Since the uniform asymptotic stability of the trivial solution of (3) guarantees the boundedness of $\|z(t)\|_2$, assuming zero initial condition for (3) we have

$$\begin{aligned} J_{zw} &= \int_0^\infty [z(t)^T z(t) - \gamma^2 w(t)^T w(t)]\, dt \\ &= \int_0^\infty \left[x(t)^T C_{CL}^T C_{CL} x(t) - \gamma^2 w(t)^T w(t)\right] dt. \end{aligned}$$

Using the same technique as in [17] and [24] and using (19), it follows that

$$\begin{aligned} J_{zw} = &\int_0^\infty \Big[x^T(t) C_{CL}^T C_{CL} x(t) - \gamma^2 w^T(t) w(t) \\ &+ \dot{V}(t, x_t)\Big]\, dt + V(t, x_t)\big|_{t=0} - V(t, x_t)\big|_{t=\infty} \\ \le &\int_0^\infty \left\{ \begin{bmatrix} x(t) \\ x(t-\tau) \\ w(t) \end{bmatrix}^T \begin{bmatrix} \begin{pmatrix} A_{CL}^T P + P A_{CL} \\ + C_{CL}^T C_{CL} + S \end{pmatrix} & P A_d & P B_1 \\ A_d^T P & -S & 0 \\ B_1^T P & 0 & -\gamma^2 I_p \end{bmatrix} \right. \\ &\left. \cdot \begin{bmatrix} x(t) \\ x(t-\tau) \\ w(t) \end{bmatrix} \right\} dt. \end{aligned} \tag{21}$$

The strictly negative inequality (4) and the quadratic form of (21) in $[x(t)\ \ x(t-\tau)\ \ w(t)]^T$ leads to $J_{zw} < 0$, i.e., the "$\mathcal{H}_\infty$ control" condition (20) is satisfied.

In conclusion, if there exist $P > 0$, $S > 0$, and $F \in \mathbf{R}^{m \times n}$ such that (4) and (5) are verified, then γ is α-suboptimal. □

References

[1] R. Bambang, E. Shimemura, and K. Uchida, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control with pole placement: State-feedback case," in *Proc. 1993 Amer. Contr. Conf.*, pp. 2777–2779.

[2] H. Bourlès, "α-stability and robustness of large-scale interconnected systems," *Int. J. Contr.*, vol. 36, pp. 2221–2232, 1987.

[3] ——, "α-stability of systems governed by a functional differential equation—Extension of results concerning linear delay systems," *Int. J. Contr.*, vol. 45, pp. 2233, 2234, 1987.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, vol. 15, 1994.

[5] M. Chilali and P. Gahinet, "$\mathcal{H}_\infty$ design with an α-stability constraint: An LMI approach," in *Proc. IFAC Workshop Robust Control Design*, Rio de Janeiro, Brazil, 1994, pp. 307–312.

[6] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, 1989.

[7] A. Feliachi and A. Thowsen, "Memoryless stabilization of linear delay-differential systems," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 586–587, 1981.

[8] E. Feron, V. Balakrishnan, and S. Boyd, "A design of stabilizing state feedback for delay systems via convex optimization," in *Proc. 31st Conf. Decision Control*, Tucson, AZ, 1992, pp. 147–148.

[9] J. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Math. Sciences, vol. 99. New York: Springer-Verlag, 1991.

[10] M. Ikeda and T. Ashida, "Stabilization of linear systems with time-varying delay," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 369–370, 1979.

[11] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*, Math. Sci. and Eng., vol. 180. New York: Academic, 1986.

[12] N. N. Krasovskii, *Stability of Motion*. Stanford, CA: Stanford Univ. Press, 1963.

[13] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless $\mathcal{H}_\infty$ controllers for state delayed systems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 159–162, 1994.

[14] M. Malek-Zavarei and M. Jamshidi, *Time Delay Systems: Analysis, Optimization and Applications*, Systems and Control Series, vol. 9. Amsterdam, 1987.

[15] T. Mori, N. Fukuma, and M. Kuwahara, "On an estimate of the decay rate for stable linear delay systems," *Int. J. Contr.*, vol. 36, pp. 95–97, 1982.

[16] S. I. Niculescu, C. E. de Souza, J.-M. Dion, and L. Dugard, "Robust stability and stabilization for uncertain linear systems with state delay: Single delay case (I)," in *Proc. IFAC Workshop Robust Control Design*, Rio de Janeiro, Brazil, 1994, pp. 469–474.

[17] ——, "Robust $\mathcal{H}_\infty$ memoryless control for uncertain linear systems with time-varying delay," in *3rd European Control Conf.*, Roma, Italy, 1995, pp. 1814–1818.

[18] S. I. Niculescu, J.-M. Dion, and L. Dugard, "α-stability criteria for linear systems with delayed state," Internal Note L.A.G. 94-150, 1994.

[19] S. I. Niculescu, "On the stability and stabilization of linear systems with delayed state in French," Ph.D. dissertation (in French), INPG, Laboratoire d'Automatique de Grenoble, France, Feb. 1996.

[20] A. W. Olbrot, "A sufficiently large time delay in feedback loop must destroy exponential stability of any decay rate," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 367–368, 1984.

[21] I. R. Petersen, "Disturbance attenuation and H_∞ optimization: A design method based on the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 427–429, 1987.

