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The Lambert W Function Approach to Time Delay Systems and the *LambertW_DDE* Toolbox

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Abstract: An overview of the recently developed Lambert W function approach for analysis and control of linear time invariant time delay systems (TDS) with a single known delay is provided. A solution via the Lambert W function is first presented for systems of order one, then extended to higher order systems using the matrix Lambert W function. Free and forced solutions, stability, observability, controllability, and observer and controller design via eigenvalue assignment can all be studied based on this framework. The use of the Matlab-based open source software in the *LambertW_DDE* Toolbox is introduced using examples.

Keywords: time delay, delay differential equations, Lambert W function, software.

1. INTRODUCTION

Time delay systems (TDS) arise in numerous natural and engineered systems, such as processes with transport delays, traffic flow problems, biological systems, teleoperation and many others. The literature on TDS is quite extensive, and includes several excellent books and review papers, e.g. (Bellman & Cooke 1963; Hale & Lunel 1963; Stepan 1989; Kolmanovskii & Myshkis 1999; Richard 2003; Gu & Niculescu 2006; Sipahi et al 2011). This paper focuses on a specific and recently developed approach, based on the classical Lambert W function, for analysis and control of linear time invariant TDS with a single known delay (Yi, Nelson & Ulsoy 2010).

1.1 Motivation and Background

For a given system of linear time invariant (LTI) ordinary differential equations (ODEs), without delay, in standard state equation form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

one can obtain the closed-form free and forced solutions in terms of a state transition matrix and a convolution integral (Chen 1998). As the system of ODEs in (1) has a finite spectrum, the stability can be determined by examining the locations of this finite number of eigenvalues in the s-plane.

Controllability and observability can be determined by the rank of the controllability and observability matrices respectively, and by the evaluation of the controllability/observability Gramians. If the system is controllable, a closed-loop controller can be designed by a variety of methods, including state feedback control and eigenvalue assignment. Similarly, if a system is observable, then a state estimator, or observer, can be designed, e.g., using eigenvalue assignment.

While such system analysis and control techniques are standard for LTI systems described by ODEs as in Eq. (1), they are often difficult to achieve for LTI TDS described by delay differential equations because of their infinite spectrum arising from the delays. Recently methods, based on the Lambert W function, have been proposed, developed and demonstrated for the analysis and control of LTI TDS, which enable the analysis and control design steps outlined above to be applied in a manner analogous to LTI systems of ODEs (Asl & Ulsoy 2003; Yi, Nelson & Ulsoy 2010).

There are numerous natural and engineered systems where time delays are significant (e.g., biological systems, economic models, supply chains, traffic flow, teleoperation, networked control systems, automotive control systems) and the extension of the system analysis and control tools, which are standard for ODEs, to systems described by DDEs can have a major impact.

1.2 Purpose and Scope

The purpose of this paper is to provide a succinct overview of the Lambert W function approach to the analysis and control of LTI TDS with a single known delay and to introduce, via simple examples, the use of the open source software

LambertW_DDE Toolbox, which is available for downloading from the web (Duan 2010).

2. THEORY, EXAMPLES & NUMERICAL SIMULATION

2.1 Lambert W Function

By definition (Lambert 1758; Euler 1777), every function $W(s)$ that satisfies:

$$W(s)e^{W(s)} = s \tag{2}$$

is called a Lambert W function (Corless et al 1996). The Lambert W function, with complex argument s, is a complex valued function with infinite branches, $k = 0, \pm 1, \pm 2, .., \pm\infty$, where s is either a scalar (i.e., scalar Lambert W function) or a matrix (i.e., matrix Lambert W function). The scalar Lambert W function is available as an embedded function in many computational software systems, e.g., see the function *lambertw* in MATLAB. The matrix Lambert W function (Yi, Nelson & Ulsoy 2010) can be obtained using a similarity transformation and can be readily evaluated using the *LambertW_DDE* Toolbox (Duan 2010). These functions are useful in combinatorics (e.g., the enumeration of trees) as well as relativity and quantum mechanics. They can be used to solve various equations involving exponentials (e.g. the maxima of the Planck, Bose-Einstein, and Fermi-Dirac distributions) as well as in the solution of delay differential equations as discussed here.

