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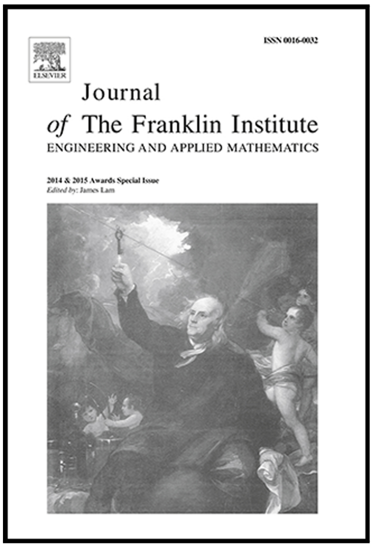

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Exponential stability analysis for singular switched positive systems under dwell-time constraints

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Abstract

The article addresses the exponential stability analysis for singular switched positive systems(SSPSs) under dwell-time constraints which include mode-dependent minimum dwell time(MDMDT), mode-dependent constant dwell time(MDCDT) and mode-dependent ranged dwell time(MDRDT) constraints. For SSPSs in both delay-free case and time-varying delay case, a sufficient exponential stability condition is proposed with MDMDT, MDCDT and MDRDT constraints, respectively, and the exponential decay rate can be set as a free parameter based on diverse circumstances. To analyze the dwell-time stability, a novel discretized linear copositive Lyapunov function(DLCLF) approach is introduced in the article, and compared with the general copositive and homogeneous Lyapunov function approach, the main advantage of the DLCLF technique is itself affine dependence on conditions of system matrix, which will be extended and utilized to systems with uncertainties or/and time-varying parameters quite easily. Meanwhile, the proposed condition of exponential stability under MDMDT(or MDCDT or MDRDT) constraint will be degenerated into the one under minimum dwell time(or constant dwell time or ranged dwell time) constraint for some specific situations, which implies that the considered MDMDT(or MDCDT or MDRDT) case is more general and practical. Finally, the validity and significance of the results are illustrated by seven examples.

Keywords:
Singular switched positive systems, Exponential stability, Dwell-time constraints, Time-varying delays, Discretized linear copositive Lyapunov function

1. Introduction

In nature, there are some essential nonnegative variables like the number of ecological population, the concentration of chemical reactants, the transmission rate of communication

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signals, etc. The dynamic system represented by such nonnegative variables can usually be described by a positive system [1]. The practical applications of positive systems involve many fields such as economics, sociology and ecology [2–5], and its definition was first published in "Introduction to dynamics systems" by Luenberger in 1979. Positive system has such a unique property: the state trajectory of the system starting from any nonnegative state is always nonnegative, which is closely related to the theory of nonnegative matrix in mathematics. The copositive Lyapunov function(CLF), compared with the quadratic Lyapunov function, becomes a basic tool for analyzing the stability of positive systems, and the fundamental reason lies in its simple structure, which can fully grasp the nonnegative characteristics of the state, and can further solve the related problems more effectively. Positive system has been greatly favored by scholars [6–9].

With the development of positive systems, more and more attention has been paid to switched positive systems(SPSs), since many positive systems are equipped with switching characteristics to a certain extent in practice. SPS refers to a class of switched systems whose state trajectories starting from any nonnegative initial states can remain positive under any switching signal, and the switching rule determines the operation law of the subsystems [10–13]. In recent years, more and more scholars have devoted themselves to the study of SPSs, such as network communication systems [14], air traffic systems [15], medical systems [16] and other fields [17–20]. Due to the physical constraints, the system cannot switch arbitrarily fast and has to stay in each subsystem for at least a period of time, so the switching signal limited by dwell-time bounds is preferable. Dwell-time constraints including mode-dependent minimum dwell time(MDMDT), mode-dependent constant dwell time(MDCDT) and mode-dependent ranged dwell time(MDRDT) constraints, often appear in traffic systems, tele-operation robot systems and other practical systems [21]. With dwell-time constraints, some results on switched systems and SPSs have been reported [22–25].

The preceding productions are about standard SPSs, i.e., every switched subsystem is positive and standard, while there are few studies on singular SPSs, i.e., every switched subsystem is positive and singular, and singular SPSs include standard SPSs as special cases. In fact, with the need of large-scale engineering technology, many practical systems such as restricted robot system, nuclear reactor system and non-causal system can only be described by singular systems, but not by standard systems [26, 27]. Noted that singular system, compared with standard system, can offer many advantages like describing a wider scope of systems more naturally, representing the relation within internal variables more accurately, and dealing with many large-scaled complex systems more efficiently [28]. Recently, singular switched positive systems(SSPSs) are found significant applications in many fields, such as electrical engineering, computer engineering, chemical engineering and so on. Due to the combination of singularity, i.e., the complex structural characteristics of singular systems, positivity and switching compatibility, the study of SSPSs becomes more difficult and challenging than that of standard ones. This kind of system has been widely concerned by scholars, but there exist only a few results [29–32] which are mainly about the issues of stability, finite-time stability and stochastic stability.

As we can notice, above research results on SSPSs [29–32] only focus on asymptotical stability issues. In practical applications, it is not enough to only require the asymptotical stability of the system, more importantly, people pay more attentions to study whether the system can converge at a faster rate like the one in exponential form. In order to discuss how stability changes especially along with the exponential decay rate(DR), it becomes significant to study the exponential stability for SSPSs. In general, the convergence rate of asymptotical stability may be very slow and has a certain conservatism, while the convergence rate of exponential stability can be relatively faster and the exponential DR can be described. In the analysis of exponential stability, it is a major difficulty to determine the DR for systems with singularity constraints. Hence, the first goal is to analyze the exponential stability of SSPSs by exploring some effective methods of determining the DR.

In addition, the research on SSPSs in the above literatures [29–32] is only concerned with average dwell time(ADT) constraint or arbitrary switching law, while rarely focused on dwell-time constraints including MDMDT, MDCDT and MDRDT constraints. Compared with the previous two, the dwell-time constraints are more consistent with the nature of switched systems where each subsystem has its own dwell-time bounds(including lower or/and upper bounds), which reduces the conservatism of the results. Besides, dwell-time constraints rather than ADT constraint or arbitrary switching law require completely different analyzing techniques and methods, and the corresponding results can't be generalized directly from each other. Then, studying SSPSs with dwell-time constraints becomes greatly significant. With dwell-time constraints, it is a huge challenge to explore an appropriate method of stability analysis and manage some specific mathematical difficulties accordingly, and the existing dwell-time theory has not reached the quantitative level. Hence, the second goal is to investigate the dwell-time stability for SSPSs with MDMDT, MDCDT and MDRDT constraints, respectively.

