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On controllability for a class of second order semilinear stochastic systems

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ABSTRACT

Many dynamical systems are modeled as second order differential systems. In this paper we address the problem of controllability for second order semilinear stochastic systems in finite dimensional spaces. Sufficient criteria for the complete controllability are formulated under some appropriate assumptions. The results are obtained by using the Banach fixed point theorem. Finally an illustrative example is provided.

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1. Introduction

It is well known that second order dynamical systems can be successfully used to describe the dynamical processes of many physical plants, such as electrical power systems (Badawy and Youssef, 1987; Narasimhamurthi, 1987), long transmission lines (Butyrin and Tichomirowa, 1987), vibrating systems with discrete parameters (Margolis and Young, 1977) and distributed parameter systems (Serbin, 1985). Similar problems appear in studies of biological and social systems (Marcguk and Dymnikov, 1985), as well as in quantum mechanics (Ashcroft and Mermin, 1976). There are some works concerned with the analysis of the deterministic nature of these models (Adegas and Stoustrup, 2015; Diwekar and Yedavalli, 1999; Hughes and Skelton, 1980; Kalenova and Morozov, 2012; Laub and Arnold, 1984; Murugesan, Dhayabaran, & Evans, 1999). Deterministic models, however, are unsuitable for modeling complex systems that involve fluctuating or noise sources and therefore considering some inherent randomness in the modeling leads to more realistic stochastic systems. There are some interesting works that address the analysis of second order stochastic systems (Patanarapeelert, Frank, Friedrich, Beek, & Tang, 2006; Rong, Meng, & Fang, 2000; Stanzhitskii, Krenevich, & Novak, 2011).

In control theoretic applications it is interesting to consider the so called controllability problem, that is, the possibility of driving a nonlinear system to a desired target by a suitable choice of the control. There are numerous works addressing the controllability of linear and nonlinear stochastic systems of first order in finite dimensional spaces. For related results, we refer the reader to Enrhardt and Kliemann (1982), Mahmudov and Denker (2000), Zabczyk (1981) and Mahmudov and Zorlu (2003). As far as second

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order evolution equations are concerned there are some works addressing the controllability problems. Mahmudov and McKibben (2006) established the approximate and exact controllability of mild solutions for a class of abstract second order stochastic evolution equations in a real separable Hilbert space. Balasubramaniam and Muthukumar (2009, 2010) derived sufficient conditions for the approximate controllability of a class of second-order stochastic semilinear abstract functional differential equations with infinite delay and nonlinear stochastic functional differential equations of McKean–Vlasov type. Zhang, Ding, Wang, Hu, and Hao (2012) investigated the controllability of impulsive neutral second-order stochastic functional evolution equations using the Sadovskii fixed point theorem and the theory of strongly continuous cosine families of operators. Muthukumar and Rajivganthi (2015) considered the approximate controllability of a class of second-order neutral stochastic differential equations subjected to infinite delay and Poisson jumps by using the cosine family of operators. A new set of sufficient conditions for the approximate controllability of a second order semilinear stochastic system with nonlocal conditions is derived by Arora and Sukavanam (2015). However it should be emphasized that most of the works in this direction are mainly concerned with infinite dimensional systems and there have been no attempts made to study the controllability of second order stochastic systems in finite dimensional spaces.

With this background in mind, the present paper is devoted to study the complete controllability for semilinear stochastic system of second order expressed in the form

$$\left.\begin{aligned} &\mathrm{d}\dot{x}(t) + A^2 x(t)\mathrm{d}t = Bu(t)\mathrm{d}t + \sigma(t, x(t))\mathrm{d}w(t),\\ &x(0) = x_0 \text{ and } \dot{x}(0) = y_0, \end{aligned}\right\} \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ is the state vector of the system and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is given. The constant non-singular matrix $A \in \mathbb{R}^{n\times n}$ is called the second-order state matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $u(t) \in \mathbb{R}^m$ is a feedback control input and w is a standard n-dimensional Wiener process representing the noise of the system. Due to the coupling of randomness and nonlinearity, the dynamic response of a stochastic system is a random variable or a random process. Here we show that the nonlinear stochastic system (1) is completely controllable under the natural assumption that the associated linear control system is completely controllable. Moreover we analyze the controllability of the original system without reducing the second order system into a first order systems and by obtaining an integral representation of its solution in terms of sine and cosine matrices. Motivation for this kind of deterministic systems can be found in Trzaska (1990).

