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ON CONCEPTS OF CONTROLLABILITY FOR DETERMINISTIC AND STOCHASTIC SYSTEMS*

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Abstract. The new necessary and sufficient conditions, which are formulated in terms of convergence of a certain sequence of operators involving the resolvent of the negative of the controllability operator, are found for deterministic linear stationary control systems to be completely and approximately controllable, respectively. These conditions are applied to study the S-controllability (a property of attaining an arbitrarily small neighborhood of each point in the state space with a probability arbitrarily close to one) and C-controllability (the S-controllability fortified with some uniformity) of stochastic systems. It is shown that the S-controllability (the C-controllability) of a partially observable linear stationary control system with an additive Gaussian white noise disturbance on all the intervals $[0,T]$ for $T>0$ is equivalent to the approximate (complete) controllability of its deterministic part on all the intervals $[0,T]$ for $T>0$.

Key words. complete controllability, approximate controllability, stochastic controllability, deterministic linear system, partially observable linear system

AMS subject classifications. Primary, 93B, 93E; Secondary, 60G

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1. Introduction.

Theory of controllability originates from the famous work [1] done by Kalman. At present this theory is almost complete for deterministic linear control systems (see, for example, Curtain and Pritchard [2]; Curtain and Zwart [3]; Balakrishnan [4]; Bensoussan et al. [5]; Zabczyk [6]).

The natural extension of the complete and approximate controllability concepts to stochastic control systems is meaningless. In Bashirov [7] and Bashirov and Kerimov [8] these concepts were weakened and, for stochastic control systems, the concept of S-controllability was defined. Briefly, an S-controllable stochastic control system is a system attaining an arbitrarily small neighborhood of each point in the state space with a probability arbitrarily close to one. We also found it useful to define the concept of C-controllability for stochastic control systems as S-controllability fortified with some uniformity.

The main results of [7, 8] concern a partially observable linear stationary control system with an additive Gaussian white noise disturbance (the system (S)) and its deterministic part (the system (D)). From the results of [8] (the necessity part of Theorems 4 and 5(b)), it follows that if the system (S) is C-controllable (S-controllable) on the interval $[0,T]$, where $T>0$ is fixed, then the system (D) is completely (approximately) controllable on the same interval $[0,T]$. A sufficient condition of C-controllability (which is a sufficient condition of S-controllability as well) for the system (S) on the fixed interval $[0,T]$ is also found in [8]. This sufficient condition is based on Lemma 7 in [8] which is proved under the complete controllability condition of the system (D) on all the intervals $[0,t]$ for $0<t\le T$. Thus, more precisely than in [8], this sufficient condition is the complete controllability of the system (D) on all the intervals $[0,t]$ for $0<t\le T$.

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A discussion of an example presented in [8] leads us to expect that a weaker sufficient condition of S-controllability on the interval $[0,T]$ for the system (S) could exist as the approximate controllability of the system (D) on all the intervals $[0,t]$ for $0 < t \leq T$. This was conjectured in [7, 8], wherein this conjecture is settled positively.

Discussing the S- and C-controllability concepts, we found the new necessary and sufficient conditions for deterministic linear stationary control systems to be completely and approximately controllable. These conditions are formulated in terms of uniform and strong convergence of a certain sequence of operators involving the resolvent of the negative of the controllability operator and clearly distinguish complete and approximate controllabilities.

Studying sources about theory of controllability, we did not find the analogues of these conditions which prompted us to consider them as new. For convenience, we call the above-mentioned conditions the resolvent conditions of complete and approximate controllabilities.

A verification of the resolvent conditions for a concrete control system requires a computation of the respective resolvent and then studying the convergence of the above-mentioned sequence involving this resolvent. This is illustrated in the examples of controlled one-dimensional heat and wave equations. We expect that the resolvent conditions will play a significant role in theoretical investigations of theory of controllability because after a first application, they have allowed us to settle the conjecture mentioned above.

2. General notations. In this paper X and Y are real separable Hilbert spaces. R^k denotes the k-dimensional real Euclidean space. As usual, $R^1 = R$. The closure of the set D is denoted by $\overline{D}$. The space of all linear bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$. The brief notation $\mathcal{L}(X) = \mathcal{L}(X,X)$ is used as well. A^* denotes the adjoint of the operator A. The trace of the operator A is denoted by $\operatorname{tr} A$. If $A \in \mathcal{L}(X)$ is self-adjoint and $\langle h, Ah\rangle \geq 0$ (respectively, $\langle h, Ah\rangle \geq c\|h\|^2$, where $c = \text{const.} > 0$) for all $h \in X$, then we write $A \geq 0$ (respectively, $A > 0$), where $\langle \cdot , \cdot \rangle$ is an inner product and $\|\cdot\|$ is a norm. For $A \geq 0$, the square root of A is denoted by $A^{1/2}$. The symbol I denotes an identity operator. Zero operator, zero vector, and the number zero are denoted by 0; it is clear which is meant from the context.

It is always supposed that two time moments are given. The initial time moment is identified with zero and is fixed. The terminal moment is denoted by T $(T > 0)$ and is considered variable. The notation $\mathbf{T}$ is used for the finite time interval $[0,T]$. $L_2(\mathbf{T}, X)$ and $L_2(0,T;X)$ denote the space of equivalence classes of all functions from $\mathbf{T} = [0,T]$ to X that are Lebesgue measurable and square integrable with respect to the Lebesgue measure. As usual, we use the brief notation $L_2(0,T) = L_2(0,T;R)$. The notation $\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}$ is used for the triangular set over $\mathbf{T}$. $B_2(\Delta, \mathcal{L}(X,Y))$ denotes the class of all $\mathcal{L}(X,Y)$-valued functions on Δ that are strongly measurable and square integrable with respect to the Lebesgue measure on Δ (see, for example, [2, 3]).

All integrals of vector-valued functions are considered in the Bochner sense. For probability, expectation, and conditional expectation, the notations $\mathbf{P}$, $\mathbf{E}$, and $\mathbf{E}(\cdot\,|\cdot\,)$, respectively, are used. $\operatorname{cov}(x,y)$ is the covariance operator of the random variables x and y. The brief notation $\operatorname{cov} x = \operatorname{cov}(x,x)$ is used as well. The integrals of operator-valued functions (except stochastic integrals) are in the strong Bochner sense.

