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Analytical stability bound for delayed second-order systems with repeating poles using Lambert function W ☆

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Abstract

By using Lambert function, the analytical stability bound is obtained in this paper for delayed second-order systems with repeatable poles. An example is presented to illustrate the obtained analytical result. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Delay; Stability bound; Analytical solutions; Lambert function

1. Introduction

In real world systems, delay is everywhere. Delay complicates the analysis and design of control systems and motivates numerous research efforts (Gorecki, Fuksa, Grabowski, & Korytowski, 1989; Marshall, Gorecki, Walton, & Korytowski 1992).

Delay differential equations (DDE), initially introduced in the 18th century by Laplace and Condorcet (Gorecki et al., 1989), are used to describe dynamic systems with time delays, also known as dead-time or transport-lag which were also studied in terms of difference differential equations in (Bellman & Cooke, 1963). The analysis of DDE stability has been a long history efforts for applied mathematicians as well as control engineers. Although Bellman and Cooke tried to get a closed form solution for the first-order DDE with some tedious infinite series, a concise analytical solution for the first-order DDE with constant coefficients has not been possible until the rediscovery of Lambert [1] W function (Lambert, 1758). Lambert function $W(x)$ satisfies

$$W(x)\mathrm{e}^{W(x)} = x.$$

Actually, Euler noticed the importance of Lambert function $W(x)$ in (Euler, 1777, 1779), according to the introductory paper on Lambert function $W(x)$ (Corless et al., 1996). Lambert function got its name during the implementation of Maple (Corless, Gonnet, Hare, & Jeffrey, 1993; Corless, Gonnet, Hare, Jeffrey, & Knuth, 1997), a symbolic computation system. Among many interesting applications for Lambert function (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996; Valluri, Jeffrey, & Corless, 2000), getting an analytical stability bound for the first-order DDE has been an exciting one (Wright, 1949). Actually, for a simple DDE,

$$\dot{y}(t) = ay(t-1)$$

with its characteristic equation

$$s = a\mathrm{e}^{-s},$$

one may immediately get

$$s = W_k(a),$$

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[1] Johann Heinrich Lambert was born on 1728 August 26 in M hausen, Alsace; died 1777 September 25 in Berlin, Prussia. Lambert was a colleague of Euler and Lagrange at the Berlin Academy of Sciences. One of his achievements was to first provide a rigorous proof that π is irrational.

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where k is the index for branches (Jeffrey, Hare, & Corless, 1996). For a with a real value, only the principal branch $W_0(a)$, or simply $W(a)$, is to be considered. In symbolic software packages such as Maple, MATLAB Symbolic Toolbox, etc. $W(x)$ is a standard function now. Recently, a new extension for the above result on DDE stability bound was presented in (Asl & Ulsoy, 2000) where the following formula is used:

$$\text{Given}: (a+bx)\mathrm{e}^{(cx)}+d=0;$$

$$\text{one gets}: x=\frac{1}{c}W\left(-\frac{cd}{b}\mathrm{e}^{ac/b}\right)-\frac{a}{b}.$$

So far, there is no effort in the literature in finding an analytical stability bound for the second-order DDE with constant coefficients. In this paper, we tried to apply Lambert W function to study a class of second-order DDEs with constant coefficients. A restriction in this paper is that the system poles without delay effect has to be the same, i.e., with a repeating pole. Our problem formulation is motivated by control systems using networks as described in Section 2. Detailed derivation for the stability bound is given in Section 3. An example is given in Section 4. Section 5 concludes this paper with some remarks on further research.

2. Problem formulation

In this communique, the following second-order DDE is investigated:

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}+2\alpha\frac{\mathrm{d}y(t)}{\mathrm{d}t}+\alpha^2 y(t)=K_\mathrm{p}y(t-\tau), \tag{1}$$

where α, K_p and τ are constants. We are interested in establishing the analytical stability bound for the above DDE.