As mentioned earlier, the results on SSPSs [29–32] simply consider the case without time-varying delays that widely exist in numerous practical systems. In many situations, a small delay may has a large impact on the system. For example, the existence of time-varying delays may lead to the aggravated oscillation of dynamic process of the system, the bifurcation of the system, and even the performance deterioration and the instability of the system. With time-varying delays, it is extremely difficult to construct a suitable Lyapunov functional and analyze the stability for systems with singularity constraints, because such system is coupled with matrix delay differential equation and matrix difference equation, which greatly increases the complexity and challenge of the research. Hence, the third goal is to further study the dwell-time stability for SSPSs in time-varying delay case.

In this paper, for SSPSs without/with time-varying delays, a sufficient exponential stability condition is proposed with constraints: MDMDT, MDCDT and MDRDT, respectively. This is not only practically important, but also theoretically challenging. The main contributions lie in:

1. It is the first time to analyze the exponential stability for SSPSs, and further for SSPSs in time-varying delay case. For the sake of proving the exponential stability, a new method based on Lyapunov functional is utilized here, and by inserting a particular exponential term into constructed functional candidate, the exponential DR can be set as a free parameter according to different situations.

2. Dwell-time constraints as MDMDT, MDCDT and MDRDT constraints are firstly considered for SSPSs in both delay-free case and time-varying delay case. To analyze the dwell-time stability, a discretized linear copositive Lyapunov function(DLCLF) approach is introduced in the article, and compared with the general copositive and homogeneous Lyapunov function approach, the main advantage of the DLCLF technique is its affine dependence on conditions of system matrix, which will be extended to systems with uncertainties or/and time-varying parameters quite easily.
3. For SSPSs in both delay-free case and time-varying delay case, the proposed condition of exponential stability under MDMDT(or MDCDT or MDRDT) constraint will be degenerated into the one under MDT(or CDT or RDT) constraint for some specific situations, that is, the considered MDMDT(or MDCDT or MDRDT) case regarded as the generalization of the MDT(or CDT or RDT) case is more general and practical.

The rest of the paper are: Section 2 gives system descriptions and preliminaries. For Section 3 in delay-free case and Section 4 in time-varying delay case, the corresponding exponential stability issue is analyzed for SSPSs with MDMDT, MDCDT and MDRDT constraints, respectively. In Section 5, seven examples are given to verify the theoretical results. Section 6 concludes the paper.

Notations For a matrix A, $rank\,(A)$ denotes its rank, $det\,(A)$ denotes its determinant, and $deg\,(\cdot)$ denotes its degree of a polynomial. $R(R_+)$ and $(Z)Z_+$ represent the group of all real(positive real) numbers and integers, respectively. $R^n(R^n_+)$ and $R^{m\times n}(R^{m\times n}_+)$ mean the collection of all n-dimensional real(positive real) vectors and $m\times n$-dimensional real(positive real) matrices, respectively. For a vector $\nu \in R^n$, ν_i denotes its ith term, $i \in \underline{n}$, and $\overline{\lambda}(\nu) = \max_{i\in\underline{n}} \nu_i$ and $\underline{\lambda}(\nu) = \min_{i\in\underline{n}} \nu_i$ imply the largest and smallest element of ν, respectively. Given a positive real number $x \in R_+$, $\lceil x \rceil$ denotes the minimum integer which is not less than x. Given a matrix $A \in R^{n\times n}$, $A \succeq 0$ indicates that each element of A is non-negative, and $A \in M$ indicates A to be a Metzler matrix.

2. System descriptions and preliminaries

A switched singular system is stated as:

$$\begin{cases} E\dot{x}(t) &= A_{\sigma(t)}x(t), \\ x(t_0) &= x_0, \end{cases} \tag{1}$$

where $x(t) \in R^n$ denotes the system's state, and x_0 is initial state. $\sigma(t)$ means the system's switching law with its total number of subsystems being N. A_i, $i \in \underline{N}$, is a known matrix, E satisfying $rank\,(E) = r$, $r < n$, is a singular matrix.

Furthermore, the time-varying delay case of system (1) is to be considered,

$$\begin{cases} E\dot{x}(t) &= A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)), \\ x(t_0+\theta) &= \varphi(\theta), \quad \theta \in [-d_u, 0], \end{cases} \tag{2}$$

where the time-varying delay $d(t)$ is differential everywhere and satisfies $0 \le d(t) \le d_u$ and $\dot{d}(t) \le d_d < 1$ for giving d_u and d_d. $\varphi(\theta)$ denotes a given continuous initial function. A_{di}, $i \in \underline{N}$, is a known matrix, and other symbols are defined in system (1).

Assumption 1. 1. *The switching instants of* $\sigma(t)$ *are denoted as* $\{t_0, t_1, t_2, \cdots, t_l, t_{l+1}, \cdots\}$, $l \in Z$*;*

2. *The ith subsystem will be activated in the interval* $[t_l, t_{l+1})$*, i.e.,* $\sigma(t) = i \in \underline{N}$*,* $t \in [t_l, t_{l+1})$*,* $l \in Z$*.*
3. *For the ith subsystem, its corresponding dwell time is the duration within period* $[t_l, t_{l+1})$*,* $l \in Z$*.*

We assume that systems (1)-(2) satisfy Assumption 1.

Definition 1. *[26]*

1. *System (1)(or system (2)) is regular if* $det(sE - A_i) \neq 0$*,* $\forall i \in \underline{N}$*;*
2. *System (1)(or system (2)) is impulse-free if* $deg(det(sE - A_i)) = rank(E)$*,* $\forall i \in \underline{N}$*.*

Definition 2. *[1] System (1)(or system (2)) is positive if the system state* $x(t)$ *starting from any non-negative initial state can always remain non-negative, i.e.,* $x(t) \succeq 0$*,* $\forall t \geq 0$*, under any switching law* $\sigma(t)$*.*

System (1) being regular and impulse-free indicates $PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ for non-singular $P \in R^{n\times n}$ and $Q \in R^{n\times n}$. Simply, take

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix} \tag{3}$$

in system (1), where $\det(A_{i4}) \neq 0$, $A_{i1} \in R^{r\times r}$, $A_{i2} \in R^{r\times(n-r)}$, $A_{i3} \in R^{(n-r)\times r}$, $A_{i4} \in R^{(n-r)\times(n-r)}$, $\forall i \in \underline{N}$. Setting $x^T(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}^T$ with $x_1(t) \in R^r$, system (1) with (3) becomes

$$\begin{cases} \dot{x}_1(t) &= \overline{A}_{i1}x_1(t), \\ x_2(t) &= \overline{A}_{i3}x_1(t), \\ x_1(t_0) &= x_{10}, \\ x_2(t_0) &= x_{20}, \end{cases} \tag{4}$$

where $\overline{A}_{i1} = A_{i1} - A_{i2}A_{i4}^{-1}A_{i3}$, $\overline{A}_{i3} = -A_{i4}^{-1}A_{i3}$, $\forall i \in \underline{N}$.