[22] J. C. Shen, B. S. Chen, and F. C. Kung, "Memoryless stabilization of uncertain dynamic delay systems: Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 638–640, 1991.

[23] K.-K. Shyu and J.-J. Yan, "Robust stability of uncertain time-delay systems and its stabilization by variable structure control," *Int. J. Contr.*, vol. 57, pp. 237–246, 1993.

[24] L. Xie and C. E. de Souza, "Robust stabilization and disturbance attenuation for uncertain delay systems," in *Proc. 2nd European Control Conf.*, Groningen, The Netherlands, 1993, pp. 667–672.

Robust Exponential Stability of Uncertain Systems with Time-Varying Delays

Silviu-Iulian Niculescu, Carlos E. de Souza, Luc Dugard, and Jean-Michel Dion

***Abstract*— This paper focuses on the problem of robust exponential stability of a class of uncertain systems described by functional differential equations with time-varying delays. The uncertainties are assumed to be continuous time-varying, nonlinear, and norm bounded. Sufficient conditions for robust exponential stability are given for both single and multiple delays cases.**

***Index Terms*—Exponential stability, robust stability, time-varying delay, uncertain systems.**

I. Introduction

Time delays are frequently encountered in the behavior of many physical processes and very often are the main cause for poor performance and instability of control systems. In view of this, the

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S.-I. Niculescu is with HEUDIASYC, Centre de Recherche de Royallieu, Université de Technologie de Compiègne, Compiègne cedex, France.

C. E. de Souza is with LNCC/CNPq, 25651-070 Petropolis, Rio de Janiero, Brazil.

L. Dugard and J.-M. Dion are with the Laboratoire d'Automatique de Grenoble, ENSIEG, BP 46, 38402, St. Martin d'Hères, France.

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robustness issue of time-delay systems is a topic of great practical importance which has attracted a great deal of interest for several decades; see, e.g., [7] and [10].

Recently, increasing attention has been devoted to the study of robust stability of uncertain linear systems with delayed state variables. Over the past few years, a number of robust stability conditions for uncertain systems with constant delays have been proposed in the literature; see, e.g. [1], [12], [13], [17], [23], and [24]. Following Mori [11], the stability criteria for time-delay systems can be classified in two classes according to their dependence on the delay size: *delay-independent* [1], [17], [24] or *delay-dependent* [12], [19]. A general treatment for these cases and also a classification of the corresponding methods can be found in [15].

In the case of systems with time-varying delays, although significant results on analysis and synthesis of control systems have been obtained in past three decades [6], [8], [9], [16], [25], to date, the problem of *robust stability* for such systems has not been fully investigated. Thus, *delay-independent* stability conditions have been considered in [5], [9], [14], and [21] using an appropriate Lyapunov–Krasovskii functional candidate (see [4] and [7]) via a Lyapunov [5] or Riccati [14], [21] equation. To the authors' best knowledge, the *delay-dependent* case has not been addressed in the time-varying delay case.

In this paper we consider the issue of *exponential stability* of a class of uncertain systems with time-varying delays. These systems are described by functional differential equations with uncertainties in both the "current" and "delayed" states. The uncertainties are assumed to be continuous time-varying, nonlinear, and cone-bounded. Uncertain systems with single as well as multiple time-varying delays have been considered. The focal point of this paper is to investigate conditions which guarantee exponential stability for all admissible uncertainties.

The robust stability conditions derived in this paper generalize the *delay-independent* results in [18] to handle systems with uncertainties, as well as the those in [23] by allowing for time-varying delays. Furthermore, the *delay-dependent* case is also treated.

The paper is organized as follows: in Section II the problem statement is given. The main results are developed in Section III. Some concluding remarks end the paper.

Notations: The following notations will be used throughout the paper. $\mathbf{R}$ denotes the set of real numbers, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space, $\mathbf{R}^{n\times m}$ denotes the set of all $n \times m$ real matrices, and $i = \overline{1,\, n}$ denotes the integers $1, 2, \cdots, n$. $\|\cdot\|$ refers to either the Euclidean vector norm or the induced matrix 2-norm and $\mu(\cdot)$ denotes the matrix measure corresponding to the induced matrix 2-norm, defined by $\mu(A) = \lim_{\varepsilon\to 0^+} \frac{\|I+\varepsilon A\|-1}{\varepsilon}$.

II. Problem Statement

Consider uncertain systems with time-varying delay described by the following functional differential equation:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + \sum_{i=1}^{n_d} A_{d_i} x(t-\tau_i(t)) + f(x(t),t)\\ &\quad + \sum_{i=1}^{n_d} f_{d_i}(x(t-\tau_i(t)),t) \qquad (1)\end{aligned}$$

with the initial condition

$$x(\theta) = \phi(\theta), \qquad \forall \theta \in \mathcal{E}_{t_0} \qquad (2)$$

where $\phi : \mathcal{E}_{t_0} \mapsto \mathbf{R}^n$ is a continuous norm-bounded initial function (see also [3]) and

$$\mathcal{E}_{t_0} = \bigcup_{i=1}^{n_d} \{t \in \mathbf{R} : t = \eta - \tau_i(\eta) \le 0, \eta \ge t_0\}$$

where $x(t) \in \mathbf{R}^n$ is the state, A and A_{d_i} are known real constant matrices, $\tau_i(t)$ are time-varying delays, and $f : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ and $f_{d_i} : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ $(i = \overline{1, n_d})$ are unknown nonlinear functions which represent time-varying state-dependent uncertainties.

