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Preservation of controllability under sampling

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Preservation of controllability under sampling

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In this paper we consider sampled-data systems realized by linear time-invariant continuous systems whose inputs are piecewise constant. The problem dealt with is the following: assuming that the continuous system is controllable, what are the necessary and sufficient conditions for the sampled-data system to be controllable? We derive a necessary condition for the multidimensional case and show that a weaker condition than that required by Kalman *et al.* (1962) is needed for sufficiency.

1. Introduction

An important class of discrete control systems is that of sampled-data systems which are realized by inserting a digital component into the control loop. In other words, we consider a dynamical continuous system whose input is piecewise constant. While the controller can change only at given instances of time, the sampling instants, the state vector, being a solution of the system's differential equations, changes continuously.

Controllability of discrete-time systems has been treated rather extensively in the literature (e.g. Kalman *et al.* 1962, Sarachik and Kreindler 1965, Katkovnik and Poluektov 1966, Grammaticos 1969, Panda 1970). However, the problem of controllability of sampled-data systems received very little attention. Kalman *et al.* (1962) found a sufficient condition for controllability of sampled-data systems which is also necessary if the dimension of the control space is one. The same results were repeated by Katkovnik and Poluektov (1966) and Chen (1970).

In this paper we deal with the following problem: assuming that the continuous system is controllable, what are the criteria for the sampled-data system to be controllable? We give a necessary condition for controllability of sampled-data systems with *any number of controls* and show that a weaker condition than Kalman's is required for sufficiency.

Most of the theorems and lemmas are stated without proof for the sake of brevity. A complete version of this paper is available from the authors (Bar-Ness and Langholz 1974).

2. Statement of the problem

Consider the linear time-invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{C}\mathbf{u}(t) \tag{1}$$

where $\mathbf{x}$ is an n-vector, the state vector; $\mathbf{u}$ is an r-vector, the control vector; $\mathbf{A}$ and $\mathbf{C}$ are constant matrices of appropriate dimensions.

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Assume that $\mathbf{u}(t)=\mathbf{u}(t_i)$, $t_i \leqslant t \leqslant t_{i+1}$, where t_i is the ith sampling time, and let $T \triangleq t_{i+1}-t_i$ $\forall i$, $T>0$, denote the sampling period. Solving eqn. (1) yields

$$\mathbf{x}(t)=\mathbf{\Phi}(t, t_0)\mathbf{x}(t_0)+\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{C}\mathbf{u}(\tau)\, d\tau \tag{2}$$

where $\mathbf{\Phi}(t, t_0)$ is the $(n \times n)$ fundamental matrix associated with eqn. (1). At the sampling times, we get from eqn. (2)

$$\mathbf{x}(t_{i+1})=\mathbf{\Phi}(t_{i+1}, t_i)\mathbf{x}(t_i)+\int_{t_i}^{t_{i+1}} \mathbf{\Phi}(t_{i+1}, \tau)\mathbf{C}d\tau\mathbf{u}(t_i) \tag{3}$$

Since we are dealing with the time-invariant case, $\mathbf{\Phi}(t_{i+1}, t_i)=\mathbf{\Phi}(t_{i+1}-t_i)=\mathbf{\Phi}(T)$, hence, rewriting eqn. (3) in a difference equation form yields

$$\mathbf{x}_{i+1}=\mathbf{\Phi}(T)\mathbf{x}_i+\mathbf{B}\mathbf{u}_i \tag{4}$$

where $\mathbf{u}_i=\mathbf{u}(t_i)$, $\mathbf{x}_i=\mathbf{x}(t_i)$ $\forall i$, and

$$\mathbf{\Phi}(T)=\mathbf{e}^{\mathbf{A}T} \tag{5}$$

$$\mathbf{B}=\int_0^T \mathbf{e}^{\mathbf{A}\tau}\mathbf{C}d\tau \tag{6}$$

Definition

Let $\mathbf{P}$ and $\mathbf{Q}$ be $(n \times n)$ and $(n \times r)$ constant matrices respectively. Then, the pair $(\mathbf{P}, \mathbf{Q})$ is called a controllable pair if $\rho[\mathbf{Q} \vdots \mathbf{PQ} \vdots \mathbf{P}^2\mathbf{Q} \vdots \ldots \vdots \mathbf{P}^{\nu-1}\mathbf{Q}]=n$, where ν is the degree of the minimal polynomial of $\mathbf{P}$.

