DCDIS A Supplement, Advances in Dynamical Systems, Vol. 14(S2) 296--301
Copyright@2007 Watam Press

Survey on Analysis of Time Delayed Systems via the Lambert W Function

Sun Yi[1], P. W. Nelson[2] and A. G. Ulsoy[1]

[1]Department of Mechanical Engineering, University of Michigan, Email: syjo@umich.edu

[2]Department of Mathematics, University of Michigan

Abstract. This paper summarizes recent research on an approach for the analytical solution to systems of delay differential equations developed using the Lambert W function [26, 29], and its applications [25, 28]. Also, several outstanding research problems associated with the solution of systems of delay differential equations using the matrix Lambert W function are highlighted. The solution has the form of an infinite series of modes written in terms of the matrix Lambert W function. The essential advantage of this approach is the similarity to the concept of the state transition matrix in linear ordinary differential equations, enabling its use for general classes of linear delay differential equations. This similarity makes it possible to extend some of control methods for ordinary differential equations to delay differential equations.

Keywords. Time delayed systems, Lambert W function, delay differential equation, stability, controllability and observability.
AMS (MOS) subject classification: 34K06

1 Introduction

This paper summarizes recent results by the authors on the analytical solution of systems of linear delay differential equations (DDEs) using the matrix Lambert W function. All the results presented in this paper in summary form have previously been presented in detail in [1,25-30]. These include the solution approach based upon the Lambert W function, the definition of the matrix Lambert W function, both time-domain and Laplace domain free and forced solutions, observations and results on stability, examples, observability and controllability of DDEs and topics for future research. We hope this summary paper will interest applied mathematics researchers in the method, and in several currently outstanding fundamental research problems:

(1) The method using the matrix Lambert W function hinges on the determination of a matrix, $\mathbf{Q}_k$. As discussed in section 3, we have always been able to find $\mathbf{Q}_k$ for the problems we have considered. However, mathematical conditions for the existence and uniqueness of $\mathbf{Q}_k$ are lacking and greatly needed.

(2) As discussed in section 4, we have observed in all our examples using DDEs that stability is determined by the principal branch (i.e., $k = 0$) of the matrix Lambert W function. This has been proven to be correct in the scalar case and for some special forms of the vector case, however a general proof is lacking.

2 Background

Time delayed systems (TDS) are systems in which a significant time delay exists between the applications of input to the system and their resulting effect. Such systems arise from an inherent time delay in the components of the system or a deliberate introduction of time delay into the system for control purposes. TDS can be represented by delay differential equations, which belong to the class of functional differential equations, and have been extensively studied over the past decades [21].

Delay differential equations were initially introduced in the 18th century by Laplace and Condorcet. The basic theory concerning stability of systems described by equations of this type was developed by Pontryagin in 1942. A Fourier-like analysis of the existence of the solution and its properties for the nonlinear DDEs is studied by Wright, 1946. Similar approaches to linear and nonlinear DDEs are also reported by Bellman in 1963. The uniqueness of the solution and its properties for the linear DDEs with varying coefficients and solution properties for the linear DDE with asymptotically constant coefficients are also studied by Wright in 1948. Important works have been written by Bellman and Cooke [2], Smith in 1957, Pinney in 1958, Halanay in 1966, El'sgol'c and Norkin in 1971, Myshkis in 1972, Yanushevski in 1978, Marshal in 1979, and Hale in 1977, 1993 [10]. The reader is referred to the detailed review in [9, 21]. The principal difficulty in studying delay differential equations is that such equations always lead to an infinite spectrum of eigenvalues. The determination of this spectrum requires a corresponding determination of zeros of certain analytic functions. One of the well-known approximation methods is the Padé approximation[12] and "Direct method" [16], which result in a shortened repeating fraction for the approximation of the characteristic equation of the delay.

