Chapter 3

Modeling with First-Order Differential Equations

3.1 Linear Models

1. Let $P = P(t)$ be the population at time t, and P_0 the initial population. From $dP/dt = kP$ we obtain $P = P_0e^{kt}$. Using $P(5) = 2P_0$ we find $k = \frac{1}{5}\ln 2$ and $P = P_0e^{(\ln 2)t/5}$. Setting $P(t) = 3P_0$ we have $3 = e^{(\ln 2)t/5}$, so

$$\ln 3 = \frac{(\ln 2)t}{5} \qquad \text{and} \qquad t = \frac{5\ln 3}{\ln 2} \approx 7.9 \text{ years.}$$

Setting $P(t) = 4P_0$ we have $4 = e^{(\ln 2)t/5}$, so

$$\ln 4 = \frac{(\ln 2)t}{5} \qquad \text{and} \qquad t = 10 \text{ years.}$$

2. From Problem 1 the growth constant is $k = \frac{1}{5}\ln 2$. Then $P = P_0e^{(1/5)(\ln 2)t}$ and $10{,}000 = P_0e^{(3/5)\ln 2}$. Solving for P_0 we get $P_0 = 10{,}000e^{-(3/5)\ln 2} = 6{,}597.5$. Now

$$P(10) = P_0e^{(1/5)(\ln 2)(10)} = 6{,}597.5e^{2\ln 2} = 4P_0 = 26{,}390.$$

The rate at which the population is growing is

$$P'(10) = kP(10) = \frac{1}{5}(\ln 2)26{,}390 = 3658 \text{ persons/year.}$$

3. Let $P = P(t)$ be the population at time t. Then $dP/dt = kP$ and $P = ce^{kt}$. From $P(0) = c = 500$ we see that $P = 500e^{kt}$. Since 15% of 500 is 75, we have
$P(10) = 500e^{10k} = 575$. Solving for k, we get $k = \frac{1}{10}\ln\frac{575}{500} = \frac{1}{10}\ln 1.15$. When $t = 30$,

$$P(30) = 500e^{(1/10)(\ln 1.15)30} = 500e^{3\ln 1.15} \approx 760 \text{ years}$$

and

$$P'(30) = kP(30) \approx \frac{1}{10}(\ln 1.15)760 \approx 10.62 \text{ persons/year.}$$

4. Let $P = P(t)$ be bacteria population at time t and P_0 the initial number. From $dP/dt = kP$ we obtain $P = P_0e^{kt}$. Using $P(3) = 400$ and $P(10) = 2000$ we find $400 = P_0e^{3k}$ or $e^k = (400/P_0)^{1/3}$. From $P(10) = 2000$ we then have $2000 = P_0e^{10k} = P_0(400/P_0)^{10/3}$, so

$$\frac{2000}{400^{10/3}} = P_0^{-7/3} \qquad \text{and} \qquad P_0 = \left(\frac{2000}{400^{10/3}}\right)^{-3/7} \approx 201.$$

5. Let $A = A(t)$ be the amount of lead present at time t. From $dA/dt = kA$ and $A(0) = 1$ we obtain $A = e^{kt}$. Using $A(3.3) = 1/2$ we find $k = \frac{1}{3.3}\ln(1/2)$. When 90% of the lead has decayed, 0.1 grams will remain. Setting $A(t) = 0.1$ we have $e^{t(1/3.3)\ln(1/2)} = 0.1$, so

$$\frac{t}{3.3}\ln\frac{1}{2} = \ln 0.1 \qquad \text{and} \qquad t = \frac{3.3\ln 0.1}{\ln(1/2)} \approx 10.96 \text{ hours}.$$

6. Let $A = A(t)$ be the amount present at time t. From $dA/dt = kA$ and $A(0) = 100$ we obtain $A = 100e^{kt}$. Using $A(6) = 97$ we find $k = \frac{1}{6}\ln 0.97$. Then $A(24) = 100e^{(1/6)(\ln 0.97)24} = 100(0.97)^4 \approx 88.5$ mg.

7. Setting $A(t) = 50$ in Problem 6 we obtain $50 = 100e^{kt}$, so

$$kt = \ln\frac{1}{2} \qquad \text{and} \qquad t = \frac{\ln(1/2)}{(1/6)\ln 0.97} \approx 136.5 \text{ hours}.$$

8. **(a)** The solution of $dA/dt = kA$ is $A(t) = A_0e^{kt}$. Letting $A = \frac{1}{2}A_0$ and solving for t we obtain the half-life $T = -(\ln 2)/k$.

(b) Since $k = -(\ln 2)/T$ we have

$$A(t) = A_0e^{-(\ln 2)t/T} = A_0 2^{-t/T}.$$

(c) Writing $\frac{1}{8}A_0 = A_0 2^{-t/T}$ as $2^{-3} = 2^{-t/T}$ and solving for t we get $t = 3T$. Thus, an initial amount A_0 will decay to $\frac{1}{8}A_0$ in three half-lives.

9. Let $I = I(t)$ be the intensity, t the thickness, and $I(0) = I_0$. If $dI/dt = kI$ and $I(1) = 0.25I_0$, then $I = I_0e^{kt}$, $k = \ln 0.25$, and $I(5) = 0.000977I_0$.

10. From $dS/dt = rS$ we obtain $S = S_0e^{rt}$ where $S(0) = S_0$.

(a) If $S_0 = \$5000$ and $r = 5.75\%$ then $S(5) = \$6665.45$.

(b) If $S(t) = \$10{,}000$ then $t = 12$ years.

(c) $S \approx \$6651.82$

11. Using 5730 years as the half-life of C-14 we have from Example 3 in the text $A(t) = A_0e^{-0.00012097t}$. Since 85.5% of the C-14 has decayed, $1 - 0.855 = 0.145$ times the original amount is now present, so

$$0.145A_0 = A_0e^{-0.00012097t}, \quad e^{-0.00012097t} = 0.145, \quad \text{and} \quad t = -\frac{\ln 0.145}{0.00012097} \approx 15{,}968 \text{ years}$$

is the approximate age.

12. From Example 3 in the text, the amount of carbon present at time t is $A(t) = A_0e^{-0.00012097t}$. Letting $t = 660$ and solving for A_0 we have $A(660) = A_0e^{-0.00012097(660)} = 0.923264A_0$. Thus, approximately 92% of the original amount of C-14 remained in the cloth as of 1988.

13. Assume that $dT/dt = k(T - (-12))$ so that $T = -12 + ce^{kt}$. If $T(0) = 21°$ and $T(1/2) = 10°$ then $c = 33$ and $k = 2\ln(2/3)$ so that $T(1) = 2.67°$. If $T(t) = -9°$ then $t = 2.96$ minutes.

14. Assume that $dT/dt = k(T - (-15))$ so that $T = -15 + ce^{kt}$. If $T(1) = 13°$ and $T(5) = -1°$ then $k = -\frac{1}{4}\ln 2$ and $c = 33.298$ so that $T(0) = 18°$.

15. Assume that $dT/dt = k(T - 100)$ so that $T = 100 + ce^{kt}$. If $T(0) = 20°$ and $T(1) = 22°$, then $c = -80$ and $k = \ln(39/40)$ so that $T(t) = 90°$, which implies $t = 82.1$ seconds. If $T(t) = 98°$ then $t = 145.7$ seconds.

16. The differential equation for the first container is $dT_1/dt = k_1(T_1 - 0) = k_1T_1$, whose solution is $T_1(t) = c_1e^{k_1t}$. Since $T_1(0) = 100$ (the initial temperature of the metal bar), we have $100 = c_1$ and $T_1(t) = 100e^{k_1t}$. After 1 minute, $T_1(1) = 100e^{k_1} = 90°\text{C}$, so $k_1 = \ln 0.9$ and $T_1(t) = 100e^{t\ln 0.9}$. After 2 minutes, $T_1(2) = 100e^{2\ln 0.9} = 100(0.9)^2 = 81°\text{C}$.

The differential equation for the second container is $dT_2/dt = k_2(T_2 - 100)$, whose solution is $T_2(t) = 100 + c_2e^{k_2t}$. When the metal bar is immersed in the second container, its initial temperature is $T_2(0) = 81$, so

$$T_2(0) = 100 + c_2e^{k_2(0)} = 100 + c_2 = 81$$

and $c_2 = -19$. Thus, $T_2(t) = 100 - 19e^{k_2t}$. After 1 minute in the second tank, the temperature of the metal bar is 91°C, so

$$\begin{aligned} T_2(1) &= 100 - 19e^{k_2} = 91 \\ e^{k_2} &= \frac{9}{19} \\ k_2 &= \ln\frac{9}{19} \end{aligned}$$

and $T_2(t) = 100 - 19e^{t\ln(9/19)}$. Setting $T_2(t) = 99.9$ we have

$$\begin{aligned} 100 - 19e^{t\ln(9/19)} &= 99.9 \\ e^{t\ln(9/19)} &= \frac{0.1}{19} \\ t &= \frac{\ln(0.1/19)}{\ln(9/19)} \approx 7.02. \end{aligned}$$

Thus, from the start of the “double dipping” process, the total time until the bar reaches 99.9°C in the second container is approximately 9.02 minutes.

17. Using separation of variables to solve $dT/dt = k(T - T_m)$ we get $T(t) = T_m + ce^{kt}$. Using $T(0) = 21$ we find $c = 21 - T_m$, so $T(t) = T_m + (21 - T_m)e^{kt}$. Using the given observations, we obtain

$$\begin{aligned} T\left(\frac{1}{2}\right) &= T_m + (21 - T_m)e^{k/2} = 43 \\ T(1) &= T_m + (21 - T_m)e^{k} = 63. \end{aligned}$$

Then, from the first equation, $e^{k/2} = (43 - T_m)/(21 - T_m)$ and

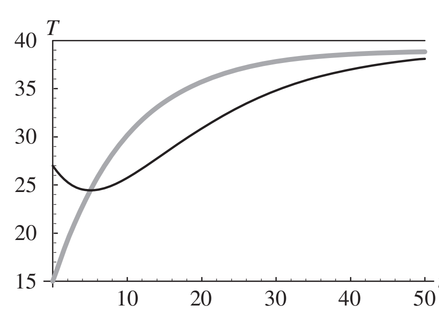
$$\begin{aligned} e^k = (e^{k/2})^2 = \left(\frac{43 - T_m}{21 - T_m}\right)^2 &= \frac{63 - T_m}{21 - T_m} \\ 1849 - 86T_m + T_m^2 &= 1323 - 84T_m + T_m^2 \\ T_m &= 263. \end{aligned}$$

The temperature in the oven is 263°.

18. **(a)** The initial temperature of the bath is $T_m(0) = 16°$, so in the short term the temperature of the chemical, which starts at 27°, should decrease or cool. Over time, the temperature of the bath will increase toward 38° since $e^{-0.1t}$ decreases from 1 toward 0 as t increases from 0. Thus, in the long term, the temperature of the chemical should increase or warm toward 38°.

(b) Adapting the model for Newton’s law of cooling, we have

$$\frac{dT}{dt} = -0.1(T - 38 + 22e^{-0.1t}), \quad T(0) = 27.$$



Writing the differential equation in the form

$$\frac{dT}{dt} + 0.1T = 3.8 - 2.2e^{-0.1t}$$

we see that it is linear with integrating factor $e^{\int 0.1\,dt} = e^{0.1t}$.

Thus

$$\frac{d}{dt}[e^{0.1t}T] = 3.8e^{0.1t} - 2.2$$
$$e^{0.1t}T = 38e^{0.1t} - 2.2t + c$$

and

$$T(t) = 38 - 2.2te^{-0.1t} + ce^{-0.1t}.$$

Now $T(0) = 27$ so $38 + c = 27$, $c = -11$ and

$$T(t) = 38 - 2.2te^{-0.1t} - 11e^{-0.1t} = 38 - (2.2t + 11)e^{-0.1t}.$$

The thinner curve verifies the prediction of cooling followed by warming toward 38°. The wider curve shows the temperature T_m of the liquid bath.

19. According to Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_m).$$

Separating variables we have

$$\frac{dT}{T - T_m} = k\,dt \qquad \text{so} \qquad \ln|T - T_m| = kt + c \qquad \text{and} \qquad T = T_m + c_1e^{kt}.$$

Setting $T(0) = T_0$ we find $c_1 = T_0 - T_m$. Thus

$$T(t) = T_m + (T_0 - T_m)e^{kt}.$$

In this problem we use $T_0 = 37$ and $T_m = 21$. Now, let n denote the number of hours elapsed before the body was found. Then $T(n) = 29$ and $T(n + 1) = 27$. Using this information, we have

$$21 + (37 - 21)e^{kn} = 29 \qquad \text{and} \qquad 21 + (37 - 21)e^{k(n+1)} = 27$$

or

$$15e^{kn} = 8 \qquad \text{and} \qquad 15e^{kn+k} = 15e^{kn}e^k = 6.$$

The last equation is the same as $8e^k = 6$. Solving for k, we have $k = \ln\frac{3}{4} \approx -0.288$. Finally, solving $e^{-0.4055n} = 15/28.6$ for n, we have

$$-0.288n = \ln\left(\frac{8}{15}\right)$$
$$n = \frac{1}{-0.288}\ln\left(\frac{8}{15}\right) \approx 2.18.$$

Thus, about 2.18 hours elapsed before the body was found.

20. Solving the differential equation $dT/dt = kS(T - T_m)$ subject to $T(0) = T_0$ gives

$$T(t) = T_m + (T_0 - T_m)e^{kSt}.$$

The temperatures of the coffee in cups A and B are, respectively,

$$T_A(t) = 21 + 45e^{kSt} \quad \text{and} \quad T_B(t) = 21 + 45e^{2kSt}.$$

Then $T_A(30) = 21 + 45e^{30kS} = 38$, which implies $e^{30kS} = 0.378$. Hence

$$\begin{aligned} T_B(30) &= 21 + 45e^{60kS} = 21 + 45\left(e^{30kS}\right)^2 \\ &= 21 + 45\,(0.378)^2 = 21 + 45\,(0.143) = 27.43°\text{C}. \end{aligned}$$

21. From $dA/dt = 4 - A/50$ we obtain $A = 200 + ce^{-t/50}$. If $A(0) = 30$ then $c = -170$ and $A = 200 - 170e^{-t/50}$.

22. From $dA/dt = 0 - A/50$ we obtain $A = ce^{-t/50}$. If $A(0) = 30$ then $c = 30$ and $A = 30e^{-t/50}$.

23. From $dA/dt = 5 - A/100$ we obtain $A = 500 + ce^{-t/100}$. If $A(0) = 0$ then $c = -500$ and $A(t) = 500 - 500e^{-t/100}$.

24. From Problem 23 the number of kilograms of salt in the tank at time t is $A(t) = 500 - 500e^{-t/100}$. The concentration at time t is $c(t) = A(t)/2000 = 0.25 - 0.25e^{-t/100}$. Therefore $c(5) = 0.25 - 0.25e^{-1/20} = 0.0122\,\text{kg/L}$ and $\lim_{t\to\infty} c(t) = 0.25$. Solving $c(t) = 0.125 = 0.25 - 0.25e^{-t/100}$ for t we obtain $t \approx 69.3\,\text{min}$.

25. From

$$\frac{dA}{dt} = 5 - \frac{40A}{2000 - (40 - 20)t} = 5 - \frac{2A}{100 - t}$$

we obtain $A = 500 - 5t + c(100 - t)^2$. If $A(0) = 0$ then $c = -0.05$. The tank is empty in 100 minutes.

26. With $c_{in}(t) = 2 + \sin(t/4)\,\text{kg/L}$, the initial-value problem is

$$\frac{dA}{dt} + \frac{1}{100}A = 6 + 3\sin\frac{t}{4}, \qquad A(0) = 25.$$

The differential equation is linear with integrating factor $e^{\int dt/100} = e^{t/100}$, so

$$\begin{aligned} \frac{d}{dt}[e^{t/100}A(t)] &= \left(6 + 3\sin\frac{t}{4}\right)e^{t/100} \\ e^{t/100}A(t) &= 600e^{t/100} + \frac{150}{313}e^{t/100}\sin\frac{t}{4} - \frac{3750}{313}e^{t/100}\cos\frac{t}{4} + c, \end{aligned}$$

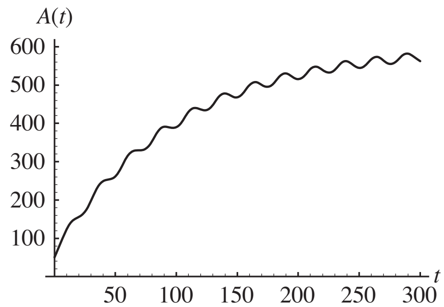
and

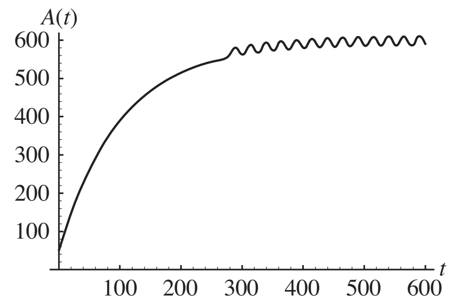
$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} + ce^{-t/100}.$$

Letting $t = 0$ and $A = 25$ we have $600 - 3750/313 + c = 25$ and $c = -8/313$. Then

$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} - \frac{84200}{313}e^{-t/100}.$$

The graphs on $[0, 300]$ and $[0, 600]$ below show the effect of the sine function in the input when compared with the graph in Figure 2.7.4(a) in the text.





