Linear Matrix Inequalities in Control

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Presentation

1. Introduction and some simple examples
2. Fundamental Properties and Basic Structure of Linear Matrix Inequalities (LMIs)
3. LMI-problems
4. Tricks in Matrix Inequalities - Approaches to create LMIs from Matrix Inequalities
   (a) Congruence Transformation
   (b) Change of Variables
   (c) Projection Lemma
   (d) S-procedure
   (e) Schur Complement
5. Examples ($\mathscr{L}_2$-gain computation, non-linearities, etc)
6. Conclusions

Introduction - A Simple Example

A linear system

$$\dot{x} = Ax$$

is stable if and only if there is a positive definite P for

$$V(x) = x^T P x \quad (i.e. \quad V(x) > 0 \ for \ x \neq 0)$$

and

$$x^T PAx + x^T A^T Px < 0 \quad \forall x \neq 0$$

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The two matrix inequalities involved here are

$$PA + A^T P < 0$$

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$$P > 0.$$

The matrix problem here is to find P so that these inequalities are satisfied. The inequalities are linear in P.

Introduction - LQR-optimal control

We would like to compute a state feedback controller $u = Kx$ controlling

$$\dot{x} = Ax + Bu$$

with an initial condition of $x(0) = x_0$.
The cost function

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt$$

is to be minimized. We know that the solution to this problem is

$$K = -R^{-1}B^T P, \qquad A^T P + PA + PBR^{-1}B^T P + Q = 0$$

$$\textit{and} \quad J = \min_u \int_0^\infty (x^T Qx + u^T Ru)dt = x_0^T Px_0.$$

How can we express this problem in terms of an LMI?

Introduction

In control the requirements for controller design are usually

1. Closed Loop Stability
2. Robustness
3. Performance
4. Robust Performance

• Control design requirements are usually best encoded in form of an optimization criterion

(e.g. robustness in terms of $\mathscr{L}_2$/small gain-requirements, performance via linear quadratic control, $\mathscr{H}^\infty$ -requirements etc.)

Introduction

- We have seen that stability of a linear autonomous system can be easily expressed via a linear matrix inequality

- We will see that linear quadratic control problems can be expressed in terms of LMIs

- $\mathscr{L}_2$ or $\mathscr{H}^\infty$ analysis/design problems can be expressed as LMI-problems

- Some classes of nonlinearities are easily captured via matrix inequalities

- This creates a synergy which allows to express a control design problem via different ‘seemingly contradictive’ requirements

- For LMIs, very reliable numerical solution tools are available

Fundamental LMI properties

A matrix Q is defined to be *positive definite* if it is symmetric and

$$x^T Q x > 0 \quad \forall x \neq 0$$

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A matrix Q is *negative definite* if it is symmetric and

$$x^T Q x < 0 \quad \forall x \neq 0 \quad thus \quad Q < 0$$

or *negative semi-definite* if it is symmetric and

$$x^T Q x \leq 0 \quad \forall x \neq 0 \quad thus \quad Q \leq 0$$

The Basic Structure of an LMI

Any linear matrix inequality *(LMI)* can be easily rewritten as

$$F(v) = F_0 + \sum_{i=1}^{m} v_i F_i > 0$$

where $v \in \mathbb{R}^m$ is a variable and F_0, F_i are given constant symmetric matrices.

This matrix inequality is linear in the variables v_i.

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This matrix inequality is linear in the variables v_i*.*

For instance for the simple linear matrix inequality in the symmetric P

$$PA + A^T P < 0$$

the variables $v \in \mathbb{R}^m$ are defined via $P \in \mathbb{R}^{n \times n}$. Hence, $m = \frac{n(n+1)}{2}$ in this case!

The Basic Structure of an LMI

Another very generic way of writing down an LMI is

$$\begin{aligned} F(V_1, V_2, \ldots, V_n) &= F_0 + G_1 V_1 H_1 + G_2 V_2 H_2 + \ldots \\ &= F_0 + \sum_{i=1}^{n} G_i V_i H_i > 0 \end{aligned}$$

where the unstructured $V_i \in \mathbb{R}^{q_i \times p_i}$ are matrix variables, $\sum_{i=1}^{n} q_i \times p_i = m$. We seek to find V_i as they are variables.

The matrices F_0, G_i, H_i are given.

From now on, we will mainly consider LMIs of this form.

System of LMIs

A system of LMIs is

$$\begin{aligned} F_1(V_1,\ldots,V_n) &> 0 \\ \vdots &> 0 \\ F_p(V_1,\ldots,V_n) &> 0 \end{aligned}$$

where

$$F_j(V_1,\ldots,V_n) = F_{0j} + \sum_{i=1}^{n} G_{ij} V_i H_{ij}$$

This can be easily changed into a single LMI ...

System of LMIs

Let's define $\tilde{F}_0, \tilde{G}_i, \tilde{H}_i, \tilde{V}_i$ as

$$
\begin{aligned}
\tilde{F}_0 &= \begin{bmatrix} F_{01} & 0 & 0 & 0 \\ 0 & F_{02} & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & F_{0p} \end{bmatrix} = \mathrm{diag}(F_{01}, \ldots, F_{0p}) \\
\tilde{G}_i &= \mathrm{diag}(G_{i1}, \ldots, G_{ip}) \\
\tilde{H}_i &= \mathrm{diag}(H_{i1}, \ldots, H_{ip}) \\
\tilde{V}_i &= \mathrm{diag}(V_i, \ldots V_i)
\end{aligned}
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\tilde{V}_i &= \operatorname{diag}(V_i, \dots V_i)
\end{aligned}$$

We then have the inequality

$$F_{big}(V_1, \dots, V_n) := \tilde{F}_0 + \sum_{i=1}^{n} \tilde{G}_i \tilde{V}_i \tilde{H}_i > 0$$

which is just one single LMI.