3. Main definitions. Consider a control system on $\mathbf{T}$. Let x_t^u be its (random or not) state value at time $t \in \mathbf{T}$ corresponding to the control u taken from the set of the admissible controls U. If the control system under consideration is stochastic, then by $\mathcal{F}^u$ we denote the smallest σ-algebra generated by the observations on the time interval $\mathbf{T}$ corresponding to the control u. Suppose that X is the state space. For $0 \le \varepsilon < \infty$ and for $0 \le p \le 1$, introduce the sets

$$D = \{x_T^u : u \in U\}, \tag{1}$$

$$S(\varepsilon, p) = \big\{h \in X : \exists u \in U \ \mathbf{P}\big(\|\mathbf{E}(x_T^u|\mathcal{F}^u) - h\|^2 > \varepsilon\big) \le 1 - p\big\}, \tag{2}$$

$$C(\varepsilon, p) = \big\{h \in X : \exists u \in U \ h = \mathbf{E}x_T^u,\ \mathbf{P}\big(\|\mathbf{E}(x_T^u|\mathcal{F}^u) - h\|^2 > \varepsilon\big) \le 1 - p\big\}. \tag{3}$$

DEFINITION 1. *A deterministic control system will be called*
(a) D^c-controllable on $\mathbf{T}$ *if* $D = X$;
(b) D^a-controllable on $\mathbf{T}$ *if* $\overline{D} = X$.

It is clear that the D^c- and D^a-controllabilities are the well-known complete and approximate controllabilities for deterministic control systems, respectively. Originally, the D^c-controllability was introduced in Kalman [1] as a concept for finite dimensional deterministic control systems, so the natural extension of this concept is too strong for many infinite dimensional control systems. Therefore, the D^a-controllability was introduced as a weakened version of the D^c-controllability. It is also clear that neither D^c- nor D^a-controllabilities can be a property of stochastic control systems, so there is a need to further weaken these concepts in order to extend them to stochastic control systems.

The following definition will be used as a step in discussing the main concepts of controllability for stochastic systems.

DEFINITION 2. *Given* $\varepsilon \ge 0$ *and* $0 \le p \le 1$, *a control system will be called*
(a) $S^c_{\varepsilon,p}$-controllable on $\mathbf{T}$ *if* $S(\varepsilon, p) = X$;
(b) $S^a_{\varepsilon,p}$-controllable on $\mathbf{T}$ *if* $\overline{S(\varepsilon, p)} = X$;
(c) $C^c_{\varepsilon,p}$-controllable on $\mathbf{T}$ *if* $C(\varepsilon, p) = X$;
(d) $C^a_{\varepsilon,p}$-controllable on $\mathbf{T}$ *if* $\overline{C(\varepsilon, p)} = X$;
(e) $S^0_{\varepsilon,p}$-controllable on $\mathbf{T}$ *if* $0 \in S(\varepsilon, p)$.

The geometric interpretation of the $S^c_{\varepsilon,p}$-controllability ($S^a_{\varepsilon,p}$-controllability) is as follows. If a control system with the initial state x_0 is $S^c_{\varepsilon,p}$-controllable ($S^a_{\varepsilon,p}$-controllable) on $\mathbf{T}$, then with probability not less than p it can pass from x_0 for the time T into the $\sqrt{\varepsilon}$-neighborhood of an arbitrary point in the state space (in a set that is dense in the state space). The geometric interpretation of the $C^c_{\varepsilon,p}$- and $C^a_{\varepsilon,p}$-controllabilities differs from the same of the $S^c_{\varepsilon,p}$- and $S^a_{\varepsilon,p}$-controllabilities since among the controls, with the help of which the $\sqrt{\varepsilon}$-neighborhood of any point h is achieved, there exists one with a property that the expectation of the target state, corresponding to this control, coincides with h. Obviously, a $C^c_{\varepsilon,p}$-controllable ($C^a_{\varepsilon,p}$-controllable) control system is $S^c_{\varepsilon,p}$-controllable ($S^a_{\varepsilon,p}$-controllable), but the converse is not true.

The smaller ε is and the larger p is for a control system, the more controllable it is; i.e., it is possible to hit into a smaller neighborhood with a higher probability. One can observe that all control systems are $S^c_{\varepsilon,p}$-, $S^a_{\varepsilon,p}$-, $C^c_{\varepsilon,p}$-, and $C^a_{\varepsilon,p}$-controllable on any interval with $\varepsilon \ge 0$ and $p = 0$ or $\varepsilon = \infty$ and $0 \le p \le 1$, if we admit ∞ as a value for ε. At the same time, it is clear that a D^c-controllable (D^a-controllable) deterministic system is $S^c_{0,1}$- and $C^c_{0,1}$-controllable ($S^a_{0,1}$- and $C^a_{0,1}$-controllable) with parameters $\varepsilon = 0$ and $p = 1$ since, for deterministic systems, $D = S(0, 1) = C(0, 1)$.

Also, each kind of controllability from Definition 2 with a smaller ε and a greater p implies the same kind of controllability with a greater ε and a smaller p.

Summarizing, we can give the following easy necessary and sufficient conditions for the D^c- and D^a-controllabilities.

PROPOSITION 1. *For a deterministic control system the following three conditions are equivalent*:

(a) D^c*-controllability on* $\mathbf{T}$;
(b) $S^c_{\varepsilon,p}$*-controllability on* $\mathbf{T}$ *for all* $\varepsilon \geq 0$ *and for all* $0 \leq p \leq 1$;
(c) $C^c_{\varepsilon,p}$*-controllability on* $\mathbf{T}$ *for all* $\varepsilon \geq 0$ *and for all* $0 \leq p \leq 1$.

PROPOSITION 2. *For a deterministic control system the following three conditions are equivalent*:

(a) D^a*-controllability on* $\mathbf{T}$;
(b) $S^a_{\varepsilon,p}$*-controllability on* $\mathbf{T}$ *for all* $\varepsilon \geq 0$ *and for all* $0 \leq p \leq 1$;
(c) $C^a_{\varepsilon,p}$*-controllability on* $\mathbf{T}$ *for all* $\varepsilon \geq 0$ *and for all* $0 \leq p \leq 1$.

