H-infinity Norm Calculation via a State Space Formulation

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(ABSTRACT)

There is much interest in the design of feedback controllers for linear systems that minimize the H_∞ norm of a specific closed-loop transfer function. The H_∞ optimization problem initiated by Zames (1981), [23], has received a lot of interest since its formulation. In H_∞ control theory one uses the H_∞ norm of a stable transfer function as a performance measure. One typically uses approaches in either the frequency domain or a state space formulation to tackle this problem. Frequency domain approaches use operator theory, J-spectral factorization or polynomial methods while in the state space approach one uses ideas similar to LQ theory and differential games. One of the key computational issues in the design of H_∞ optimal controllers is the determination of the optimal H_∞ norm. That is, determining the infimum of r for which the H_∞ norm of the associated transfer function matrix is less than r. Doyle *et al*, [7], presented a state space characterization for the sub-optimal H_∞ control problem. This characterization requires that the unique stabilizing solutions to two Algebraic Riccati Equations are positive semi definite as well as satisfying a spectral radius coupling condition. In this work, we describe an algorithm by Lin *et al*, [16], used to calculate the H_∞ norm for the state feedback and output feedback control problems. This algorithm only relies on standard assumptions and divides the problem into three sub-problems. The first two sub-problems rely on algorithms for the state feedback problem formulated in the frequency domain as well as a characterization of the optimal value in terms of the singularity of the upper-half of a matrix created by the stacked basis vectors of the invariant sub-space of the associated Hamiltonian matrix. This characterization is verified through a bisection or secant method. The third sub-problem relies on the geometric nature of the spectral radius of the product of the two solutions to the Algebraic Riccati Equations associated with the first two sub-problems. Doyle makes an intuitive argument that the spectral radius condition will fail before the conditions involving the Algebraic Riccati Equations fail. We present numerical results where we demonstrate that the Algebraic Riccati Equation conditions fail before the spectral radius condition fails.

Dedication

To my grandparents.

Acknowledgments

Completing this project would have been an impossible task without the support of numerous others along the way. It would be an gross understatement to say that without the guidance and enthusiasm of my advisor, Dr. Lizette Zietsman this thesis would not have come to fruition. Her insight, patience and ability to keep me motivated are several reasons I remained working steadily.

I would like to thank my fellow graduate students who served as sound boards when I needed to piece together difficult topics. Specifically Alan Lattimer and Steven Boyce played indispensable roles in helping me synthesize the many topics encountered in this research.

Dr. Vasil Skenderi was the most influential undergraduate mathematics professor I had the pleasure of taking a course with. Unknowingly, he sparked my interest in research that instilled a desire to attend graduate school.

Above all however, I am grateful to my wife Lianne for her unwavering support, encouragement and understanding throughout this entire process.

George Kuster

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Chapter 1

Introduction

One of the many challenges in control theory is the design of controllers that result in sufficient performance of a plant under various types of inputs and disturbances. H_∞ control theory was designed to reduce modeling errors and unknown disturbances in a system, while providing quantifiable optimization of large scale multivariable problems. In application most solutions to H_∞ control problems are actually sub-optimal controllers. That is, controllers which achieve a predetermined bound on the H_∞ norm of an associated transfer function. In this formulation of the problem, the solution is robust for a predetermined bound. However, if one can find the H_∞ norm, one can answer the question as to the existence of a controller for a certain range of perturbations to the system.

George Zames, [23] introduced the formal H-infinity control theory by formulating the probloem of sensitivity reduction as an optimization problem with an operator norm, in particular, the H-infinity norm [15]. Where H_∞ is the space of all bounded analytic matrix valued functions in the open right half complex plane. This formulation was based entirely in the frequency domain. Zames suggested that using the H-infinity norm as a performance measure would better satisfy the demands of applications in comparison to the popular Linear Quadratic Gaussian control design [25]. This brought about the design of H-infinity optimal control, where one aims to find a controller which stabilizes a system while minimizing the effects of disturbances in the system. This norm is now used to numerically evaluate the controller's sensitivity, robustness and performance of the closed loop feedback system.

Shortly afterwards John Doyle developed tools for testing robust stability in an effort to gauge model uncertainties and achieve the H-infinity control objective of minimizing the effect of these perturbations on the regulated outputs [18]. In 1984 Doyle, using state space methods, offered the first solution to a general multivariable H-infinity optimal control problem [6].

While these frequency domain formulations dealt with the robustness issues, the tools developed were both notedly complicated and provided limited insight concerning the nature of the stabilizing solutions. State space formulations were initially developed by Doyle, Francis and Glover in the mid 1980's, but were only successful to a degree, that is the solutions to these early formulations had solutions of high order, see [8] and [11].

In the late 1980's work by Khargonekar, Petersen, Rotea and Zhou, [13], [14], and [20], presented the solution to the H-infinity state feedback problem in terms of a constant feedback gain matrix, and provided a formula for finding this gain matrix in terms of the solution to a Riccati equation. Furthermore, they established a relationship between linear quadratic game theory and H-infinity optimal control and gave the solution to the state-feedback problem by solving an algebraic Riccati equation.

In 1989 Doyle, Glover, Khargonekar and Francis, [7], presented the first general state space solution to the H-infinity control problem in their now classic paper *State space solutions to standard H_2 and H-infinity control problems*. Results in this paper, give necessary and sufficient conditions for the existence of an admissible controller in terms of solutions of algebraic Riccati equations and a coupling condition involving a spectral radius condition. Thus formulation resulted in algorithms that uses concave criteria as well as Newton-like methods, see [9], [10], and [21].

This study focuses on a collection of algorithms used for H_∞-control purposes, in particular for state feedback, [3], [4], and output feedback problems. The latter though the algorithm presented by Lin, Wang and Xu [16].

The outline of the remainder is as follows. Chapter 2 provides the necessary background such as controllability, observability, detectabilty, stabilizability as well as other topics such as transfer functions and introduces the H_∞ norm. Chapter 3 discusses two algorithms based on frequency domain tools to calculate the H_∞ norm of a transfer function associated with the state feedback case. The first algorithm is the bisection algorithm presented by [3] and the second algorithm is a two step algorithm originally presented by [4].

The main discussion is found in Chapter 4 where we discuss and implement the algorithm Lin, Wang and Xu [16] proposed to calculate the H_∞ norm of the state and output feedback control problems. The algorithm uses a state space formulation while appealing to frequency domain tools to find the infimum of r for which the H_∞ norm of the associated transfer function matrix is less than r. The problem of finding the H_∞ norm is broken into three sub-problems. The first two rely heavily on the algorithms discussed in Chapter 3 and the third relies on the geometric nature of the spectral radius of the product of the two solutions to the Algebraic Riccati Equations associated with the first two sub-problems. Concluding remarks are found in Chapter 5.

Chapter 2

Background

2.1 State Space Formulation

For the initial discussion consider the basic time invariant, continuous time system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{2.1.1}$$

Here A is an $n \times n$ matrix, B is $n \times m$, C is $r \times n$ and D is $r \times m$. The vector x represents the state, u the control and y is the measured output. The basic block diagram associated with this system is given in Figure 1.1.

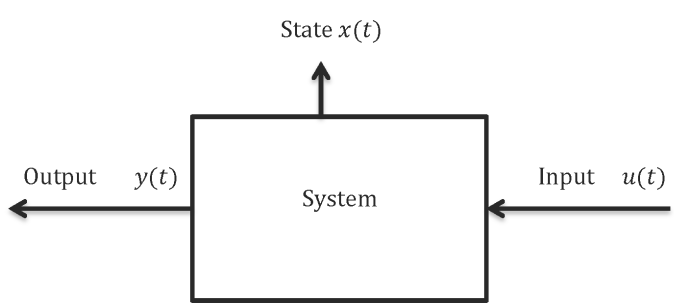

Figure 2.1: Block Diagram: Continuous time state space model.

Example 2.1.1. *To better illustrate the state space formulation, consider the simple example where two masses are connected by a spring and damper. The control force u is applied to a mass M_1, which directly effects the position, z of second mass M_2. The input of this system is the force, and the output is the change of position of the second mass M_2. Here we would like to control the displacement of M_2.*

Using Newton's Second Law, we attain the following equations:

$$
\begin{aligned}
M_1\ddot{w} &= -b(\dot{w}-\dot{z})-k(w-z)+u && (2.1.2)\\
M_2\ddot{z} &= b(\dot{w}-\dot{z})+k(w-z). && (2.1.3)
\end{aligned}
$$

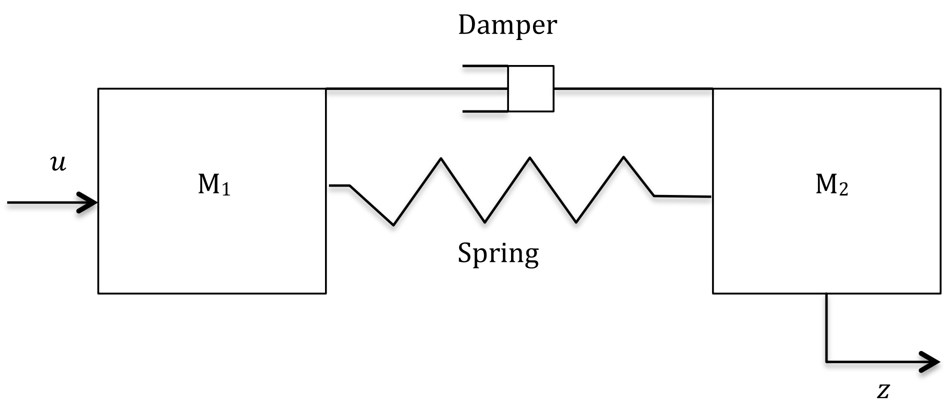

Figure 2.2: Spring mass damper system.

In this example, b denotes the damping coefficient, w is the displacement of M_1, z is the displacement of M_2, and k is the spring constant. This second order system are rewritten as a system of first order equations:

Let $x_1 = w,\ \dot{x_1} = \dot{w} = x_2,\ x_3 = z,\ \dot{x_3} = \dot{z} = x_4$. Then,

$$
\begin{aligned}
\dot{x_2} &= -\frac{b}{m_1}(x_2-x_4)-\frac{k}{m_1}(x_1-x_3)+u, && (2.1.4)\\
\dot{x_4} &= \frac{b}{m_2}(x_2-x_4)+\frac{k}{m_1}(x_1-x_3)+u, && (2.1.5)\\
y &= x_3. && (2.1.6)
\end{aligned}
$$

In matrix-vector notation (2.1.4)-(2.1.6) become

$$
\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\\ \dot{x_3}\\ \dot{x_4}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0\\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & \frac{b}{m_1}\\ 0 & 0 & 0 & 1\\ \frac{k}{m_2} & \frac{b}{m_2} & -\frac{k}{m_2} & -\frac{b}{m_2}\end{bmatrix} \cdot \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m_1}\\ 0\\ 0\end{bmatrix} \cdot u, \qquad (2.1.7)
$$

$$
y = \begin{bmatrix} 0 & 0 & 1 & 0\end{bmatrix} \cdot \begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}. \qquad (2.1.8)
$$

This spring mass damper system is now of the form (2.1.1)

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{b}{m_2} & -\frac{k}{m_2} & -\frac{b}{m_2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \quad D = 0.$$

2.2 State Space Solutions

If the control u is known, the system (2.1.1) can be solved explicitly. Before we discuss this in more detail, we review a few basic concepts and follow the discussion in [5].

