We consider the undamped control system

$$\ddot{y}(t) = Ay(t) + Bu(t) \qquad (1)$$

Denote $z(t) = \dot{y}(t)$, we obtain

$$\begin{cases} \dot{z}(t) = Ay(t) + Bu(t) \\ \dot{y}(t) = z(t) \end{cases}$$

$$\Leftrightarrow \frac{d}{dt}\begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} z(t) \\ y(t) \end{bmatrix}}_{X(t)} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t)$$

The explicit solution form is

$$X(t) = \exp\left(\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} t \right) \begin{bmatrix} z_0 \\ y_0 \end{bmatrix} + \int_0^t \exp\left(\begin{bmatrix} 0 & A \\ I_n & 0 \end{bmatrix} (t-s) \right) \begin{bmatrix} B \\ 0 \end{bmatrix} u(s)ds \qquad (2)$$

In case $u \equiv 0$, we obtain

$$X_0(t) := \exp\left(\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix} t \right) \begin{bmatrix} z_0 \\ y_0 \end{bmatrix} \qquad (3)$$

Problem 1: Find necessary & sufficient condition of A, B s.t. system (1) is positive, i.e., $\forall z_0 = \dot{y}(0) \geq 0$, $\forall y_0 = y(0) \geq 0$, $\forall u(t) \geq 0 \; \forall t$ we have $y(t) \geq 0$ $\forall t \geq 0$.

Notice that we only require $y(t)$ but not $\dot{y}(t) \geq 0 \; \forall t$.

Solution: let $M = \begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}$ we use the identity

$$e^{Mt} = \sum_{n=0}^{\infty} M^n \cdot \frac{t^n}{n!}$$

By induction we can directly prove that

$$M^{2n} = \begin{bmatrix} A^n & 0 \\ 0 & A^n \end{bmatrix} \quad \forall n \geq 0$$

$$M^{2n+1} = \begin{bmatrix} 0 & A^{n+1} \\ A^n & 0 \end{bmatrix} \quad \forall n \geq 0.$$

Consequently, from (3) we have

$$X_0(t) = \left(\sum_{n=0}^{\infty} \begin{bmatrix} A^n & 0 \\ 0 & A^n \end{bmatrix} \frac{t^{2n}}{(2n)!} + \begin{bmatrix} 0 & A^{n+1} \\ A^n & 0 \end{bmatrix} \frac{t^{2n+1}}{(2n+1)!} \right) \begin{bmatrix} z_0 \\ y_0 \end{bmatrix}$$

Thus, for $u \equiv 0$ we have

$$y(t) = \begin{bmatrix} 0 & I_n \end{bmatrix} X_0(t)$$

$$= \left(\sum_{n=0}^{\infty} \begin{bmatrix} 0 & A^n \end{bmatrix} \frac{t^{2n}}{(2n)!} + \begin{bmatrix} A^n & 0 \end{bmatrix} \frac{t^{2n+1}}{(2n+1)!} \right) \begin{bmatrix} z_0 \\ y_0 \end{bmatrix}$$

Choose the input $u \equiv 0$ & the initial condition $y_0 = \vec{0}$, we have

$$y(t) = \sum_{n=0}^{\infty} \cancel{[A^n \ 0] \, t} \; A^n \frac{t^{2n+1}}{(2n+1)!} z_0$$

Since $y(t) \geq 0 \ \forall t \geq 0, \forall z_0 \geq 0$ we see that

$A \geq 0$ (choose $t \simeq 0$ & disproof)
element wise

Now let us consider (2) & the inital condition $y_0 = z_0 = \vec{0}$ we have

$$y(t) = \int_0^t \begin{bmatrix} 0 & I_n \end{bmatrix} \exp\left(\begin{bmatrix} 0 & A \\ I_n & 0 \end{bmatrix} (t-s) \right) \begin{bmatrix} B \\ 0 \end{bmatrix} u(s)\, ds$$

$$= \int_0^t \left(\sum_{n=0}^{\infty} [0 \quad A^n] \frac{(t-s)^{2n}}{(2n)!} + [A^n \quad 0] \frac{(t-s)^{2n+1}}{(2n+1)!} \right) \cdot \begin{bmatrix} B \\ 0 \end{bmatrix} u(s)\, ds$$

Since $[0 \quad A^n] \begin{bmatrix} B \\ 0 \end{bmatrix} = 0$ & $[A^n \quad 0] \begin{bmatrix} B \\ 0 \end{bmatrix} = A^n$,

we have

$$y(t) = \int_0^t \sum_{n=0}^{\infty} A^n B \frac{(t-s)^{2n+1}}{(2n+1)!} u(s)\, ds$$

Again, using disproof & choosing suitable test function we see that for $n=0$ then $A^n B \geq 0 \Leftrightarrow B \geq 0$.

Thus, the desired condition for problem 1 is that

$$A \geqslant 0 \ \& \ B \geq 0.$$

<u>Problem 2:</u> State bounding estimation? Open.

<u>Problem 3:</u> l_1 (l_∞) gain control?