{\|\hat{A}_{ij}\gamma^{-\hat{d}_{ij}}\|\right\}\gamma^{k-k_0}.
\end{aligned} \tag{A.10}$$

Note that for every i, the worst case of $k_{ij}\notin\mathcal{O}_{k_0}$ occurs in the case that $k_{ij}\notin\mathcal{O}_{k_0}$ for all j between 0 and p^i-1. Then, similar to the derivation of Eq. (A.6), it can be verified that

$$\sum_{j=0}^{p^i-1}\left\{\|\hat{A}_{ij}\gamma^{-\hat{d}_{ij}}\|\right\} \le \left[\sum_{s=1}^{p}\|A^{22}_{ds}\gamma^{-\bar{d}_s}\|\right]^i. \tag{A.11}$$

Since $0<\underline{\vartheta}<\gamma^2<1$, $\gamma^{-\bar{d}_s}<\underline{\vartheta}^{-\bar{d}_s/2}$ holds. Then, from Eqs. (A.10) and (A.11), the term $\underbrace{\cdot}_{2}$ in Eq. (A.1) can be finally bounded by

$$\begin{aligned}
&\epsilon\|\psi(k_0)\|_{\bar{d}}\|A^{21}\|\sum_{i=0}^{\mathcal{K}-1}\left[\sum_{s=1}^{p}\|A^{22}_{ds}\underline{\vartheta}^{-\bar{d}_s/2}\|\right]^{i}\gamma^{k-k_0}\\
&\quad\le\epsilon\|A^{21}\|(1-\Sigma^{\mathcal{K}}_{\underline{\vartheta}})/(1-\Sigma_{\underline{\vartheta}})\|\psi(k_0)\|_{\bar{d}}\gamma^{k-k_0}.
\end{aligned}\tag{A.12}$$

Part 3. (Upper bound for the term $\underbrace{\cdot}_{3}$ in Eq. (A.1)). Define $\|A^{21}_{d}\|=\max_{s=1,\ldots,p}\|A^{21}_{ds}\|$. From Eqs. (19) and (20), we have

$$\begin{aligned}
\|\hat{A}_{iv_j}\|\gamma^{k_{(i+1)j}-k_0}&\le\|\hat{A}_{(i-1)v_{v_j}}A^{22}_{d(\kappa_p^{v_j}+1)}\gamma^{k_{iv_j}-k_0}\gamma^{-d_{(\kappa_p^{j}+1)}(k_{iv_j})}\|\\
&\le\gamma^{-\bar{d}}\|\hat{A}_{(i-1)v_{v_j}}\gamma^{k_{(i-1)v_{v_j}}-k_0}A^{22}_{d(\kappa_p^{v_j}+1)}\gamma^{-d_{(\kappa_p^{v_j}+1)}(k_{(i-1)v_{v_j}})}\|\\
&\le\gamma^{-\bar{d}}\|\hat{A}_{(i-1)v_{v_j}}\gamma^{k_{(i-1)v_{v_j}}-k_0}A^{22}_{d(\kappa_p^{v_j}+1)}\gamma^{-\bar{d}_{(\kappa_p^{v_j}+1)}}\|\\
&\triangleq\gamma^{-\bar{d}}\|\widetilde{A}_{(i-1)v_{v_j}}\|.
\end{aligned}\tag{A.13}$$

Noting that $\sum_{j=0}^{p^{i+1}-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|$ equals

$$\begin{aligned}
&\sum_{j=0}^{p-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|+\cdots+\sum_{j=p^2-p}^{p^2-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|\\
&\quad+\sum_{j=p^2}^{p^2+p-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|+\cdots+\sum_{j=2p^2-p}^{2p^2-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|\\
&\quad+\cdots+\sum_{j=p^{i+1}-p^2}^{p^{i+1}-p^2+p-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|+\cdots+\sum_{j=p^{i+1}-p}^{p^{i+1}-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|
\end{aligned}$$

and in view of that v_{v_j} remains constant for $j\in\{hp^2,hp^2+1,\ldots,(h+1)p^2-1\}$, $h=0,1,2,\ldots,$ we have

$$\sum_{j=0}^{p^{i+1}-1}\|\widetilde{A}_{(i-1)v_{v_j}}\|\le\sum_{j=0}^{p^{i-1}-1}\left\{\|\hat{A}_{(i-1)j}\gamma^{k_{(i-1)j}-k_0}\|\right\}\left[p\sum_{s=1}^{p}\|A^{22}_{ds}\gamma^{-\bar{d}_s}\|\right].\tag{A.14}$$