Definition 2.2.1. *Given an $n \times n$ matrix A, the **matrix exponential** e^{At} is defined as:*

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

We note two properties of the matrix exponential:

1. $e^{A0} = A^0 = I$.
2. $$\frac{d}{dt}e^{At} = \sum_{k=0}^{\infty} \frac{d}{dt}\frac{A^k t^k}{k!} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} A\frac{A^k t^k}{k!} = Ae^{At}.$$

It is well known that the solution to the initial value problem

$$\dot{x}(t) = Ax(t), \qquad x(t_0) = x_0, \tag{2.2.1}$$

is given by $x(t) = e^{A(t-t_0)}x_0$.

Considering the more general formulation for given u,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t_0) = x_0 \tag{2.2.2}$$

$$y(t) = Cx(t) + Du(t), \tag{2.2.3}$$

the solution is obtained through variation of parameters or using the Laplace transform. We first review the definition of the Laplace transform and the inverse Laplace transform.

Definition 2.2.2. *The **Laplace Transform** of a function $f(x)$, $x \in [0, \infty)$ is defined by the integral*

$$\mathcal{L}(f(x))(s) = \hat{f}(s) = \int_0^{\infty} f(x)e^{-sx}dx,$$

for values of $s \in \mathbb{C}$ for which this integral is defined.

The inverse Laplace transform of $\hat{f}$, denoted by $\mathcal{L}^{-1}(\hat{f})$, is the function f whose Laplace transform is $\hat{f}$.

To determine the solution to (2.2.2)–(2.2.3), take the Laplace transform of (2.2.2) and solve for $\hat{x}$. This results in

$$\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s). \tag{2.2.4}$$

Taking the inverse Laplace of (2.2.4) gives

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left((sI - A)^{-1}x_0\right) + \mathcal{L}^{-1}\left((sI - A)^{-1}Bu(t)\right) \\ &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds. \end{aligned} \tag{2.2.5}$$

Substituting (2.2.5) into (2.2.3) yields $y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-s)}Bu(s)ds + Du(t)$.

Definition 2.2.3. *The matrix $e^{A(t-t_0)}$ is called the* ***state transition matrix****.*

In the event that the control u is not predetermined, various control strategies are used to determine the control that would result in a desired outcome. In this thesis our focus will be on H_∞ control which is described in Section ???

2.3 Transfer Function

In this section we introduce the *transfer function* associated with a linear time-invariant system such as (2.1.1) with $x(0) = x_0$. The transfer function gives a relationship between the input and output of the system and plays an important role in control theory in general and provides insight in how disturbances in the system will affect the output.

To derive the transfer function associated with (2.1.1)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.3.1}$$

take the Laplace transform as before in (2.2.4) to get

$$\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s) \tag{2.3.2}$$

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s). \tag{2.3.3}$$

Substituting (2.3.2) into (2.3.3), and setting $G(s) = \dfrac{\hat{y}(s)}{\hat{u}(s)}$ Equation (2.3.3) becomes

$$\hat{y}(s) = G(s)\hat{u}(s)$$

or

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)}. \tag{2.3.4}$$

The function G(s) reflects the ratio between the system output $\hat{y}(s)$ and the system input $\hat{u}(s)$ and is called the **matrix transfer function** or simply the **transfer matrix.** Entry (i, j) in the matrix represents the transfer function from the j^{th} input to the i^{th} output. This transfer function represents properties of the system and is not representative of the magnitude or nature of the inputs. It also does not serve to provide any information about the structure of the system.

One main advantage to using transfer functions is they allow for taking a complicated system in a time domain and represent it as a function of a single variable in frequency domain. Unfortunately, this approach only works for single input single output systems. This means, for multiple input-multiple output systems, there are multiple transfer functions used to represent each input-output relationship. It should be noted that transfer functions are not unique, that is many systems can share the same transfer function. They are unrelated to the systems magnitude or the nature of the input.

Definition 2.3.1. *The points p at which the transfer function $G(p) = \infty$ are called the poles of the system.*

If $G(\infty)$ is a constant matrix the transfer function is called **proper** and if $G(\infty) = 0$, the transfer function is called **strictly proper**.

Let us use the spring mass damper system from Example 2.1.1 to illustrate how one would find the transfer function of a given system described by a state space representation.

Example 2.3.1.

From Example 2.1.1 we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{b}{m_1} & \frac{k}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & \frac{b}{m_2} & -\frac{k}{m_2} & -\frac{b}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \ 0.$$

Thus

$$G(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -2 & 4 & 2 \\ 0 & 0 & 0 & 1 \\ 4 & 2 & -4 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Simplification leads to

$$G(s) = \frac{s+2}{s^4 + 3s^3 + 6s^2}.$$

Definition 2.3.2. *For the continuous time state space representation given by (2.3.1), the matrix*

$$G(i\omega) = C(i\omega I - A)^{-1}B + D \tag{2.3.5}$$

*is called the **frequency response matrix**, where $\omega \in \mathbb{R}$ is the **frequency**.*

2.4 Controllability and Observability

Controllability and observability are fundamental ideas of control theory. We follow the discussion in [5] to introduce these concepts.

Definition 2.4.1. *The system described in (2.1.1),*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

is called controllable if the system can be driven to any final state $x_1 = x(t_1)$, in finite time t_1, from any initial state $x(0)$ by selecting the control $u(t), 0 \leq t \leq t_1$ accordingly.

The controllability of the system (2.1.1), is referred to as the controllability of the pair (A, B). The following theorem provides verifiable conditions as to whether or not a system is controllable.

Theorem 2.4.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m} (m \leq n)$ then the following statements are equivalent:*

(i) The system (2.1.1) is controllable.

(ii) The $n \times nm$ controllability matrix $C_M = [B, AB, A^2B, \ldots, A^{n-1}B]$ has full rank.

(iii) The matrix

$$W_C = \int_0^{t_1} e^{At}BB^Te^{A^Tt}dt$$

is nonsingular for any $t_1 > 0$.

(iv) If (λ, x) is an eigenpair of A^T, then $x^TB \neq 0$.

(v) Rank$(A - \lambda I, B) = n$ for every eigenvalue λ of A.

(vi) The eigenvalues for $A - BK$ can be arbitrarily assigned by an appropriate choice of K.

Observability and controllability of a system are dual problems and observability is defined as follows:

Definition 2.4.2. *The continuous time system (2.1.1) is said to be **observable** if there exists a $t_1 > 0$ such that the initial state $x(t_0)$ can be uniquely determined from knowledge of $u(t)$ and $y(t)$ for all t, where $0 \leq t \leq t_1$.*

The observability of the system (2.1.1) is referred to as the observability of the pair (A, C). The duality means that (A, C) is observable if $(A,^T C^T)$ is controllable. Due to the duality of observability and controllability, observability has similar characterizations as controllability in Theorem 2.4.1.

Theorem 2.4.2. *The following statements are equivalent:*

(i) The system (2.1.1) is observable.

(ii) The observability matrix $O_M = [C \; CA \; \ldots \; CA^{n-1}]^T$ *has full rank.*

(iii) The matrix

$$W_O = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$$

is nonsingular for any $t_1 > 0$.

(iv) The matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ *has rank* n *for all eigenvalues of* A.

(v) If (λ, y) *is an eigenpair of* A*, then* $Cy \neq 0$.

(vi) The eigenvalues for $A - LC$ *can be arbitrarily assigned by an appropriate choice of* L.

Controllability and observability play important roles in the existence of positive definite and positive semidefinite solutions to Lyapunov equations.

Definition 2.4.3. *Let* A *be a stable matrix, then the matrix:*

$$C_G = \int_0^{t_1} e^{At} B B^T e^{A^T t} dt \tag{2.4.1}$$

is called the ***controllability Grammian****.*

Definition 2.4.4. *Let* A *be a stable matrix, then the matrix:*

$$O_G = \int_0^{\infty} e^{A^T t} C^T C e^{At} dt \tag{2.4.2}$$

is called the ***observability Grammian****.*

Theorem 2.4.3. *Let* A *be a stable matrix. The controllability Grammian* C_G *satisfies the Lyapunov equation*

$$AC_G + C_G A^T = -BB^T \tag{2.4.3}$$

and is symmetric positive definite if and only if (A, B) *is controllable.*

Theorem 2.4.4. *Let* A *be a stable matrix. The observability Grammian* O_G *satisfies the Lyapunov equation*

$$O_G A + A^T O_G = -C^T C \tag{2.4.4}$$

and is symmetric positive definite if and only if (A, C) *is observable.*

The following is an example illustrating the computation of both the observability and controllability Grammians.

Example 2.4.1. *Consider the simple system:*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned}$$

where

$$A = \begin{bmatrix} -4 & -8 & -12 \\ 0 & -8 & -4 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = 0.$$

The matrix A is stable and using the MATLAB *command* `lyap` *to solve the Lyapunov equations in Theorems 2.4.3 and 2.4.4 we arrive at*

$$C_G = \begin{bmatrix} 0.1458 & 0.0208 & 0.0208 \\ 0.0208 & 0.0833 & 0.0833 \\ 0.0208 & 0.0833 & 0.0833 \end{bmatrix} = O_G.$$

We note that this matrix is singular, and thus we can conclude that (A, B) is not controllable and (A, C) is not observable.

2.5 Stabilizability and Detectability

A controller is often designed to stabilize a linear system. This is done by choosing an appropriate control u such that the closed loop matrix $A - BK$ is stable where K is the feedback matrix where $u(t) = Kx(t)$. In this section we define stabilizability of a system and present characterizations of systems that can be stabilized through feedback control.

First consider the uncontrolled system:

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{2.5.1}$$

Definition 2.5.1. *The equilibrium state of the system (2.5.1), is the vector x_e satisfying*

$$Ax_e = 0 \tag{2.5.2}$$

If A is a nonsingular matrix, then the only solution is the trivial solution, $x_e = 0$.

Definition 2.5.2. *Let x_e be an equilibrium state of (2.5.1), then x_e is called*

1. *Stable if for each $\epsilon > 0$ there is a $\delta > 0$ such that $||x(t) - x_e|| < \epsilon$ for all $t \geq 0$ if $||x(t_0) - x_e|| < \delta$.*

2. *Asymptotically stable if the equilibrium state is stable and $\delta > 0$ exists such that $||x(t_0) - x_e|| < \delta$ implies $\lim\limits_{t\to\infty} x(t) = x_e$.*

This means that all solutions which start sufficiently close to an asymptotically stable equilibrium point will not only remain near that point, but will also approach equilibrium as time increases. Thus if $x_e = 0$ and A is nonsingular, the uncontrolled system above is asymptotically stable if $x(t) \to 0$ as $t \to \infty$.

The eigenvalues if the matrix A in (2.5.1) characterizes the stability of the system.

Theorem 2.5.1. *System (2.5.1) is said to be asymptotically stable if and only if all of the eigenvalues of the matrix A have strictly negative real parts.*

Definition 2.5.3. *The matrix A is called a* ***stable matrix*** *if all of the eigenvalues of A have negative real parts.*

As in the case of controllability, the solution to a Lyapunov equation are also characterized in terms of the stability of (2.5.1).

