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On the Convergence of the Matrix Lambert W Approach to Solution of Systems of Delay Differential Equations

by

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Abstract

Convergence of the matrix Lambert W function method for solving systems of delay differential equations (DDEs) is considered. Recent research shows that convergence problems occur with certain DDEs when using the well-established **Q**-iteration approach. A complementary, and recently proposed, **W**-iteration approach is shown to converge even on systems where **Q**-iteration fails. Furthermore, the role played by the branch numbers $k = -\infty \,..\, -1, 0, 1, \,..\, \infty$ of the matrix Lambert W function, $\mathbf{W}_k$, in terms of initializing the iterative solutions, is also discussed and elucidated. Several second order examples, known to have convergence problems with **Q**-iteration, are readily solved by **W**-iteration. Examples of third and fourth order DDEs show that **W**-iteration is also effective on higher-order systems.

1. Introduction

This paper considers the matrix Lambert W function approach, developed during 2003-2006, for solution of systems of delay differential equations (DDEs) [1, 2]. The method solves linear systems of DDEs with a single known delay. It can be used to assess stability, obtain free and forced solutions, determine observability and controllability, and for controller/observer design via eigenvalue assignment [3]. The method relies on a numerical search for an unknown matrix **Q** that satisfies certain conditions. In this paper this original numerical iteration is referred to as ***Q-iteration***, and has been successfully applied to a variety of time delay systems [3-5]. Recent studies have, however, demonstrated convergence problems with **Q**-iteration on some specific problems [6, 7]. Researchers have also recently proposed a complementary approach, which replaces an iteration to find the unknown matrix **Q** with iteration of the matrix Lambert W function itself [8]. This complementary approach will be denoted in this paper as ***W-iteration***. Recent research results also demonstrate that it is not necessary to use all the branches of the multi-valued matrix Lambert W function [7]. The complex multi-valued matrix Lambert W

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function, $\mathbf{W}_k$, has an infinite number of branches denoted by the branch numbers k = -∞ .. -1, 0, 1, .. ∞. The work by Cepeda-Gomes and Michiels shows that if the initial value of **Q** is chosen appropriately, then it is sufficient to only use the branches k = 0 and -1.

The purpose of this paper is to understand these convergence issues and their potential resolution, as well as to elucidate the role that the branches of the multi-valued matrix Lambert W function play in these iterative solutions. This paper demonstrates, for the first time, that the well-documented convergence problems in the literature can be resolved by using the complementary **W**-iteration rather than the original **Q**-iteration. Furthermore, the role of multiple branches, rather than only the k = 0 and -1 branches, in initializing these iterative solutions is explained based upon the multi-valued nature of the Lambert W function.

2. The Lambert W Function and Related Approaches to Solving DDEs

2.1 The Lambert W function

The Lambert W function is any function, $W(H)$, that satisfies [9]:

$$W(H)e^{W(H)} = H \tag{1}$$

The argument H as well as the function W_k, which has an infinite number of branches denoted by the branch numbers k = -∞ .. -1, 0, 1, .. ∞, are complex valued. The W function is similar to the complex logarithm function, $P(H)$ = log H, and its inverse, $H = e^{P(H)}$, which has an infinite number of branches, k, and has branch cuts which are horizontal lines in the complex P – plane, defined by $P = (2k+1)\pi i$, where i = -∞ .. -1, 0, 1, .. ∞. The curves which define the branch cuts for the Lambert W function are a subset of the so-called Quadratrix of Hippias [9]. Commercial software packages, such as MATLAB, Maple, and Mathematica, have embedded commands for numerically computing the Lambert W function [9]. Plots of the complex Lambert W function, showing its repetitive nature are given in Fig. 1 for branches k = -2, -1, 0, 1, 2. Extension of the Lambert W function to a matrix Lambert W function, **W**(**H**), has been presented based on the use of the Jordan canonical form of the complex matrix argument, **H** [1, 3, 10]. Figure 2 is a simplified 2-dimensional plot, based on Fig. 1 (a), that shows the real part of the complex scalar Lambert W function, Re(W_k), versus the real part of its complex scalar argument, x = Re(H_k), for the branches k = -2, -1, 0, 1 and 2. Because of the multi-valued nature of this function a single value of the function (e.g., Re(W_k) ≈ 2) can be obtained from various branches with different arguments (e.g., $W_0(10)$, $W_1(35)$, $W_{-1}(-40)$, $W_2(65)$, $W_{-2}(-70)$, etc.). A range of values for Re(W_k), such as 1.5 to 3, can be obtained by either: (1) using a single branch (e.g., W_0) with a range of arguments (e.g., H_0 = 5 to 40), or (2) using multiple branches (e.g., k = 0, 1, 2) with a single argument (e.g., H ≈ 40). This multi-valued nature of the Lambert W function is relevant to the