The above DDE can be the special case of the one-mass system controlled over the network. Consider the dynamical control system

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}+a_m\frac{\mathrm{d}y(t)}{\mathrm{d}t}+b_m y(t)=u(t), \tag{2}$$

where $u(t)$ is the control signal while the output is $y(t)$. The position $y(t)$ has a time delay τ due to the communication network used in the control system such as the web-based manipulation of the mass position. Suppose the control law used is a simple proportional controller, i.e.,

$$u(t)=K_\mathrm{p}y(t-\tau), \tag{3}$$

where K_p is the proportional controller gain, then system (2) under controller (3) is actually governed by a second-order DDE

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}+a_m\frac{\mathrm{d}y(t)}{\mathrm{d}t}+b_m y(t)-K_\mathrm{p}y(t-\tau)=0. \tag{4}$$

The question here is how parameters a_m, b_m, K_p and τ affect the overall stability property of DDE (4). More specifically, for a given system parameters a_m, b_m, and τ, how to determine the stability bound on controller gain K_p? It is desirable to have an analytical result concerning the stability bound. Unfortunately, there is no such analytical solution found in the literature. No symbolic solution exists to the best of our knowledge. However, in this paper, we are able to obtain an analytical solution for (1), a special case of (4) by assuming that the poles in system (2) are repeating. That is, when assuming $b_m=a_m^2/4 \triangleq \alpha^2$ the Laplacian transformation becomes

$$(s+\alpha)^2-K_\mathrm{p}\mathrm{e}^{-\tau s}=0. \tag{5}$$

Remark 2.1. Here the delay τ may be the sum of τ_c, the communication delay for certain protocol and τ_p, the plant delay due to physical transportation delay intrinsic to the system under control. We cannot change τ_p but may be able to adjust τ_c. Therefore, τ may become a design parameter to shape the stability bound.

From (5), in the next section, an analytical solution for s in terms of symbolic α, K_p and τ can be obtained by using the Lambert function W.

3. Analytical stability bound using Lambert function *W*

To obtain the stability bound, if possible, directly solving the transcendental equation (5) is the simplest way. This is possible by using the $W(\cdot)$ function discussed above. Multiplying $\mathrm{e}^{\tau s}$ to both side of (5) gives

$$(s+\alpha)^2\mathrm{e}^{\tau s}=K_\mathrm{p} \tag{6}$$

and putting the square root for both the sides of the above yields

$$(s+\alpha)\mathrm{e}^{(\tau/2)s}=\pm\sqrt{K_\mathrm{p}}. \tag{7}$$

Now, let

$$s_2=\frac{\tau}{2}s,\qquad \alpha_2=\frac{\tau\alpha}{2},\qquad K_2=\frac{\tau}{2}\sqrt{K_\mathrm{p}} \tag{8}$$

and (7) becomes

$$(s_2+\alpha_2)\mathrm{e}^{s_2+\alpha_2}=\pm K_2\mathrm{e}^{\alpha_2}. \tag{9}$$

Immediately, by using the Lambert W function,

$$s_2+\alpha_2=W(\pm K_2\mathrm{e}^{\alpha_2}), \tag{10}$$

i.e.,

$$s=\frac{2}{\tau}W\left(\frac{\tau}{2}\mathrm{e}^{((\tau/2)\alpha)}\right)(\pm\sqrt{K_\mathrm{p}})-\alpha, \tag{11}$$

which is what we called the "analytical stability bound" for DDE (4). Obviously, the stability condition is that for all possible τ, α and K_p,

$$\frac{2}{\tau}W\left(\frac{\tau}{2}\mathrm{e}^{((\tau/2)\alpha)}(\pm\sqrt{K_\mathrm{p}})\right)-\alpha\leqslant 0. \tag{12}$$

When $\tau = 0$, i.e., there is no delay, we have

$$\lim_{\tau\to 0} \frac{2}{\tau} W\left(\frac{\tau}{2}\mathrm{e}^{((\tau/2)\alpha)}(\pm\sqrt{K_\mathrm{p}})\right)$$

$$=\lim_{\tau\to 0} 2W'\left(\frac{\tau}{2}\mathrm{e}^{((\tau/2)\alpha)}(\pm\sqrt{K_\mathrm{p}})\right)\left(\pm\frac{\sqrt{K_\mathrm{p}}}{2}\right)$$

$$\times\mathrm{e}^{((\tau/2)\alpha)}\left(1+\frac{\alpha}{2}\tau\right)$$

$$=W'(0)(\pm\sqrt{K_\mathrm{p}}) = \pm\sqrt{K_\mathrm{p}}, \tag{13}$$

where $W'(x) = \mathrm{d}W(x)/\mathrm{d}x = W(x)/[x(1+W(x))]$, $W(0) = 0$ and $W'(0) = 1$. Therefore, the stability bound (12) for DDE with zero delay is that

$$\pm\sqrt{K_\mathrm{p}} - \alpha \leqslant 0, \tag{14}$$

which is in accordance with the classical stability result.

