Positivity and stability of coupled differential–difference equations with time-varying delays

Article *in* Automatica · July 2015

DOI: 10.1016/j.automatica.2015.04.007

2 authors:

Jun Shen
Nanjing University of Aeronautics & Astronautics

Wei Xing Zheng
Western Sydney University

Automatica 57 (2015) 123–127

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Positivity and stability of coupled differential–difference equations with time-varying delays[☆]

Jun Shen [a,b], Wei Xing Zheng [a,1]

[a] School of Computing, Engineering and Mathematics, University of Western Sydney, Sydney, NSW 2751, Australia

[b] Department of Mechanical Engineering, The University of Hong Kong, Pokfulam, Hong Kong

ARTICLE INFO

Article history:
Received 9 August 2014
Received in revised form
17 March 2015
Accepted 24 March 2015
Available online 16 May 2015

Keywords:
Coupled differential–difference equations
Positive systems
Time-delay systems

ABSTRACT

This paper studies the asymptotic stability of a special class of coupled delay differential–difference equations with internal positive property. An explicit characterization on the positivity of coupled differential–difference equations is firstly given. Then, based on the positivity of coupled differential–difference equations with constant delays, we investigate the entrywise monotonicity and asymptotic property of their state trajectories starting from appropriately chosen initial conditions. Furthermore, the time-varying delay system is analyzed through comparing with the corresponding constant delay system. It turns out that an internally positive coupled differential–difference equation with bounded time-varying delays is asymptotically stable as long as the corresponding delay-free system is asymptotically stable.

1. Introduction

Recent years have witnessed a growing interest in the analysis of dynamic systems captured by coupled differential–difference equations. Such systems, frequently encountered in lossless propagation model in circuits and fluid systems (Niculescu, 2001; Răsvan & Niculescu, 2002), cover a large class of time-delay systems, including delay systems of neutral type and some singular time-delay systems (Gu & Liu, 2009). On the other hand, plenty of practical models in a variety of disciplines, such as economics, systems biology (Haddad, Chellaboina, & Hui, 2010), pharmacokinetics (Jacquez, 1985) and ecology (Caswell, 2001), involve quantities that are intrinsically nonnegative. This naturally gives rise to a class of dynamic systems whose state variables are always nonnegative. Such systems, commonly referred to as positive systems (Farina & Rinaldi, 2000) or nonnegative systems (Haddad et al., 2010), possess the property that the state and output are always confined in the first orthant whenever the initial conditions and input are nonnegative. A standard input–output positive linear forward system with delayed output feedback results in a closed-loop system described by a coupled delay differential–difference equation with internal positivity. A massive literature on the behavioral analysis and controller synthesis of positive linear system is available in Kaczorek (2002), Li, Lam, and Shu (2010), Li, Lam, Wang, and Date (2011), AitRami and Napp (2012, 2014), Zhao, Zhang, Shi, and Liu (2012), Zhao, Liu, Yin, and Li (2014), Zhu, Han, and Zhang (2014).

The Lyapunov–Krasovskii functional approach is well developed in the stability analysis of coupled differential–difference equations, see for example Pepe and Verriest (2003), Pepe (2005), Gu and Liu (2009), Li and Gu (2010), Gu, Zhang, and Xu (2011). A linear co-positive Lyapunov–Krasovskii functional is also suggested in Aleksandrov and Mason (2014) by taking into account the positivity of coupled delay differential–difference equations. However, these approaches are no longer suitable for the stability analysis of positive coupled differential–difference equations with time-varying delays. The techniques employed in some existing works on stability analysis of positive linear systems with time-varying delays (Liu & Dang, 2011; Liu, Yu, & Wang, 2010; Ngoc, 2013) are partly based on the explicit expression of system trajectories, which are handicapped when coupled differential–difference equations are concerned. These technical difficulties motivate the current work.

In this paper, we aim at investigating the asymptotic stability of internally positive coupled differential–difference equations with time-varying delays. The essential idea is to resort to the comparison system theory and make use of the entrywise

☆ This work was supported in part by the Australian Research Council under Grant DP120104986 and by the postgraduate studentship from the University of Hong Kong. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Chunjiang Qian under the direction of Editor André L. Tits.

E-mail addresses: junshen2009@gmail.com (J. Shen), w.zheng@uws.edu.au (W.X. Zheng).