Theorem 2.5.2. *The system:*

$$\dot{x}(t) = Ax(t),$$

is asymptotically stable if and only if, for any symmetric positive definite matrix M, there exists a unique solution to the equation

$$XA + A^T X = -M.$$

Consider the linear system (2.1.1),

$$\begin{aligned} \dot{x}(t) &= \quad Ax(t) + Bu(t), \\ y(t) &= \quad Cx(t) + Du(t). \end{aligned}$$

We consider the problem of finding a controller u which drives this system to a desired state. That is, find a feedback matrix K such that the **feedback matrix** $(A - BK)$ is stable. Not every system can be driven to a specific state, so the importance of this section is that is presents necessary and sufficient conditions for when a stabilizing feedback matrix exists.

Also suppose we have full information from the state, $x(t)$, that is, at any time t information about any variable from the state is available. Using linear control, the controller $u(t)$ is of the form,

$$u(t) = -Kx(t), \tag{2.5.3}$$

where K is the feedback matrix.

Substituting $u(t) = -Kx(t)$ into (2.1.1) the **closed-loop system** is

$$\begin{aligned} \dot{x}(t) &= \quad (A - BK)x(t), \\ y(t) &= \quad (C - DK)x(t). \end{aligned} \tag{2.5.4}$$

Recall that the uncontrolled system (2.5.1) is stable when the matrix A is stable. Thus stabilizing the system (2.5.4) reduces to finding a matrix K such that $(A - BK)$ is stable. If such a K exists, it is called a **stabilizing feedback matrix** and we the system is stabilizable. The pair of matrices (A,B) is known as the **stabilizable pair**.

The following theorem provides necessary and sufficient conditions for determining if a stabilizing feedback matrix exists for a particular system.

Theorem 2.5.3. *Just as in the case of stabilizability, the following statements are equivalent:*

1. *The matrix pair (A,B) is stabilizable.*
2. *Rank*$(A - \lambda I, B) = n$ *for all Re*$(\lambda) \geq 0$.
3. *For all* λ *and* $x \neq 0$ *such that* $x^* A = \lambda x^*$ *and Re*$(\lambda) \geq 0$ *one have* $x^* B \neq 0$.

Corollary 2.5.1. *If the pair (A,B) is controllable, then the pair is also stabilizable.*

We note that controllability of a system implies stabilizability of a system but stabilizability does not imply controllability.

Just as observability is the dual of controllability, so is detectability of a system the dual problem of stabilizability. These system properties plays a important role in control theory in general and in particular in the results that we will use in H_∞ control.

Definition 2.5.4. *A system is said to be detectable if there exists a matrix L such that that given the matrix pair* (A, C), $A - LC$ *is stable.*

Theorem 2.5.4. *The following statements are equivalent:*

1. (A, C) *is detectable.*
2. *The matrix* $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ *has full rank for all Re*$(\lambda) \geq 0$.
3. *For all* λ *and* $x \neq 0$ *such that* $Ax = \lambda x$ *and Re*$(\lambda) \geq 0$, *we have* $Cx \neq 0$.
4. (A^T, C^T) *is stabilizable.*

It can be shown that is a system is controllable then the closed loop eigenvalues can be assigned arbitrarily (eigenvalue assignment problem). The issue with assigning the eigenvalues arbitrarily is that there is no methodology to finding an optimal controller such that the design characteristics are met as efficiently as possible.

This led to the development of numerous other optimal control strategies such as Linear Quadratic optimal control. The objective of the linear quadratic control is to find a control u which minimizes some predetermined quadratic cost function. Quadratic optimal control is beyond the scope of this study. However, note that [7] presents a parallel development of H_∞ control and H_2 optimal control which is known as Linear Quadratic Gaussian control.

2.6 H_∞ norm

First note that the transfer function of any system modeled by linear time invariant ordinary differential equations is rational and has real valued coefficients as noted by [17] and [1]. Thus, transfer functions of any such system have poles contained entirely in the left half plane and are analytic in the right half plane. A space of such functions is called a Hardy space. We shall denote this space as H_∞. For any multivariable system, the transfer function $G(i\omega)$ is a matrix. The singular values of a matrix A, $\sigma_j(A)$ are definted as

$$\sigma_j(A) = \lambda_j(AA^T),$$

where $\lambda_j(E)$ denotes the jth eigenvalue of the matrix E.

Note the Euclidean norm of a matrix is defined to be

$$||A|| = \max_i \sigma_i(A).$$

Then we have

$$||G(i\omega)|| = \max_i \sigma_i(G(i\omega)) = \sigma_{\max}(G(i\omega)).$$

Definition 2.6.1. *The H_∞ (induced) norm of the transfer function $G(s)$, denoted as $||G||_\infty$ is defined as*

$$||G||_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{max}(G(i\omega)), \tag{2.6.1}$$

where $\sup\limits_{\omega\in\mathbb{R}}$ denotes the supremum or least upper bound over all real valued frequencies ω.

This provides another way to characterize the stability of a system, namely if the systems transfer function $G(s) \in H_\infty$ then the system is stable. As described in [15] for the case when G is rational, $G \in H_\infty$ if and only if G has no poles in the closed right half plane.

Chapter 3

Algorithms: Frequency Domain

3.1 Computing the H_∞ norm

In this section we present two algorithms for finding the H_∞-norm of a system of linear ordinary differential equations. While the algorithms use frequency domain tools, they are important for understanding the state space algorithm presented later. Approximating the H_∞-norm of a system requires computing the supremum of the frequency response over all frequencies, thus the use of iterative process is required. The connection between the H_∞-norm of a stable transfer function (i.e no poles in the right half plane) and its associated Hamiltonian matrix plays an important role in the algorithm for the state feedback case presented later.

Consider the system:

$$\begin{aligned} \dot{x}(t) = &\quad Ax(t) + Bu(t), \\ y(t) = &\quad Cx(t) + Du(t). \end{aligned} \tag{3.1.1}$$

Definition 3.1.1. *From the coefficients in the system above, we define M_r, the Hamiltonian matrix*

$$M_r = \begin{bmatrix} A + BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I + DR^{-1}D^T)C & -(A + BR^{-1}D^TC)^T) \end{bmatrix} \tag{3.1.2}$$

where $R = r^2I - D^TD$.

Recall the transfer function $G(s)$ defined from coefficient matrices of the system (3.1.1)

$$G(s) = C(si - A)^{-1}B + D.$$

Theorem 3.1.1. *Let $G(s)$ be a stable transfer function and let $r > 0$. Then $||G||_\infty < r$ if and only if $\sigma_{max}(D) < r$ and M_r has no purely imaginary eigenvalues.*

Theorem 3.1.1 relates the eigenvalues of M_r to the singular values of $G(s)$. This theorem implies that for r values larger than $r^* = ||G(s)||_\infty$, M_r will have no purely imaginary eigenvalues. Further, this implies that if $r < r^*$, M_r will have at least one purely imaginary eigenvalue.

3.2 Bisection Algorithm

This result leads directly to the bisection algorithm presented by Boyd *et al* [3] where in each iteration one checks each eigenvalue of M_r to determine the if there exists a purely imaginary eigenvalue or not. It should be noted that this algorithm converges linearly and the algorithm is also noted to approximate the H_∞-norm with a relative accuracy of ϵ. To start the algorithm one needs to compute upper and lower bounds of the r-interval. One could use $r_{lb} = 0$ and set r_{ub} sufficiently large and immediately proceed with the bisection strategy. However to obtain tighter bounds one could use Hankel singular values of the system.

Definition 3.2.1. *The Hankel singular values are the square roots of the eigenvalues of the matrix $C_G O_G$, where C_G and O_G are the controllability and observability grammians, defined by Equations 2.4.3 and 2.4.4 respectively.*

Hankel singular values, denoted σH_i following convention, are ordered in descending order such that σH_1 is the largest singular value.

The method for calculating r_{lb} and r_{ub} from Step 1 (3.2.2), is known as the **Enns-Glover formula**. We note that the authors suggest alternative formulas for computing the boundary values for the bisection algorithm which can be computed with the following formulas:

$$\begin{aligned} r_{lb} &= max\left\{\sigma_{max}(D), \sqrt{\text{Trace}(O_G C_G)/n}\right\}, \\ r_{ub} &= \sigma_{max}(D) + 2\sqrt{n\text{Trace}(C_G O_G)} \end{aligned} \tag{3.2.1}$$

Algorithm 3.2.1. *Given system (3.1.1) with coefficient matrices A,B,C,D where A is stable and error tolerance $\epsilon > 0$.*
Step 1: *Compute the bounds for the bisection algorithm, r_{lb} and r_{ub}, where*

$$\begin{aligned} r_{lb} = &\quad max\{\sigma_{max}(D), \sigma H_1\} \\ r_{ub} = &\quad \sigma_{max}(D) + 2\textstyle\sum_{j=1}^{n} \sigma H_i \end{aligned} \tag{3.2.2}$$

Step 2: *Set $r = (r_{ub} + r_{lb})/2$*
If $2(r_{ub} - r_{lb}) < \epsilon$, end.
Step 3: *Compute M_r.*
Step 4: *Check for purely imaginary eigenvalues of M_r.*
If M_r has a purely imaginary eigenvalue, then set $r_{lb} = r$.
Else set $r_{ub} = r$.

By Theorem 3.1.1 if we find that M_r has no purely imaginary eigenvalues, we can take r to be the new upper bound for the algorithm. Similarly if we find that M_r does have purely imaginary eigenvalues, we can take r as the new lower bound. In order to ensure the computed value for r^* is truly an accurate approximation, we stop the iterations when $2(r_{ub} - r_{lb}) < \epsilon$.

It should be pointed out that computing the starting bounds for this algorithm requires solving two Lyapunov equations (2.4.3) and (2.4.4), multiplying two $n \times n$ matrices and computing the eigenvalues of this product. For large matrices these computations are computationally expense and are impractical. Also, this algorithm relies on very accurate computations of eigenvalues, which requires very specific eigenvalue solvers, see for example [2].

Example 3.2.1. *Consider system 3.1.1 where*

$$A = \begin{bmatrix} -4 & -8 & 12 \\ 0 & -8 & 0 \\ 0 & 0 & -16 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = 0.$$

Take $\epsilon = 10^{-6}$.

We must first compute O_G *and* C_G*. Using the* `lyap` *command in* MATLAB *, we attain:*

$$O_G = \begin{bmatrix} 0.1250 & 0 & 0.1250 \\ 0 & 0.0625 & 0 \\ 0.1250 & 0 & 0.1250 \end{bmatrix}, C_G = \begin{bmatrix} 0.1250 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using these in Equations 3.2.1 yields:

$$r_{lb} = 0.0722, \quad r_{ub} = 0.4330,$$

which gives $r = 0.2526$ *for the first iteration and*

$$M_r = \begin{bmatrix} -4.0000 & -8.0000 & 12.0000 & 15.6735 & 0 & 0 \\ 0 & -8.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & -16.0000 & 0 & 0 & 0 \\ -1.0000 & -1.0000 & -1.0000 & 4.0000 & 0 & 0 \\ -1.0000 & -1.0000 & -1.0000 & 8.0000 & 8.0000 & 0 \\ -1.0000 & -1.0000 & -1.0000 & -12.0000 & 0 & 16.0000 \end{bmatrix}$$

The eigenvalues of M_r *are* $\{8.0000, 16.0000, -0.5714, 0.5714, -8.0000, -16.0000\}$ *Since none of the eigenvalues are purely imaginary we set* $r = r_{ub}$ *and continue the iterations.*

After 20 iterations we find that $r_{lb} = 0.2499$ *and* $r_{ub} = 0.2500$*, thus the error bound is satisfied and the process is terminated. We arrive at* $||G(s)||_\infty \approx .2500$.