Excepting the limit values $\varepsilon = 0$ and $p = 1$ from the above-mentioned necessary and sufficient conditions for the D^c- and D^a-controllabilities, one can obtain the weakened versions of these concepts. For a moment call a given stochastic system

(a) S^c*-controllable on* $\mathbf{T}$ if it is $S^c_{\varepsilon,p}$-controllable on $\mathbf{T}$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$;
(b) S^a*-controllable on* $\mathbf{T}$ if it is $S^a_{\varepsilon,p}$-controllable on $\mathbf{T}$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$;
(c) C^c*-controllable on* $\mathbf{T}$ if it is $C^c_{\varepsilon,p}$-controllable on $\mathbf{T}$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$;
(d) C^a*-controllable on* $\mathbf{T}$ if it is $C^a_{\varepsilon,p}$-controllable on $\mathbf{T}$ for all $\varepsilon > 0$ and for all $0 \leq p < 1$.

In [7, 8] it is shown that the concepts of S^c- and S^a-controllabilities are equivalent. It will also be shown that for partially observable linear stationary control systems with additive Gaussian white noise disturbance the C^a-controllability on all the intervals $[0, T]$ with $T > 0$ is equivalent to the S^c-and S^a-controllabilities on all the intervals $[0, T]$ with $T > 0$. Thus, we can define two basic and one additional concepts of controllability for stochastic systems.

DEFINITION 3. *A control system will be called*

(a) S-controllable on $\mathbf{T}$ *if it is* $S^c_{\varepsilon,p}$*-controllable on* $\mathbf{T}$ *or, equivalently,* $S^a_{\varepsilon,p}$*-controllable on* $\mathbf{T}$ *for all* $\varepsilon > 0$ *and for all* $0 \leq p < 1$;
(b) C-controllable on $\mathbf{T}$ *if it is* $C^c_{\varepsilon,p}$*-controllable on* $\mathbf{T}$ *for all* $\varepsilon > 0$ *and for all* $0 \leq p < 1$;
(c) S^0-controllable on $\mathbf{T}$ *if it is* $S^0_{\varepsilon,p}$*-controllable on* $\mathbf{T}$ *for all* $\varepsilon > 0$ *and for all* $0 \leq p < 1$.

Geometrically, the S-controllability can be interpreted as follows: an S-controllable on $\mathbf{T}$ control system can attain for the time T an arbitrarily small neighborhood of each point in the state space with a probability arbitrarily close to one. The C-controllability is the S-controllability fortified with some uniformity. The S^0-controllability is useful in discussing S- and C-controllabilities.

Finally, notice that the abbreviations D, S, C, c, and a in the previously introduced controllability concepts mean deterministic, stochastic, combined, complete, and approximate, respectively.

4. Preliminaries. In this paper it is always supposed that A is the infinitesimal generator of a strongly continuous semigroup $\mathcal{U}$, $B \in \mathcal{L}(Y, X)$, $C \in \mathcal{L}(X, R^k)$; x_0 is a Gaussian random variable with $\operatorname{cov} x_0 = P_0$; m and n are X- and R^k-valued Wiener processes, respectively; $n_0 = 0$, $m_0 = 0$, $\mathbf{E}n_t = 0$, $\mathbf{E}m_t = 0$, $\operatorname{cov} n_t = It$, $\operatorname{cov} m_t = Mt$, M is a nuclear operator on X; and x_0, n, m are mutually independent. Let $f \in L_2(\mathbf{T}, X)$ and consider the linear partially observable stochastic control system

$$\text{(4)} \qquad \begin{cases} dx_t^u = (Ax_t^u + Bu_t + f_t)dt + dm_t, \ 0 < t \le T, \ x_0^u = x_0, \\ d\xi_t^u = Cx_t^u dt + dn_t, \ 0 < t \le T, \ \xi_0^u = 0, \end{cases}$$

where x, u, and ξ are state, control, and observation processes. Under the set U of admissible controls we consider the set of all controls in the linear feedback form

$$u_t = \bar{u}_t + \int_0^t K_{t,s}\, d\xi_s^u,$$

where $K \in B_2\big(\Delta, \mathcal{L}\big(R^k, Y\big)\big)$ and $\bar{u} \in L_2(\mathbf{T}, Y)$.

To the system (4) one can associate two control systems. The first is the deterministic control system

$$\text{(5)} \qquad \frac{d}{dt} y_t^v = Ay_t^v + Bv_t + f_t, \ 0 < t \le T, \ y_0^v = y_0 = \mathbf{E}x_0,$$

where v is a control in $V = L_2(\mathbf{T}, Y)$. The second is the partially observable stochastic control system

$$\text{(6)} \qquad \begin{cases} dz_t^w = (Az_t^w + Bw_t)dt + dm_t, \ 0 < t \le T, \ z_0^w = z_0 = x_0 - \mathbf{E}x_0, \\ d\eta_t^w = Cz_t^w dt + dn_t, \ 0 < t \le T, \ \eta_0^w = 0, \end{cases}$$

where w is a control in W consisting of all controls in the linear feedback form

$$w_t = \int_0^t K_{t,s}\, d\eta_s^w,$$

where $K \in B_2\big(\Delta, \mathcal{L}\big(R^k, Y\big)\big)$.

Note that solutions of the equations in (4), (5), and (6) are meant in the mild sense, i.e.,

$$\begin{aligned} x_t^u &= \mathcal{U}_t x_0 + \int_0^t \mathcal{U}_{t-s}(Bu_s + f_s)\, ds + \int_0^t \mathcal{U}_{t-s}\, dm_s, \ 0 \le t \le T, \\ y_t^v &= \mathcal{U}_t y_0 + \int_0^t \mathcal{U}_{t-s}(Bv_s + f_s)\, ds, \ 0 \le t \le T, \\ z_t^w &= \mathcal{U}_t z_0 + \int_0^t \mathcal{U}_{t-s} Bw_s\, ds + \int_0^t \mathcal{U}_{t-s}\, dm_s, \ 0 \le t \le T. \end{aligned}$$

Denote

$$\text{(7)} \qquad \Gamma_{T-t} = \int_t^T \mathcal{U}_{T-s} BB^* \mathcal{U}_{T-s}^*\, ds, \ 0 \le t \le T.$$

For $0 \le t < T$, the operator Γ_{T-t} is called a controllability operator. One can see that $\Gamma_{T-t} \ge 0$ and, hence, the resolvent $R(\lambda, -\Gamma_{T-t}) = (\lambda I + \Gamma_{T-t})^{-1}$ is well defined for all $\lambda > 0$ and for all $0 \le t \le T$. If $\Gamma_{T-t} > 0$, then $R(\lambda, -\Gamma_{T-t})$ is defined for $\lambda = 0$ as well.