27. From

$$\frac{dA}{dt} = 10 - \frac{15A}{400 + (20 - 15)t} = 10 - \frac{3A}{80 + t}$$

we obtain $A = 80 + 2.5t + c(80 + t)^{-2}$. If $A(0) = 5$ then $c = -480{,}000$ and $A(30) = 64.38$ kg.

28. (a) Initially the tank contains 300 gallons of solution. Since brine is pumped in at a rate of 3 gal/min and the mixture is pumped out at a rate of 2 gal/min, the net change is an increase of 1 gal/min. Thus, in 100 minutes the tank will contain its capacity of 400 gallons.

(b) The differential equation describing the amount of salt in the tank is $A'(t) = 6 - 2A/(300 + t)$ with solution

$$A(t) = 600 + 2t - (4.95 \times 10^7)(300 + t)^{-2}, \qquad 0 \le t \le 100,$$

as noted in the discussion following Example 5 in the text. Thus, the amount of salt in the tank when it overflows is

$$A(100) = 800 - (4.95 \times 10^7)(400)^{-2} = 490.625 \text{ lbs}.$$

(c) When the tank is overflowing the amount of salt in the tank is governed by the differential equation

$$\begin{aligned}\frac{dA}{dt} &= (3 \text{ gal/min})(2 \text{ lb/gal}) - \left(\frac{A}{400} \text{ lb/gal}\right)(3 \text{ gal/min}) \\ &= 6 - \frac{3A}{400}, \qquad A(100) = 490.625.\end{aligned}$$

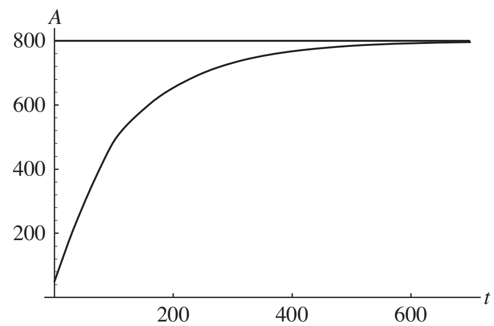
Solving the equation, we obtain $A(t) = 800 + ce^{-3t/400}$. The initial condition yields $c = -654.947$, so that

$$A(t) = 800 - 654.947e^{-3t/400}.$$

When $t = 150$, $A(150) = 587.37$ lbs.

(d) As $t \to \infty$, the amount of salt is 800 lbs, which is to be expected since (400 gal)(2 lb/gal)= 800 lbs.

(e)



29. Assume $L\,di/dt + Ri = E(t)$, $L = 0.1$, $R = 50$, and $E(t) = 50$ so that $i = \frac{3}{5} + ce^{-500t}$. If $i(0) = 0$ then $c = -3/5$ and $\lim_{t\to\infty} i(t) = 3/5$.

30. Assume $L\,di/dt + Ri = E(t)$, $E(t) = E_0 \sin \omega t$, and $i(0) = i_0$ so that

$$i = \frac{E_0 R}{L^2\omega^2 + R^2} \sin \omega t - \frac{E_0 L\omega}{L^2\omega^2 + R^2} \cos \omega t + ce^{-Rt/L}.$$

Since $i(0) = i_0$ we obtain $c = i_0 + \dfrac{E_0 L\omega}{L^2\omega^2 + R^2}$.

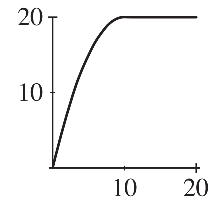
31. Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 200$, $C = 10^{-4}$, and $E(t) = 100$ so that $q = 1/100 + ce^{-50t}$. If $q(0) = 0$ then $c = -1/100$ and $i = \frac{1}{2}e^{-50t}$.

32. Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 1000$, $C = 5 \times 10^{-6}$, and $E(t) = 200$. Then $q = \frac{1}{1000} + ce^{-200t}$ and $i = -200ce^{-200t}$. If $i(0) = 0.4$ then $c = -\frac{1}{500}$, $q(0.005) = 0.0003$ coulombs, and $i(0.005) = 0.1472$ amps. We have $q \to \frac{1}{1000}$ as $t \to \infty$.

33. For $0 \le t \le 20$ the differential equation is $20\,di/dt + 2i = 120$. An integrating factor is $e^{t/10}$, so $(d/dt)[e^{t/10}i] = 6e^{t/10}$ and $i = 60 + c_1e^{-t/10}$. If $i(0) = 0$ then $c_1 = -60$ and $i = 60 - 60e^{-t/10}$. For $t > 20$ the differential equation is $20\,di/dt + 2i = 0$ and $i = c_2e^{-t/10}$. At $t = 20$ we want $c_2e^{-2} = 60 - 60e^{-2}$ so that $c_2 = 60\left(e^2 - 1\right)$. Thus

$$i(t) = \begin{cases} 60 - 60e^{-t/10}, & 0 \le t \le 20 \\ 60\left(e^2 - 1\right)e^{-t/10}, & t > 20 \end{cases}$$

34. We first solve $(1 - t/10)di/dt + 0.2i = 4$. Separating variables we obtain $di/(40 - 2i) = dt/(10 - t)$. Then



$$-\frac{1}{2}\ln|40 - 2i| = -\ln|10 - t| + c \quad \text{or} \quad \sqrt{40 - 2i} = c_1(10 - t).$$

Since $i(0) = 0$ we must have $c_1 = 2/\sqrt{10}$. Solving for i we get $i(t) = 4t - \frac{1}{5}t^2$, $0 \le t < 10$. For $t \ge 10$ the equation for the current becomes $0.2i = 4$ or $i = 20$. Thus

$$i(t) = \begin{cases} 4t - \dfrac{1}{5}t^2, & 0 \le t < 10 \\ 20, & t \ge 10. \end{cases}$$

The graph of $i(t)$ is given in the figure.

35. (a) From $m\,dv/dt = mg - kv$ we obtain $v = mg/k + ce^{-kt/m}$. If $v(0) = v_0$ then $c = v_0 - mg/k$ and the solution of the initial-value problem is

$$v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

(b) As $t \to \infty$ the limiting velocity is mg/k.

(c) From $ds/dt = v$ and $s(0) = 0$ we obtain

$$s(t) = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-kt/m} + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right).$$

36. (a) Integrating $d^2s/dt^2 = -g$ we get $v(t) = ds/dt = -gt + c$. From $v(0) = 90$ we find $c = 90$, and we are given $g = 9.8$, so the velocity is $v(t) = -9t + 90$.

(b) Integrating again and using $s(0) = 0$ we get $s(t) = -4.9t^2 + 90t$. The maximum height is attained when $v = 0$, that is, at $t_a = 9.184$. The maximum height will be $s(9.184) = 413.27\,\text{m}$.

37. When air resistance is proportional to velocity, the model for the velocity is $m\,dv/dt = -mg - kv$ (using the fact that the positive direction is upward.) Solving the differential equation using separation of variables we obtain $v(t) = -mg/k + ce^{-kt/m}$. From $v(0) = 90$ we get

$$v(t) = -\frac{mg}{k} + \left(9 + \frac{mg}{k}\right)e^{-kt/m}.$$

Integrating and using $s(0) = 0$ we find

$$s(t) = -\frac{mg}{k}t + \frac{m}{k}\left(90 + \frac{mg}{k}\right)(1 - e^{-kt/m}).$$

Setting $k = 0.0025$, $m = 75/9.8 = 7.653$, and $g = 9.8$ we have

$$s(t) = 9.211^7 - 30{,}000t - 9.211^7e^{-0.000327t}$$

and

$$v(t) = -30{,}000 + 30{,}090e^{-0.000327t}.$$

The maximum height is attained when $v = 0$, that is, at $t_a = 9.161$. The maximum height will be $s(9.161) = 412.44\,\text{m}$, which is less than the maximum height in Problem 36.

38. Assuming that the air resistance is proportional to velocity and the positive direction is downward with $s(0) = 0$, the model for the velocity is $m\,dv/dt = mg - kv$. Using separation of variables to solve this differential equation, we obtain $v(t) = mg/k + ce^{-kt/m}$. Then, using $v(0) = 0$, we get $v(t) = (mg/k)(1 - e^{-kt/m})$. Letting $k = 7.25$, $m = (550+160)/9.8 = 72.5$, and $g = 9.8$, we have $v(t) = 98(1 - e^{-0.1t})$. Integrating, we find $s(t) = 98t + 980e^{-0.1t} + c_1$. Solving $s(0) = 0$ for c_1 we find $c_1 = -980$, therefore $s(t) = 98t + 980e^{-0.1t} - 980$. At $t = 15$, when the parachute opens, $v(15) = 76.13$ and $s(15) = 708.67$. At this time the value of k changes to $k = 145$ and the new initial velocity is $v_0 = 76.13$. With the parachute open, the skydiver's velocity is $v_p(t) = mg/k + c_2e^{-kt/m}$, where t is reset to 0 when the parachute opens. Letting $m = 72.5$, $g = 9.8$, and $k = 145$, this gives $v_p(t) = 4.9 + c_2e^{-2t}$. From $v(0) = 76.13$ we find $c_2 = 71.23$, so $v_p(t) = 4.9 + 71.23e^{-2t}$. Integrating, we get $s_p(t) = 4.9t - 35.615e^{-2t} + c_3$. Solving $s_p(0) = 0$ for c_3, we find $c_3 = 35.615$, so $s_p(t) = 4.9t - 35.615e^{-2t} + 35.615$. Twenty seconds after leaving the plane is five seconds after the parachute opens. The skydiver's velocity at this time is $v_p(5) = 4.903$ m/s and she has fallen a total of $s(15) + s_p(5) = 708.67 + 60.113 = 768.78\,\text{m}$. Her terminal velocity is $\lim_{t\to\infty} v_p(t) = 4.9$, so she has very nearly reached her terminal velocity five seconds after the parachute opens. When the parachute opens, the distance to the ground is $4500 - s(15) = 4500 - 708.67 = 3791.33$ m. Solving $s_p(t) = 3791.33$ we get $t = 766.47$ s $= 12.77$ min. Thus, it will take her approximately 12.77 minutes to reach the ground after her parachute has opened and a total of $(766 + 15)/60 = 13.02$ minutes after she exits the plane.

39. (a) With the values given in the text the initial-value problem becomes

$$\frac{dv}{dt} + \frac{2}{200-t}v = -9.8 + \frac{2000}{200-t}, \quad v(0) = 0.$$

This is a linear differential equationwith integrating factor

$$e^{\int [2/(200-t)]\,dt} = e^{-2\ln|200-t|} = (200-t)^{-2}.$$

Then

$$\begin{aligned}
\frac{d}{dt}\left[(200-t)^{-2}v\right] &= -9.8(200-t)^{-2} + 2000(200-t)^{-3} && \longleftarrow \text{integrate}\\
(200-t)^{-2}v &= -9.8(200-t)^{-1} + 1000(200-t)^{-2} + c && \longleftarrow \text{multiply by}\,(200-t)^2\\
v &= -9.8(200-t) + 1000 + c(200-t)^2\\
&= -960 + 9.8t + c(200-t)^2.
\end{aligned}$$

Using the initial condition we have

$$0 = v(0) = -960 + 40{,}000c \qquad \text{so} \qquad c = \frac{960}{40{,}000} = 0.024.$$

Thus

$$\begin{aligned} v(t) &= -960 + 9.8t + 0.024(200 - t)^2 \\ &= -960 + 9.8t + 0.024(40,000 - 400t + t^2) \\ &= -960 + 9.8t + 960 - 9.6t + 0.024t^2 \\ &= 0.024t^2 + 0.2t. \end{aligned}$$

(b) Integrating both sides of

$$\frac{ds}{dt} = v(t) = 0.024t^2 + 0.2t$$

we find

$$s(t) = 0.008t^3 + 0.1t^2 + c_1.$$

We assume that the height of the rocket is measured from $s = 0$, so that $s(0) = 0$ and $c_1 = 0$. Then the height of the rocket at time t is $s(t) = 0.008t^3 + 0.1t^2$.

40. (a) From Problem 22 in Exercises 1.3, $t_b = m_f(0)/\lambda$. In this case $m_f(0) = 50$ and $\lambda = 1$, so the time of burnout is 50 s.

(b) The velocity at burnout time is

$$v(50) = 0.024(50)^2 + 0.2(50) = 70 \text{ m/s}.$$

(c) The height of the rocket at burnout is

$$s(50) = 0.008(50)^3 + 0.1(50)^2 = 1250 \text{ m}.$$

(d) At burnout the rocket will have upward momentum which will carry it higher.

(e) After burnout the total mass of the rocket is a constant $200 - 50 = 150$ kg. By Problem 22 in Exercises 1.3 the velocity for a rocket with variable mass due to fuel consumption is

$$m\frac{dv}{dt} + v\frac{dm}{dt} + kv = -mg + R.$$

Here m is the total mass of the rocket, and in this case m is constant after burnout, so $dm/dt = 0$ and the velocity of the rocket satisfies

$$m\frac{dv}{dt} + kv = -mg + R.$$

We identify $m = 150, k = 3, g = 9.8$ and $R = 0$, since the thrust is 0 after burnout. Then

$$150\frac{dv}{dt} + 3v = -150(9.8) = -1470 \quad \text{or} \quad \frac{dv}{dt} + \frac{1}{50}v = -9.8.$$

41. **(a)** The differential equation is first-order and linear. Letting $b = k/\rho$, the integrating factor is $e^{\int 3b\,dt/(bt+r_0)} = (r_0 + bt)^3$. Then

$$\frac{d}{dt}[(r_0 + bt)^3 v] = g(r_0 + bt)^3 \quad \text{and} \quad (r_0 + bt)^3 v = \frac{g}{4b}(r_0 + bt)^4 + c.$$

The solution of the differential equation is $v(t) = (g/4b)(r_0 + bt) + c(r_0 + bt)^{-3}$. Using $v(0) = 0$ we find $c = -gr_0^4/4b$, so that

$$v(t) = \frac{g}{4b}(r_0 + bt) - \frac{gr_0^4}{4b(r_0 + bt)^3} = \frac{g\rho}{4k}\left(r_0 + \frac{k}{\rho}t\right) - \frac{g\rho r_0^4}{4k(r_0 + kt/\rho)^3}\,.$$

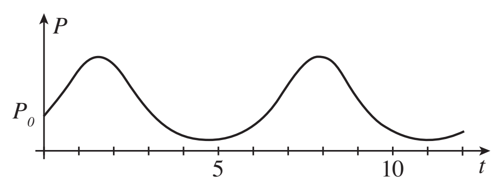
(b) Integrating $dr/dt = k/\rho$ we get $r = kt/\rho + c$. Using $r(0) = r_0$ we have $c = r_0$, so $r(t) = kt/\rho + r_0$.

(c) If $r = 2\,\text{mm}$ when $t = 10\,\text{s}$, then solving $r(10) = 2$ for k/ρ, we obtain $k/\rho = -0.1$ and $r(t) = 3 - 0.1t$. Solving $r(t) = 0$ we get $t = 30$, so the raindrop will have evaporated completely at 30 seconds.

42. Separating variables, we obtain $dP/P = k\cos t\,dt$, so

$$\ln|P| = k\sin t + c \qquad \text{and} \qquad P = c_1 e^{k\sin t}.$$

If $P(0) = P_0$, then $c_1 = P_0$ and $P = P_0 e^{k\sin t}$.



43. **(a)** From $dP/dt = (k_1 - k_2)P$ we obtain $P = P_0 e^{(k_1 - k_2)t}$ where $P_0 = P(0)$.

(b) If $k_1 > k_2$ then $P \to \infty$ as $t \to \infty$. If $k_1 = k_2$ then $P = P_0$ for every t. If $k_1 < k_2$ then $P \to 0$ as $t \to \infty$.