3.3 Two Step Algorithm

Shortly after the Bisection algorithm was presented, a two step algorithm was proposed by Bruinsma and Steinbuch [4] in which only the lower bound needs to be computed. It should be noted that a similar algorithm was presented by Boyd and Balakrishnan [3] at around the same time. It is important to note that the lower bound computation does not require finding O_G or C_G, and is thus less expensive to calculate. Bruinsma and Steinbuch's algorithm also relies on Theorem 3.1.1 as well as the following theorem.

Theorem 3.3.1. *Suppose $r > \sigma_{max}(D)$ and $\omega \in \mathbb{R}$. Then the $det(M_r - \omega i I) = 0$ if and only if for some n, $\sigma_n(G(\omega i)) = r$.*

As a consequence of this theorem, the imaginary eigenvalues of M_r are exactly the frequencies where some singular value of $G(\omega i) = r$. That is ωi is an eigenvalue of M_r if and only if r is a singular value of $G(\omega i)$ where $\omega \in \mathbb{R}$. In Figure 3.1 ωi represents the frequency, r represents the iterations and the m_i represent the midpoint between each ω_i and ω_{i+1}. Here we see Theorem 3.3.1 allows for relating the purely imaginary eigenvalues back to multiple values of r. Specifically each m_i relates back to a value for r. In Figure 3.1 we can see that the next iteration of r is taken to be the maximum of the singular value of $G(s)$ for each m_i.

The algorithm by Bruinsma and Steinbuch builds on the ideas of the bisection algorithm, but uses Theorem 3.3.1 to search for $||G(s)||_\infty$ using multiple values of ω. The bisection algorithm only searches at one frequency per iteration, where as the two step algorithm searches multiple frequencies per iteration. This is the main advantage to using this algorithm. The authors claim this algorithm converges quadratically (they also claim $r_i < r_{i+1}$, though no proof of this could be determined).

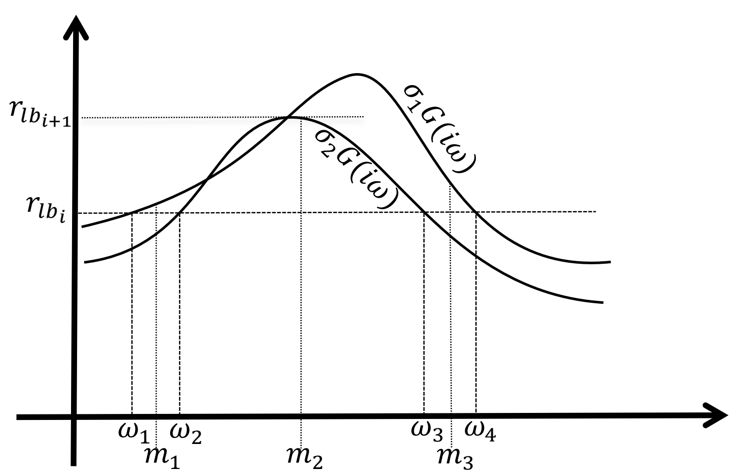

Figure 3.1: Relationship between singular values and eigenvalues

To begin the algorithm let

$$r_{lb} = \max\{\sigma_{max}G(0), \sigma_{max}G(\omega_p i), \sigma_{max}(D)\}, \tag{3.3.1}$$

for finding the lower bound, where $\omega_p = |\lambda_i|$ and λ_i is a pole of the transfer function $G(s)$ chosen based on the following criteria:

If $G(s)$ has one or more poles with imaginary parts, then λ_i is the pole that maximizes the value

$$\left|\frac{Im(\lambda_i)}{Re(\lambda_i)}\frac{1}{\lambda_i}\right|. \tag{3.3.2}$$

If $G(s)$ has poles which are strictly real valued, then λ_i is the pole which maximizes $|\frac{1}{\lambda_i}|$.

As is the case with the bisection algorithm, this method is for systems where A is stable.

Algorithm 3.3.1. *Given system (3.1.1) with coefficient matrices A,B,C,D where A is stable one can find the H_∞ norm using the following steps.*
Step 1.1: *Compute the transfer function G(s) of the given system.*
Step 1.2: *Find the poles of G(s).*
If the poles are strictly real valued, set $\omega_p = max|\lambda_i|$.
Else set $\omega_p = \lambda_i$, where λ_i maximizes (3.3.2).
Step 1.3: *Compute r_{lb} using (3.3.1).*
Step 2.1: *Compute $r = (1 + 2\epsilon)r_{lb}$.*
Step 2.2: *Compute M_r from (3.1.2).*
Sort all of the purely imaginary eigenvalues of M_r, and label them in descending order $\omega_1, \ \ldots \ , \omega_k$.
If M_r has no purely imaginary eigenvalues, set $r_{ub} = r$ and proceed to Step 3.
Step 2.3: *For $i = 1$ to $k - 1$*
(i) Compute $m_i = \frac{1}{2}(\omega_i + \omega_{i+1})$.
(ii) Compute $\sigma_{max}(G(m_i i)) = svd_i$.
Set $r_{lb} = max(svd_i)$.
Step 3: *Compute $\|G\|_\infty = \frac{1}{2}(r_{lb} + r_{ub})$.*

Since the H_∞ norm is defined as the maximum singular value along the imaginary axis, we can sample along the imaginary axis to find the lower bound for the H_∞ norm. To ensure $\|G\|_\infty$ can be calculated using the two step algorithm, one must carefully choose a lower bound for the algorithm.

After computing the eigenvalues of M_r, one must determine if r is an upper or lower bound, this is accomplished using Theorem 3.1.1, similar to the bisection algorithm. However, if one finds purely imaginary eigenvalues $\omega_1, \omega_2, \ldots, \omega_n$, by Theorem 3.3.1 we know r is a singular value for each $G(\omega_i)$. This implies there exists frequencies $m_1, \ldots, m_{n-1}$, each m_i between its respective ω_i, ω_{i+1} such that for some m_i, $G(\omega i) < G(m_i)$. By computing $\sigma_{max}(G(m_i)$ for all i, the algorithm searches over a much larger area than the bisection algorithm. After finding $\sigma_{max}(G(m_i)$ for all i, we set this value equal to r_{lb}, and start the next iteration.

Example 3.3.1. *Let*

$$A = \begin{bmatrix} -1.0000 + 1.0000i & -1.0000 & -1.0000 \\ -1.0000 & -2.0000 + 1.0000i & -1.0000 \\ -1.0000 & -1.0000 & -2.0000 - 1.0000i \end{bmatrix}.$$

The poles of $G(s)$ are the eigenvalues of A. We find the eigenvalues of A to be $\{-3.3589 + 0.2858i, -0.3628 + 0.9534i, -1.2784 - 0.2392i\}$. Thus A is stable.

Next we find that $-0.3628 + 0.9534i$, maximizes (3.3.2) and using (3.3.1) we calculate $r_{lb} = 0.9907$.

Using the above value for r_{lb} we find,

$$M_r = \begin{bmatrix} -1+i & -1 & -1 & 1.0188 & 0 & 0 \\ -1 & -2+i & -1 & 0 & 0 & 0 \\ -1 & -1 & -2-i & 0 & 0 & 0 \\ -1 & -1 & -1 & 1+i & 1 & 1 \\ -1 & -1 & -1 & 1 & 2+i & 1 \\ -1 & -1 & -1 & 1 & 1 & 2-i \end{bmatrix}.$$

Computing the eigenvalues of M_r we get:

$$\begin{array}{rcl} -3.1927 & + & 0.2164i, \\ 3.1927 & + & 0.2164i, \\ -1.3224 & - & 0.1563i, \\ 1.3224 & - & 0.1563i, \\ -0.0000 & + & 0.8597i, \\ 0.0000 & + & 1.0201i \end{array}$$

We find and order the purely imaginary eigenvalues, $w_1 = -0.0000 + 0.8597i$ and $w_2 = 0.0000 + 1.0201i$. Finding the average of these two gives $m_1 = 0.0000 + 0.9399i$. The maximum singular value of $G(m_1 i) = 1.0122 = r_{lb}$.

We then start the next iteration since there are strictly imaginary eigenvalues of M_r. After a total of four iterations we arrive at $\|G\|_\infty = 1.0121$.

Remark: We note one major issue which arrises in implementing this algorithm is the need for an accurate eigenvalue solver since the depends on the decision if the eigenvalues are purely imaginary or not.

Chapter 4

Algorithms: State space Approach

Doyle *et al* [7] presented a state space characterization for the suboptimal H_∞ control problem. This characterization includes the requirement of the unique stabilizing solutions to two Algebraic Riccati Equations to be positive semidefinite as well as a spectral radius coupling condition.

4.1 The Hamiltonian and Associated Riccati Equation

We first review the relationship between an Algebraic Riccati Equation and the associated Hamiltonian matrix and introduce some notation.

The Hamiltonian matrix,

$$H = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix},$$

where $A, Q, S \in \mathbb{R}^{n \times n}$ and both Q and S are symmetric has an associated Algebraic Riccati Equation (ARE),

$$XA + A^T X + Q - XSX = 0.$$

If H has no eigenvalues on the imaginary axis, H will have n eigenvalues with positive real parts and n eigenvalues with negative real parts. Denote the n-dimensional stable invariant subspace of H, by $\chi_-(H)$, which corresponds to the eigenvalues in the open left half plane $\mathbb{C}_-$. We write

$$\chi_-(H) = Im \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$

where T_1 and $T_2 \in \mathbb{R}^{n \times n}$ and $Im(A)$ denotes the column space of A.

Theorem 4.1.1. *Let (A, B) be stabilizable and (A, Q) be detectable, then the Hamiltonian matrix H, has n eigenvalues with negative real parts, no eigenvalues on the imaginary axis and n eigenvalues with positive real parts. Further the associated Riccati equation has a* *unique stabilizing (symmetric) solution X, and the eigenvalues of $A - BK$ are the stable eigenvalues of H.*

If H satisfies the following two properties,

1. $\Lambda(H) \cap \mathbb{C}_0 = \emptyset$ (stability property) where $\Lambda(H)$ denotes the spectrum of H.

2. The two spaces
$$\chi_-(H) = Im \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$
and
$$Im \begin{bmatrix} 0 \\ I \end{bmatrix}$$
are complimentary, in other words T_1 is nonsingular;

we can set $X = T_2 T_1^{-1}$. Then X is uniquely determined by H and we say that $H \in dom(Ric)$. If $H \in dom(Ric)$ then H has no purely imaginary eigenvalues and the matrix T_1 is nonsingular. That is $Ric : H \to X$ and $Ric(H) = X$. One can verify a given H exists in $dom(Ric)$ with the following lemma from Doyle *et al* [7].

Lemma 4.1.1. *Suppose $H \in dom(Ric)$ and $X = Ric(H)$. Then:*

1. *X is symmetric;*
2. *X satisfies the ARE $XA + A^T X + Q - XSX = 0$;///*
3. *$A - SX$ is stable.*

Recall the system,
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{4.1.1}$$

with the Hamiltonian matrix associated with the system
$$H(r) = \begin{bmatrix} A & -\frac{1}{r^2}BB^T \\ -C^T C & -A^T \end{bmatrix}. \tag{4.1.2}$$

Note $H(r)$ is equal to M_r given by (3.1.2), where $D = 0$.