We shall make the following assumption for the time-delays $\tau_i(t)$.

Assumption 1: The time-varying delays $\tau_i(t)$ are positive continuous differentiable functions satisfying

$$\tau_i(t) \le \tau(t) \le \bar{\tau}, \qquad \forall t \ge t_0, \qquad i = \overline{1, n_d} \qquad (3)$$

$$\dot{\tau}(t) \le \alpha < 1, \qquad \forall t \ge t_0 \qquad (4)$$

where $\tau(t)$ is a given strictly positive continuous differentiable function, and $\bar{\tau} > 0$ and $\alpha \ge 0$ are given real numbers.

The admissible uncertainties f and f_{d_i} $(i = \overline{1, n_d})$ are assumed to satisfy the following boundedness conditions.

Assumption 2: There exist nonnegative numbers β and β_{d_i}, $i = \overline{1, n_d}$ such that for all $x \in \mathbf{R}^n$ and for all t

$$\|f(x,t)\| \le \beta \|x\| \qquad (5)$$

$$\|f_{d_i}(x,t)\| \le \beta_{d_i} \|x\|, \qquad i = \overline{1, n_d}. \qquad (6)$$

Remark 1: We observe that Assumptions 1 and 2 are sufficient conditions for the existence and uniqueness of a solution to the functional differential equation (1).

Notice that this system model including time-varying delay and norm bounded uncertainty describes, for example, the behavior of a nonlinear chemical process including several coupled tanks. In this case, the *delays* are of *transport* type (see, e.g., [7]). A complete description of the model and simulation results can be found in [20].

Throughout this paper we will use the following concept of *robust exponential* stability.

Definition 1: The uncertain time delay system (1) is said to be *robustly exponentially stable with a decay rate* λ if the trivial solution $x(t) \equiv 0$ is exponentially stable with a decay rate λ for all admissible uncertainties, i.e., there exist $k \ge 1$ and $\lambda > 0$ such that

$$\|x(t)\| \le k \sup_{\theta \in \mathcal{E}_{t_0}} \{\|x(\theta)\|\} e^{-\lambda(t-t_0)}.$$

Without loss of generality, in the sequel we shall consider $t_0 \equiv 0$.

The main aim of this paper is to investigate conditions for the robust exponential stability of the class of uncertain systems described by (1) and (2). In fact, we address the *delay-independent* as well as the *delay-dependent* exponential stability cases. In order to have a delay-independent stability condition one needs *supplementary* restrictions on the linear part of the system model: the *Hurwitz stability* of the nondelayed matrix A (see also [3]).

For the delay-dependent case, we have considered here the *simplest* case, i.e., to give a bound on the delay which guarantees the stability, when the system *free* of delays is stable. Other comments on the delay-dependent stability-type results and some comparisons with the delay-independent ones may be found in [15]. Notice that all the criteria proposed here are *only* sufficient but *easy to check* for a numerical example.

III. Robust Stability Results

In the sequel we shall present results concerning the robust exponential stability of uncertain systems with time-varying delays of the form (1). In a first step, the analysis is given for a single delay

$n_d = 1$. We will consider two cases, depending on the stability of the matrices A and $A + A_d$.

First, we will deal with the case when the state matrix A is Hurwitz stable. Under such a condition we have the following *delay-independent* stability result.

Theorem 1: Consider the system (1) and (2) with $n_d = 1$ and assume that A is a Hurwitz stable matrix satisfying

$$\|\exp(At)\| \le k_A \cdot \exp(-\eta_A t) \tag{7}$$

for some real numbers $k_A \ge 1$ and $\eta_A > 0$. If the inequality

$$\frac{k_A}{\eta_A}(\|A_d\| + \beta + \beta_d) < 1 \tag{8}$$

holds, then the transient response of $x(t)$ satisfies

$$\|x(t)\| \le M \sup_{\theta \in \mathcal{E}_0}\{\|\phi(\theta)\|\} \exp\left(-\sigma \int_0^t \frac{d\theta}{\tau(\theta)}\right), \quad \forall t \ge 0,\ M \ge 1 \tag{9}$$

where $\sigma > 0$ is the unique positive solution of the transcendental equation

$$1 - \frac{k_A}{\eta_A}\beta - \frac{\sigma}{\eta_A \tau(0)} = \frac{k_A}{\eta_A}(\|A_d\| + \beta_d)\exp\left(\frac{\sigma}{1-\alpha}\right). \tag{10}$$

Furthermore, system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$.

A proof of Theorem 1 is given in Appendix I.

Remark 2: Notice that the transcendental equation (10) has a unique positive solution because the left-hand side is a continuous decreasing function, the right-hand side is a continuous increasing function, and for $\sigma = 0$, by virtue of (8), the right-hand side is less than the left-hand side.

Remark 3: For the particular case of a constant time-delay, i.e., $\tau(t) \equiv \tau$ and $\alpha = 0$, (9) and (10) become, respectively

$$\|x(t)\| \le M \sup_{\theta \in [-\tau, 0]} \|\{\phi(\theta)\|\} \exp\left(-\frac{\sigma}{\tau}t\right), \qquad \forall t \ge 0$$

and

$$1 - \frac{k_A}{\eta_A}\beta - \frac{\sigma}{\eta_A \tau} = \frac{k_A}{\eta_A}(\|A_d\| + \beta_d)\exp(\sigma).$$

It follows from the proof of Theorem 1 in Appendix I that in this case we can choose $M = k_A$. If in addition we denote $\sigma_1 = \frac{\sigma}{\tau}$ and $r_0 = \frac{k_A}{\eta_A}(\|A_d\| + \beta_d)$, Theorem 1 recovers the results of [23].