Notations. The notations used in this paper are fairly standard. Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with probability measure $\mathbb{P}$ on Ω with a filtration $\{\mathcal{F}_t | t \in [t_0, T]\}$ generated by m-dimensional Wiener process $\{w(s) : t_0 \le s \le t\}$ defined on the probability space; $L_2(\Omega, \mathcal{F}_t, \mathbb{R}^n)$ denotes the Hilbert space of all $\mathcal{F}_t$-measurable square-integrable random variables with values in $\mathbb{R}^n$; $L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^n)$ denotes the Hilbert space of all square-integrable and $\mathcal{F}$-adapted processes with values in $\mathbb{R}^n$; $\mathcal{U}_{ad} := L_2^{\mathcal{F}}([t_0, T], \mathbb{R}^m)$, the set of admissible controls; $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$; $\mathbb{E}$ denotes the mathematical expectation operator of the stochastic process with respect to the given probability measure $\mathbb{P}$.

2. System description and preliminaries

Consider the second order linear stochastic system

$$\left.\begin{aligned} &\mathrm{d}\dot{x}(t) + A^2x(t)\mathrm{d}t = Bu(t)\mathrm{d}t + \tilde{\sigma}(t)\mathrm{d}w(t),\\ &x(0) = x_0 \text{ and } \dot{x}(0) = y_0, \end{aligned}\right\} \tag{2}$$

where $x = x(t) \in \mathbb{R}^n$ is the system state vector and $\tilde{\sigma} : [0, T] \to \mathbb{R}^{n\times n}$ is a given function. The solution of the linear system (2) is given by Sharma and George (2007) and Trzaska (1990)

$$\begin{aligned} x(t) = \cos(At)x_0 + A^{-1}\sin(At)y_0 + \int_0^t A^{-1}\sin(A(t-s))Bu(s)\mathrm{d}s\\ + \int_0^t A^{-1}\sin(A(t-s))\tilde{\sigma}(s)\mathrm{d}w(s). \end{aligned}$$

Now let us introduce the following matrices and sets:

The linear bounded operator $\mathcal{K}_T \in \mathcal{L}(L_2^{\mathcal{F}}([0, T], \mathbb{R}^m), L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$ is defined by

$$\mathcal{K}_T u = \int_0^T A^{-1}\sin(A(T-s))Bu(s)\mathrm{d}s$$

and its adjoint linear bounded operator

$$(\mathcal{K}_T)^* : L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) \to L_2^{\mathcal{F}}([0, T], \mathbb{R}^m)$$

is defined by

$$[(\mathcal{K}_T)^* z](t) = B^*(A^{-1}\sin(A(T-t)))^*\mathsf{E}\{z \mid \mathcal{F}_t\}, \quad t \in [0, T]$$

and the set of all states attainable from x_0 in time $t > 0$ using admissible controls is given by

$$\begin{aligned} \mathcal{R}_t(\mathcal{U}_{ad}) &= \{x(t; x_0, u) \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n) : u(\cdot) \in \mathcal{U}_{ad}\}\\ &= \cos(At)x_0 + A^{-1}\sin(At)y_0 + \operatorname{Im}\mathcal{K}_T + \int_0^t A^{-1}\sin(A(t-s))\tilde{\sigma}(s)\mathrm{d}w(s). \end{aligned}$$

The linear controllability operator $\mathcal{W}_T \in \mathcal{L}(L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n), L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$ which is associated with the operator $\mathcal{K}_T$ is defined by

$$\begin{aligned} \mathcal{W}_T &= \mathcal{K}_T\,(\mathcal{K}_T)^*\{\cdot\}\\ &= \int_0^T A^{-1}\sin(A(T-s))BB^*(A^{-1}\sin(A(T-s)))^*\mathsf{E}\{\cdot \mid \mathcal{F}_s\}\mathrm{d}s \end{aligned}$$

and the deterministic controllability matrix $\Gamma_s^T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is

$$\Gamma_s^T = \int_s^T A^{-1}\sin(A(T-\tau))BB^*(A^{-1}\sin(A(T-\tau)))^*\mathrm{d}\tau,$$

$\tau \in [0, T]$, where the $*$ indicates the matrix transpose.

The following definitions will be used in the rest of this paper.