But be aware that this time the new variable $\tilde{V}_i$ is structured, i.e. not all elements of $\tilde{V}_i$ are free parameters!

Different classes of LMI-problems: Feasibility Problem

We seek a *feasible* solution $\{V_1, \ldots, V_n\}$ such that

$$F(V_1, \ldots, V_n) > 0$$

We are not interested in the optimality of the solution, only in finding a solution, which satisfies the LMI and may not be unique.

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Example: A linear system

$$\dot{x} = Ax$$

is stable if and only if there is a matrix P satisfying

$$PA + A^T P < 0$$

and

$$P > 0.$$

LMI-problems: Linear Objective Minimization

Minimization (or maximization) of a *linear scalar* function, $\alpha(.)$, of the matrix variables V_i, subject to LMI constraints:

$$\min \alpha(V_1, \ldots, V_n) \qquad \underbrace{\text{s.t.}}_{\text{'such that', 'subject to'}} \qquad F(V_1, \ldots, V_n) > 0$$

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Example: Calculating the $\mathscr{H}^\infty$ norm of a linear system.

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \end{aligned}$$

the $\mathscr{H}^\infty$ norm of the transfer function matrix T_{zw} from w to z is computed by:

$$\min \gamma \quad \text{s.t.} \quad \begin{bmatrix} A^TP + PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0, \qquad P > 0.$$

The LMI variables are P and γ! The value of γ is unique, P is not.

LMI-problems: Generalized eigenvalue problem

$$\begin{aligned} \min \lambda \quad \text{s.t.} \qquad F_1(V_1,\ldots,V_n) + \lambda F_2(V_1,\ldots,V_n) &< 0 \\ F_2(V_1,\ldots,V_n) &< 0 \\ F_3(V_1,\ldots,V_n) &< 0 \end{aligned}$$

Note that in some cases, a GEVP problem can be reduced to a linear objective minimization problem, through an appropriate change of variables.

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Example: Bounding the decay rate of a linear system.
The decay rate is the largest α such that

$$\|x(t)\| \le \exp(-\alpha t)\beta\|x(0)\|, \quad \beta >, \quad \forall x(t)$$

Let's choose the Lyapunov function $V(x) = x^T P x > 0$ and ensure that $\dot{V}(x) \le -2\alpha V(x)$.
The problem of finding the decay rate could be posed as

$$\begin{aligned} \min\ -\alpha \quad \text{s.t.} \qquad A^T P + PA + 2\alpha P &< 0, \\ -P &< 0, \end{aligned}$$

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$$\begin{aligned} \min\ -\alpha \quad \text{s.t.} \qquad A^T P + PA + 2\alpha P &< 0, \ i.e.\ F_1(P) := A^T P + PA \\ -P &< 0, \ i.e.\ F_2(P) := -2P \\ & \qquad\ \ i.e.\ F_3(P) := -I \end{aligned}$$

Tricks: Congruence transformation

We know that for $Q \in \mathbb{R}^{n \times n}$

$$Q > 0$$

and a real $W \in \mathbb{R}^{n \times n}$ such that $\mathrm{rank}(W) = n$, the following inequality holds

$$WQW^T > 0$$

Definiteness of a matrix is invariant under pre and post-multiplication by a full rank real matrix, and its transpose, respectively.

Often W is chosen to have a diagonal structure.

Tricks: Change of variables

By defining new variables, it is sometimes possible to 'linearise' nonlinear MIs

Example: State feedback control synthesis
Find F such that the eigenvalues of $A+BF$ are in the open left-half complex plane

This is equivalent to finding a matrix F and $P >$ for

$$(A+BF)^T P + P(A+BF) < 0 \quad \textit{or} \quad A^T P + PA + F^T B^T P + PBF < 0$$

! Terms with products of F and P are 'nonlinear' or 'bilinear' !

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Multiply with $Q := P^{-1} > 0$ (*A very simple case of congruence transformation*):

$$QA^T + AQ + QF^T B^T + BFQ < 0$$

This is a new matrix inequality in the variables $Q > 0$ and F (still non-linear).

Tricks: Change of variables

$$QA^T + AQ + QF^T B^T + BFQ < 0$$

Define a second new variable $L = FQ$

$$QA^T + AQ + L^T B^T + BL < 0$$

We now have an LMI feasibility problem in the new variables $Q > 0$ and L.

Recovery of F and P by

$$F = LQ^{-1}, \qquad P = Q^{-1}.$$

Tricks: Schur complement

Schur's formula says that the following statements are equivalent:

$$i. \qquad \Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^T & \Phi_{22} \end{bmatrix} < 0$$

$$ii. \qquad \begin{aligned} \Phi_{22} &< 0 \\ \Phi_{11} - \Phi_{12}\Phi_{22}^{-1}\Phi_{12}^T &< 0 \end{aligned}$$

The main use is to transform quadratic matrix inequalities into linear matrix inequalities.

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Example: Making a LQR-type quadratic inequality linear (Riccati inequality)

$$A^T P + PA + PBR^{-1}B^T P + Q < 0$$

where $P > 0$ is the matrix variable and $Q, R > 0$ are constant.