Remark 4: When there are no uncertainties in (1), i.e., $\beta = 0$ and $\beta_d = 0$, Theorem 1 gives the same result as that in [18]. Furthermore, in this case (8) becomes

$$\frac{k_A}{\eta_A}\|A_d\| < 1. \tag{11}$$

By comparing (8) and (11) we can conclude that if (1) without the uncertainties f and f_d is exponentially stable, then the exponential stability of this system is preserved in the presence of any uncertainties f and f_d satisfying Assumption 2 and with

$$\beta + \beta_d < \frac{\eta_A}{k_A} - \|A_d\|.$$

Recall that the matrix measure $\mu(\cdot)$ satisfies the inequality (see, e.g., [2])

$$\|\exp(At)\| \le \exp(\mu(A)t), \qquad \forall t \ge 0. \tag{12}$$

This allows us to restate Theorem 1 in term of the measure of the matrix A by letting $k_A = 1$ and $\eta_A = -\mu(A)$ in (8) and (10). This is summarized in the following corollary.

Corollary 1: Consider (1) and (2) with $n_d = 1$ and assume that A is a Hurwitz stable matrix. If the inequality

$$\mu(A) + \|A_d\| + \beta + \beta_d < 0$$

holds, then system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$, where $\sigma > 0$ is the unique positive solution of the transcendental equation

$$\mu(A) + \beta + \frac{\sigma}{\tau(0)} + (\|A_d\| + \beta_d)\exp\left(\frac{\sigma}{1-\alpha}\right) = 0.$$

Remark 5: From the corollary, it follows that one implicitly needs

$$\mu(A) + \|A_d\| < 0$$

as condition which implies the Hurwitz stability property of $A + A_d$. Indeed, for *delay-independent* stability, one needs that the matrices A and $A + A_d$ are simultaneously stable (see [15] and the references therein).

As specified before, the assumption of Hurwitz stability of the matrix A is relatively restrictive. An obvious necessary condition for the robust exponential stability of the uncertain system (1) and (2) is the exponential stability of the trivial solution of this system without time-delay and uncertainties, i.e., the asymptotic stability of the system

$$\dot{x}(t) = (A + A_d)x(t).$$

We have the following *delay-dependent* result.

Theorem 2: Consider the system (1) and (2) with $n_d = 1$ and assume that $A + A_d$ is a Hurwitz stable matrix satisfying

$$\|\exp((A + A_d)t)\| \le k\exp(-\eta t) \tag{13}$$

for some real numbers $k \ge 1$ and $\eta > 0$. If the inequality

$$\frac{k}{\eta}\left[\bar{\tau}\left(\|A_d A\| + \left\|A_d^2\right\| + \|A_d\|\beta + \|A_d\|\beta_d\right) + \beta + \beta_d\right] < 1 \tag{14}$$

holds, then the transient response of $x(t)$ satisfies

$$\|x(t)\| \le M \sup_{\theta \in \mathcal{E}_0}\{\|\phi(\theta)\|\} \exp\left(-\sigma \int_0^t \frac{d\theta}{\tau(\theta)}\right), \quad \forall t \ge 0,\ M \ge 1 \tag{15}$$

where $\sigma > 0$ is the unique positive solution of the transcendental equation

$$\begin{aligned} 1-\frac{k}{\eta}\beta - \frac{\sigma}{\eta\tau(0)} &= \frac{k}{\eta}\exp\left(\frac{\sigma}{1-\alpha}\right)\Big[\bar{\tau}(\|A_d A\| + \|A_d\|\beta) + \beta_d \\ &\quad + \bar{\tau}\left(\left\|A_d^2\right\| + \|A_d\|\beta_d\right)\exp\left(\frac{\sigma}{1-\alpha}\right)\Big]. \end{aligned} \tag{16}$$

Furthermore, system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$.

A sketch of the proof of Theorem 2 is given in Appendix II.

Remark 6: Similarly as in Remark 2, we note that the transcendental equation (16) is guaranteed to have a unique positive solution.

Remark 7: For the particular case of a constant and known time delay, i.e., $\tau(t) \equiv \tau = \bar{\tau}$ and $\alpha = 0$, (14) becomes

$$\frac{k}{\eta}\left[\tau\left(\|A_d A\| + \left\|A_d^2\right\| + \|A_d\|\beta + \|A_d\|\beta_d\right) + \beta + \beta_d\right] < 1$$

which is in general less conservative than the so-called delay-dependent condition given in [23]

$$\frac{k}{\eta}[\tau\|A_d\|(\|A\| + \|A_d\| + \beta + \beta_d) + \beta + \beta_d] < 1.$$

As in the case when A is a Hurwitz stable matrix, using the matrix measure of $A + A_d$ we have the following result which can be easily obtained from Theorem 2 by letting $k = 1$ and $\eta = -\mu(A + A_d)$.