[1] Tel.: +61 2 4736 0608; fax: +61 2 4736 0374.

http://dx.doi.org/10.1016/j.automatica.2015.04.007
0005-1098/

monotonicity of positive coupled differential–difference equations (with appropriately chosen initial conditions) subject to constant delays. It is worth noting that the selection of initial conditions is rather important since not all the state trajectories of positive coupled differential–difference equations with constant delays are monotonic. Indeed, the state trajectory of a positive coupled differential–difference system with time-varying delays is entrywise upper bounded by that of the corresponding constant delay system with suitable initial conditions. Our results reveal that the asymptotic stability of positive coupled differential–difference equations is robust against bounded time-varying delays.

2. Preliminaries

The notations in this paper are generally standard. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and nonnegative real numbers, respectively. $\mathbb{R}^{m\times n}$ represents the set of all $m \times n$ real matrices while $\mathbb{R}_+^{m\times n}$ denotes the set of $m\times n$ real matrices with nonnegative entries. All the matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $\mathbf{1}$ stands for a column vector with each entry equal to 1. A real square matrix A is called a Schur matrix if all its eigenvalues lie in the open unit disk. The notions of nonnegative matrices, which can be found in the book Berman and Plemmons (1994), are introduced in the following. For two matrices $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}^{m\times n}$, $A \succeq B$ (respectively, $A \succ B$) means that $a_{ij} \geq b_{ij}$ (respectively, $a_{ij} > b_{ij}$) for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. A matrix $A \in \mathbb{R}^{m\times n}$ with all of its entries nonnegative is called a nonnegative matrix and is denoted by $A \succeq 0$. A square matrix $A \in \mathbb{R}^{n\times n}$ with all its off-diagonal entries nonnegative is called a Metzler matrix. The ∞-norm of a matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ is the maximal absolute row sum, that is, $\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$. For a given scalar $\tau > 0$, $\mathcal{PC}([-\tau, 0), \mathbb{R}^m)$ denotes the set of bounded, right continuous and piecewise continuous vector-valued functions defined on $[-\tau, 0)$, which is endowed with norm $\|\phi\|_\infty = \sup_{-\tau\leq s<0} \|\phi(s)\|_\infty$. The matrix exponential e^{At} is a nonnegative matrix for all $t \geq 0$ if and only if A is a Metzler matrix. Furthermore, e^{At} $(t \geq 0)$ has positive diagonal entries if A is a Metzler matrix. The following lemma on a Hurwitz stable Metzler matrix is well known in the literature.

Lemma 1 (Farina & Rinaldi, 2000)**.** *Suppose that matrix A is Metzler. Then A is Hurwitz if and only if*

(i) *there exists a column vector $\lambda \succ 0$, such that $A\lambda \prec 0$.*
(ii) *A is nonsingular and $-A^{-1}$ is a nonnegative matrix.*

3. Main results

In this paper, we consider the following linear coupled differential–difference equation with time-varying delays:

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t - \tau(t)), \\ y(t) = Cx(t) + Dy(t - \tau(t)), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$. The time-varying delay $\tau(t)$ is assumed to be continuous and bounded, that is, $0 \leq \tau(t) \leq \tau$ for some positive constant τ. The initial condition of system (1) is given by $x(0) = \psi$ and $y(s) = \phi(s)$, $(s \in [-\tau, 0))$, where $\psi \in \mathbb{R}^n$ and $\phi \in \mathcal{PC}([-\tau, 0), \mathbb{R}^m)$. In the sequel, we denote the state trajectory of system (1) by $x(t; \psi, \phi)$ and $y(t; \psi, \phi)$, when we would like to specify the initial conditions. As noted in Gu and Liu (2009), Aleksandrov and Mason (2014), a necessary condition for the asymptotic stability of system (1) is that D is a Schur matrix. Therefore, throughout this paper, we always assume that D is a Schur matrix. The internal positivity of coupled differential–difference (1) is defined as follows.

Remark 1. The delay function $\tau(t)$ is not assumed to be differentiable or to have its derivative less than 1 since this is not needed in this paper. Such an assumption often arises when the Lyapunov–Krasovskii functional method is utilized. On the other hand, when D is Schur and $\tau(t) = 0$ $(t \geq 0)$, one can easily calculate that $y(t) = (I-D)^{-1}Cx(t)$ and thus $\dot{x}(t) = (A+B(I-D)^{-1}C)x(t)$, whose solution exists and is asymptotically stable when $A + B(I - D)^{-1}C$ is Hurwitz.