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The Riccati inequality can be transformed into

$$\begin{bmatrix} A^T P + PA + Q & PB \\ \star & -R \end{bmatrix} < 0$$

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$$A^TP+PA+PBR^{-1}B^TP+Q<0 \quad \textit{is equivalent to} \quad \begin{bmatrix} A^TP+PA+Q & PB \\ \star & -R \end{bmatrix} <0$$

where $P>0$ is the matrix variable and $Q,R>0$ are constant. This inequality can be used to minimize the cost function

$$J=\int_0^{\infty}(x^TQx+u^TRu)dt$$

for the computation of the state feedback controller $u=Kx$ controlling

$$\dot{x}=Ax+Bu$$

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for the computation of the state feedback controller $u = Kx$ controlling

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with an initial condition of $x(0) = x_0$. We know that the solution to this problem is

$$K = -R^{-1}B^T \tilde{P}, \qquad A^T \tilde{P} + \tilde{P}A + \tilde{P}BR^{-1}B^T \tilde{P} + Q = 0$$

$$\textit{and} \quad J = \min_u \int_0^\infty (x^T Qx + u^T Ru)dt = x_0^T \tilde{P} x_0.$$

Tricks: Schur complement

The alternative solution to the optimization problem is given by the following LMI-problem:

$$
\begin{aligned}
\min x_0^T P x_0 \quad \text{s.t.}& \\
\begin{bmatrix} A^T P + PA + Q & PB \\ \star & -R \end{bmatrix} &< 0 \\
-P &< 0
\end{aligned}
$$

for which the optimal controller is given by $K = -R^{-1}B^T P$.

Tricks: The S-procedure

We would like to guarantee that a single quadratic function of $x \in \mathbb{R}^m$ is such that

$$F_0(x) \leq 0 \quad F_0(x) := x^T A_0 x + 2b_0 x + c_0$$

whenever certain other quadratic functions are positive semi-definite

$$F_i(x) \geq 0 \quad F_i(x) := x^T A_i x + 2b_0 x + c_0, \quad i \in \{1, 2, \ldots, q\}$$

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Illustration:

Consider $i = 1$. We need to ensure $F_0(x) \leq 0$ for all x such that $F_1(x) \geq 0$.

If there is a scalar, $\tau > 0$, such that

$$F_{aug}(x) := F_0(x) + \tau F_1(x) \leq 0 \quad \forall x \quad s.t. F_1(x) \geq 0$$

then our goal is achieved.

$F_{aug}(x) \leq 0$ implies that $F_0(x) \leq 0$ if $\tau F_1(x) \geq 0$ because $F_0(x) \leq F_{aug}(x)$ if $F_1(x) \geq 0$.

Tricks: The S-procedure

$$F_{aug}(x) := F_0(x) + \tau F_1(x) \leq 0 \quad \forall x \quad s.t. F_1(x) \geq 0$$

Extending this idea to q inequality constraints:

$$F_0(x) \leq 0 \quad \text{whenever} \quad F_i(x) \geq 0 \qquad (**)$$

holds if

$$F_0(x) + \sum_{i=1}^{q} \tau_i F_i(x) \leq 0, \quad \tau_i \geq 0 \qquad (***)$$

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- The S-procedure is conservative; inequality $(**)$ implies inequality $(***)$
- Equivalence is only guaranteed for $i = 1$.
- The τ_i's are usually variables in an LMI problem.

Tricks: The Projection Lemma

We sometimes encounter inequalities of the form

$$\Psi(V) + G(V)\Lambda H^T(V) + H(V)\Lambda^T G^T(V) < 0 \qquad (**)$$

where V and Λ are the matrix variables, Λ is an unstructured matrix variable.

$\Psi(.), G(.), H(.)$ are (normally affine) functions of V.

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$\Psi(.), G(.), H(.)$ are (normally affine) functions of V.

Inequality $(**)$ is satisfied for some V if and only if

$$\begin{cases} W_{G(V)}^T \Psi(V) W_{G(V)} & < & 0 \\ W_{H(V)}^T \Psi(V) W_{H(V)} & < & 0 \end{cases}$$

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where $W_{G(V)}$ and $W_{H(V)}$ are the *orthogonal complements* of $G(V)$ and $H(V)$, i.e.

$$W_{G(V)} G(V) = 0 \quad W_{H(V)} H(V) = 0.$$

and $[W_{G(V)}^T G(V)]$, $[W_{H(V)}^T H(V)]$ are both full rank.

Tricks: The Projection Lemma

The main point is that we can transform a matrix inequality which is a function of *two* variables, V and Λ, into two inequalities which are functions of *one* variable:

(i) It can facilitate the derivation of an LMI.

(ii) There are less variables for computation.

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(ii) There are less variables for computation.

It is often the approach is to solve for V using

$$\begin{cases} W_{G(V)}^T \Psi(V) W_{G(V)} & < & 0 \\ W_{H(V)}^T \Psi(V) W_{H(V)} & < & 0 \end{cases}$$

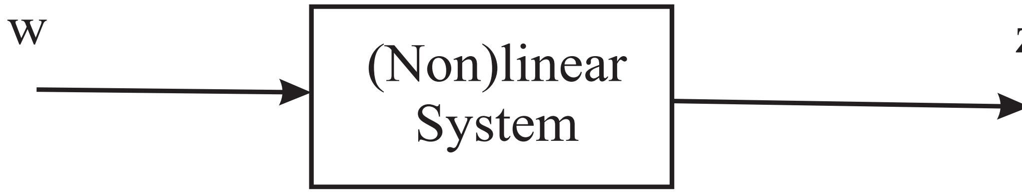
and then for Λ using

$$\Psi(V) + G(V)\Lambda H^T(V) + H(V)\Lambda^T G^T(V) < 0$$

Note that this can be numerically unreliable!!