Corollary 2: Consider the system (1) and (2) with $n_d = 1$ and assume that $A + A_d$ is a stable Hurwitz matrix. If the inequality

$$\mu(A + A_d) + \bar{\tau}\left(\|A_d A\| + \left\|A_d^2\right\| + \|A_d\|\beta + \|A_d\|\beta_d\right) + \beta + \beta_d < 0$$

holds, then system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$, where σ is the unique positive solution of the transcendental equation

$$\mu(A + A_d) + \beta + \frac{\sigma}{\tau(0)} + \exp\left(\frac{\sigma}{1-\alpha}\right)\left[\bar{\tau}(\|A_d A\| + \|A_d\|\beta) + \beta_d + \bar{\tau}\left(\left\|A_d^2\right\| + \|A_d\|\beta_d\right)\exp\left(\frac{\sigma}{1-\alpha}\right)\right] = 0.$$

Let us consider now the general case $n_d > 1$. Using the same ideas, Theorem 1 becomes the following.

Theorem 3: Consider system (1) and (2) and assume that A is a Hurwitz stable matrix satisfying

$$\|\exp(At)\| \le k_A \cdot \exp(-\eta_A t) \tag{17}$$

for some real numbers $k_A \ge 1$ and $\eta_A > 0$. If the following inequality:

$$\frac{k_A}{\eta_A}\sum_{i=1}^{n_d}(\|A_{d_i}\| + \beta_{d_i}) + \beta < 1 \tag{18}$$

holds, then the transient response of $x(t)$ satisfies

$$\|x(t)\| < M \sup_{\theta \in \bar{\mathcal{E}}_0}\{\|\phi(\theta)\|\}\exp\left(-\sigma\int_0^t \frac{d\theta}{\tau(\theta)}\right), \quad \forall t \ge 0 \;\; M \ge 1 \tag{19}$$

where $\sigma > 0$ is the unique positive solution of the transcendental equation

$$1 - \frac{k_A}{\eta_A}\beta - \frac{\sigma}{\eta_A \tau(0)} = \frac{k_A}{\eta_A}\sum_{i=1}^{n_d}(\|A_{d_i}\| + \beta_{d_i})\exp\left(\frac{\sigma}{1-\alpha}\right). \tag{20}$$

Furthermore, system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$.

Proof: It can be established using the arguments as in the single time-varying case, where $\|A_d\|$ and β_d are replaced by $\sum_{i=1}^{n_d}\|A_{d_i}\|$ and $\sum_{i=1}^{n_d}\beta_{d_i}$, respectively. □

For the particular case of constant time delays, i.e., $\tau_i(t) \equiv \tau_i$ and $\alpha = 0$, it is easy to verify that the Theorem 3 specializes to the corollary as below.

Corollary 3: Consider the system (1) and (2) and assume that A is a Hurwitz stable matrix satisfying

$$\|\exp(At)\| \le k_A \cdot \exp(-\eta_A t)$$

for some real numbers $k_A \ge 1$ and $\eta_A > 0$. If the following inequality:

$$\frac{k_A}{\eta_A}\sum_{i=1}^{n_d}(\|A_{d_i}\| + \beta_{d_i}) + \beta < 1$$

holds, then the transient response of $x(t)$ satisfies

$$\|x(t)\| < k_A \sup_{\theta \in [-\tau, 0]}\{\|\phi(\theta)\|\}\exp\left(-\frac{\sigma}{\tau}t\right), \qquad \forall t \ge 0$$

where $\tau = \max\{\tau_i, i = \overline{1, n_d}\}$ and $\sigma > 0$ is the unique positive solution of the transcendental equation

$$1 - \frac{k_A}{\eta_A}\beta - \frac{\sigma}{\eta_A \tau} = \frac{k_A}{\eta_A}\sum_{i=1}^{n_d}(\|A_{d_i}\| + \beta_{d_i})\exp(\sigma).$$

Remark 8: Similarly to the single-delay case, when there are no uncertainties in system (1) and (2), i.e., $\beta = 0$ and $\beta_{d_i} = 0, i = \overline{1, n_d}$, Theorem 3 recovers the result of [18].

Remark 9: As in the previous section, we observe that in view of (12), Theorem 3 can be restated in terms of the matrix measure of A by simply letting $k_A = 1$ and $\eta_A = -\mu(A)$ in (18) and (20).

Consider now the *delay-dependent* case. The matrix $A + \sum_{i=1}^{n_d} A_{d_i}$ is assumed to be Hurwitz stable. Note that the latter assumption is a necessary condition for the exponential stability of the system (1) and (2) in the absence of time delays and uncertainties.

Theorem 4: Consider the system (1) and (2) and assume that $A + \sum_{i=1}^{n_d} A_{d_i}$ is a Hurwitz stable matrix satisfying

$$\left\|\exp\left(\left(A + \sum_{i=1}^{n_d} A_{d_i}\right)t\right)\right\| \le k \cdot \exp(-\eta t) \tag{21}$$

for some real numbers $k \ge 1$ and $\eta > 0$. If the inequality

$$\frac{k}{\eta}\left[\bar{\tau}\sum_{i=1}^{n_d}\left(\|A_{d_i}A\| + \sum_{j=1}^{n_d}\|A_{d_i}A_{d_j}\| + \|A_{d_i}\|\beta + \|A_{d_i}\|\sum_{j=1}^{n_d}\beta_{d_j}\right) + \beta + \sum_{i=1}^{n_d}\beta_{d_i}\right] < 1 \tag{22}$$

