Eigenvalue Assignment via the Lambert W Function for Control of Time-delay Systems

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Eigenvalue Assignment via the Lambert W Function for Control of Time-delay Systems

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Abstract: In this paper, we consider the problem of feedback controller design via eigenvalue assignment for linear time-invariant systems of linear delay differential equations (DDEs) with a single delay. Unlike ordinary differential equations (ODEs), DDEs have an infinite eigenspectrum, and it is not feasible to assign all closed-loop eigenvalues. However, we can assign a critical subset of them using a solution to linear systems of DDEs in terms of the matrix Lambert W function. The solution has an analytical form expressed in terms of the parameters of the DDE, and is similar to the state transition matrix in linear ODEs. Hence, one can extend controller design methods developed based upon the solution form of systems of ODEs to systems of DDEs, including the design of feedback controllers via eigenvalue assignment. We present such an approach here, illustrate it using some examples, and compare with other existing methods.

Keywords: Delay differential equation, feedback control, eigenvalue assignment, Lambert W function.

1. INTRODUCTION

Using the classical pole-placement method, if a system of linear ordinary differential equations (ODEs) is completely controllable, the eigenvalues can be arbitrarily assigned via state feedback (Chen, 1984). However, delay differential equations (DDEs) always lead to an infinite spectrum of eigenvalues, and the determination of this spectrum requires a corresponding determination of roots of certain analytic functions. Moreover, an analytical solution of systems of DDEs has been lacking. Thus, such a pole-placement method for controller design, as in systems of ODEs, cannot be applied directly to systems of DDEs.

During recent decades, the stabilization of systems of linear DDEs using feedback control has been studied extensively. The problem of robust stabilization of time-delayed systems, or the stabilization problem via delayed feedback control, is most frequently solved

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via Finite Spectrum Assignment (Manitius and Olbrot, 1979; Wang et al., 1995; Brethé and Loiseau, 1998), which shifts an arbitrary, but finite, number of eigenvalues. The stabilization problem can also be approached using stability conditions, as expressed by solving a Riccati equation (Lien et al., 1999), or by the feasibility of a set of linear matrix inequalities (Li and deSouza, 1998; Niculescu, 2001). A stability analysis called the "Direct Method", in which a simplifying substitution is used for the transcendental terms in the characteristic equation (Olgac and Sipahi, 2002), was applied for active vibration suppression by Sipahi and Olgac (2003). The act-and-wait control concept was introduced for continuous-time control systems with feedback delay by Stépán and Insperger (2006). The study showed that if the duration of waiting is larger than the feedback delay, the system can be represented by a finite-dimensional monodromy matrix and, thus, the infinite-dimensional pole-placement problem is reduced to a finite-dimensional one. In addition, variants of the Smith predictor method have been developed to decrease errors enabling one to use Proportional–Integral–Derivative (PID) control in time-delayed systems (Fliess et al., 2002; Sharifi et al., 2003). A numerical stabilization method was developed by Michiels et al. (2002) using a simulation package that computes the rightmost eigenvalues of the characteristic equation. The approach is similar to the classical pole-assignment method for ODEs in determining the rightmost eigenvalues of a linear time-delay system using analytical and numerical methods.

Recently, an approach for the solution of linear time-invariant systems of DDEs has been developed using the Lambert W function (Asl and Ulsoy, 2003; Yi et al., 2007a). The approach using the Lambert W function provides a solution form for DDEs similar to that of the matrix exponential for ODEs (see Table 1 in Yi et al., 2007a). Unlike results obtained using other existing methods, the solution has an analytical form expressed with the parameters of the DDE. One can determine how the parameters are involved in the solution, and, furthermore, how each parameter affects each eigenvalue and the solution. In addition, each eigenvalue corresponds to a branch of the Lambert W function. Hence, the concept of the state transition matrix in ODEs can be generalized to DDEs using the matrix Lambert function. This suggests that some analyses used in systems of ODEs, based upon the concept of the state transition matrix, can potentially be extended to systems of DDEs. For example, concepts of observability and controllability with their Gramians become tractable (Yi et al., 2008).

In this paper, we apply the matrix Lambert W function-based approach for the solution to DDEs to stabilize linear systems of DDEs. Thus, we present a new approach for controller design via eigenvalue assignment, and illustrate the method with several examples. Using the proposed method, we can move a subset of the eigenvalues to desired locations in a manner similar to pole placement for systems of ODEs. For a given system, which can be represented by DDEs, the solution to the system is obtained based on the Lambert W function, and stability is determined. If the system is unstable, after controllability of the system is checked, a stabilizing feedback is designed by assigning eigenvalues, and finally the closed-loop system of DDEs can be stabilized. All of these results are based upon the Lambert W function-based approach.

2. EIGENVALUE ASSIGNMENT FOR TIME-DELAYED SYSTEMS

2.1. Stability

Consider a linear time-invariant (LTI) real system of DDEs, with a single constant delay, h,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{A}_{\mathrm{d}}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t), && t > 0, \\ \mathbf{x}(t) &= \mathbf{g}(t), && t \in [-h, 0), \\ \mathbf{x}(t) &= \mathbf{x}_0, && t = 0, \end{aligned} \tag{1}$$

where $\mathbf{A}$ and $\mathbf{A}_{\mathrm{d}}$ are $n \times n$ matrices, $\mathbf{x}(t)$ is an $n \times 1$ state vector, $\mathbf{B}$ is an $n \times r$ matrix, and $\mathbf{u}(t)$, an $r \times 1$ vector, is a function representing the external excitation. The coefficient matrix $\mathbf{C}$ is $p \times n$ and $\mathbf{y}(t)$ is a $p \times 1$ measured output vector. The solution to equation 1 in terms of the matrix Lambert W function is (Yi et al., 2007a)

$$\mathbf{x}(t) = \underbrace{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^{\mathrm{I}}}_{\text{free}} + \underbrace{\int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k (t-\xi)}\mathbf{C}_k^{\mathrm{N}}\mathbf{B}\mathbf{u}(\xi)\mathrm{d}\xi}_{\text{forced}}, \tag{2}$$

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k(\mathbf{A}_{\mathrm{d}}h\mathbf{Q}_k) + \mathbf{A}. \tag{3}$$

The coefficient $\mathbf{C}_k^{\mathrm{I}}$ in equation 2 is a function of $\mathbf{A}$, $\mathbf{A}_{\mathrm{d}}$, h and the preshape function, $\mathbf{g}(t)$, and the initial point, $\mathbf{x}_0$, while $\mathbf{C}_k^{\mathrm{N}}$ is a function of $\mathbf{A}$, $\mathbf{A}_{\mathrm{d}}$ and h, and does not depend on $\mathbf{g}(t)$ or $\mathbf{x}_0$. The methods for computing $\mathbf{C}_k^{\mathrm{I}}$ and $\mathbf{C}_k^{\mathrm{N}}$ were developed by Asl and Ulsoy (2003) and Yi et al. (2007a). The matrix, $\mathbf{Q}_k$ in equation 3 can be obtained from the following condition:

$$\mathbf{W}_k(\mathbf{A}_{\mathrm{d}}h\mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A}_{\mathrm{d}}h\mathbf{Q}_k)+\mathbf{A}h} = \mathbf{A}_{\mathrm{d}}h. \tag{4}$$

Conditions for convergence of the infinite series in equation 2 are presented in Yi et al. (2007a) and Yi et al. (2008). For example, for a bounded external excitation $\mathbf{u}(t)$, if the coefficient matrix, $\mathbf{A}_{\mathrm{d}}$, is nonsingular, the infinite series converges to the solution. Note that $\mathbf{W}_k$ in equations 3 and 4 denotes the matrix Lambert W function, as detailed in Appendix A.