Definition 1 (Farina & Rinaldi, 2000)**.** System (1) is said to be (internally) positive, if for any initial condition $\psi \succeq 0$ and $\phi(s) \succeq 0$, $(s \in [-h, 0))$, the state trajectory of system (1) satisfies that $x(t; \psi, \phi) \succeq 0$ and $y(t; \psi, \phi) \succeq 0$ for all $t \geq 0$.

The following lemma provides a characterization on when the state trajectory of system (1) is nonnegative for all nonnegative initial conditions and nonnegative inputs.

Lemma 2. *Assume that D is a Schur matrix. For all delays $\tau(t)$ satisfying $0 \leq \tau(t) \leq \tau$, the state trajectory of system*

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t - \tau(t)) + w(t), \\ y(t) = Cx(t) + Dy(t - \tau(t)) + u(t), \end{cases} \tag{2}$$

is nonnegative for all nonnegative initial conditions and piecewise continuous nonnegative inputs $w(t) \succeq 0$, $u(t) \succeq 0$, $(t \geq 0)$, if and only if A is Metzler, B, C and D are nonnegative.

Proof. Taking into account the fact that e^{At} is nonnegative for all $t \geq 0$ if and only if A is Metzler, the necessity is obvious by considering the constant delay case. For the sufficiency part, we first prove that for any given $T > 0$, $x(t)$ and $y(t)$ $(t \in [0, T])$ are nonnegative for all nonnegative initial conditions and all inputs $w(t) \succ 0$, $u(t) \succ 0$. This amounts to proving that the set

$$\mathcal{S} = \{t \in [0, T] : [x(t)^T, y(t)^T]^T \not\succeq 0\}$$

is empty. Suppose on the contrary that $\mathcal{S}$ is nonempty and hence we can define $t_0 = \inf \mathcal{S}$. Then,

$$x(t_0) = e^{At_0}x(0) + \int_0^{t_0} e^{A(t_0-s)}(y(s - \tau(s)) + w(s))\mathrm{d}s \succ 0.$$

Note that

$$y(t_0) = Cx(t_0) + Dy(t_0 - \tau(t_0)) + u(t_0) \succ 0$$

if $\tau(t_0) \neq 0$, while

$$y(t_0) = (I - D)^{-1}(Cx(t_0) + u(t_0)) \succ 0$$

if $\tau(t_0) = 0$ (since $(I - D)^{-1}$ is a nonnegative matrix and has no zero rows). Therefore, we can conclude that $x(t_0) \succ 0$ and $y(t_0) \succ 0$, which contradicts the definition of t_0. Hence, for any given $T > 0$, $x(t)$ and $y(t)$ $(t \in [0, T])$ are nonnegative for all nonnegative initial condition and all inputs $w(t) \succ 0$, $u(t) \succ 0$. Then, by the continuous dependence of the solution of system (2) on its parameters, one can conclude that $x(t)$ and $y(t)$ $(t \in [0, T])$ are nonnegative for all nonnegative initial conditions and all inputs $w(t) \succeq 0$ and $u(t) \succeq 0$. This completes the proof. □

It is now clear that system (1) is a positive system if and only if A is Metzler, B, C and D are nonnegative matrices. Note that any initial condition ψ and ϕ can be rewritten as $\psi = \psi^+ - \psi^-$ and $\phi = \phi^+ - \phi^-$, where ψ^+, ψ^-, ϕ^+ and ϕ^- are nonnegative. Since system (1) is linear, its asymptotic stability with respect to initial conditions in the first orthant is equivalent to asymptotic stability with respect to arbitrary initial conditions. Also note that if system (1) is positive, then it is asymptotically stable only if $A+B(I-D)^{-1}C$ is Hurwitz, which is proved in the following proposition.

Proposition 1. *Suppose that A is Metzler, B, C and D are nonnegative. If for any initial condition ψ and ϕ, the state trajectory of system (1) satisfies that*

$$\lim_{t\to+\infty} x(t;\psi,\phi) = \lim_{t\to+\infty} y(t;\psi,\phi) = 0,$$

then $A + B(I-D)^{-1}C$ is Hurwitz.