An analytic approach to obtain the complete solution to systems of DDEs based on the concept of the Lambert W function, that has been known to be useful to analyze DDEs [6], was developed by Asl and Ulsoy in 2003 [1]. Unlike results by other existing methods, the solution has an analytical form expressed with the parameters of the DDE. One can explicitly determine how the parameters are involved in the solution and, furthermore, how each parameter affects each eigenvalue and the solution. Also, each eigenvalue is distinguished by the branch of the Lambert W function. However, their approach was only correct in the scalar case and in the special case where certain matrices in the systems of DDEs commute. In [26, 29], the analytical approach was extended to general systems of DDEs and to non-homogeneous DDEs, and compared with the results obtained by numerical integration. The method has been validated, for stability [25], and for free and forced response, by comparison to numerical integration. As shown in Table 1, the approach using the Lambert W function provides a solution form for DDEs similar to that to the free and forced solution of linear constant coefficient ordinary differential equations (ODEs) in terms of the state transition matrix. This analogy enables extensions of the method as shown by our recent results, to topics such as observability and controllability.

This paper summarizes recent research on an approach for the analytical solution to systems of delay differential equations developed using the Lambert W function [26, 29], and applications based of the similarity to the concept of the state transition matrix in linear ordinary differential equations [25, 28].

3 Solution Using the Matrix Lambert W function

Consider a linear real system of delay differential equations with a single constant delay, h

$$\begin{array}{rcll} \dot{\mathbf{x}}(t) &=& \mathbf{A}\mathbf{x}(t)+\mathbf{A_d}\mathbf{x}(t-h)+\mathbf{B}\mathbf{u}(t) & t>0 \\ \mathbf{x}(t) &=& \mathbf{g}(t) & t\in[-h,0) \\ \mathbf{x}(t) &=& \mathbf{x}_0 & t=0 \\ \mathbf{y}(t) &=& \mathbf{C}\mathbf{x}(t) & \end{array} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{A_d}$ are $n\times n$ coefficient matrices, and $\mathbf{x}(t)$ is an $n\times 1$ state vector, $\mathbf{B}$ is an $n\times r$ matrix, $\mathbf{u}(t)$, an $r\times 1$ vector, is a function representing the external excitation, and $\mathbf{g}(t)$ and $\mathbf{x}_0$ are a specified preshape function and an initial state respectively. The output matrix $\mathbf{C}$ is $p\times n$ and $\mathbf{y}(t)$ is a $p\times 1$ measured output vector. The existence and uniqueness of the solution for the system of linear DDEs in (1) was studied in [2]. First we assume a free solution form, i.e., $\mathbf{u}(t)=\mathbf{0}$, as

$$\mathbf{x}(t)=e^{\mathbf{S}t}\mathbf{x}_0 \tag{2}$$

where $\mathbf{S}$ is $n\times n$ matrix. In the usual case, the characteristic equation for (1) is obtained from the equation by looking for nontrivial solution of the form $e^{st}\mathbf{C}$ where 's' is a scalar variable and $\mathbf{C}$ is constant [10]. However, such an approach does not lead to an interesting result, nor does it help in deriving a solution to systems of DDEs in (1). Alternatively, one could assume the form of (2) to derive the solution to systems of DDEs in (1) using the matrix Lambert W function. Substitution of (2) into (1) enables one to obtain a homogeneous solution to (1) [29],

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I \tag{3}$$

where

$$\mathbf{S}_k=\frac{1}{h}\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)+\mathbf{A} \tag{4}$$

The constant matrices $\mathbf{C}_k^I$ in (3) are computed from a given preshape function $\mathbf{g}(t)$, and an initial condition, $\mathbf{x}_0$ [29]. The matrix, $\mathbf{Q}_k$ is obtained from the following condition, that can be used to solve for the unknown matrix $\mathbf{Q}_k$ [29],

$$\mathbf{W}_k(\mathbf{A}_d h\mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A_d}h\mathbf{Q}_k)+\mathbf{A}h}=\mathbf{A_d}h \tag{5}$$

Our result, in numerous examples studied to date, always yield a unique solution $\mathbf{Q}_k$ from the numerical solution to Eq. (5) for each k. However, conditions for existence and uniqueness of such a solution have not been established. Note that $\mathbf{W}_k$ in (4) denotes the matrix Lambert W function which satisfies [1]

$$\mathbf{W}_k(\mathbf{H}_k)e^{\mathbf{W}_k(\mathbf{H}_k)}=\mathbf{H}_k \tag{6}$$