We will use the following operator Riccati equations:

$$\text{(8)} \qquad \frac{d}{dt} Q_t + Q_t A + A^* Q_t - \lambda^{-1} Q_t BB^* Q_t = 0, \ 0 \le t < T, \ Q_T = I, \ \lambda > 0,$$

$$\frac{d}{dt}P_t - AP_t - P_tA^* - M + P_tC^*CP_t = 0,\ 0 < t \leq T,\ P_0 = \operatorname{cov} z_0. \tag{9}$$

LEMMA 1. *There exist the unique strongly continuous solutions* (*in scalar product sense*) Q^λ *and* P *of* (8) *and* (9), *respectively, satisfying* $Q_t^\lambda \geq 0$ *and* $P_t \geq 0$ *for all* $t \in \mathbf{T}$. *Moreover, the solution of* (8) *has the explicit form*

$$Q_t^\lambda = \lambda \mathcal{U}_{T-t}^* R(\lambda, -\Gamma_{T-t})\mathcal{U}_{T-t},\ 0 \leq t \leq T,\ \lambda > 0. \tag{10}$$

For the proof of existence and uniqueness part of this lemma, see [2]. For the proof of representation (10), see [8, 9].

Consider the linear regulator problem consisting of minimizing the cost functional

$$J(v) = \|y_T^v - h\|^2 + \lambda \int_0^T \|v_t\|^2\, dt, \tag{11}$$

where y^v is a state process, defined by (5); v is a control in $V = L_2(\mathbf{T}, Y)$; and $h \in X$ and $\lambda > 0$ are parameters.

LEMMA 2. *For given* $h \in X$ *and* $\lambda > 0$, *there exists a unique optimal control* v^λ *in* $L_2(\mathbf{T}, Y)$ *at which the functional* (11) *takes on its minimum value and*

$$v_t^\lambda = -B^*\mathcal{U}_{T-t}^* R(\lambda, -\Gamma_T)(\mathcal{U}_T y_0 - h + g) \text{ almost everywhere (a.e.) on } \mathbf{T}, \tag{12}$$

$$y_T^{v^\lambda} - h = \lambda R(\lambda, -\Gamma_T)(\mathcal{U}_T y_0 - h + g), \tag{13}$$

where

$$g = \int_0^T \mathcal{U}_{T-t} f_t\, dt.$$

Proof. The existence and the uniqueness of an optimal control follows from a general theorem about linear regulator problems (see [2]). We will prove the formulae (12) and (13). By computing the variation of the functional (11), one can easily obtain

$$v_t^\lambda = -\lambda^{-1} B^* \mathcal{U}_{T-t}^* \left(y_T^{v^\lambda} - h\right) \text{ a.e. on } \mathbf{T}. \tag{14}$$

Using this in (5), we have

$$\begin{aligned} y_T^{v^\lambda} &= \mathcal{U}_T y_0 + \int_0^T \mathcal{U}_{T-t}\left(Bv_t^\lambda + f_t\right) dt \\ &= \mathcal{U}_T y_0 + g - \lambda^{-1} \int_0^T \mathcal{U}_{T-t} BB^* \mathcal{U}_{T-t}^* \left(y_T^{v^\lambda} - h\right) dt \\ &= \mathcal{U}_T y_0 + g - \lambda^{-1}\Gamma_T \left(y_T^{v^\lambda} - h\right). \end{aligned}$$

Hence,

$$\lambda y_T^{v^\lambda} = \lambda(\mathcal{U}_T y_0 + g) - \Gamma_T\left(y_T^{v^\lambda} - h\right),$$

which implies

$$(\lambda I + \Gamma_T) y_T^{v^\lambda} = \lambda(\mathcal{U}_T y_0 + g) + \Gamma_T h$$

and, consequently,

$$\begin{aligned} y_T^{v^\lambda} &= \lambda(\lambda I+\Gamma_T)^{-1}(\mathcal{U}_T y_0+g)+(\lambda I+\Gamma_T)^{-1}(\lambda I+\Gamma_T-\lambda I)h \\ &= \lambda R(\lambda,-\Gamma_T)(\mathcal{U}_T y_0+g-h)+h. \end{aligned}$$

Thus, (13) holds. Substituting (13) in (14), we obtain (12). The lemma is proven.

5. Resolvent conditions for D^c- and D^a-controllabilities. In this section, the necessary and sufficient conditions of D^c - and D^a -controllabilities are discussed.

THEOREM 1. *The following statements are equivalent*:

(1a) *the control system* (5) *is* D^c*-controllable on* $\mathbf{T}$;

(1b) *the complete controllability condition for the system* (5) *holds, i.e.,* $\Gamma_T > 0$;

(1c) $R(\lambda,-\Gamma_T)$ *converges as* $\lambda \to 0$ *in uniform operator topology*;

(1d) $R(\lambda,-\Gamma_T)$ *converges as* $\lambda \to 0$ *in strong operator topology*;

(1e) $R(\lambda,-\Gamma_T)$ *converges as* $\lambda \to 0$ *in weak operator topology*;

(1f) $\lambda R(\lambda,-\Gamma_T)$ *converges to zero operator as* $\lambda \to 0$ *in uniform operator topology.*

Proof. The equivalence (1a) $\Leftrightarrow$ (1b) is well known. For the implication (1b) $\Rightarrow$ (1c), suppose $\Gamma_T > 0$. Then for all $x \in X$ and for all $\lambda \geq 0$,

$$\langle x, (\lambda I+\Gamma_T)x\rangle \geq (\lambda+k)\|x\|^2,$$

where $k > 0$ is a constant. Therefore, for all $\lambda \geq 0$,

$$\|R(\lambda,-\Gamma_T)\| = \left\|(\lambda I+\Gamma_T)^{-1}\right\| \leq \frac{1}{\lambda+k} \leq \frac{1}{k}.$$