initialization of the numerical solution methods (i.e, **Q**-iteration and **W**-iteration) discussed in the following sections.

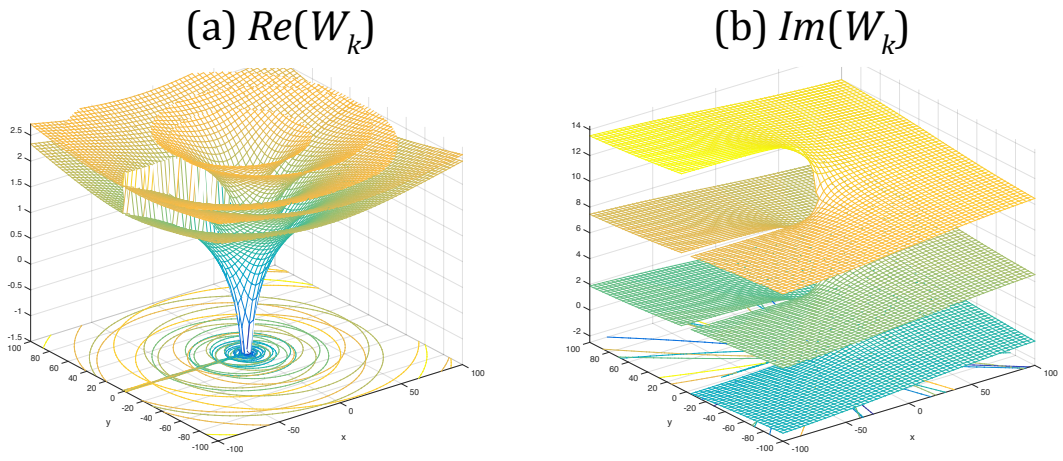

Figure 1. Three-Dimensional Plot of the (a) Real Part and (b) Imaginary Part of the scalar Lambert W Function, $W_k(x+yi)$ for Branches k = -2, -1, 0, 1, 2

