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Complement to method of analysis of time delay systems via the Lambert W function[☆]

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ABSTRACT

In this note we propose a complement to known method of analysis of linear time invariant delay systems, which expands applicability of the method. Applicability of the modified method is illustrated by an example.

1. Introduction

In Asl and Ulsoy (2003) a method of analysis of linear time delay systems has been given using a matrix version of the Lambert W function. In Yi, Nelson, and Ulsoy (2010) and Yi and Ulsoy (2006) the generalization of the method for the case when system matrices are not commuting, is made. The goal of this paper is to give to the generalized method the complement, which expands applicability of the method.

The paper is organized as follows. In Section 2, for completeness, we explain the essence of the known generalized method (we shall call it the base method). In Section 3 the complement to the base method is presented (the base method with the complement we shall call the modified method). In Section 4 we give an example of system with non commuting matrices A and B and show that the modified method gives correct results.

2. The method of analysis of time delay systems via the Lambert W function

In this section we explain essence of the base method, presented in Yi et al. (2010) and Yi and Ulsoy (2006).

Consider system of DDEs in the matrix–vector form

$$\begin{aligned} &\dot{x}(t) + Ax(t) + Bx(t-\tau) = 0, \quad t > 0, \\ &x(t) = \varphi(t), \quad t \in [\tau, 0], \end{aligned} \tag{1}$$

where $A, B \in C^{n\times n}$ are non commuting numerical matrices. Assume that the solution of (1) has the form

$$x(t) = e^{St}x_0, \tag{2}$$

where $S \in C^{n\times n}$, $x_0 \in C^{n\times 1}$.

Substituting (2) into (1) yields

$$\left(S + Be^{-S\tau} + A\right)e^{St}x_0 = 0,$$

from which we have

$$S + Be^{-S\tau} + A = 0. \tag{3}$$

Multiply (3) by $\tau e^{A\tau}$ and rearrange to obtain

$$\tau(S + A)e^{S\tau}e^{A\tau} = -B\tau e^{A\tau}. \tag{4}$$

It is shown in Yi and Ulsoy (2006) that when A and B not commute, then A and S also not commute, therefore

$$\tau(S + A)e^{S\tau}e^{A\tau} \neq \tau(S + A)e^{(S+A)\tau}. \tag{5}$$

To write the solution in terms of the Lambert W function, introduce an unknown matrix Q that satisfies

$$\tau(S + A)e^{\tau(S+A)} = -B\tau Q. \tag{6}$$

Every function $W(H)$ that satisfies

$$W(H)e^{W(H)} = H \tag{7}$$

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is called a Lambert W function (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996). Using (7), from (6) we get

$$S = \frac{1}{\tau} W(-B\tau Q) - A. \tag{8}$$

Taking into account that the Lambert W function has a countable infinite number of branches, we have

$$S_k = \frac{1}{\tau} W_k(-B\tau Q_k) - A, \quad k \in Z. \tag{9}$$

Substituting (9) into (3) yields condition, which can be used to solve for the unknown matrix Q_k:

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$$W_k(-B\tau Q_k)\, e^{W_k(-B\tau Q_k) - A\tau} = -B\tau, \quad k \in Z. \tag{10}$$

Finally, the Q_k obtained from (10) can be substituted into (9) to obtain S_k. Substituting obtained S_k into (2) we would get infinite number of particular solutions to (1). The linear combination of all these solutions gives the general solution

$$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k; \tag{11}$$

here C_k is a numerical vector-column, elements of which are arbitrary constants. Taking into account initial function $x(t) = \varphi(t)$, $t \in [-\tau, 0]$, we can obtain concrete values for elements of C_k (Yi et al., 2010; Yi & Ulsoy, 2006).

3. Main result: complement to known method

In this section we propose a complement to known method, presented in Section 2. This proposal expands applicability of the method.