The solution form in equation 2 reveals that the stability condition for the system of equation 1 depends on the eigenvalues of the matrix $\mathbf{S}_k$, and thus also on the matrix $e^{\mathbf{S}_k}$. A time-delayed system characterized by equation 1 is asymptotically stable if and only if:

All the Eigenvalues of $\mathbf{S}_k$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, have negative real parts,

or, equivalently, in the sense of Lyapunov:

All the Eigenvalues of $e^{\mathbf{S}_k}$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, lie within the unit circle.

Computing the matrices $\mathbf{S}_k$ or $e^{\mathbf{S}_k}$ for an infinite number of branches, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$ is not feasible. However, if the coefficient matrix, $\mathbf{A}_\mathrm{d}$, does not have repeated zero eigenvalues, then, we have observed that the characteristic roots of equation 1, see equation 40, obtained using only the principal branch ($k = 0$) are the rightmost ones in the complex plane and determine the stability of the system in equation 1. This gives (Yi et al., 2007b)

Conjecture:

$$\begin{aligned} \max\left[\mathrm{Re}\left\{\text{eigenvalues of } \mathbf{S}_0\right\}\right] \quad &\geq \quad \mathrm{Re}\left\{\text{all other eigenvalues of } \mathbf{S}_k\right\}, \\ &\text{for } k = -\infty, \cdots, -1, \ 0, \ 1, \cdots, \ \infty. \end{aligned} \tag{5}$$

For the general matrix DDE in equation 1, this is a conjecture, and there is no proof. However, for the scalar DDE case, it has been proven that the root obtained with the principal branch, $k = 0$, of the Lambert W function always determines stability (Shinozaki and Mori, 2006) using the monotonicity of the real part of the Lambert W function with respect to its branch k. Such a proof can readily be extended to systems of DDEs where $\mathbf{A}$ and $\mathbf{A}_\mathrm{d}$ are simultaneously triangularizable and, thus, commute with each other (Radjavi and Rosenthal, 2000). Even though such a proof is not available in the case of the general matrix-vector DDEs in equation 1, we have observed consistency with the Conjecture in all the examples we have considered. With this important and useful observation, we formulate the above Conjecture as the basis not only to determine the stability of systems of DDEs, but also to place a subset of the eigenspectrum at desired locations.

In designing a feedback controller for a delayed system, represented by DDEs in equation 1, because there exists an infinite number of solution matrices $\mathbf{S}_k$, $k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$, and the number of control parameters is finite, it is not feasible to assign all of them at once (Manitius and Olbrot, 1979). Placing a selected finite number of eigenvalues using the classical pole-placement method for ODEs (Chen, 1984) may cause other uncontrolled eigenvalues to move to RHP (Michiels et al., 2002). However, the subsequent approach for control design using the matrix Lambert W function, based on the above Conjecture, provides proper control laws without such loss of stability.

2.2. Eigenvalue Assignment

First, consider a scalar, autonomous, DDE,

$$\dot{x}(t) = ax(t) + a_\mathrm{d}x(t-h). \tag{6}$$

The solution to equation 6 can be obtained using the Lambert W function as (Corless et al., 1996; Asl and Ulsoy, 2003)

$$x(t) = \sum_{k=-\infty}^{\infty} e^{\left(\frac{1}{h}W_k\left(a_\mathrm{d}he^{-ah}\right)+a\right)t} C_k^\mathrm{I}. \tag{7}$$

Table 1. Corresponding values of a_d for each desired pole. By adjusting the parameter a_d, we can assign the rightmost pole of the system in equation 6 to the desired values.

$\lambda_{\mathrm{desired}}$	-0.5	0	0.5
a_d	0.3033	1.0000	2.4731

The roots, λ_k, of the characteristic equation of equation 6, $\lambda - a - a_\mathrm{d}e^{-\lambda h} = 0$ are

$$\lambda_k = \frac{1}{h} W_k \left(a_\mathrm{d} h e^{-ah} \right) + a, \quad \text{for} \quad k = -\infty, \cdots, -1, 0, 1, \cdots, \infty. \tag{8}$$

This solution is exact and analytical. In this scalar case, we can compute all of the roots, the rightmost pole among them is always obtained by using the principal branch ($k = 0$), and the pole determines stability of equation 6 (Shinozaki and Mori, 2006), that is,

$$\max \; [\mathrm{Re}\{W_k(H)\}] = \mathrm{Re}\{W_0(H)\}. \tag{9}$$

In designing a control law for the delayed system, it is crucial to handle the rightmost poles among an infinite number of ones. In this regard, the property in equation 9 of the Lambert W function is powerful. By adjusting the parameters, a, a_d and/or the delay time, h, we can assign the rightmost pole of the equation to be at the desired values in the complex plane. First, decide on the desired location of the rightmost pole, $\lambda_{\mathrm{desired}}$, that is,

$$\lambda_{\mathrm{desired}} = \lambda_0 = \frac{1}{h} W_0 \left(a_\mathrm{d} h e^{-ah} \right) + a. \tag{10}$$

Equation 10 can be solved using numerical methods, e.g. “fsolve”, and a command for the Lambert W function, “lambertw”, which are already embedded in Matlab.

Example 1: Consider the scalar DDE in equation 6 with $a = -1$ and $h = 1$. Table 1 shows the corresponding values of a_d required to move the rightmost pole of equation to the exact desired locations. As seen in Figure 1, each rightmost pole locates on the desired position corresponding to a_d.

However, each branch of the Lambert W function has its own range and, especially, the value of the principal branch has the range (Corless et al., 1996):

$$\mathrm{Re}\{W_0(H)\} \geq -1. \tag{11}$$

Therefore, depending on the structure or parameters of a given system, there exist limitations on assigning the rightmost pole. Although general research on the limitation is lacking so far, in the example above we can conclude that (using equation 11):

$$\mathrm{Re}\{S_0\} = \frac{1}{h} \underbrace{\mathrm{Re}\left\{ W_0(a_\mathrm{d} h e^{-ah}) \right\}}_{\geq -1} + a \geq -\frac{1}{h} + a. \tag{12}$$

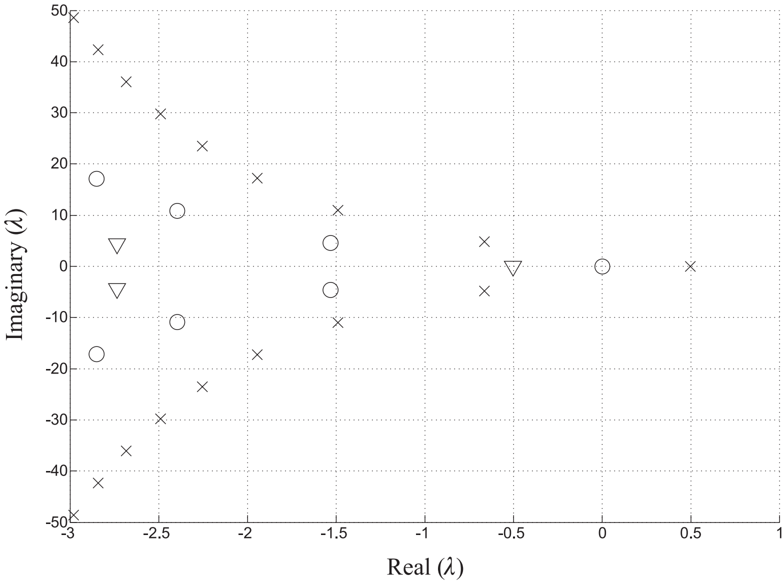

Figure 1. eigenspectra of equation 6 with $a = -1$, $h = 1$ and a_d in Table 1. Using the values of a_d, we can make the rightmost poles move exactly to the desired locations $(-0.5(\triangledown); 0(\mathrm{O}); 0.5(\times))$.

Thus, the rightmost eigenvalue cannot be smaller than –2 with any value of a_d.