Examples: $\mathscr{L}_2$ gain- Continuous-time systems

Linear systems: The $\mathscr{H}^\infty$ norm is equivalent to the maximum RMS (Root-Mean-Square) energy gain, the $\mathscr{H}^\infty$ -gain of a linear system, the $\mathscr{L}_2$ gain.



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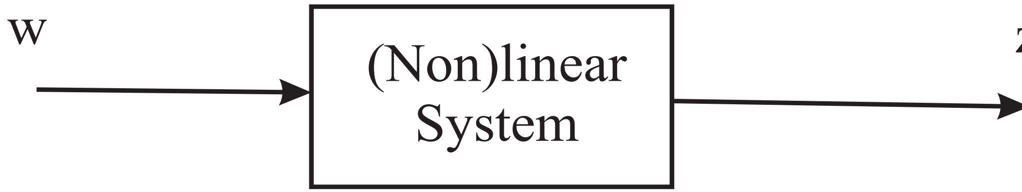

A system with input $w(t)$ and output $z(t)$ is said to have an $\mathscr{L}_2$ gain of γ if

$$\|z\|_2 < \gamma\|w\|_2 + \beta, \quad \beta > 0$$

where $\|w\|_2 = \sqrt{\int_{t=0}^{\infty} w'(t)w(t)dt}$.

The $\mathscr{L}_2$ gain is a 'measure' of the output relative to the size of its input.

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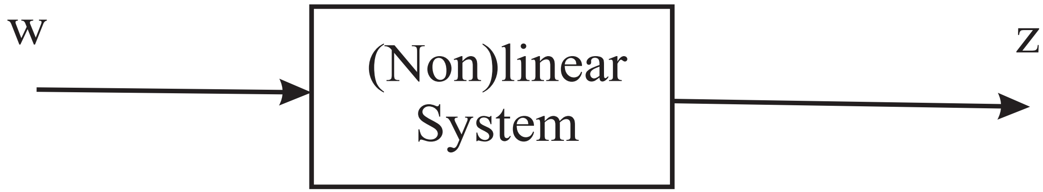

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The $\mathscr{L}_2$ gain is a 'measure' of the output relative to the size of its input.

The $\mathscr{H}^\infty$ norm of $\quad \dot{x} = Ax + Bw, \quad z = Cx + Dw \quad$ is given by:

$$\min \gamma \quad \text{s.t.} \quad \begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0, \;\; P > 0.$$

Examples: $\mathscr{L}_2$ gain- Continuous-time systems

$$\begin{bmatrix} A^TP+PA & PB & C^T \\ B^TP & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0, \quad P > 0.$$

The Schur complement gives

$$\begin{bmatrix} A^TP+PA+\frac{1}{\gamma}C^TC & PB+\frac{1}{\gamma}C^TD \\ B^TP+\frac{1}{\gamma}D^TC & -\gamma I+\frac{1}{\gamma}D^TD \end{bmatrix}$$

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In terms of $\left[x^T \ w^T\right]^T$, it follows that we need to find the minimum of γ so that

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \frac{1}{\gamma} C^T C & PB + \frac{1}{\gamma} C^T D \\ B^T P + \frac{1}{\gamma} D^T C & -\gamma I + \frac{1}{\gamma} D^T D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \quad < \quad 0, \quad \forall \left[x^T \ w^T\right]^T \neq 0$$

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or

$$x^TA^TPx+x^TPAx+\frac{1}{\gamma}x^TC^TCx+x^T(PB+\frac{1}{\gamma}C^TD)w+w^T(B^TP+\frac{1}{\gamma}D^TC)x+w^T\frac{1}{\gamma}D^TDw-\gamma w^Tw$$

$$= x^TA^TPx+x^TPA^Tx+2x^TPBw+\frac{1}{\gamma}z^Tz-\gamma w^Tw < 0$$

Examples: $\mathscr{L}_2$ gain- Continuous-time systems

Defining $V = x^T P x$

$$\dot{V} = x^T A^T P x + x^T P A x + 2x^T P B w$$

Thus, we require:

$$x^T A^T P x + x^T P A^T x + 2x^T P B w + \frac{1}{\gamma} z^T z - \gamma w^T w = \dot{V} + \frac{1}{\gamma} z^T z - \gamma w^T w < 0$$

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and integration in the interval $[0, \infty)$ implies

$$\begin{aligned}
V(t=\infty) - V(t=0) \; + \; \int_{t=0}^{\infty} \frac{1}{\gamma} z^T(s) z(s) ds - \int_{t=0}^{\infty} \gamma w^T(s) w(s) ds < 0 \\
\int_{t=0}^{\infty} z^T(s) z(s) ds \quad < \quad \int_{t=0}^{\infty} \gamma^2 w^T(s) w(s) ds + \gamma(V(t=0) - V(t=\infty)) \\
\sqrt{\int_{t=0}^{\infty} z^T(s) z(s) ds} \quad < \quad \sqrt{\int_{t=0}^{\infty} \gamma^2 w^T(s) w(s) ds + \gamma V(t=0)} \\
\sqrt{\int_{t=0}^{\infty} z^T(s) z(s) ds} \quad < \quad \gamma \sqrt{\int_{t=0}^{\infty} w^T(s) w(s) ds} + \sqrt{\gamma V(t=0)}
\end{aligned}$$