holds, then the transient response of $x(t)$ satisfies

$$\|x(t)\| \le M \sup_{\theta \in \mathcal{E}_{01}}\{\|\phi(\theta)\|\}\exp\left(-\sigma\int_0^t \frac{d\theta}{\tau(\theta)}\right), \quad \forall t \ge 0, \; M \ge 1 \tag{23}$$

where $\sigma > 0$ is the unique positive solution of the transcendental equation

$$1 - \frac{k}{\eta}\beta - \frac{\sigma}{\eta\tau(0)} = \frac{k}{\eta}\exp\left(\frac{\sigma}{1-\alpha}\right) \cdot \left[\bar{\tau}\sum_{i=1}^{n_d}(\|A_{d_i}A\| + \|A_{d_i}\|\beta) + \sum_{i=1}^{n_d}\beta_{d_i} + \bar{\tau}\sum_{i=1}^{n_d}\sum_{j=1}^{n_d}(\|A_{d_i}A_{d_j}\| + \|A_{d_i}\|\beta_{d_j})\exp\left(\frac{\sigma}{1-\alpha}\right)\right]. \tag{24}$$

Furthermore, the system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$.

The proof of Theorem 4 follows the same ideas as in the proof of Theorem 2 and makes use of the approach given in [18].

IV. Conclusion

In this paper, we have derived a number of criteria of robust exponential stability for a class of uncertain systems with time-varying delays. Both cases of single and multiple time-varying delays have been tackled. The proposed stability criteria are relatively simple to be checked numerically and generalize the results in [18] to handle systems with uncertainties and those in [23] by allowing for time-varying delays.

Appendix I
Proof of Theorem 1

The proof uses ideas given in [18]. We first introduce the following differential equation:

$$\dot{y}(t) = -(\eta_A - k_A\beta)y(t) + q(t)y(t-\tau(t)) \tag{25}$$

where

$$q(t) = \left(\eta_A - k_A\beta - \frac{\sigma}{\tau(t)}\right)\exp\left(-\sigma\int_{t-\tau(t)}^{t}\frac{d\theta}{\tau(\theta)}\right). \tag{26}$$

A direct verification shows that

$$y(t) = C_0\,\exp\left(-\sigma\int_0^t\frac{d\theta}{\tau(\theta)}\right) \tag{27}$$

where C_0, a real constant, is a solution of (25).

Next observe that in view of (4), by applying twice the mean-value theorem to $\int_{t-\tau(t)}^{t}\frac{d\theta}{\tau(\theta)}$, it follows that there exist real numbers θ_1 and θ_2 satisfying $0 < \theta_2 < \theta_1 < 1$ such that

$$\begin{aligned}\int_{t-\tau(t)}^{t}\frac{d\theta}{\tau(\theta)} &= \frac{\tau(t)}{\tau(t)-\theta_1\tau(t)\dot{\tau}(t-\theta_2\tau(t))}\\ &= \frac{1}{1-\theta_1\dot{\tau}(t-\theta_2\tau(t))} \le \frac{1}{1-\alpha}.\end{aligned} \tag{28}$$

Also, considering (28), it results from (26) that for $\sigma > 0$ satisfying (10) we have

$$\begin{aligned}q(t) &\ge \left(\eta_A - k_A\beta - \frac{\sigma}{\tau(0)}\right)\exp\left(-\frac{\sigma}{1-\alpha}\right)\\ &= k_A(\|A_d\| + \beta_d).\end{aligned} \tag{29}$$

We will now show that for $\sigma > 0$ satisfying (10) and for a particular choice of C_0, we can ensure that the solution of (25) is an upper bound for the solution of (1) and (2).

Let us choose C_0 such that the following inequalities hold simultaneously:

$$\begin{aligned}&y(t) \ge \|\phi(t)\|, \quad \forall t \in \mathcal{E}_0\\ &C_0 \ge k_A \sup_{\theta\in\mathcal{E}_0}\|\phi(\theta)\|.\end{aligned} \tag{30}$$

The solution of the functional differential equation (1) can be written as

$$\begin{aligned}x(t) = {} & \exp(At)\phi(0) + \int_0^t \exp(A(t-\theta))A_d x(\theta-\tau(\theta))\,d\theta\\ &+ \int_0^t \exp(A(t-\theta))[f(x(\theta),\theta) + f_d(x(\theta-\tau(\theta)),\theta)]\,d\theta.\end{aligned}$$

Hence, taking into account (5)–(7), we have that

$$\begin{aligned}\|x(t)\| \le {} & k_A\exp(-\eta_A t)\sup_{\theta\in\mathcal{E}_0}\|\phi(\theta)\|\\ &+ \int_0^t k_A\beta\exp(-\eta_A(t-\theta))\|x(\theta)\|\,d\theta\\ &+ \int_0^t k_A(\|A_d\|+\beta_d)\,\exp(-\eta_A(t-\theta))\\ &\times\|x(\theta-\tau(\theta))\|\,d\theta, \qquad \forall t \ge 0.\end{aligned} \tag{31}$$

Next, considering the term $k_A\beta y(t) + q(t)y(t-\tau(t))$ in (25) as an inhomogeneous term, we can write the solution of this equation as

$$\begin{aligned}y(t) = {} & C_0\exp(-\eta_A t) + \int_0^t k_A\beta\exp(-\eta_A(t-\theta))y(\theta)\,d\theta.\\ &+ \int_0^t \exp(-\eta_A(t-\theta))q(\theta)y(\theta-\tau(\theta))\,d\theta.\end{aligned} \tag{32}$$