Similar to the above process, take

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, \quad A_{di} = \begin{bmatrix} A_{di1} & A_{di2} \\ A_{di3} & A_{di4} \end{bmatrix} \tag{5}$$

in system (2), where $A_{di1} \in R^{r\times r}$, $A_{di2} \in R^{r\times(n-r)}$, $A_{di3} \in R^{(n-r)\times r}$, $A_{di4} \in R^{(n-r)\times(n-r)}$, $\forall i \in \underline{N}$, and other symbols are defined in system (3). System (2) with (5) yields, $\forall i \in \underline{N}$,

$$\begin{cases} \dot{x}_1(t) &= \overline{A}_{i1}x_1(t) + \overline{A}_{di1}x_1(t-d(t)) + \overline{A}_{di2}x_2(t-d(t)), \\ x_2(t) &= \overline{A}_{i3}x_1(t) + \overline{A}_{di3}x_1(t-d(t)) + \overline{A}_{di4}x_2(t-d(t)), \\ x_1(t_0+\theta) &= \varphi_1(\theta), \quad \theta = [-d_u, 0], \\ x_2(t_0+\theta) &= \varphi_2(\theta), \quad \theta = [-d_u, 0], \end{cases} \tag{6}$$

where $\overline{A}_{di1} = A_{di1} - A_{i2}A_{i4}^{-1}A_{di3}$, $\overline{A}_{di2} = A_{di2} - A_{i2}A_{i4}^{-1}A_{di4}$, $\overline{A}_{di3} = -A_{i4}^{-1}A_{di3}$, $\overline{A}_{di4} = -A_{i4}^{-1}A_{di4}$, and other symbols are defined in system (4).

Lemma 1. *[33]*

1. *System (1) with (3) is positive iff* $\overline{A}_{i1} \in M$, $\overline{A}_{i3} \succeq 0$, $\forall i \in \underline{N}$;
2. *System (2) with (5) is positive iff* $\overline{A}_{i1} \in M$, $\overline{A}_{i3} \succeq 0$, $\overline{A}_{di1} \succeq 0$, $\overline{A}_{di2} \succeq 0$, $\overline{A}_{di3} \succeq 0$, $\overline{A}_{di4} \succeq 0$, $\forall i \in \underline{N}$.

Definition 3. *[8] If there are constants* $c > 0$ *and* $\alpha > 0$ *so that,*

1. *with any initial condition* $x(t_0) = x_0$, $x(t)$ *satisfies*
$$\|x(t)\|_1 \leq ce^{-\alpha(t-t_0)}\|x_0\|_1, \quad \forall t \geq 0, \tag{7}$$
then system (1) is exponentially stable(ES);
2. *with any initial condition* $x(t_0 + \theta) = \varphi(\theta)$, $\theta \in [-d_u, 0]$, $x(t)$ *satisfies*
$$\|x(t)\|_1 \leq ce^{-\alpha(t-t_0)}\|\varphi\|_{1c} = ce^{-\alpha(t-t_0)} \sup_{-d_u \leq \theta \leq 0} \|\varphi(\theta)\|_1, \quad \forall t \geq 0, \tag{8}$$
then system (2) is ES.

In Definition 3, c denotes the decay coefficient and α denotes the DR.

Definition 4. *For any switching signal* $\sigma(t)$ *satisfying* $\sigma(t) = i \in \underline{N}$, $t \in [t_l, t_{l+1})$, $l \in Z$,

1. *if* $T_{li} \geq T_i^*$, $T_{li} = t_{l+1} - t_l$, $T_i^* \in R_+$, *then it has a MDMDT constraint;*
2. *if* $T_{li} \equiv T_i^*$, $T_{li} = t_{l+1} - t_l$, $T_i^* \in R_+$, *then it has a MDCDT constraint;*
3. *if* $\underline{T}_i^* \leq T_{li} \leq \overline{T}_i^*$, $T_{li} = t_{l+1} - t_l$, $\underline{T}_i^* \in R_+$, $\overline{T}_i^* \in R_+$, *then it has a MDRDT constraint.*

Remark 1. *In Definition 4, the proposed MDMDT(or MDCDT) constraint is degenerated into the mode-independent one, i.e., minimum dwell time(MDT)(or constant dwell time(CDT)) case when* $T_i^* = T_j^*$, *and the proposed MDRDT constraint is degenerated into the mode-independent one, i.e., ranged dwell time(RDT) case when* $\underline{T}_i^* = \underline{T}^*$, $\overline{T}_i^* = \overline{T}^*$. *That is, the considered MDMDT(or MDCDT or MDRDT) constraint regarded as the generalization of MDT(or CDT or RDT) case is more general and practical. This has been discussed particularly in Sections 3-4.*

The goals of this paper are:

1. Considering system (1) in delay-free case, present the exponential stability condition under MDMDT, MDCDT and MDRDT constraints, respectively, and further give the corresponding ones under MDT, CDT and RDT constraints;
2. Considering system (2) in time-varying delay case, propose the delay-dependent exponential stability condition under MDMDT, MDCDT and MDRDT constraints, respectively, and further give the corresponding ones under MDT, CDT and RDT constraints;

3. Exponential stability in delay-free case

Considering system (1) in delay-free case, Section 3 is focused on the exponential stability analysis under MDMDT, MDCDT and MDRDT constraints, respectively.

3.1. MDMDT case

For system (1), on the basis of DLCLF method, Subsection 3.1 presents an exponential stability condition under MDMDT constraint, and further gives a degenerated one under MDT constraint.