We obtain that $\|R(\lambda,-\Gamma_T)\|$ is bounded with respect to $\lambda \geq 0$. Furthermore,

$$\begin{aligned} \left\|R(\lambda,-\Gamma_T)-\Gamma_T^{-1}\right\| &= \left\|(\lambda I+\Gamma_T)^{-1}-\Gamma_T^{-1}\right\| \\ &= \left\|\Gamma_T^{-1}(\Gamma_T-\lambda I-\Gamma_T)(\lambda I+\Gamma_T)^{-1}\right\| \\ &\leq \lambda\left\|\Gamma_T^{-1}\right\|\left\|(\lambda I+\Gamma_T)^{-1}\right\| \\ &\leq \lambda k^{-2}. \end{aligned}$$

Thus, $R(\lambda,-\Gamma_T)$ converges to Γ_T^{-1} as $\lambda \to 0$ in uniform operator topology. The implications (1c) $\Rightarrow$ (1d) $\Rightarrow$ (1e) are obvious. The implication (1e) $\Rightarrow$ (1f) follows from the boundedness of a weakly convergent sequence of operators. For the implication (1f) $\Rightarrow$ (1b), suppose

$$\lambda\|R(\lambda,-\Gamma_T)\| = \lambda\left\|(\lambda I+\Gamma_T)^{-1}\right\| \to 0,\ \lambda \to 0.$$

Then $\lambda^{1/2}\left\|(\lambda I+\Gamma_T)^{-1/2}\right\| \to 0$ as $\lambda \to 0$. For sufficiently small $\lambda_0 > 0$, we can write

$$\lambda_0^{1/2}\left\|(\lambda_0 I+\Gamma_T)^{-1/2}\right\| \leq 1/\sqrt{2}.$$

Thus, for all $x \in X$ we have

$$\begin{aligned} \|x\|^2 &= \left\|\left(\lambda_0^{1/2}(\lambda_0 I+\Gamma_T)^{-1/2}\right)\left(\lambda_0^{-1/2}(\lambda_0 I+\Gamma_T)^{1/2}\right)x\right\|^2 \\ &\leq \frac{1}{2}\left\|\lambda_0^{-1/2}(\lambda_0 I+\Gamma_T)^{1/2}x\right\|^2 \\ &= \frac{1}{2}\left\langle\lambda_0^{-1}(\lambda_0 I+\Gamma_T)x, x\right\rangle, \end{aligned}$$

which implies

$$\left\langle \lambda_0^{-1}(\lambda_0 I + \Gamma_T)x, x\right\rangle \geq 2\|x\|^2$$

and, consequently,

$$\langle \Gamma_T x, x\rangle \geq \lambda_0 \|x\|^2.$$

Thus, $\Gamma_T > 0$. The theorem is proven.

THEOREM 2. *The following statements are equivalent*:

(2a) *the control system* (5) *is D^a-controllable on* $\mathbf{T}$;

(2b) *the approximate controllability condition for the system* (5) *holds, i.e., if $B^*\mathcal{U}_t^* x = 0$ for all $t \in \mathbf{T}$, then $x = 0$*;

(2c) *$\lambda R(\lambda, -\Gamma_T)$ converges to zero operator as $\lambda \to 0$ in strong operator topology*;

(2d) *$\lambda R(\lambda, -\Gamma_T)$ converges to zero operator as $\lambda \to 0$ in weak operator topology.*

Proof. The equivalence (2a) $\Leftrightarrow$ (2b) is well known. For the implication (2c) $\Rightarrow$ (2a), suppose $\lambda R(\lambda, -\Gamma_T) \to 0$ as $\lambda \to 0$ in strong operator topology. Consider arbitrary $h \in X$ and the functional (11) with this h. By (13), selecting λ sufficiently small, we can make $y_T^{v^\lambda}$ close to h, so the control system (5) is D^a-controllable. For the implication (2a) $\Rightarrow$ (2c), let the control system (5) be D^a-controllable. Then for arbitrary $h \in X$, there exists a sequence $\{\bar{v}^i\}$ in $L_2(\mathbf{T}, Y)$ such that $\left\|y_T^{\bar{v}^i} - h\right\| \to 0$ as $i \to \infty$. We have

$$\left\|y_T^{v^\lambda} - h\right\|^2 \leq \left\|y_T^{v^\lambda} - h\right\|^2 + \lambda \int_0^T \left\|v_t^\lambda\right\|^2 dt \leq \left\|y_T^{\bar{v}^i} - h\right\|^2 + \lambda \int_0^T \left\|\bar{v}_t^i\right\|^2 dt,$$

where v^λ is the control at which the functional (11) takes on its minimum value. If $\varepsilon > 0$ is given, then we can make $\|y_T^{\bar{v}^i} - h\| < \varepsilon/\sqrt{2}$ for some sufficiently large i and then we can select $\delta > 0$ to be sufficiently small so that for all $0 < \lambda < \delta$,

$$\lambda \int_0^T \left\|\bar{v}_t^i\right\|^2 dt < \frac{\varepsilon^2}{2}.$$

Thus, $\left\|y_T^{v^\lambda} - h\right\| < \varepsilon$ for all $0 < \lambda < \delta$. By (13) and the arbitrariness of h, the convergence of $\lambda R(\lambda, -\Gamma_T)$ to zero operator is implied as $\lambda \to 0$ in strong operator topology. Finally, the equivalence (2c) $\Leftrightarrow$ (2d) is a consequence of $\lambda R(\lambda, -\Gamma_T) \geq 0$. The theorem is proven.

The conditions (1f) and (2c) in Theorems 1 and 2 clearly distinguish the D^c- and D^a-controllabilities of the control system (5) showing that the distinction between them is in a kind of convergence of $\lambda R(\lambda, -\Gamma_T)$ to zero operator as $\lambda \to 0$. We call these conditions the resolvent conditions for the control system (5) to be D^c- and D^a-controllable, respectively.

An application of the resolvent conditions to a concrete system requires a computation of the respective resolvent and then a verification of the respective convergence. These are illustrated below in the examples of controlled one-dimensional heat and wave equations.