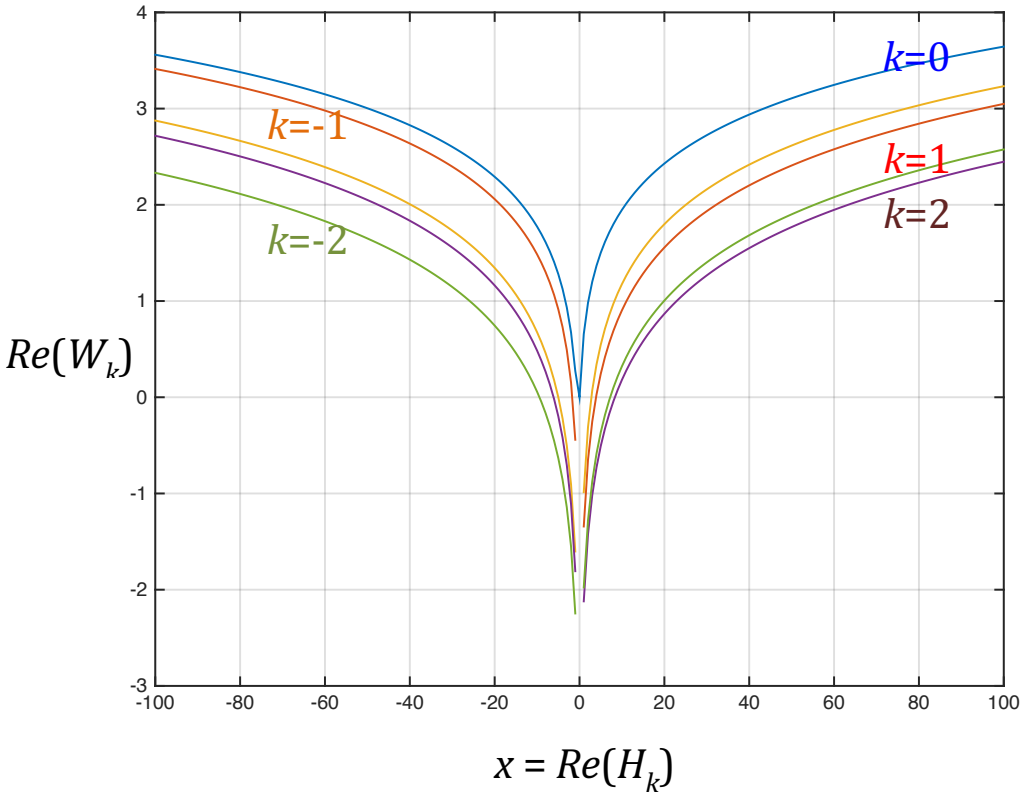

Figure 2. Re(W_k) vs x = Re(H_k) for Branches k = -2, -1, 0, 1, 2

*2.2 Solution of DDEs and the **Q** iteration approach [3]*

The Lambert W function in Eq. (1) can be used to solve the scalar DDE:

$$\dot{x}(t) = ax(t) + a_d x(t-h) \tag{2}$$

where *h* is a single known delay. Assuming the exponential solution form:

$$x(t) = e^{st}C \tag{3}$$

and substituting Eq. (3) into Eq. (2) one obtains the characteristic equation:

$$s - a - a_d e^{sh} = 0 \tag{4}$$

Multiply both sides of Eq. (4) by $he^{sh}e^{-ah}$ to obtain:

$$h(s-a)e^{h(s-a)} = ha_d e^{-ha} \tag{5}$$

Note that Eq. (5) is of the same form as Eq. (1) if one defines

$$W = h(s-a) \tag{6}$$

and the argument $H = ha_d e^{-ha}$. The eigenvalues, i.e. the solutions to the characteristic equation, are found by solving Eq. (6) for *s* for each branch *k* = -∞ .. -1, 0, 1, .. ∞.

$$s_k = (1/h)W_k(ha_d e^{-ha}) + a \tag{7}$$

It has been proven [11] that the infinite eigenspectrum of Eq. (2) is given by Eq. (7) and that the principal (i.e., *k* = 0) branch gives the rightmost eigenvalue in the *s*-plane and, thus, determines system stability.

For a system of DDEs, analogous to Eq. (2), represented as:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_d\mathbf{x}(t-h) \tag{8}$$

following the same approach, and corresponding to Eq. (5), one obtains:

$$h(\mathbf{S}-\mathbf{A})e^{h\mathbf{S}}e^{-h\mathbf{A}} = h\mathbf{A}_d e^{-h\mathbf{A}} \tag{9}$$

If **S** and **A** commute, then $e^{h\mathbf{S}}e^{-h\mathbf{A}} = e^{h(\mathbf{S}-\mathbf{A})}$ and we can write the solution, analogous to Eq. (7) in terms of the matrix Lambert W function, as:

$$\mathbf{S} = (1/h)\mathbf{W}(h\mathbf{A}_d e^{-h\mathbf{A}}) + \mathbf{A} \tag{10}$$

However, in general **S** and **A** do not commute, and Eq. (10) is not a valid solution to Eq. (9). To address this issue, one can introduce an unknown matrix **Q** to satisfy:

$$h(\mathbf{S}-\mathbf{A})e^{h(\mathbf{S}-\mathbf{A})} = h\mathbf{A}_d\mathbf{Q} \tag{11}$$

Thus, the solution matrix, **S**, comparing Eq. (1) to Eq. (11), can be obtained as:

$$\mathbf{S} = (1/h)\mathbf{W}(h\mathbf{A}_d\mathbf{Q}) + \mathbf{A} \tag{12}$$

Substituting Eq. (12) into Eq. (9), one obtains the following condition for **Q**:

$$\mathbf{W}(h\mathbf{A}_d\mathbf{Q})e^{\mathbf{W}(h\mathbf{A}_d\mathbf{Q})+h\mathbf{A}} = h\mathbf{A}_d \tag{13}$$

The system of nonlinear equations in Eq. (13) can be initialized as $\mathbf{Q} = \mathbf{Q}_{ini}$ and solved iteratively for **Q**. Here we refer to this solution method as **Q**-iteration.