Proof. First note that $A + B(I-D)^{-1}C$ is nonsingular, since otherwise one can find a nonzero vector θ such that $(A + B(I-D)^{-1}C)\theta = 0$, which results in that $x(t) \equiv \theta$, $y(t) \equiv (I-D)^{-1}C\theta$ is an equilibrium point (contradicting with $x(t;\theta,(I-D)^{-1}C\theta) \to 0$ as $t \to +\infty$). Given a positive vector $v \succ 0$, define

$$\tilde{x}(t) \triangleq x(t) - (A + B(I-D)^{-1}C)^{-1}v,$$

$$\tilde{y}(t) \triangleq y(t) - (I-D)^{-1}C(A + B(I-D)^{-1}C)^{-1}v.$$

Then, $\tilde{x}$ and $\tilde{y}$ satisfy

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{y}(t-\tau(t)) + v,$$
$$\tilde{y}(t) = C\tilde{x}(t) + D\tilde{y}(t-\tau(t)).$$

By Lemma 2, it can be deduced that $\tilde{x}(t;\tilde{\psi},\tilde{\phi}) \succeq 0$ $(t \geq 0)$ for nonnegative initial conditions $\tilde{\psi}$ and $\tilde{\phi}$. Since

$$\lim_{t\to+\infty} \tilde{x}(t;\tilde{\psi},\tilde{\phi}) = -(A + B(I-D)^{-1}C)^{-1}v \succeq 0,$$

by defining

$$\lambda = -(A + B(I-D)^{-1}C)^{-1}v + \sigma\mathbf{1},$$

one can conclude that for sufficiently small $\sigma > 0$, it holds that $\lambda \succ 0$ and

$$(A + B(I-D)^{-1}C)\lambda = -v + \sigma(A + B(I-D)^{-1}C)\mathbf{1} \prec 0.$$

This, by Lemma 1, implies that $A + B(I-D)^{-1}C$ is Hurwitz. □

Remark 2. It is possible that general linear system (1) with delays is asymptotically stable while the corresponding delay-free system is not. However, this will never happen when positivity is imposed on system (1).

Following the above discussion, in the sequel, it is natural to make the following assumptions on system (1) and (2).

Assumption 1. (i) A is Metzler, B and C are nonnegative matrices, D is a nonnegative Schur matrix.
(ii) $A + B(I-D)^{-1}C$ is Hurwitz.
(iii) There exists a unique solution in $[0,+\infty)$ for system (2).

Remark 3. The assumption on the existence and uniqueness of the solution of coupled differential–difference equations is also adopted in the existing literature (see, e.g., Gu and Liu (2009); Pepe (2005); Pepe and Verriest (2003)). How to relax this assumption would be an interesting topic for future research.

Our purpose is to prove that as long as $A + B(I-D)^{-1}C$ is Hurwitz, positive system (1) with bounded time-varying delays is asymptotically stable regardless of the magnitude of delays. To this end, some properties are firstly unveiled for the constant delay system:

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{y}(t-\tau), \\ \bar{y}(t) = C\bar{x}(t) + D\bar{y}(t-\tau). \end{cases} \quad (3)$$

Several simple facts can be readily obtained as follows. Since D is a nonnegative Schur matrix, $D - I$ is Metzler and Hurwitz. By Lemma 1, there exists $\eta \succ 0$, such that $(I-D)\eta \succ 0$. Again by Lemma 1, it can be concluded that there exists a column vector $\lambda \succ 0$, such that

$$(A + B(I-D)^{-1}C)\lambda \prec 0. \quad (4)$$

Define

$$\xi \triangleq (I-D)^{-1}C\lambda + \epsilon\eta, \quad (5)$$

then for sufficiently small $\epsilon > 0$, we have that $\xi \succ 0$ and $A\lambda + B\xi \prec 0$ hold simultaneously.

In the following, we focus on constant delay system (3) with special constant initial condition $\psi = \lambda$, $\phi(s) = \xi$, $(s \in [-\tau, 0))$. We aim to unravel the monotonicity and asymptotic behaviors of its state trajectory. Before moving on, the following lemma is needed.

Lemma 3. *Suppose that $\lambda \succ 0$ satisfies inequality (4) and that $\xi \succ 0$ defined in (5) satisfies $A\lambda + B\xi \prec 0$. Then, the state trajectory of constant delay system (3) satisfies that*

(i) *$\bar{x}(t;\lambda,\xi) \preceq \lambda$ and $\bar{y}(t;\lambda,\xi) \preceq \xi$ for $t \geq 0$.*
(ii) *$\bar{x}(t;\lambda,\xi) \preceq \bar{x}(s;\lambda,\xi)$ and $\bar{y}(t;\lambda,\xi) \preceq \bar{y}(s;\lambda,\xi)$ for any $t \geq s \geq 0$.*