Example 1. Consider a controlled one-dimensional heat equation

$$\frac{\partial}{\partial t} y_{t,\theta} = \frac{\partial^2}{\partial \theta^2} y_{t,\theta} + v_{t,\theta},\ 0 \leq \theta \leq 1,\ 0 < t \leq T, \tag{15}$$

with the initial and boundary conditions

$$y_{0,\theta} = y_{t,0} = y_{t,1} = 0,\ 0 \leq \theta \leq 1,\ 0 \leq t \leq T. \tag{16}$$

Let $X = L_2(0,1)$. In the system (15)–(16), the second-order differential operator $d^2/d\theta^2$ stands for the operator A with the domain

$$D(A) = \{h \in X : (d^2/d\theta^2)h \in X,\ h_0 = h_1 = 0\},$$

and it generates the strongly continuous semigroup $\mathcal{U}$ defined by

$$[\mathcal{U}_t h]_\theta = \sum_{i=1}^{\infty} 2e^{-i^2\pi^2 t} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha,\ 0 \le \theta \le 1,\ t \ge 0,\ h \in X.$$

If v is considered as a control action taken from the set of admissible controls $V = L_2(\mathbf{T}, X)$, then it is easily shown that $B = B^* = I$ and, since $\mathcal{U}_t$ is self-adjoint,

$$\Gamma_T = \int_0^T \mathcal{U}_{T-s} BB^* \mathcal{U}_{T-s}^*\, ds = \int_0^T \mathcal{U}_{2s}\, ds.$$

Therefore, for $h \in X$,

$$\begin{aligned}
[\Gamma_T h]_\theta &= \left[\int_0^T \mathcal{U}_{2s} h\, ds\right]_\theta \\
&= \sum_{i=1}^{\infty} \int_0^T 2e^{-2i^2\pi^2 s} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha\, ds \\
&= \sum_{i=1}^{\infty} \frac{1 - e^{-2i^2\pi^2 T}}{i^2\pi^2} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha.
\end{aligned}$$

The half-range Fourier sine expansion of $h \in X$ is

$$h_\theta = \sum_{i=1}^{\infty} 2\sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha,\ 0 \le \theta \le 1.$$

Using this, we obtain

$$[(\lambda I + \Gamma_T)h]_\theta = \sum_{i=1}^{\infty} \frac{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}}{i^2\pi^2} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha.$$

Let $(\lambda I + \Gamma_T)h = g$. If we use the half-range Fourier sine expansion of $g \in X$, then

$$\begin{aligned}
&\sum_{i=1}^{\infty} \frac{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}}{i^2\pi^2} \sin(i\pi\theta) \int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha \\
&\quad = \sum_{i=1}^{\infty} 2\sin(i\pi\theta) \int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^1 h_\alpha \sin(i\pi\alpha)\, d\alpha \\
&\quad = \frac{2i^2\pi^2}{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}} \int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha,\ i = 1, 2, \ldots.
\end{aligned}$$

Therefore,

$$\begin{aligned} h_\theta &= \left[(\lambda I + \Gamma_T)^{-1} g\right]_\theta = [R(\lambda, -\Gamma_T)g]_\theta \\ &= \sum_{i=1}^{\infty} \frac{4i^2\pi^2}{2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T}} \sin(i\pi\theta) \int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha. \end{aligned}$$

If $g_\alpha \equiv 1$, then by Parseval's identity,

$$\begin{aligned} \|R(\lambda, -\Gamma_T)g\|_X^2 &= \frac{1}{2} \sum_{i=1}^{\infty} \frac{(4i^2\pi^2)^2}{(2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T})^2} \left(\int_0^1 \sin(i\pi\alpha)\, d\alpha \right)^2 \\ &= \sum_{i=1}^{\infty} \frac{8i^2\pi^2(1-(-1)^i)^2}{(2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T})^2} \\ &\geq \sum_{i=1}^{\infty} \frac{8i^2\pi^2(1-(-1)^i)^2}{(2i^2\pi^2\lambda + 1)^2} = \sum_{i=1,3,5,\ldots} \frac{32i^2\pi^2}{(2i^2\pi^2\lambda + 1)^2}. \end{aligned}$$

One can verify that the inequality

$$\frac{i}{2i^2\pi^2\lambda + 1} > \frac{i+1}{2(i+1)^2\pi^2\lambda + 1}$$

holds whenever i is an integer that is greater than the number $1/\sqrt{2\lambda}\pi$. Let N_λ be the smallest odd integer that is greater than $1/\sqrt{2\lambda}\pi$. Then the sequence

$$\left\{ i^2\pi^2 / \left(2i^2\pi^2\lambda + 1\right)^2 \right\}_{i=1,2,\ldots}$$

is decreasing for $i \geq N_\lambda$. The following limits are obvious:

$$N_\lambda \to \infty \text{ and } \lambda N_\lambda^2 \to \frac{1}{2\pi^2} \text{ as } \lambda \to 0.$$

Using these, for $g_\alpha \equiv 1$, we obtain

$$\begin{aligned} \|R(\lambda, -\Gamma_T)g\|_X^2 &\geq \sum_{i=N_\lambda}^{\infty} \frac{16i^2\pi^2}{(2i^2\pi^2\lambda + 1)^2} \geq \int_{N_\lambda}^{\infty} \frac{16\pi^2 t^2}{(2\pi^2\lambda t^2 + 1)^2}\, dt \\ &\geq \int_{N_\lambda}^{\infty} \frac{4\pi^2 t}{(2\pi^2\lambda t^2 + 1)^2}\, dt = \frac{1}{\lambda(2\pi^2\lambda N_\lambda^2 + 1)} \to \infty \end{aligned}$$

as $\lambda \to 0$. Therefore, by (1a) $\Leftrightarrow$ (1d) in Theorem 1, the system (15)–(16) is not D^c-controllable. At the same time, for all $g \in X$,

$$\begin{aligned} &\|\lambda R(\lambda, -\Gamma_T)g\|_X^2 \\ &= \sum_{i=1}^{\infty} \frac{8i^4\pi^4\lambda^2}{(2i^2\pi^2\lambda + 1 - e^{-2i^2\pi^2 T})^2} \left(\int_0^1 g_\alpha \sin(i\pi\alpha)\, d\alpha \right)^2 \to 0 \end{aligned}$$

as $\lambda \to 0$ and hence, by (2a) $\Leftrightarrow$ (2c) in Theorem 2, the system (15)–(16) is D^a-controllable.