2.2 Scalar Case

Consider the first-order TDS (Asl & Ulsoy 2003):

$$\dot{x}(t) = ax(t) + a_d x(t-h) + bu(t) \tag{3}$$

with constant known parameters a, b and a_d, and where h is the constant known delay. The initial condition $x(t=0) = x_0$ and preshape function $x(t) = g(t)$ for $-h \le t < 0$, must be specified. The Lambert W function is applied to solve the transcendental characteristic equation of Eq. (3), which can be written as:

$$(s-a)e^{sh} = a_d \tag{4}$$

Multiplying both sides of Eq. (4) by he^{-ah} yields:

$$h(s-a)e^{h(s-a)} = a_d he^{-ah} \tag{5}$$

Based on the definition of the Lambert W function in Eq. (2) it is clear that

$$W(a_d he^{-ah})e^{W(a_d he^{-ah})} = a_d he^{-ah} \tag{6}$$

Comparing Eqs. (5) and (6)

$$h(s-a) = W(a_d he^{-ah}) \tag{7}$$

Thus, the solution of the characteristic equation in Eq. (4) can be written in terms of the Lambert W function as:

$$s = \frac{1}{h}W(a_d he^{-ah}) + a \tag{8}$$

The infinite spectrum of the scalar DDE in (3) is, thus, obtained using the infinite branches of the Lambert W function, and is given explicitly in terms of parameters a, a_d and h of the system. The roots of the characteristic equation (4), for $k = 0, \pm 1, \pm 2, \pm 3, .., \pm\infty$, are:

$$s_k = \frac{1}{h}W_k(a_d he^{-ah}) + a \tag{9}$$

Furthermore, stability is determined by the rightmost eigenvalue in the s-plane, which Shinozaki and Mori (2006) have shown, for Eq. (3), can be obtained using only the principal (i.e., $k = 0$) branch of the Lambert W function.

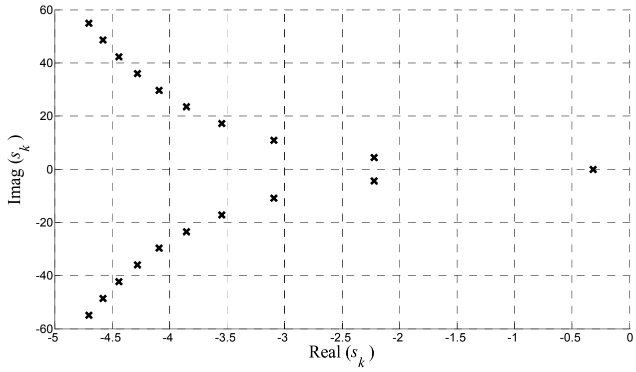

Figure 1. Eigenvalues of Eq. (3) when $a = -1$, $a_d = 0.5$ and $h = 1$. The rightmost eigenvalue is for $k = 0$, the next pair are for $k = \pm 1$, the next for $k = \pm 2$, to $k = \pm 9$.

2.3 Example 1

For $a = -1$, $a_d = 0.5$, and $h = 1$ the characteristic roots are obtained using Eq. (9) and the function *lambertw* in MATLAB, and are plotted in Figure 1. It can also be shown (Yi, Nelson & Ulsoy 2010) that the total (i.e., free plus forced) solution to Eq. (3), can be represented in terms of an infinite series based on the eigenvalues in Eq. (9) as:

$$x(t) = \sum_{k=-\infty}^{+\infty} e^{s_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{+\infty} e^{s_k (t-\eta)} C_k^N bu(\eta)d\eta \tag{10}$$

where

$$C_k^I = \frac{x_0 + a_d e^{-s_k h} \int_0^h e^{-s_k t} g(t-h)\,dt}{1 + a_d he^{-s_k h}} \tag{11}$$

and

$$C_k^N = \frac{1}{1 + a_d he^{-s_k h}} \tag{12}$$

Note that the coefficients C_k^I are determined from the preshape function *g(t)* and the initial state x_0, and the coefficients C_k^N are determined only in terms of the system parameters *a*, a_d and *h*. Thus, the total solution in Eq. (10) can be viewed as the sum of the free and forced solutions. Conditions for convergence of such a series solution form are discussed in Bellman and Cook (1963). A very practically important and useful aspect of this particular series representation of the solution for *x(t)* is that truncating the series, e.g., $k = 0, \pm1, \pm2, .., \pm n$, yields an approximation of the solution in terms of the $(2n+1)$ rightmost, or most dominant, eigenvalues.