44. **(a)** The solution of the differential equation is $P(t) = c_1 e^{kt} + h/k$. If we let the initial population of fish be P_0 then $P(0) = P_0$ which implies that

$$c_1 = P_0 - \frac{h}{k} \quad \text{and} \quad P(t) = \left(P_0 - \frac{h}{k}\right)e^{kt} + \frac{h}{k}.$$

(b) For $P_0 > h/k$ all terms in the solution are positive. In this case $P(t)$ increases as time t increases. That is, $P(t) \to \infty$ as $t \to \infty$. For $P_0 = h/k$ the population remains constant for all time t:

$$P(t) = \left(\frac{h}{k} - \frac{h}{k}\right)e^{kt} + \frac{h}{k} = \frac{h}{k}.$$

For $0 < P_0 < h/k$ the coefficient of the exponential function is negative and so the function decreases as time t increases.

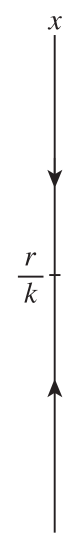
(c) For $0 < P_0 < h/k$ the function decreases and is concave down, therefore the graph of $P(t)$ crosses the t-axis. That is, there exists a time $T > 0$ such that $P(T) = 0$. Solving

$$\left(P_0 - \frac{h}{k}\right) e^{kT} + \frac{h}{k} = 0$$

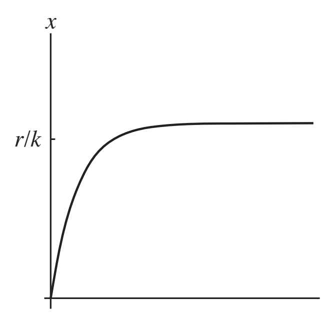
for T shows that the time of extinction is

$$T = \frac{1}{k} \ln \left(\frac{h}{h - kP_0} \right).$$

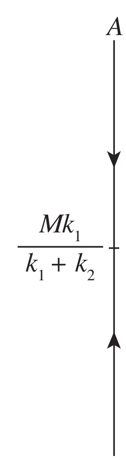
45. **(a)** Solving $r - kx = 0$ for x we find the equilibrium solution $x = r/k$. When $x < r/k$, $dx/dt > 0$ and when $x > r/k$, $dx/dt < 0$. From the phase portrait we see that $\lim_{t\to\infty} x(t) = r/k$.



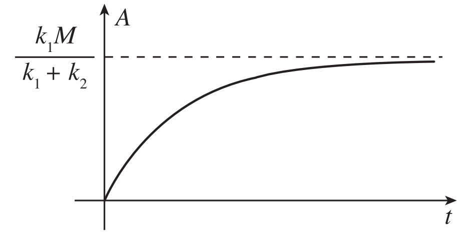
(b) From $dx/dt = r - kx$ and $x(0) = 0$ we obtain $x = r/k - (r/k)e^{-kt}$ so that $x \to r/k$ as $t \to \infty$. If $x(T) = r/2k$ then $T = (\ln 2)/k$.



46. **(a)** Solving $k_1(M - A) - k_2 A = 0$ for A we find the equilibrium solution $A = k_1 M/(k_1 + k_2)$. From the phase portrait we see that $\lim_{t\to\infty} A(t) = k_1 M/(k_1 + k_2)$. Since $k_2 > 0$, the material will never be completely memorized and the larger k_2 is, the less the amount of material will be memorized over time.



(b) Write the differential equation in the form $dA/dt + (k_1 + k_2)A = k_1 M$. Then an integrating factor is $e^{(k_1+k_2)t}$, and



$$\frac{d}{dt}\left[e^{(k_1+k_2)t}A\right] = k_1 M e^{(k_1+k_2)t}$$

$$e^{(k_1+k_2)t}A = \frac{k_1 M}{k_1+k_2}e^{(k_1+k_2)t} + c$$

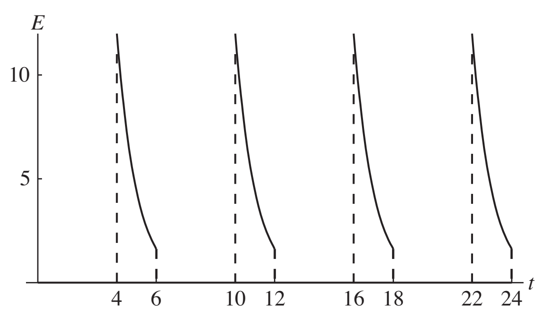
$$A = \frac{k_1 M}{k_1+k_2} + ce^{-(k_1+k_2)t}.$$

Using $A(0) = 0$ we find $c = -\dfrac{k_1 M}{k_1+k_2}$ and $A = \dfrac{k_1 M}{k_1+k_2}\left(1 - e^{-(k_1+k_2)t}\right)$. As $t \to \infty$, $A \to \dfrac{k_1 M}{k_1+k_2}$.

47. (a) For $0 \le t < 4$, $6 \le t < 10$ and $12 \le t < 16$, no voltage is applied to the heart and $E(t) = 0$. At the other times, the differential equation is $dE/dt = -E/RC$. Separating variables, integrating, and solving for e, we get $E = ke^{-t/RC}$, subject to $E(4) = E(10) = E(16) = 12$. These intitial conditions yield, respectively, $k = 12e^{4/RC}$, $k = 12e^{10/RC}$, $k = 12e^{16/RC}$, and $k = 12e^{22/RC}$. Thus

$$E(t) = \begin{cases} 0, & 0 \le t < 4,\ 6 \le t < 10,\ 12 \le t < 16 \\ 12e^{(4-t)/RC}, & 4 \le t < 6 \\ 12e^{(10-t)/RC}, & 10 \le t < 12 \\ 12e^{(16-t)/RC}, & 16 \le t < 18 \\ 12e^{(22-t)/RC}, & 22 \le t < 24 \end{cases}$$

(b)



48. (a) (i) Using Newton's second law of motion, $F = ma = m\,dv/dt$, the differential equation for the velocity v is

$$m\frac{dv}{dt} = mg\sin\theta \qquad \text{or} \qquad \frac{dv}{dt} = g\sin\theta,$$

where $mg\sin\theta$, $0 < \theta < \pi/2$, is the component of the weight along the plane in the direction of motion.

(*ii*) The model now becomes

$$m\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta,$$

where $\mu mg\cos\theta$ is the component of the force of sliding friction (which acts perpendicular to the plane) along the plane. The negative sign indicates that this component of force is a retarding force which acts in the direction opposite to that of motion.

(*iii*) If air resistance is taken to be proportional to the instantaneous velocity of the body, the model becomes

$$m\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta - kv,$$

where k is a constant of proportionality.

(b) (*i*) The differential equation is

$$\frac{dv}{dt} = (g)\cdot\frac{1}{2} \qquad \text{or} \qquad \frac{dv}{dt} = g/2.$$

Integrating the last equation gives $v(t) = gt/2 + c_1$. Since $v(0) = 0$, we have $c_1 = 0$ and so $v(t) = gt/2$.

(*ii*) The differential equation is

$$\frac{dv}{dt} = (g)\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}\cdot(g)\cdot\frac{\sqrt{3}}{2} \qquad \text{or} \qquad \frac{dv}{dt} = g/8.$$

In this case $v(t) = gt/8$.

(*iii*) When the retarding force due to air resistance is taken into account, the differential equation for velocity v becomes

$$\frac{dv}{dt} = (g)\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}\cdot(g)\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}v \qquad \text{or} \qquad \frac{dv}{dt} = g/8 - \frac{1}{4}v.$$

The last differential equation is linear and has solution $v(t) = g/2 + c_1e^{-t/4}$. Since $v(0) = 0$, we find $c_1 = -g/2$, so $v(t) = g/2 - (g/2)e^{-t/4}$.

49. (a) (*i*) If $s(t)$ is distance measured down the plane from the highest point, then $ds/dt = v$. Integrating $ds/dt = (g/2)t$ gives $s(t) = (g/4)t^2 + c_2$. Using $s(0) = 0$ then gives $c_2 = 0$. Now the length L of the plane is $L = 15/\sin 30° = 30\,\text{ft}$. The time it takes the box to slide completely down the plane is the solution of $s(t) = 30$ or $t^2 = 12.245$, so $t \approx 3.5\,\text{s}$.

(*ii*) Integrating $ds/dt = gt/8$ gives $s(t) = (g/16)t^2 + c_2$. Using $s(0) = 0$ gives $c_2 = 0$, so $s(t) = (g/16)t^2$ and the solution of $s(t) = 30$ is now $t \approx 7\,\text{s}$.

(*iii*) Integrating $ds/dt = g/2 - (g/2)e^{-t/4}$ and using $s(0) = 0$ to determine the constant of integration, we obtain $s(t) = (g/2)t + 2ge^{-t/4} - 2g$. With the aid of a CAS we find that the solution of $s(t) = 30$, or

$$30 = (g/2)t + 2ge^{-t/4} - 2g$$

is now $t \approx 9.78\,\text{s}.92.1$

(b) The differential equation $m\,dv/dt = mg\sin\theta - \mu mg\cos\theta$ can be written

$$m\frac{dv}{dt} = mg\cos\theta(\tan\theta - \mu).$$

If $\tan\theta = \mu$, $dv/dt = 0$ and $v(0) = 0$ implies that $v(t) = 0$. If $\tan\theta < \mu$ and $v(0) = 0$, then integration implies $v(t) = g\cos\theta(\tan\theta - \mu)t < 0$ for all time t.

(c) Since $\tan 23° = 0.4245$ and $\mu = \sqrt{3}/4 = 0.4330$, we see that $\tan 23° < 0.4330$. The differential equation is $dv/dt = 9.8\cos 23°(\tan 23° - \sqrt{3}/4) = -0.077$. Integration and the use of the initial condition gives $v(t) = -0.077t+0.3$. When the box stops, $v(t) = 0$ or $0 = -0.077t+0.3$ or $t = 3.896$ s. From $s(t) = -0.0385t^2+0.3t$ we find $s(3.869) = 0.584$ m.

(d) With $v_0 > 0$, $v(t) = -0.077t + v_0$ and $s(t) = -0.0385t^2 + v_0t$. Because two real positive solutions of the equation $s(t) = 30$, or $0 = -0.0385t^2 + v_0t - 30$, would be physically meaningless, we use the quadratic formula and require that $b^2 - 4ac = 0$ or $v_0^2 - 4.62 = 0$. From this last equality we find $v_0 \approx 2.15$ m/s. For the time it takes the box to traverse the entire inclined plane, we must have $0 = -0.0385t^2 + 2.15t - 30$.

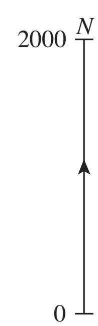
The roots are $t = 27.2727$ s and $t = 28.5714$ s. So if $v_0 > 2.15$, we are guaranteed that the box will slide completely down the plane.

50. (a) We saw in part (a) of Problem 36 that the ascent time is $t_a = 9.375$. To find when the cannonball hits the ground we solve $s(t) = -16t^2 + 300t = 0$, getting a total time in flight of $t = 18.75$ s. Thus, the time of descent is $t_d = 18.75 - 9.375 = 9.375$. The impact velocity is $v_i = v(18.75) = -300$, which has the same magnitude as the initial velocity.

(b) We saw in Problem 37 that the ascent time in the case of air resistance is $t_a = 9.162$. Solving $s(t) = 1{,}340{,}000 - 6{,}400t - 1{,}340{,}000e^{-0.005t} = 0$ we see that the total time of flight is 18.466 s. Thus, the descent time is $t_d = 18.466 - 9.162 = 9.304$. The impact velocity is $v_i = v(18.466) = -290.91$, compared to an initial velocity of $v_0 = 300$.

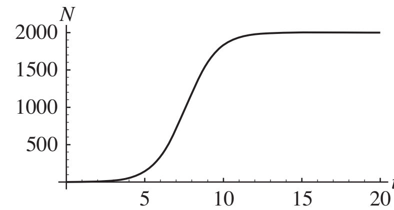
3.2 Nonlinear Models

1. (a) Solving $N(1 - 0.0005N) = 0$ for N we find the equilibrium solutions $N = 0$ and $N = 2000$. When $0 < N < 2000$, $dN/dt > 0$. From the phase portrait we see that $\lim_{t\to\infty} N(t) = 2000$. A graph of the solution is shown in part (b).



(b) Separating variables and integrating we have

$$\frac{dN}{N(1-0.0005N)} = \left(\frac{1}{N} - \frac{1}{N-2000}\right) dN = dt$$



and

$$\ln N - \ln|N-2000| = t + c.$$

Solving for N we get $N(t) = 2000e^{c+t}/(1+e^{c+t}) = 2000e^ce^t/(1+e^ce^t)$. Using $N(0)=1$ and solving for e^c we find $e^c = 1/1999$ and so $N(t) = 2000e^t/(1999+e^t)$. Then $N(10) = 1833.59$, so 1834 companies are expected to adopt the new technology when $t = 10$.

2. From $dN/dt = N(a-bN)$ and $N(0) = 500$ we obtain

$$N = \frac{500a}{500b + (a-500b)e^{-at}}.$$

Since $\lim_{t\to\infty} N = a/b = 50,000$ and $N(1) = 1000$ we have $a = 0.7033$, $b = 0.00014$, and $N = 50{,}000/(1+99e^{-0.7033t})$.

3. From $dP/dt = P\left(10^{-1} - 10^{-7}P\right)$ and $P(0) = 5000$ we obtain $P = 500/(0.0005 + 0.0995e^{-0.1t})$ so that $P \to 1{,}000{,}000$ as $t\to\infty$. If $P(t) = 500{,}000$ then $t = 52.9$ months.

4. (a) We have $dP/dt = P(a-bP)$ with $P(0) = 3.929$ million. Using separation of variables we obtain

$$P(t) = \frac{3.929a}{3.929b + (a-3.929b)e^{-at}} = \frac{a/b}{1+(a/3.929b-1)e^{-at}}$$
$$= \frac{c}{1+(c/3.929-1)e^{-at}},$$

where $c = a/b$. At $t = 60(1850)$ the population is 23.192 million, so

$$23.192 = \frac{c}{1+(c/3.929-1)e^{-60a}}$$

or $c = 23.192 + 23.192(c/3.929-1)e^{-60a}$. At $t = 120(1910)$,

$$91.972 = \frac{c}{1+(c/3.929-1)e^{-120a}}$$

or $c = 91.972 + 91.972(c/3.929-1)(e^{-60a})^2$. Combining the two equations for c we get

$$\left(\frac{(c-23.192)/23.192}{c/3.929-1}\right)^2 \left(\frac{c}{3.929}-1\right) = \frac{c-91.972}{91.972}$$

or

$$91.972(3.929)(c-23.192)^2 = (23.192)^2(c-91.972)(c-3.929).$$

The solution of this quadratic equation is $c = 197.274$. This in turn gives $a = 0.0313$. Therefore,

$$P(t) = \frac{197.274}{1+49.21e^{-0.0313t}}.$$

(b)

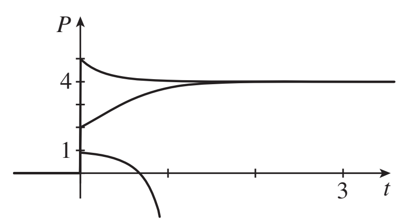
Year	Census Population	Predicted Population	Error	% Error
1790	3.929	3.929	0.000	0.00
1800	5.308	5.334	–0.026	–0.49
1810	7.240	7.222	0.018	0.24
1820	9.638	9.746	–0.108	–1.12
1830	12.866	13.090	–0.224	–1.74
1840	17.069	17.475	–0.406	–2.38
1850	23.192	23.143	0.049	0.21
1860	31.433	30.341	1.092	3.47
1870	38.558	39.272	–0.714	–1.85
1880	50.156	50.044	0.112	0.22
1890	62.948	62.600	0.348	0.55
1900	75.996	76.666	–0.670	–0.88
1910	91.972	91.739	0.233	0.25
1920	105.711	107.143	–1.432	–1.35
1930	122.775	122.140	0.635	0.52
1940	131.669	136.068	–4.399	–3.34
1950	150.697	148.445	2.252	1.49

5. **(a)** The differential equation is $dP/dt = P(5-P)-4$. Solving $P(5-P)-4=0$ for P we obtain equilibrium solutions $P=1$ and $P=4$. The phase portrait is shown on the right and solution curves are shown in part (b). We see that for $P_0 > 4$ and $1 < P_0 < 4$ the population approaches 4 as t increases. For $0 < P < 1$ the population decreases to 0 in finite time.