The $(n \times r\nu)$ matrix $[\mathbf{Q} \vdots \mathbf{PQ} \vdots \mathbf{P}^2\mathbf{Q} \vdots \ldots \vdots \mathbf{P}^{\nu-1}\mathbf{Q}]$ is called the controllability matrix.

Remark. By Definition 1, for the pair $(\mathbf{P}, \mathbf{Q})$ to be controllable, the inequality $r\nu \geqslant n$ must be satisfied.

Assuming now that the linear time-invariant system of eqn. (1) is controllable, the problem of finding criteria for the sampled-data system of eqn. (4) to be controllable is solved in § 4. Some results necessary for the solution are outlined in the following section.

3. Mathematical preliminaries

Lemma 1

Let $\mathbf{G}$ be any $(n \times n)$ non-singular constant matrix. Then, the pair $(\mathbf{P}, \mathbf{Q})$ is controllable iff the pair $(\mathbf{G}^{-1}\mathbf{PG}, \mathbf{G}^{-1}\mathbf{Q})$ is controllable.

Remark. Following Lemma 1, it can be assumed, without loss of generality, that $\mathbf{P}$ is in a Jordan canonical form.

Lemma 2

Consider the linear system of eqn. (1) and assume that the sampling time $T \neq 2\pi\beta/\lambda_i$, $\beta = \pm 1, \pm 2, \ldots$, whenever $\operatorname{Re}\{\lambda_i\}=0$, where the λ_i's are the eigenvalues of $\mathbf{A}$. Then the pair $(\mathbf{A}, \mathbf{B})$, where $\mathbf{B}$ is given by eqn. (6), is controllable iff the pair $(\mathbf{A}, \mathbf{C})$ is controllable.

For the matrix function $\mathbf{e}^{\mathbf{A}t}$ we have (Zadeh and Desoer 1963)

$$\mathbf{e}^{\mathbf{A}t}=\sum_{k=0}^{\nu-1} \alpha_k(t)\mathbf{A}^k \tag{7}$$

where ν is the degree of the minimal polynomial of $\mathbf{A}$. The algebraic equations which determine the α_k's are of the form

$$\mathbf{M}\boldsymbol{\alpha}(t)=\boldsymbol{\psi}(t) \tag{8}$$

where $\mathbf{M}$ is a non-singular matrix, $\boldsymbol{\alpha}(t)=[\alpha_0(t)\ \alpha_1(t)\ \dots\ \alpha_{\nu-1}(t)]'$ and $\boldsymbol{\psi}(t)$ is a vector whose elements are of the form $t^l \exp(\lambda_k t)$, $k=1, 2, \dots, \delta$; $l=0, 1, 2, \dots, m_k-1$, with the λ_k's and m_k's being the distinct eigenvalues of $\mathbf{A}$ and their corresponding multiplicities. Following Zadeh and Desoer (1963), the components $\alpha_k(t)$ of $\boldsymbol{\alpha}(t)$ form a set of time functions that are linearly independent over any interval of positive length.

4. Criteria for controllability

Kalman *et al.* (1962) gave a sufficient condition for controllability of a sampled-data system which is also necessary if the dimension of the control space is one. The same results were again stated by Katkovnik and Poluektov (1966) and Chen (1970). In this section we derive a necessary condition for controllability of a sampled-data system with any number of controls (Theorem 2) and show that a weaker condition than Kalman's is required for sufficiency (Theorem 1).

Define the $(\nu\times\nu)$ matrix $\mathbf{W}$:

$$\mathbf{W}=\begin{bmatrix} \alpha_0(0) & \alpha_1(0) & \dots & \alpha_{\nu-1}(0) \\ \alpha_0(T) & \alpha_1(T) & \dots & \alpha_{\nu-1}(T) \\ \vdots & \vdots & & \vdots \\ \alpha_0((\nu-1)T) & \alpha_1((\nu-1)T) & \dots & \alpha_{\nu-1}((\nu-1)T) \end{bmatrix} \tag{9}$$