Definition 1: The second order stochastic system (2) is controllable on $[0, T]$ if, for any $x_0 \in \mathbb{R}^n$ and $x_T \in \mathbb{R}^n$, there exists an admissible control function $u(t) \in L_2^{\mathcal{F}}([0, T], \mathbb{R}^m)$ such that the solution of the system (2) satisfies $x(T) = x_T$.

Definition 2: The second order stochastic system (2) is completely controllable on $[0, T]$ if

$$\mathcal{R}_T(\mathcal{U}_{ad}) = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

that is, if all the points in $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ can be exactly reached at time T from any arbitrary initial point $x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ at time $T > 0$.

Definition 3: The second order stochastic system (2) is said to be approximately controllable on $[0, T]$ if

$$\overline{\mathcal{R}_T(\mathcal{U}_{ad})} = L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n),$$

that is, if all the points in $L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ can be approximately reached at time T from any arbitrary initial point $x_0 \in L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n)$ at time $T > 0$.

3. Linear stochastic system

In this section, we derive some important results to establish the controllability of the linear second order stochastic system (2).

Consider the corresponding deterministic system of the following form

$$\left.\begin{aligned} &\ddot{x}(t) + A^2 x(t) = Bv(t), \\ &x(0) = x_0 \text{ and } \dot{x}(0) = y_0, \end{aligned}\right\} \tag{3}$$

where the admissible controls $v \in L_2([t_0, T], \mathbb{R}^m)$.

Lemma 1: (Sharma and George, 2007)

The following four statements are equivalent:

(a) *The linear system* (3) *is controllable on* $[0, T]$.
(b) *The rows of* $A^{-1} \sin(At)B$ *are linearly independent.*
(c) *The Controllability Grammian,*

$$\Gamma_0^T = \int_0^T A^{-1} \sin(A(T-s))BB^*(A^{-1}\sin(A(T-s)))^* \mathrm{d}s$$

is nonsingular.

(d) $\mathrm{Rank}[B : A^2B : (A^2)^2B : \cdots : (A^2)^{n-1}B] = n$.

The following lemma shows that the controllability of the associated deterministic linear system (3) is equivalent to complete and approximate controllability of the linear stochastic system (2).

Lemma 2: *The following conditions are equivalent:*

(a) *Deterministic system* (2) *is controllable on* $[0, T]$.
(b) *Stochastic system* (2) *is completely controllable on* $[0, T]$.
(c) *Stochastic system* (2) *is approximately controllable on* $[0, T]$.

Proof: The proof is quite similar to that of Theorem 3 in Mahmudov and Denker (2000) and is omitted. □

Note that, if the linear stochastic system (2) is completely controllable, then, for some $\gamma > 0$,

$$\mathbb{E}\,\langle \mathcal{W}_T z, z\rangle \geq \gamma \mathbb{E}\|z\|^2, \quad \text{for all } z \in L_2(\Omega, \mathcal{F}_{t_1}, R^n).$$

Hence, in the operator sense, we have the following inequality $\mathcal{W}_T \geq \gamma I$ which means that the operator $\mathcal{W}_T$ is strictly positive definite and thus the inverse linear operator $\mathcal{W}_T^{-1}$ is bounded and consequently (Mahmudov and Denker, 2000),

$$\|\mathcal{W}_T^{-1}\| \leq \frac{1}{\gamma} = M_W.$$

The following lemma gives a formula for a control steering the linear second order stochastic system from initial state x_0 to an arbitrary final state x_T.

Lemma 3: *Assume that the linear stochastic system (2) is completely controllable. Then, for arbitrary $x_T \in \mathbb{R}^n$ and $\tilde{\sigma}(\,\cdot\,) \in L_2^{\mathcal{F}}([0,T], \mathbb{R}^{n\times n})$, the control*

$$\begin{aligned} u(t) = B^*(A^{-1}\sin(A(T-t)))^* \mathsf{E}\Big\{ \mathcal{W}_T^{-1}\Big(& x_T - \cos(AT)x_0 - A^{-1}\sin(AT)y_0 \\ & - \int_0^T A^{-1}\sin(A(t-s))\tilde{\sigma}(s)\mathrm{d}w(s)\Big) \,\Big|\, \mathcal{F}_t \Big\} \end{aligned} \tag{4}$$

transfers the system

$$\begin{aligned} x(t) = \cos(At)x_0 + A^{-1}\sin(At)y_0 + \int_0^t A^{-1}\sin(A(t-s))Bu(s)\mathrm{d}s \\ + \int_0^t A^{-1}\sin(A(t-s))\tilde{\sigma}(s)\mathrm{d}w(s) \end{aligned} \tag{5}$$

from $x_0 \in \mathbb{R}^n$ to $x_T \in \mathbb{R}^n$ at time T.