Theorem 1. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars* $\alpha > 0$, $S \in Z_+$, $T_i^* \in R_+$, $i \in \underline{N}$, *if there exist vectors* $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, *so that*, $\forall (i,j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,

$$A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \tag{9}$$

$$A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \tag{10}$$

$$A_i^T \nu_{i,S} + \alpha E^T \nu_{i,S} \preceq 0, \tag{11}$$

$$\nu_{i,0} - \nu_{j,S} \preceq 0, \tag{12}$$

then SSPS (1) is ES with MDMDT constraint meeting

$$T_{li} \geq T_i^*, \quad T_{li} = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z. \tag{13}$$

Proof. Construct the DLCLF candidate in the following:

$$V(x(t)) = x^T(t) E^T \nu_{\sigma(t)}(t), \tag{14}$$

where $\nu_{\sigma(t)}(t) \succ 0$ varies over time.

Due to that $\sigma(t)$ meets the MDMDT constraint (13), the interval $[t_l, t_{l+1})$ can be divided into $[t_l, t_l + T_i^*)$ and $[t_l + T_i^*, t_{l+1})$.

Divide the interval $[t_l, t_l + T_i^*)$ into S segments described as $S_{is} = [t_l + \frac{sT_i^*}{S}, t_l + \frac{(s+1)T_i^*}{S})$, $s = \{0, 1, \cdots, S-1\}$, and the size of every segment is $\frac{T_i^*}{S}$. Set $\nu_i(t)$ to be liner during each segment S_{is} and let $\nu_{i,s} = \nu_i(t_l + \frac{sT_i^*}{S})$, then $\nu_i(t)$ is piecewise linear during $[t_l, t_l + T_i^*)$, and $\nu_i(t)$ could be expressed as:

$$\nu_i(t) = \begin{cases} (1 - \rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}, & t \in S_{is}, \\ \nu_{i,S}, & t \in [t_l + T_i^*, t_{l+1}), \end{cases} \tag{15}$$

which leads to

$$\dot{\nu}_i(t) = \begin{cases} \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}), & t \in S_{is}, \\ 0, & t \in [t_l + T_i^*, t_{l+1}), \end{cases} \tag{16}$$

where $\rho(t) = \frac{t - (t_l + \frac{sT_i^*}{S})}{\frac{T_i^*}{S}} = \frac{S(t - t_l - \frac{sT_i^*}{S})}{T_i^*}$.

Firstly, considering the case $t \in [t_l, t_l + T_i^*)$, from (1), (14) and (9)-(10), we get

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &= x^T(t)A_i^T \nu_i(t) + x^T(t)E^T \dot{\nu}_i(t) + \alpha x^T(t)E^T \nu_i(t) \\
&= x^T(t)A_i^T((1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}) + x^T(t)E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) \\
&\quad + \alpha x^T(t)E^T((1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}) \\
&= x^T\{(1-\rho(t))[A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s})] \\
&\quad + \rho(t)[A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s})]\} \\
&\leq 0. \qquad (17)
\end{aligned}$$

Next, the case $t \in [t_l + T_i^*, t_{l+1})$ is considered, one can get from (1), (14) and (11) that

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &= x^T(t)A_i^T \nu_i(t) + x^T(t)E^T \dot{\nu}_i(t) + \alpha x^T(t)E^T \nu_i(t) \\
&= x^T(t)A_i^T \nu_{i,S} + \alpha x^T(t)E^T \nu_{i,S} \\
&= x^T(t)[A_i^T \nu_{i,S} + \alpha E^T \nu_{i,S}] \\
&\leq 0. \qquad (18)
\end{aligned}$$

Combining (17) and (18) results in

$$\dot{V}_i(x(t)) + \alpha V_i(x(t)) \leq 0, \quad t \in [t_l, t_{l+1}), \qquad (19)$$

which yields

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_l)} V_{\sigma(t_l)}(x(t_l)), \quad t \in [t_l, t_{l+1}). \qquad (20)$$

Then, taking the switching instant t_l into account, where $\sigma(t) = j$ for $t \in [t_{l-1}, t_l)$ and $\sigma(t) = i$ for $t \in [t_l, t_{l+1})$, $\forall i, j \in \underline{N}$, (1) and (14) lead to

$$\begin{aligned}
V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) &= x^T(t_l^+)E^T \nu_{\sigma(t_l^+)}(t_l^+) - x^T(t_l^-)E^T \nu_{\sigma(t_l^-)}(t_l^-) \\
&= x^T(t_l)E^T(\nu_{i,0} - \nu_{j,S}), \qquad (21)
\end{aligned}$$

and from (12), we have

$$V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) = V_{\sigma(t_l)}(x(t_l^+)) - V_{\sigma(t_{l-1})}(x(t_l^-)) \leq 0, \quad \forall l \in Z_+. \qquad (22)$$

Then (20) and (22) yields

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_0)} V_{\sigma(t_0)}(x(t_0)). \qquad (23)$$

From the other side, set $\nu_i(t) = \begin{bmatrix} \nu_i^{(1)}(t) \\ \nu_i^{(2)}(t) \end{bmatrix}$ with $\nu_i^{(1)}(t) \in R^r$ and $\nu_i^{(2)}(t) \in R^{n-r}$, then DLCLF (14) leads to

$$\begin{aligned} V_{\sigma(t)}(t) &= x^T(t)E^T\nu_i(t) \\ &= \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \nu_i^{(1)}(t) \\ \nu_i^{(2)}(t) \end{bmatrix} \\ &= x_1^T(t)\nu_i^{(1)}(t). \end{aligned} \tag{24}$$

Based on (14) and (24), it is obvious that

$$V_{\sigma(t)}(t) \geq \min_{\forall i \in \underline{N}, s \in \underline{S}} \underline{\lambda}(\nu_{i,s}^{(1)})\|x_1(t)\|_1 = \beta_1\|x_1(t)\|_1, \tag{25}$$

$$V_{\sigma(t_0)}(t_0) = x_1^T(t_0)\nu_i^{(1)}(t_0) \leq \max_{\forall i \in \underline{N}} \overline{\lambda}(\nu_{i,0}^{(1)})\|x_1(t_0)\|_1 = \beta_2\|x_1(t_0)\|_1, \tag{26}$$

where $\beta_1 = \min_{\forall i \in \underline{N}, s \in \underline{S}} \underline{\lambda}(\nu_{i,s}^{(1)})$, $\beta_2 = \max_{\forall i \in \underline{N}} \overline{\lambda}(\nu_{i,0}^{(1)})$. Then, combining (23) and (25)-(26) leads to

$$\|x_1(t)\|_1 \leq \beta e^{-\alpha(t-t_0)}\|x_1(t_0)\|_1 \leq \beta e^{-\alpha(t-t_0)}\|x(t_0)\|_1, \tag{27}$$

where $\beta = \beta_2/\beta_1$, and obviously, $x_1(t)$ is ES with DR α.