and the $(\nu\times\nu)$ matrix $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega}=\left[\begin{array}{cccccc}
1 & \exp(\lambda_1 T) & \exp(2\lambda_1 T) & \exp(3\lambda_1 T) & \dots & \exp[(\nu-1)\lambda_1 T] \\
0 & \exp(\lambda_1 T) & 2\exp(2\lambda_1 T) & 3\exp(3\lambda_1 T) & \dots & (\nu-1)\exp[(\nu-1)\lambda_1 T] \\
\vdots & \vdots & \vdots & \vdots & & \\
0 & \exp(\lambda_1 T) & 2^{m_1-1}\exp(2\lambda_1 T) & 3^{m_1-1}\exp(3\lambda_1 T) & \dots & (\nu-1)^{m_1-1}\exp[(\nu-1)\lambda_1 T] \\
\hdashline
1 & \exp(\lambda_2 T) & \exp(2\lambda_2 T) & \exp(3\lambda_2 T) & \dots & \exp[(\nu-1)\lambda_2 T] \\
0 & \exp(\lambda_2 T) & 2\exp(2\lambda_2 T) & 3\exp(3\lambda_2 T) & \dots & (\nu-1)\exp[(\nu-1)\lambda_2 T] \\
\vdots & \vdots & \vdots & \vdots & & \\
0 & \exp(\lambda_2 T) & 2^{m_2-1}\exp(2\lambda_2 T) & 3^{m_2-1}\exp(3\lambda_2 T) & \dots & (\nu-1)^{m_2-1}\exp[(\nu-1)\lambda_2 T] \\
\hdashline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hdashline
1 & \exp(\lambda_\delta T) & \exp(2\lambda_\delta T) & \exp(3\lambda_\delta T) & \dots & \exp[(\nu-1)\lambda_\delta T] \\
0 & \exp(\lambda_\delta T) & 2\exp(2\lambda_\delta T) & 3\exp(3\lambda_\delta T) & \dots & (\nu-1)\exp[(\nu-1)\lambda_\delta T] \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \exp(\lambda_\delta T) & 2^{m_\delta-1}\exp(2\lambda_\delta T) & 3^{m_\delta-1}\exp(3\lambda_\delta T) & \dots & (\nu-1)^{m_\delta-1}\exp[(\nu-1)\lambda_\delta T]
\end{array}\right] \tag{10}$$

where $T > 0$, λ_k and m_k, $k = 1, 2, \ldots, \delta$, are the distinct eigenvalues of $\mathbf{A}$ and their corresponding minimal polynomial multiplicities, respectively. Following these definitions, we have

Lemma 3

Let $\mathbf{W}$ and $\mathbf{\Omega}$ be the matrices defined by eqns. (9) and (10) respectively. Then, $\rho(\mathbf{W}) = \rho(\mathbf{\Omega})$.

Theorem 1

Consider the linear time-invariant system of eqn. (1) and assume that (*a*) the pair $(\mathbf{A}, \mathbf{C})$ is controllable, and (*b*) $T \neq 2\pi\beta/\lambda_i$, $\beta = \pm 1, \pm 2, \ldots$, for $\mathrm{Re}\,\{\lambda_i\} = 0$, where the λ_1's are the eigenvalues of $\mathbf{A}$. Then, the pair $(\mathbf{\Phi}, \mathbf{B})$ is controllable if the matrix $\mathbf{\Omega}$, defined by eqn. (10), is non-singular. This condition is also necessary if the dimension of the control space is one.

For proof of this theorem see the Appendix.

Example 1

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The degree of the minimal polynomial is $\nu = n = 3$. Since $\mathrm{Re}\,\{\lambda_i\} \neq 0$. $i = 1, 2$, condition (*b*) of Theorem 1 is satisfied for any $T > 0$. Using eqn. (10) we have

$$\mathbf{\Omega} = \begin{bmatrix} 1 & \exp(T) & \exp(2T) \\ 0 & \exp(T) & 2\exp(2T) \\ 1 & \exp(2T) & \exp(4T) \end{bmatrix}$$

and it can be readily shown that $\mathbf{\Omega}$ is non-singular for any $T > 0$. Hence, the sampled-data system $(\mathbf{\Phi}, \mathbf{B})$ is controllable for any sampling time $T > 0$, if the pair $(\mathbf{A}, \mathbf{C})$ is controllable.