Examples: $\mathscr{L}_2$ gain- Continuous-time systems

$$
\begin{aligned}
V(t=\infty)-V(t=0) \; + \; \int_{t=0}^{\infty}\frac{1}{\gamma}z^T(s)z(s)ds-\int_{t=0}^{\infty}\gamma w^T(s)w(s)ds<0 \\
\sqrt{\int_{t=0}^{\infty}z^T(s)z(s)ds} \quad < \quad \gamma\sqrt{\int_{t=0}^{\infty}w^T(s)w(s)ds}+\sqrt{\gamma V(t=0)} \\
\|z\|_2 \quad < \quad \gamma\|w\|_2+\underbrace{\beta}_{\sqrt{\gamma V(t=0)}}
\end{aligned}
$$

Thus, the linear system (A,B,C,D) has indeed an $\mathscr{L}_2$ gain γ.

Examples: Discrete-time systems

A linear discrete system

$$x(k+1) = Ax(k)$$

is asymptotically stable if and only if there is

$$V(x) = x^T Px, \quad P > 0.$$

and

$$V(x(k+1)) - V(x(k)) = \Delta V(x(k+1)) = x^T(k)A^T PAx(k) - x^T(k)Px(k) < 0 \quad \forall x(k) \neq 0$$

or

$$A^T PA - P < 0.$$

Examples: l_2 gain- Discrete-time systems

A system with input $w(t)$ and output $z(t)$ is said to have an $\mathscr{L}_2$ gain of γ if

$$\|z\|_2 < \gamma\|w\|_2 + \beta, \quad \beta > 0$$

where $\boxed{\boxed{\|w\|_2 = \sqrt{\sum_{k=0}^{\infty} w^T(k)w(k)}}}$.

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For linear systems

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k) \\ y &= Cx(k) + Dw(k) \end{aligned}$$

the value of the finite l_2-gain, γ, ($\mathscr{H}^\infty$ norm; the maximum RMS energy gain) is:

$$\begin{aligned} &\min \gamma \quad \text{s.t.} \\ \begin{bmatrix} A^TPA - P + \frac{1}{\gamma}C^TC & A^TPB + \frac{1}{\gamma}C^TD \\ \frac{1}{\gamma}D^TC & -\gamma I + B^TPB + \frac{1}{\gamma}D^TD \end{bmatrix} &< 0 \\ -P &< 0 \end{aligned}$$

for $P > 0$.

Examples: l_2 gain- Discrete-time systems

The l_2 gain relationship readily follows for $V = x^T P x$ from:

$$\Delta V(x(k+1)) + \frac{1}{\gamma} y^T(k) y(k) - \gamma w^T(k) w(k) < 0$$

$$\underbrace{\sum_{k=0}^{\infty} \Delta V(x(k+1))}_{V(x(\infty)) - V(x(0))} + \frac{1}{\gamma} \sum_{k=0}^{\infty} y^T(k) y(k) - \gamma \sum_{k=0}^{\infty} w^T(k) w(k) < 0$$

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Problem : The matrix inequality

$$\begin{bmatrix} A^T P A - P + \frac{1}{\gamma} C^T C & A^T P B + \frac{1}{\gamma} C^T D \\ \frac{1}{\gamma} D^T C & -\gamma I + B^T P B + \frac{1}{\gamma} D^T D \end{bmatrix} < 0$$

is not linear. $P > 0$ and $\gamma > 0$ are variables.

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\Delta V(x(k+1)) &+ \frac{1}{\gamma} y^T(k)y(k) - \gamma w^T(k)w(k) < 0 \\
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$$\begin{bmatrix} A^T PA - P + \frac{1}{\gamma}C^T C & A^T PB + \frac{1}{\gamma}C^T D \\ \frac{1}{\gamma}D^T C & -\gamma I + B^T PB + \frac{1}{\gamma}D^T D \end{bmatrix} < 0$$

is not linear. $P > 0$ and $\gamma > 0$ are variables.

The Schur Complement implies

$$\begin{bmatrix} A^T PA - P & A^T PB & C^T \\ B^T PA & -\gamma I + B^T PB & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

Examples: l_2 gain- Discrete-time systems

$$\begin{bmatrix} A^TPA - P + \frac{1}{\gamma}C^TC & A^TPB + \frac{1}{\gamma}C^TD \\ \frac{1}{\gamma}D^TC & -\gamma I + B^TPB + \frac{1}{\gamma}D^TD \end{bmatrix} < 0$$

Congruence transformation & Change of variable approach:

$$\begin{bmatrix} \gamma A^TPA - \gamma P + C^TC & \gamma A^TPB + C^TD \\ D^TC & -\gamma^2 I + \gamma B^TPB + D^TD \end{bmatrix} < 0$$

Defining $Q = P\gamma$ and $\mu = \gamma^2$:

$$\min \ \mu \quad \text{s.t.}$$

$$\begin{aligned} \begin{bmatrix} A^TQA - Q + C^TC & A^TQB + C^TD \\ D^TC & -\mu I + B^TQB + D^TD \end{bmatrix} &< 0 \\ -Q &< 0 \end{aligned}$$

$Q > 0$ and the scalar $\mu > 0$ are variables.

The l_2-gain is readily computed with $\gamma = \sqrt{\mu}$.