The Riccati equation associated with the Hamiltonian matrix $H(r)$ is
$$XA + A^T X + Q - XSX = 0,$$
where $S = \frac{1}{r^2}BB^T$ and $Q = -C^T C$.

Recall that the Hamiltonian matrix M_r, is connected to $||G||_\infty$ through Theorem 3.1.1. Thus, $H(r)$ has no purely imaginary eigenvalues when $||G||_\infty < r$. Theorem 4.1.1 connects the Hamiltonian matrix $H(r)$ with its associated ARE and provides conditions under which $H(r) \in dom(Ric)$, or has no purely imaginary eigenvalues. Theorem 4.1.1 further connects $||G||_\infty$ with the existence of a stabilizing solution X. Specifically it lays the groundwork for determining necessary and sufficient conditions for the existence of an admissible controller in a state space formulation.

That is if (A, B) is stabilizable and (A, Q) is detectable, then $H(r) \in dom(Ric)$, hence there exists matrices T_1 and T_2 such that,

$$\chi_-(H) = Im \begin{bmatrix} T_1 \\ T_2 \end{bmatrix},$$

and $X = T_2 T_1^{-1}$ is the stabilizing solution to the associated ARE with $X \geq 0$.

Since $H(r) \in dom(Ric)$, $H(r)$ has no purely imaginary eigenvalues and by Theorem 3.1.1, $||G||_\infty < r$. Combining Theorems 3.1.1 and 4.1.1, we have if the stabilizing solution to the associated ARE exists, then $||G||_\infty < r$. Thus given system (4.1.1) the problem of finding the smallest value of r such that $H(r)$ has no purely imaginary eigenvalues becomes finding the smallest value of r such that a stabilizing solution to the ARE exists where $X = T_2 T_1^{-1}$ with $X \geq 0$.

Output Feedback

We now consider the output feedback case. A finite-dimensional time-invariant linear continuous-time dynamical system may be described using the following system of first-order ordinary differential equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t). \end{aligned} \tag{4.1.3}$$

Here $x(t)$, $y(t)$, $z(t)$ are the state, sensed output and controlled output vectors respectively and $u(t)$, $w(t)$ are the control and disturbance inputs respectively.
Assume throughout the discussion that follows that the matrices $A, B_1, B_2, C_1, C_2, D_{12}, D_{21}$ are real and satisfy the following assumptions:

(A1) (A, B_1) is stabilizable and (C_1, A) is detectable;

(A2) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A3) $D_{12}^T [C_1 \; D_{12}] = [0 \; I]$;

(A4) $D_{21}^T [B_1^T \; D_{21}^T] = [0 \; I]$.

The two Hamiltonian matrices associated with system (4.1.3) are

$$H_\infty(r) := \begin{bmatrix} A & \frac{1}{r^2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}, \tag{4.1.4}$$

and its dual,

$$J_\infty(r) := \begin{bmatrix} A^T & \frac{1}{r^2}C_1^TC_1 - C_2^TC_2 \\ -B_1^TB_1 & -A \end{bmatrix}. \tag{4.1.5}$$

The two Algebraic Riccati Equations associated with the system are:

$$XA + A^TX - X(B_2B_2^T - \frac{1}{r^2}B_1B_1^T)X + C_1^TC_1 = 0, \tag{4.1.6}$$

and its dual,

$$AY + Y^TA - Y(C_2^TC_2 - \frac{1}{r^2}C_1^TC_1)Y + B_1B_1^T = 0. \tag{4.1.7}$$

The four assumptions (A1)—(A4) are standard for control applications. They were originally formulated for H_2 control and since H_∞ control was developed in parallel with H_2 control these assumptions followed [7]. Assumption (A1) is necessary for the existence of a stabilizing controller in the form $u = Ky$. Assumption (A2) is sufficient for the closed loop system to be both stabilizable and detectable. If the closed loop system is not stabilizable, then it is certainly not controllable and therefore the H_∞ optimal control problem has no solution. Assumptions (A3) and (A4) imply there is no overlap between the cost of the control input and the state. In other words the cost penalty applied to z contains a non-singular and normalized cost penalty on the control input u. Assumptions (A3) and (A4) can be relaxed but they are made in an effort to obtain cleaner formulas.

Recall the Hamiltonian matrix $H(r)$ given by (4.1.2), associated with system (4.1.1). It is important to note the difference between $H(r)$ and $H_\infty(r)$ given by (4.1.4). Specifically the ability to make $\frac{1}{r^2}B_1B_1^T$ arbitrarily large or small no longer guarantees a sign definite matrix in the $[1,2]$ block of $H_\infty(r)$. As a consequence one is not guaranteed the existence of a stabilizing solution to the ARE (4.1.6) even when $H_\infty(r) \in dom(Ric)$. In other words $H_\infty(r) \in dom(Ric)$ is still a necessary condition for the existence of a stabilizing solution, but it is now no longer a sufficient condition.

Doyle *et al* [7] shows that system (4.1.3) which satisfies (A1)—(A4), has an admissible controller K, such that $||T_{zw}||_\infty < r$ if and only if the following necessary and sufficient conditions are met:

(C1) $H_\infty(r) \in dom(Ric)$ and $X(r) := Ric(H_\infty(r)) \geq 0$;

(C2) $J_\infty(r) \in dom(Ric)$ and $Y(r) := Ric(J_\infty(r)) \geq 0$;

(C3) $\rho(\mathrm{X}(r)\mathrm{Y}(r)) < r^2$.

where T_{zw} denotes the transfer function from the disturbance w to the controlled output z, such that

$$T_{zw} = C_1(sI - A + B_2K)^{-1}B_1 + D12.$$

Lin, Wang, Xu [16] present an algorithm to find $||T_{zw}||$ using a state space approach. The algorithm breaks the three conditions (C1)—(C3) into three seperate sub-problems.

1. Finding the smallest r value for which (C1) holds r_x, is performed first. In doing so, r_x is distinguished by the singularity of the T_1 matrix created by the basis vectors of the stable invariant subspace of $H_\infty(r)$. Using a reformulation (in terms of singularities) of (C1) a secant method is applied (rather than a Newton type method) directly utilizing the stable invariant subspace of $H_\infty(r)$.

2. The same technique is then applied to $J_\infty(r)$, to find r_y, the smallest value for r such that (C2) holds.

3. The geometric nature (strictly decreasing) of $\rho(X(r)Y(r))$ is then exploited to find r^*, the value for which all three conditions (C1)—(C3), hold.

To find r_x the smallest value such that condition (C1) holds, we must first find the interval such that $H_\infty(r)$ has no purely imaginary eigenvalues. It is a well known result that there is only one such interval, (r_{lb}, ∞) [10], [12], [21]. Furthermore it is a well known result that $r_{lb} = ||G_Z||_\infty$, where

$$G_Z(s) = B_1^T(sI - (A - ZC_1^TC_1)^T)^{-1}C_1^T,$$

with $Z = Ric(H_2)$ and

$$H_2 = \begin{bmatrix} A^T & -C_1C_1^T \\ -B_2B_2^T & -A \end{bmatrix}.$$

Thus to find r_{lb} we can apply either the bisection or two step method to the transfer function

$$G(i\omega) = B_1^T(i\omega I - (A - ZC_1^TC_1)^T)^{-1}C_1^T. \tag{4.1.8}$$

Alternatively a bisection approach could be used directly on $H_\infty(r)$. Upon calculating the eigenvalues of $H_\infty(r)$, one could verify $Re(\lambda_i) \neq 0$ and $im(\lambda_i) = 0$ for $1 \leq i \leq n$. If $\Lambda(H_\infty(\tilde{r})) \cap \mathbb{C}_0 = \emptyset$ then $\tilde{r} < r_{lb}$ and if $\Lambda(H_\infty(r)) \cap \mathbb{C}_0 \neq \emptyset$ then $\tilde{r} < r_{lb}$.

To satisfy the second condition of (C1), $X(r) \geq 0$ we note that as a consequence of the existence of only one interval (r_{lb}, ∞) such that $\Lambda(H_\infty(r)) \cap \mathbb{C}_0 = \emptyset$ then once r_{lb} is found $H_\infty(r)$ has no purely imaginary eigenvalue for all $r > r_{lb}$. Then one can find matrices T_1 and T_2 which correspond to the stable eigenvalues of $H_\infty(r)$ where

$$\chi_-(H_\infty)(r) = Im \begin{bmatrix} T_1(r) \\ T_2(r) \end{bmatrix},$$

and if $T_1(r)$ is non-singular set $X(r) = T_2(r)T_1(r)^{-1}$. When such an X exists and is positive semi-definite for some $\tilde{r}$ then it positive semi-definite for all $r > \tilde{r}$, and is monotonic decreasing by the following theorem from [12].

Theorem 4.1.1. *If a non-zero $X(\tilde{r}) = Ric(H_\infty(\tilde{r}))$ exists and is positive semidefinite for some $\tilde{r} > 0$, then $X(r) = Ric(H_\infty(r))$ exists and is positive semidefinite for all $r \in [\tilde{r}, \infty)$ and obeys the partial order $0 \leq X(r_2) \leq X(r_1)$ for $\tilde{r} \leq r_1 \leq r_2 < \infty$.*

Assume $0 < r_x < \infty$, Theorem 4.1.1 implies the existence of a least r such that $X(r) \geq 0$, where $A \geq 0$ denotes the matrix A is positive semi-definite. As a result, one could implement a bisection algorithm to find such an r noting if $X(r) \geq 0$ then $r > r_x$ and if $X(r) \ngeq 0$ then $r < r_x$. The following theorem from Lin *et al* (1998), is the keystone to this, as it provides the means in which one can implement a bisection or secant method to search for this r. In fact the following theorem is a reformulation of (C1) from [7].

Theorem 4.1.2. *For $r > r_{lb}$, let $X(r) = Ric(H_\infty(r))$ whenever $H_\infty(r) \in dom(Ric)$. Suppose that $X(r_2) \geq 0$ and $X(r_1) \ngeq 0$ such that $r_2 > r_1 > r_{lb}$. Then the following are true:*

(i) There exists at least one $r \in (r_1, r_2)$ such that $T_1(r)$ is singular.
(ii) Let

$$r_s = sup\{r | T_1(r) \text{ is singular}\}. \tag{4.1.9}$$

Then $X(r) \geq 0$ for $r > r_s$ and $X(r) \ngeq 0$ for $r_{lb} < r < r_s$, when $X(r)$ exists.

Proof(outline)

(i) Assume that $T_1(r)$ is nonsingular for all $r \in (r_1, r_2)$.

- Since $X(r_2) \geq 0$ and $X(r_1) \ngeq 0$, by continuity of eigenvalues there exists $\tilde{r} \in (r_1, r_2)$ such that $X(\tilde{r})$ is singular and $X(r) \geq 0$ for all $r > \tilde{r}$.
- By the monotonicity result in Theorem 4.1.1, we have $0 \leq X(r) \leq X(\tilde{r})$, which implies the smallest eigenvalue of $X(r)$ is zero and further this applies to all $r \geq \tilde{r}$.
- However this contradicts the fact that $X(r_2) \geq 0$, thus there exists an $r \in (r_1, r_2)$ such that $T_1(r)$ is singular.

