Simulation and High-Performance Computing

Part 6: Method of Lines for Heat and Wave Equations

Steffen Börm

Christian-Albrechts-Universität zu Kiel

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Example: Heat equation

Model problem: Dissipation of heat in a two-dimensional domain.

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= c\Delta u(t,x) + g(t,x) && \text{for all } t \in \mathbb{R},\ x \in \Omega, \\ u(t,x) &= 0 && \text{for all } t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned}$$

where $u(t,x)$ is the temperature at time t in a point x and $g(t,x)$ describes external heating or cooling.

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Method of lines: Replace functions by grid functions and differential operators by finite difference operators.

$$\begin{aligned}
\frac{\partial u_h}{\partial t}(t,x) &= c\Delta_h u_h(t,x) + g(t,x) && \text{for all } t \in \mathbb{R},\ x \in \Omega_h,\\
u_h(t,x) &= 0 && \text{for all } t \in \mathbb{R},\ x \in \partial\Omega_h.
\end{aligned}$$

Reformulation as an ODE

Method of lines yields semi-discrete system

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u_h(t,x) &= 0 && \text{for all } t \in \mathbb{R},\ x \in \partial\Omega_h.
\end{aligned}$$

In order to be able to apply timestepping methods, we rewrite $u_h(t,x) = u_h(t)(x)$ and obtain a system of ordinary differential equations

$$u_h'(t) = c\Delta_h u_h(t) + g_h(t) \qquad \text{for all } t \in \mathbb{R}$$

with grid functions $u_h(t)\colon \Omega_h \to \mathbb{R}$ for all $t \in \mathbb{R}$.

Adjustments: Δ_h is modified to take care of missing boundary points, and the grid functions $g_h(t)\colon \Omega_h \to \mathbb{R}$ are defined by

$$g_h(t)(x) := g(t,x) \qquad \text{for all } t \in \mathbb{R},\ x \in \Omega_h.$$

Timestepping for the heat equation

Heat equation as explicit ordinary differential equation:

$$u_h(0) = u_{h,0}, \qquad u_h'(t) = c\Delta_h u_h(t) + g_h(t) \qquad \text{for all } t \in \mathbb{R}.$$

Explicit Euler method applied to this equation yields

$$\begin{aligned}
\tilde{u}_h(t_0) &= u_{h,0}, \\
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta(c\Delta_h \tilde{u}_h(t_k) + g_h(t_k)) \qquad \text{for all } k \in \mathbb{N}_0.
\end{aligned}$$

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Implementation in C:

```
for(k=0; k<n; k++) {
  heatsource(t, du);                /* Write g_h to du */
  addeval_laplace(-c, u, du);       /* Add c Delta_h u */

  add(delta, du, u);                /* Euler step */
  t += delta;
}
```

CFL condition

Eigenvectors of $-\Delta_h$: For all $\nu, \mu \in [1 : N]$, the grid function

$$e_{\nu\mu}(x) := \sin(\pi\nu\, x_1) \sin(\pi\mu\, x_2) \qquad\qquad \text{for all } x \in \Omega_h$$

satisfies

$$-\Delta_h e_{\nu\mu} = \lambda_{\nu\mu} e_{\nu\mu}, \qquad \lambda_{\nu\mu} := \frac{4}{h^2}\bigl(\sin^2(\pi\nu\, h/2) + \sin^2(\pi\mu\, h/2)\bigr).$$

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Explicit Euler with $g = 0$ and $u_h(0) = e_{\nu\mu}$ yields

$$\tilde{u}_h(t_{k+1}) = \tilde{u}_h(t_k) + \delta c \Delta_h \tilde{u}_h(t_k) = (1 - \delta c \lambda_{\nu\mu})\tilde{u}_h(t_k).$$

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Courant-Friedrichs-Levy condition: To avoid oscillations, we ensure

$$\delta c\, \lambda_{\nu\mu} < 1 \qquad \Longleftarrow \qquad \delta c \frac{8}{h^2} \le 1 \qquad \Longleftrightarrow \qquad \delta \le \frac{h^2}{8c}.$$

Experiment: Explicit methods for the heat equation

Approach: $g = 0$, $u_h(0) = e_{\nu\mu}$, error measured at time $t = 1$.