Example 2

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The degree of the minimal polynomial is $\nu = 2 < n = 3$. Since $\mathrm{Re}\,\{\lambda\} \neq 0$, condition (*b*) of Theorem 1 is satisfied for any $T > 0$. Using eqn. (10) we have

$$\mathbf{\Omega} = \begin{bmatrix} 1 & \exp(T) \\ 0 & \exp(T) \end{bmatrix}$$

and it is clear that $\mathbf{\Omega}$ is non-singular for any $T > 0$. Hence, if the pair $(\mathbf{A}, \mathbf{C})$ is controllable, so is the pair $(\mathbf{\Phi}, \mathbf{B})$ for any $T > 0$. (Note: in this example, the pair $(\mathbf{A}, \mathbf{C})$ is controllable only if $r > 1$.)

Remark. It is possible to prove that $\rho(\mathbf{\Omega}) = \nu$ iff $\mathrm{Im}\,\{\lambda_i - \lambda_j\} \neq 2\pi\beta/T$, $\beta = \pm 1, \pm 2, \ldots$, for $\mathrm{Re}\,\{\lambda_i - \lambda_j\} = 0$, where λ_i and λ_j are the distinct eigenvalues of $\mathbf{A}$. Now, assume that all the Jordan blocks of $\mathbf{A}$ are distinct, then $\nu = n$, $\mathbf{\Omega}$ is an $(n \times n)$ matrix and $\rho(\mathbf{\Omega}) = n$. Note that this condition is precisely the condition of Kalman *et al.* (1962). Hence, the sufficient condition for controllability (Theorem 1) is weaker than that required by Kalman.

Theorem 2

Consider the linear time-invariant system of eqn. (1) and assume that (*a*) the pair $(\mathbf{A}, \mathbf{C})$ is controllable, and (*b*) $T \neq 2\pi\beta/\lambda_i$, $\beta = \pm 1, \pm 2, \ldots$, whenever $\mathrm{Re}\{\lambda_i\} = 0$, where the λ_i's are the distinct eigenvalues of $\mathbf{A}$. Then, for the pair $(\mathbf{\Phi}, \mathbf{B})$ to be controllable, it is necessary that $\rho(\mathbf{\Omega}) \geqq [n/r]$†, where n and r are the dimensions of the state and control spaces respectively.

For proof of this theorem see the Appendix.

Example 3

Let

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $\alpha \triangleq 1 + 2j$, $\beta \triangleq 1 - 2j$. It can be readily shown that the pair $(\mathbf{A}, \mathbf{C})$ is controllable and since $\mathrm{Re}\{\lambda_i\} \neq 0$, $i = 1, 2, 3$, conditions (*a*) and (*b*) of Theorem 2 are both satisfied. The degree of the minimal polynomial of $\mathbf{A}$ is $\nu = n = 3$. Using eqn. (10) we have

$$\mathbf{\Omega} = \begin{bmatrix} 1 & \exp(-2T) & \exp(-4T) \\ 1 & \exp(-\alpha T) & \exp(-2\alpha T) \\ 1 & \exp(-\beta T) & \exp(-2\beta T) \end{bmatrix}$$

and it can be shown that if, for example, $T = \frac{1}{2}\pi\gamma$, $\gamma = \pm 1, \pm 2, \ldots$, $\mathbf{\Omega}$ is singular $(\rho(\mathbf{\Omega}) = 2)$. Hence, by Theorem 1, the pair $(\mathbf{\Phi}, \mathbf{B})$ is not controllable.

Consider now the case when $r = 2$ and let

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Again, conditions (*a*) and (*b*) of Theorem 2 are satisfied and we intend to show that now the sampled-data system is controllable for $T = \frac{1}{2}\pi\gamma$, $\gamma = \pm 1, \pm 2, \ldots$. Using direct computation, we have

$$\mathbf{\Phi} = \mathbf{e}^{\mathbf{A}T} = \begin{bmatrix} \exp(-2T) & 0 & 0 \\ 0 & \exp(-\alpha T) & 0 \\ 0 & 0 & \exp(-\beta T) \end{bmatrix}$$

and

$$\mathbf{B} = \int_0^T \mathbf{e}^{\mathbf{A}\tau}\, d\tau\, \mathbf{C} = \begin{bmatrix} \frac{1}{2}[1 - \exp(-2T)] & \frac{1}{2}[1 - \exp(-2T)] \\ (1/\alpha)[1 - \exp(-\alpha T)] & 0 \\ (1/\beta)[1 - \exp(-\beta T)] & (1/\beta)[1 - \exp(-\beta T)] \end{bmatrix}$$

† $[a]$ denotes the largest integer which is smaller than a.