The matrix Lambert W function, $\mathbf{W}_k(\mathbf{H}_k)$, is complex valued, with a complex argument $\mathbf{H}_k$, and has an infinite number of branches $\mathbf{W}_k(\mathbf{H}_k)$, where $k=-\infty,\cdots,-1,0,1,\cdots,\infty$ [6]. Corresponding to each branch, k, of the matrix Lambert W function, $\mathbf{W}_k$, there is a solution $\mathbf{Q}_k$ from (5), and for $\mathbf{H}_k=\mathbf{A_d}h\mathbf{Q}_k$, we compute the Jordan canonical form $\mathbf{J}_k$ from $\mathbf{H}_k=\mathbf{Z}_k\mathbf{J}_k\mathbf{Z}_k^{-1}$. $\mathbf{J}_k=\mathrm{diag}(J_{k1}(\hat\lambda_1),J_{k2}(\hat\lambda_2),\ldots,J_{kp}(\hat\lambda_p))$, where $J_{ki}(\hat\lambda_i)$ is $m\times m$ Jordan block and m is multiplicity of the eigenvalue $\hat\lambda_i$. Then, the matrix Lambert W function can be computed as [20]

$$\begin{aligned}&\mathbf{W}_k(\mathbf{H}_k)=\\&\mathbf{Z}_k\left\{\mathrm{diag}\left(\mathbf{W}_k(J_{k1}(\hat\lambda_1)),\ldots,\mathbf{W}_k(J_{kp}(\hat\lambda_p))\right)\right\}\mathbf{Z}_k^{-1}\end{aligned} \tag{7}$$

where

$$\mathbf{W}_k(J_{ki}(\hat\lambda_i)=\begin{bmatrix} W_k(\hat\lambda_i) & W_k'(\hat\lambda_i) & \cdots & \frac{1}{(m-1)!}W_k^{(m-1)}(\hat\lambda_i) \\ 0 & W_k(\hat\lambda_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k(\hat\lambda_i)\end{bmatrix} \tag{8}$$

The principal and other branches of the Lambert W function in (8) can be calculated analytically using a series expansion [6], or alternatively, using commands already embedded in the various commercial software packages, such as Matlab, Maple, and Mathematica.

When $\mathbf{u}(t)\neq\mathbf{0}$ in (1), the solution in (3) can be extended to the form [26]

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I+\int_0^t\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^N\mathbf{B}\mathbf{u}(\xi)d\xi \tag{9}$$

The coefficient $\mathbf{C}_k^I$ in (9) is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and the preshape function $\mathbf{g}(t)$ and the initial condition $\mathbf{x}_0$, while $\mathbf{C}_k^N$ is a function of $\mathbf{A}$, $\mathbf{A_d}$, h and does not depend on $\mathbf{g}$ or $\mathbf{x}_0$. The methods for computing $\mathbf{C}_k^I$ and $\mathbf{C}_k^N$ were developed in [26, 27]. With the assumption that any pair of roots of the characteristic equation of (1) have minimum value of distance, it can be shown that the infinite series in (9) always converges during any finite interval,

Table 1: Comparison of the solutions to ODEs and DDEs. The solution to DDEs in terms of the Lambert W function shows a formal semblance to that of ODEs [26, 29]

ODEs	DDEs
Scalar Case	
$\dot{x}(t) = ax(t) + bu(t), \quad t > 0$	$\dot{x}(t) = ax(t) + a_d x(t-h) + bu(t), \quad t > 0$
$x(t) = x_0, \quad t = 0$	$x(t) = g(t), \quad t \in [-h, 0)$; $x(t) = x_0, \quad t = 0$
$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\xi)} bu(\xi) d\xi$	$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{S_k (t-\xi)} C_k^N bu(\xi) d\xi$ where, $S_k = \frac{1}{h} W_k(a_d h e^{-ah}) + a$
Matrix-Vector Case	
$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad t > 0$	$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t), \quad t > 0$
$\mathbf{x}(t) = \mathbf{x}_0, \quad t = 0$	$\mathbf{x}(t) = \mathbf{g}(t), \quad t \in [-h, 0)$; $\mathbf{x}(t) = \mathbf{x}_0, \quad t = 0$
$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\xi)} \mathbf{B}\mathbf{u}(\xi) d\xi$	$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t} \mathbf{C}_k^I + \int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k (t-\xi)} \mathbf{C}_k^N \mathbf{B}\mathbf{u}(\xi) d\xi$ where, $\mathbf{S}_k = \frac{1}{h} \mathbf{W}_k(\mathbf{A_d} h \mathbf{Q}_k) + \mathbf{A}$

or converges at any time when all the roots have negative real parts [2]. The solution to DDEs in terms of the matrix Lambert W function, and its analogy to that of ODEs are summarized in Table 1.