Example 2. Consider a controlled wave equation

$$\frac{\partial^2}{\partial t^2}\xi_{t,\theta} = \frac{\partial^2}{\partial \theta^2}\xi_{t,\theta} + b_\theta v_t,\ 0 \leq \theta \leq 1,\ 0 < t \leq T, \tag{17}$$

with the initial and boundary conditions

$$\xi_{0,\theta} = f_\theta,\ \frac{\partial}{\partial t}\xi_{t,\theta}\Big|_{t=0} = g_\theta,\ \xi_{t,0} = \xi_{t,1} = 0,\ 0 \le \theta \le 1,\ 0 \le t \le T, \tag{18}$$

where v is a control action taken from the set of admissible controls $V = L_2(0,T)$, i.e., $Y = R$. We assume that f, g, and b are functions in $L_2(0,1)$. For these functions, we will use the half-range Fourier sine expansions

$$f_\theta = \sum_{i=1}^{\infty} \alpha_i \sin(i\pi\theta),\ g_\theta = \sum_{i=1}^{\infty} \beta_i \sin(i\pi\theta),\ b_\theta = \sum_{i=1}^{\infty} \gamma_i \sin(i\pi\theta)$$

and suppose that

$$\sum_{i=1}^{\infty} i^2\alpha_i^2 < \infty.$$

Let X be a Hilbert space of all functions

$$h = \begin{bmatrix} f \\ g \end{bmatrix} : [0,1] \to R,$$

where f and g satisfy the above-mentioned conditions endowed with the scalar product

$$\langle h, \tilde{h}\rangle = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \tilde{f} \\ \tilde{g} \end{bmatrix} \right\rangle = \sum_{i=1}^{\infty} \left(i^2\pi^2\alpha_i\tilde{\alpha}_i + \beta_i\tilde{\beta}_i \right),$$

where $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are the respective Fourier coefficients of $\tilde{f}$ and $\tilde{g}$. This space X is suitable for the problem (17)–(18) (see Curtain and Zwart [3, p. 149] and Zabczyk [6, p. 180]). For the operator

$$A = \begin{bmatrix} 0 & I \\ d^2/d\theta^2 & 0 \end{bmatrix}, \tag{19}$$

where I is the identity operator on $L_2(0,1)$ and $d^2/d\theta^2$ has the domain

$$D\big(d^2/d\theta^2\big) = \big\{\eta \in L_2(0,1) : \big(d^2/d\theta^2\big)\eta \in L_2(0,1),\ \eta_0 = \eta_1 = 0\big\},$$

and for $B \in \mathcal{L}(R, X)$ defined by

$$[Bv]_\theta = \begin{bmatrix} 0 \\ b_\theta v \end{bmatrix},\ 0 \le \theta \le 1,\ v \in R,$$

the problem (17)–(18) can be formulated in the abstract form

$$\frac{d}{dt}y_t = Ay_t + Bv_t,\ t > 0, \tag{20}$$

where

$$[y_t]_\theta = \begin{bmatrix} \xi_{t,\theta} \\ (\partial/\partial t)\xi_{t,\theta} \end{bmatrix}, 0 \le \theta \le 1,\ 0 < t \le T;\ y_0 = \begin{bmatrix} f \\ g \end{bmatrix}.$$

It is known that the operator A defined by (19) generates a continuous group $\mathcal{U}$ (see Curtain and Zwart [3] and Zabczyk [6]) as defined by

$$[\mathcal{U}_t h]_\theta = \sum_{i=1}^{\infty} \begin{bmatrix} \cos(i\pi t) & (i\pi)^{-1}\sin(i\pi t) \\ -i\pi\sin(i\pi t) & \cos(i\pi t) \end{bmatrix} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \sin(i\pi\theta),\ 0 \le \theta \le 1,\ t \in R,$$

where

$$h = \begin{bmatrix} f \\ g \end{bmatrix} \in X$$

and α_i and β_i are Fourier coefficients of f and g, respectively. Since $\mathcal{U}$ is a group, we have $\mathcal{U}_t^* = \mathcal{U}_{-t}$. Therefore, the controllability operator Γ_T of the system (20) is

$$\Gamma_T h = \int_0^T \mathcal{U}_{T-t} B B^* \mathcal{U}_{T-t}^* h\, dt = \int_0^T \mathcal{U}_t B B^* \mathcal{U}_{-t} h\, dt,\ h \in X.$$

We have

$$[\mathcal{U}_{-t} h]_\theta = \sum_{i=1}^{\infty} \begin{bmatrix} \alpha_i \cos(i\pi t) - \beta_i (i\pi)^{-1}\sin(i\pi t) \\ \alpha_i i\pi \sin(i\pi t) + \beta_i \cos(i\pi t) \end{bmatrix} \sin(i\pi\theta).$$

One can calculate that

$$B^* h = \sum_{i=1}^{\infty} \gamma_i \beta_i,\ h \in X.$$

Hence,

$$B^* \mathcal{U}_{-t} h = \sum_{i=1}^{\infty} \gamma_i (\alpha_i i\pi \sin(i\pi t) + \beta_i \cos(i\pi t))$$

and, consequently,

$$[\mathcal{U}_t B B^* \mathcal{U}_{-t} h]_\theta = \sum_{i=1}^{\infty} \begin{bmatrix} \gamma_i (i\pi)^{-1} \sin(i\pi t) \\ \gamma_i \cos(i\pi t) \end{bmatrix} \sin(i\pi\theta)$$
$$\times \sum_{j=1}^{\infty} \gamma_j (\alpha_j j\pi \sin(j\pi t) + \beta_j \cos(j\pi t)).$$

Thus, for $T = 2$,

$$[\Gamma_2 h]_\theta = \int_0^2 [\mathcal{U}_t B B^* \mathcal{U}_{-t} h]_\theta\, dt = \sum_{i=1}^{\infty} \begin{bmatrix} \gamma_i^2 \alpha_i \\ \gamma_i^2 \beta_i \end{bmatrix} \sin(i\pi\theta).$$