Proof: By substituting (4) into (5) and putting $t = T$ we observe that

$$\begin{aligned} x(T) = {} & \cos(AT)x_0 + A^{-1}\sin(AT)y_0 \\ & + \int_0^T A^{-1}\sin(A(T-s))BB^*(A^{-1}\sin(A(T-s)))^* \mathsf{E}\Big\{\mathcal{W}_T^{-1}\Big(x_T - \cos(AT)x_0 \\ & - A^{-1}\sin(AT)y_0 - \int_0^T A^{-1}\sin(A(t-\tau))\tilde{\sigma}(\tau)\mathrm{d}w(\tau)\Big) \,\Big|\, \mathcal{F}_s\Big\}\mathrm{d}s \\ & + \int_0^T A^{-1}\sin(A(T-s))\tilde{\sigma}(s)\mathrm{d}w(s) \\ = {} & x_T \end{aligned}$$

Thus, the control $u(t)$ transfers the system (5) from x_0 to x_T. □

4. Semilinear stochastic system

Taking into account the above notations, definitions and lemmas, we shall derive controllability conditions for the semilinear stochastic system (1) by using the contraction mapping principle.

For the proof of the main result, we impose the following assumptions on the data of the problem:

(H1) The function σ satisfies the following Lipschitz condition: there exist a constant $L_1 > 0$ for $x, y \in \mathbb{R}^n$ and $0 \le t \le T$ such that

$$\|\sigma(t,x) - \sigma(t,y)\|^2 \le L_1 \|x - y\|^2.$$

(H2) The function σ is continuous and satisfies the usual linear growth condition, that is there exists a constant $L_2 > 0$ such that

$$\|\sigma(t,x)\|^2 \le L_2(1 + \|x\|^2)$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$.

By a solution of the system (1), we mean a solution of the nonlinear integral equation

$$\begin{aligned} x(t) &= \cos(At)x_0 + A^{-1}\sin(At)y_0 + \int_0^t A^{-1}\sin(A(t-s))Bu(s)\mathrm{d}s \\ &\quad + \int_0^t A^{-1}\sin(A(t-s))\sigma(s,x(s))\mathrm{d}w(s). \end{aligned} \tag{6}$$

It is easy to see that, under assumptions (H1) and (H2), for given $u(t) \in L_2^{\mathcal{F}}([0,T],\mathbb{R}^m)$, system (1) admits a unique solution (Murge and Pachpatte, 1990). Under these existence and uniqueness conditions, we will further prove that the system (1) is also controllable.

Let $\mathcal{B}_2$ denote the Banach space $C([0,T], L_2(\Omega, \mathcal{F}_T, \mathbb{R}^n))$ of all square integrable and $\mathcal{F}_t$-adapted processes $\varphi(t)$ endowed with the norm by

$$\|\varphi\|^2 := \sup_{t\in[0,T]} \mathsf{E}\|\varphi(t)\|^2.$$

To apply the contraction mapping principle, define the nonlinear operator $\mathcal{P}$ from $\mathcal{B}_2$ to $\mathcal{B}_2$ as follows:

$$\begin{aligned} (\mathcal{P}x)(t) &= \cos(At)x_0 + A^{-1}\sin(At)y_0 + \int_0^t A^{-1}\sin(A(t-s))Bu(s)\mathrm{d}s \\ &\quad + \int_0^t A^{-1}\sin(A(t-s))\sigma(s,x(s))\mathrm{d}w(s) \end{aligned} \tag{7}$$

and the control u given by

$$\begin{aligned} u(t) &= B^*(A^{-1}\sin(A(T-t)))^*\mathsf{E}\Big\{\mathcal{W}_T^{-1}\Big(x_T - \cos(AT)x_0 - A^{-1}\sin(AT)y_0 \\ &\quad - \int_0^T A^{-1}\sin(A(T-s))\sigma(s,x(s))\mathrm{d}w(s)\Big)\Big|\mathcal{F}_t\Big\}. \end{aligned} \tag{8}$$