From the Laplace transform of the system (1), the solution to (1) in the Laplace domain [27] is

$$\mathbf{X}(s) = \underbrace{\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1} \left\{\mathbf{x}_0 + \mathbf{A_d}\mathbf{G}(s)e^{-sh}\right\}}_{\text{free}} + \underbrace{\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)^{-1} \left\{\mathbf{B}\mathbf{U}(s)\right\}}_{\text{forced}} \tag{10}$$

Comparing the solution in the Laplace domain in (10) with the solution in the time domain in terms of the matrix Lambert W function in (9) yields [27],

$$\mathfrak{L}^{-1}\left\{\left(s\mathbf{I} - \mathbf{A} - \mathbf{A_d}e^{-sh}\right)\right\}^{-1} = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^N \tag{11}$$

4 Stability of Time Delayed Systems

The solution form in equations (9) with (4) reveals that the stability condition of the systems of (1) depend on the eigenvalues of the matrix $\mathbf{S}_k$, and thus also on the matrix $e^{\mathbf{S}_k}$. A time delayed system characterized by (9) is *asymptotically stable* in the sense of Lyapunov if and only if all the eigenvalue of $\mathbf{S}_k$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, have negative real parts or, equivalently, all the eigenvalue of $\mathbf{S}_k$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, lie within the unit circle. However, computing the matrix $\mathbf{S}_k$ or $e^{\mathbf{S}_k}$ for an infinite number of branches, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$ is not practical. We have observed, in numerous examples, that the eigenvalues obtained using the principal branch ($k = 0$) are closest to the imaginary axis thus determine stability of the system. That is, [25]

$$\max\{\Re\{\text{eigenvalues for } k = 0\}\} \geq \Re\{\text{all other eigenvalues}\} \tag{12}$$

For the scalar DDE case, it has been proven that the root obtained with the principal branch always determines stability [22] using monotonousness of the Lambert W function in given area as an image of circle. Such a proof can readily be extended to systems of DDEs where $\mathbf{A}$ and $\mathbf{A_d}$ commute thus are simultaneously triangularizable. Even though such a proof is not available in the case of general matrix-vector DDEs, we have observed such behavior in all the examples we have considered. That is, the eigenvalues obtained using the principal branch are closest to the imaginary axis, and their real parts determine stability. With this useful observation, the approach introduced in previous section can be applied to solve a problem in a machining process.

Example: regenerative chatter in the turning process[25]: The linearized chatter equation can be expressed in state space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-T) \tag{13}$$

where $\mathbf{x} = \{x \quad \dot{x}\}^T$, ^{T} indicates transpose, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\left(1 + \frac{k_c}{k_m}\right)\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad \mathbf{A_d} = \begin{bmatrix} 0 & 0 \\ \frac{k_c}{k_m}\omega_n^2 & 0 \end{bmatrix}. \tag{14}$$

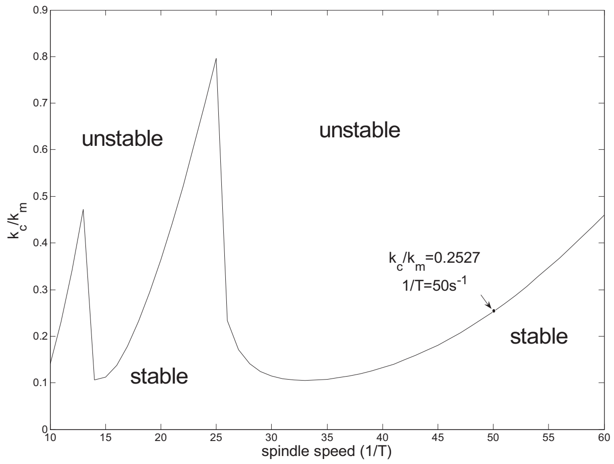

Figure 1: Stability lobes for the chatter equation [25]