In the case of systems of DDEs, by adjusting the elements in the coefficient matrices $\mathbf{A}$ and $\mathbf{A}_\mathrm{d}$ in equation 1, one can assign the eigenvalues of a single matrix, $\mathbf{S}_0$, corresponding to the principal branch, $k = 0$, by solving equations 3, 4 and

$$\text{Eigenvalues of } \mathbf{S}_0 = \text{desired values} \tag{13}$$

simultaneously using numerical methods embedded in software packages, such as "fsolve" in Matlab. This approach is based upon the Conjecture presented previously in Section 2.1, is applied here to design feedback controllers and is validated with examples. In the subsequent section, the approach presented above is applied to the design of feedback control laws for both the scalar DDE and systems of DDEs.

3. DESIGN OF A FEEDBACK CONTROLLER

Delay terms can arise in two different ways: i) delays in the control, that is, $\mathbf{u}(t-h)$, or ii) delays in the state variables, $\mathbf{x}(t-h)$. In both cases, the resulting feedback operators contain integrals over the past values of control or state trajectory (Manitius and Olbrot, 1979). For systems without time delay, a time-delayed control is often used for various special purposes, often motivated by intuition. The most common examples are the vibration absorber with delayed feedback control, with which it is possible to absorb an external force of unknown frequencies (Olgac et al., 1997), and delayed feedback to stabilize unstable periodic orbits without any information of the periodic trajectory, except the period by constructing a control force from the difference of the current state to the state one period before (Pyragas, 1992; Hövel and Schöll, 2005).

3.1. Scalar Case

For a scalar DDE with state feedback:

$$\dot{x}(t) = ax(t) + a_{\mathrm{d}}x(t-h) + u(t) \quad \text{and} \quad u = kx(t). \tag{14}$$

One may approach this problem with the first-order Padé approximation as:

$$e^{-hs} \approx \frac{1 - hs/2}{1 + hs/2}. \tag{15}$$

Then, the characteristic equation of 14 becomes a simple second-order polynomial:

$$s^2h + s(2 - ah - kh + a_{\mathrm{d}}h) - 2(a + k) - 2a_{\mathrm{d}} = 0. \tag{16}$$

Then, for a desired pole, we can derive the control gain. For example, with parameters $a = 1$, $a_d = -2$ and $h = 1$, equation 16 becomes

$$s^2 - s(1 + k) + 2 - 2k = 0. \tag{17}$$

For the value $k = -1.1$, equation has two stable poles. However, this control gain is applied to the original system of equation 14 will fail to stabilize the system. The resulting eigenspectrum is shown in Figure 2. Even though higher order Padé approximations, or other advanced rational approximations, can be used to approximate the exponential term in the characteristic equation, such approaches constitute a limitation in accuracy, and at worst may lead to instability of the original system (Silva et al., 2001; Richard, 2003).

Alternatively, from the characteristic equation of 14 with the desired pole, the linear equation for k can be derived as:

$$\lambda_{\text{desired}} - a - a_{\mathrm{d}}e^{-\lambda_{\text{desired}}h} - k = 0. \tag{18}$$

For example, with the parameters $a = 1$, $a_d = -1$ and $h = 1$, just by substituting the variable as $\lambda_{\text{desired}} = -1$, then, the obtained gain, k, using equation 18 is 0.7183. However, this control gain is also applied to the original system in equation 14 and fails to stabilize the system, because the desired pole is not guaranteed to be the rightmost pole. The resulting eigenspectrum is shown in Figure 3.

While one of the poles, not the rightmost, is placed at the desired location, -1, the rightmost one is in RHP. Although, for the desired pole, the control gain, k, is derived, when the gain is applied to equation 18, there exists other infinite number of poles to satisfy the equation and some of them can have real parts larger than that of the desired pole.

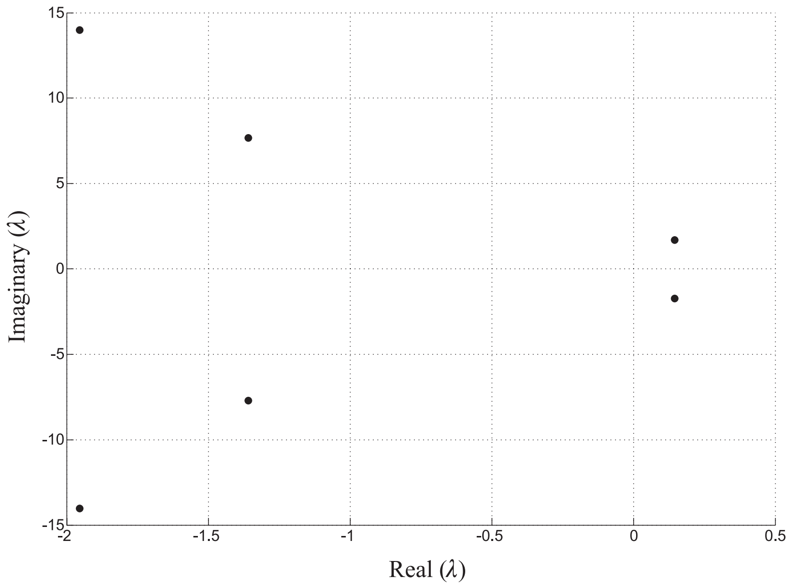

Figure 2. Eigenspectrum with the feedback controller designed using the Padé approximation showing the failure to move the pole to the desired value and thus stabilize the system.

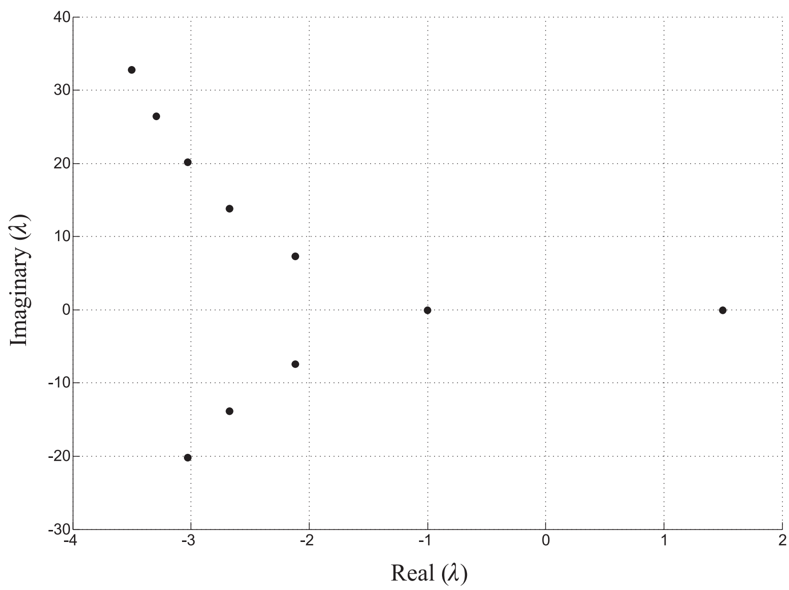

Figure 3. Eigenspectrum with the feedback controller designed using the linear approach in equation 18 showing the failure to move the pole to the desired value and thus stabilize the system.