		Expl. Euler		Runge	
δ	$8c\delta/h^2$	error	ratio	error	ratio
$1/16$	2.1_{+3}	1.97_{+39}		3.11_{+87}	
$1/32$	1.1_{+3}	4.58_{+82}		4.12_{+169}	
$1/64$	5.3_{+2}	8.03_{+159}		∞	
$1/16384$	2.1_{+0}	∞		∞	
$1/32768$	1.0_{+0}	2.73_{-8}		5.49_{-12}	
$1/65536$	5.2_{-1}	1.36_{-8}	2.0	1.37_{-12}	4.0
$1/131072$	2.6_{-1}	6.83_{-9}	2.0	3.43_{-13}	4.0
$1/262144$	1.3_{-1}	3.41_{-9}	2.0	8.57_{-14}	4.0

Observation: No convergence while CFL condition $8c\delta/h^2 \leq 1$ violated.

Implicit methods

Problem: The CFL condition forces us to use very small timesteps with explicit timestepping methods.

Implicit Euler with $g = 0$ and $u_h(0) = e_{\nu\mu}$ yields

$$\tilde{u}_h(t_{k+1}) = \tilde{u}_h(t_k) + \delta c \Delta_h \tilde{u}_h(t_{k+1}),$$
$$\tilde{u}_h(t_{k+1}) = \frac{1}{1 + \delta c \lambda_{\nu\mu}} \tilde{u}_h(t_k).$$

Proper exponential decay without oscillations, since $\lambda_{\nu\mu} > 0$.

Crank-Nicolson with $g = 0$ and $u_h(0) = e_{\nu\mu}$ yields

$$\tilde{u}_h(t_{k+1}) = \tilde{u}_h(t_k) + \frac{\delta}{2}\big(c\Delta_h \tilde{u}_h(t_k) + c\Delta_h \tilde{u}_h(t_{k+1})\big),$$
$$\tilde{u}_h(t_{k+1}) = \left(\frac{1 - \frac{\delta}{2} c \lambda_{\nu\mu}}{1 + \frac{\delta}{2} c \lambda_{\nu\mu}}\right) \tilde{u}_h(t_k).$$

Experiment: Implicit methods for the heat equation

Approach: $g = 0$, $u_h(0) = e_{\nu\mu}$, error measured at time $t = 1$.

		Impl. Euler		Crank-Nicolson	
δ	$8c\delta/h^2$	error	ratio	error	ratio
$1/16$	2.1_{+3}	4.46_{-3}		4.43_{-6}	
$1/32$	1.1_{+3}	3.56_{-4}	12.5	2.23_{-6}	2.0
$1/64$	5.3_{+2}	5.33_{-5}	6.7	6.74_{-7}	3.3
$1/128$	2.6_{+2}	1.37_{-5}	3.9	1.77_{-7}	3.8
$1/256$	1.3_{+2}	4.88_{-6}	2.8	4.48_{-8}	4.0
$1/512$	6.6_{+1}	2.06_{-6}	2.4	1.12_{-8}	4.0
$1/1024$	3.3_{+1}	9.50_{-7}	2.2	2.81_{-9}	4.0
$1/2048$	1.7_{+1}	4.56_{-7}	2.1	7.02_{-10}	4.0

Observation: Expected convergence rates although CFL condition violated.

Example: Wave equation

Model problem: Propagation of waves in a two-dimensional domain.

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) &= c\Delta u(t,x) + g(t,x) && \text{for all } t \in \mathbb{R},\ x \in \Omega, \\ u(t,x) &= 0 && \text{for all } t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned}$$

where $u(t,x)$ is the displacement at time t in a point x and $g(t,x)$ describes external forces.

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where $u(t,x)$ is the displacement at time t in a point x and $g(t,x)$ describes external forces.

Method of lines: Replace functions by grid functions and differential operators by finite difference operators.

$$u_h''(t) = c\Delta_h u_h(t) + g_h(t) \qquad \text{for all } t \in \mathbb{R}.$$

Timestepping for the wave equation

Idea: Introduce the velocity $v_h(t) := u_h'(t)$ to obtain the familiar form

$$\begin{aligned} u_h'(t) &= v_h(t), \\ v_h'(t) &= c\Delta_h u_h(t) + g_h(t) \qquad \text{for all } t \in \mathbb{R}. \end{aligned}$$

Explicit Euler:

$$\begin{aligned} \tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta \tilde{v}_h(t_k), \\ \tilde{v}_h(t_{k+1}) &= \tilde{v}_h(t_k) + \delta c \Delta_h \tilde{u}_h(t_k) + \delta g_h(t_k). \end{aligned}$$

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$$\tilde{u}_h(t_{k+1}) = \tilde{u}_h(t_k) + \delta \tilde{v}_h(t_k) + \delta^2 g_h(t_{k+1}) + \delta^2 c \Delta_h \tilde{u}_h(t_{k+1})$$

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Implicit Euler: Requires us to solve two linear systems per timestep.