Proof. (i) Define $e_x(t) \triangleq \lambda - \bar{x}(t;\lambda,\xi)$, and $e_y(t) \triangleq \xi - \bar{y}(t;\lambda,\xi)$. Then, e_x and e_y satisfy a new coupled differential–difference equation subject to a special constant input:

$$\begin{cases} \dot{e}_x(t) = Ae_x(t) + Be_y(t-\tau) - (A\lambda + B\xi), \\ e_y(t) = Ce_x(t) + De_y(t-\tau) + (I-D)\xi - C\lambda. \end{cases} \quad (6)$$

Noting that

$$-(A\lambda + B\xi) \succ 0 \quad \text{and} \quad (I-D)\xi - C\lambda = \epsilon(I-D)\eta \succ 0,$$

by Lemma 2, it immediately follows that $e_x(t) \succeq 0$ and $e_y(t) \succeq 0$ for all $t \geq 0$. This implies that $\bar{x}(t;\lambda,\xi) \preceq \lambda$ and $\bar{y}(t;\lambda,\xi) \preceq \xi$ for $t \geq 0$, which completes the proof of (i).

(ii) For any given $h > 0$, we define $\tilde{x}(t) \triangleq \bar{x}(t;\lambda,\xi) - \bar{x}(t+h;\lambda,\xi)$, $\tilde{y}(t) \triangleq \bar{y}(t;\lambda,\xi) - \bar{y}(t+h;\lambda,\xi)$. Clearly, $\tilde{x}$ and $\tilde{y}$ satisfy

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{y}(t-\tau), \qquad \tilde{y}(t) = C\tilde{x}(t) + D\tilde{y}(t-\tau).$$

Note that

$$\tilde{x}(0) = \bar{x}(0;\lambda,\xi) - \bar{x}(h;\lambda,\xi) = \lambda - \bar{x}(h;\lambda,\xi) \succeq 0$$

due to statement (i). Similarly, for $-\tau \leq t < 0$, we have

$$\tilde{y}(t) = \bar{y}(t;\lambda,\xi) - \bar{y}(t+h;\lambda,\xi) = \xi - \bar{y}(t+h;\lambda,\xi) \succeq 0.$$

Therefore, by Lemma 2, it can be deduced that $\tilde{x}(t) \succeq 0$ and $\tilde{y}(t) \succeq 0$ for all $t \geq 0$. This completes the proof. □

In the light of the above two lemmas, the convergence of constant delay system (3) is presented below. Unlike most existing works, we do not employ the Lyapunov–Krasovskii functional method, but resort to the Final Value Theorem in the convergence analysis.

Lemma 4. *Assume that $\lambda \succ 0$ satisfies inequality (4) and that $\xi \succ 0$ defined in (5) satisfies $A\lambda + B\xi \prec 0$. Then, the state trajectory of system (3) satisfies that*

$$\lim_{t\to\infty} \bar{x}(t;\lambda,\xi) = \lim_{t\to\infty} \bar{y}(t;\lambda,\xi) = 0.$$

Proof. From Lemma 3, it is clear that $\bar{x}(t;\lambda,\xi)$ and $\bar{y}(t;\lambda,\xi)$ are entrywise monotonically non-increasing and have a lower bound. This, in turn, implies that $\lim_{t\to\infty} \bar{x}(t;\lambda,\xi)$ and $\lim_{t\to\infty} \bar{y}(t;\lambda,\xi)$ exist. Suppose that $\lim_{t\to+\infty} \bar{y}(t;\lambda,\xi) = M$, and thus $\lim_{t\to+\infty} \bar{x}(t;\lambda,\xi) = (I-D)^{-1}CM$ from the second equation of system (3). Since $\bar{x}(t;\lambda,\xi)$ and $\bar{y}(t;\lambda,\xi)$ are bounded on $[0,+\infty)$ due to Lemma 3 and thus $\dot{\bar{x}}(t;\lambda,\xi)$ is bounded as well, the Laplace transforms of $\bar{x}(t;\lambda,\xi)$, $\bar{y}(t;\lambda,\xi)$ and $\dot{\bar{x}}(t;\lambda,\xi)$ exist. By the Final Value Theorem, one has

$$\lim_{s\to 0} s\mathcal{L}(\bar{x}(t;\lambda,\xi)) = (I-D)^{-1}CM,$$
$$\lim_{s\to 0} s\mathcal{L}(\bar{y}(t;\lambda,\xi)) = M.$$

Taking Laplace transform on both sides of the first equation of system (3), we arrive at

$$\begin{aligned}(A+B(I-D)^{-1}C)M &= \lim_{s\to 0} s\mathcal{L}(\dot{\bar{x}}(t;\lambda,\xi))\\ &= \lim_{s\to 0} s(s\mathcal{L}(\bar{x}(t;\lambda,\xi))-\lambda)\\ &= \lim_{s\to 0} s((I-D)^{-1}CM-\lambda)=0.\end{aligned}$$

Since $A+B(I-D)^{-1}C$ is Hurwitz and thus invertible, it immediately follows that $M=0$. The proof is complete. □

In order to further facilitate the analysis of system (1), the relationship between the trajectory of time-varying delay system (1) and that of constant delay system (3) is provided in the following lemma.