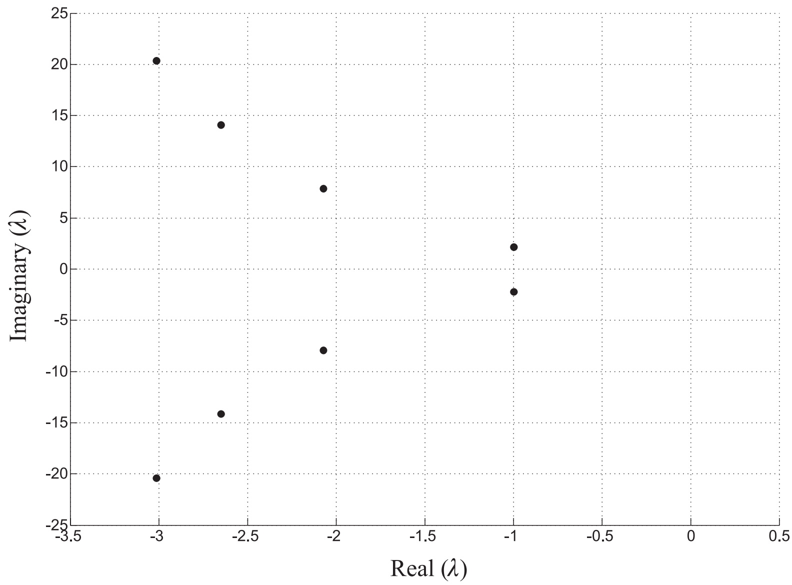

Figure 4. Eigenspectrum with the feedback controller designed using the Lambert W function approach. The rightmost eigenvalues are placed at the exact desired location.

On the other hand, using the Lambert W function, one can safely assign the real part of the rightmost pole exactly. For example, for system with $a = 1$, $a_d = -1$ and $h = 1$,

$$\mathrm{Re}\left(S_0 = \frac{1}{h}W_0\left(a_\mathrm{d}he^{-(a+k)h}\right) + a + k\right) = -1. \tag{19}$$

Then, the resulting value of k is -3.5978. As seen in Figure 4, the eigenvalues are placed at the exact desired location. Compared with the results in Figures 2 and 3, the approach using the Lambert W function provides the exact result and stabilizes the unstable system safely.

3.2. Systems with Control Delays

In systems of controllable ODEs, the major result is that, with full state feedback, one can specify all the closed-loop eigenvalues by selecting the gains. However, DDEs have an infinite number of eigenvalues, and it is not feasible to specify all of them. Furthermore, research on the relation between controllability and eigenvalue assignment is lacking so far. Nevertheless, in Sections 3.2 and 3.3, for the controllable system of DDEs, the Lambert W function approach is used to specify the first matrix, $\mathbf{S}_0$, corresponding to the principal branch, $k = 0$, and is observed to be critical in the solution form in equation 2, by choosing the feedback gain and designing a feedback controller.

First, consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-h). \tag{20}$$

Then, the feedback

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) \tag{21}$$

yields the closed-loop form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{x}(t-h). \tag{22}$$

The gain, $\mathbf{K}$, to assign the rightmost eigenvalues is determined as follows. First, select desired eigenvalues, $\lambda_{i,\text{desired}}$, for $i = 1, \ldots, n$, and set an equation so that the selected eigenvalues become those of the matrix $\mathbf{S}_0$ as

$$\lambda_i(\mathbf{S}_0) = \lambda_{i,\text{desired}}, \quad \text{for} \quad i = 1, \ldots, n, \tag{23}$$

where $\lambda_i(\mathbf{S}_0)$ is the i^{th} eigenvalue of the matrix $\mathbf{S}_0$. Second, apply the two new coefficient matrices $\mathbf{A} = \mathbf{A}, \mathbf{A}_{\text{d}} = \mathbf{BK}$ in equation 22 to equation 4 and solve numerically to obtain the matrix $\mathbf{Q}_0$ for the principal branch ($k = 0$). Note that $\mathbf{K}$ is an unknown matrix with all unknown elements in it, and the matrix $\mathbf{Q}_0$ is a function of the unknown $\mathbf{K}$. Then, for the third step, substitute the matrix $\mathbf{Q}_0$ from equation 4 into equation 3 to obtain $\mathbf{S}_0$ and its eigenvalues as the function of the unknown matrix $\mathbf{K}$. Finally, equation 23 with the matrix $\mathbf{S}_0$ is solved for the unknown $\mathbf{K}$ using numerical methods such as "fsolve" in Matlab. As mentioned in Section 2.2, depending on the structure or parameters of a given system, there exists a limitation of the rightmost eigenvalues, and some values are not proper for the rightmost eigenvalues. In that case, the above approach does not yield any solution for $\mathbf{K}$. To resolve the problem, one may try with fewer desired eigenvalues, or different values of the desired rightmost eigenvalues. Then, the solution, $\mathbf{K}$, is obtained numerically for a variety of initial conditions by an iterative trial and error procedure.

Example 2: Consider the van der Pol equation, which has become a prototype for systems with self-excited limit cycle oscillations and has the form

$$\ddot{x}(t) + f(x,t)\dot{x}(t) + x(t) = g(x,t;h), \tag{24}$$

with

$$f(x,t) = \varepsilon\left(x^2(t) - 1\right). \tag{25}$$

For the dynamics of the van der Pol equation under the effects of linear position and velocity time-delayed feedback, the left-hand side of equation 24 can be written as

$$g(x, t; h) = k_1 x(t-h) + k_2 \dot{x}(t-h). \tag{26}$$

Then, with the damping coefficient function in equation 25 and feedback in equation 26, equation 24 becomes

$$\ddot{x}(t) + x(t) = \varepsilon \left(1 - x^2(t)\right) \dot{x}(t) + k_1 x(t-h) + k_2 \dot{x}(t-h). \tag{27}$$

Linearizing equation 27 about the zero equilibrium yields the equation for infinitesimal perturbations,

$$\ddot{x}(t) + x(t) = \varepsilon \dot{x}(t) + k_1 x(t-h) + k_2 \dot{x}(t-h). \tag{28}$$

Or, equivalently, by defining $x_1 = x$ and $x_2 = \dot{x}$, one obtains the state equations

$$\left\{ \begin{array}{c} \dot{x}_1(t) \\ \dot{x}_2(t) \end{array} \right\} = \left[\begin{array}{cc} 0 & 1 \\ -1 & \varepsilon \end{array} \right] \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} + \left[\begin{array}{cc} 0 & 0 \\ k_1 & k_2 \end{array} \right] \left\{ \begin{array}{c} x_1(t-h) \\ x_2(t-h) \end{array} \right\}, \tag{29}$$

which can also be expressed in the form of equation 22 as:

$$\dot{\mathbf{x}}(t) = \underbrace{\left[\begin{array}{cc} 0 & 1 \\ -1 & \varepsilon \end{array} \right]}_{\mathbf{A}} \mathbf{x}(t) + \underbrace{\left[\begin{array}{c} 0 \\ 1 \end{array} \right]}_{\mathbf{B}} \underbrace{\left[\begin{array}{cc} k_1 & k_2 \end{array} \right]}_{\mathbf{K}} \mathbf{x}(t-h). \tag{30}$$

Equations of this type were investigated using the asymptotic perturbation method (Maccari, 2001), bifurcation methods (Reddy et al., 2000; Wirkus and Rand, 2002; Xu and Chung, 2003) and the Taylor expansion with averaging (Li et al., 2006) to show that vibration control and quasi-periodic motion suppression are possible for appropriate choices of the time delay and feedback gains. The effect of time delay under external excitation, with various practical examples, was considered by Maccari (2003), demonstrating the importance of this oscillator in engineering science.

Controllability and stabilizability of the system of equations in equation 20 have been studied during recent decades (Olbrot, 1972; Frost, 1982; Mounier, 1998). According to the definition and the corresponding simple rank condition in equation 49, presented by Olbrot (1972), also in Appendix B, the system of equation 29 is *controllable*. Therefore, using the pole-placement method introduced in the previous section, we can design an appropriate feedback controller to stabilize the system and choose the gains k_1 and k_2 to locate the eigenvalues at desired positions in the complex plane.