(b) The differential equation is

$$\frac{dP}{dt} = P(5-P)-4 = -(P^2-5P+4) = -(P-4)(P-1).$$

Separating variables and integrating, we obtain



$$\frac{dP}{(P-4)(P-1)} = -dt$$

$$\left(\frac{1/3}{P-4} - \frac{1/3}{P-1}\right) dP = -dt$$

$$\frac{1}{3}\ln\left|\frac{P-4}{P-1}\right| = -t + c$$

$$\frac{P-4}{P-1} = c_1 e^{-3t}.$$

Setting $t=0$ and $P=P_0$ we find $c_1 = (P_0-4)/(P_0-1)$. Solving for P we obtain

$$P(t) = \frac{4(P_0-1)-(P_0-4)e^{-3t}}{(P_0-1)-(P_0-4)e^{-3t}}\,.$$

(c) To find when the population becomes extinct in the case $0 < P_0 < 1$ we set $P = 0$ in

$$\frac{P-4}{P-1} = \frac{P_0-4}{P_0-1}\, e^{-3t}$$

from part (a) and solve for t. This gives the time of extinction

$$t = -\frac{1}{3}\ln\frac{4(P_0-1)}{P_0-4}.$$

6. Solving $P(5-P) - \frac{25}{4} = 0$ for P we obtain the equilibrium solution $P = \frac{5}{2}$. For $P \neq \frac{5}{2}$, $dP/dt < 0$. Thus, if $P_0 < \frac{5}{2}$, the population becomes extinct (otherwise there would be another equilibrium solution.) Using separation of variables to solve the initial-value problem, we get

$$P(t) = [4P_0 + (10P_0 - 25)t]/[4 + (4P_0 - 10)t].$$

To find when the population becomes extinct for $P_0 < \frac{5}{2}$ we solve $P(t) = 0$ for t. We see that the time of extinction is $t = 4P_0/5(5-2P_0)$.

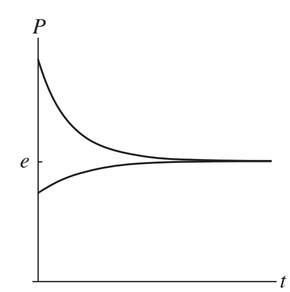
7. Solving $P(5-P)-7 = 0$ for P we obtain complex roots, so there are no equilibrium solutions. Since $dP/dt < 0$ for all values of P, the population becomes extinct for any initial condition. Using separation of variables to solve the initial-value problem, we get

$$P(t) = \frac{5}{2} + \frac{\sqrt{3}}{2}\tan\left[\tan^{-1}\left(\frac{2P_0-5}{\sqrt{3}}\right) - \frac{\sqrt{3}}{2}t\right].$$

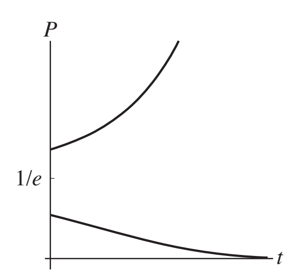
Solving $P(t) = 0$ for t we see that the time of extinction is

$$t = \frac{2}{3}\left(\sqrt{3}\tan^{-1}(5/\sqrt{3}) + \sqrt{3}\tan^{-1}\left[(2P_0-5)/\sqrt{3}\right]\right).$$

8. (a) The differential equation is $dP/dt = P(1-\ln P)$, which has the equilibrium solution $P = e$. When $P_0 > e$, $dP/dt < 0$, and when $P_0 < e$, $dP/dt > 0$.



(b) The differential equation is $dP/dt = P(1+\ln P)$, which has the equilibrium solution $P = 1/e$. When $P_0 > 1/e$, $dP/dt > 0$, and when $P_0 < 1/e$, $dP/dt < 0$.



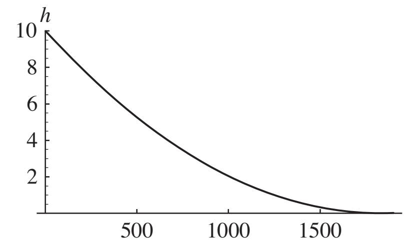
9. Let $X = X(t)$ be the amount of C at time t and $dX/dt = k(120-2X)(150-X)$. If $X(0) = 0$ and $X(5) = 10$, then

$$X(t) = \frac{150 - 150e^{180kt}}{1 - 2.5e^{180kt}},$$

where $k = .0001259$ and $X(20) = 29.3$ g. Now by L'Hôpital's rule, $X \to 60$ as $t \to \infty$, so that the amount of $A \to 0$ and the amount of $B \to 30$ as $t \to \infty$.

10. From $dX/dt = k(150 - X)^2$, $X(0) = 0$, and $X(5) = 10$ we obtain $X = 150 - 150/(150kt + 1)$ where $k = .000095238$. Then $X(20) = 33.3$ g and $X \to 150$ as $t \to \infty$ so that the amount of $A \to 0$ and the amount of $B \to 0$ as $t \to \infty$. If $X(t) = 75$ then $t = 70$ minutes.

11. **(a)** The initial-value problem is $dh/dt = -4.427 A_h\sqrt{h}/A_w$, $h(0) = H$. Separating variables and integrating we have



$$\frac{dh}{\sqrt{h}} = -\frac{4.427A_h}{A_w}\,dt \quad \text{and} \quad 2\sqrt{h} = -\frac{4.427A_h}{A_w}t + c.$$

Using $h(0) = H$ we find $c = 2\sqrt{H}$, so the solution of the initial-value problem is $\sqrt{h(t)} = (A_w\sqrt{H} - 2.214A_h t)/A_w$, where $A_w\sqrt{H} - 2.214A_h t \geq 0$. Thus,

$$h(t) = (A_w\sqrt{H} - 2.214A_h t)^2/A_w^2 \quad \text{for} \quad 0 \leq t \leq A_w\sqrt{H}/2.214A_h.$$

(b) Identifying $H = 3$, $A_w = 0.36\pi$, and $A_h = \pi/6944$ we have $h(t) = t^2/1.275^6 - 0.00307t + 3$. Solving $h(t) = 0$ we see that the tank empties in 1956 seconds or 32.6 minutes.

12. To obtain the solution of this differential equation we use $h(t)$ from Problem 13 in Exercises 1.3. Then $h(t) = (A_w\sqrt{H} - 2.214cA_h t)^2/A_w^2$. Solving $h(t) = 0$ with $c = 0.6$ and the values from Problem 11 we see that the tank empties in 3259.45 seconds or 54.3 minutes.

13. **(a)** Separating variables and integrating gives

$$h^{3/2}\,dh = -0.0266t \quad \text{and} \quad 0.4h^{5/2} = -0.0266t + c.$$

Using $h(0) = 20$ we find $c = 35.27$, so the solution of the initial-value problem is $h(t) = (88.175 - 0.166t)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in 531.17 seconds or 8.85 minutes.

(b) When the height of the water is h, the radius of the top of the water is $r = h\tan 30° = h/\sqrt{3}$ and $A_w = \pi h^2/3$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(5/100)^2}{\pi h^2/3}\sqrt{19.6h} = -\frac{0.02}{h^{3/2}}.$$

Separating variables and integrating gives

$$h^{3/2}\,dh = -0.02\,dt \quad \text{and} \quad 0.4h^{5/2} = -0.02t + c.$$

Using $h(0) = 3$ we find $c = 6.235$, so the solution of the initial-value problem is $h(t) = (15.588 - 0.05t)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in 311.76 seconds or 5.196 minutes.

14. When the height of the water is h, the radius of the top of the water is $\frac{1}{2}(6-h)$ and $A_w = 0.25\pi(6-h)^2$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(5/100)^2}{0.25\pi(6-h)^2}\sqrt{19.6h} = -0.0266\frac{\sqrt{h}}{(6-h)^2}\,.$$

Separating variables and integrating we have

$$\frac{(6-h)^2}{\sqrt{h}}\,dh = -0.0266\,dt \quad \text{and} \quad 72\sqrt{h} - 8h^{3/2} + \frac{2}{5}h^{5/2} = -0.0266t + c.$$

Using $h(0) = 6$ we find $c = 94.06$, so an implicit solution of the initial-value problem is

$$72\sqrt{h} - 8h^{3/2} + \frac{2}{5}h^{5/2} = -0.0266t + 94.06\,.$$

To find the time it takes the tank to empty we set $h = 0$ and solve for t. The tank empties in 3536 seconds or 58.93 minutes. Thus, the tank empties more slowly when the base of the cone is on the bottom.

15. (a) After separating variables we obtain

$$\begin{aligned}
\frac{m\,dv}{mg - kv^2} &= dt\\
\frac{1}{g}\,\frac{dv}{1 - (\sqrt{k}\,v/\sqrt{mg}\,)^2} &= dt\\
\frac{\sqrt{mg}}{\sqrt{k}\,g}\,\frac{\sqrt{k/mg}\,dv}{1 - (\sqrt{k}\,v/\sqrt{mg}\,)^2} &= dt\\
\sqrt{\frac{m}{kg}}\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg}} &= t + c\\
\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg}} &= \sqrt{\frac{kg}{m}}\,t + c_1.
\end{aligned}$$

Thus the velocity at time t is

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t + c_1\right).$$

Setting $t = 0$ and $v = v_0$ we find $c_1 = \tanh^{-1}(\sqrt{k}\,v_0/\sqrt{mg}\,)$.

(b) Since $\tanh t \to 1$ as $t \to \infty$, we have $v \to \sqrt{mg/k}$ as $t \to \infty$.

(c) Integrating the expression for $v(t)$ in part (a) we obtain an integral of the form $\int du/u$:

$$s(t) = \sqrt{\frac{mg}{k}} \int \tanh\left(\sqrt{\frac{kg}{m}}t + c_1\right) dt = \frac{m}{k} \ln\left[\cosh\left(\sqrt{\frac{kg}{m}}t + c_1\right)\right] + c_2.$$

Setting $t = 0$ and $s = 0$ we find $c_2 = -(m/k)\ln(\cosh c_1)$, where c_1 is given in part (a).

16. The differential equation is $m\,dv/dt = -mg - kv^2$. Separating variables and integrating, we have

$$\frac{dv}{mg + kv^2} = -\frac{dt}{m}$$

$$\frac{1}{\sqrt{mgk}} \tan^{-1}\left(\frac{\sqrt{k}\,v}{\sqrt{mg}}\right) = -\frac{1}{m}t + c$$

$$\tan^{-1}\left(\frac{\sqrt{k}\,v}{\sqrt{mg}}\right) = -\sqrt{\frac{gk}{m}}\,t + c_1$$

$$v(t) = \sqrt{\frac{mg}{k}} \tan\left(c_1 - \sqrt{\frac{gk}{m}}\,t\right)$$

Setting $v(0) = 90$, $m = \frac{75}{9.8} = 7.653$, $g = 9.8$, and $k = 0.0003$, we find $v(t) = 500\tan(c_1 - 0.0196t)$ and $c_1 = 3.318$. Integrating

$$v(t) = 500\tan(3.318 - 0.0196t)$$

we get

$$s(t) = 25510\ln|\cos(3.318 - 0.0196t)| + c_2.$$

Using $s(0) = 0$ we find $c_2 = 399$. Solving $v(t) = 0$ we see that the maximum height is attained when $t = 9$. The maximum height is $s(9) = 360.63\,\text{m}$.

17. **(a)** Let ρ be the weight density of the water and V the volume of the object. Archimedes' principle states that the upward buoyant force has magnitude equal to the weight of the water displaced. Taking the positive direction to be down, the differential equation is

$$m\frac{dv}{dt} = mg - kv^2 - \rho V.$$

(b) Using separation of variables we have

$$\frac{m\,dv}{(mg - \rho V) - kv^2} = dt$$

$$\frac{m}{\sqrt{k}} \frac{\sqrt{k}\,dv}{(\sqrt{mg - \rho V}\,)^2 - (\sqrt{k}\,v)^2} = dt$$

$$\frac{m}{\sqrt{k}} \frac{1}{\sqrt{mg - \rho V}} \tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg - \rho V}} = t + c.$$

Thus

$$v(t) = \sqrt{\frac{mg - \rho V}{k}}\tanh\left(\frac{\sqrt{kmg - k\rho V}}{m}t + c_1\right).$$

(c) Since $\tanh t \to 1$ as $t \to \infty$, the terminal velocity is $\sqrt{(mg - \rho V)/k}$.

18. (a) Writing the equation in the form $(x - \sqrt{x^2 + y^2}\,)dx + y\,dy = 0$ we identify $M = x - \sqrt{x^2 + y^2}$ and $N = y$. Since M and N are both homogeneous functions of degree 1 we use the substitution $y = ux$. It follows that

$$\begin{aligned}
\left(x - \sqrt{x^2 + u^2x^2}\right) dx + ux(u\,dx + x\,du) &= 0\\
x\left[1 - \sqrt{1 + u^2} + u^2\right] dx + x^2u\,du &= 0\\
-\frac{u\,du}{1 + u^2 - \sqrt{1 + u^2}} &= \frac{dx}{x}\\
\frac{u\,du}{\sqrt{1 + u^2}\,(1 - \sqrt{1 + u^2}\,)} &= \frac{dx}{x}.
\end{aligned}$$

Letting $w = 1 - \sqrt{1 + u^2}$ we have $dw = -u\,du/\sqrt{1 + u^2}$ so that

$$\begin{aligned}
-\ln\left|1 - \sqrt{1 + u^2}\right| &= \ln|x| + c\\
\frac{1}{1 - \sqrt{1 + u^2}} &= c_1x\\
1 - \sqrt{1 + u^2} &= -\frac{c_2}{x} \qquad\qquad (-c_2 = 1/c_1)\\
1 + \frac{c_2}{x} &= \sqrt{1 + \frac{y^2}{x^2}}\\
1 + \frac{2c_2}{x} + \frac{c_2^2}{x^2} &= 1 + \frac{y^2}{x^2}.
\end{aligned}$$

Solving for y^2 we have

$$y^2 = 2c_2x + c_2^2 = 4\left(\frac{c_2}{2}\right)\left(x + \frac{c_2}{2}\right)$$

which is a family of parabolas symmetric with respect to the x-axis with vertex at $(-c_2/2, 0)$ and focus at the origin.

(b) Let $u = x^2 + y^2$ so that

$$\frac{du}{dx} = 2x + 2y\frac{dy}{dx}.$$

Then

$$y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - x$$

and the differential equation can be written in the form

$$\frac{1}{2}\frac{du}{dx} - x = -x + \sqrt{u} \quad \text{or} \quad \frac{1}{2}\frac{du}{dx} = \sqrt{u}.$$

Separating variables and integrating gives

$$\begin{aligned}\frac{du}{2\sqrt{u}} &= dx\\ \sqrt{u} &= x + c\\ u &= x^2 + 2cx + c^2\\ x^2 + y^2 &= x^2 + 2cx + c^2\\ y^2 &= 2cx + c^2.\end{aligned}$$

19. (a) From $2W^2 - W^3 = W^2(2 - W) = 0$ we see that $W = 0$ and $W = 2$ are constant solutions.

(b) Separating variables and using a CAS to integrate we get

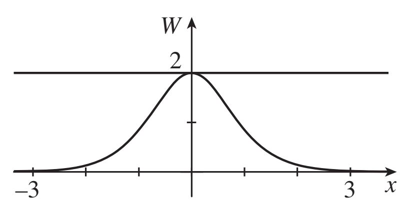
$$\frac{dW}{W\sqrt{4 - 2W}} = dx \quad \text{and} \quad -\tanh^{-1}\left(\frac{1}{2}\sqrt{4 - 2W}\right) = x + c.$$

Using the facts that the hyperbolic tangent is an odd function and $1 - \tanh^2 x = \operatorname{sech}^2 x$ we have

$$\begin{aligned}\frac{1}{2}\sqrt{4 - 2W} &= \tanh(-x - c) = -\tanh(x + c)\\ \frac{1}{4}(4 - 2W) &= \tanh^2(x + c)\\ 1 - \frac{1}{2}W &= \tanh^2(x + c)\\ \frac{1}{2}W &= 1 - \tanh^2(x + c) = \operatorname{sech}^2(x + c)\end{aligned}$$

Thus, $W(x) = 2\operatorname{sech}^2(x + c)$.

(c) Letting $x = 0$ and $W = 2$ we find that $\operatorname{sech}^2(c) = 1$ and $c = 0$.



20. (a) Solving $r^2 + (10 - h)^2 = 10^2$ for r^2 we see that $r^2 = 20h - h^2$. Combining the rate of input of water, π, with the rate of output due to evaporation, $k\pi r^2 = k\pi(20h - h^2)$, we have $dV/dt = \pi - k\pi(20h - h^2)$. Using $V = 10\pi h^2 - \frac{1}{3}\pi h^3$, we see also that $dV/dt = (20\pi h - \pi h^2)dh/dt$. Thus,

$$(20\pi h - \pi h^2)\frac{dh}{dt} = \pi - k\pi(20h - h^2) \quad \text{and} \quad \frac{dh}{dt} = \frac{1 - 20kh + kh^2}{20h - h^2}.$$

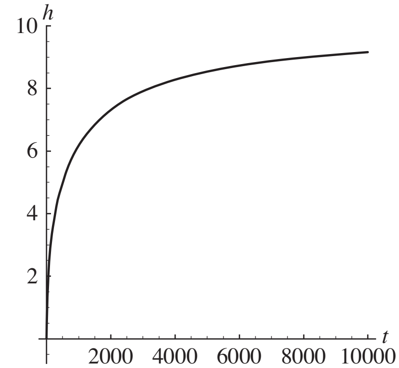
(b) Letting $k = 1/100$, separating variables and integrating (with the help of a CAS), we get

$$\frac{100h(h-20)}{(h-10)^2}\,dh = dt$$

and

$$\frac{100(h^2-10h+100)}{10-h} = t + c.$$

Using $h(0) = 0$ we find $c = 1000$, and solving for h we get $h(t) = 0.005\left(\sqrt{t^2+4000t} - t\right)$, where the positive square root is chosen because $h \geq 0$.