Since Eq. (13) is a nonlinear system of equations with multiple solutions, the choice of initial condition, $\mathbf{Q}_{\text{ini}}$, will determine the specific solution **Q** that is obtained. In many problems reported in the literature $\mathbf{Q}_{ini} = e^{-h\mathbf{A}}$ is used, and this corresponds to the value that **Q** would take if **S** and **A** did commute. If the same argument (i.e., same value of $\mathbf{Q}_{ini}$) is used with different branches then one can obtain different values of the argument $\mathbf{H}_k$ and the function $\mathbf{W}_k$ corresponding to different eigenvalues. Alternatively, one can obtain those different eigenvalues by using only one branch, but varying the argument. This follows from the multi-valued nature of the Lambert W function and from our previous discussion of the simplified plot in Fig. 2. For example, using only the fixed argument $\text{Re}(H_k) = 20$, and all the branches (i.e., k = -2, -1, 0, 1, 2) one obtains values for $\text{Re}(W_k)$ in the range of 0.8 to 2.5. Alternatively, using only branch $k = 0$ one can obtain the same range of values for $\text{Re}(W_0)$ by using different arguments $\text{Re}(H_0)$ in the range 5 to 20.

When $\mathbf{A}_d$ is rank deficient, which is often the case in natural and engineered time delay systems, the matrix **Q** obtained as the solution of Eq. (13) will not be unique. This can lead to numerical convergence issues in some particular cases, as documented in the literature [6, 7]. Corresponding to each branch, k, of the Lambert W function, $\mathbf{W}_k$, there is a solution $\mathbf{Q}_k$ of Eq. (13), and an argument $\mathbf{H}_k = h\mathbf{A}_d\mathbf{Q}_k$. The infinite eigenspectrum of the system in Eq. (8) is obtained from the eigenvalues of the solution matrices:

$$\mathbf{S}_k = (1/h)\mathbf{W}_k(h\mathbf{A}_d\mathbf{Q}_k) + \mathbf{A} \tag{14}$$

For the scalar case of Eq. (2), rightmost eigenvalues are determined by the principal branch, $k = 0$, of the scalar Lambert W function [11]. The proof can be extended to the matrix case when **A** and $\mathbf{A}_\text{d}$ in Eq. (8) commute [10]. No such proof is currently available for the general case of matrix DDEs. However, in all examples considered in the literature, it is observed that the rightmost eigenvalue is always obtained when using only the first $k = -m, -(m-1), .., 0, .., (m-1), m$ branches, where m is the nullity of $\mathbf{A}_d$ [3, 12].

*2.3 Complementary **W**-iteration approach [8]*

The approach proposed in [8], which will be denoted here as **W**-iteration, is that Eq. (13) can be simplified and written as:

$$\mathbf{D}_k e^{\mathbf{D}_k + h\mathbf{A}} = h\mathbf{A}_d \tag{15}$$

and iteratively solved for an unknown matrix $\mathbf{D}_k$, rather than an unknown matrix $\mathbf{Q}_k$, for each branch k. The iteration is initialized, using the matrix Lambert W function for each branch k, as $\mathbf{D}_{ini} = \mathbf{W}_k(h\mathbf{A}_d e^{-h\mathbf{A}})$. As noted previously, this choice of initial condition matrix corresponds to the value that $\mathbf{D} = \mathbf{W}$ would take if **S** and **A** did commute (which they do not). The iteration for $\mathbf{D}_k$ converges to the matrix $\mathbf{W}_k$ in Eq. (14). This complement to the original **Q**-iteration is referred to here as **W**-iteration and has the following advantages [8]:

1. The initial matrix $\mathbf{D}_{ini}$ is always chosen in the same way as $\mathbf{D}_{ini} = \mathbf{W}_k(h\mathbf{A}_d e^{-h\mathbf{A}})$. The issues of how to initialize **Q** and whether it will converge are avoided.
2. The problem of non-uniqueness of **Q** in the original **Q**-iteration method, when the matrix $\mathbf{A}_d$ is rank deficient, is also avoided.