$$\begin{aligned} \tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta \tilde{v}_h(t_{k+1}) \\ \tilde{v}_h(t_{k+1}) &= \tilde{v}_h(t_k) + \delta c \Delta_h \tilde{u}_h(t_{k+1}) + \delta g_h(t_{k+1}). \\ (1 - \delta^2 c \Delta_h) \tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta \tilde{v}_h(t_k) + \delta^2 g_h(t_{k+1}) \\ (1 - \delta^2 c \Delta_h) \tilde{v}_h(t_{k+1}) &= \tilde{v}_h(t_k) + \delta c \Delta_h \tilde{u}_h(t_k) + \delta g_h(t_{k+1}) \end{aligned}$$

Second-order timestepping

Leapfrog:

$$\begin{aligned}
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta\, \tilde{v}_h(t_{k+1/2}),\\
\tilde{v}_h(t_{k+3/2}) &= \tilde{v}_h(t_{k+1/2}) + \delta\, c\Delta_h \tilde{u}_h(t_{k+1}) + \delta\, g_h(t_{k+1}).
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\end{aligned}$$

Crank-Nicolson:

$$\begin{aligned}
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \tfrac{\delta}{2}\big(\tilde{v}_h(t_k) + \tilde{v}_h(t_{k+1})\big),\\
\tilde{v}_h(t_{k+1}) &= \tilde{v}_h(t_k) + \tfrac{\delta}{2} c\Delta_h\big(\tilde{u}_h(t_k) + \tilde{u}_h(t_{k+1})\big) + \tfrac{\delta}{2}\big(g_h(t_k) + g_h(t_{k+1})\big).
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$$\begin{aligned}\left(1 - c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{u}_h(t_{k+1}) &= \left(1 + c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{u}_h(t_k) + \delta\,\tilde{v}_h(t_k)\\ &\quad + \tfrac{\delta^2}{4}\bigl(g_h(t_k) + g_h(t_{k+1})\bigr),\end{aligned}$$

Second-order timestepping

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$$\tilde{u}_h(t_{k+1}) = \tilde{u}_h(t_k) + \delta\, \tilde{v}_h(t_{k+1/2}),$$
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Crank-Nicolson:

$$\tilde{u}_h(t_{k+1}) = \tilde{u}_h(t_k) + \tfrac{\delta}{2}\big(\tilde{v}_h(t_k) + \tilde{v}_h(t_{k+1})\big),$$
$$\tilde{v}_h(t_{k+1}) = \tilde{v}_h(t_k) + \tfrac{\delta}{2} c\Delta_h\big(\tilde{u}_h(t_k) + \tilde{u}_h(t_{k+1})\big) + \tfrac{\delta}{2}\big(g_h(t_k) + g_h(t_{k+1})\big).$$
$$\begin{aligned}\left(1 - c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{u}_h(t_{k+1}) &= \left(1 + c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{u}_h(t_k) + \delta\, \tilde{v}_h(t_k)\\ &\quad + \tfrac{\delta^2}{4}\big(g_h(t_k) + g_h(t_{k+1})\big),\end{aligned}$$

Second-order timestepping

Leapfrog:

$$\begin{aligned}
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta\,\tilde{v}_h(t_{k+1/2}),\\
\tilde{v}_h(t_{k+3/2}) &= \tilde{v}_h(t_{k+1/2}) + \delta\,c\Delta_h\tilde{u}_h(t_{k+1}) + \delta\,g_h(t_{k+1}).
\end{aligned}$$

Crank-Nicolson:

$$\begin{aligned}
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \tfrac{\delta}{2}\big(\tilde{v}_h(t_k) + \tilde{v}_h(t_{k+1})\big),\\
\tilde{v}_h(t_{k+1}) &= \tilde{v}_h(t_k) + \tfrac{\delta}{2}c\Delta_h\big(\tilde{u}_h(t_k) + \tilde{u}_h(t_{k+1})\big) + \tfrac{\delta}{2}\big(g_h(t_k) + g_h(t_{k+1})\big).
\end{aligned}$$

$$\begin{aligned}
\left(1 - c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{u}_h(t_{k+1}) &= \left(1 + c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{u}_h(t_k) + \delta\,\tilde{v}_h(t_k)\\
&\quad + \tfrac{\delta^2}{4}\big(g_h(t_k) + g_h(t_{k+1})\big),\\
\left(1 - c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{v}_h(t_{k+1}) &= \left(1 + c\tfrac{\delta^2}{4}\Delta_h\right)\tilde{v}_h(t_k) + \delta\,c\Delta_h\tilde{u}_h(t_k)\\
&\quad + \tfrac{\delta}{2}\big(g_h(t_k) + g_h(t_{k+1})\big).
\end{aligned}$$