From Lemma 3, the control (8) transfers the system (6) from the initial state x_0 to the final state x_T provided that the operator $\mathcal{P}$ defined by (7) has a fixed point. So, if the operator $\mathcal{P}$ has a fixed point, then the system (1) is completely controllable. Now, for our convenience, let us introduce the following notations:

$$\begin{aligned}
M_B &= \|B\|^2,\ M_G = \max\{\|\Gamma_s^T\|^2 : s \in [0,T]\},\\
M_S &= \{\|A^{-1}\sin(At)\|^2 : t \in [0,T]\},\\
M_C &= \{\|\cos(At)\|^2 : t \in [0,T]\}.
\end{aligned}$$

Theorem 1: *Assume that the conditions* (H1) *and* (H2) *hold and suppose that the linear second order stochastic system* (2) *is completely controllable. If the inequality*

$$2M_S L_1(1 + M_W M_G)T < 1 \tag{9}$$

is satisfied, then the system (1) *is completely controllable.*

Proof: To prove the complete controllability of the system (1), it is enough to show that $\mathcal{P}$ has a fixed point in $\mathcal{B}_2$. To do this, we employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps $\mathcal{B}_2$ into itself.

By Lemma 3, we get

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{P}x)(t)\|^2\\
&= \mathsf{E}\Big\| \cos(At)x_0 + A^{-1}\sin(At)y_0 + \int_0^t A^{-1}\sin(A(t-s))Bu(s)\mathrm{d}s\\
&\quad + \int_0^t A^{-1}\sin(A(t-s))\sigma(s,x(s))\mathrm{d}w(s)\Big\|^2\\
&\le 4M_C\|x_0\|^2 + 4M_S\|y_0\|^2 + 4M_S\mathsf{E}\int_0^t \|\sigma(s,x(s))\|^2\mathrm{d}s\\
&\quad + 4\Big\|\int_0^t A^{-1}\sin(A(t-s))BB^*(A^{-1}\sin(A(T-s)))^*\mathsf{E}\Big\{\mathcal{W}_T^{-1}\Big(x_T - \cos(AT)x_0\\
&\quad - A^{-1}\sin(AT)y_0 - \int_0^T A^{-1}\sin(A(T-s))\sigma(s,x(s))\mathrm{d}w(s)\Big)\ \Big|\ \mathcal{F}_s\Big\}\mathrm{d}s\Big\|^2\\
&\le 4M_C\|x_0\|^2 + 4\,M_S\|y_0\|^2 + 4M_S\mathsf{E}\int_0^t \|\sigma(s,x(s))\|^2\mathrm{d}s\\
&\quad + 16\,M_G M_W(\|x_T\|^2 + M_C\|x_0\|^2 + M_S\|y_0\|^2)\\
&\quad + 16M_G M_W M_S\mathsf{E}\int_0^T \|\sigma(s,x(s))\|^2\mathrm{d}s.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathsf{E}\|(\mathcal{P}x)(t)\|^2 &\le 4M_C\|x_0\|^2 + 4\,M_S\|y_0\|^2\\
&\quad + 16\,M_G M_W(\|x_T\|^2 + M_C\|x_0\|^2 + M_S\|y_0\|^2)\\
&\quad + 4(1 + 4M_G M_W)M_S L_2\int_0^T (1 + \mathsf{E}\|x(s)\|^2)\mathrm{d}s.
\end{aligned} \tag{10}$$

It follows, from (10) and condition (H1), that there exists $C_1 > 0$ such that

$$\begin{aligned}\mathsf{E}\|(\mathcal{P}x)(t)\|^2 &\leq C_1\left(1 + \int_0^T \mathsf{E}\|x(s)\|^2\mathrm{d}s\right)\\ &\leq C_1\left(1 + T\sup_{s\in[0,T]}\mathsf{E}\|x(s)\|^2\right).\end{aligned}$$

Therefore $\mathcal{P}$ maps $\mathcal{B}_2$ into itself.