Similar to the derivation of Eqs. (A.9) and (A.11), it can be obtained that

$$\begin{aligned}
&\|\hat{A}_{(i-1)j}\gamma^{k_{(i-1)j}-k_0}\|\le\|\hat{A}_{(i-1)j}\gamma^{-\hat{d}_{(i-1)j}}\|\gamma^{k-k_0},\\
&\sum_{j=0}^{p^{i-1}-1}\|\hat{A}_{(i-1)j}\gamma^{-\hat{d}_{(i-1)j}}\|\le\left[\sum_{s=1}^{p}\|A^{22}_{ds}\gamma^{-\bar{d}_s}\|\right]^{i-1}.
\end{aligned}\tag{A.15}$$

Noting $0<\underline{\vartheta}<\gamma^2<1$, then, from Eqs. (23), (A.13)–(A.15), the term $\underbrace{\cdot}_{3}$ in Eq. (A.1) can be bounded by

$$
\begin{aligned}
&\|A_d^{21}\| \sum_{i=0}^{\mathcal{K}-1} \sum_{\substack{j=0 \\ k_{iv_j} \notin \mathcal{O}_{k_0}}}^{p^{i+1}-1} \|\hat{A}_{iv_j}\| \|x_1(k_{(i+1)j})\| \\
&\leq \epsilon \|\psi(k_0)\|_{\bar{d}} \|A_d^{21}\| \sum_{i=0}^{\mathcal{K}-1} \sum_{\substack{j=0 \\ k_{iv_j} \notin \mathcal{O}_{k_0}}}^{p^{i+1}-1} \|\hat{A}_{iv_j}\| \gamma^{k_{(i+1)j}-k_0} \\
&\leq \epsilon \|\psi(k_0)\|_{\bar{d}} \gamma^{-\bar{d}} \|A_d^{21}\| \sum_{i=0}^{\mathcal{K}-1} \left[\sum_{s=1}^{p} \|A_{ds}^{22}\gamma^{-\bar{d}_s}\|\right]^{i-1} \left[p\sum_{s=1}^{p} \|A_{ds}^{22}\gamma^{-\bar{d}_s}\|\right] \gamma^{k-k_0} \\
&\leq \epsilon p \gamma^{-\bar{d}} \|A_d^{21}\| (1-\Sigma_{\underline{\vartheta}}^{\mathcal{K}})/(1-\Sigma_{\underline{\vartheta}}) \|\psi(k_0)\|_{\bar{d}} \gamma^{k-k_0}. \qquad \text{(A.16)}
\end{aligned}
$$

Now, from Eqs. (A.8), (A.12) and (A.16), inequality Eq. (22) holds.

(3). (Proof of Eq. (25)). Similar to the derivation of Eq. (A.2), $\hat{A}_{ij}$ can be re-written as: $\hat{A}_{ij} = \left(-A_{dk_i}^{22}\right)\bar{\vartheta}^{-\underline{d}_{k_i}/2} \ldots \left(-A_{dk_1}^{22}\right)\bar{\vartheta}^{-\underline{d}_{k_1}/2}\bar{\vartheta}^{\check{d}_{ij}/2}$, where $k_1, \ldots, k_i$ are positive integers between 1 and p, and $\check{d}_{ij} = \sum_{\varepsilon=1}^{i} \underline{d}_{k_\varepsilon}$. When $k_{ij} \in \mathcal{O}_{k_0}$, where k_{ij} follows the same definition in Eq. (A.2), $k_0 - \bar{d} \leq k - (d_{k_1} + \ldots + d_{k_i}) \leq k_0$, which implies that $\check{d}_{ij} \leq d_{k_1} + \ldots + d_{k_i} < k - k_0 + \bar{d}$. Thus, using the definition of $\check{d}_{ij}$ and upper bound of $\check{d}_{ij}$, and noting $\vartheta > 1$, the term $\underbrace{\cdot}_{1}$ in Eq. (A.1) can be bounded by

$$
\sum_{i=1}^{\mathcal{K}} \sum_{\substack{j=0 \\ k_{ij} \in \mathcal{O}_{k_0}}}^{p^i-1} \left\{\|\hat{A}_{ij}\|\right\} \|\psi_2(k_0)\|_{\bar{d}} \leq \sum_{i=1}^{\mathcal{K}} \sum_{\substack{j=0 \\ k_{ij} \in \mathcal{O}_{k_0}}}^{p^i-1} \left\{\|\hat{A}_{ij}\bar{\vartheta}^{-\check{d}_{ij}/2}\|\right\} \|\psi_2(k_0)\|_{\bar{d}} \bar{\vartheta}^{\bar{d}/2} \bar{\vartheta}^{(k-k_0)/2}.
$$

The remaining proof is similar to that in Eq. (2), and thus it is omitted due to space limitation.

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