3. Numerical Examples

Numerical examples are considered to show that: (1) **W**-iteration works well on problems that were previously shown to have convergence problems with **Q**-iteration, and (2) **W**-iteration works well, not only on second order, but also higher order systems of DDEs. MATLAB code for these examples is given in the Appendix.

Phương pháp lặp W không những hoạt động tốt với hệ bậc 2 mà còn với cả những hệ bậc cao hơn.

3.1 Example from [7]

Consider the system of DDEs, as in Eq. (8), with a delay $h = 5$ and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -1 \end{bmatrix}; \quad \mathbf{A}_d = \begin{bmatrix} 0 & 0 \\ -3 & -0.6 \end{bmatrix} \tag{16}$$

For this system **Q**-iteration was shown to not converge, with initialization $\mathbf{Q}_{ini} = e^{-h\mathbf{A}}$ [7]. However, if the solution matrix **S** is known *a priori*, then an appropriate initialization of **Q** can be found for which the **Q**-iteration method converges. Unfortunately, if the solution **S** is known *a priori* then, of course, there is no need for the **Q**-iteration at all! When **W**-iteration, with initialization $\mathbf{D}_{ini} = \mathbf{W}_k(h\mathbf{A}_d e^{-h\mathbf{A}})$ is used, the solution is readily obtained. The rightmost eigenvalues of the system are the complex conjugate pair located at $\lambda_{1,2} = 0.0377 \pm 1.7911i$. In all examples considered in the literature it has been observed that the rightmost eigenvalue is always obtained by using only the first $k = -m, -(m-1), .., 0, .., (m-1), m$ branches, where m is the nullity of $\mathbf{A}_d$ [3, 12]. Since rank($\mathbf{A}_d$) = 1, i.e., m = nullity of $\mathbf{A}_d$ is 1, we consider only $k = -1, 0,$ and 1 to ensure that these rightmost eigenvalues are found. The MATLAB code in

the Appendix quickly converges to the results in Table 1. These results show that the convergence problems observed in [7] when using **Q**-iteration have been eliminated by using **W**-iteration. Furthermore, the observation that only $k = -m, .. 0. .., m$ branches of $\mathbf{W}_k$ are needed to obtain the rightmost eigenvalues is confirmed, since $m = 1$ and the dominant eigenvalues are $0.038 \pm 1.791i$. The initial $\mathbf{D}_{ini} = \mathbf{W}_k(h\mathbf{A}_d e^{-h\mathbf{A}})$ matrix is complex, as is the resulting solution matrix, $\mathbf{S}_k$, for branch $k = i$ and their conjugates are found for $k = -i$. The system eigenvalues are either real or occur in complex conjugate pairs, thus, both branches i and $-i$ must be considered to find those complex conjugate pairs. The eigenvalues from right to left in the s-plane are $0.038 \pm 1.791i$, $-0.020 \pm 2.771i$, and $-0.628 \pm 2.403i$.

k	Initial Condition for $\mathbf{W}_k$	Solution Matrix, $\mathbf{S}_k$	Eig($\mathbf{S}_k$)
-1	$\begin{bmatrix} 0 & 0 \\ 0.66+1.28i & -2.20-4.23i \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -4.962+0.141i & 0.017-0.979i \end{bmatrix}$	$0.038 + 1.791i$ $-0.020 - 2.771i$
0	$\begin{bmatrix} 0 & 0 \\ -0.11 & 0.37 \end{bmatrix}$	$\begin{bmatrix} 15.140 & 3.512 \\ -72.433 & -16.3953 \end{bmatrix}$	$-0.628 \pm 2.403i$
+1	$\begin{bmatrix} 0 & 0 \\ 0.66-1.28i & -2.20+4.23i \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -4.962-0.141i & 0.017+0.979i \end{bmatrix}$	$0.038 - 1.791i$ $-0.020 + 2.771i$