Without the delayed feedback term (i.e. $k_1 = k_2 = 0$), the system in equation 29 is unstable when $\varepsilon = 0.1$, and its eigenvalues are 0.0500 ± 0.9987i. For example, when $h = 0.2$, if the desired eigenvalues are -1 and -2, which are arbitrarily selected, then, the required gains are found to be $k_1 = -0.0469$ and $k_2 = -1.7663$. As seen in Figure 5, the response without feedback control is unstable. Applying the designed feedback controller stabilizes the system. Figure 6 shows the eigenspectra of systems without feedback and with feedback. The rightmost eigenvalues are moved exactly to the desired locations and all the other eigenvalues are to the left. If the desired eigenvalues are -1.0000 ± 2.0000i,

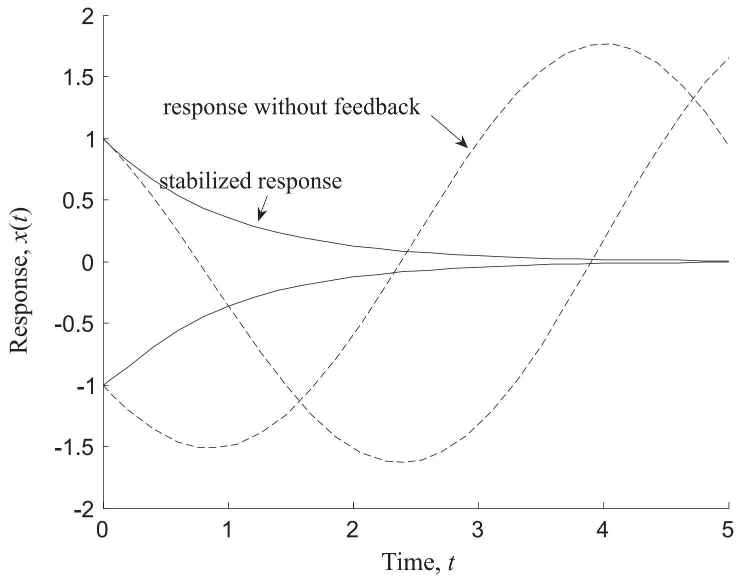

Figure 5. Comparison of responses before (dashed) and after (solid) applying feedback with $\mathbf{K} = [-0.0469 \quad -1.7663]$. The chosen feedback gain stabilizes the system.

or $-1.0000 \pm 1.0000\text{i}$, then the corresponding gains are $\mathbf{K} = [-1.9802 \quad -1.8864]$, or $\mathbf{K} = [-0.2869 \quad -1.5061]$, respectively.

In Michiels et al. (2002), a numerical stabilization method was developed using a simulation package that computes the rightmost eigenvalues of the characteristic equation. For the obtained finite number of eigenvalues, the eigenvalues can be moved to the LHP using sensitivities with respect to changes in the feedback gain, k (see Figures 7 and 8). Compared with this approach, the matrix Lambert W function-based method yields the equation for assignment of the rightmost eigenvalues with the parameters of the system. Using the analytical expression, one can obtain the control gain to move the critical eigenvalues to the desired positions without starting with their initial unstable positions or computing the rightmost eigenvalues and their sensitivities after every small movement in a quasi-continuous way. Using the Lambert W function, we can find the control gain independently of the path of the rightmost eigenvalues. Without planning the path, only from their destination, the control laws for the system are obtained.

For the system in equation 20 with equation 21, the control law using the Finite Spectrum Assignment (FSA) method based on prediction,

$$\mathbf{u}(t) = \mathbf{K}e^{\mathbf{A}h}\mathbf{x}(t) + \mathbf{K}\int_0^h e^{\mathbf{A}(h-\theta)}\mathbf{B}\mathbf{u}(t+\theta-h)\mathrm{d}\theta, \tag{31}$$

can make the system finite-dimensional and assign the finite eigenvalues to the desired values (Manitius and Olbrot, 1979; Wang et al., 1995; Brethé and Loiseau, 1998). However, such a method requires model-based calculation, which may cause unexpected errors when

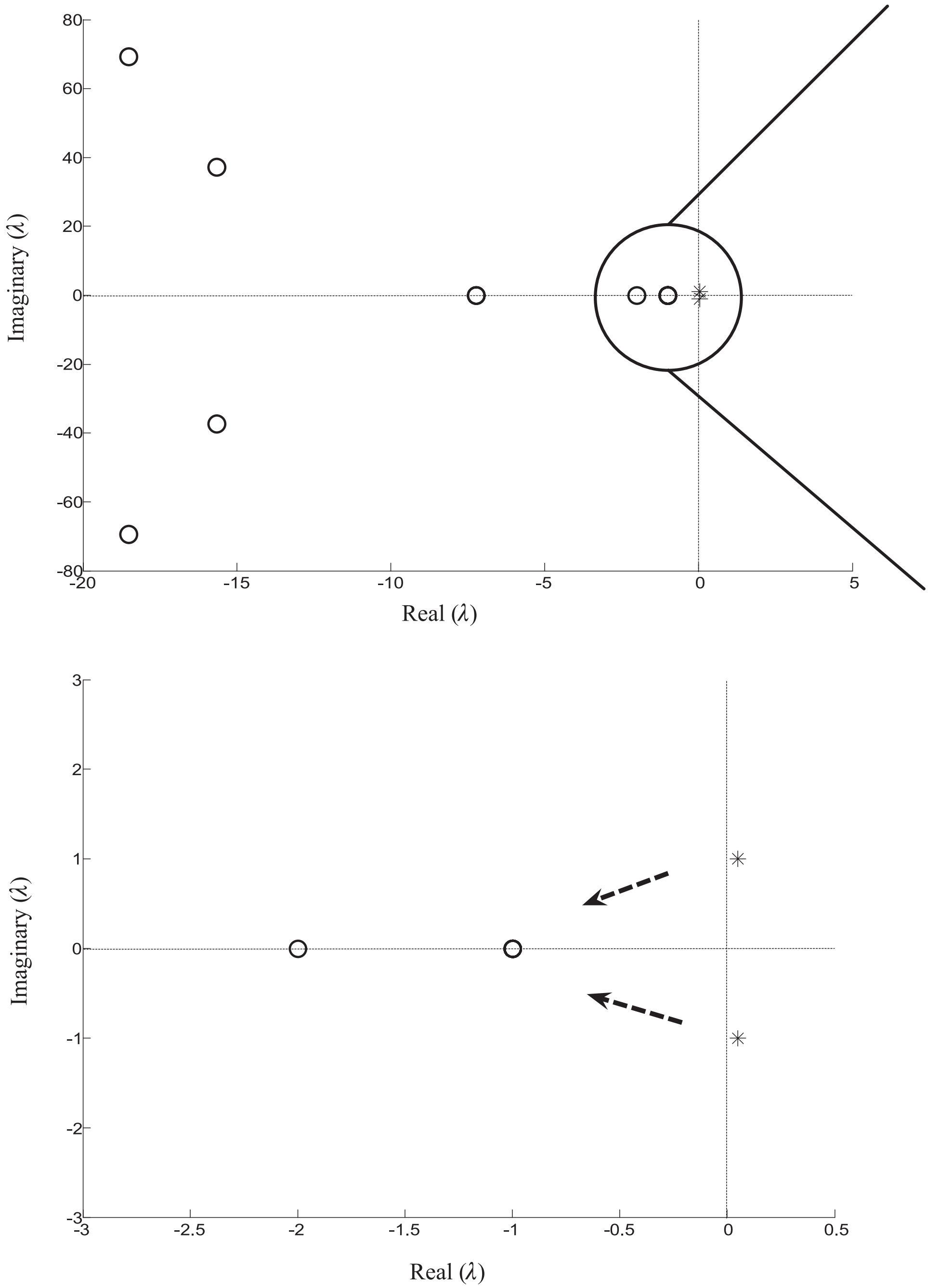

Figure 6. Movement of eigenvalues after applying the feedback (* without feedback; o with feedback). The rightmost eigenvalues are located at the exact desired location –1 and –2.