2.4 Example 2

For $a = -1$, $a_d = 0.5$, and $h = 1$ one can obtain the values of s_k using Eq. (9) and the function *lambertw* as in Ex. 1, and the values of C_k^I and C_k^N using Eqs. (11)-(12), where $x_0 = 1$ and $g(t) = 1$ for $-h \le t < 0$. These are given in Table 1. Figure 2 shows the total response to $u(t) = \sin(t)$ and a comparison between the Lambert W function-based method (using the 7 terms in Table 1) and a numerical solution (using the function *dde23* in MATLAB). The two plots are essentially indistinguishable.

Table 1. The eigenvalues and coefficients in the solution for Ex. 2.

k	s_k	C^I_k	C^N_k
0	-0.3149	0.9422	0.5934
±1	-2.2211±4.4442*i*	0.0197±0.0111*i*	-0.0112±0.2245*i*
±2	-3.0915±10.8044*i*	0.0038±0.0015*i*	-0.0093±0.0916*i*
±3	-3.5450±17.1313*i*	0.0016±0.0005*i*	-0.0052±0.0579*i*

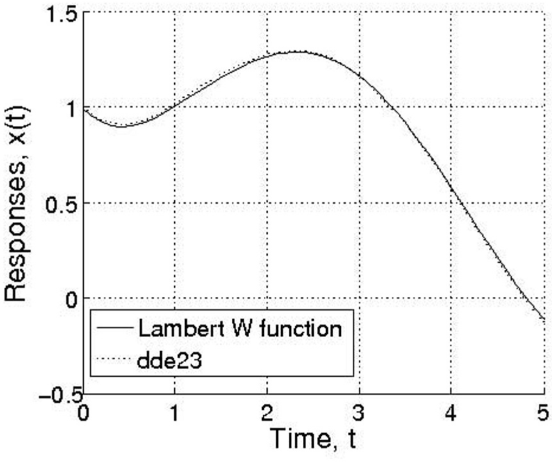

Figure 2. Total response to $u(t) = \sin(t)$, with $x_0 = 1$ and $g(t) = 1$ for $-h \le t < 0$, and comparison between the 7-term (see Table 1) Lambert W function-based method and the numerical method (function *dde23* in MATLAB). Parameters are $a = -1$, $a_d = 0.5$, $b = 1$, and $h = 1$.

2.5 General Case

The approach presented in the previous section has been generalized in (Yi, Nelson & Ulsoy 2010) to LTI TDS of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{13}$$

where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ is the input vector, $\mathbf{y}$(t) is the output vector, $\mathbf{A}$, $\mathbf{A_d}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are coefficient matrices, and *h* is the constant known scalar delay. The initial condition $\boldsymbol{x}(t=0) = \boldsymbol{x}_0$ and preshape function $\boldsymbol{x}(t) = \boldsymbol{g}(t)$ for $-h \le t < 0$, must also be specified. The total solution for the states is now given as:

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k (t-\eta)}\mathbf{C}_k^N \mathbf{B}\mathbf{u}(\eta) d\eta$$

(14)

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A}_d h \mathbf{Q}_k) + \mathbf{A} \tag{15}$$

and $\mathbf{Q}_k$ is obtained from numerical solution (e.g., using *fsolve* in MATLAB) of:

$$\mathbf{W}_k(\mathbf{A}_d h \mathbf{Q}_k) e^{\mathbf{W}_k(\mathbf{A}_d h \mathbf{Q}_k) + \mathbf{A}h} = \mathbf{A}_d h \tag{16}$$