Table 1. **W**-iteration Solution for Example from [7]

k	Initial Condition for $\mathbf{W}_k$	Solution Matrix, $\mathbf{S}_k$	Eig($\mathbf{S}_k$)
-1	$\begin{bmatrix} 0 & 0 \\ -0.252+2.804i & 0.432-4.802i \end{bmatrix}$	$\begin{bmatrix} 0.710 & 0 \\ 11.266+0.823i & -15.156-1.159i \end{bmatrix}$	0.710 $-15.156 - 1.159i$
0	$\begin{bmatrix} 0 & 0 \\ -0.911 & 1.560 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0.710 \end{bmatrix}$	0 0.710
+1	$\begin{bmatrix} 0 & 0 \\ -0.252-2.804i & 0.432+4.802i \end{bmatrix}$	$\begin{bmatrix} 0.710 & 0 \\ 11.266-0.823i & -15.156+1.159i \end{bmatrix}$	0.710 $-15.156 + 1.159i$

Table 2. **W**-iteration Solution for Example from [6]

3.2 Example from [6]

Consider, with $h = 1$, the system of DDEs defined by:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2.5 & 2.5 \end{bmatrix}; \quad \mathbf{A}_d = \begin{bmatrix} 0 & 0 \\ 2.5 & 0 \end{bmatrix} \tag{17}$$

This system corresponds to a region in the parameter plane where **Q**-iteration does not converge [6]. However, **W**-iteration converges quickly to the results in Table 2.

For this example m = 1, so k = -1, 0, 1. The dominant eigenvalue is 0.710 as found from all three branches considered. As in the previous example, the initial condition matrix and the solution matrix are complex, and those for branch k = 1 are the conjugates of the ones for branch k = -1. Thus, the eigenvalues obtained from right to left in the s-plane are 0.710, 0, and -15.156 $\pm$ 1.159i.

3.3 Higher order examples from [3, 13, 14]

This section includes examples of third and fourth order systems of DDEs [3, 13, 14]. First, consider, with h = 0.06, the system of DDEs defined by:

$$\mathbf{A} = \begin{bmatrix} -27 & -0.0097 & 6 \\ 9.5999 & -40.2750 & -40.6578 \\ 0 & 18.0608 & 4.1480 \end{bmatrix}; \quad \mathbf{A}_d = \begin{bmatrix} 0 & 0 & 0 \\ 21 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

This third order system corresponds to closed loop control of a diesel engine as described in [14]. The rank of $\mathbf{A}_d$ is 1, so m = 2. The eigenvalues of the solution matrix $\mathbf{S}_k$, for each branch k = -2, -1, 0, 1, 2 are in Table 3. The eigenvalues, ordered from right to left in the s-plane, are -10, -21.56 $\pm$ 23.71i , -114.4 $\pm$ 90.52i, and -145.5 $\pm$ 208.33i. The rightmost eigenvalue, -10, is found for the principal branch, k = 0. The initial condition matrix and solution matrix are complex, and those for branches k = 1, 2 are the conjugates of those for k = -1, -2 respectively.

Branch, k	Eigenvalues of $\mathbf{S}_k$
-2	-21.56 $\pm$ 23.71i, -145.5 – 208.33i
-1	-21.56 $\pm$ 23.71i, -114.4 – 90.52i
0	-10.0, -21.56 $\pm$ 23.71i
1	-21.56 $\pm$ 23.71i, -114.4 + 90.52i
2	-21.56 $\pm$ 23.71i, -145.5 + 208.33i

Table 3 **W**-iteration Solution for Example from [14]

Next, consider, with h = 0.01, the system of DDEs defined by:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & 0 & -117.8 & -67.1 \\ 1 & -1 & 0.1 & -0.1 \end{bmatrix}; \quad \mathbf{A}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

This fourth order system corresponds to closed loop control of a two-degree-of-freedom mechanical vibration problem [13]. The rank of $\mathbf{A}_d$ is 1, so m = 3 and the eigenvalues of the solution matrix $\mathbf{S}_k$, for the branches k = -3, -2, -1, 0, 1, 2, 3 are in Table 4. The rightmost eigenvalue is -1.75$\pm$0.34i, and is found for all branches

considered. The initial condition matrix and solution matrix are complex, and those for branches k = 1, 2, 3 are conjugates of the ones for k = -1, -2, -3 respectively.