Remark 3.1. The analytical stability bound (12) can be extended to delayed systems with multiple repeating poles, i.e., for

$$(s+\alpha)^n - K_\mathrm{p}\mathrm{e}^{-\tau s} = 0, \tag{15}$$

its stability bound, in general, is that

$$\frac{n}{\tau} W\left(\frac{\tau}{n}\mathrm{e}^{((\tau/n)\alpha)}(\sqrt[n]{K_\mathrm{p}})\right) - \alpha \leqslant 0. \tag{16}$$

Remark 3.2. In many process control systems such as mixing systems of noninteracting series structure (Martin, 1995, pp. 156–165), the open-loop transfer function can be given by

$$G(s) = \frac{K\mathrm{e}^{-\tau s}}{(s+\alpha)^n}. \tag{17}$$

Its closed-loop characteristic equation under proportional control with gain K_p/K is the same as (15). Therefore, the stability bound analytically given in (16) is practically useful.

4. An example

We can get a 3D plot for $s(\tau, K_\mathrm{p})$ for any given α. In Fig. 1 where $\alpha = 1$ the stability bound can be clearly seen from the intersection of the surface $s(\tau, K_\mathrm{p})$ and surface $s = 0$. Similarly, the stability bound is obtained in Fig. 2 for $\alpha = 2$. Qualitatively, we know that when α gets larger, i.e., the stability margin of the nominal system (the ODE system without time delay term) of DDE increases, the stability bound for (τ, K_p) becomes bigger. This can be clearly observed by comparing Figs. 1 and 2.

In practice, we are more interested in determining a suitable range for K_p, given an α and a range for possible delays $[\tau_{\min}, \tau_{\max}]$. Fig. 3 shows a sample run using the obtained analytical stability bound (12). Clearly, from Fig. 3, K_p should be < 0.25. Note that the maximal K_p is restricted by $\tau_{\max}$. This is true when a stability margin is to be considered.

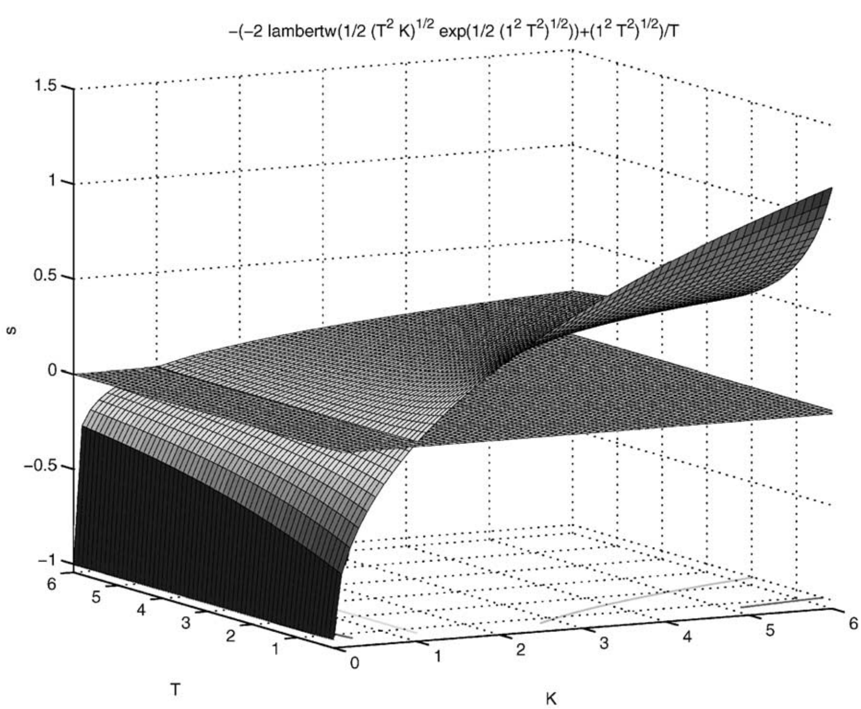

Fig. 1. The stability bound when $\alpha = 1$ (K-axis is for K_p and T-axis is for τ).