Lemma 5. *Suppose that $\lambda \succ 0$ satisfies inequality (4) and that $\xi \succ 0$ defined in (5) satisfies $A\lambda + B\xi \prec 0$. Then, it holds that $x(t;\lambda,\xi) \preceq \bar{x}(t;\lambda,\xi)$ and $y(t;\lambda,\xi) \preceq \bar{y}(t;\lambda,\xi)$, where $x(t)$, $y(t)$ and $\bar{x}(t)$, $\bar{y}(t)$ are the state trajectories of systems (1) and (3), respectively.*

Proof. Define

$$e_x(t) \triangleq \bar{x}(t;\lambda,\xi) - x(t;\lambda,\xi),$$
$$e_y(t) \triangleq \bar{y}(t;\lambda,\xi) - y(t;\lambda,\xi).$$

Simple manipulations yield that

$$\begin{aligned}\dot{e}_x(t) &= Ae_x(t) + Be_y(t-\tau(t))\\ &\quad + B(\bar{y}(t-\tau;\lambda,\xi) - \bar{y}(t-\tau(t);\lambda,\xi)),\\ e_y(t) &= Ce_x(t) + De_y(t-\tau(t))\\ &\quad + D(\bar{y}(t-\tau;\lambda,\xi) - \bar{y}(t-\tau(t);\lambda,\xi)).\end{aligned}$$

Then it follows from Lemma 3 that

$$\bar{y}(t-\tau;\lambda,\xi) - \bar{y}(t-\tau(t);\lambda,\xi) \succeq 0, \quad \forall t \geq 0.$$

Therefore, by Lemma 2, one can deduce that $e_x(t) \succeq 0$ and $e_y(t) \succeq 0$ for all $t \geq 0$. □

The following result is a direct consequence of the linearity and positivity of system (1).

Lemma 6. *For any initial conditions satisfying $\psi_1 \preceq \psi_2$ and $\phi_1(s) \preceq \phi_2(s)$ for $s \in [-\tau, 0)$, we have $x(t;\psi_1,\phi_1) \preceq x(t;\psi_2,\phi_2)$ and $y(t;\psi_1,\phi_1) \preceq y(t;\psi_2,\phi_2)$.*

Now we are in a position to present the main theorem, which states that the asymptotic stability of positive system (1) is insensitive to the magnitude of time-varying delays.

Theorem 1. *Under Assumption 1, for any delays satisfying $0 \leq \tau(t) \leq \tau$, system (1) is asymptotically stable.*

Proof. Under Assumption 1, one can always find $\lambda \succ 0$ and $\xi \succ 0$ (as defined in (5)), such that inequalities $(A+B(I-D)^{-1}C)\lambda \prec 0$ and $A\lambda + B\xi \prec 0$ hold. Since λ and ξ are strictly positive vectors, for any initial condition $x(0) = \psi \succeq 0$ and $y(s) = \phi(s) \succeq 0$, $(s \in [-\tau, 0))$, for sufficiently large scalar $k > 0$, one has $\psi \prec k\lambda$ and $\phi(s) \prec k\xi$, $(s \in [-\tau, 0))$. By Lemmas 3, 5 and 6, it can be concluded that

$$x(t;\psi,\phi) \preceq x(t;k\lambda,k\xi) \preceq \bar{x}(t;k\lambda,k\xi) \preceq k\lambda.$$

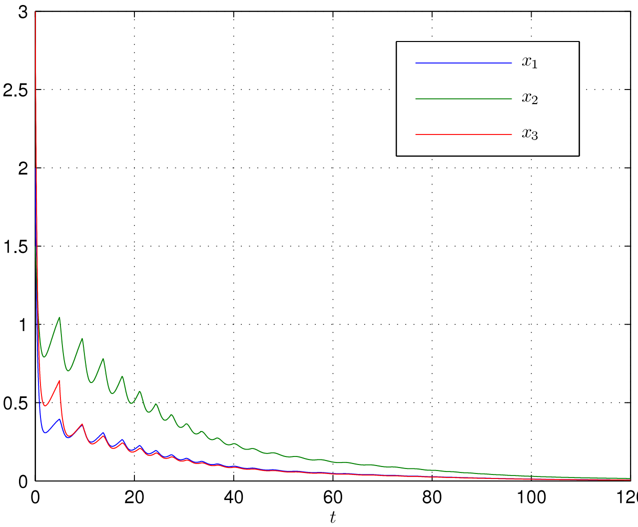

Fig. 1. The evolution of $x(t)$ of system (1).