(ii) Suppose there exists $\alpha > r_s$ where $X(\alpha) \ngeq 0$, then by Theorem 4.1.1, we have $\alpha < r_2$. Thus from (i) there must exist $\beta \in (\alpha, r_2)$ with $X(\beta)$ singular. This contradicts the definition of r_s, thus $X(r) \geq 0$ for all $r > r_s$.

On the other hand, suppose there exists $\gamma \in (r_{lb}, r_s)$ with $X(\gamma) \geq 0$, then by Theorem 4.1.1 we have $X(r)$ exists for all $r \geq \gamma$. However this is a contradiction because $X(r_s)$ does not exist. Thus $X(r) \ngeq 0$ for $r \in (r_{lb}, r_s)$.

Assume that $r_x < \infty$, and $r_{lb} < r_s$.

Then there exists at least one $r \in (r_{lb}, \infty)$ such that $T_1(r)$ in,

$$\chi_-(H_\infty(r)) = Im \begin{bmatrix} T_1(r) \\ T_2(r) \end{bmatrix},$$

is singular and $r_x = r_s$, where

$$r_s = sup\{r|T_1(r) \text{ is singular}\}.$$

Thus the problem of finding the smallest value for r such that (C1) holds is now reformulated into finding r_s, the smallest value such that $X(r) \geq 0$ for $r > r_s$ where $r_s \geq r_{lb} > 0$.

Note $r_{lb} \leq r_s$ and $H_\infty(r) \cap \mathbb{C}_0 = \emptyset$ for all $r > r_{lb}$, then by assumptions (A1) and (A2) and Theorems 4.1.1 and 4.1.2, the matrices T_1 and T_2 are defined, T_1 is non singular and $T_2(r)T_1^{-1}(r) = X(r) \geq 0$ for all $r > r_s$. Thus r_s can be approximated with a bisection strategy, checking the positive semi-definiteness of the matrix $T_2(r)T_1^{-1}(r)$. Explicitly, if $T_2(r)T_1^{-1}(r) \geq 0$ then $r > r_s$ and if $T_2(r)T_1^{-1}(r) \ngeq 0$ then $r < r_s$.

To increase the rate of convergence,one could apply a variant secant method to search for r_s as well. By Theorem 4.1.2, r_s is the largest value such that $T_1(r)$ is singular, then our goal is to find the largest r such that $f(r) = 0$, where

$$f(r) = \sigma_{min}(T_1(r)).$$

However singular value plots are non-negative functions and by Theorem 4.1.2 $f(r)$ may have multiple zeros. For this reason the secant method is not guaranteed to converge using $f(r)$.

Thus we actually apply the secant method to the new smooth function

$$f(r) = \begin{cases} -\sigma_{min}(T_1(r)) & \text{if } r \leq r_s \\ \sigma_{min}(T_1(r)) & \text{if } r \geq r_s \end{cases},$$

that is, f along with its reflection over the r axis for $r < r_s$. One issue which arrises from this new function is now one must deduce which case to use, that is f or $-f$. To make this determination we appeal once more to Theorem 4.1.2, and check if $X(r) = T_2(r)T_1^{-1}(r) \geq 0$ by computing the eigenvalues of the matrix $T_1^T(r)T_2(r)$ which is congruent to $T_2(r)T_1^{-1}(r)$ and avoids the inverse calculation for $T_1(r)$.

This leads to an algorithm for the state feedback case. The following is an outline of the algorithm.

Algorithm 4.1.1. *This algorithm computes r_x, the smallest value for r such that condition (C1) holds.*
Step 1:
Attain values for $r_-$and r_+, using equations 4.1.10 - 4.1.12.
Compute $\Lambda H_\infty(\tilde{r})$, the eigenvalues of $H_\infty(\tilde{r})$, where $\tilde{r} = \frac{1}{2}(r_- + r_+)$

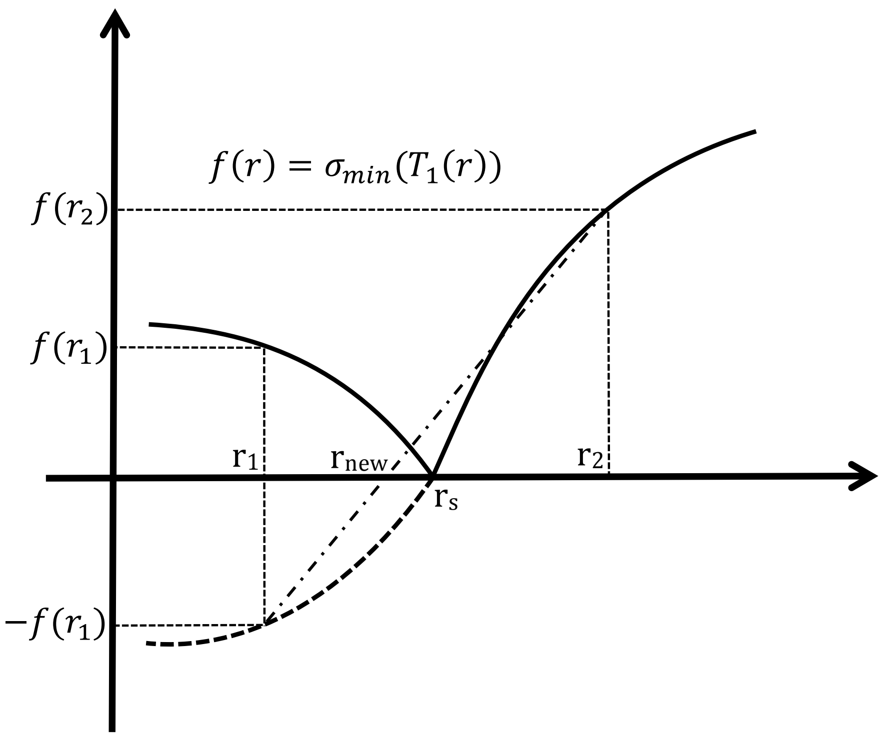

Figure 4.1: Variant secant method

Step 2:
Determine the next iterations interval $[r_-, r_+]$.

Step 2.1
If *the matrix* $H_\infty(\tilde{r})$ *has purely imaginary eigenvalues then set* $r_- = \tilde{r}$ *and move to Step 3.*

Step 2.2
If *the matrix* $H_\infty(\tilde{r})$ *has no purely imaginary eigenvalues then compute the eigen-matrix* $[T_1\ T_2]^T$ *and set case* $= 1$.
If $T_2T_1^{-1}$ *is positive semidefinite, then set* $r_+ = \tilde{r}$.
If $T_2T_1^{-1}$ *is not positive semidefinite, then set* $r_- = \tilde{r}$.

Step 3: *Update* $\tilde{r}$.

If $r_+ - r_- \leq Tol$, *then use the bisection method.*
If $r_+ - r_- > Tol$, *then use the secant method method, with functions based on the case value from Step 2.*

Step 4: *Stop Criteria:*

If *any of the stop criteria hold, then accept* r_* *accordingly*

Else *return to Step 1.*

Step 5: *Double check:*

Perform Step 1 and Step 2 with $\tilde{r} = r_*$.
If $T_2 T_1^{-1}$ *is positive semi definite find* $\Delta r > 0$ *such that* $T_1(r_* + \Delta r)$ *is well conditioned.*
If $T(r_* + \Delta r)_2 T_1(r_* + \Delta r)^{-1}$ *is positive semi-definite then accept* $r_* + \Delta r = r_*$.
Else *set* $r_- = r_* + \Delta r$ *and return to Step 1.*

If $T_2 T_1^{-1}$ *is not positive semi definite set* r_+ *large and return to Step 1.*

We will now explore the complexity of each of the steps above, one at a time.

Step 1

To begin the algorithm one must find an initial interval on which to search for r_x, the smallest value for r for which (C1) holds. In practical cases assuming B_2 is not singular, to compute the upper bound of this interval we can set

$$r_+ = \frac{\sigma_{max}(B_1)}{\sigma_0(B_2)}, \tag{4.1.10}$$

where $\sigma_{\max}$ is the largest singular value and $\sigma_{\min}$ is the smallest singular value.

Much like the previous algorithms, one could use the Hankel singular values for the upper and lower bounds as well, where

$$r_- = \sigma_{H1}, \tag{4.1.11}$$
$$r_+ = 2\sum_i \sigma_{Hi}. \tag{4.1.12}$$

Note However depending on the system it might be more efficient to set $r_- = 0$ use 4.1.10 for r_+.

Note, if one computes r_{lb} using the two step method or the bisection method on the system response matrix $G_Z(i\omega)$, 4.1.8 then Step 2.1 below can be avoided since $||G_Z||_\infty = r_{lb}$, thus establishing the interval on which $\Lambda H_\infty(r) \cap \mathbb{C}_0 = \emptyset$ before starting this algorithm.

Step 2

In this step we determine the next upper bound or lower bound of the interval by checking if the value for $\tilde{r}$ is above or below r_x. This involves testing values of r to verify they satisfy condition (C1), $\Lambda H_\infty(r) \cap \mathbb{C}_0 \neq \emptyset$ and $Ric(H) \geq 0$. We must first find r_{lb}, then we seek r_s. We note that r_s may or not be the same value as r_{lb}.

Step 2.1, $\Lambda H_\infty(r) \cap \mathbb{C}_0 \neq \emptyset$

Since there exists only one such interval, if $\Lambda H_\infty(r) \cap \mathbb{C}_0 \neq \emptyset$ then $\tilde{r} < r_{lb}$ and we set $\tilde{r} = r_-$. Doing so requires accurately computing the eigenvalues of a $2n \times 2n$ Hamiltonian matrix. Error in these calculations could severely effect the choice of r_- since it must be determined if $Re(\lambda_i) < Tol$ and $im(\lambda_i) > Tol$ for all $0 \leq i \leq n$.

Step 2.2, $\Lambda H_\infty(r) \cap \mathbb{C}_0 = \emptyset$

Since there can be values for r such that the Hamiltonian matrix $H_\infty(r)$ may have no eigenvalues on $\mathbb{C}_0$, but $X(r) \ngeq 0$, we must test the positive semi-definite property of $T_2 T_1^{-1}(\tilde{r})$. Here we use the singular values of $T_1(\tilde{r})$ to determine if the value for $\tilde{r}$ is greater than or less than r_x. Theorem 4.1.1 implies that one can impliment a bisection algorithm in the search for r_x and Theorem 4.1.2 provides the means to implement this search, namely searching over the singular values of $T_1(\tilde{r})$.

There are two major components to this step. First one must compute the $2n \times n$ eigenmatrix

$$\begin{bmatrix} T_1(\tilde{r}) \\ T_2(\tilde{r}) \end{bmatrix},$$

then it must be determined if $T_2(\tilde{r})T_1^{-1}(\tilde{r}) \geq 0$.

To do so, one can calculate the n eigenvectors of $H_\infty(\tilde{r})$ which correspond to the stable eigenvalues, order them in a matrix of size $2n \times n$ and divide the matrix such that T_1 is the top half and T_2 is the bottom half.

Now we must judge the positive semi-definiteness of $T_2 T_1^{-1}$. Since it is best to avoid the computation of an inverse matrix, we note that $T_2 T_1^{-1}$ and$T_1^T T_2$ are congruent matrices, so therefore we check the positive semi-definiteness of $T_1^T T_2$ instead. To do so we verify that $Re(\lambda_i) \geq 0$ for all $0 \leq i \leq n$. Again note this requires accurate computations of eigenvalues.