We obtain that

$$[(\lambda I + \Gamma_2) h]_\theta = \sum_{i=1}^{\infty} \left(\lambda + \gamma_i^2\right) \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \sin(i\pi\theta),$$

which implies

$$[R(\lambda, -\Gamma_2) h]_\theta = \left[(\lambda I + \Gamma_2)^{-1} h\right]_\theta = \sum_{i=1}^{\infty} \left(\lambda + \gamma_i^2\right)^{-1} \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} \sin(i\pi\theta).$$

Finally, for all $h \in X$,

$$\|\lambda R(\lambda, -\Gamma_2)h\|^2 = \sum_{i=1}^{\infty} \frac{\lambda^2}{\left(\lambda + \gamma_i^2\right)^2}\left(i^2\pi^2\alpha_i^2 + \beta_i^2\right) \to 0$$

as $\lambda \to 0$ if $\gamma_i \neq 0$ for all $i = 1, 2, \ldots$. Thus, by (2a) $\Leftrightarrow$ (2c) in Theorem 2, we obtain the following sufficient condition for the approximate controllability of the system (17)–(18) which agrees with Theorem 2.10 in Zabczyk [6]: if $T \geq 2$ and b is such that

$$\gamma_i = 2\int_0^1 b_\theta \sin(i\pi\theta)\, d\theta \neq 0, \;\; i = 1, 2, \ldots,$$

then the system (17)–(18) is D^a-controllable.

6. Necessary and sufficient conditions for C- and S-controllabilities. The following two lemmas are proven in [7, 8].

LEMMA 3. *For $\varepsilon > 0$ and for $0 \leq p < 1$, the control system* (4) *is $C^c_{\varepsilon,p}$-controllable* ($C^a_{\varepsilon,p}$*-controllable*) *on* $\mathbf{T}$ *if and only if the control system* (5) *is D^c-controllable* (D^a*-controllable*) *on* $\mathbf{T}$ *and the control system* (6) *is $S^0_{\varepsilon,p}$-controllable on* $\mathbf{T}$.

By this lemma, the study of the $C^c_{\varepsilon,p}$-controllability (the $C^a_{\varepsilon,p}$-controllability) of the control system (4) is separated into the study of the D^c-controllability (D^a-controllability) and the $S^0_{\varepsilon,p}$-controllability of the control systems (5) and (6), respectively.

LEMMA 4. *The following statements hold:*

(a) *there exists a finite limit*

$$a_T = \lim_{\lambda\to 0}\int_0^T \operatorname{tr} CP_sQ_s^\lambda P_sC^*ds,$$

where Q^λ and P are solutions of (8) *and* (9), *respectively*;

(b) *the control system* (6) *is $S^0_{\varepsilon,p}$-controllable on* $\mathbf{T}$ *if $a_T < \varepsilon(1-p)$*;

(c) *the system* (6) *is S^0-controllable on* $\mathbf{T}$ *if $a_T = 0$.*

It turns out that the condition $a_T = 0$, which is sufficient for the system (6) to be S^0-controllable, is weaker than the D^a-controllability (particularly, the D^c-controllability) of the control system (5) on all the intervals $[0, t]$ with $0 < t \leq T$.

LEMMA 5. *If the control system* (5) *is D^a-controllable on all the intervals $[0, t]$ with $0 < t \leq T$, then $a_T = 0$, where a_T is defined in Lemma* 4(a).

Proof. From (2a) $\Leftrightarrow$ (2c) (see Theorem 2), we obtain that $\lambda R(\lambda, -\Gamma_{T-t})$ strongly converges to zero operator as $\lambda \to 0$ for all $0 \leq t < T$. Hence, by Lemma 1, Q_t^λ strongly converges to zero operator as $\lambda \to 0$ for all $0 \leq t < T$. Furthermore, substituting $h = \lambda^{1/2}(\lambda I + \Gamma_{T-t})^{-1/2}x$ in

$$\left\langle \lambda^{-1}(\lambda I + \Gamma_{T-t})h, h\right\rangle \geq \langle h, h\rangle,$$

we obtain

$$\left\langle \lambda(\lambda I + \Gamma_{T-t})^{-1}x, x\right\rangle \leq \|x\|^2.$$

Thus, $\lambda R(\lambda, -\Gamma_{T-t}) \leq I$ and by Lemma 1, $Q_t^\lambda \leq \mathcal{U}^*_{T-t}\mathcal{U}_{T-t}$ for all $\lambda > 0$ and for all $0 \leq t \leq T$. Hence, we can change the places of limit, integral, and trace in definition of the number a_T in Lemma 4(a) to obtain $a_T = 0$. The lemma is proven.

THEOREM 3. *The control system* (4) *is C-controllable on all the intervals* $[0,T]$ *with* $T>0$ *if and only if the control system* (5) *is D^c-controllable on all the intervals* $[0,T]$ *with* $T>0$.

This follows from Lemmas 3, 4(c), and 5.

THEOREM 4. *The control system* (4) *is S-controllable on all the intervals* $[0,T]$ *with* $T>0$ *if and only if the control system* (5) *is D^a-controllable on all the intervals* $[0,T]$ *with* $T>0$.

The necessity follows from Theorem 5(b) in [8]. The sufficiency follows from Lemmas 3, 4(c), and 5.

COROLLARY 1. *The control system* (4) *is S-controllable on all the intervals* $[0,T]$ *with* $T>0$ *if it is C^a-controllable on all the intervals* $[0,T]$ *with* $T>0$.

Proof. From Lemmas 3 and 5 and Theorem 4, one can see that each of the S- and C^a- controllabilities of the control system (4) on all the intervals $[0,T]$ with $T>0$ is equivalent to the D^a-controllability of the control system (5) on all the intervals $[0,T]$ with $T>0$.

Example 3. Consider the control system (4) with the operators A and B as defined in Example 1. It was shown that the deterministic part of this system is D^a-controllable on all the intervals $[0,T]$ with $T>0$. Hence, this system is S-controllable on all the intervals $[0,T]$ with $T>0$.

Example 4. Consider the control system (4) with the operators A and B as defined in Example 2. It was shown that the deterministic part of this system is D^a-controllable on all the intervals $[0,T]$ with $T>2$ if some additional condition holds. However, Theorem 4 does not guarantee the S-controllability of this system.

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