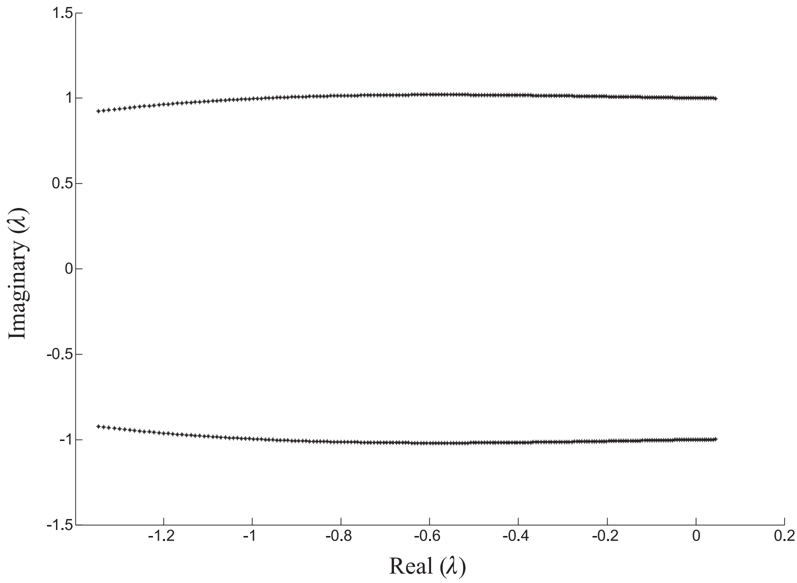

Figure 7. Movement of eigenvalues from their original positions, $0.0500 \pm 0.9987i$, to LHP when the method in Michiels et al. (2002), is applied to the system in equation 30.

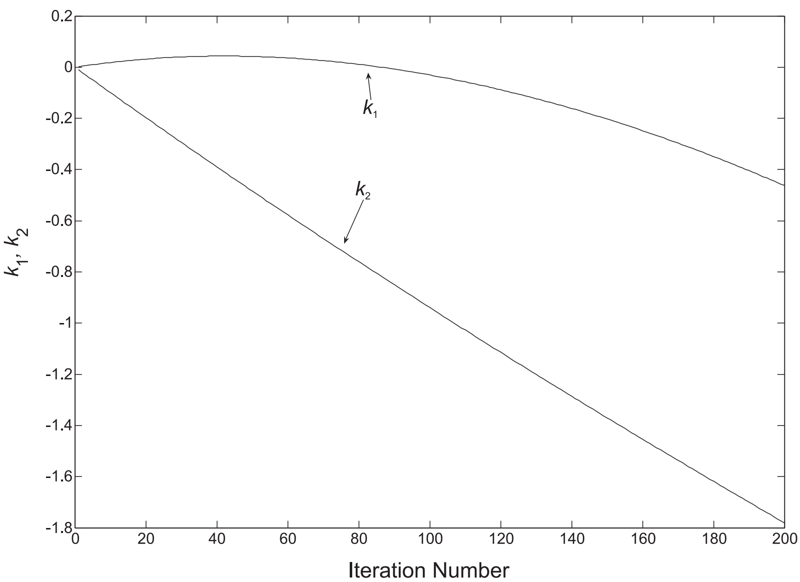

Figure 8. The value of $\mathbf{K}$ in equation 30 corresponding to the number of iteration to compute for the movement of eigenvalues in Figure 7.

applied to a real system. Limitations on the FSA have been studied, with several examples by Engelborghs et al. (2001) and Van Assche et al. (1999), the implementation of such a controller is still an open problem (Richard, 2003).

3.3. Systems with State Delays

Consider the following time-delayed system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_{\mathrm{d}}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t) \tag{32}$$

and a generalized feedback containing current and delayed states:

$$\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t) + \mathbf{K}_{\mathrm{d}}\mathbf{x}(t-h). \tag{33}$$

Then, the closed-loop system becomes

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}(t) + (\mathbf{A}_{\mathrm{d}} + \mathbf{B}\mathbf{K}_{\mathrm{d}})\mathbf{x}(t-h). \tag{34}$$

The gains, $\mathbf{K}$ and $\mathbf{K}_d$ are determined as follows. First, select the desired eigenvalues, $\lambda_{i,\mathrm{desired}}$ for $i = 1, \ldots, n$, and set an equation so that the selected eigenvalues become those of the matrix $\mathbf{S}_0$ as

$$\lambda_i(\mathbf{S}_0) = \lambda_{i,\mathrm{desired}}, \quad \text{for} \quad i = 1, \ldots, n, \tag{35}$$

where, $\lambda_i(\mathbf{S}_0)$ is the i^{th} eigenvalue of the matrix $\mathbf{S}_0$. Second, apply the new two coefficient matrices $\mathbf{A} = \mathbf{A}+\mathbf{B}\mathbf{K}$ and $\mathbf{A}_{\mathrm{d}} = \mathbf{A}_{\mathrm{d}}+\mathbf{B}\mathbf{K}_{\mathrm{d}}$ in equation 34 to equation 4 and solve numerically to obtain the matrix $\mathbf{Q}_0$ for the principal branch ($k = 0$). Note that $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$ are unknown matrices with all unknown elements, and the matrix $\mathbf{Q}_0$ is a function of the unknown $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$. For the third step, substitute the matrix $\mathbf{Q}_0$ from equation 4 into equation 3 to obtain $\mathbf{S}_0$ and its eigenvalues as the function of the unknown matrix $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$. Finally, equation 35 with the matrix, $\mathbf{S}_0$, is solved for the unknown $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$ using numerical methods, such as “fsolve” in Matlab. As mentioned in Section 2.2, depending on the structure or parameters of a given system, there exists a limitation of the rightmost eigenvalues, and some values are not proper for the rightmost eigenvalues. In that case, the above approach does not yield any solution for $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$. To resolve the problem, one may try again with fewer desired eigenvalues, or different values of the desired rightmost eigenvalues. Then, the solution, $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$, is obtained numerically for a variety of initial conditions by an empirical trial and error procedure.

The controllability of such a system, using the solution form of equation 2 was studied by Yi et al. (2008), as briefly summarized in Appendix B. In the case of LTI systems of ODEs, if it is completely controllable, then the eigenvalues can arbitrarily be assigned by choosing feedback gain. Here, examples considering controllability and eigenvalue assignment in DDEs are considered.

Example 3: Consider the following system of DDEs:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 1.1000 & -0.1732 \\ -0.0577 & 1.1000 \end{bmatrix} \mathbf{x}(t) \\ &+ \begin{bmatrix} 0.3500 & 0.2598 \\ 0.0866 & 0.3500 \end{bmatrix} \mathbf{x}(t-h) + \begin{bmatrix} 1.0000 \\ -0.5774 \end{bmatrix} \mathbf{u}(t).\end{aligned} \tag{36}$$

When the coefficients are applied to the condition in equation 50 for controllability (Yi et al., 2008, also see Appendix B), the corresponding matrix is

$$\left(s\mathbf{I} - \mathbf{A} - \mathbf{A}_{\mathrm{d}} e^{-sh}\right)^{-1} \mathbf{B} = \begin{bmatrix} \dfrac{5}{5s - 6 - e^{-sh}} \\ \dfrac{-5/\sqrt{3}}{5s - 6 - e^{-sh}} \end{bmatrix}. \tag{37}$$

Obviously, the two elements in equation 37 are linearly dependent and equation 36 fails the rank condition in equation 50. Thus, the system in equation 36 is not *point-wise controllable*, and we cannot find any appropriate feedback control in the form of equation 33 to stabilize it.