Branch, k	Eigenvalues of $\mathbf{S}_k$
-3	$-1.75 \pm 0.34i$, -285 - 1718i, -1728 – 1.426i
-2	$-1.75 \pm 0.34i$, -2.69 – 0.70i, -239.3 - 1088.4i
-1	$-1.75 \pm 0.34i$, -2.69 – 0.70i, -153.6 - 463.5i
0	$-1.75 \pm 0.34i$, $-2.69 \pm 7.00i$
1	$-1.75 \pm 0.34i$, -2.69 + 0.70i, -153.6 + 463.5i
2	$-1.75 \pm 0.34i$, -2.69 + 0.70i, -239.3 + 1088.4i
3	$-1.75 \pm 0.34i$, -285 + 1718i, -1728 + 1.426i

Table 4 **W**-iteration Solution for Example from [13]

In addition, **W**-iteration was also evaluated on other examples from [6-8] and found to readily converge, and to always yield the dominant eigenvalues for branches k = -m, .., -1, 0, 1, .., m. Due to space limitations, those results are not included here.

4. Summary, Conclusions and Future Work

The solution of systems of DDEs using the matrix Lambert W function was considered. The original **Q**-iteration and the complementary **W**-iteration methods were applied to problems where **Q**-iteration was known to have convergence problems. On those non-converging example problems, and on others, **W**-iteration was shown to converge to the expected results. Consequently, the solution of systems of DDEs using the matrix W function is shown to be a reliable engineering tool when **W**-iteration is employed. Furthermore, considering the multi-valued nature of the complex Lambert W function, it was illustrated that the necessary iterations can be initialized by using only the principal branch or by using multiple branches of the function. However, using only the principal branch requires that the desired solution be known *a priori*, thus, making that approach impractical. Using multiple branches allows for multiple initializations of the iterative solution method, which then converge to different solutions in the infinite eigenspectrum.

The examples presented indicate that **W**-iteration works when **Q**-iteration does not, and should be preferred. Furthermore, the long-standing observation that at most m (where m = nullity of $\mathbf{A}_d$) branches are needed to ensure that one finds the rightmost eigenvalues is confirmed in all the examples considered. These are important and encouraging results, which demonstrate that the **W**-iteration approach to solving systems of DDEs, with multiple branches used in the initialization, can be a powerful and reliable engineering tool. Nevertheless, some basic research questions remain: (1) whether the selection of different branches to initialize **W**-iteration can ensure that all solutions will indeed be found, and (2) the important and useful observation

that only m branches are needed to ensure that one finds the rightmost eigenvalues has not been proven for general $\mathbf{A}$ and $\mathbf{A}_d$.

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Appendix: MATLAB Code

```
% Solve the equation xdot(t) = A*x(t) + Ad*x(t-h) using the W-iteration method,
% with branches in N and initial condition W0 = W(h*Ad*expm(h*A))
global h A Ad
% Define system of DDEs for Example in Section 3.1
h=5; A=[0 1; -5 -1]; Ad=[0 0; -3 -0.6]; N=-1:1;
% Determine matrix W for all branches in N
for k=N
   k
   W0=lambertw_matrix(k,h*Ad*expm(h*A));
   W=fsolve(@witer,W0);
   S=(1/h)*W + A;
   lambda=eig(S)
end

% Define function witer used in fsolve
function eps = witer(W)
global h A Ad
eps=W*expm(W+h*A)-h*Ad;
end
```

Notes:

1. The function, *lambertw_matrix*, is from the MATLAB LambertWDDE Toolbox [15] and is available for downloading on the web site http://www-personal.umich.edu/~ulsoy/TDS_Supplement.htm. It implements the matrix Lambert W function as described in [1, 3, 10].
2. The function *fsolve*, which solves systems of nonlinear equations of several variables, is part of the MATLAB optimization toolbox.

Figure and Table Captions

Figure 1. Three-Dimensional Plot of the (a) Real Part and (b) Imaginary Part of the scalar Lambert W Function, $W_k(x+yi)$ for bBranches k = -2, -1, 0, 1, 2

Figure 2. Re(W_k) vs x = Re(H_k) for Branches k = -2, -1, 0, 1, 2

Table 1. **W**-iteration Solution for Example from [7]

Table 2. **W**-iteration Solution for Example from [6]

Table 3 **W**-iteration Solution for Example from [14]

Table 4 **W**-iteration Solution for Example from [13]