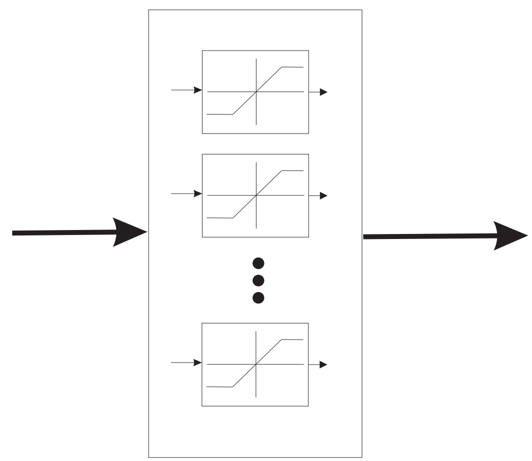
Examples: Sector boundedness

The saturation function is defined as

$$sat(u) = [sat_1(u_1), \ldots, sat_2(u_m)]^T$$

and $sat_i(u_i) = \mathrm{sign}(u_i) \times \min\{|u_i|, \bar{u_i}\}, \quad \bar{u_i} > 0 \quad \forall i \in \{1, \ldots, m\}$

$\bar{u_i}$ is the i'th saturation limit



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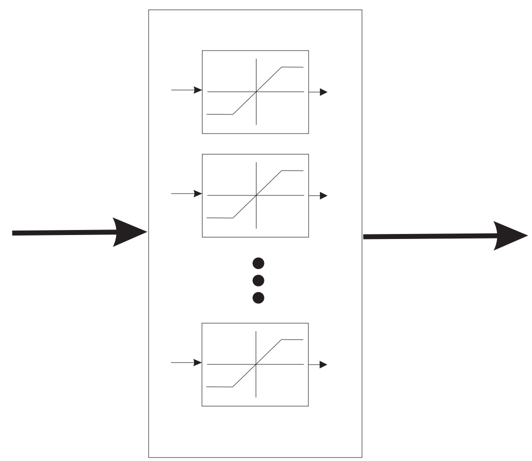

It is easy to verify that the saturation function, $\mathrm{sat}_i(u_i)$ satisfies the following inequality

$$u_i \mathrm{sat}_i(u_i) \geq \mathrm{sat}_i^2(u_i)$$

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$$\text{sat}_i(u_i)[u_i - \text{sat}_i(u_i)]w_i \geq 0$$

for some $w_i > 0$.

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for some $w_i > 0$. We can write

$$\mathrm{sat}(u)^T W[u - \mathrm{sat}(u)] \geq 0$$

for some diagonal $W = diag(w_1, w_2, \ldots) > 0$.

This inequality can be easily used in an S procedure approach where the elements of the diagonal matrix W act as free parameters if necessary/possible.

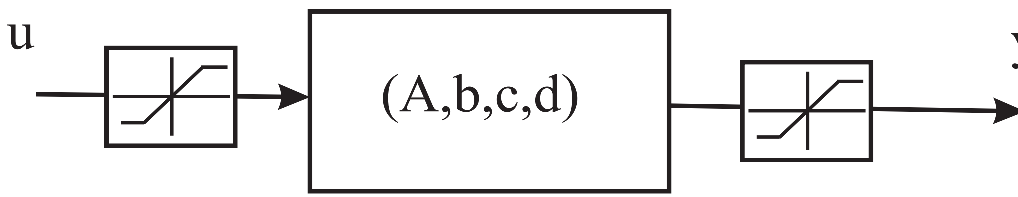
Examples: A slightly more detailed example

Consider the $\mathscr{L}_2$-gain for the SISO-system with saturated input signal u:

$$\dot{x} = Ax + b\,\mathrm{sat}(u),\ x \in \mathbb{R}^n$$

and a limited measurement range of the output y:

$$y = \mathrm{sat}(cx + d\,\mathrm{sat}(u)).$$



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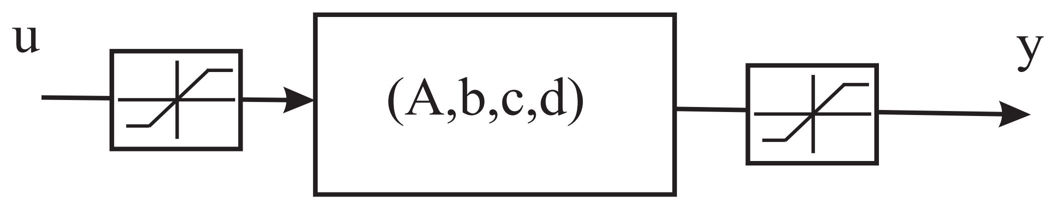
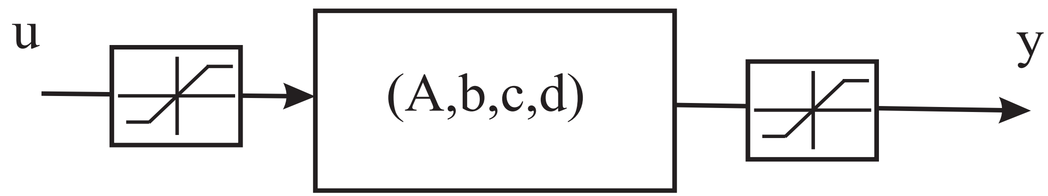


The limits at the actuator inputs u can be due to mechanical limits (e.g. valves) or due to digital-to-analogue converter voltage signal limits.

Output signals can be constrained due to sensor voltage range limits or simply by analogue-to-digital converter limits.

The analysis of such systems is vital to practical control systems and will be pursued in greater detail later. We may consider here an $\mathscr{L}_2$ gain analysis.