Since the degree of the minimal polynomial of $\mathbf{A}$ is $\nu = 3$, we consider

$$\mathbf{D} = [\mathbf{B} \vdots \mathbf{\Phi B} \vdots \mathbf{\Phi}^2 \mathbf{B}]$$

$$= \left[\begin{array}{ccc}
\frac{1}{2}[1-\exp(-2T)] & \frac{1}{\alpha}[1-\exp(-\alpha T)] & \frac{1}{\beta}[1-\exp(-\beta T)] \\
\frac{1}{2}[1-\exp(-2T)] & 0 & \frac{1}{\beta}[1-\exp(-\beta T)] \\
\hdashline
\frac{1}{2}[1-\exp(-2T)]\exp(-2T) & \frac{1}{\alpha}[1-\exp(-\alpha T)]\exp(-\alpha T) & \frac{1}{\beta}[1-\exp(-\beta T)]\exp(-\beta T) \\
\frac{1}{2}[1-\exp(-2T)]\exp(-2T) & 0 & \frac{1}{\beta}[1-\exp(-\beta T)]\exp(-\beta T) \\
\hdashline
\frac{1}{2}[1-\exp(-2T)]\exp(-4T) & \frac{1}{\alpha}[1-\exp(-\alpha T)]\exp(-2\alpha T) & \frac{1}{\beta}[1-\exp(-\beta T)]\exp(-2\beta T) \\
\frac{1}{2}[1-\exp(-2T)]\exp(-4T) & 0 & \frac{1}{\beta}[1-\exp(-\beta T)]\exp(-2\beta T)
\end{array}\right]$$

The 1st, 2nd and 4th columns of $\mathbf{D}$ are linearly independent for $T=\frac{1}{2}\pi\gamma$, $\gamma=\pm 1, \pm 2, \dots$ and, hence $\rho(\mathbf{D})=3$ and the pair $(\mathbf{\Phi}, \mathbf{B})$ is controllable. Now, recall that $\rho(\mathbf{\Omega})=2$ and since $[n/r]=1$, the inequality $\rho(\mathbf{\Omega}) \geqslant [n/r]$ is satisfied.

Remark. This example shows that, as a result of Theorem 2, it is possible to have a multidimensional controllable sampled-data system with conditions that are less restrictive than in the case of $r=1$.

Theorem 3

Consider the linear time-invariant system of eqn. (1) and assume that (*a*) $T \neq 2\pi\beta/\lambda_i$, $\beta=\pm 1, \pm 2, \dots$, whenever $\operatorname{Re}\{\lambda_i\}=0$, where the λ_i's are the eigenvalues of $\mathbf{A}$, and (*b*) the pair $(\mathbf{A}, \mathbf{C})$ is such that

$$\rho[\mathbf{C} \vdots \mathbf{AC} \vdots \mathbf{A}^2\mathbf{C} \vdots \dots \vdots \mathbf{A}^{l-1}\mathbf{C}]=n$$

where $l \leqslant n$. Then the pair $(\mathbf{\Phi}, \mathbf{B})$ is controllable if $\rho(\mathbf{\Omega})=l$.

Corollary 1

If $l=[n/r]$, then $\rho(\mathbf{\Omega})=l$ is a necessary and sufficient condition for the sampled-data system to be controllable.

Remark. Example 3 demonstrates Corollary 1.

Corollary 2

If $\mathbf{C}$ is an $(n \times n)$ non-singular matrix, then the sampled-data system is always controllable.

5. Conclusions

The problem of controllability of a sampled-data system realized by a linear time-invariant continuous system with piecewise constant input was considered. A necessary condition was derived for controllability of multidimensional sampled-data systems and it was shown that a weaker condition than that required by Kalman *et al.* (1962) is needed for sufficiency.

Appendix

A 1. *Proof of Theorem* 1

Assumptions (*a*) and (*b*) imply, by Lemma 2, that the pair $(\mathbf{A}, \mathbf{B})$ is controllable, i.e.

$$\rho(\mathbf{L}_\nu)=\rho(\mathbf{B} \vdots \mathbf{AB} \vdots \mathbf{A}^2\mathbf{B} \vdots \dots \vdots \mathbf{A}^{\nu-1}\mathbf{B})=n \tag{11}$$

We intend to prove that

$$\rho(\mathbf{D})=\rho(\mathbf{B} \vdots \mathbf{\Phi B} \vdots \mathbf{\Phi}^2\mathbf{B} \vdots \dots \vdots \mathbf{\Phi}^{\nu-1}\mathbf{B})=n$$

It should be noted (Gantmacher 1959, p. 158) that both $\mathbf{A}$ and $\mathbf{\Phi}$ have the same degree for their minimal polynomial.