This generalization is dependent on the introduction of a matrix Lambert W function, $\mathbf{W}_k$, as described in (Yi, Nelson & Ulsoy 2010). The quantities $\boldsymbol{Q}_k$, $\boldsymbol{W}_k$, $\mathbf{S}_k$ $\boldsymbol{C}^I_k$ and $\boldsymbol{C}^N_k$ in Eqs. (14)-(16) can all be computed using the software in the *LambertW_DDE* Toolbox in terms of given *h*, $\mathbf{A}$, $\mathbf{A}_d$, $\mathbf{g}(t)$, $\mathbf{x}_0$, $\mathbf{B}$ and $\mathbf{u}(t)$. The main functions of the *LambertW_DDE* Toolbox (Duan 2010) are summarized in Table 2.

Table 2 Main functions of the *LambertW_DDE* Toolbox (Duan 2010)

Name	Description
lambertw_matrix	Calculate matrix Lambert W function
find_Sk	Find $\mathbf{S_k}$ and $\mathbf{Q_k}$ for a given branch
find_CI	Calculate $\mathbf{C^I}$ under specific initial conditions for a given branch
find_CN	Calculate $\mathbf{C^N}$ for a given branch
pwcont_test	Controllability test for DDEs
pwobs_test	Observability test for DDEs
cont_gramian_dde	Controllability Gramian for DDEs
obser_gramian_dde	Observability Gramian for DDEs
place_dde	Rightmost eigenvalue assignment for DDEs
stabilityradius_dde	Calculate stability radius for DDEs
examples	Lists examples for using this toolbox; each cell is a short example and can be evaluated separately (Ctr+Enter)

2.5 Example 3

Given the argument $\mathbf{H}_k = \mathbf{A}_d h \mathbf{Q}_k$, then the matrix Lambert W function $\mathbf{W}_k(\mathbf{H}_k)$ can be found by first transforming $\mathbf{H}_k$ to Jordan canonical form, by using the function *jordan* in MATLAB. Then, by using the function *lambertw_matrix,* $\mathbf{W}_k$ is obtained. The steps for calculating the matrix Lambert W function are carried out in the function *lambertw_matrix*, which is then used in the function *find_Sk* to solve Eq. (15) for a particular branch, *k*. For example, given

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix}; \ \mathbf{A_d} = \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.33 \end{bmatrix}; \ h = 1$$

For the principal branch, $k = 0$, one obtains:

$$\mathbf{S}_0 = \begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix}$$

The eigenvalues of which are -1.0119 and -1.9841. Next, using *find_CI* and *find_CN*, one can obtain the coefficients for the series solution in Eq. (14). For example, if $u(t) = 0$, $\mathbf{g}(t) = \mathbf{0}$, and we have an abrupt change at $t = 0$ to $\boldsymbol{x}_0 = [1 \ 1]^T$, we can obtain (using *find_CI*) the coefficients for the free response for $k = 0$ as:

$$\mathbf{C}_0^I = \begin{bmatrix} 0.2635 \\ 0.4290 \end{bmatrix}$$

Thus, the single branch approximation, for $k = 0$, for the free response is:

$$\mathbf{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = e^{\begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix} t} \begin{Bmatrix} 0.2635 \\ 0.4290 \end{Bmatrix}$$

Note that in MATLAB the matrix exponential is evaluated using the function *expm*, not the scalar exponential function *exp*. To improve the approximation, this process can be repeated for additional branches, *k*, then an approximate series solution can be obtained using Eq. (14) with a finite number of *k* (see Fig. 3). For example, including the branches $k = \pm 1$ gives the additional complex conjugate $\mathbf{S}_k$ matrices:

$$\mathbf{S}_{-1,+1} = \begin{bmatrix} -0.399 \pm 4.980i & -1.6253 \pm 0.1459i \\ 2.4174 \pm 0.1308i & -5.1048 \pm 4.5592i \end{bmatrix}$$

with complex conjugate coefficients for the free response for $k = \pm 1$ as:

$$\mathbf{C}_{-1,+1}^I = \begin{bmatrix} 0.0909 \pm 0.1457i \\ 0.0435 \pm 0.1938i \end{bmatrix}$$