By Theorem 4.1.2 if $T_2(\tilde{r})T_1^{-1}(\tilde{r}) \geq 0$ then $\tilde{r}$ is an upper bound for r_s so we set $\tilde{r} = r_+$, however if it is determined that $T_2(\tilde{r})T_1^{-1}(\tilde{r}) \ngeq 0$ then $\tilde{r}$ is an a lower bound for r_s so then we set $\tilde{r} = r_-$.

Step 3

In step three of the algorithm we update the next iterate of $\tilde{r}$. Here we use either the bisection method or secant method. We determine which method to use based on whether or not

$(r_+ - \tilde{r})$ is less than some predetermined tolerance, *tol*. If the difference is less than *tol*, we use the secant method and if the difference is larger than *tol* we use the bisection method to speed up the iterative process.

The bisection algorithm is standard, that is if $(r_+ - r_-) > tol$ set $\tilde{r} = \frac{1}{2}(r_- + r_+)$ and proceed to Step 4 to check the stop criteria.

The secant method can be applied to both Step 2.1 (case = 0) and Step 2.2 (case =1), however it is applied to different functions for the for the two different cases since the goal for each case is different. For the Step 2.1, that is $\Lambda(H_\infty(\tilde{r})) \cap \mathbb{C}_0 \neq \emptyset$, where we let $\tilde{r} = r_+$ such that,

$$\tilde{r} = \frac{(r_+ - r_-)\min\limits_{i \neq j, i,j \in I_0} |\lambda_i(H_\infty(r_-)) - \lambda_j(H_\infty(r_-))|}{\min\limits_{1 \leq k \leq 2n} |\mathcal{R}(\lambda_k(H_\infty(r_+)))| +) \min\limits_{i \neq j, i,j \in I_0} |(\lambda_i(H_\infty(r_-)) - \lambda_j(H_\infty(r_-))|}. \tag{4.1.13}$$

This is the secant method for the function

$$g(r) = \begin{cases} -\min\limits_{i \neq j, i,j \in I_0} |\lambda_i(H_\infty(r)) - \lambda_j(H_\infty(r))| & \text{if } r \leq r_{lb}, \\ \min_k |\mathcal{R}(\lambda_k(H_\infty(r)))| & \text{if } r \geq r_{lb}, \end{cases} \tag{4.1.14}$$

where $\lambda_k(H_\infty(r))$ is the k^{th} eigenvalue of $H_\infty(r)$ and

$$I_0 = \{i | 1 \leq i \leq 2n, \lambda_1(H_\infty(r)) \in \Lambda(H_\infty(r)) \cap \mathbb{C}_0\}.$$

The main reasons for choosing $g(r)$ as such are, as $r \to r_{lb}$ from the right, the real parts of $\lambda_i(H_\infty(r))$ go to zero since some of the eiegnvalues of $H_\infty(r)$ are on $\mathbb{C}_0$ for $r < r_{lb}$. Also as $r \to r_{lb}$ from the left the distance between the imaginary eigenvalues of $H_\infty(r)$ goes to zero since the eigenvalues are moving off the imaginary axis.

If $case = 1$ that is, $\Lambda(H_\infty(\tilde{r})) \cap \mathbb{C}_0 = \emptyset$ we let $\tilde{r} = r_-$ where,

$$\tilde{r} = r_- + \frac{(r_+ - r_-)\sigma_{min}(T_1(r_-))}{\sigma_{min}(T_1(r_+)) + \sigma_{min}(T_1(r_-))}. \tag{4.1.15}$$

We note that this is the secant method applied to the function

$$f(r) = \begin{cases} -\sigma_{min}(T_1(r)) & if \quad r \leq r_s, \\ \sigma_{min}(T_1(r)) & if \quad r \geq r_s. \end{cases} \tag{4.1.16}$$

It is also important to note that $f(r)$ was chosen so that f is smooth and the secant method will converge about r_s.

Step 4

There are two types of stop criteria used in this algorithm for each case. The first stop criterion deals with the distance between the previous and the next iteration of r, $\Delta\tilde{r}$. The second stop criterion deals with the magnitude of the functions 4.1.14 and 4.1.15. The two stop criteria are necessary since $\Delta\tilde{r} \to 0$ does not directly imply $r \to r_x$. Thus we combine the convergence of $\Delta\tilde{r}$ with the value of equations 4.1.14 or 4.1.15 accordingly.

Suppose the $\hat{r} = \tilde{r} + \Delta\tilde{r}$ is the updated iterate of $\tilde{r}$. For both cases the stop criteria $\Delta r < tol$ was used, were $tol = 10^{-6}$.

For the function convergence in case=0, we are concerned with the eigenvalues of the Hamiltonian matrix, $H_\infty(\tilde{r})$. The secant method uses equation 4.1.14. Since we have $g(r) \to 0$ as $r \to r_{lb}$ then we want $|g(r)| < tol$ and move to Step 5.

For case =1 we are concerned with the singular vales of T_1. For this case we use equation 4.1.15 for the secant method. Since we want to find the smallest value of r such that T_1 is singular we want $f(r) = 0$, thus we stop when $|f(r)| < tol$ and move to Step 5.

Step 5

It is important to note that the vast majority of the computations in the algorithm rely on the various matrices being well conditioned. The computation of $T_2(r)T_1(r)^{-1}$, is not trivial and relies on the numerical computations being accurate. This accuracy depends on the condition of the matrices $T_2(r)$ and $T_1(r)$. Further the secant method applied to 4.1.15 may have converged to an incorrect value of r. Thus we must double check the algorithm converged to the correct value and also verify the condition of $T_1(r)$ since it must be inverted. In doing so the actual value for r_x may not be attained, but rather a more conservative value will found.

To ensure convergence of the algorithm to $r_x = r_s$ we must verify if $T_2(\tilde{r})T_1^{-1}(\tilde{r}) \geq 0$. If this holds then we check if $T_1(\tilde{r})$ is well conditioned. If the matrix is well conditioned we stop and accept $\tilde{r} = r_x$.

On the other hand if $T_1(\tilde{r})$ is ill-conditioned, we must find the smallest perturbation of $\tilde{r}$ such that $T_1(r)$ is well conditioned.

If $T_2(\tilde{r})T_1^{-1}(\tilde{r}) \ngeq 0$ then we know $\tilde{r} < r_x$ thus set $r_- = \tilde{r}$ and r_+ sufficiently large and return to Step 1.

It is for this reason that in implementing this algorithm, it would be immensely beneficial to accurately calculate r_{lb} before hand. The bisection algorithm presented earlier would provide an approximation for r_{lb} which could then be used as the lower bound in Algorithm 4.1.1.

It should be noted that (C1) and (C2) are dual problems of each other thus the above algorithm is also used to find the value of r_y, that is the smallest value of r such that condition (C2) holds, by replacing A, B_1, B_2 and C_1 with A^T, C_1^T, C_2^T and B_1^T respectively.

Let $r_m = max\{r_x, r_y\}$, then by Theorem 4 and the definitions of r_x and r_y, $X(r) := Ric(H_\infty(r))$ and $Y(r) := Ric(J_\infty(r))$ both exist and further $X(r), Y(r) \geq 0$ for all $r \geq r_m$.

From this we see that r_m is a lower bound for r^*, the value for which (C1)—(C3) hold simultaneously.

Recall: (C3) $\rho(\mathrm{X}(r)\mathrm{Y}(r)) < r^2$
Define,

$$\rho(r) = \rho(\mathrm{X}(r)\mathrm{Y}(r)). \tag{4.1.17}$$

To find r^* we take advantage of the geometrical natures of $\rho(r)$ and r^2. Specifically r^2 is and increasing function of r and $\rho(r)$ is decreasing function of r [12]. One important advantage of using this geometrical nature is that an upper bound can be easily found.
In implementing a bisection algorithm to approximate the value of r such that (C3) holds, we note that $\rho(r) > r^2$ for all $r < r^*$ and $\rho(r) < r^2$ for all $r > r^*$. Thus if $\rho(r_m) \leq r_m^2$, then $r^* = r_m$ and we stop. If however $\rho(r_m) > r_m^2$, by considering the nature of the two curves we have $r^* > r_m$ and further $\sqrt{\rho(r_m)}$ must be an upper bound for r^* thus we set $r_+ = r_m$.

To numerically implement this, we use the standard bisection or secant methods on the function

$$f(r) = \rho(r) - r^2.$$

We stop the iterations when $-f(r) < tol$, that is when $r^2 - \rho(r)$ is positive and sufficiently small. The following algorithm can be used to compute r^*, the smallest value of r such that conditions (C1)—(C3) hold simultaneously.

Algorithm 4.1.2.

Step 1: *Find r_x and r_y using Algorithm 4.1.1.*

Step 2: *Set $\tilde{r} = r_m$.*

Step 3: *Compute $\rho(\tilde{r})$.*
    If *$\rho(\tilde{r}) \leq r^2$, set $r^* = \tilde{r}$ and stop.*
    If *$\rho(\tilde{r}) > r^2$, then set $r_- = \tilde{r}$ and $r_+ = \sqrt{\rho(\tilde{r})}$.*

Step 4: *Update $\tilde{r}$.*
    If *$r_+ - r_- \geq tol$, use the bisection method.*
    If *$r_+ - r_- < tol$ use the secant method applied to the function $\rho(r) - r^2$.*

Step 5: *Stop Criteria*

    If *$-f(\tilde{r}) < tol$ holds then set $r^* = \tilde{r}$ and stop*
    Else *return to Step 4.*

4.2 Numerical Examples

The following numerical examples are taken directly from [16]. It should be noted the first two were originally from [19]. All times were computed using MATLAB's 'tic toc' command.

We note that the number of iterations for the examples is much higher than those found in [16]. This is due to their use much more efficient and accurate numerical solvers, in comparison to the use of basic MATLAB commands here. It is noted however that although the tools used here could be considered inferior, the algorithm still converged to the correct values.

Example 1
Consider the system,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t),\\ z(t) &= C_1 x(t) + D_{12} u(t),\\ y(t) &= C_2 x(t) + D_{21} w(t),\end{aligned}$$

Table 4.1: Numerical example 1

	Time	Iterations	Computed value
r_{lb_x}			0.707106
r_x	.029 sec	39	0.707106
r_{lb_y}			0.707106
r_y	.314 sec	39	0.707106
r^*	.015 sec	40	0.73205
	0.358 sec		

This shows the importance of using accurate solvers quite remarkedly. Even though r_{lb} is approximately equal to r_x and r_y, in both cases the algorithm required a significant number of iterations to converge.

Example 2
Consider system 4.1.3 with the following coefficient matrices,

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & I \end{bmatrix}, B_2 = I,$$

$$C_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, C_2 = I, D_{12} = \begin{bmatrix} I \\ 0 \end{bmatrix}, D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}.$$

Here we see the value of r_{lb_x} need not equal r_x. This also shows that the r value with satisfies (C1) and (C2) is a lower bound for r^*, but need not be close to r^*.

Table 4.2: Numerical example 2

	Time	Iterations	Computed value
r_{lb_x}			0.71837
r_x	.0297 sec	39	$.9\bar{9}$
r_{lb_x}			0.71837
r_y	.2686 sec	32	$.9\bar{9}$
r^*	.049 sec	88	8.14
	.298 sec		

The following example was presented as a counter example to the arguement in [7] that (C3) would fail before (C1) or (C2).