Now we are ready to compare $\|x(t)\|$ with $y(t)$. Letting $z(t) = \|x(t)\| - y(t)$ and using (31) and (32) we obtain

$$\begin{aligned}z(t) \le {} & \left(k_A\sup_{\theta\in\mathcal{E}_0}\|\phi(\theta)\| - C_0\right)\exp(-\eta_A t)\\ &+ k_A\int_0^t \exp(-\eta_A(t-\theta))[(\|A_d\|+\beta_d)z(\theta-\tau(\theta))\\ &+ \beta z(\theta)]\,d\theta + \int_0^t \exp(-\eta_A(t-\theta))[k_A(\|A_d\|\\ &+ \beta_d) - q(\theta)]y(\theta-\tau(\theta))\,d\theta, \qquad \forall t \ge 0.\end{aligned}$$

In view of the inequalities of (29) and (30), it results that

$$\begin{aligned}z(t) \le k_A\int_0^t & \exp(-\eta(t-\theta))[(\|A_d\|+\beta_d)z(\theta-\tau(\theta))\\ &+ \beta z(\theta)]\,d\theta, \qquad \forall t \ge 0.\end{aligned} \tag{33}$$

Also, note that the inequalities of (30) imply that

$$z(t) \le 0, \qquad \forall t \in \mathcal{E}_0. \tag{34}$$

Furthermore, since $z(t)$ is continuous, the inequality of (34) also holds in some neighborhood of zero.

We will now prove that $z(t) \le 0$ for all $t > 0$. By contradiction, assume that this is not true and let $t^* > 0$ be the smallest t such that $z(t^*) > 0$. In view of (34) and considering that $z(\theta) \le 0$ for any $0 < \theta < t^*$, it follows from (33) that $z(t^*) \le 0$ which cannot be true due to the hypothesis made. Hence, we have that $z(t) \le 0$ for all $t \ge 0$ and in view of (27) we obtain

$$\begin{aligned}\|x(t)\| &\le C_0\exp\left(-\sigma\int_0^t\frac{d\theta}{\tau(\theta)}\right)\\ &= M\sup_{\theta\in\mathcal{E}_0}\{\|\phi(\theta)\|\}\exp\left(-\sigma\int_0^t\frac{d\theta}{\tau(\theta)}\right), \qquad \forall t \ge 0\end{aligned}$$

where

$$M = \frac{C_0}{\sup_{\theta\in\mathcal{E}_0}\|\phi(\theta)\|} \ge 1.$$

Finally, from the boundedness of $\tau(t)$ it follows that

$$\|x(t)\| \le M\sup_{\theta\in\mathcal{E}_0}\{\|\phi(\theta)\|\}\exp\left(-\frac{\sigma}{\bar{\tau}}t\right), \qquad \forall t \ge 0$$

and thus we conclude that the system (1) and (2) is robustly exponentially stable with a decay rate $\frac{\sigma}{\bar{\tau}}$. □

Appendix II
Proof of Theorem 2

The proof uses similar arguments as in the proof of Theorem 1 given in Appendix I.

We consider the following functional differential equation:

$$\begin{aligned}\dot{\xi}(t) = {} & (A + A_d)\xi(t) + f(\xi(t),t) + f_d(\xi(t-\tau(t),t)\\ &+ \int_{t-\tau(t)}^{t}\big[A_dA\xi(\theta) + A_d^2\xi(\theta-\tau(\theta))\\ &+ A_d f(\xi(\theta),\theta) + A_d f_d(\xi(\theta-\tau(\theta)),\theta)\big]\,d\theta\end{aligned} \tag{35}$$

obtained from (1) by using the Leibniz–Newton formula

$$\xi(t) - \xi(t-\tau(t)) = \int_{t-\tau(t)}^{t} \dot{\xi}(\theta)\, d\theta.$$

The initial condition for (35) is given by the vector-valued function ϕ on the set

$$\mathcal{E}_{01} = \{t \in \mathbf{R} : t = \theta - \tau(\theta) \le 0, \theta \ge 0\} \\ \cup \{t \in \mathbf{R} : t = \theta - \tau(\theta) - \tau(\theta - \tau(\theta)) \le 0,\ \theta \ge 0\}.$$

It should be observed that each solution of (1) is also a solution of (35); see, e.g., [18].

We also introduce the following differential equation:

$$\dot{y}(t) = -(\eta - k\beta)y(t) + q(t)y(t-\tau(t)) \tag{36}$$

where

$$q(t) = \left(\eta - k\beta - \frac{\sigma}{\tau(t)}\right) \exp\left(-\sigma \int_{t-\tau(t)}^{t} \frac{d\theta}{\tau(\theta)}\right) \tag{37}$$

and the proof runs similarly to the proof of Theorem 1 for the "new" functional differential equation (35). □

References

[1] E. Cheres, S. Gutman, and Z. J. Palmor, "Robust stabilization of uncertain dynamic systems including state delay," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1199–1203, 1989.

[2] C. A. Desoer and M. Vidyasagar, *Feedback System: Input–Output Properties*. New York: Academic, 1975.

[3] L. E. Els'gol'ts and S. B. Norkin, "Introduction to the theory and applications of differential equations with deviating arguments," *Math. Sci. Eng.*, vol. 105, 1973.