Then, $x_2(t)$ being ES with DR α is to be proved. One has from the second equation of system (4) that

$$\|x_2(t)\|_1 \leq \|\overline{A}_{i3}\|_1\|x_1(t)\|_1 \leq \|\overline{A}_{i3}\|_1\beta e^{-\alpha(t-t_0)}\|x(t_0)\|_1. \tag{28}$$

Finally, equations (27)-(28) yield $\|x(t)\|_1 \leq \varpi e^{-\alpha(t-t_0)}\|x(t_0)\|_1$, $\varpi = \max\{\beta, \|\overline{A}_{i3}\|_1\beta\}$, $\forall t \geq 0$, that is, in light of Definition 3, SSPS (1) is ES with DR α.

The proof is completed. □

Remark 2. *In view of Definition 3 on exponential stability, the larger the value of α is, the more quickly the system can converge. However, from the conditions in Theorem 1, a smaller α is obviously favorable to the feasibility of (9)-(12). Given a positive scalar α, if no feasible solution can be found for (9)-(12), one can regulate the scalar α to be a smaller one such that feasible solution may be found for (9)-(12). Following the principle, a feasible solution with respect to the conditions in Theorem 1 can be found. This view also applies to the remaining results in Theorems 2-6, and will be illustrated in detail in Example 6.*

Remark 3. *For Theorem 1, as noted in [22], the scalar S is specified in advance, and various S can result in different stability results. In general, the larger the scalar S is, the finer the division for interval $[t_l, t_l + T_i^*)$ will be, and consequently, the wider the range of corresponding feasible results will be. This view also applies to the remaining conditions in Theorems 2-6, and will be illustrated in detail in Example 6.*

Remark 4. *When $E = I$, where I is an appropriate identity matrix, SSPS (1) will become a standard SPS, and correspondingly, the stability criteria of SSPS (1) will be degenerated into the one of standard SPS. That is, the stability conditions for standard SPS can be easily obtained if the derivative matrix E is replaced by the identity matrix I in conditions (9)-(12) of Theorem 1. Thus, the singular system includes the standard system as a specific case.*

When $T_i^* = T_j^*$, $\forall i, j \in \underline{N}$, $i \neq j$, the MDMDT constraint is degenerated into MDT constraint, and exponential stability issue with MDT constraint is presented accordingly.

Corollary 1. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T^* \in R_+$, if there exist vectors $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,*

$$\begin{aligned} A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \\ A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \\ A_i^T \nu_{i,S} + \alpha E^T \nu_{i,S} \preceq 0, \\ \nu_{i,0} - \nu_{j,S} \preceq 0, \end{aligned}$$

then SSPS (1) is ES with MDT constraint meeting

$$T_l \geq T^*, \quad T_l = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z.$$

In addition, when T^* approaches 0 in Corollary 1, the MDT constraint is degenerated into arbitrary switching case. In light of Corollary 1, choose $S = 0$ so as to make the term $\frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s})$ independent of T^*, and in this case, it is interesting to see that $\nu_{i,0} = \nu_{j,0} = \nu$. Then, the existing theory based on common CLF can be recovered.

Corollary 2. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given a scalar $\alpha > 0$, if there exists a vector $\nu \in R_+^n$, so that*

$$A_i^T \nu + \alpha E^T \nu \preceq 0,$$

then SSPS (1) is ES with arbitrary switching law.

3.2. MDCDT case

For system (1), on the basis of DLCLF method, Subsection 3.2 presents an exponential stability condition under MDCDT constraint, and further gives a degenerated one under CDT constraint.

Theorem 2. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T_i^* \in R_+$, $i \in \underline{N}$, if there exist vectors $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in$*

$\{0,1,\cdots,S\}$, *so that,* $\forall(i,j)\in\underline{N}\times\underline{N}$, $i\neq j$, $s\in\{0,1,\cdots,S\}$,

$$A_i^T\nu_{i,s}+\alpha E^T\nu_{i,s}+E^T\frac{S}{T_i^*}(\nu_{i,s+1}-\nu_{i,s})\preceq 0,\quad s\neq S, \tag{29}$$

$$A_i^T\nu_{i,s+1}+\alpha E^T\nu_{i,s+1}+E^T\frac{S}{T_i^*}(\nu_{i,s+1}-\nu_{i,s})\preceq 0,\quad s\neq S, \tag{30}$$

$$\nu_{i,0}-\nu_{j,S}\preceq 0, \tag{31}$$

then SSPS (1) is ES with MDCDT constraint meeting

$$T_{li}\equiv T_i^*,\quad T_{li}=t_{l+1}-t_l,\quad \sigma(t)=i,\quad t\in[t_l,t_{l+1}),\quad l\in Z. \tag{32}$$

Proof. Select the DLCLF candidate (14) where $\nu_{\sigma(t)}(t)\succ 0$ varies over time.

Due to that $\sigma(t)$ meets the MDCDT constraint (32), the interval $[t_l,t_{l+1})$ can be divided into S segments described as $S_{is}=[t_l+\frac{sT_i^*}{S},t_l+\frac{(s+1)T_i^*}{S})$, $s=\{0,1,\cdots,S-1\}$, and the size of every segment is $\frac{T_i^*}{S}$. Set $\nu_i(t)$ to be liner during each segment S_{is} and let $\nu_{i,s}=\nu_i(t_l+\frac{sT_i^*}{S})$, then $\nu_i(t)$ is piecewise linear during $[t_l,t_{l+1})$, and $\nu_i(t)$ could be expressed as:

$$\nu_i(t)=(1-\rho(t))\nu_{i,s}+\rho(t)\nu_{i,s+1},\quad t\in S_{is}, \tag{33}$$

which leads to

$$\dot{\nu}_i(t)=\frac{S}{T_i^*}(\nu_{i,s+1}-\nu_{i,s}),\quad t\in S_{is}, \tag{34}$$

where $\rho(t)=\frac{t-(t_l+\frac{sT_i^*}{S})}{\frac{T_i^*}{S}}=\frac{S(t-t_l-\frac{sT_i^*}{S})}{T_i^*}$.