Suppose that there exists a vector $\mathbf{x}$ such that $\mathbf{x}'\mathbf{D}=\mathbf{0}$. Substituting $\mathbf{e}^{\mathbf{A}T}$ for $\mathbf{\Phi}$ and using eqn. (7), $\mathbf{x}'\mathbf{D}=\mathbf{0}$ implies that

$$\alpha_0(jT)\mathbf{x}'\mathbf{B}_i+\alpha_1(jT)\mathbf{x}'(\mathbf{AB})_i+\alpha_2(jT)\mathbf{x}'(\mathbf{A}^2\mathbf{B})_i+\dots + \alpha_{\nu-1}(jT)\mathbf{x}'(\mathbf{A}^{\nu-1}\mathbf{B})_i=0 \tag{12}$$

$j=0, 1, \ldots, \nu-1$; $i=1, 2, \ldots, r$, where $\mathbf{B}_i$, $(\mathbf{AB})_i$, $(\mathbf{A}^2\mathbf{B})_i$, ..., $(\mathbf{A}^{\nu-1}\mathbf{B})_i$ are the ith columns of $\mathbf{B}$, $\mathbf{AB}$, $\mathbf{A}^2\mathbf{B}$, ..., $\mathbf{A}^{\nu-1}\mathbf{B}$ respectively. Hence, writing eqn. (12) in a matrix notation, we have

$$\begin{bmatrix} \mathbf{x}'\mathbf{B}_i \\ \mathbf{x}'(\mathbf{AB})_i \\ \mathbf{x}'(\mathbf{A}^2\mathbf{B})_i \\ \vdots \\ \mathbf{x}'(\mathbf{A}^{\nu-1}\mathbf{B})_i \end{bmatrix} \mathbf{W}=\mathbf{0}, \quad i=1, 2, \ldots, r \tag{13}$$

where $\mathbf{W}$ is given by eqn. (9). But, by Lemma 3, $\mathbf{\Omega}$ being non-singular implies that $\mathbf{W}$ is non-singular and, therefore, the only solution of eqn. (13) is $\mathbf{x}'\mathbf{B}_i=0$, $\mathbf{x}'(\mathbf{AB})_i=0$, ..., $\mathbf{x}'(\mathbf{A}^{\nu-1}\mathbf{B})_i=0$, $i=1, 2, \ldots, r$. Hence, $\mathbf{x}$ is orthogonal to all the columns of the matrix, $\mathbf{L}_\nu$ and it follows from eqn. (11) that $\mathbf{x}=\mathbf{0}$. Therefore, the null space of $\mathbf{D}$ is an empty set and hence, $\rho(\mathbf{D})=n$.

Now, let $r=1$, i.e. $\mathbf{B}$ is an n-dimensional vector and assume that $\rho(\mathbf{W})<n$. Hence, $\mathbf{Wy}=\mathbf{0}$ has at least one solution $\mathbf{y}\neq\mathbf{0}$. We can therefore solve the equation

$$\begin{bmatrix} \mathbf{x}'\mathbf{B} \\ \mathbf{x}'\mathbf{AB} \\ \mathbf{x}'\mathbf{A}^2\mathbf{B} \\ \vdots \\ \mathbf{x}'\mathbf{A}^{\nu-1}\mathbf{B} \end{bmatrix} = \mathbf{y}'$$

which can be rewritten as $\mathbf{L}_\nu\mathbf{x}=\mathbf{y}$. Since $\rho(\mathbf{L}_\nu)=n$, $\mathbf{y}\neq\mathbf{0}$ implies $\mathbf{x}\neq\mathbf{0}$. Hence, there exists a vector $\mathbf{x}\neq\mathbf{0}$ satisfying eqn. (13). Equivalently, we have $\mathbf{x}'\mathbf{D}=\mathbf{0}$ for $\mathbf{x}\neq\mathbf{0}$, i.e. $\rho(\mathbf{D})<n$ and the pair $(\mathbf{\Phi}, \mathbf{B})$ is not controllable. Therefore, the condition is also necessary when $r=1$.