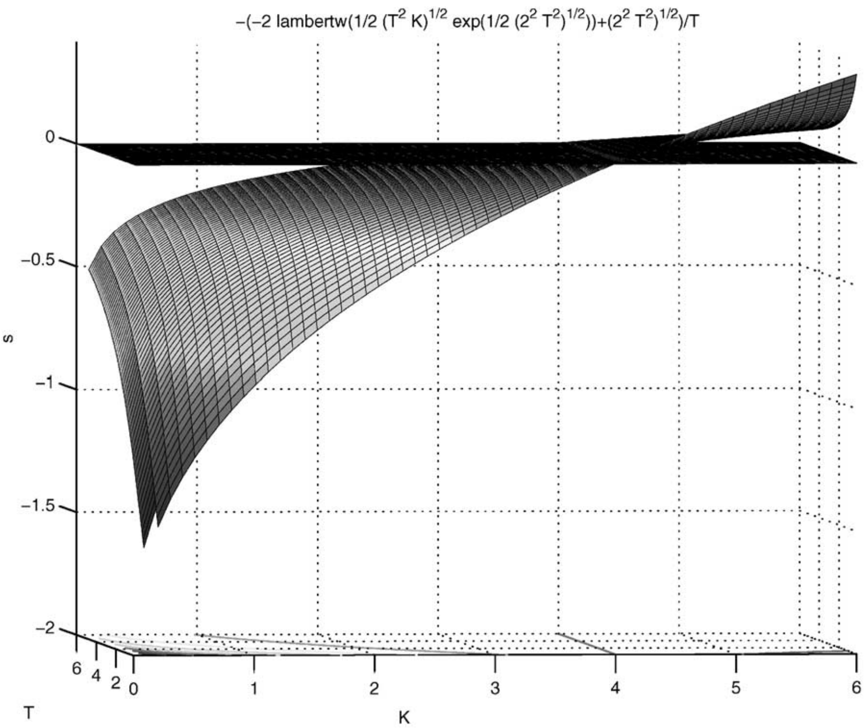

Fig. 2. The stability bound when $\alpha = 2$ (K-axis is for K_p and T-axis is for τ).

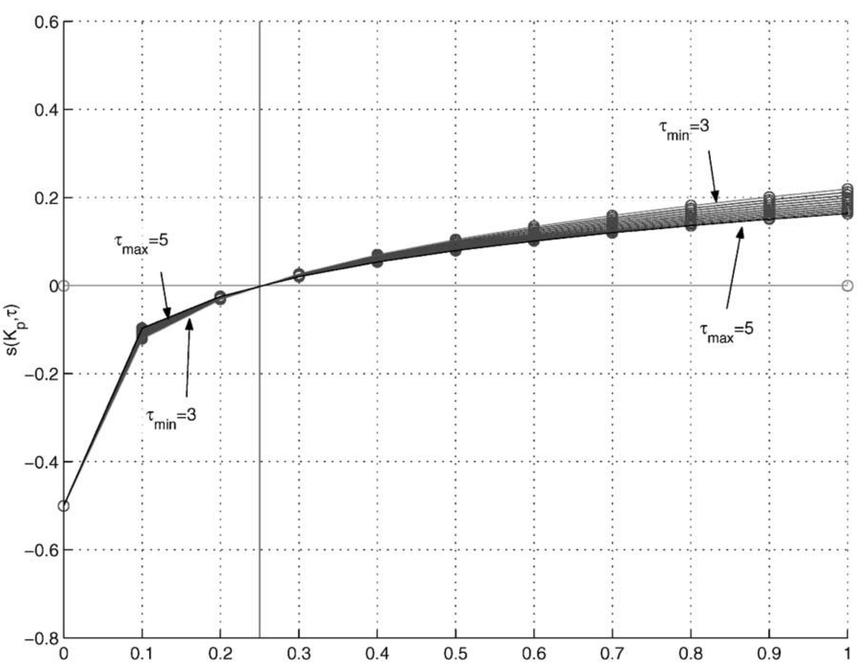

Fig. 3. The stabilizing range for K_p when $\alpha = 0.5$ and $\tau \in [3, 5]$.

5. Concluding remarks

In this paper, an analytical stability bound has been obtained for a class of second-order delay differential equations with constant coefficients. The results are generalizable to high-order systems. An example is given for illustration. Further efforts are to remove the restriction on the repeating pole assumption in this paper. This seems to be an open problem at this moment.

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