Here, $\mathbf{A}$ and $\mathbf{A_d}$ are the linearized coefficient matrices of the process model and are functions of the machine-tool and workpiece structural parameters such as natural frequency, damping, and stiffness, T, equivalent to h in (1), is multiplicative inverse of spindle speed. If we observe the roots obtained using the principal branch, we can find the critical point when the roots cross the imaginary axis. For example, when spindle speed $(1/T) = 50$, $\omega_n = 150(\text{sec}^{-2})$ and $\zeta = 0.05$, the critical ratio of gains (k_c/k_m) is 0.2527. This value agrees with the result obtained by the Lyapunov method [17], the Nyquist criterion and the computational method of [4]. The stability lobes by this method are depicted in Figure 1 with respect to the spindle speed (*rps*, revolution per second).

In obtaining the result shown in the Figure 1, we note that the roots obtained using the principal branch always determine stability. One of the advantages of using the matrix Lambert function over other methods appears to be the observation that the stability of the system can be obtained from only the principal branch among an infinite number of roots.

5 Controllability and Observability [28]

Controllability and observability of linear time delay systems has been studied, and various definitions and criteria have been presented since the 1960s. For a detailed review, refer to Lee & Olbrot (1981)[13], Malek-Zavarei et al. (1987)[17], Richard (2003)[21], and Yi et al. (2007)[28]. However, the lack of an analytical solution approach has limited the applicability of the existing theory. Using the solution form in terms of the matrix Lambert W function, algebraic conditions and Gramians for observability and controllability of DDEs were derived by Yi, Ulsoy, & Nelson [28] in a manner analogous to the well-known observability and controllability results for the ODE case. (See Table 2)

Definition 1: The system (1) is *point-wise controllable* if, for any given initial conditions $\mathbf{g}$ and $\mathbf{x}_0$, there exist a time t_1, $0 < t_1 < \infty$, and an admissible (i.e., measurable and bounded on a finite time interval) control segment $\mathbf{u}_{[0,t_1+h]}$ such that $\mathbf{x}(t; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}) = \mathbf{x}_1$ at $t = t_1$ for all $\mathbf{x}_1 \in \Re^n$ with initial conditions $\mathbf{g}(t)$, $\mathbf{x}_0$ and control $\mathbf{u}(t)$. If a system (1) is point-wise complete [5], there exist a control which results in *point-wise controllability* in finite time of the solution of (1) for any initial conditions $\mathbf{g}$ and $\mathbf{x}_0$, if and only if the controllability Gramian, $\mathcal{C}$, computed with the kernel has a full rank. Using the result in (11), the rank condition can be expressed as

$$\text{rank}[\mathcal{C}(0,t_1)] \equiv \text{rank}\left[\int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \times \mathbf{B}\mathbf{B}^T \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi\right] = n \tag{15}$$

Similarly, a rank criteria for observability was developed.

Definition 2: The system of (1) is *point-wise observable*, (or *observable*) in $[0, t_1]$ if the point $\mathbf{x}_0$ can be uniquely determined from a knowledge of $\mathbf{u}(t)$, $\mathbf{g}(t)$, and $\mathbf{y}(t)$. Then, if and only if the observability Gramian $\mathcal{O}(0, t_1)$ computed with the kernel defined in (11) satisfies the condition, i.e.,

$$\text{rank}[\mathcal{O}(0,t_1)] \equiv \text{rank}\left[\int_0^{t_1} \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T \times \mathbf{C}^T\mathbf{C} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi\right] = n \tag{16}$$

the system of (1) is *point-wise observable*. This condition was applied to determine whether a time-delayed system is controllable/observable with examples, and to derive other algebraic conditions for controllability and observability.