Example 4: Consider the following time-delay model from Mahmoud and Ismail (2005):

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} \mathbf{x}(t-h) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(t). \tag{38}$$

Before applying the feedback, the rightmost eigenvalues are –1.1183 and 0.1098, and, thus, the system is unstable when the delayed time $h = 0.1$ (see Figures 9 and 10). When the coefficients are applied to the condition in equation 50, the system in equation 38 satisfies the criterion, and, thus, is *point-wise controllable*. Then, using the pole-placement method, we can design an appropriate feedback controller to stabilize the system and choose the gains $\mathbf{K}$ and $\mathbf{K}_{\mathrm{d}}$ to locate the eigenvalues at desired positions in the complex plane. For example, when the desired eigenvalues are -1.0000 and -6.0000, which are chosen arbitrarily, the computed gains are $\mathbf{K} = [-0.1391 \quad -1.8982]$; $\mathbf{K}_{\mathrm{d}} = [-0.1236 \quad -1.8128]$, or $\mathbf{K} = [-0.1687 \quad -3.6111]$; $\mathbf{K}_{\mathrm{d}} = [1.6231 \quad -0.9291]$ for -2.0000; -4.0000. By applying the obtained feedback gains to equation 33, we can stabilize the system (see Figure 9) and place the eigenvalues at desired positions in the complex plane (see Figure 10).

4. CONCLUSION

In this paper, new results for feedback controller design for a class of time-delayed systems are presented. For a given system, which can be represented by DDEs, based on the Lambert

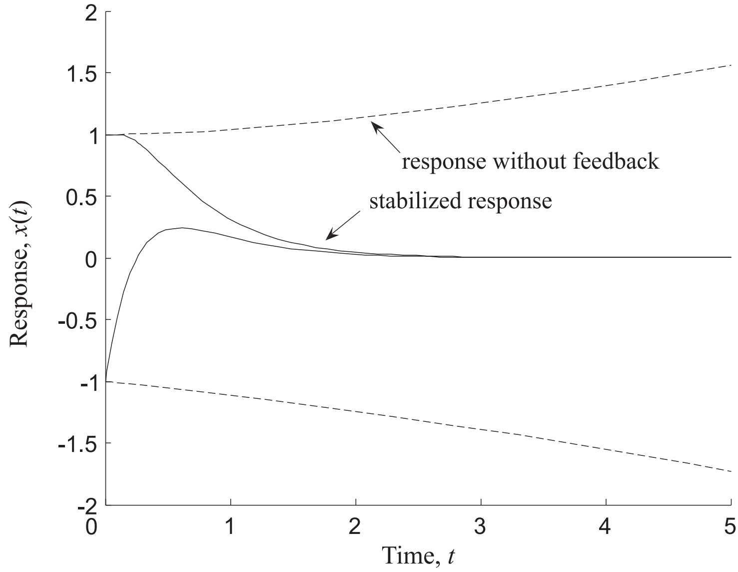

Figure 9. Comparison of responses before (dashed) and after (solid) applying feedback 33 with feedback gains $\mathbf{K} = [-0.1687 \quad -3.6111]$ and $\mathbf{K}_d = [1.6231 \quad -0.9291]$. The chosen feedback gain stabilizes the system.

W function, the solution to the system is obtained, and stability is determined. If the system is unstable, after the controllability of the system is checked, a stabilizing feedback is designed by assigning eigenvalues, and finally the closed-loop system of DDEs can be stabilized. All of these results are based upon the Lambert W function-based approach. Numerical examples are presented to illustrate the approach. Although DDEs have an infinite eigenspectrum, and it is not possible to assign all closed-loop eigenvalues, we assign a subset of them that are critical in determining the stability for the system of DDEs.

We compare the proposed method, based upon the Lambert W function, with other approaches (see examples in Section 3). Many of these are ad-hoc, and can fail on certain problems (Silva et al., 2001; Richard, 2003). The FSA method is based upon prediction, and known to have implementation problems (Van Assche et al., 1999; Engelborghs et al, 2001). The method of Michiels et al. (2002) is the most effective, but is an iterative method based upon the sensitivity of eigenvalues to the control gains. The Lambert W function-based method is direct and effective in all problems evaluated.

In future research, we plan to use this approach for the design of systems with observer-based feedback controller for systems of DDEs. Problems of robust controller design and time-domain specifications are also being considered.

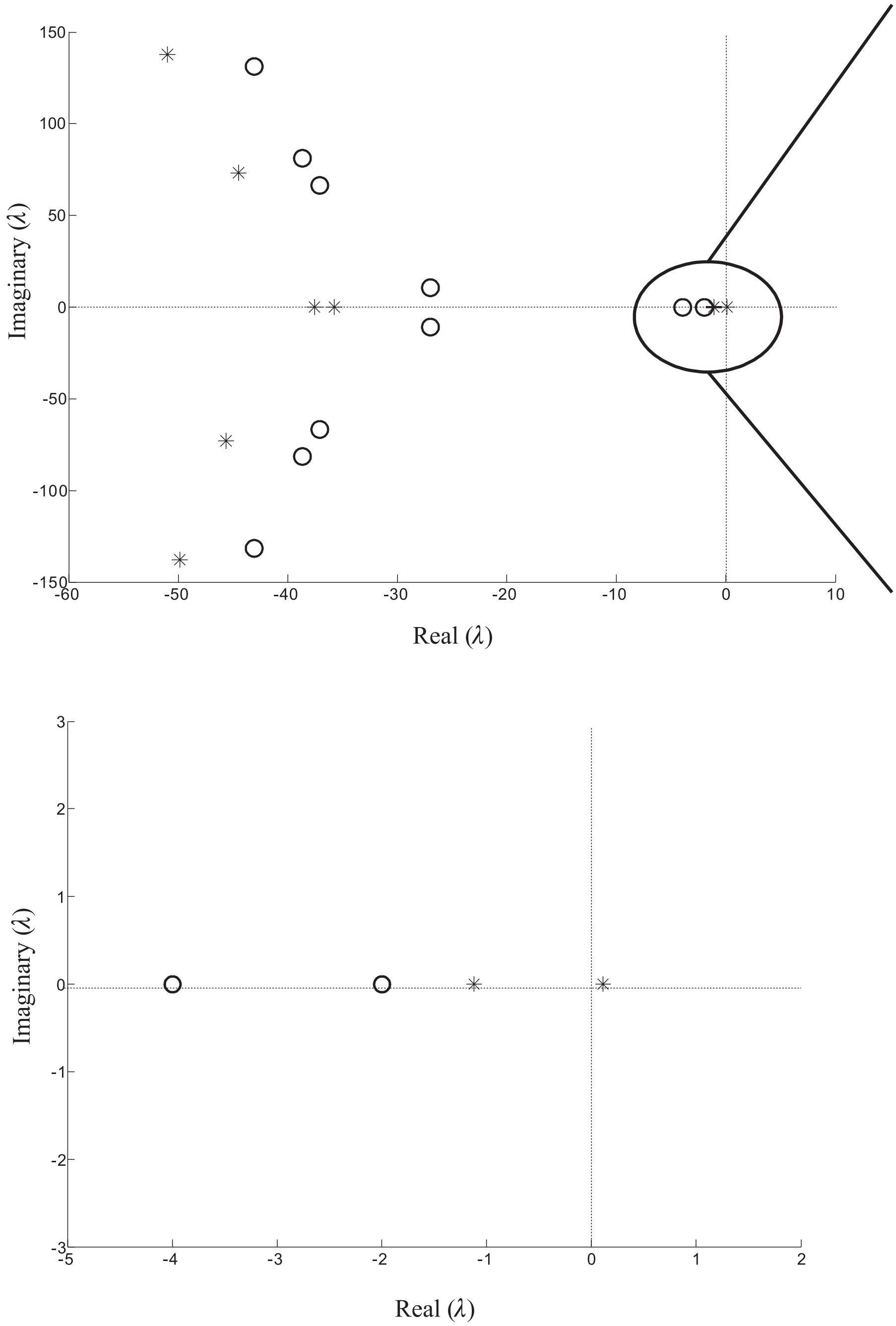

Figure 10. Movement of eigenvalues after applying feedback (* without feedback; o with feedback). The rightmost eigenvalues are located at the exact desired location -2.0000; -4.0000 using the feedback gains $\mathbf{K} = [-0.1687 \quad -3.6111]$ and $\mathbf{K}_\mathrm{d} = [1.6231 \quad -0.9291]$.