Consider $t\in[t_l,t_{l+1})$, and from (1), (14) and (29)-(30), we have

$$\begin{aligned}\dot{V}_i(x(t))+\alpha V_i(x(t))&=x^T(t)A_i^T\nu_i(t)+x^T(t)E^T\dot{\nu}_i(t)+\alpha x^T(t)E^T\nu_i(t)\\&=x^T(t)A_i^T((1-\rho(t))\nu_{i,s}+\rho(t)\nu_{i,s+1})+x^T(t)E^T\frac{S}{T_i^*}(\nu_{i,s+1}-\nu_{i,s})\\&\quad+\alpha x^T(t)E^T((1-\rho(t))\nu_{i,s}+\rho(t)\nu_{i,s+1})\\&=x^T(t)\{(1-\rho(t))[A_i^T\nu_{i,s}+\alpha E^T\nu_{i,s}+E^T\frac{S}{T_i^*}(\nu_{i,s+1}-\nu_{i,s})]\\&\quad+\rho(t)[A_i^T\nu_{i,s+1}+\alpha E^T\nu_{i,s+1}+E^T\frac{S}{T_i^*}(\nu_{i,s+1}-\nu_{i,s})]\}\\&\leq 0,\end{aligned} \tag{35}$$

which yields

$$V_{\sigma(t)}(x(t))\leq e^{-\alpha(t-t_l)}V_{\sigma(t_l)}(x(t_l)),\quad t\in[t_l,t_{l+1}). \tag{36}$$

Then, taking the switching instant t_l into account, where $\sigma(t)=j$ for $t\in[t_{l-1},t_l)$ and $\sigma(t)=i$ for $t\in[t_l,t_{l+1})$, $\forall i,j\in\underline{N}$, (1) and (14) lead to

$$\begin{aligned}V_{\sigma(t_l^+)}(x(t_l^+))-V_{\sigma(t_l^-)}(x(t_l^-))&=x^T(t_l^+)E^T\nu_{\sigma(t_l^+)}(t_l^+)-x^T(t_l^-)E^T\nu_{\sigma(t_l^-)}(t_l^-)\\&=x^T(t_l)E^T(\nu_{i,0}-\nu_{j,S}),\end{aligned} \tag{37}$$

and from (31), we have

$$V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) = V_{\sigma(t_l)}(x(t_l^+)) - V_{\sigma(t_{l-1})}(x(t_l^-)) \leq 0, \quad \forall l \in Z_+. \tag{38}$$

Then it follows from (36) and (38) that

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_0)} V_{\sigma(t_0)}(x(t_0)). \tag{39}$$

The remaining proof is similar to that in Theorem 1.

The proof is completed. □

When $T_i^* = T_j^*$, $\forall i, j \in \underline{N}$, $i \neq j$, the MDCDT constraint is equal to CDT constraint, and exponential stability issue under CDT case is presented accordingly.

Corollary 3. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T^* \in R_+$, if there exist vectors $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i,j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,*

$$\begin{aligned} A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \\ A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \\ \nu_{i,0} - \nu_{j,S} \preceq 0, \end{aligned}$$

then SSPS (1) is ES with CDT constraint meeting

$$T_l \equiv T^*, \quad T_l = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z.$$

3.3. MDRDT case

For system (1), on the basis of DLCLF method, Subsection 3.3 presents an exponential stability result under MDRDT constraint, and further gives a degenerated one under RDT constraint.

Theorem 3. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $\overline{T}_i^* \in R_+$, $\underline{T}_i^* \in R_+$, $i \in \underline{N}$, if there exist vectors $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i,j) \in \underline{N} \times \underline{N}$, $i \neq j$, $m \in \{\lceil \frac{S\underline{T}_i^*}{\overline{T}_i^*} \rceil, \lceil \frac{S\underline{T}_i^*}{\overline{T}_i^*} \rceil + 1, \lceil \frac{S\underline{T}_i^*}{\overline{T}_i^*} \rceil + 2, \cdots, S\}$, $s \in \{0, 1, \cdots, S\}$,*

$$A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \tag{40}$$

$$A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \tag{41}$$

$$\nu_{i,0} - \nu_{j,m} \preceq 0, \tag{42}$$

then SSPS (1) is ES with MDRDT constraint meeting

$$\underline{T}_i^* \leq T_{li} \leq \overline{T}_i^*, \quad T_{li} = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z. \tag{43}$$

Proof. Select the DLCLF candidate (14) where $\nu_{\sigma(t)}(t) \succ 0$ varies over time.

Due to that $\sigma(t)$ meets the MDRDT constraint (43), the interval $[t_l, t_l + \overline{T}_i^*)$ can be divided into S segments described as $S_{is} = [t_l + \frac{s\overline{T}_i^*}{S}, t_l + \frac{(s+1)\overline{T}_i^*}{S})$, $s = \{0, 1, \cdots, S-1\}$, and the size of every segment is $\frac{\overline{T}_i^*}{S}$. Set $\nu_i(t)$ to be liner during each segment S_{is} and let $\nu_{i,s} = \nu_i(t_l + \frac{s\overline{T}_i^*}{S})$, then $\nu_i(t)$ is piecewise linear during $[t_l, t_l + \overline{T}_i^*)$, and $\nu_i(t)$ could be expressed as:

$$\nu_i(t) = (1 - \rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}, \quad t \in S_{is}, \tag{44}$$

which leads to

$$\dot{\nu}_i(t) = \frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s}), \quad t \in S_{is}, \tag{45}$$

where $\rho(t) = \frac{t-(t_l+\frac{s\overline{T}_i^*}{S})}{\frac{\overline{T}_i^*}{S}} = \frac{S(t-t_l-\frac{s\overline{T}_i^*}{S})}{\overline{T}_i^*}$.