A 2. *Proof of Theorem* 2

Assumptions (*a*) and (*b*) imply by Lemma 2 that

$$\rho(\mathbf{L}_\nu)=\rho[\mathbf{B} \vdots \mathbf{AB} \vdots \mathbf{A}^2\mathbf{B} \vdots \ldots \vdots \mathbf{A}^{\nu-1}\mathbf{B}]=n \tag{14}$$

Consider the matrix $\mathbf{D}=[\mathbf{B} \vdots \mathbf{\Phi B} \vdots \mathbf{\Phi}^2\mathbf{B} \vdots \ldots \vdots \mathbf{\Phi}^{\nu-1}\mathbf{B}]$. Substituting eqn. (7) for $\mathbf{\Phi}=\mathbf{e}^{T\mathbf{A}}$ yields

$$\begin{aligned} \mathbf{D} &= [\alpha_0(0)\mathbf{B}+\alpha_1(0)\mathbf{AB}+\ldots+\alpha_{\nu-1}(0)\mathbf{A}^{\nu-1}\mathbf{B} \vdots \alpha_0(T)\mathbf{B}+\alpha_1(T)\mathbf{AB}+\ldots \\ &\quad +\alpha_{\nu-1}(T)\mathbf{A}^{\nu-1}\mathbf{B} \vdots \ldots \vdots \alpha_0((\nu-1)T)\mathbf{B}+\alpha_1((\nu-1)T)\mathbf{AB}+\ldots \\ &\quad +\alpha_{\nu-1}((\nu-1)T)\mathbf{A}^{\nu-1}\mathbf{B}] \\ &= [\mathbf{B} \vdots \mathbf{AB} \vdots \ldots \vdots \mathbf{A}^{\nu-1}\mathbf{B}]\mathbf{R} \end{aligned}$$

where

$$\mathbf{R}=\begin{bmatrix} \alpha_0(0)\mathbf{I}_r & \alpha_0(T)\mathbf{I}_r & \ldots & \alpha_0((\nu-1)T)\mathbf{I}_r \\ \alpha_1(0)\mathbf{I}_r & \alpha_1(T)\mathbf{I}_r & \ldots & \alpha_1((\nu-1)T)\mathbf{I}_r \\ \vdots & \vdots & & \vdots \\ \alpha_{\nu-1}(0)\mathbf{I}_r & \alpha_{\nu-1}(T)\mathbf{I}_r & \ldots & \alpha_{\nu-1}((\nu-1)T)\mathbf{I}_r \end{bmatrix} \tag{15}$$

and $\mathbf{I}_r$ is the $(r \times r)$ identity matrix. Hence

$$\mathbf{D} = \mathbf{L}_\nu \mathbf{R} \tag{16}$$

where the dimensions of $\mathbf{D}$, $\mathbf{L}_\nu$ and $\mathbf{R}$ are $(n \times r\nu)$, $(n \times r\nu)$ and $(r\nu \times r\nu)$ respectively.

Applying Sylvester's inequality (Gantmacher, 1959, p. 66) to eqn. (16) we have

$$\rho(\mathbf{D}) \leqslant \min \{\rho(\mathbf{L}_\nu), \rho(\mathbf{R})\}$$

But, by eqn. (14), $\rho(\mathbf{L}_\nu) = n$ and therefore

$$\rho(\mathbf{D}) \leqslant \min \{n, \rho(\mathbf{R})\}$$

Now, if the sampled-data system is controllable, $\rho(\mathbf{D}) = n$ and we have

$$n \leqslant \rho(\mathbf{R}) \tag{17}$$

By elementary row and column manipulations, it can be readily shown that the matrix $\mathbf{R}$ (eqn. (15)) is transformed to a diagonal matrix $\mathbf{R}^*$ which has r blocks on its main diagonal, each one of these blocks being the $(\nu \times \nu)$ matrix $\mathbf{W}'$ defined by eqn. (9). Hence, $\rho(\mathbf{R}) = \rho(\mathbf{R}^*) = r\rho(\mathbf{W}')$, and, following Lemma 3, $\rho(\mathbf{R}) = r\rho(\mathbf{\Omega})$. Substituting this result into eqn. (17) yields $n \leqslant r\rho(\mathbf{\Omega})$, and since the rank of a matrix is an integer, we have

$$\rho(\mathbf{\Omega}) \geqslant [n/r]$$

which is necessary for the pair $(\mathbf{\Phi}, \mathbf{B})$ to be controllable.

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