Let we have got the condition (10), which we use to find the unknown matrix Q_k. We denote

Matlab fsolve

$$D_k = W_k(-B\tau Q_k), \tag{12}$$

which yields (see (9) and (10))

$$S_k = \frac{1}{\tau} D_k - A, \quad k \in Z \tag{13}$$

and

$$D_k e^{D_k - A\tau} = -B\tau, \quad k \in Z. \tag{14}$$

Essence of the offered complement is that we search not for a matrix Q_k, but for a matrix D_k. For the search of the matrix D_k we implement the following algorithm:

Algorithm 1. 1. Consider an objective function $f(D_k) = D_k e^{D_k - A\tau} + B\tau$ and choose an accuracy of calculations ε (stopping criteria).

2. Fix a branch k ($k \in -N, \ldots, N$) of the Lambert W function (here $2N+1$ is a number of branches of the Lambert W function, used at calculation of the solution of (1)).
3. Set $i = 0$ and choose an initial matrix $D_k^{(0)} = W_k\left(-B\tau e^{A\tau}\right)$.
4. Compute $f(D_k^{(i)})$ (here $f(D_k^{(i)}) = D_k^{(i)} e^{D_k^{(i)} - A\tau} + B\tau$).
5. If $\left\| f(D_k^{(i)}) \right\| < \varepsilon$, then jump to step 8, otherwise continue downwards (here $\|A\|$ is a norm of a matrix A).
6. Set $i = i + 1$ and find $D_k^{(i)} = D_k^{(i-1)} + h^{(i-1)}$ (here $h^{(i-1)}$ is a step, found using Gauss–Newton–Powell's Dog-Leg algorithm (Rosen, Michael, & Leonard, 2014, Algorithm 4)).
7. Return to step 4.
8. $D_k = D_k^{(i)}$.
9. Return to step 2 (fix a new branch of the Lambert W function).

Remark 1. Initial matrix $D_k^{(0)}$ is expressed through the matrix Lambert W function: $D_k^{(0)} = W_k(H)\left(H = -B\tau e^{A\tau}\right)$, $k \in Z$. The matrix Lambert W function we shall define in standardized way (see, e.g. Higham, 2006, Horn & Johnson, 1991 and Jarlebring & Damm, 2007). Let matrix H have s eigenvalues $\lambda_1, \ldots, \lambda_s$ with multiplicities $n_1, \ldots, n_s$, respectively. Then the Jordan form of the matrix H will be

$$J = \operatorname{diag}\left(J_{n_1}(\lambda_1), \ldots, J_{n_s}(\lambda_s)\right);$$

here $J_n(\lambda)$ is the $n \times n$ Jordan block corresponding to eigenvalue λ. By similarity transformation we have $H = TJT^{-1}$. Then, the Lambert W function of the matrix H can be defined as follows:

$$W_k(H) = TW_k(J)T^{-1},$$

$$W_k(J) = \operatorname{diag}(W_{k_1}(J_{n_1}(\lambda_1)), \ldots, W_{k_s}(J_{n_s}(\lambda_s)))),$$

$$W_k(J_n(\lambda)) = \begin{pmatrix} W_k(\lambda) & W_k'(\lambda) & \cdots & \dfrac{1}{(n-1)!} W_k^{(n-1)}(\lambda) \\ 0 & W_k(\lambda) & \cdots & \dfrac{1}{(n-2)!} W_k^{(n-2)}(\lambda) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & W_k(\lambda) \end{pmatrix}.$$

Since the principal branch $W_0(z)$ is not differentiable at point $z = -e^{-1}$, we assume that $-e^{-1}$ is not an eigenvalue corresponding to a Jordan block of dimension $n \geq 2$ (Jarlebring & Damm, 2007).

Remark 2. A special situation is in the case, when some eigenvalue of the matrix H is equal to zero: $W_k(\lambda_i) = W_k(0)$. It is known that Corless et al. (1996)

$$W_k(0) = -\infty \quad \text{if } k \neq 0, \tag{15}$$

and

$$W_k(0) = 0 \quad \text{if } k = 0. \tag{16}$$

We change values of $W_k(0)$ at $k \neq 0$ as follows:

$$W_k(0) = 0 \quad \text{if } k \neq 0. \tag{17}$$

This assumption is made on the base of definition of the Lambert W function $W_k(z)e^{W_k(z)} = z$, according which

$$\begin{aligned} W_k(0)e^{W_k(0)} &= \lim_{x \to -\infty} xe^x \\ &= \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{x'}{(e^{-x})'} \\ &= \lim_{x \to -\infty} \frac{1}{-e^{-x}} = -\lim_{x \to +\infty} \frac{1}{e^x} = -0, \end{aligned} \tag{18}$$

if we take into account (15), and

$$W_k(0)e^{W_k(0)} = 0e^0 = 0, \tag{19}$$

if we take into account (17).