Considering $t \in [t_l, t_{l+1})$, from (1), (14) and (40)-(41), we have

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &= x^T(t)A_i^T\nu_i(t) + x^T(t)E^T\dot{\nu}_i(t) + \alpha x^T(t)E^T\nu_i(t) \\
&= x^T(t)A_i^T((1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}) + x^T(t)E^T\frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s}) \\
&\quad + \alpha x^T(t)E^T((1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}) \\
&= x^T(t)\{(1-\rho(t))[A_i^T\nu_{i,s} + \alpha E^T\nu_{i,s} + E^T\frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s})] \\
&\quad + \rho(t)[A_i^T\nu_{i,s+1} + \alpha E^T\nu_{i,s+1} + E^T\frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s})]\} \\
&\leq 0,
\end{aligned} \tag{46}$$

which yields

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_l)}V_{\sigma(t_l)}(x(t_l)), \quad t \in [t_l, t_{l+1}). \tag{47}$$

Then, taking the switching instant t_l into account, where $\sigma(t) = j$ for $t \in [t_{l-1}, t_l)$ and $\sigma(t) = i$ for $t \in [t_l, t_{l+1})$, $\forall i, j \in \underline{N}$, (1) and (14) lead to

$$\begin{aligned}
V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) &= x^T(t_l^+)E^T\nu_{\sigma(t_l^+)}(t_l^+) - x^T(t_l^-)E^T\nu_{\sigma(t_l^-)}(t_l^-) \\
&= x^T(t_l)E^T(\nu_{i,0} - \nu_{j,m}),
\end{aligned} \tag{48}$$

and from (42), we have

$$V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) = V_{\sigma(t_l)}(x(t_l^+)) - V_{\sigma(t_{l-1})}(x(t_l^-)) \leq 0, \quad \forall l \in Z_+. \tag{49}$$

Then it follows from (47) and (49) that

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_0)}V_{\sigma(t_0)}(x(t_0)). \tag{50}$$

The remaining proof is similar to that in Theorem 1.

The proof is completed. □

When $\overline{T}_i^* = \overline{T}_j^*$, $\underline{T}_i^* = \underline{T}_j^*$, $i, j \in \underline{N}$, $i \neq j$, the MDRDT constraint is turned into RDT constraint, and exponential stability issue with RDT constraint is presented accordingly.

Corollary 4. *For system (1), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $\overline{T}^* \in R_+$, $\underline{T}^* \in R_+$, if there exist vectors $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $m \in \{\lceil \frac{S\underline{T}^*}{\overline{T}^*} \rceil, \lceil \frac{S\underline{T}^*}{\overline{T}^*} \rceil + 1, \lceil \frac{S\underline{T}^*}{\overline{T}^*} \rceil + 2, \cdots, S\}$, $s \in \{0, 1, \cdots, S\}$,*

$$\begin{aligned}
A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{\overline{T}^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \\
A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{\overline{T}^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \\
\nu_{i,0} - \nu_{j,m} \preceq 0,
\end{aligned}$$

then SSPS (1) is ES with RDT constraint meeting

$$\underline{T}^* \leq T_l \leq \overline{T}^*, \quad T_l = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z.$$

3.4. Discussions

In the framework of DLCLF approach, the stability conditions under MDMDT, MDCDT and MDRDT constraints have been proposed in Subsections 3.1-3.3, then the advantages and limitations of this DLCLF approach will be discussed in this part. We start with a linear switched positive system in the following form,

$$\begin{cases} \dot{x}(t) &= A_{\sigma(t)} x(t), \\ x(t_0) &= x_0. \end{cases} \tag{51}$$

By using the DLCLF approach, the stability of system (51) under MDMDT constraint can be easily obtained by following the proof line of Theorem 1.

Lemma 2. *Given scalars $S \in Z_+$, $T_i^* \in R_+$, $i \in \underline{N}$, if there exist vectors $\nu_{i,s} \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,*

$$A_i^T \nu_{i,s} + \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \tag{52}$$

$$A_i^T \nu_{i,s+1} + \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) \preceq 0, \quad s \neq S, \tag{53}$$

$$A_i^T \nu_{i,S} \preceq 0, \tag{54}$$

$$\nu_{i,0} - \nu_{j,S} \preceq 0, \tag{55}$$

then system (51) is globally stable with MDMDT constraint meeting (13).

Actually, dwell-time stability issue has been discussed in some existing literatures. Based on general copositive and homogeneous Lyapunov functions, the stability of system (51) under MDMDT constraint can be directly achieved through Theorem 1 in [34].

Lemma 3. *Given scalars* $T_i^* \in R_+$, $i \in \underline{N}$, *if there exist vectors* $\nu_i \in R_+^n$, $i \in \underline{N}$, *such that,* $\forall (i,j) \in \underline{N} \times \underline{N}$, $i \neq j$,

$$A_i^T \nu_i \preceq 0, \tag{56}$$

$$e^{A_i^T T_i^*} \nu_i - \nu_j \preceq 0, \tag{57}$$

then system (51) is globally stable with MDMDT constraint meeting (13).

Interestingly, above two lemmas can be proved to be equivalent under certain circumstances with the similar proof line of Theorem 2 in [24], and is formulated as follows.

Lemma 4. *Consider system (51), there always exists a sufficiently large* $S^* \geq 1$, *so that two conditions below are equivalent,* $\forall S \geq S^*$:

- *(a) There exist* $N(S+1)$ *vectors* $\nu_{i,s} \in R_+^n$, $s \in \{0, 1, \cdots, S\}$, *such that (52)-(55) hold.*
- *(b) There exist* N *vectors* $\nu_i \in R_+^n$, $i \in \underline{N}$, *such that (56)-(57) hold.*

Remark 5. *Lemma 4(b) seems to be simple and convenient to utilize, while the system matrix* A_i *of condition (57) is non-affine because of the term* $e^{A_i^T T_i^*}$. *The main disadvantage of such conditions is the existence of exponential terms, which makes it quite difficult to generalize to systems with uncertainties since there are no efficient solutions to handle matrix uncertainties in the exponential terms. Also, the second disadvantage of this approach is the limitation to apply to systems with time-varying parameters and its control design, since the exponential term is not suitable for systems with time-varying parameters, or else the nonconvex terms will be generated strongly in the design conditions accordingly.*

On the other hand, A_i *in Lemma 4(a) is affine in (52)-(55), which is an important character of many further extensions like the gain performance characterization. The main advantage of the DLCLF technique, which leads to LP conditions devoid of exponential terms, is its affine dependence on conditions of system matrix which will be extended to systems with uncertainties or/and time-varying parameters quite easily. Therefore, this DLCLF approach is applicable for a wider scope of systems.*

Remark 6. *It is worth mentioning that, in the case of DLCLF technique, high-order divisions would be needed to obtain accurate results, but would also dramatically increase the number of decision variables and may lead to numerical problems. Therefore, although Lemma 4(a) is equivalent to Lemma 4(b) under certain circumstances, the cost is to increase the computational complexity caused by taking finer divisions of* $[t_l, t_l + T_i^*)$, *i.e., by setting a larger* S, *shown in Table 1. It becomes a matter of future research to see how the complexity of the problem addressed in this technical note can be reduced.*

A useful lemma of positive systems is presented below.