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APPENDIX A. DERIVATION OF SOLUTION TO EQUATION 1 USING THE MATRIX LAMBERT W FUNCTION

The analytical solution, equation 2, which is similar to that of the matrix exponential for ODEs, to systems of DDEs in equation 1 using the Lambert W function was developed by Yi et al. (2007a). Here, we briefly summarize the derivation of the solution. First, we assume a free solution form as

$$\mathbf{x}(t) = e^{\mathbf{S}t}\mathbf{x}_0, \tag{39}$$

where $\mathbf{S}$ is an $n \times n$ matrix. Typically, the characteristic equation for equation 1 is obtained by assuming a nontrivial solution of the form $e^{st}\mathbf{C}$ as

$$\det\left(s\mathbf{I} - \mathbf{A} - \mathbf{A}_{\mathrm{d}}e^{-sh}\right) = 0, \tag{40}$$

where s is a scalar variable and $\mathbf{C}$ is constant (Hale and Lunel, 1993). However, such an approach leads to a solution to systems of DDEs in equation. Alternatively, one can assume the form of to derive the solution to systems of DDEs in equation 1 using the matrix Lambert W function. Substitution of equation 39 into equation 1 enables one to derive a homogeneous solution to equation 1

$$\mathbf{x}(t) = \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^{\mathrm{I}}, \tag{41}$$

where

$$\mathbf{S}_k = \frac{1}{h}\mathbf{W}_k\left(\mathbf{A}_{\mathrm{d}}h\mathbf{Q}_k\right) + \mathbf{A}. \tag{42}$$

The constant matrices $\mathbf{C}_k^{\mathrm{I}}$ are computed from a given preshape function $\mathbf{g}(t)$ and initial point $\mathbf{x}_0$. The matrix $\mathbf{Q}_k$ is obtained from the following condition, which can be used to solve for the unknown matrix $\mathbf{Q}_k$,

$$\mathbf{W}_k(\mathbf{A}_{\mathrm{d}}h\mathbf{Q}_k)e^{\mathbf{W}_k(\mathbf{A}_{\mathrm{d}}h\mathbf{Q}_k)+\mathbf{A}h} = \mathbf{A}_{\mathrm{d}}h. \tag{43}$$

Note that $\mathbf{W}_k(\mathbf{H}_k)$denotes the matrix Lambert W function that satisfies the definition (Asl and Ulsoy, 2003),

$$\mathbf{W}_k(\mathbf{H}_k)e^{\mathbf{W}_k(\mathbf{H}_k)} = \mathbf{H}_k. \tag{44}$$

The Lambert W function is complex-valued, with a complex argument $\mathbf{H}_k$, and has an infinite number of branches $\mathbf{W}_k(\mathbf{H}_k)$, where $k = -\infty, \cdots, -1,\ 0,\ 1, \cdots, \infty$ (Corless et al., 1996). Corresponding to each branch, k, of the Lambert function, $\mathbf{W}_k$, there is a solution $\mathbf{Q}_k$ from equation 43, and, for $\mathbf{H}_k = \mathbf{A}_\mathrm{d} h \mathbf{Q}_k$, we compute the Jordan canonical form $\mathbf{J}_k$ from $\mathbf{H}_k = \mathbf{Z}_k \mathbf{J}_k \mathbf{Z}_k^{-1}$. The block diagonal matrix $\mathbf{J}_k$ can be expressed as diag $\left(J_{k1}(\hat{\lambda}_1), J_{k2}(\hat{\lambda}_2), \ldots, J_{kp}(\hat{\lambda}_p)\right)$, where $J_{ki}(\hat{\lambda}_i)$ is an $m \times m$ Jordan block; m is the multiplicity of the eigenvalue $\hat{\lambda}_i$ and $\mathbf{Z}_k$ is an invertible matrix. Then, the matrix Lambert W function can be computed as (Pease, 1965):

$$\mathbf{W}_k(\mathbf{H}_k) = \mathbf{Z}_k \left\{ \text{diag} \left(\mathbf{W}_k \left(J_{k1}(\hat{\lambda}_1) \right), \mathbf{W}_k \left(J_{k2}(\hat{\lambda}_2) \right), \ldots, \mathbf{W}_k \left(J_{kp}(\hat{\lambda}_p) \right) \right) \right\} \mathbf{Z}_k^{-1}, \quad (45)$$

where

$$\mathbf{W}_k(J_{ki}(\hat{\lambda}_i)) = \begin{bmatrix} W_k(\hat{\lambda}_i) & W_k'(\hat{\lambda}_i) & \cdots & \frac{1}{(m-1)!} W_k^{(m-1)}(\hat{\lambda}_i) \\ 0 & W_k(\hat{\lambda}_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k(\hat{\lambda}_i) \end{bmatrix}. \quad (46)$$

The principal ($k = 0$) and other ($k \neq 0$) branches of the Lambert W function can be calculated analytically using a series expansion (Corless et al., 1996), or alternatively, using commands already embedded in the various commercial software packages, such as Matlab, Maple and Mathematica.

In the many examples we have studied, equation 43 always has a unique solution $\mathbf{Q}_k$ for each branch, k. The solution is obtained numerically, for a variety of initial conditions, e.g. using the "fsolve" function in Matlab. When $\mathbf{u}(t) \neq \mathbf{0}$ in equation 1, the solution in equation 41 can be extended to (Yi et al., 2007a)

$$\mathbf{x}(t) = \underbrace{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t} \mathbf{C}_k^\mathrm{I}}_{\text{free}} + \underbrace{\int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k (t-\xi)} \mathbf{C}_k^\mathrm{N} \mathbf{B} \mathbf{u}(\xi) \mathrm{d}\xi}_{\text{forced}}.$$

APPENDIX B. CONTROLLABILITY OF SYSTEMS OF DDES

The problems of controllability of the system descried by equation 20 were considered by Olbrot (1972). The linear system of equation 20,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-h), \quad (48)$$

is said to be *controllable* on $[0, t_1]$ if there exists an admissible (that is, measurable and bounded on a finite-time interval) control $\mathbf{u}(t)$, such that $\mathbf{x}(t_1) = 0$, where $t_1 > h$. According to the criterion presented by Olbrot (1972), the system is controllable on $[0, t_1]$ if and only if

$$\text{rank}\,[\mathbf{B} \vdots \mathbf{AB}] = n. \tag{49}$$

Controllability and observability of linear time delay systems described by equation 1 have also been studied, and various definitions and criteria have been presented since the 1960s. However, the lack of an analytical solution approach has limited the applicability of the existing theory. Using the solution form in terms of the matrix Lambert W function, algebraic conditions and Gramians for controllability and observability of DDEs were recently derived by Yi et al. (2008) in a manner analogous to the well-known observability and controllability results for the ODE case. Here, we summarize the definition and the corresponding condition. The system of equation 1 is *point-wise controllable* if, for any given initial conditions $\mathbf{g}(t)$ and $\mathbf{x}_0$, there exist a time t_1, $0 < t_1 < \infty$, and an admissible control segment $\mathbf{u}(t)$, such that $\mathbf{x}(t; 0, \mathbf{g}, \mathbf{x}_0, \mathbf{u}) = \mathbf{0}$ at $t = t_1$. If and only if all rows of

$$\left(s\mathbf{I} - \mathbf{A} - \mathbf{A}_{\mathrm{d}} e^{-sh}\right)^{-1} \mathbf{B} \tag{50}$$

are linearly independent, over the field of complex numbers, then the system in equation 1 is *point-wise controllable*.

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