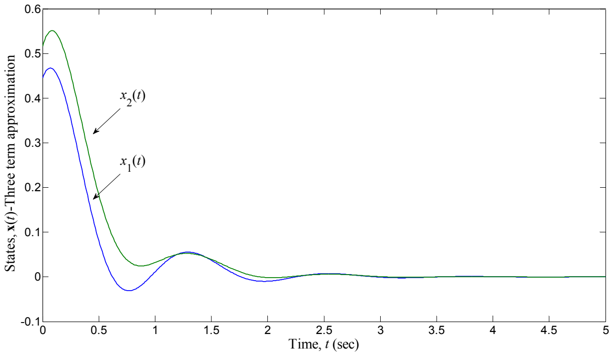

Figure 3. Approximate (3-term) free response for the system in Ex. 3

2.6 Observability and Controllability

<u>*Point-wise Controllability:*</u> The system of DDEs in Eq. (13) is *point-wise controllable* if, for any given initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, there exists a time t_1, $0 < t_1 < \infty$, and an admissible (i.e., measurable and bounded on a finite time interval) control segment $\mathbf{u}(t)$ for $t \in [0, t_1]$ such that $\mathbf{x}(t_1; \mathbf{g}, \mathbf{x}_0, \mathbf{u}(t)) = \mathbf{0}$ (Weiss 1967). For the scalar DDE in Eq. (3) it is point-wise controllable if and only if, for all s not at the poles of the system, $(s - a - a_d e^{-sh})^{-1} b \neq 0$; similarly for Eq. (13) one must have linearly independent rows of $(s\mathbf{I} - \mathbf{A} - \mathbf{A}_d e^{-sh})^{-1}\mathbf{B}$. Furthermore, the controllability Gramian for Eq. (13) must be full rank (Yi, Nelson & Ulsoy 2010).

<u>*Point-wise Observability:*</u> The system of DDEs in Eq. (13) is *point-wise observable* in $[0, t_1]$ if the initial point $\mathbf{x}_0$ can be uniquely determined from the knowledge of $\mathbf{u}(t)$, $\mathbf{g}(t)$, and $\mathbf{y}(t)$ (Delfour & Mitter, 1972). For the scalar DDE in Eq. (3), it is point-wise observable if and only if, for all s not at the poles of the system, $c(s - a - a_d e^{-sh})^{-1} \neq 0$; similarly, for Eq. (13) one must have linearly independent columns of $\mathbf{C}(s\mathbf{I} - \mathbf{A} - \mathbf{A}_d e^{-sh})^{-1}$. Furthermore, the observability Gramian for Eq. (13) must be full rank (Yi, Nelson & Ulsoy 2010).

2.7 Example 4

Consider Eq. (13), with $\mathbf{A}$, $\mathbf{A}_d$ and h as given in Ex. 3, and

$$\mathbf{B} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The function *pwcontr_test* can be used to establish that the system is piecewise controllable. Furthermore, piecewise observability can be established using the function *pwobs_test*, and the controllability and observability Gramians can be approximately computed, for $k = n$ branches, using the functions *contr_gramian_dde* and *obs_gramian_dde* respectively.

2.8 Placement of Dominant Poles

Consider the scalar LTI TDS in Eq. (3) with the generalized feedback structure

$$u(t) = -Kx(t) - K_d x(t-h)$$

The closed-loop system becomes

$$\dot{x}(t) = (a - bK)x(t) + (a_d - bK_d)x(t-h)$$

One can use the Lambert W function approach to assign the rightmost eigenvalues of the system. The procedure for selecting the gains K and K_d can be described as:

1 Select desired rightmost eigenvalue $\lambda_{desired}$

2 Set initial conditions $K = K_0$ and $K_d = K_{d0}$

3 While $\lambda(S_{0\_new}) - \lambda_{desired} \geq \text{tolerance}$

4 Select $K = K_{new}$ and $K_d = K_{d\_new}$

5 Let $a_{new} = (a - bK_{new})$, $a_{d\_new} = (a_d - bK_{d\_new})$
calculate $S_{0\_new} = \frac{1}{h} W_0(a_{d\_new} h e^{-a_{new} h}) + a_{new}$