(c) The volume of the tank is $V = \frac{2}{3}\pi(10)^3$ m^3, so at a rate of π cubic meter per minute, the tank will fill in $\frac{2}{3}(10)^3 \approx 666.67$ minutes ≈ 11.11 hours.

(d) At 666.67 minutes, the depth of the water is $h(666.67) = 5.486$ m. From the graph in (b) we suspect that $\lim_{t\to\infty} h(t) = 10$, in which case the tank will never completely fill. To prove this we compute the limit of $h(t)$:

$$\lim_{t\to\infty} h(t) = 0.005 \lim_{t\to\infty}\left(\sqrt{t^2+4000t} - t\right) = 0.005 \lim_{t\to\infty} \frac{t^2+4000t-t^2}{\sqrt{t^2+4000t}+t}$$

$$= 0.005 \lim_{t\to\infty} \frac{4000t}{t\sqrt{1+4000/t}+t} = 0.005\,\frac{4000}{1+1} = 0.005(2000) = 10.$$

21. (a) With $c = 0.01$ the differential equation is $dP/dt = kP^{1.01}$. Separating variables and integrating we obtain

$$\begin{aligned}
P^{-1.01}\,dP &= k\,dt \\
\frac{P^{-0.01}}{-0.01} &= kt + c_1 \\
P^{-0.01} &= -0.01kt + c_2 \\
P(t) &= (-0.01kt + c_2)^{-100} \\
P(0) &= c_2^{-100} = 10 \\
c_2 &= 10^{-0.01}.
\end{aligned}$$

Then

$$P(t) = \frac{1}{(-0.01kt + 10^{-0.01})^{100}}$$

and, since P doubles in 5 months from 10 to 20,

$$P(5) = \frac{1}{(-0.01k(5) + 10^{-0.01})^{100}} = 20$$

so

$$\left(-0.01k(5) + 10^{-0.01}\right)^{100} = \frac{1}{20}$$

$$-0.01k = \frac{\left[\left(\frac{1}{20}\right)^{1/100} - \left(\frac{1}{10}\right)^{1/100}\right]}{5}$$

$$= -0.001350\,.$$

Thus $P(t) = 1/\left(-0.001350t + 10^{-0.01}\right)^{100}$.

(b) Define $T = \left(\frac{1}{10}\right)^{1/100}/0.001350 \approx 724$ months $= 60$ years. As $t \to 724$ (from the left), $P \to \infty$.

(c)

$$P(50) = 1/\left[-0.001350\,(50) + 10^{-0.01}\right]^{100} \approx 12,839 \qquad \text{and}$$

$$P(100) = 1/\left[-0.001350\,(100) + 10^{-0.01}\right]^{100} \approx 28,630,966$$

22. (a) From the phase portrait we see that $P = 0$ is an attractor for $0 < P_0 < K = a/b$ and $P = K$ is a repeller for $P_0 > K$.

(b) Letting $a = 0.1$, $b = 0.0005$ and using separation of variables gives

$$\left(-\frac{1}{P} + \frac{b}{bP - a}\right) dP = a\,dt$$

Integrating we have

$$-\ln P + \ln(bP - a) = at + c_1$$

$$\ln\left(\frac{bP - a}{P}\right) = at + c_1$$

$$\frac{bP - a}{P} = c_2 e^{at}$$

$$P = \frac{a}{b - c_2 e^{at}}.$$

Since $P(0) = 300$,

$$c_2 = \frac{300b - a}{300} \qquad \text{and} \qquad P(t) = \frac{300a}{300b - (300b - a)\,e^{at}}.$$

Then, with $b = 0.0005$ and $a = 0.1$,

$$P(t) = \frac{300(0.1)}{00(0.0005) - [300(0.0005) - 0.1]\, e^{0.1t}} = \frac{30}{0.15 - 0.05e^{0.1t}} = \frac{600}{3 - e^{0.1t}}$$

and

$$3 - e^{0.1} = 0 \qquad \text{implies} \qquad 0.1t \ln 3 \qquad \text{so} \qquad t = 10 \ln 3.$$

This is doomsday in finite time, since $P(t) \to \infty$ as $t \to 10 \ln 3$ (from the left) ≈ 10.99.

(c) For $P_0 = 100$

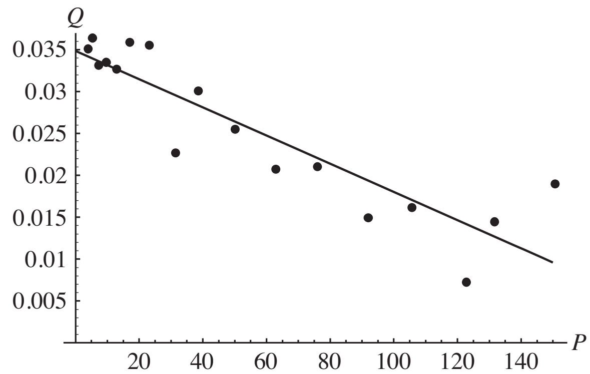
$$P(t) = \frac{100a}{100b - (100b - a)\, e^{at}} = \frac{100(0.1)}{100(0.0005) - [100(0.0005) - 0.1]\, e^{0.1t}}$$
$$= \frac{10}{0.05 + 0.05e^{0.1t}} = \frac{200}{1 + e^{0.1t}},$$

and $P(t) \to 0$ as $t \to \infty$.

23. (a)

t	P(t)	Q(t)
0	3.929	0.035
10	5.308	0.036
20	7.240	0.033
30	9.638	0.033
40	12.866	0.033
50	17.069	0.036
60	23.192	0.036
70	31.433	0.023
80	38.558	0.030
90	50.156	0.026
100	62.948	0.021
110	75.996	0.021
120	91.972	0.015
130	105.711	0.016
140	122.775	0.007
150	131.669	0.014
160	150.697	0.019
170	179.300	

(b) The regression line is $Q = 0.0348391 - 0.000168222P$.



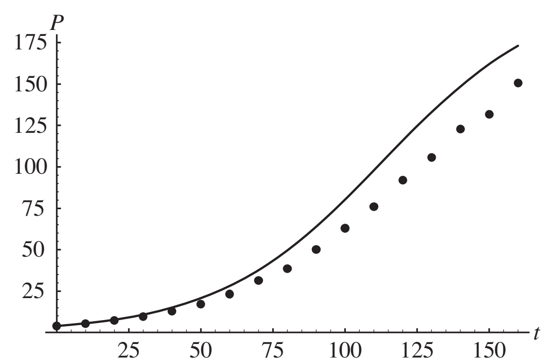
(c) The solution of the logistic equation is given in Equation (5) in the text. Identifying $a = 0.0348391$ and $b = 0.000168222$ we have

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

(d) With $P_0 = 3.929$ the solution becomes

$$P(t) = \frac{0.136883}{0.000660944 + 0.0341781e^{-0.0348391t}}.$$

(e)



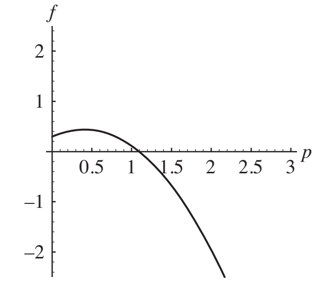
(f) We identify $t = 180$ with 1970, $t = 190$ with 1980, and $t = 200$ with 1990. The model predicts $P(180) = 188.661$, $P(190) = 193.735$, and $P(200) = 197.485$. The actual population figures for these years are 203.303, 226.542, and 248.765 millions. As $t \to \infty$, $P(t) \to a/b = 207.102$.

24. (a) Using a CAS to solve $P(1 - P) + 0.3e^{-P} = 0$ for P we see that $P = 1.09216$ is an equilibrium solution.

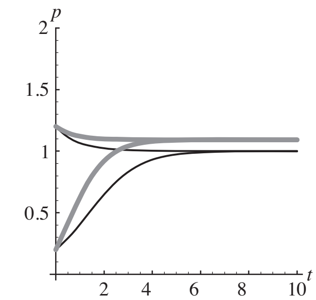
(b) Since $f(P) > 0$ for $0 < P < 1.09216$, the solution $P(t)$ of

$$dP/dt = P(1-P) + 0.3e^{-P}, \quad P(0) = P_0,$$

is increasing for $P_0 < 1.09216$. Since $f(P) < 0$ for $P > 1.09216$, the solution $P(t)$ is decreasing for $P_0 > 1.09216$. Thus $P = 1.09216$ is an attractor.



(c) The curves for the second initial-value problem are thicker. The equilibrium solution for the logic model is $P = 1$. Comparing 1.09216 and 1, we see that the percentage increase is 9.216



25. To find t_d we solve

$$m\,\frac{dv}{dt} = mg - kv^2, \qquad v(0) = 0$$

using separation of variables. This gives

$$v(t) = \sqrt{\frac{mg}{k}}\,\tanh\sqrt{\frac{kg}{m}}\,t.$$

Integrating and using $s(0) = 0$ gives

$$s(t) = \frac{m}{k}\ln\left(\cosh\sqrt{\frac{kg}{m}}\,t\right).$$

To find the time of descent we solve $s(t) = 360.63$ and find $t_d = 8.599$. The impact velocity is $v(t_d) = 83.481$, which is positive because the positive direction is downward.

26. **(a)** Solving $v_t = \sqrt{mg/k}$ for k we obtain $k = mg/v_t^2$. The differential equation then becomes

$$m\,\frac{dv}{dt} = mg - \frac{mg}{v_t^2}\,v^2 \quad \text{or} \quad \frac{dv}{dt} = g\left(1 - \frac{1}{v_t^2}\,v^2\right).$$

Separating variables and integrating gives

$$v_t \tanh^{-1}\frac{v}{v_t} = gt + c_1.$$

The initial condition $v(0) = 0$ implies $c_1 = 0$, so

$$v(t) = v_t \tanh\frac{gt}{v_t}\,.$$

We find the distance by integrating:

$$s(t) = \int v_t \tanh \frac{gt}{v_t}\,dt = \frac{v_t^2}{g}\ln\left(\cosh\frac{gt}{v_t}\right) + c_2.$$

The initial condition $s(0) = 0$ implies $c_2 = 0$, so

$$s(t) = \frac{v_t^2}{g}\ln\left(\cosh\frac{gt}{v_t}\right).$$

In 25 seconds she has fallen $6000 - 4500 = 1500$ m. Using a CAS to solve

$$1500 = (v_t^2/9.8)\ln\left(\cosh\frac{9.8(25)}{v_t}\right)$$

for v_t gives $v_t \approx 76.53$ m/s. Then

$$s(t) = \frac{v_t^2}{g}\ln\left(\cosh\frac{gt}{v_t}\right)$$

(b) At $t = 15$, $s(15) = 746.39$ m and $v(15) = s'(15) = 142.9$ m/sec.

27. While the object is in the air its velocity is modeled by the linear differential equation $m\,dv/dt = mg - kv$. Using $m = 71.43$, $k = 2.5$, and $g = 9.8$, the differential equation becomes $dv/dt + (1/28.572)v = 9.8$. The integrating factor is $e^{\int dt/28.572} = e^{t/28.572}$ and the solution of the differential equation is $e^{t/28.572}v = \int 9.8e^{t/28.572}\,dt = 280e^{t/28.572} + c$. Using $v(0) = 0$ we see that $c = -280$ and $v(t) = 280 - 280e^{-t/28.572}$. Integrating we get $s(t) = 280t + 8000e^{-t/28.572} + c$. Since $s(0) = 0$, $c = -8000$ and $s(t) = -8000 + 280t + 8000e^{-t/28.572}$. To find when the object hits the liquid we solve $s(t) = 150 - 25 = 125$, obtaining $t_a = 5.204$. The velocity at the time of impact with the liquid is $v_a = v(t_a) = 46.623$. When the object is in the liquid its velocity is modeled by the nonlinear differential equation $m\,dv/dt = mg - kv^2$. Using $m = 71.43$, $g = 9.8$, and $k = 0.15$ this becomes $dv/dt = (4666.76 - v^2)/476.2$. Separating variables and integrating we have

$$\frac{dv}{4666.76 - v^2} = \frac{dt}{476.2} \quad \text{and} \quad 0.007319\ln\left|\frac{v - 68.3137}{v + 68.3137}\right| = \frac{1}{476.2}t + c.$$

Solving $v(0) = v_a = 46.623$ we obtain $c = -0.0122$. Then, for $v < 68.3137$,

$$\left|\frac{v - 68.3137}{v + 67.3137}\right| = e^{\sqrt{2}t/5 - 1.8443} \quad \text{or} \quad -\frac{v - 68.3137}{v + 68.3137} = e^{\sqrt{2}t/5 - 1.8443}.$$

Solving for v we get

$$v(t) = \frac{1081 - 170.9e^{\sqrt{2}t/5}}{15.82 + 2.502e^{\sqrt{2}t/5}}.$$

Integrating we find

$$s(t) = 68.33t - 483.08\ln\left(6.323 + e^{\sqrt{2}t/5}\right) + c.$$

Solving $s(0) = 0$ we see that $c = 961.83$, so

$$s(t) = 691.83 + 68.33t - 483.08 \ln\left(6.323 + e^{\sqrt{2}t/5}\right).$$

To find when the object hits the bottom of the tank we solve $s(t) = 25$, obtaining $t_b = 0.5158$. The time from when the object is dropped from the helicopter to when it hits the bottom of the tank is $t_a + t_b = 5.72$ seconds.

28. The velocity vector of the swimmer is

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r = (-v_s \cos\theta, -v_s \sin\theta) + (0, v_r) = (-v_s \cos\theta, -v_s \sin\theta + v_r) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Equating components gives

$$\frac{dx}{dt} = -v_s \cos\theta \qquad \text{and} \qquad \frac{dy}{dt} = v_s \sin\theta + v_r$$

so

$$\frac{dx}{dt} = -v_s \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \frac{dy}{dt} = -v_s \frac{y}{\sqrt{x^2 + y^2}} + v_r.$$

Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-v_s y + v_r \sqrt{x^2 + y^2}}{-v_s x} = \frac{v_s y - v_r \sqrt{x^2 + y^2}}{v_s x}.$$

29. **(a)** With $k = v_r/v_s$,

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$

is a first-order homogeneous differential equation (see Section 2.5). Substituting $y = ux$ into the differential equation gives

$$u + x\frac{du}{dx} = u - k\sqrt{1 + u^2} \quad \text{or} \quad \frac{du}{dx} = -k\sqrt{1 + u^2}.$$

Separating variables and integrating we obtain

$$\int \frac{du}{\sqrt{1 + u^2}} = -\int k\, dx \quad \text{or} \quad \ln\left(u + \sqrt{1 + u^2}\right) = -k \ln x + \ln c.$$

This implies

$$\ln x^k \left(u + \sqrt{1 + u^2}\right) = \ln c \quad \text{or} \quad x^k \left(\frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x}\right) = c.$$

The condition $y(1) = 0$ gives $c = 1$ and so $y + \sqrt{x^2 + y^2} = x^{1-k}$. Solving for y gives

$$y(x) = \frac{1}{2}\left(x^{1-k} - x^{1+k}\right).$$

(b) If $k = 1$, then $v_s = v_r$ and $y = \frac{1}{2}(1 - x^2)$. Since $y(0) = \frac{1}{2}$, the swimmer lands on the west beach at $(0, \frac{1}{2})$. That is, $\frac{1}{2}$ km north of $(0,0)$. If $k > 1$, then $v_r > v_s$ and $1 - k < 0$. This means $\lim_{x\to 0^+} y(x)$ becomes infinite, since $\lim_{x\to 0^+} x^{1-k}$ becomes infinite. The swimmer never makes it to the west beach and is swept northward with the current. If $0 < k < 1$, then $v_s > v_r$ and $1 - k > 0$. The value of $y(x)$ at $x = 0$ is $y(0) = 0$. The swimmer has made it to the point $(0,0)$.