The same argument applies to $y(t;\psi,\phi)$. It is then not hard to deduce that system (1) is asymptotically stable by taking into account the fact that

$$\lim_{t\to+\infty} \bar{x}(t;\lambda,\xi) = \lim_{t\to+\infty} \bar{y}(t;\lambda,\xi) = 0$$

and the linearity of system (1). □

Remark 4. Among the earliest works which pointed out that positive systems are robust against time-varying delays, we would like to mention Ait Rami (2009), which considered the asymptotic stability of positive linear systems with bounded time-varying delays. In this paper, we generalize the delay-independent stability results to coupled differential–difference equations under positivity constraints.

4. Numerical example

Consider coupled delay differential–difference system (1) with system matrices given by

$$A = \begin{bmatrix} -2.5 & 0.3 & 0\\ 0.5 & -2 & 0.1\\ 0.4 & 0.6 & -3\end{bmatrix}, \qquad B = \begin{bmatrix} 0.2 & 0.1\\ 0.5 & 0.3\\ 0 & 0.4\end{bmatrix},$$
$$C = \begin{bmatrix} 0.3 & 0.4 & 0.1\\ 0.2 & 0.2 & 0\end{bmatrix}, \qquad D = \begin{bmatrix} 0.6 & 0.3\\ 0.1 & 0.2\end{bmatrix}.$$

The delay is given as $\tau(t) = 4 + \cos(0.1t)$. Note that A is Metzler, B, C and D are nonnegative matrices, which implies that system (1) is internally positive due to Lemma 2. It can be readily verified that D is Schur and $A + B(I-D)^{-1}C$ is Hurwitz, by which we can conclude that system (1) is asymptotically stable for any bounded delays owing to Theorem 1. The state trajectories of system (1) with initial conditions $\psi = [2 \quad 1.5 \quad 3]^T$ and $\phi = [1+\cos(0.1\pi s) \quad 3+2\sin(0.05\pi s)]^T$ $(s \in [-5, 0))$ are depicted in Figs. 1 and 2, respectively. It can be observed that the state trajectory is always nonnegative and system (1) is asymptotically stable, which confirms the conclusion in Theorem 1.

5. Conclusion

In this paper, we have provided a necessary and sufficient condition on the asymptotic stability of a class of coupled delay differential–difference equations with time-varying delays, which possess internal positive characteristics. A characterization on the

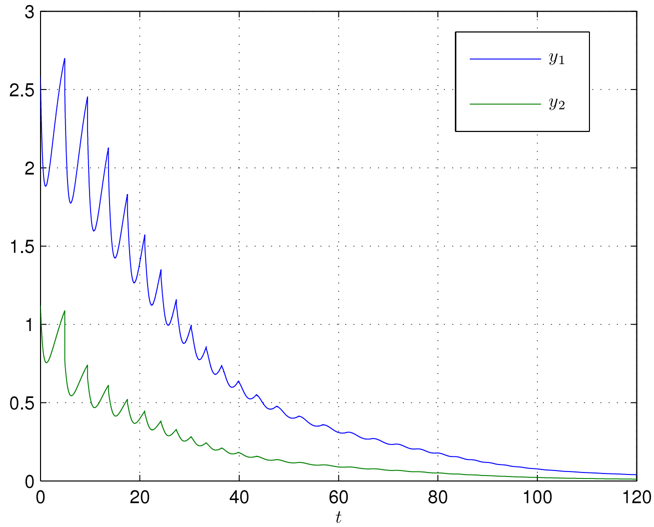

Fig. 2. The evolution of $y(t)$ of system (1).

positivity of coupled delay differential–difference equations has been given. Then, we have pointed out several properties on the monotonicity and asymptotic behaviors of coupled differential–difference equations with constant delays and appropriate initial conditions. Finally, by virtue of a comparison argument, we have shown that the asymptotic stability of coupled differential–difference equations is insensitive to the magnitude of time-varying delays.