Second-order timestepping

Leapfrog:

$$\begin{aligned}
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \delta\, \tilde{v}_h(t_{k+1/2}),\\
\tilde{v}_h(t_{k+3/2}) &= \tilde{v}_h(t_{k+1/2}) + \delta\, c\Delta_h \tilde{u}_h(t_{k+1}) + \delta\, g_h(t_{k+1}).
\end{aligned}$$

Crank-Nicolson: Requires us to solve two linear systems per timestep.

$$\begin{aligned}
\tilde{u}_h(t_{k+1}) &= \tilde{u}_h(t_k) + \tfrac{\delta}{2}\big(\tilde{v}_h(t_k) + \tilde{v}_h(t_{k+1})\big),\\
\tilde{v}_h(t_{k+1}) &= \tilde{v}_h(t_k) + \tfrac{\delta}{2} c\Delta_h\big(\tilde{u}_h(t_k) + \tilde{u}_h(t_{k+1})\big) + \tfrac{\delta}{2}\big(g_h(t_k) + g_h(t_{k+1})\big).
\end{aligned}$$

$$\begin{aligned}
\left(1 - c\tfrac{\delta^2}{4}\Delta_h\right) \tilde{u}_h(t_{k+1}) &= \left(1 + c\tfrac{\delta^2}{4}\Delta_h\right) \tilde{u}_h(t_k) + \delta\, \tilde{v}_h(t_k)\\
&\quad + \tfrac{\delta^2}{4}\big(g_h(t_k) + g_h(t_{k+1})\big),\\
\left(1 - c\tfrac{\delta^2}{4}\Delta_h\right) \tilde{v}_h(t_{k+1}) &= \left(1 + c\tfrac{\delta^2}{4}\Delta_h\right) \tilde{v}_h(t_k) + \delta\, c\Delta_h \tilde{u}_h(t_k)\\
&\quad + \tfrac{\delta}{2}\big(g_h(t_k) + g_h(t_{k+1})\big).
\end{aligned}$$

Experiment: Timestepping for the wave equation

Approach: $g = 0$, $u_h(0) = e_{\nu\mu}$, $v_h(0) = 0$, error measured at time $t = 1$.

	Leapfrog		Impl. Euler		Crank-Nicol.	
δ	error	ratio	error	ratio	error	ratio
1/16	3.10_{+19}		1.07_{+2}		4.64_{+1}	
1/32	1.63_{+33}		8.60_{+1}	1.2	1.17_{+1}	4.0
1/64	4.19_{+35}		5.50_{+1}	1.6	2.94_{+0}	4.0
1/128	3.68_{-1}		3.11_{+1}	1.8	7.36_{-1}	4.0
1/256	9.20_{-2}	4.0	1.66_{+1}	1.9	1.84_{-1}	4.0
1/512	2.30_{-2}	4.0	8.53_{+0}	1.9	4.60_{-2}	4.0
1/1024	5.75_{-3}	4.0	4.33_{+0}	2.0	1.15_{-2}	4.0
1/2048	1.44_{-3}	4.0	2.18_{+0}	2.0	2.87_{-3}	4.0

Observation: CFL condition required for Leapfrog, but not for both implicit methods.

Conservation of energy

Reminder: The energy of the mass-spring system is conserved in the differential equation and the Crank-Nicolson method.

Energy of the wave equation:

$$\tilde{E}_h(t_k) := \frac{1}{2}\langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle - \frac{c}{2}\langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle.$$

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$$\tilde{E}_h(t_k) := \frac{1}{2}\langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle - \frac{c}{2}\langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle.$$

Crank-Nicolson conserves the energy (if $g_h = 0$):

$$\begin{aligned}\tilde{E}(t_{k+1}) - E(t_k) &= \tfrac{1}{2}\left(\langle \tilde{v}_h(t_{k+1}), \tilde{v}_h(t_{k+1})\rangle - \langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle\right) \\ &\quad - \tfrac{c}{2}\left(\langle \tilde{u}_h(t_{k+1}), \Delta_h \tilde{u}_h(t_{k+1})\rangle - \langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle\right)\end{aligned}$$

Conservation of energy

Reminder: The energy of the mass-spring system is conserved in the differential equation and the Crank-Nicolson method.