Our presumption: the adjustment

$$\begin{aligned} &(W_k(0) = -\infty \text{ if } k \neq 0) \Rightarrow \\ &(W_k(0) = 0 \text{ if } k \neq 0) \end{aligned} \tag{20}$$

has influence only on order of numbering of eigenvalues, but not on all the set. This presumption, in general, must be proved. In the example, presented in Section 4, it is confirmed.

Remark 3. The modified method has following advantages if to compare it with the base method.

(1) The initial matrix $D_k^{(0)}$ in the modified method always is chosen in the same way ($D_k^{(0)} = W_k(-B\tau e^{A\tau})$), while in a base method to find a suitable initial matrix $Q_k^{(0)}$ is a great problem: not always it is possible to find such initial matrix that corresponding iterative process would be convergent.

(2) Using the modified method we avoid complicated search of Q_k from equation

$$W_k(-B\tau Q_k)e^{W_k(-B\tau Q_k)-A\tau} = -B\tau,$$

which requires calculation of values of the matrix Lambert W function. In the modified method we calculate D_k using simpler equation $D_k e^{D_k - A\tau} = -B\tau$.

4. Example

The modified method we shall apply to the problem, taken from Jarlebring and Damm (2007). We shall show that taking non commuting pair of matrices

$$A = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\beta \\ 0 & 0 \end{pmatrix}$$

$$(\alpha, \beta \in R, \alpha > 0,\ \beta > 0), \tag{21}$$

considered in the counter-example of Jarlebring and Damm (2007), and applying the modified method, we shall get the correct result. To validate our reasoning we shall perform calculations of step responses of the system, governed by (1), with A and B given in (21), using three methods: (1) exact method of consequent integration, (2) modified method via the Lambert W function, (3) numerical method based on the dde23 function in MATLAB.

Step responses matrix. The response of a component $x_i(t)$ of a vector $x(t)$ to a unit jump in the component $x_j(t)$ will be referred as a step response $h_{ij}(t)$. The set of the step responses $h_{ij}(t)$ $(i, j = 1, 2)$ form 2×2 matrix $h(t) = (h_{ij}(t))$. This matrix will be called a step responses matrix of the system. The jth column of the step responses matrix $h(t)$ we shall denote $h_j(t)$.

Differential equation for step responses. Differential equation with respect to column-vector $h_j(t)$ (for the system, governed by (1), with A and B given in (21)) can be presented as follows:

$$\dot{h}_j(t) + Ah_j(t) + Bh_j(t-\tau) = \delta(t)I^{(j)}, \quad j = 1, 2, t > 0, \tag{22}$$
$$h_j(t) = 0, \quad t \in [-\tau, 0];$$

here $I^{(j)}$ is a column-vector with jth element equal to 1 and other elements equal to zero, $\delta(t)$ is the Dirac delta function,

$$h_j(t) = \begin{pmatrix} h_{1j}(t) \\ h_{2j}(t) \end{pmatrix}, \quad j = 1, 2.$$

Solution by method of consequent integration. This method (sometimes called method of "steps") is described in Bellman and Cooke (1963) and Kalmanovskii and Myshkis (1992). Using method of consequent integration and applying the Laplace transform the exact solution of (22) can be presented as follows (for details of similar calculations see, e.g. Rimas, 2004):

$$h_1(t) = \begin{pmatrix} h_{11}(t) \\ h_{21}(t) \end{pmatrix}, \qquad h_2(t) = \begin{pmatrix} h_{12}(t) \\ h_{22}(t) \end{pmatrix},$$

$$h_{11}(t) = h_{22}(t) = \sum_{n=0}^{\infty} \alpha^n \beta^n \frac{(t-n\tau)^{2n}}{(2n)!} 1(t-n\tau), \tag{23}$$

$$h_{21}(t) = \sum_{n=0}^{\infty} \alpha^n \beta^{n+1} \frac{(t-n\tau)^{2n+1}}{(2n+1)!} 1(t-n\tau), \tag{24}$$

$$h_{12}(t) = \sum_{n=0}^{\infty} \alpha^{n+1} \beta^n \frac{(t-(n+1)\tau)^{2n+1}}{(2n+1)!} 1(t-(n+1)\tau); \tag{25}$$

here $1(t)$ is the unit step function (Heaviside step function).