Example 3

Now consider the system with the following coefficient matrices,

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{10} \end{bmatrix}, B_2 = \begin{bmatrix} \frac{1}{5} & \frac{3}{2} \\ -\frac{3}{2} & 0 \end{bmatrix}, C_2 = \begin{matrix} -\frac{7}{5} & -\frac{7}{10} \\ -\frac{1}{2} & -\frac{7}{10} \end{matrix},$$

$$B_1 = C_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D_{12} = D_{21}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Table 4.3: Numerical example 3

	Time	Iterations	Computed value
r_{lb_x}			0.69641
r_x	.277 sec	12	0.6964
r_{lb_y}			2.1804
r_y	.432 sec	16	2.767
r^*	.022 sec	29	3.413
	.731 sec		

Note that when $r = 2.25$, we have

$$\rho(2.25) \approx 4.7916 < 5.0625 = r^2.$$

However the eigenvalues of $Y(r)$ are .44486 and -6.11457. Thus (C2) fails before (C3). This is important to note because some algorithms such as that presented by [19] only checks condition (C3).

Chapter 5

Conclusions

5.1 Conclusion on State Space Formulation

The most intriguing aspect of the paper by Lin, Wang and Xu is their use of frequency domain tools in the state space formulation. The state space formulation was originally developed in an effort to move away from the frequency domain approach to H_∞-control. Yet, Lin Wang and Xu reformulate some of the state space necessary and sufficient conditions into a statement about singular values.

A major issue which arrises from this reformulation is the accurate calculation of eigenvalues. It should be noted that for the purposes of this study, MatLab functions were used to compute all eigenvalues and singular values. This however was not the case in [16] where accurate eigen-solvers were used.

Accurate computations of eigenvalues is paramount here. The reason for this is the decision making process inherent in the algorithm presented in Chapter 4. The symmetric properties of Hamiltonian matrices is the key factor in how the state space formulation operates. Many eigen-solvers are not capable of preserving this property and as a result loose accuracy in their computations. In turn because the algorithm here relies on answering yes or no type questions and removing entire intervals from a search, false negatives or positives could have devastating results. For instance suppose the algorithm converges to a value such that it indicates a controller exists for a larger error disturbance than it can actually accommodate. One may waste valuable resources implementing a control only to find it doesn't work when applied.

Future research plans include implementing more accurate eigen-solvers and applying this algorithm to a large scale model from application type setting. It would be beneficial to see if this algorithm could be made competitive when compared to other methods currently in use for estimating the H_∞ norm of a system.

5.2 Extending to Mathematics Education

Since I will be pursuing my Doctorate degree in Mathematics Education, it is pertinent to view this project as a learning experience not only pertaining to the content of this project but also how the learning itself occurred. Reflecting back on the process of performing research on more or less my own, there are a few interesting connections noted with various learning theories. Piaget believed that learning occurred through the complimentary processes, assimilation and accommodation. While studying the various topics which encompass H_∞ control, there were numerous occasions which I felt I understood a topic only to find, upon explaining it to a more knowledgeable other, that the way I assimilated the topic was not consistent with H_∞ control theory. In this instance I was presented with a state of dis-equilibrium which required accommodation, that is modification to existing schemes.

Often times however it was noted that certain concepts would repeatedly be sources of confusion, some continuing until late in the research process. The confusion persisted, even through repeated efforts to clarify the misunderstandings. It was evident during the multiple discussions about these topics, that the way in which I understood them was not correct, yet I (perhaps subconsciously) did not accommodate for this new information. There is a plethora of possibilities which could explain why this did not happen, ranging from the inability of the more knowledgable sources to relate to my previously established knowledge, to me not paying close enough attention. However, it might be interesting to apply Tall and Vinners idea of Concept Image and Concept Definition [22] to this phenomenon. That is, even though these topics appeared to consistently be sources of confusion, it might be possible that only the under developed pieces of the overall concept image were being evoked at any given time.

While it is impossible for a person's entire concept image to be evoked at any given time, it has been interesting looking back and reflecting on how I once thought of certain topics presented in this thesis and comparing this to how I now think about them. For instance when I first began research on H_∞ control, when I read the words "transfer function" I would simply relate it to a matrix. Now the image is much more developed consisting of concepts such as linear mappings.

Building on Sfard's work concerning processes which work on previously established objects, Zandieh defined *process-object pairs* [24] in which a series develops. That is roughly, a process is established as an object which can then be acted on by other processes. While this was initially developed around topics covered in calculus, it could certainly be extended to what occurred throughout this research project. Take for instance the transfer function, it could simply be seen as an object, that is a matrix developed from the coefficients of our system matrices. It can also be viewed as a ratio, specifically the ratio of the output to the input. At arguably its deepest level, it could also be seen as a function mapping the inputs to the outputs. It would be interesting to see how these layers developed during specific processes conducted in working on this thesis.

Upon reflecting on the research experience there are numerous benefits to completing a Masters thesis which can be applied to the field Mathematics Education. All too often there is an impression that researching areas in applied mathematics isn't beneficial for one who will ultimately study mathematics education. After this experience, I realize this couldn't be further from the truth. Not only does completing a Masters Thesis play a key role in preparing a student for independent research, but further it instills a true appreciation for what it means to learn and understand a concept. Further, I have gained an appreciation for my own learning process which is certainly beneficial to understanding how others learn.

Constructivism in Research
Research at the Masters level is a far more independent venture than one typically gets to experience as an undergraduate. The advisor's role is to ensure the student stays on track and their work is appropriate for the task presented. The advisor does not explain or lecture the student topics concerning the material past the point of redirection or filling gaps in the student's understanding. Hopefully, input from the student is met with interest, appreciation and consideration. That is, the student's input is valued. The use of more guidance and less lecture results in the student forming much more personal knowledge concerning the subject, compared to an assimilation of the lecturer's knowledge.

Knowing Verus Understanding
One can know that a certain continent is on the other side of the earth, but without experiencing the trip, one don't have first hand understanding of what this means. Research provides first hand knowledge of a specific concept, while capturing ones motivation and building on previous experiences. The learner is actively engaged in constructing their own knowledge, while simultaneously building confidence in their ability to learn. This increase in confidence in turn leads to a more actively engaged learner. The important experience from research is not reading about or getting to know the concepts being researched. The value of research comes from being actively immersed in the topic of interest. The knowledge built from participating in this study is no different. The largest amount of knowledge did not come from reading relevant topics, it came from discussions with those involved in it, actively implementing algorithms, troubleshooting those algorithms and motivation to seek connections.

Deeper Understanding
Often times courses are structured in a linear fashion. That is, the course has been designed to present topics in a predetermined order. This order may or may not be consistent with the order in which topics would naturally fall and as a consequence may lack motivation. For illustration purposes only, it is customary to teach integration before infinite series in many university calculus sequences. Though it may be more natural to learn sequence and series before integration. While actively participating in research one must learn topics as they arise, not as they are presented. This provides a significant increase in motivation to learn these topics

Overall completing this Masters Thesis had a large impact on not only my current understanding of control theory, but also on my future work as a mathematics education researcher.

Bibliography

[1] T. Başar and P. Bernhard. *H-infinity optimal control and related minimax design problems: a dynamic game approach.* Birkhäuser Boston, 2008.

[2] P. Benner, V. Mehrmann, and H. Xu. A numerically stable, structure preserving method for computing the eigenvalues of real hamiltonian or symplectic pencils. *Numerische Mathematik*, 78(3):329–358, 1998.

[3] S. Boyd, V. Balakrishnan, and P. Kabamba. A bisection method for computing the H_∞ norm of a transfer matrix and related problems. *Mathematics of Control, Signals, and Systems (MCSS)*, 2(3):207–219, 1989.

[4] NA Bruinsma and M. Steinbuch. A fast algorithm to compute the H_∞ norm of a transfer function matrix. *Systems & Control Letters*, 14(4):287–293, 1990.

[5] B. Datta. *Numerical methods for linear control systems.* Academic Press, 2003.

[6] J.C. Doyle et al. Advances in multivariable control. In *Lecture Notes at ONR/Honeywell Workshop. Minneapolis, MN*, 1984.

[7] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State-space solutions to standard H_2 and H_∞ control problems. *Automatic Control, IEEE Transactions on*, 34(8):831–847, 1989.

[8] B.A. Francis and J.C. Doyle. Linear control theory with an H_∞ optimality criterion. *SIAM Journal on Control and Optimization*, 25(4):815–844, 1987.

[9] P. Gahinet. On the game riccati equations arising in H_∞ control problem. *SIAM Journal on Control and Optimization*, 32(3):635–647, 1994.

[10] P.M. Gahinet and P. Pandey. Fast and numerically robust algorithm for computing the H_∞ optimum. In *Decision and Control, 1991., Proceedings of the 30th IEEE Conference on*, pages 200–205. IEEE, 1991.

[11] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L_∞ error bounds. *International Journal of Control*, 39(6):1115–1193, 1984.

[12] G. Hewer. Existence theorems for positive semidefinite and sign indefinite stabilizing solutions of H_∞ riccati equations. *SIAM journal on control and optimization*, 31(1):16–29, 1993.

[13] P.P. Khargonekar, I.R. Petersen, and M.A. Rotea. H_∞-optimal control with state-feedback. *Automatic Control, IEEE Transactions on*, 33(8):786–788, 1988.

[14] P.P. Khargonekar, I.R. Petersen, and K. Zhou. Robust stabilization of uncertain linear systems: quadratic stabilizability and H_∞ control theory. *Automatic Control, IEEE Transactions on*, 35(3):356–361, 1990.

[15] D.J.N. Limebeer and M. Green. *Linear robust control.* Prentice-Hall, Englewood Clis, NJ, 1995.

[16] W.W. Lin, C.S. Wang, and Q.F. Xu. On the computation of the optimal H_∞ norms for two feedback control problems. *Linear algebra and its applications*, 287(1):223–255, 1999.

[17] K.A. Morris. *Introduction to feedback control.* Academic Press, Inc., 2000.

[18] A. Packard, M.K.H. Fan, and J. Doyle. A power method for the structured singular value. In *Decision and Control, 1988., Proceedings of the 27th IEEE Conference on*, pages 2132–2137. IEEE, 1988.

[19] P. Pandey, C. Kenney, A. Packard, and AJ Laub. A gradient method for computing the optimal H_∞-norm. *Automatic Control, IEEE Transactions on*, 36(7):887–890, 1991.

[20] I. Petersen. Disturbance attenuation and H_∞ optimization: a design method based on the algebraic riccati equation. *Automatic Control, IEEE Transactions on*, 32(5):427–429, 1987.

[21] C. Scherer. H_∞-control by state-feedback and fast algorithms for the computation of optimalH_∞-norms. *Automatic Control, IEEE Transactions on*, 35(10):1090–1099, 1990.

[22] D. Tall and S. Vinner. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, 12(2):151–169, 1981.

[23] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *Automatic Control, IEEE Transactions on*, 26(2):301–320, 1981.

[24] M. Zandieh. A theoretical framework for analyzing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education. IV. CBMS Issues in Mathematics Education*, pages 103–127, 2000.

[25] K. Zhou, J.C. Doyle, K. Glover, et al. *Robust and optimal control*, volume 40. Prentice Hall Upper Saddle River, NJ, 1996.