Examples: A slightly more detailed example



We may define

$$s = \mathrm{sat}(u).$$

Hence, it follows

$$sw_1(u-s) \geq 0, \qquad w_1 > 0$$

For the output signal y:

$$yw_2(cx+ds-y) \geq 0, \qquad w_2 > 0$$

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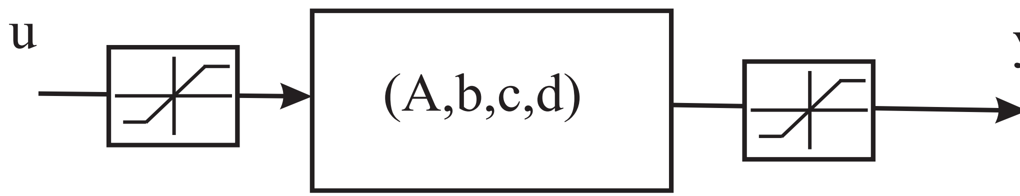

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$$yw_2(cx + ds - y) \geq 0, \qquad w_2 > 0$$

We know that from

$$\dot{V} + \frac{1}{\gamma}y^2 - \gamma u^2 \leq 0$$

follows that our system has the $\mathscr{L}_2$-gain γ.

We have to consider the two saturation nonlinearities!

Examples: A slightly more detailed example

With the S-procedure

$$\dot{V} + \frac{1}{\gamma}y^2 - \gamma u^2 + 2sw_1(u-s) + 2yw_2(cx+ds-y) < 0 \quad for \qquad \begin{bmatrix} x^T & u & y \end{bmatrix} \neq 0$$

the system has also an $\mathscr{L}_2$-gain of γ.

The expression $\dot{V}$ implies:

$$\begin{aligned} & x^T A^T Px + xPAx + x^T Pbs + sb^T Px \\ & \quad + \frac{1}{\gamma}y^2 - \gamma u^2 + 2sw_1(u-s) + 2yw_2(cx+ds-y) \leq 0 \end{aligned}$$

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Rewriting:

$$\begin{bmatrix} x \\ s \\ u \\ y \end{bmatrix}^T \begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & -2w_1 & w_1 & dw_2 \\ 0 & w_1 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} x \\ s \\ u \\ y \end{bmatrix} < 0$$

for $\begin{bmatrix} x^T & s & u & y \end{bmatrix} \neq 0.$

Examples: A slightly more detailed example

This is equivalent to

$$\begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & -2w_1 & w_1 & dw_2 \\ 0 & w_1 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix} < 0.$$

We would like to minimize γ, while P, w_1, w_2 are variables. Not an LMI!

Examples: A slightly more detailed example

This is equivalent to

$$\begin{bmatrix} A^TP+PA & Pb & 0 & c^Tw_2 \\ b^TP & -2w_1 & w_1 & dw_2 \\ 0 & w_1 & -\gamma & 0 \\ w_2c & w_2d & 0 & -2w_2+\frac{1}{\gamma} \end{bmatrix} < 0.$$

We would like to minimize γ, while P, w_1, w_2 are variables. Not an LMI!

Using Projection Lemma twice, we can derive a significantly simpler matrix inequality which delivers the L_2-gain.

First Step:

$$\begin{bmatrix} A^TP+PA & Pb & 0 & c^Tw_2 \\ b^TP & 0 & 0 & dw_2 \\ 0 & 0 & -\gamma & 0 \\ w_2c & w_2d & 0 & -2w_2+\frac{1}{\gamma} \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w_1 \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} w_1 \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} < 0.$$

Examples: A slightly more detailed example

First Step:

$$\begin{bmatrix} A^TP+PA & Pb & 0 & c^Tw_2 \\ b^TP & 0 & 0 & dw_2 \\ 0 & 0 & -\gamma & 0 \\ w_2c & w_2d & 0 & -2w_2+\frac{1}{\gamma} \end{bmatrix}$$

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Defining the matrices

$$g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ h_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},\ \Psi_1 = \begin{bmatrix} A^TP+PA & Pb & 0 & c^Tw_2 \\ b^TP & 0 & 0 & dw_2 \\ 0 & 0 & -\gamma & 0 \\ w_2c & w_2d & 0 & -2w_2+\frac{1}{\gamma} \end{bmatrix},$$

allows us to write $\Psi_1 + g_1 w_1 h_1^T + h_1 w_1^T g_1^T < 0$.

Examples: A slightly more detailed example

$$g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ h_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},\ \Psi_1 = \begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & 0 & 0 & d w_2 \\ 0 & 0 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix},$$

The null space matrices W_{g_1} and W_{h_1} satisfy

$$\left[W_{g_1}^T\ g_1\right]\ \&\ \left[W_{h_1}^T\ h_1\right]\quad full\ rank;\quad W_{g_1} g_1 = 0,\ W_{h_1} h_1 = 0$$

Examples: A slightly more detailed example

$$g_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; h_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \; \Psi_1 = \begin{bmatrix} A^T P + PA & Pb & 0 & c^T w_2 \\ b^T P & 0 & 0 & d w_2 \\ 0 & 0 & -\gamma & 0 \\ w_2 c & w_2 d & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix},$$

The null space matrices W_{g_1} and W_{h_1} satisfy

$$\left[W_{g_1}^T \; g_1\right] \; \& \; \left[W_{h_1}^T \; h_1\right] \quad full\ rank; \quad W_{g_1} g_1 = 0, \; W_{h_1} h_1 = 0$$

Hence,

$$W_{g_1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; W_{h_1} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples: A slightly more detailed example