30. The velocity vector of the swimmer is

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r = (-v_s, 0) + (0, v_r) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Equating components gives

$$\frac{dx}{dt} = -v_s \quad \text{and} \quad \frac{dy}{dt} = v_r$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_r}{-v_s} = -\frac{v_r}{v_s}.$$

31. The differential equation

$$\frac{dy}{dx} = -\frac{30x(1-x)}{2}$$

separates into $dy = 15(-x + x^2)dx$. Integration gives $y(x) = -\frac{15}{2}x^2 + 5x^3 + c$. The condition $y(1) = 0$ gives $c = \frac{5}{2}$ and so $y(x) = \frac{1}{2}(-15x^2 + 10x^3 + 5)$. Since $y(0) = \frac{5}{2}$, the swimmer has to walk 2.5 km back down the west beach to reach $(0,0)$.

32. This problem has a great many components, so we will consider the case in which air resistance is assumed to be proportional to the velocity. By Problem 35 in Section 3.1 the differential equation is

$$m\frac{dv}{dt} = mg - kv,$$

and the solution is

$$v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

If we take the initial velocity to be 0, then the velocity at time t is

$$v(t) = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m}.$$

The mass of the raindrop is about $m = 1000 \times 0.0000000042 \approx 4.2^{-6}$ and $g = 9.8$, so the velocity at time t is

$$v(t) = \frac{4.116 \times 10^{-5}}{k} - \frac{4.116 \times 10^{-5}}{k}e^{-238095kt}$$

If we let $k = 0.000009$, then $v(100) \approx 4.57$ m/s. In this case 100 is the time in seconds. Since 11 km/h $\approx$ 3 m/s, the assertion that the average velocity is 11 km/h is not unreasonable. Of course, this assumes that the air resistance is proportional to the velocity, and, more importantly, that the constant of proportionality is 0.000009. The assumption about the constant is particularly suspect.

33. (a) Letting $c = 0.6$, $A_h = \pi(8^{-4})^2$, $A_w = \pi \cdot 0.3^2 = 0.09\pi$, and $g = 9.8$, the differential equation in Problem 12 becomes $dh/dt = -0.00001889\sqrt{h}$. Separating variables and integrating, we get $2\sqrt{h} = -0.00001889t + c$, so $h = (c_1 - 0.000009445t)^2$. Setting $h(0) = 0.6$, we find $c = 0.7746$, so $h(t) = (0.7746 - 0.000009445t)^2$, where h is measured in m and t in seconds.

(b) One hour is 3,600 seconds, so the hour mark should be placed at

$$h(3600) = [0.7746 - 0.000009445(3600)]^2 \approx 0.548\,\text{m} \approx 54.8\,\text{cm}.$$

up from the bottom of the tank. The remaining marks corresponding to the passage of 2, 3, 4, ..., 12 hours are placed at the values shown in the table. The marks are not evenly spaced because the water is not draining out at a uniform rate; that is, $h(t)$ is not a linear function of time.

Time (seconds)	Height (cm)
0	60
1	54.8
2	49.93
3	45.24
4	40.78
5	36.55
6	32.57
7	28.79
8	25.26
9	21.96
10	18.89
11	16.05
12	13.44

34. (a) In this case $A_w = \pi h^2/4$ and the differential equation is

$$\frac{dh}{dt} = -\frac{1}{147,055}h^{-3/2}.$$

Separating variables and integrating, we have

$$h^{3/2}\,dh = -\frac{1}{147,055}\,dt$$

$$\frac{2}{5}h^{5/2} = -\frac{1}{147,055}t + c_1.$$

Setting $h(0) = 0.6$ we find $c_1 = 0.1115$, so that

$$\frac{2}{5}h^{5/2} = -\frac{1}{147055}t + 0.1115,$$

$$h^{5/2} = 0.2788 - \frac{1}{58822}t,$$

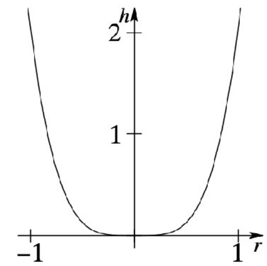
and

$$h = \left(0.2788 - \frac{1}{58822}t\right)^{2/5}.$$

(b) In this case $h(4\text{ hr}) = h(14{,}400\text{ s}) = 25.86$ cm and $h(5\text{ hr}) = h(18{,}000\text{ s})$ is not a real number. Using a CAS to solve $h(t) = 0$, we see that the tank runs dry at $t \approx 16{,}400$ s $\approx$ 4.55 hr. Thus, this particular conical water clock can only measure time intervals of less than 4.55 hours.

35. If we let r_h denote the radius of the hole and $A_w = \pi[f(h)]^2$, then the differential equation $dh/dt = -k\sqrt{h}$, where $k = cA_h\sqrt{2g}/A_w$, becomes

$$\frac{dh}{dt} = -\frac{c\pi r_h^2\sqrt{2g}}{\pi[f(h)]^2}\sqrt{h} = -\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}.$$



For the time marks to be equally spaced, the rate of change of the height must be a constant; that is, $dh/dt = -a$. (The constant is negative because the height is decreasing.) Thus

$$-a = -\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}, \qquad [f(h)]^2 = \frac{8cr_h^2\sqrt{h}}{a}, \qquad \text{and} \qquad r = f(h) = 2r_h\sqrt{\frac{2c}{a}}\,h^{1/4}.$$

Solving for h, we have

$$h = \frac{a^2}{64c^2r_h^4}\,r^4.$$

The shape of the tank with $c = 0.6$, $a = 2$ ft/12 hr $= 1$ ft/21,600 s, and $r_h = 1/32(12) = 1/384$ is shown in the above figure.

3.3 Modeling with Systems of First-Order DEs

1. The linear equation $dx/dt = -\lambda_1 x$ can be solved by either separation of variables or by an integrating factor. Integrating both sides of $dx/x = -\lambda_1 dt$ we obtain $\ln|x| = -\lambda_1 t + c$ from which we get $x = c_1e^{-\lambda_1 t}$. Using $x(0) = x_0$ we find $c_1 = x_0$ so that $x = x_0e^{-\lambda_1 t}$. Substituting this result into the second differential equation we have

$$\frac{dy}{dt} + \lambda_2 y = \lambda_1 x_0 e^{-\lambda_1 t}$$

which is linear. An integrating factor is $e^{\lambda_2 t}$ so that

$$\frac{d}{dt}\left[e^{-\lambda_2 t}y\right] + \lambda_2 y = \lambda_1 x_0 e^{(\lambda_2-\lambda_1)t}$$

$$y = \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)t}e^{-\lambda_2 t} + c_2e^{-\lambda_2 t} = \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + c_2e^{-\lambda_2 t}.$$

Using $y(0) = 0$ we find $c_2 = -\lambda_1 x_0/(\lambda_2-\lambda_1)$. Thus

$$y = \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

Substituting this result into the third differential equation we have

$$\frac{dz}{dt} = \frac{\lambda_1\lambda_2 x_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

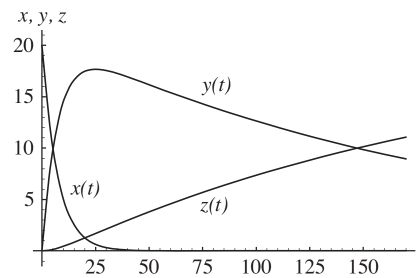
Integrating we find

$$z = -\frac{\lambda_2 x_0}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}e^{-\lambda_2 t} + c_3.$$

Using $z(0) = 0$ we find $c_3 = x_0$. Thus

$$z = x_0\left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}e^{-\lambda_2 t}\right).$$

2. We see from the graph that the half-life of A is approximately 4.7 days. To determine the half-life of B we use $t = 50$ as a base, since at this time the amount of substance A is so small that it contributes very little to substance B. Now we see from the graph that $y(50) \approx 16.2$ and $y(191) \approx 8.1$. Thus, the half-life of B is approximately 141 days.



3. The amounts x and y are the same at about $t = 5$ days. The amounts x and z are the same at about $t = 20$ days. The amounts y and z are the same at about $t = 147$ days. The time when y and z are the same makes sense because most of A and half of B are gone, so half of C should have been formed.

4. Suppose that the series is described schematically by $W \Longrightarrow -\lambda_1 X \Longrightarrow -\lambda_2 Y \Longrightarrow -\lambda_3 Z$ where $-\lambda_1$, $-\lambda_2$, and $-\lambda_3$ are the decay constants for W, X and Y, respectively, and Z is a stable element. Let $w(t)$, $x(t)$, $y(t)$, and $z(t)$ denote the amounts of substances W, X, Y, and Z, respectively. A model for the radioactive series is

$$\frac{dw}{dt} = -\lambda_1 w$$

$$\frac{dx}{dt} = \lambda_1 w - \lambda_2 x$$

$$\frac{dy}{dt} = \lambda_2 x - \lambda_3 y$$

$$\frac{dz}{dt} = \lambda_3 y.$$

5. (a) Since the third equation in the system is linear and containts only the variable $K(t)$ we have

$$\frac{dK}{dt} = -\left(\lambda_1 + \lambda_2\right)P$$

$$K(t) = c_1 e^{-(\lambda_1+\lambda_2)t}$$

Using $K(0) = K_0$ yields $K(t) = K_0 e^{-(\lambda_1+\lambda_2)t}$.

We can now solve for $A(t)$ and $C(t)$:

$$\frac{dA}{dt} = \lambda_2 P(t) = \lambda_2 K_0 e^{-(\lambda_1+\lambda_2)t}$$

$$A(t) = -\frac{\lambda_2}{\lambda_1 + \lambda_2}K_0 e^{-(\lambda_1+\lambda_2)t} + c_2$$

Using $A(0) = 0$ implies $c_2 = \dfrac{\lambda_2}{\lambda_1+\lambda_2}K_0$. Therefore,

$$A(t) = \frac{\lambda_2}{\lambda_1+\lambda_2}K_0\left[1 - e^{-(\lambda_1+\lambda_2)t}\right].$$

We use the same approach to solve for $C(t)$:

$$\frac{dC}{dt} = \lambda_1 K(t) = \lambda_1 K_0 e^{-(\lambda_1+\lambda_2)t}$$
$$C(t) = -\frac{\lambda_1}{\lambda_1+\lambda_2}K_0 e^{-(\lambda_1+\lambda_2)t} + c_3$$

Using $C(0) = 0$ implies $c_3 = \dfrac{\lambda_1}{\lambda_1+\lambda_2}K_0$. Therefore,

$$C(t) = \frac{\lambda_1}{\lambda_1+\lambda_2}K_0\left[1 - e^{-(\lambda_1+\lambda_2)t}\right].$$

(b) It is known that $\lambda_1 = 4.7526 \times 10^{-10}$ and $\lambda_2 = 0.5874 \times 10^{-10}$ so

$$\lambda_1 + \lambda_2 = 5.34 \times 10^{-10}$$
$$K(t) = K_0 e^{-0.000000000534t}$$
$$K(t) = \frac{1}{2}K_0$$
$$t = \frac{\ln\frac{1}{2}}{-0.000000000534} \approx 1.3 \times 10^9 \text{ years}$$

or the half-life of K-40 is about 1.3 billion years.

(c) Using the solutions $A(t)$, $C(t)$, and the values of λ_1 and λ_2 from part (b) we see that

$$\begin{aligned}\lim_{t\to\infty} A(t) &= \frac{\lambda_2}{\lambda_1+\lambda_2}\lim_{t\to\infty}\left[K_0\left(1 - e^{-(\lambda_1+\lambda_2)t}\right)\right] \\ &= \frac{\lambda_2}{\lambda_1+\lambda_2}K_0 = \frac{0.5874\times 10^{-10}}{5.34\times 10^{-10}}K_0 = 0.11K_0 \text{ or } 11\% \text{ of } K_0 \\ \lim_{t\to\infty} C(t) &= \frac{\lambda_1}{\lambda_1+\lambda_2}\lim_{t\to\infty}\left[K_0\left(1 - e^{-(\lambda_1+\lambda_2)t}\right)\right] \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}K_0 = \frac{4.7526\times 10^{-10}}{5.34\times 10^{-10}}K_0 = 0.89K_0 \text{ or } 89\% \text{ of } K_0\end{aligned}$$

6. (a) From part (a) of Problem 5:

$$\frac{A(t)}{K(t)} = \frac{\frac{\lambda_2}{\lambda_1+\lambda_2}K_0\left(1 - e^{-(\lambda_1+\lambda_2)t}\right)}{K_0 e^{-(\lambda_1+\lambda_2)t}} = \frac{\lambda_2}{\lambda_1+\lambda_2}\left(e^{(\lambda_1+\lambda_2)t} - 1\right)$$

(b) Solving for t in part (a) we get

$$\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)\frac{A(t)}{K(t)} = e^{(\lambda_1+\lambda_2)t} - 1$$

$$e^{(\lambda_1+\lambda_2)t} = 1 + \left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)\frac{A(t)}{K(t)}$$

$$t = \frac{1}{\lambda_1+\lambda_2}\ln\left[1 + \left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)\frac{A(t)}{K(t)}\right]$$

(c) From part (b)

$$t = \frac{1}{5.34\times 10^{-10}}\ln\left[1 + \left(\frac{5.34\times 10^{-10}}{0.5874\times 10^{-10}}\right)\frac{8.5\times 10^{-7}}{5.4\times 10^{-6}}\right] \approx 1.66 \text{ billion years}$$

7. The system is

$$x_1' = 2\cdot 3 + \frac{1}{50}x_2 - \frac{1}{50}x_1\cdot 4 = -\frac{2}{25}x_1 + \frac{1}{50}x_2 + 6$$

$$x_2' = \frac{1}{50}x_1\cdot 4 - \frac{1}{50}x_2 - \frac{1}{50}x_2\cdot 3 = \frac{2}{25}x_1 - \frac{2}{25}x_2\,.$$

8. Let x_1, x_2, and x_3 be the amounts of salt in tanks A, B, and C, respectively, so that

$$x_1' = \frac{1}{100}x_2\cdot 2 - \frac{1}{100}x_1\cdot 6 = \frac{1}{50}x_2 - \frac{3}{50}x_1$$

$$x_2' = \frac{1}{100}x_1\cdot 6 + \frac{1}{100}x_3 - \frac{1}{100}x_2\cdot 2 - \frac{1}{100}x_2\cdot 5 = \frac{3}{50}x_1 - \frac{7}{100}x_2 + \frac{1}{100}x_3$$

$$x_3' = \frac{1}{100}x_2\cdot 5 - \frac{1}{100}x_3 - \frac{1}{100}x_3\cdot 4 = \frac{1}{20}x_2 - \frac{1}{20}x_3.$$

9. (a) A model is

$$\frac{dx_1}{dt} = 3\cdot\frac{x_2}{100-t} - 2\cdot\frac{x_1}{100+t}\,, \qquad x_1(0) = 100$$

$$\frac{dx_2}{dt} = 2\cdot\frac{x_1}{100+t} - 3\cdot\frac{x_2}{100-t}\,, \qquad x_2(0) = 50.$$

(b) Since the system is closed, no salt enters or leaves the system and $x_1(t) + x_2(t) = 100 + 50 = 150$ for all time. Thus $x_1 = 150 - x_2$ and the second equation in part (a) becomes

$$\frac{dx_2}{dt} = \frac{2(150-x_2)}{100+t} - \frac{3x_2}{100-t} = \frac{300}{100+t} - \frac{2x_2}{100+t} - \frac{3x_2}{100-t}$$

or

$$\frac{dx_2}{dt} + \left(\frac{2}{100+t} + \frac{3}{100-t}\right)x_2 = \frac{300}{100+t}\,,$$

which is linear in x_2. An integrating factor is

$$e^{2\ln(100+t) - 3\ln(100-t)} = (100+t)^2(100-t)^{-3}$$

so

$$\frac{d}{dt}[(100+t)^2(100-t)^{-3}x_2] = 300(100+t)(100-t)^{-3}.$$

Using integration by parts, we obtain

$$(100+t)^2(100-t)^{-3}x_2 = 300\left[\frac{1}{2}(100+t)(100-t)^{-2} - \frac{1}{2}(100-t)^{-1} + c\right].$$

Thus

$$\begin{aligned} x_2 &= \frac{300}{(100+t)^2}\left[c(100-t)^3 - \frac{1}{2}(100-t)^2 + \frac{1}{2}(100+t)(100-t)\right] \\ &= \frac{300}{(100+t)^2}[c(100-t)^3 + t(100-t)]. \end{aligned}$$

Using $x_2(0) = 50$ we find $c = 5/3000$. At $t = 30$, $x_2 = (300/130^2)(70^3c + 30 \cdot 70) \approx 47.4$ kg.