[4] J. Hale and S. M. Verduyn Lunel, "Introduction to functional differential equations," *Appl. Math. Sci.*, vol. 99, 1991.

[5] M. Ikeda and T. Ashida, "Stabilization of linear systems with time-varying delay," *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 369–370, 1979.

[6] R. A. Johnson, "Functional equations, approximations and dynamic response of systems with variable time-delay," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 398–401, 1972.

[7] V. B. Kolmanovskii and V. R. Nosov, "Stability of functional differential equations," *Math. Science Eng.*, vol. 180, 1986.

[8] B. Lehman and K. Shujaee, "Dealy independent stability conditions and decay estimates for time-varying functional differential equations," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1673–1676, 1994.

[9] J. Louisell, "A stability analysis for a class of differential-delay equations having time-varying delay," in *Delay Differential Equations and Dynamical Systems*, Lecture Notes in Math., vol. 1475, S. Busenberg and M. Martelli, Eds. Berlin, Germany: Springer Verlag, 1991, pp. 225–242.

[10] M. Malek-Zavarei and M. Jamshidi, *Time-Delays Systems: Analysis, Optimization and Applications*. North-Holland, 1987.

[11] T. Mori, "Criteria for asymptotic stability of linear time-delay systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 158–160, 1985.

[12] S. I. Niculescu, C. E. de Souza, J. M. Dion, and L. Dugard, "Robust stability and stabilization of uncertain linear systems with state delay: Single delay case (I)," in *Proc. Workshop Robust Control Design*, Rio de Janeiro, Brazil, 1994, pp. 469–474.

[13] ——, "Robust stability and stabilization for uncertain linear systems with state delay: Multiple delays case (II)," in *Proc. Workshop Robust Control Design*, Rio de Janeiro, Brazil, 1994, pp. 475–480.

[14] ——, "Robust $\mathcal{H}_\infty$ memoryless control for uncertain linear systems with time-varying delays," in *Proc. 3rd European Control Conf.*, Rome, Italy, 1995, pp. 1802–1808.

[15] S. I. Niculescu, "On the stability and stabilization of linear systems with delayed state (in French)," Ph.D. dissertation, INPG, Laboratoire d'Automatique de Grenoble, Feb. 1996.

[16] M. T. Nihtila, "Finite pole assignements for systems with time-varying input delay," in *Proc. 30th IEEE Conf. Decision and Control*, Brighton, U.K., 1991, pp. 927–928.

[17] S. Phoojaruenchanachai and K. Furuta, "Memoryless stabilization of uncertain linear systems including time-varying delay," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1022–1026, 1992.

[18] V. I. Rozkhov and A. M. Popov, "Inequalities for solutions of certain systems of differential equations with large time-lag," *Diff. Eq.*, vol. 7, pp. 271–278, 1971.

[19] T. J. Su and C. G. Huang, "Robust stability of delay dependence for linear uncertain systems," *IEEE Trans. Automat. Control*, vol. 37, pp. 1656–1659, 1992.

[20] R. Ştefan, "The development of toolbox for time-delay systems modelization. Application to the control of a tank system (in French)," Int. Rep. L.A.G., 1994.

[21] E. I. Verriest, "Robust stability of time varying systems with unknown bounded delays," in *Proc. 33rd IEEE Conf. Decision and Contr.*, Lake Buena Vista, FL, 1994, pp. 417–422.

[22] J. A. Walker, *Dynamical Systems and Evolution Equations*. New York: Plenum, 1980.

[23] S. S. Wang, B. S. Chen, and T. P. Lin, "Robust stability of uncertain time-delay systems," *Int. J. Contr.*, vol. 46, pp. 963–976, 1987.

[24] L. Xie and C. E. de Souza, "Robust stabilization and disturbance attenuation for uncertain delay system," in *Proc. 1993 European Contr. Conf.*, Groningen, The Netherlands, pp. 667–672.

[25] L. A. Zadeh, "Operational analysis of variable-delay systems," in *Proc. IRE*, 1952, vol. 40, pp. 564–568.

A Discrete Iterative Learning Control for a Class of Nonlinear Time-Varying Systems

Chiang-Ju Chien

***Abstract*—A discrete iterative learning control is presented for a class of discrete-time nonlinear time-varying systems with initial state error, input disturbance, and output measurement noise. A feedforward learning algorithm is designed under a stabilizing controller and is updated by more than one past control data in the previous trials. A systematic approach is developed to analyze the convergence and robustness of the proposed learning scheme. It is shown that the learning algorithm not only solves the convergence and robustness problems but also improves the learning rate for discrete-time nonlinear time-varying systems.**

***Index Terms*—Discrete-time, iterative learning control, nonlinear time-varying system.**

I. Introduction

The iterative learning control (ILC) method has been proposed by Arimoto *et al.* [1] for the control systems which can perform the same task repetitively. To date, most of the proposed learning algorithms have been used in applications on robot control where the robot system is required to execute the same motion repetitively, with a certain periodicity. The basic learning controller for generating the present control input is based on the previous control history and a learning mechanism. A recent textbook [2] about ILC for

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The author is with the Department of Electronic Engineering, Hua Fan University, Shihtin, Taipei Hsien, Taiwan, R.O.C. (e-mail: cjc@huafan.hfu.edu.tw).

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