6 End

Due to the range limitation of the branches of the Lambert W function, the rightmost poles cannot be assigned to any arbitrary location in the s-plane. For scalar time-delay systems, as in Eq. (3), this can be easily seen by examining the principal branch (Yi, Nelson & Ulsoy 2010).

$$\mathrm{Re}\left\{\frac{1}{h} W_0(a_d h e^{-ah}) + a\right\} \geq \mathrm{Re}\left\{-\frac{1}{h} + a\right\} \geq -\frac{1}{h} + a$$

since $\mathrm{Re}\{W_0(H)\} \geq -1$. Thus the rightmost pole cannot be less than $-\frac{1}{h} + a$. This feasibility constraint has to be considered in the design process (e.g., in the selection of $\lambda_{desired}$) for the method to succeed. The generalization of this approach to systems of DDEs, as in Eq. (13), is presented in (Yi, Nelson & Ulsoy 2010) and applied to both controller and observer design problems.

2.9 Example 5

For $a = -1$, $a_d = 0.5$, $b = 1$, and $h = 1$ the rightmost eigenvalue can be assigned to any value > -2. Here we consider $\lambda_{desired}$ = -1.5, and use the function *place_dde* to obtain the controller gains:

$$K = 1.1378$$
$$K_d = 0.3576$$

Thus, the closed-loop LTI TDS becomes:

$$\dot{x}(t) = -2.1378x(t) + 0.1424x(t-1)$$

and the rightmost eigenvalue can be found, using $k = 0$ in the function *lambertw*, as in Ex. 1, to now be located at -1.4998 as desired.

2.10 Robust Control and Time Domain Specifications

The assignment of rightmost eigenvalues for LTI TDS can also be used for observer design, and extended to robust design in the presence of structured model uncertainties. Since the response of the LTI TDS is dominated by the rightmost eigenvalues, approximate specification of time domain characteristics (e.g., settling time, overshoot) can also be achieved (Yi, Nelson & Ulsoy 2010). The function *stabilityradius_dde* in the Toolbox can be used to calculate the stability radius for DDEs as described in (Hu & Davison 2003).

2.11 Decay Function for TDS

Accurate estimation of the decay function for time delay systems has been a long-standing problem, which has recently been addressed using the Lambert W function based approach (Duan, Ni & Ulsoy 2012). The goal is to find a tight upper bound for the decay rate, which is referred to as α -stability, as well as an upper bound for the factor K , such that the norm of the states is bounded:

$$\|\mathbf{x}(t)\| \leq K e^{\alpha t} \Phi(h, t_0)$$

where $\Phi(h, t_0) = \sup_{t_0 - h \leq t \leq t_0} \{\|\mathbf{x}(t)\|\}$ and $\|\cdot\|$ denotes the 2-norm. A less conservative estimate of the decay function leads to a more accurate description of the transient response, and more efficient control strategies based on the decay model (Duan, Ni & Ulsoy 2012).

2.12 Example 6

Consider the system in Eq. (13) with the same coefficients as in Ex. 3. From Eq. (15), with $k = 0$, the rightmost pole is found to be:

$$\alpha = \max\left\{\mathrm{Re}(eig(\mathbf{S}_0))\right\}$$
$$= \max\left\{\mathrm{Re}(eig(\frac{1}{h}\mathbf{W}_0(-\mathbf{A}_d h\mathbf{Q}_0)-\mathbf{A}))\right\} = -1.012$$

Thus the decay rate is obtained and, using the solution in Eq. (14), one can also obtain a bound on ***K***. As shown in Table 3, the results are less conservative when compared to other methods (Duan, Ni & Ulsoy 2012).

Table 3. Comparison of Results for Ex. 6

	Factor, K	Decay Rate, α
Matrix Measure Approach (Hale, 1993)	8.019	3.053
Lyapunov Approach (Mondié, 2005)	9.33	-0.907
Lambert-W Approach (Duan et al 2012)	3.8	-1.012

3. CONCLUDING REMARKS

This paper provides a succinct overview of the Lambert W function approach for analysis and control of LTI TDS with constant delay, which is described in detail in (Yi, Nelson & Ulsoy 2010). Simple examples are used here to illustrate the use of the *lambertw* function in MATLAB, as well as other useful functions available in the open source *LambertW_DDE* Toolbox software package for LTI DDEs (Duan 2010).