10. A model is

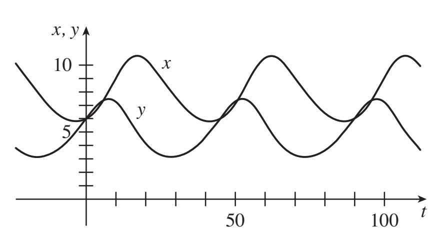
$$\begin{aligned} \frac{dx_1}{dt} &= (4 \text{ L/min})(0 \text{ kg/gal}) - (4 \text{ gal/min})\left(\frac{1}{200}x_1 \text{ kg/L}\right) \\ \frac{dx_2}{dt} &= (4 \text{ L/min})\left(\frac{1}{200}x_1 \text{ kg/L}\right) - (4 \text{ gal/min})\left(\frac{1}{150}x_2 \text{ kg/L}\right) \\ \frac{dx_3}{dt} &= (4 \text{ L/min})\left(\frac{1}{150}x_2 \text{ kg/L}\right) - (4 \text{ gal/min})\left(\frac{1}{100}x_3 \text{ kg/L}\right) \end{aligned}$$

or

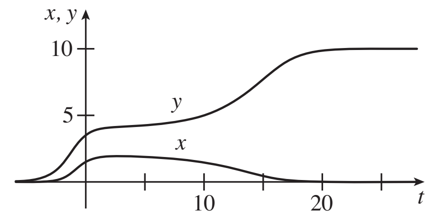
$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{1}{50}x_1 \\ \frac{dx_2}{dt} &= \frac{1}{50}x_1 - \frac{2}{75}x_2 \\ \frac{dx_3}{dt} &= \frac{2}{75}x_2 - \frac{1}{25}x_3. \end{aligned}$$

Over a long period of time we would expect x_1, x_2, and x_3 to approach 0 because the entering pure water should flush the salt out of all three tanks.

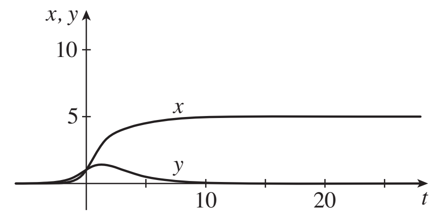
11. Zooming in on the graph it can be seen that the populations are first equal at about $t = 5.6$. The approximate periods of x and y are both 45.



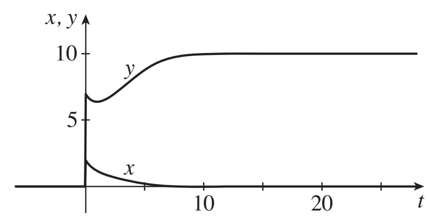
12. **(a)** The population $y(t)$ approaches 10,000, while the population $x(t)$ approaches extinction.



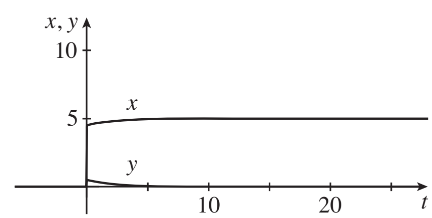
(b) The population $x(t)$ approaches 5,000, while the population $y(t)$ approaches extinction.



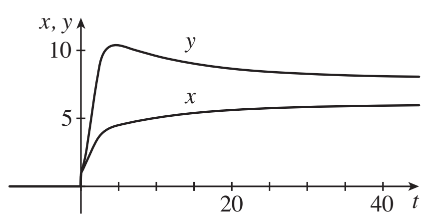
(c) The population $y(t)$ approaches 10,000, while the population $x(t)$ approaches extinction.



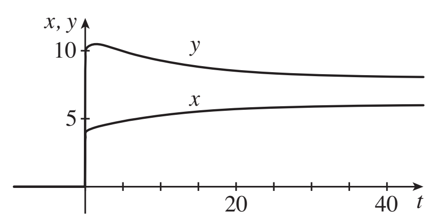
(d) The population $x(t)$ approaches 5,000, while the population $y(t)$ approaches extinction.



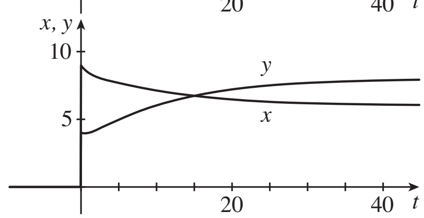
13. **(a)**



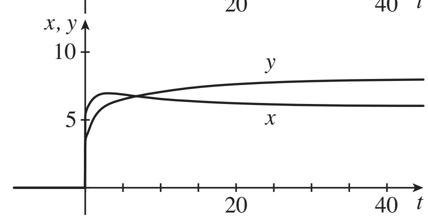
(b)



(c)



(d)



In each case the population $x(t)$ approaches 6,000, while the population $y(t)$ approaches 8,000.

14. By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Li_1' + R_1 i_2$ and $E(t) = Li_1' + R_2 i_3 + q/C$ so that $q = CR_1 i_2 - CR_2 i_3$. Then

$i_3 = q' = CR_1 i_2' - CR_2 i_3$ so that the system is

$$Li_2' + Li_3' + R_1 i_2 = E(t)$$
$$-R_1 i_2' + R_2 i_3' + \frac{1}{C} i_3 = 0.$$

15. By Kirchhoff's first law we have $i_1 = i_2 + i_3$. Applying Kirchhoff's second law to each loop we obtain

$$E(t) = i_1 R_1 + L_1 \frac{di_2}{dt} + i_2 R_2$$

and

$$E(t) = i_1 R_1 + L_2 \frac{di_3}{dt} + i_3 R_3.$$

Combining the three equations, we obtain the system

$$L_1 \frac{di_2}{dt} + (R_1 + R_2) i_2 + R_1 i_3 = E$$
$$L_2 \frac{di_3}{dt} + R_1 i_2 + (R_1 + R_3) i_3 = E.$$

16. By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Li_1' + Ri_2$ and $E(t) = Li_1' + q/C$ so that $q = CRi_2$. Then $i_3 = q' = CRi_2'$ so that system is

$$Li' + Ri_2 = E(t)$$
$$CRi_2' + i_2 - i_1 = 0.$$

17. We first note that $s(t) + i(t) + r(t) = n$. Now the rate of change of the number of susceptible persons, $s(t)$, is proportional to the number of contacts between the number of people infected and the number who are susceptible; that is, $ds/dt = -k_1 si$. We use $-k_1 < 0$ because $s(t)$ is decreasing. Next, the rate of change of the number of persons who have recovered is proportional to the number infected; that is, $dr/dt = k_2 i$ where $k_2 > 0$ since r is increasing. Finally, to obtain di/dt we use

$$\frac{d}{dt}(s + i + r) = \frac{d}{dt} n = 0.$$

This gives

$$\frac{di}{dt} = -\frac{dr}{dt} - \frac{ds}{dt} = -k_2 i + k_1 si.$$

The system of differential equations is then

$$\frac{ds}{dt} = -k_1 si$$
$$\frac{di}{dt} = -k_2 i + k_1 si$$
$$\frac{dr}{dt} = k_2 i.$$

A reasonable set of initial conditions is $i(0) = i_0$, the number of infected people at time 0, $s(0) = n - i_0$, and $r(0) = 0$.

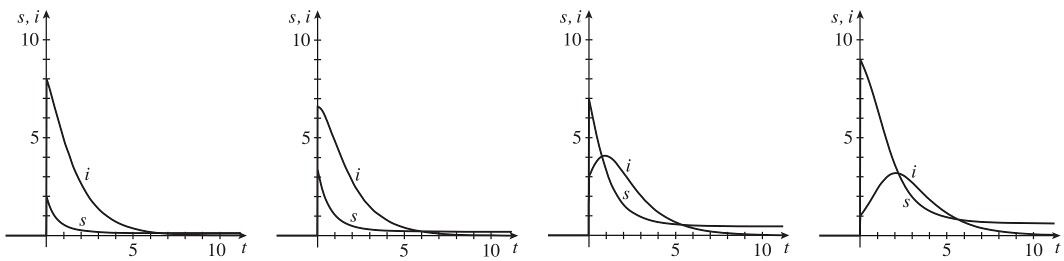
18. **(a)** If we know $s(t)$ and $i(t)$ then we can determine $r(t)$ from $s + i + r = n$.

(b) In this case the system is

$$\frac{ds}{dt} = -0.2si$$

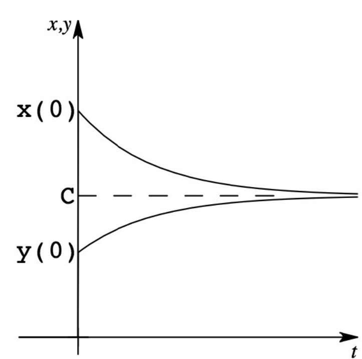
$$\frac{di}{dt} = -0.7i + 0.2si.$$

We also note that when $i(0) = i_0$, $s(0) = 10 - i_0$ since $r(0) = 0$ and $i(t) + s(t) + r(t) = 0$ for all values of t. Now $k_2/k_1 = 0.7/0.2 = 3.5$, so we cconsider initial conditions $s(0) = 2$, $i(0) = 8$; $s(0) = 3.4$, $i(0) = 6.6$; $s(0) = 7$, $i(0) = 3$; and $s(0) = 9$, $i(0) = 1$.



We see that an initial susceptible population greater than k_2/k_1 results in an epidemic in the sense that the number of infected persons increases to a maximum before decreasing to 0. On the other hand, when $s(0) < k_2/k_1$, the number of infected persons decreases from the start and there is no epidemic.

19. Since $x_0 > y_0 > 0$ we have $x(t) > y(t)$ and $y - x < 0$. Thus $dx/dt < 0$ and $dy/dt > 0$. We conclude that $x(t)$ is decreasing and $y(t)$ is increasing. As $t \to \infty$ we expect that $x(t) \to C$ and $y(t) \to C$, where C is a constant common equilibrium concentration.



20. We write the system in the form

$$\frac{dx}{dt} = k_1(y - x)$$

$$\frac{dy}{dt} = k_2(x - y),$$

where $k_1 = \kappa/V_A$ and $k_2 = \kappa/V_B$. Letting $z(t) = x(t) - y(t)$ we have

$$\frac{dx}{dt} - \frac{dy}{dt} = k_1(y - x) - k_2(x - y)$$

$$\frac{dz}{dt} = k_1(-z) - k_2 z$$

$$\frac{dz}{dt} + (k_1 + k_2)z = 0.$$

This is a linear first-order differential equation with solution $z(t) = c_1 e^{-(k_1+k_2)t}$. Now

$$\frac{dx}{dt} = -k_1(y - x) = -k_1 z = -k_1 c_1 e^{-(k_1+k_2)t}$$

and

$$x(t) = c_1 \frac{k_1}{k_1 + k_2} e^{-(k_1+k_2)t} + c_2.$$

Since $y(t) = x(t) - z(t)$ we have

$$y(t) = -c_1 \frac{k_2}{k_1 + k_2} e^{-(k_1+k_2)t} + c_2.$$

The initial conditions $x(0) = x_0$ and $y(0) = y_0$ imply

$$c_1 = x_0 - y_0 \qquad \text{and} \qquad c_2 = \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2}.$$

The solution of the system is

$$x(t) = \frac{(x_0 - y_0)k_1}{k_1 + k_2} e^{-(k_1+k_2)t} + \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2}$$

$$y(t) = \frac{(y_0 - x_0)k_2}{k_1 + k_2} e^{-(k_1+k_2)t} + \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2}.$$

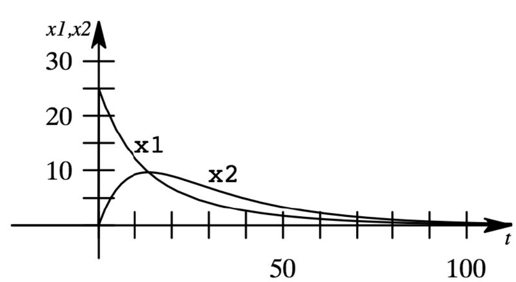
As $t \to \infty$, $x(t)$ and $y(t)$ approach the common limit

$$\frac{x_0 k_2 + y_0 k_1}{k_1 + k_2} = \frac{x_0 \kappa/V_B + y_0 \kappa/V_A}{\kappa/V_A + \kappa/V_B} = \frac{x_0 V_A + y_0 V_B}{V_A + V_B}$$

$$= x_0 \frac{V_A}{V_A + V_B} + y_0 \frac{V_B}{V_A + V_B}.$$

This makes intuitive sense because the limiting concentration is seen to be a weighted average of the two initial concentrations.

21. Since there are initially 25 pounds of salt in tank A and none in tank B, and since furthermore only pure water is being pumped into tank A, we would expect that $x_1(t)$ would steadily decrease over time. On the other hand, since salt is being added to tank B from tank A, we would expect $x_2(t)$ to increase over time. However, since pure water is being added to the system at a constant



rate and a mixed solution is being pumped out of the system, it makes sense that the amount of salt in both tanks would approach 0 over time.

22. We assume here that the temperature, $T(t)$, of the metal bar does not affect the temperature, $T_A(t)$, of the medium in container A. By Newton's law of cooling, then, the differential equations for $T_A(t)$ and $T(t)$ are

$$\frac{dT_A}{dt} = k_A(T_A - T_B), \quad k_A < 0$$

$$\frac{dT}{dt} = k(T - T_A), \quad k < 0,$$

subject to the initial conditions $T(0) = T_0$ and $T_A(0) = T_1$. Separating variables in the first equation, we find $T_A(t) = T_B + c_1e^{k_At}$. Using $T_A(0) = T_1$ we find $c_1 = T_1 - T_B$, so

$$T_A(t) = T_B + (T_1 - T_B)e^{k_At}.$$

Substituting into the second differential equation, we have

$$\frac{dT}{dt} = k(T - T_A) = kT - kT_A = kT - k[T_B + (T_1 - T_B)e^{k_At}]$$

$$\frac{dT}{dt} - kT = -kT_B - k(T_1 - T_B)e^{k_At}.$$

This is a linear differential equation with integrating factor $e^{\int -k\,dt} = e^{-kt}$. Then

$$\frac{d}{dt}[e^{-kt}T] = -kT_Be^{-kt} - k(T_1 - T_B)e^{(k_A-k)t}$$

$$e^{-kt}T = T_Be^{-kt} - \frac{k}{k_A - k}(T_1 - T_B)e^{(k_A-k)t} + c_2$$

$$T = T_B - \frac{k}{k_A - k}(T_1 - T_B)e^{k_At} + c_2e^{kt}.$$

Using $T(0) = T_0$ we find $c_2 = T_0 - T_B + \dfrac{k}{k_A - k}(T_1 - T_B)$, so

$$T(t) = T_B - \frac{k}{k_A - k}(T_1 - T_B)e^{k_At} + \left[T_0 - T_B + \frac{k}{k_A - k}(T_1 - T_B)\right]e^{kt}.$$

Chapter 3 in Review

1. The differential equation is $dP/dt = 0.15P$.

2. True. From $dA/dt = kA$, $A(0) = A_0$, we have $A(t) = A_0e^{kt}$ and $A'(t) = kA_0e^{kt}$, so $A'(0) = kA_0$. At $T = -(\ln 2)k$,

$$A'(-(\ln 2)/k) = kA(-(\ln 2)/k) = kA_0e^{k[-(\ln 2)/k]} = kA_0e^{-\ln 2} = \frac{1}{2}kA_0.$$

3. From $\dfrac{dP}{dt} = 0.018P$ and $P(0) = 4$ billion we obtain $P = 4e^{0.018t}$ so that $P(45) = 8.99$ billion.

4. Let $A = A(t)$ be the volume of CO_2 at time t. From $dA/dt = 0.03 - A/4$ and $A(0) = 0.04\,\mathrm{m}^3$ we obtain $A = 0.12 + 0.08e^{-t/4}$. Since $A(10) = 0.127\,\mathrm{m}^3$, the concentration is 0.0254%. As $t \to \infty$ we have $A \to 0.12\,\mathrm{m}^3$ or 0.06%.

5. The starting point is $A(t) = A_0e^{-0.00012097t}$. With $A(t) = 0.53A_0$ we have

$$-0.00012097t = \ln 0.53 \quad \text{or} \quad t = \frac{\ln 0.53}{-0.00012097} \approx 5248 \text{ years.}$$

This represents the iceman's age in 1991, so the approximate date of his death would be

$$1991 - 5248 = -3257 \qquad \text{or} \qquad 3257 \text{ BC.}$$

6. **(a)** We assume that the rate of disintegration of Iodine-131 is proportional to the amount remaining. If $A(t)$ is the amount of Iodine-131 remaining at time t then

$$\frac{dA}{dt} = kA, \quad A(0) = A_0,$$

where k is the constant of proportionality and A_0 is the initial amount. The solution of the initial-value problem is $A(t) = A_0e^{kt}$. After one day the amount of Iodine-131 is

$$(1 - 0.083)A_0 = 0.917A_0 \qquad \text{so} \qquad A(1) = A_0e^k = 0.917A_0$$

and $e^k = 0.917$. After eight days

$$A(8) = A_0e^{8k} = A_0(e^k)^8 = A_0(0.917)^8 \approx 0.49998A_0.$$

(b) Since $A(8) \approx \frac{1}{2}A_0$, the half-life of Iodine-131 is approximately 8 days.