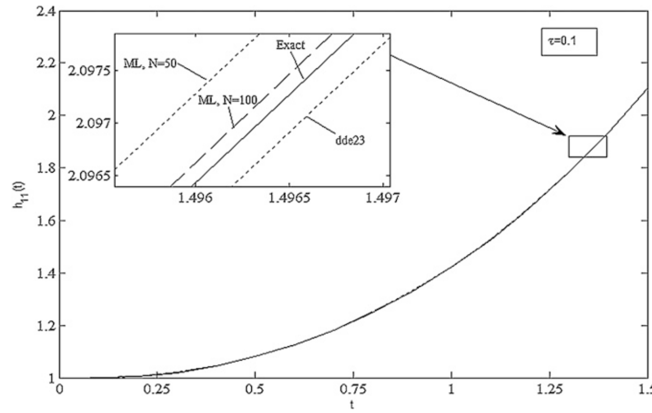

Fig. 1. The step response $h_{11}(t)$, calculated by three methods: consequent integration (Exact); modified Lambert function (ML); numerical with dde23 function (dde23).

Solution by modified method. First, we find the solution of (22) on the interval $[0, \tau]$. Column-vector $h_j(t-\tau)$ is a zero column-vector on this interval due to the initial conditions (see (22)). Taking this into account, on the interval $[0, \tau]$ we get the following differential equation for $h_j(t)$:

$$\dot{h}_j(t) + Ah_j(t) = \delta(t)I^{(j)}, \quad j = 1, 2,\ t > 0, \tag{26}$$
$$h_j(t) = 0, \quad t \in [-\tau, 0].$$

Solution of matrix ODE (26) is a vector-function (Greenberg, 2012)

$$h_j(t) = e^{At}I^{(j)}, \quad j = 1, 2,\ t \in [0, \tau]. \tag{27}$$

Taking this solution as the initial vector-function, we get the following homogeneous matrix DDE for $h_j(t)$ on the interval $(\tau, +\infty)$:

$$\dot{h}_j(t) + Ah_j(t) + Bh_j(t-\tau) = 0, \quad j = 1, 2,\ t > \tau, \tag{28}$$
$$h_j(t) = e^{At}I^{(j)}, \quad t \in [0, \tau].$$

The matrices A and B do not commute, so, applying the modified method, described in Section 3, the solution of (28) we present as follows:

$$h_j(t) = \sum_{k=-\infty}^{+\infty} e^{S_k t} C_{k_j}, \quad j = 1, 2,\ t > \tau; \tag{29}$$

here C_{k_j} is a numerical vector-column, the elements of which are arbitrary constants. Taking into account initial vector-function $h_j(t) = e^{At}I^{(j)}$, $t \in [0, \tau]$ (see (28)), we shall obtain concrete values for elements of C_{k_j}.

Using (27) and (29), we write the solution of (22) on the interval $[0, +\infty)$:

$$h_j(t) = \begin{cases} e^{At}I^{(j)}, & j = 1, 2,\ t \in [0, \tau], \\ \sum_{k=-\infty}^{+\infty} e^{S_k t} C_{k_j}, & j = 1, 2,\ t \in (\tau, +\infty). \end{cases} \tag{30}$$

Results of calculations. On Fig. 1 the graphs of step response $h_{11}(t)$, calculated by three methods, when $\alpha = \beta = 1, \tau = 1$, are presented. As we see from this figure, step responses, calculated by the modified method, approach the exact graph if the number of the branches, used at calculations, are being increased. Relative errors of calculations can be evaluated using results presented in Table 1. On Fig. 2 distribution of the eigenvalues on the complex plane is given. As we see from this figure, the set of exact eigenvalues, taken from Jarlebring and Damm (2007), and the set of eigenvalues, calculated by the modified method, coincide. More detailed

Table 1
Relative errors calculated at point $t = 1.5$.