References

Ait Rami, M. (2009). Stability analysis and synthesis for linear positive systems with time-varying delays. In *Lecture notes in control and information sciences, Vol. 389* (pp. 205–215). Berlin: Springer-Verlag.

Ait Rami, M., & Napp, D. (2012). Characterization and stability of autonomous positive descriptor systems. *IEEE Transactions on Automatic Control*, *57*(10), 2668–2673.

Ait Rami, M., & Napp, D. (2014). Positivity of discrete singular systems and their stability: An LP-based approach. *Automatica*, *50*(1), 84–91.

Aleksandrov, A. Yu., & Mason, O. (2014). Absolute stability and Lyapunov–Krasovskii functionals for switched nonlinear systems with time-delay. *Journal of the Franklin Institute*, *351*(8), 4381–4394.

Berman, A., & Plemmons, R. J. (1994). *Nonnegative matrices*. Philadelphia, PA: SIAM.

Caswell, H. (2001). *Matrix population models: construction, analysis and interpretation*. Sunderland, MA: Sinauer Assoc.

Farina, L., & Rinaldi, S. (2000). *Positive linear systems: theory and applications*. New York: Wiley-Interscience.

Gu, K., & Liu, Y. (2009). Lyapunov-Krasovskii functional for uniform stability of coupled differential-functional equations. *Automatica*, *45*(3), 798–804.

Gu, K., Zhang, Y., & Xu, S. (2011). Small gain problem in coupled differential–difference equations, time-varying delays, and direct Lyapunov method. *International Journal of Robust and Nonlinear Control*, *21*(4), 429–451.

Haddad, W. M., Chellaboina, V. S., & Hui, Q. (2010). *Nonnegative and compartmental dynamical systems*. Princeton, NJ: Princeton Univ. Press.

Jacquez, J. (1985). *Compartmental analysis in biology and medicine*. Ann Arbor, MI: Univ. Michigan Press.

Kaczorek, T. (2002). *Positive 1D and 2D systems*. London: Springer Verlag.

Li, H., & Gu, K. (2010). Discretized Lyapunov–Krasovskii functional for coupled differential–difference equations with multiple delay channels. *Automatica*, *46*(5), 902–909.

Li, P., Lam, J., & Shu, Z. (2010). H_∞ positive filtering for positive linear discrete-time systems: An augmentation approach. *IEEE Transactions on Automatic Control*, *55*(10), 2337–2342.

Li, P., Lam, J., Wang, Z., & Date, P. (2011). Positivity-preserving H_∞ model reduction for positive systems. *Automatica*, *47*(7), 1504–1511.

Liu, X., & Dang, C. (2011). Stability analysis of positive switched linear systems with delays. *IEEE Transactions on Automatic Control*, *56*(7), 1684–1690.

Liu, X., Yu, W., & Wang, L. (2010). Stability analysis for continuous-time positive systems with time-varying delays. *IEEE Transactions on Automatic Control*, *55*(4), 1024–1028.

Ngoc, P. H. A. (2013). Stability of positive differential systems with delay. *IEEE Transactions on Automatic Control*, *58*(1), 203–209.

Niculescu, S.-I. (2001). Delay effects on stability: a robust control approach. In *Lecture notes in control and information science, Vol. 269*. London: Springer.

Pepe, P. (2005). On the asymptotic stability of coupled delay differential and continuous time difference equations. *Automatica*, *41*(1), 107–112.

Pepe, P., & Verriest, E. (2003). On the stability of coupled delay differential and continuous time difference equations. *IEEE Transactions on Automatic Control*, *48*(8), 1422–1427.

Răsvan, V., & Niculescu, S.-I. (2002). Oscillations in lossless propagation models: A Lyapunov–Krasovskii approach. *IMA Journal of Mathematical Control and Information*, *19*(1–2), 157–172.

Zhao, X., Liu, X., Yin, S., & Li, H. (2014). Improved results on stability of continuous-time switched positive linear systems. *Automatica*, *50*(2), 614–621.

Zhao, X., Zhang, L., Shi, P., & Liu, M. (2012). Stability of switched positive linear systems with average dwell time switching. *Automatica*, *48*(6), 1132–1137.

Zhu, S., Han, Q.-L., & Zhang, C. (2014). l_1-gain performance analysis and positive filter design for positive discrete-time Markov jump linear systems: A linear programming approach. *Automatica*, *50*(8), 2098–2107.