7. Separating variables, we have

$$\frac{\sqrt{a^2 - y^2}}{y}\,dy = -dx.$$

Substituting $y = a\sin\theta$, this becomes

$$\begin{aligned}
\frac{\sqrt{a^2 - a^2\sin^2\theta}}{a\sin\theta}(a\cos\theta)\,d\theta &= -dx \\
a\int \frac{\cos^2\theta}{\sin\theta}\,d\theta &= -\int dx \\
a\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta &= -x + c \\
a\int (\csc\theta - \sin\theta)\,d\theta &= -x + c \\
a\ln|\csc\theta - \cot\theta| + a\cos\theta &= -x + c \\
a\ln\left|\frac{a}{y} - \frac{\sqrt{a^2 - y^2}}{y}\right| + a\,\frac{\sqrt{a^2 - y^2}}{a} &= -x + c.
\end{aligned}$$

Letting $a = 10$, this is

$$10 \ln \left| \frac{10}{y} - \frac{\sqrt{100 - y^2}}{y} \right| + \sqrt{100 - y^2} = -x + c.$$

Letting $x = 0$ and $y = 10$ we determine that $c = 0$, so the solution is

$$10 \ln \left| \frac{10}{y} - \frac{\sqrt{100 - y^2}}{y} \right| + \sqrt{100 - y^2} = -x.$$

8. From $V\, dC/dt = kA(C_s - C)$ and $C(0) = C_0$ we obtain $C = C_s + (C_0 - C_s)e^{-kAt/V}$.

9. (a) The differential equation

$$\begin{aligned} \frac{dT}{dt} &= k(T - T_m) = k[T - T_2 - B(T_1 - T)] \\ &= k[(1+B)T - (BT_1 + T_2)] = k(1+B)\left(T - \frac{BT_1 + T_2}{1+B}\right) \end{aligned}$$

is autonomous and has the single critical point $(BT_1 + T_2)/(1 + B)$. Since $k < 0$ and $B > 0$, by phase-line analysis it is found that the critical point is an attractor and

$$\lim_{t\to\infty} T(t) = \frac{BT_1 + T_2}{1+B}.$$

Moreover,

$$\lim_{t\to\infty} T_m(t) = \lim_{t\to\infty} [T_2 + B(T_1 - T)] = T_2 + B\left(T_1 - \frac{BT_1 + T_2}{1+B}\right) = \frac{BT_1 + T_2}{1+B}.$$

(b) The differential equation is

$$\frac{dT}{dt} = k(T - T_m) = k(T - T_2 - BT_1 + BT)$$

or

$$\frac{dT}{dt} - k(1+B)T = -k(BT_1 + T_2).$$

This is linear and has integrating factor $e^{-\int k(1+B)dt} = e^{-k(1+B)t}$. Thus,

$$\begin{aligned} \frac{d}{dt}[e^{-k(1+B)t}T] &= -k(BT_1 + T_2)e^{-k(1+B)t} \\ e^{-k(1+B)t}T &= \frac{BT_1 + T_2}{1+B}\, e^{-k(1+B)t} + c \\ T(t) &= \frac{BT_1 + T_2}{1+B} + ce^{k(1+B)t}. \end{aligned}$$

Since $T(0) = T_1$ we find, $T(t) = \dfrac{BT_1 + T_2}{1+B} + \dfrac{T_1 - T_2}{1+B}e^{k(1+B)t}$.

(c) The temperature $T(t)$ decreases to $(BT_1+T_2)/(1+B)$, whereas $T_m(t)$ increases to $(BT_1+T_2)/(1+B)$ as $t \to \infty$. Thus, the temperature $(BT_1+T_2)/(1+B)$, (which is a weighted average,

$$\frac{B}{1+B}T_1 + \frac{1}{1+B}T_2,$$

of the two initial temperatures), can be interpreted as an equilibrium temperature. The body cannot get cooler than this value whereas the medium cannot get hotter than this value.

10. (a) By separation of variables and partial fractions,

$$\ln\left|\frac{T-T_m}{T+T_m}\right| - 2\tan^{-1}\left(\frac{T}{T_m}\right) = 4T_m^3kt + c.$$

Then rewrite the right-hand side of the differential equation as

$$\begin{aligned}\frac{dT}{dt} &= k(T^4 - T_m^4) = [(T_m + (T-T_m))^4 - T_m^4] \\ &= kT_m^4\left[\left(1+\frac{T-T_m}{T_m}\right)^4 - 1\right] \\ &= kT_m^4\left[\left(1 + 4\frac{T-T_m}{T_m} + 6\left(\frac{T-T_m}{T_m}\right)^2 + \cdots\right) - 1\right] \leftarrow \text{binomial expansion}\end{aligned}$$

(b) When $T - T_m$ is small compared to T_m, every term in the expansion after the first two can be ignored, giving

$$\frac{dT}{dt} \approx k_1(T-T_m), \quad \text{where} \quad k_1 = 4kT_m^3.$$

11. Separating variables, we obtain

$$\begin{aligned}\frac{dq}{E_0 - q/C} &= \frac{dt}{k_1+k_2t} \\ -C\ln\left|E_0 - \frac{q}{C}\right| &= \frac{1}{k_2}\ln|k_1+k_2t| + c_1 \\ \frac{(E_0-q/C)^{-C}}{(k_1+k_2t)^{1/k_2}} &= c_2.\end{aligned}$$

Setting $q(0) = q_0$ we find $c_2 = (E_0 - q_0/C)^{-C}/k_1^{1/k_2}$, so

$$\frac{(E_0 - q/C)^{-C}}{(k_1 + k_2 t)^{1/k_2}} = \frac{(E_0 - q_0/C)^{-C}}{k_1^{1/k_2}}$$

$$\left(E_0 - \frac{q}{C}\right)^{-C} = \left(E_0 - \frac{q_0}{C}\right)^{-C} \left(\frac{k_1}{k_1 + k_2 t}\right)^{-1/k_2}$$

$$E_0 - \frac{q}{C} = \left(E_0 - \frac{q_0}{C}\right) \left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

$$q = E_0 C + (q_0 - E_0 C) \left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}.$$

12. From $y\left[1 + (y')^2\right] = k$ we obtain $dx = (\sqrt{y}/\sqrt{k - y})dy$. If $y = k\sin^2\theta$ then

$$dy = 2k\sin\theta\cos\theta\, d\theta, \quad dx = 2k\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta, \quad \text{and} \quad x = k\theta - \frac{k}{2}\sin 2\theta + c.$$

If $x = 0$ when $\theta = 0$ then $c = 0$.

13. From $dx/dt = k_1 x(\alpha - x)$ we obtain

$$\left(\frac{1/\alpha}{x} + \frac{1/\alpha}{\alpha - x}\right) dx = k_1\, dt$$

so that $x = \alpha c_1 e^{\alpha k_1 t}/(1 + c_1 e^{\alpha k_1 t})$. From $dy/dt = k_2 xy$ we obtain

$$\ln|y| = \frac{k_2}{k_1}\ln\left|1 + c_1 e^{\alpha k_1 t}\right| + c \quad \text{or} \quad y = c_2\left(1 + c_1 e^{\alpha k_1 t}\right)^{k_2/k_1}.$$

14. In tank A the salt input is

$$\left(7\,\frac{\text{L}}{\text{min}}\right)\left(2\,\frac{\text{kg}}{\text{L}}\right) + \left(1\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{kg}}{\text{L}}\right) = \left(14 + \frac{1}{100}x_2\right)\frac{\text{kg}}{\text{min}}.$$

The salt output is

$$\left(3\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{kg}}{\text{L}}\right) + \left(5\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{kg}}{\text{L}}\right) = \frac{2}{25}x_1\,\frac{\text{kg}}{\text{min}}.$$

In tank B the salt input is

$$\left(5\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{kg}}{\text{L}}\right) = \frac{1}{20}x_1\,\frac{\text{kg}}{\text{min}}.$$

The salt output is

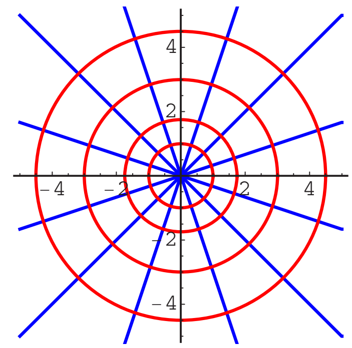
$$\left(1\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{kg}}{\text{L}}\right) + \left(4\,\frac{\text{L}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{kg}}{\text{L}}\right) = \frac{1}{20}x_2\,\frac{\text{kg}}{\text{min}}.$$

The system of differential equations is then

$$\frac{dx_1}{dt} = 14 + \frac{1}{100}x_2 - \frac{2}{25}x_1$$

$$\frac{dx_2}{dt} = \frac{1}{20}x_1 - \frac{1}{20}x_2.$$

15.



$$y = c_1 x$$

$$\frac{dy}{dx} = c_1$$

$$\frac{dy}{dx} = \frac{y}{x}$$

Therefore the differential equation of the orthogonal family is

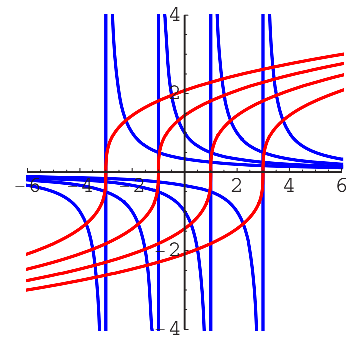
$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy + x\,dx = 0$$

$$x^2 + y^2 = c_2$$

which is a family of circles ($c_2 > 0$) centered at the origin.

16.



$$x^2 - 2y^2 = c_1$$

$$2x - 4y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{2y}$$

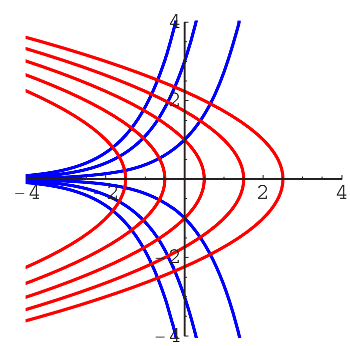
Therefore the differential equation of the orthogonal family is

$$\frac{dy}{dx} = -\frac{2y}{x}$$

$$\frac{1}{y}\,dy = -\frac{2}{x}\,dx$$

$$y = \frac{c_2}{x^2}$$

17. From $y = c_1 e^x$ we obtain $y' = y$ so that the differential equation of the orthogonal family is



$$\frac{dy}{dx} = -\frac{1}{y}.$$

Separating variables and integrating we get

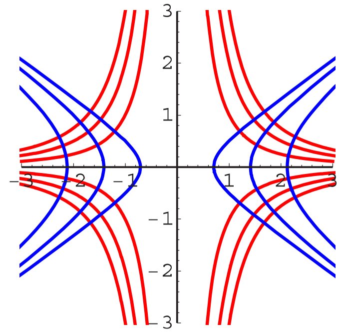
$$y\,dy = -\,dx$$

$$\frac{1}{2}y^2 = -x + c$$

$$y^2 + 2x = c_2$$

18. Differentiating the family of curves, we have

$$y' = -\frac{1}{(x+c_1)^2} = -y^2\,.$$



The differential equation for the family of orthogonal trajectories is then $y' = \dfrac{1}{y^2}$. Separating variables and integrating we get

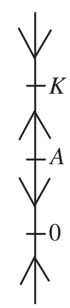
$$y^2\,dy = dx$$

$$\frac{1}{3}y^3 = x + c_2$$

$$y^3 = 3x + c_3\,.$$

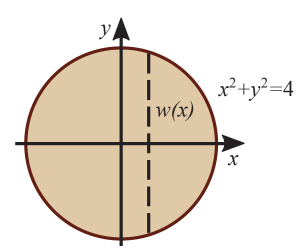
19. Critical points of the equation

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)\left(\frac{P}{A}-1\right) \qquad r>0,$$



are 0, A, and K. Here A is called the **Allee threshold** and satisfies $0 < A < K$. From the accompanying phase portrait we see that K and 0 are attractors, or asymptotically stable, but A is a repeller, or unstable. Thus, for an initial value $P_0 < A$ the population decreases over time, that is, $P \to 0$ as $t \to \infty$.

20. (a) From the cross section on the right we see in this case that $w(x) = 2\sqrt{4-x^2}$ and the initial-value problem is then



$$2\sqrt{4-x^2}\,\frac{dx}{dt} = 1, \qquad x(0) = -2.$$

Solving the differential equation by separation of variables gives $x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{1}{2}x\right) = t + c$. Using $x(0) = -2$ we have

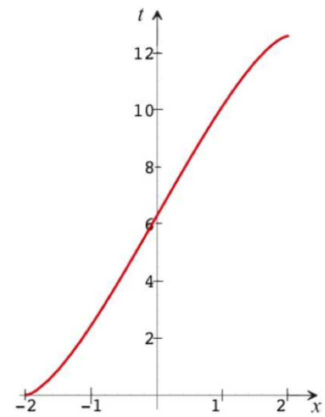
$$c = 4\sin^{-1}(-1) = 4\left(-\frac{\pi}{2}\right) = -2\pi\,.$$

Therefore an implicit solution is $x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{1}{2}x\right) = t - 2\pi$.

(b) The graph of $t(x) = 2\pi + x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{1}{2}x\right)$ on the x-interval $[-2, 2]$ is given on the right. From the graph we see that the time corresponding to $x = 2$ is approximately $t = 12.5$. The exact time to cut throug the piece of wood is

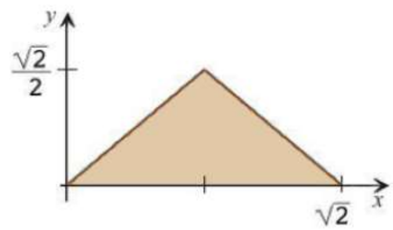
$$t(2) = 2\pi + 2\sqrt{4-2^2} + 4\sin^{-1}(1) = 2\pi + 4\left(\frac{\pi}{2}\right)$$

or $t = 4\pi$. In other words, the solution $x(t)$ defined implicitely by the equation in part (a) is defined on the t-interval $[0, 4\pi]$.



21. The piecewise-defined function $w(x)$ is now

$$w(x) = \begin{cases} x, & 0 \le x \le \dfrac{\sqrt{2}}{2} \\ \sqrt{2} - x, & \dfrac{\sqrt{}}{2} < x \le \sqrt{2} \end{cases}$$



First, we solve

$$x\frac{dx}{dt} = 1, \qquad x(0) = 0$$

by separation of variables. This yield $x(t) = \sqrt{2t}$. The time interval corresponding to $0 \le x \le \frac{\sqrt{2}}{2}$ is defined by $0 \le t \le \frac{1}{4}$. Second, we solve

$$\left(\sqrt{2} - x\right)\frac{dx}{dt} = 1, \qquad x\left(\frac{1}{4}\right) = \frac{\sqrt{2}}{2}.$$

This gives $x^2 - 2\sqrt{2}\,x + 2t + 1 = 0$. Using the quadratic formula, we have $x(t) = \sqrt{2} - \sqrt{1-2t}$. The time interval corresponding to $\frac{\sqrt{2}}{2} < x \le \sqrt{2}$ is defined by $\frac{1}{4} \le t \le \frac{1}{2}$. Thus,

$$x(t) = \begin{cases} \sqrt{2t}, & 0 \le t \le \frac{1}{4} \\ \sqrt{2} - \sqrt{1-2t}, & \frac{1}{4} < t \le \frac{1}{2}. \end{cases}$$

The time that it takes the saw to cut through the piece of wood is then $t = \frac{1}{2}$.

22. **(a)** By separation of variables

$$\begin{aligned} \frac{dA}{dt} &= -kA^2 \\ A^{-2}\,dA &= -k\,dt \\ -A^{-1} &= -kt + c_1 \\ A^{-1} &= c_2 + kt \\ A(t) &= \frac{1}{c_2 + kt} \end{aligned}$$

Then $A(0) = \dfrac{1}{c_2}$ and thus $c_2 = \dfrac{1}{A_0}$. Therefore $A(t) = \dfrac{A_0}{1 + A_0kt}$.

(b) $A(t) + B(t) = A_0$

$$B(t) = A_0 - A(t) = A_0 - \frac{A_0}{1 + A_0kt}$$

$$B(t) = \frac{kA_0^2t}{1 + A_0kt}\,.$$

(c)