It remains to show that $\mathcal{P}$ is a contraction on $\mathcal{B}_2$. For $x_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned}&\mathsf{E}\|(\mathcal{P}x_1)(t) - (\mathcal{P}x_2)(t))\|^2\\ &= \left\|\Gamma_0^T\mathcal{W}_T^{-1}\left(\int_0^T A^{-1}\sin(A(T-s))[\sigma(s,x_2(s)) - \sigma(s,x_1(s))]\mathrm{d}w(s)\right)\right.\\ &\quad \left. + \int_0^t A^{-1}\sin(A(t-s))[\sigma(s,x_1(s)) - \sigma(s,x_2(s))]\mathrm{d}w(s)\right\|^2\\ &\leq (2M_GM_WM_S + 2M_S)\left(\int_0^T \mathsf{E}\|\sigma(s,x_1(s)) - \sigma(s,x_2(s))\|^2\mathrm{d}s\right)\\ &\leq 2M_SL_1(1+M_GM_W)\int_0^T \mathsf{E}\|x_1(s) - x_2(s)\|^2\mathrm{d}s.\end{aligned}$$

It results in

$$\sup_{t\in[0,T]}\mathsf{E}\|(\mathcal{P}x_1)(t) - (\mathcal{P}x_2)(t))\|^2 \leq 2M_SL_1(1+M_WM_G)T\sup_{t\in[0,T]}\mathsf{E}\|x_1(t) - x_2(t)\|^2.$$

Therefore we conclude from (9) that $\mathcal{P}$ is a contraction mapping on $\mathcal{B}_2$. Then the mapping $\mathcal{P}$ has a unique fixed point $x(\cdot) \in \mathcal{B}_2$ which is the solution of the Equation (6). Thus the system is completely controllable on $[0, T]$. □

5. Example

Consider the matrix second order nonlinear stochastic system described by

$$\left.\begin{aligned}&\mathrm{d}\dot{x}(t) + A^2x(t)\mathrm{d}t = Bu(t)\mathrm{d}t + \sigma(t,x(t))\mathrm{d}w(t),\\ &x(0) = x_0 \text{ and } \dot{x}(0) = y_0,\end{aligned}\right\} \tag{11}$$

where $x(t) \in \mathbb{R}^2$ and $\sigma(t,x(t)) = \begin{pmatrix}\sigma_1(x_1,x_2)\\ \sigma_2(x_1,x_2)\end{pmatrix}$ with the initial conditions

$$x(0) = \begin{pmatrix}0\\1\end{pmatrix},\ \dot{x}(0) = \begin{pmatrix}-1\\1\end{pmatrix}.$$

Let

$$A = \begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix},\quad B = \begin{pmatrix}1\\1\end{pmatrix}.$$

The controllability matrix is

$$Q = [B : A^2B] = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

and Rank $(Q) = 2$. Hence the corresponding linear system is controllable. We have the following numerical estimate, taking $T = 1$,

$$M_S = \sup_{t\in[0,T]} \|A^{-1} \sin (At)\|^2 = 0.8932,$$
$$M_B = \|B\|^2 = 1.4142.$$

Let us take

$$\sigma_1(x_1, x_2) = \frac{x_1(t)}{15}, \quad \sigma_2(x_1, x_2) = \frac{\sin (x_2(t))}{10}.$$

The nonlinear function $\sigma(t, x(t))$ is Lipschitz continuous with Lipschitz constant $L_1 = 1/100$. It can be easily seen that the controllability Grammian matrix Γ_0^T is

$$\Gamma_0^T = \begin{pmatrix} 0.272676 & 0.149229 \\ 0.149229 & 0.092755 \end{pmatrix}$$

which is positive definite. It can be verified that $4M_SL_1(1 + M_WM_G)T < 1$ and hence, it satisfies all the assumptions of the Theorem 1. So the second order stochastic system (11) is controllable. It is easy to compute the steering control $u(t)$, steering the state from $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ during the time interval $[0, 1]$ using (8). The above numerical values are obtained by using *Mathematica* software.

6. Conclusion

The research presented in this paper focuses on the controllability of stochastic dynamical systems that can be described in terms of semilinear stochastic differential equations of second order. Sufficient conditions for the complete controllability are established by an application of fixed point argument. The theoretical results obtained were verified with an example. The problem of the stochastic controllability discussed can often occur in engineering practice. The efficiency of describing the real-life phenomena by fractional differential equation is more accurate than by classical differential equations. Hence it will be interesting to investigate the controllability for second order stochastic systems of fractional order in finite dimensional spaces (Balasubramaniam and Tamilalagan, 2015; Balasubramaniam, Kumaresan, Ratnavelu, & Tamilalagan, 2015) and this will be the subject of our future research.

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