The results presented agree with those obtained using the already existing algebraic methods (e.g., [7],[8],[14], [19] and [24]). However, using the method of Gramians developed in [28], one can acquire more information. The controllability and observability Gramians in (15)-(16) indicate how controllable and observable the corresponding states are [11], while algebraic conditions tell only whether a system is controllable/observable or not. With the condition using Gramian concepts, one can determine how the change in some specific parameters of the system or the delay time, h, affect the controllability and observability of the system via the changes in the Gramians. For systems of ODEs, a balanced realization in which the controllability Gramian and observability Gramian of a system are equal and diagonal was introduced in [18] and its existence was investigated in [23]. By balancing a realization we mean that we symmetrize a certain input property (controllability) with a certain output property (observability) through a suitable choice of basis [23]. The significance of the method has been established because of its desirable properties such as good error bounds, computational simplicity, stability, and its close connection to robust multi-variable control

Table 2: Comparison of the Criteria for Controllability and Observability for the systems of ODEs and DDEs [28]

ODEs	DDEs
Controllability	Point-Wise Controllability
$\mathcal{C}_{(0,t_1)} \equiv \int_0^{t_1} e^{\mathbf{A}(t_1-\xi)}\mathbf{B}\mathbf{B}^T \left\{e^{\mathbf{A}(t_1-\xi)}\right\}^T d\xi$	$\mathcal{C}_{(0,t_1)} \equiv \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N \mathbf{B}\mathbf{B}^T \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi$
$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$	$\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}\mathbf{B}$
$e^{\mathbf{A}(t-0)}\mathbf{B}$	$\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{B}$
Observability	Point-Wise Observability
$\mathcal{O}_{(0,t_1)} \equiv \int_0^{t_1} \left\{e^{\mathbf{A}(\xi-0)}\right\}^T \mathbf{C}^T\mathbf{C}e^{\mathbf{A}(\xi-0)} d\xi$	$\mathcal{O}_{(0,t_1)} \equiv \int_0^{t_1} \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T \mathbf{C}^T\mathbf{C} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi$
$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}$	$\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A_d}e^{-sh}\right)^{-1}$
$\mathbf{C}e^{\mathbf{A}(t-0)}$	$\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N$

[15]. However, for systems of DDEs, results on balanced realizations have been lacking. Using the Gramians defined in (15) and (16), the concept of the balanced realization can be extended to systems of DDEs [28].

6 Concluding Remarks and Future Research

Recent results on the solution of delay differential equation (DDEs) using the matrix Lambert W function and its applications are summarized in this paper. The main advantage of this method is that the solution form in terms of the matrix Lambert W function is similar to that of ODEs. Hence, the concept of the state transition matrix in ODEs can be generalized to DDEs using the matrix Lambert W function. This suggests that some analyses used for systems of ODEs, based upon the concept of the state transition matrix, can potentially be extended to systems of DDEs. For example, concepts of observability, controllability with their Gramians [28], controller design via eigenvalue assignment for systems of DDEs [30] and state estimator design are tractable and are being studied by the authors. Also, stability of time invariant linear DDEs can be determined using the approach and extension to stability of time-varying DDEs is also currently under investigation.

In this survey paper we have also highlighted several outstanding research problems associated with the solution of systems of DDEs using the matrix Lambert W function. First, conditions for existence and uniqueness of a solution $\mathbf{Q}_k$ to Eq. (5) are needed. Second, a proof that the stability of the systems in (1) can be determined by the principal ($k = 0$) branch is lacking. We hope that researchers in the DDE community will be interested in those problems.

7 Acknowledgements

This work was supported by NSF Grant #0555765.

8 References

[1] F. M. Asl and A. G. Ulsoy, Analysis of a system of linear delay differential equations, *J. Dyn. Syst. Meas. Control*, **125**, (2003) 215-223.

[2] R. E. Bellman and K. L. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.

[3] C. T. Chen, Linear System Theory and Design, Harcourt Brace Jovanovich, Florida, 1984.

[4] S. G. Chen, A. G. Ulsoy and Y. Koren , Analysis of a System of Linear Delay Differential Equations, *J. Dyn. Syst. Meas. Control*, **119**, (1997) 457-460.

[5] A. K. Choudhury, Necessary and Sufficient Conditions of Pointwise Completeness of Linear Time-Invariant Delay-Differential Systems, *Int. J. Control*, **1**, (1972) 1083-1100.

[6] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth, On the Lambert W function, *Adv. Comput. Math.*, **5**, (1996) 329-359.