Energy of the wave equation:

$$\tilde{E}_h(t_k) := \frac{1}{2}\langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle - \frac{c}{2}\langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle.$$

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&\quad - \frac{c}{2}\left(\langle \tilde{u}_h(t_{k+1}), \Delta_h \tilde{u}_h(t_{k+1})\rangle - \langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle\right) \\
&= \frac{1}{2}\langle \tilde{v}_h(t_{k+1}) - \tilde{v}_h(t_k), \tilde{v}_h(t_{k+1}) + \tilde{v}_h(t_k)\rangle \\
&\quad - \frac{c}{2}\langle \tilde{u}_h(t_{k+1}) - \tilde{u}_h(t_k), \Delta_h(\tilde{u}_h(t_{k+1}) + \tilde{u}_h(t_k))\rangle
\end{aligned}$$

Conservation of energy

Reminder: The energy of the mass-spring system is conserved in the differential equation and the Crank-Nicolson method.

Energy of the wave equation:

$$\tilde{E}_h(t_k) := \frac{1}{2}\langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle - \frac{c}{2}\langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle.$$

Crank-Nicolson conserves the energy (if $g_h = 0$):

$$\begin{aligned}
\tilde{E}(t_{k+1}) - E(t_k) &= \frac{1}{2}\left(\langle \tilde{v}_h(t_{k+1}), \tilde{v}_h(t_{k+1})\rangle - \langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle\right) \\
&\quad - \frac{c}{2}\left(\langle \tilde{u}_h(t_{k+1}), \Delta_h \tilde{u}_h(t_{k+1})\rangle - \langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle\right) \\
&= \frac{1}{2}\langle \tilde{v}_h(t_{k+1}) - \tilde{v}_h(t_k), \tilde{v}_h(t_{k+1}) + \tilde{v}_h(t_k)\rangle \\
&\quad - \frac{c}{2}\langle \tilde{u}_h(t_{k+1}) - \tilde{u}_h(t_k), \Delta_h(\tilde{u}_h(t_{k+1}) + \tilde{u}_h(t_k))\rangle \\
&= \delta\frac{c}{4}\langle c\Delta_h(\tilde{u}_h(t_k) + \tilde{u}_h(t_{k+1})), \tilde{v}_h(t_{k+1}) + \tilde{v}_h(t_k)\rangle \\
&\quad - \delta\frac{c}{4}\langle \tilde{v}_h(t_k) + \tilde{v}_h(t_{k+1}), \Delta_h(\tilde{u}_h(t_{k+1}) + \tilde{u}_h(t_k))\rangle
\end{aligned}$$

Conservation of energy

Reminder: The energy of the mass-spring system is conserved in the differential equation and the Crank-Nicolson method.

Energy of the wave equation:

$$\tilde{E}_h(t_k) := \frac{1}{2}\langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle - \frac{c}{2}\langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle.$$

Crank-Nicolson conserves the energy (if $g_h = 0$):

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\tilde{E}(t_{k+1}) - E(t_k) &= \frac{1}{2}\left(\langle \tilde{v}_h(t_{k+1}), \tilde{v}_h(t_{k+1})\rangle - \langle \tilde{v}_h(t_k), \tilde{v}_h(t_k)\rangle\right) \\
&\quad - \frac{c}{2}\left(\langle \tilde{u}_h(t_{k+1}), \Delta_h \tilde{u}_h(t_{k+1})\rangle - \langle \tilde{u}_h(t_k), \Delta_h \tilde{u}_h(t_k)\rangle\right) \\
&= \frac{1}{2}\langle \tilde{v}_h(t_{k+1}) - \tilde{v}_h(t_k), \tilde{v}_h(t_{k+1}) + \tilde{v}_h(t_k)\rangle \\
&\quad - \frac{c}{2}\langle \tilde{u}_h(t_{k+1}) - \tilde{u}_h(t_k), \Delta_h(\tilde{u}_h(t_{k+1}) + \tilde{u}_h(t_k))\rangle \\
&= \delta\frac{c}{4}\langle c\Delta_h(\tilde{u}_h(t_k) + \tilde{u}_h(t_{k+1})), \tilde{v}_h(t_{k+1}) + \tilde{v}_h(t_k)\rangle \\
&\quad - \delta\frac{c}{4}\langle \tilde{v}_h(t_k) + \tilde{v}_h(t_{k+1}), \Delta_h(\tilde{u}_h(t_{k+1}) + \tilde{u}_h(t_k))\rangle = 0.
\end{aligned}$$

Summary

Method of lines: Discretize in space to obtain a system of coupled ordinary differential equations. $\rightarrow$ Solve by timestepping methods.

Heat equation: Explicit methods require the CFL condition, i.e., the stepsize δ has to be quite small.

Implicit methods can avoid this requirement, but require us to solve a linear system in each step.

Wave equation: Explicit methods require the CFL condition, implicit methods can avoid it, but require us to solve two linear systems each step.

Crank-Nicolson method conserves the energy.