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Hence,

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Hence, it follows

$$W_{g_1} \Psi_1 W_{g_1}^T = \begin{bmatrix} A^T P + PA & 0 & c^T w_2 \\ 0 & -\gamma & 0 \\ w_2 c & 0 & -2w_2 + \frac{1}{\gamma} \end{bmatrix}, \quad W_{h_1} \Psi_1 W_{h_1}^T = \begin{bmatrix} A^T P + PA & Pb & c^T w_2 \\ b^T P & -\gamma & d w_2 \\ w_2 c & w_2 d & -2w_2 + \frac{1}{\gamma} \end{bmatrix}$$

Examples: A slightly more detailed example

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If $W_{h_1}\Psi_1 W_{h_1}^T < 0$ then also $W_{g_1}\Psi_1 W_{g_1}^T < 0$ (easily seen from a further analysis using the Projection lemma).
We may carry on investigating $W_{h_1}\Psi_1 W_{h_1}^T$ only

Examples: A slightly more detailed example

$$W_{g_1}\Psi_1 W_{g_1}^T = \begin{bmatrix} A^TP+PA & 0 & c^Tw_2 \\ 0 & -\gamma & 0 \\ w_2c & 0 & -2w_2+\frac{1}{\gamma} \end{bmatrix}, \quad W_{h_1}\Psi_1 W_{h_1}^T = \begin{bmatrix} A^TP+PA & Pb & c^Tw_2 \\ b^TP & -\gamma & dw_2 \\ w_2c & w_2d & -2w_2+\frac{1}{\gamma} \end{bmatrix}$$

If $W_{h_1}\Psi_1 W_{h_1}^T < 0$ then also $W_{g_1}\Psi_1 W_{g_1}^T < 0$ (easily seen from a further analysis using the Projection lemma).
We may carry on investigating $W_{h_1}\Psi_1 W_{h_1}^T$ only

$$W_{h_1}\Psi_1 W_{h_1}^T = \Psi_2 + g_2 w_2 h_2^T + h_2 w_2 g_2^T$$

where

$$g_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \, h_2 = \begin{bmatrix} c^T \\ d \\ -1 \end{bmatrix}, \, \Psi_2 = \begin{bmatrix} A^TP+PA & Pb & 0 \\ b^TP & -\gamma & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

This allows us to derive the null space matrices W_{g_2} and W_{h_2} for g_2 and h_2

$$W_{g_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \, W_{h_2} = \begin{bmatrix} I & 0 & c^T \\ 0 & 1 & d \end{bmatrix}$$

Examples: A slightly more detailed example

$$W_{g_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ W_{h_2} = \begin{bmatrix} I & 0 & c^T \\ 0 & 1 & d \end{bmatrix},\ \Psi_2 = \begin{bmatrix} A^T P + PA & Pb & 0 \\ b^T P & -\gamma & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

Thus,

$$W_{g_2} \Psi_2 W_{g_2}^T = \begin{bmatrix} A^T P + PA & Pb \\ b^T P & -\gamma \end{bmatrix},\ W_{h_2} \Psi_2 W_{h_2}^T = \begin{bmatrix} A^T P + PA + \frac{c^T c}{\gamma} & Pb + \frac{c^T d}{\gamma} \\ b^T P + \frac{dc}{\gamma} & -\gamma + \frac{d^2}{\gamma} \end{bmatrix}.$$

Examples: A slightly more detailed example

$$W_{g_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; W_{h_2} = \begin{bmatrix} I & 0 & c^T \\ 0 & 1 & d \end{bmatrix}, \; \Psi_2 = \begin{bmatrix} A^T P + PA & Pb & 0 \\ b^T P & -\gamma & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

Thus,

$$W_{g_2} \Psi_2 W_{g_2}^T = \begin{bmatrix} A^T P + PA & Pb \\ b^T P & -\gamma \end{bmatrix}, \; W_{h_2} \Psi_2 W_{h_2}^T = \begin{bmatrix} A^T P + PA + \frac{c^T c}{\gamma} & Pb + \frac{c^T d}{\gamma} \\ b^T P + \frac{dc}{\gamma} & -\gamma + \frac{d^2}{\gamma} \end{bmatrix}.$$

$W_{g_2} \Psi_2 W_{g_2}^T < 0$ is always satisfied if $W_{h_2} \Psi_2 W_{h_2}^T$.

Hence, the $\mathscr{L}_2$ gain is computed using

$$\begin{bmatrix} A^T P + PA + \frac{c^T c}{\gamma} & Pb + \frac{c^T d}{\gamma} \\ b^T P + \frac{dc}{\gamma} & -\gamma + \frac{d^2}{\gamma} \end{bmatrix} < 0, \quad P > 0$$

The $\mathscr{L}_2$-gain of the linear system (A, b, c, d) is an upper bound of the non-linear operator.

The $\mathscr{L}_2$-gain of the non-linear and linear operator are identical.

Summary

- Matrix inequalities have shown to be versatile tool to

    1. represent $\mathscr{L}_2$, $\mathscr{H}^\infty$, linear quadratic performance constraints, $\mathscr{H}_2$ etc.

    2. analyze linear parameter varying systems, mild non-linear systems

    3. combine several analysis problems in one frame work

- Matrix inequalities can often be transformed into linear matrix inequalities
by congruence transformation, change of variable approach, etc.

- Existence of a large variety of powerful tools for solving LMIs (semi-definite programming)

- LMIs have become a standard tool in the analysis and controller design of linear and non-linear control systems