The proposed approach can be used, just as for systems of LTI ODEs as in Eq. (1), for a variety of important analysis and control tasks for LTI DDEs, such as free and forced solutions, stability, observability and controllability, controller and observer design via assignment of dominant eigenvalues, robust stability, determination of the decay function, etc.

The open source software in the *LambertW_DDE* Toolbox, and the accompanying documentation and examples on the web (Duan 2010), we hope will make the lambert W function based approach more accessible and useful for those interested in applications that are well modelled as LTI TDS with a single constant delay.

REFERENCES

Asl, F.M and A.G. Ulsoy, 2003, “Analysis of a System of Linear Delay Differential Equations,” *ASME J. Dynamic Systems, Measurement and Control*, **125**, pp. 215-223.

Bellman, R., and Cooke, K.L., 1963, *Differential-Difference Equations*, Academic Press, New York.

Chen, C.T., 1998, *Linear System Theory and Design*, Oxford University Press.

Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J. and D.E. Knuth, 1996, "On the Lambert W Function," *Advances in Computational Mathematics*, **5**, pp. 329-359.

Delfour, M.C. and S.K. Mitter, 1972, “Controllability, Observability and Optimal Feedback Control of Affine Hereditary Differential Systems,” *SIAM J. Control*, **10**, pp. 298-328.

Duan, S, 2010,”Supplement for Time Delay Systems: Analysis and Control Using the Lambert W Function,” http://www.umich.edu/~ulsoy/TDS_Supplement.htm (01/01/2012).

Duan, S., Ni, J. and A.G. Ulsoy, 2012, “Decay Function Estimation for Linear Time Delay Systems via the Lambert W Function,” *Journal of Vibration and Control*, doi:10.1177/1077546311408168 (in press).

Euler, L., 1777, ‘‘Deformulis exponentialibus replicatis,’’ *Leonardo Euleri Opera Omnia*, **1**(15), pp. 268–297.

Gu, K. and S.I. Niculescu, 2006, “Stability Analysis of Time Delay Systems: A Lyapunov Aproach,” *Advanced Topics in Control Systems Theory, Lecture Notes in Control and Information Sciences*, **328**, pp. 139-170.

Hale, J.K. and S.M.V. Lunel, 1963, *Introduction to Functional Differential Equations*, Springer Verlag, New York.

Hu, G. and E.J. Davison, 2003, “Real Stability Radii of Linear Time-Invariant Time-Delay Systems,” *Systems and Control Letters*, **50**(3), pp. 209-219.

Kolmanovskii, V. B. and Myshkis, A., 1999, *Introduction to the Theory and Applications of Functional Differential Equations*, Dordrecht: Kluwer Academy.

Lambert, J.H., 1758, ‘‘Observationes Variea in Mathesin Puram,’’ *Acta Helveticae, Physico-mathematico-anatomico-botanico-medica*, **3**, pp. 128–168.

Richard, J.P., 2003, “Time Delay Systems: An Overview of Some Recent Advances and Open Problems,” *Automatica*, **39**(10), pp. 335-357.

Shinozaki, H. and T. Mori, 2006, “Robust stability analysis of linear time-delay systems by Lambert W function: Some extreme point results,” *Automatica*, **42**(1), pp. 1791-1799.

Sipahi, R., S.I. Niculescu, C.T. Abdallah, W. Michiels, and K. Gu, 2011, “Stability and Stabilization of Systems with Time Delay: Limitations and Opportunities,” *IEEE Control Systems Magazine*, **31**(1), Feb. 2011, pp 38-65.

Stepan, G., 1989, *Retarded Dynamical Systems: Stability and Characteristic Functions*, Wiley, New York.

Weiss, L., 1967, “On the Controllability of Delay-Differential Systems, *SIAM J. Control,* **5**, pp. 575-587.

Yi, S., P.W. Nelson and A.G. Ulsoy, 2010, *Time-Delay Systems: Analysis and Control Using the Lambert W Function,* World Scientific, New Jersey.