[7] M. C. Delfour and S. K. Mitter, Controllability, observability and optimal feedback control of affine hereditary differential systems, *SIAM J. Control*, **10**, (1972) 298-328.

[8] M. Fliess and H. Mounier, Interpretation and comparison of various types of delay system controllabilities, in *IFAC Conference, System Structure and Control*, (1995) 330-335.

[9] H. Gorecki, S. Fuksa, P. Grabowski and A. Korytowski, Analysis and Synthesis of Time Delay Systems, John Wiley and Sons, New York, 1989.

[10] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[11] S. Holford and P. Agathoklis, Use of model reduction techniques for designing IIR filters with linear phase in the passband, *IEEE Trans. Signal Process.*, **44**, (1996) 2396-2404.

[12] J. Lam, Model- reduction of delay systems using Padé approximations, *Int. J. Control*, **57** ,(1993) 377-391.

[13] E. B. Lee and A. Olbrot, Observability and related structural results for linear hereditary systems, *Int. J. Control*, **34**, (1981) 1061-1078.

[14] T. N. Lee and S. Dianat, Stability of time-delay systems, *IEEE Trans. Aut. Cont.*, **26**, (1981) 951-953,

[15] W. S. Lu, E. B. Lee and Q. T. Zhang, Balanced approximation of two-dimensional and delay-differential systems, *Int. J. Control*, **46**, (1987) 2199-2218.

[16] N. Macdonald, J. E. Marshall and K. Walton, "Direct stability boundary method for distributed systems with discrete delay," Int. J. Control, **47**, (1988) 711-716.

[17] M. Malek-Zavarei and M. Jamshidi, Time-Delay Systems : Analysis, Optimization, and Applications, Elsevier Science Pub., New York, 1987.

[18] B. C. Moore, Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction, *IEEE Trans. Aut. Cont.*, **26**, (1981) 17-32, .

[19] A. W. Olbrot, On controllability of linear systems with time delays in control, *IEEE Trans. Aut. Cont.*, **17**, (1972) 664-666 .

[20] M. C. Pease, Methods of Matrix Algebra, Academic Press, New York, 1965.

[21] J. P. Richard, Time-delay systems: an overview of some recent advaces and open problems, *Automatica*, **39**, (2003) 1667-1964.

[22] H. Shinozaki and T. Mori, Robust stability analysis of linear time-delay systems by Lambert W function: Some extreme point results, *Automatica*, **42**, (2006) 1791-1799, .

[23] E. I. Verriest and T. Kailath, On generalized balanced realizations, *IEEE Trans. Aut. Cont.*, **28**, (1983) 833-844.

[24] L. Weiss, An algebraic criterion for controllability of linear systems with time delay, *IEEE Trans. Aut. Cont.*, **15**, (1970) 443-444, .

[25] S. Yi, P. W. Nelson, A. G. Ulsoy, Chatter Stability Analysis Using the Matrix Lambert Function and Bifurcation Analysis, *Math. Biosci. Eng*, **4**, (2007) 355-368.

[26] S. Yi and A. G. Ulsoy, Solution of a system of linear delay differential equations using the matrix Lambert function, *Proc. 25th American Control Conference*, Minneapolis, MN, Jun. (2006) 2433-2438.

[27] S. Yi, A. G. Ulsoy and P. W. Nelson, Solution of systems of linear delay differential equations via Laplace transformation, *Proc. 45th IEEE Conf. on Decision and Control*, San Diego, CA, (2006) 2535-2540.

[28] S. Yi, A. G. Ulsoy and P. W. Nelson, Controllability and Observability of Systems of Linear Delay Differential Equations via the Matrix Lambert Function, *IEEE Trans. Aut. Cont.* (submitted).

[29] S. Yi, A. G. Ulsoy and P. W. Nelson, Analysis of Systems of Linear Delay Differential Equations Using the Matrix Lambert Function and the Laplace Transformation, *Automatica* (in press).

[30] S. Yi, P. W. Nelson and A. G. Ulsoy, Feedback Control via Eigenvalue Assignment for Time Delayed Systems Using the Lambert W Function, *J. Vib. Control* (submitted).

email:syjo@umich.edu