τ	Modified Lambert W function				dde23
	N				
	1	5	50	100	
1	0.0260	0.0029	0.0004	0.000236	0.000000002
0.1	0.0113	0.0021	0.00041	0.000092	0.000173

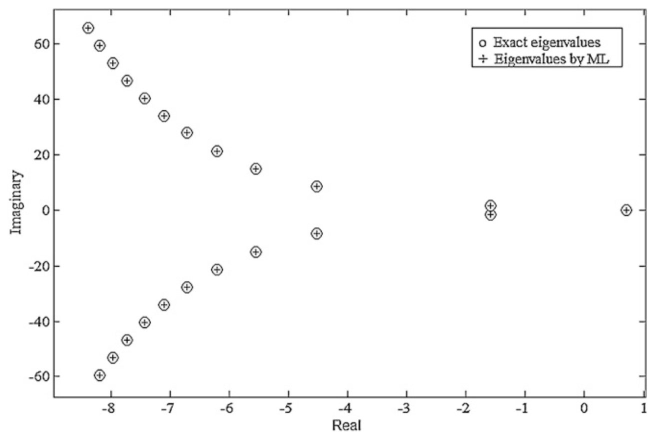

Fig. 2. Eigenvalues calculated by modified Lambert function method (Eigenvalues by ML) and exact eigenvalues obtained using formula $S_k = \frac{2}{\tau} W_k\left(\pm \frac{1}{2}\tau\sqrt{\alpha}\right)$, taken from Jarlebring and Damm (2007) ($\alpha = \beta = 1, \tau = 1$).

analysis shows that these eigenvalues differ only by order of their numbering (for example: $\begin{matrix}\lambda_{0,1}^{(ML)} = \lambda_{0,1}^{(Exact)} \\ \lambda_{0,2}^{(ML)} = \lambda_{0,2}^{(Exact)}\end{matrix}$; $\begin{matrix}\lambda_{1,1}^{(ML)} = \lambda_{1,1}^{(Exact)} \\ \lambda_{1,2}^{(ML)} = \lambda_{0,1}^{(Exact)}\end{matrix}$; $\begin{matrix}\lambda_{2,1}^{(ML)} = \lambda_{1,2}^{(Exact)} \\ \lambda_{2,2}^{(ML)} = \lambda_{0,1}^{(Exact)}\end{matrix}$; $\begin{matrix}\lambda_{3,1}^{(ML)} = \lambda_{2,1}^{(Exact)} \\ \lambda_{3,2}^{(ML)} = \lambda_{0,1}^{(Exact)}\end{matrix}$; $\begin{matrix}\lambda_{4,1}^{(ML)} = \lambda_{2,2}^{(Exact)} \\ \lambda_{4,2}^{(ML)} = \lambda_{0,1}^{(Exact)}\end{matrix} \cdots$; here $\lambda_{k,i}^{(ML)}$ is an eigenvalue, calculated by the modified Lambert W function method, $\lambda_{k,i}^{(Exact)}$ is the exact eigenvalue obtained by formula $S_k = \frac{2}{\tau} W_k\left(\pm \frac{1}{2}\tau\sqrt{\alpha}\right)$, taken from Jarlebring and Damm (2007)). Consequently, in the presented example the adjustment (20) has influenced only on the numbering of eigenvalues (all the set of eigenvalues has remained unchanged). This result confirms the presumption, made in Remark 2 of Section 3.

Remark 4. In the counter-example of Jarlebring and Damm (2007) the authors for analysis of delay system with not commuting coefficient matrices use formula which holds true only for systems with commuting coefficient matrices. For this reason they have got distribution of eigenvalues on complex plane which differs from exact distribution.

5. Conclusions

The modified method of analysis of linear delay systems via the Lambert W function can be applied in the case when coefficient matrices do not commute. The modified method as compared with the base method has advantage since the procedure of finding an auxiliary matrix is simpler and always leads to desired result. If argument of the matrix Lambert W function, involved in the solution of the transcendental characteristic equation of the system, has zero eigenvalue, the modified method can be used after some adjustment of the Lambert W function at zero argument.

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