Lemma 5. *A Metzler matrix* $A \in R^{n\times n}$ *is Hurwitz stable iff there is a vector* $\nu \in R_+^n$ *so that* $A^T \nu \preceq 0$.

Table 1: The computational complexities between Lemma 4(a) and Lemma 4(b)

	Number of variables	LP constrains
Lemma 4(a)	(S+1)nN	(n+3S)nN
Lemma 4(b)	n(N)	(n+1)nN

Remark 7. *In Theorem 1, the stability conditions (9)-(12) for the system under MDMDT switching signal are proposed. It should be pointed out that condition (11) implies $A_i^T \nu_{i,S} \preceq 0$, which reveals the necessity of requiring stability of all the subsystems due to Lemma 5. This point can also be obtained directly from the equivalent stability condition (56) in Lemma 3. However, in Theorem 3, observing the stability conditions (40)-(42) for the system under MDRDT switching signal, the necessity limitation does not exist anymore, which indicates that this approach permits the instability of the subsystems, and is applicable to a wider range. This has been discussed in detail in Example 7.*

4. Exponential stability in time-varying delay case

Considering system (2) in time-varying delay case, Section 4 is focused on the exponential stability analysis under MDMDT, MDCDT and MDRDT constraints, respectively.

4.1. MDMDT case

For system (2), on the basis of discretized linear copositive Lyapunov-Krasovskii functional(DLCLKF) method, Subsection 4.1 presents an exponential stability result under MDMDT constraint, and further gives a degenerated one under MDT constraint.

Theorem 4. *For system (2), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T_i^* \in R_+$, $i \in \underline{N}$, if there exist vectors $\nu_{i,s} \in R_+^n$, $\vartheta \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i,j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,*

$$A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \tag{58}$$

$$A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \tag{59}$$

$$A_{di}^T \nu_{i,s} - (1 - d_d) e^{-\alpha d_u} \vartheta \preceq 0, \tag{60}$$

$$A_i^T \nu_{i,S} + \alpha E^T \nu_{i,S} + \vartheta \preceq 0, \tag{61}$$

$$\nu_{i,0} - \nu_{j,S} \preceq 0, \tag{62}$$

then SSPS (2) is ES with MDMDT constraint meeting

$$T_{li} \geq T_i^*, \quad T_{li} = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z. \tag{63}$$

Proof. Construct the DLCLKF candidate in the following,

$$V(x(t)) = x^T(t)E^T\nu_{\sigma(t)}(t) + \int_{t-d(t)}^{t} e^{\alpha(-t+s)}x^T(s)\vartheta ds, \tag{64}$$

and meanwhile $\nu_{\sigma(t)}(t) \succ 0$ satisfies the conditions (15)-(16).

Due to that $\sigma(t)$ meets the MDMDT constraint (63), the interval $[t_l, t_{l+1})$ can be divided into $[t_l, t_l + T_i^*)$ and $[t_l + T_i^*, t_{l+1})$.

Firstly, considering the case $t \in [t_l, t_l + T_i^*)$, one has from (2), (64) and (58)-(60) that

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &= \dot{x}^T(t)E^T\nu_i(t) + x^T(t)E^T\dot{\nu}_i(t) - \alpha\int_{t-d(t)}^{t} e^{\alpha(-t+s)}x^T(s)\vartheta ds \\
&\quad + x^T(t)\vartheta - (1-\dot{d}(t))e^{-\alpha d(t)}x^T(t-d(t))\vartheta \\
&\quad + \alpha x^T(t)E^T\nu_i(t) + \alpha\int_{t-d(t)}^{t} e^{\alpha(-t+s)}x^T(s)\vartheta ds \\
&\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T]\nu_i(t) + x^T(t)E^T\dot{\nu}_i(t) \\
&\quad + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T][(1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}] \\
&\quad + x^T(t)E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq x^T(t)\{(1-\rho(t))[A_i^T\nu_{i,s} + \alpha E^T\nu_{i,s} + E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta] \\
&\quad + \rho(t)[A_i^T\nu_{i,s+1} + \alpha E^T\nu_{i,s+1} + E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta]\} \\
&\quad + x^T(t-d(t))\{(1-\rho(t))[A_{di}^T\nu_{i,s} - (1-d_d)e^{-\alpha d_u}\vartheta] \\
&\quad + \rho(t)[A_{di}^T\nu_{i,s+1} - (1-d_d)e^{-\alpha d_u}\vartheta]\} \\
&\leq 0.
\end{aligned} \tag{65}$$

Next, the case $t \in [t_l + T_i^*, t_{l+1})$ is to be considered, and (2), (64) and (60)-(61) lead to

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T]\nu_i(t) + x^T(t)E^T\dot{\nu}_i(t) \\
&\quad + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T]\nu_{i,S} \\
&\quad + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq x^T(t)[A_i^T\nu_{i,S} + \alpha E^T\nu_{i,S} + \vartheta] \\
&\quad + x^T(t-d(t))[A_{di}^T\nu_{i,S} - (1-d_d)e^{-\alpha d_u}\vartheta] \\
&\leq 0.
\end{aligned} \tag{66}$$

Combing (65) and (66) result in

$$\dot{V}_i(x(t)) + \alpha V_i(x(t)) \leq 0, \quad t \in [t_l, t_{l+1}), \tag{67}$$

which yields

$$V_{\sigma(t)}(x(t)) \le e^{-\alpha(t-t_l)} V_{\sigma(t_l)}(x(t_l)), \quad t \in [t_l, t_{l+1}). \tag{68}$$

Then, taking the switching instant t_l into account, where $\sigma(t) = j$ for $t \in [t_{l-1}, t_l)$ and $\sigma(t) = i$ for $t \in [t_l, t_{l+1})$, $\forall i, j \in \underline{N}$, (2) and (64) lead to

$$\begin{aligned} V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) &= x^T(t_l^+)E^T \nu_{\sigma(t_l^+)}(t_l^+) + \int_{t_l^+ - d(t_l^+)}^{t_l^+} e^{\alpha(-t_l^+ + s)} x^T(s)\vartheta ds \\ &\quad - x^T(t_l^-)E^T \nu_{\sigma(t_l^-)}(t_l^-) - \int_{t_l^- - d(t_l^-)}^{t_l^-} e^{\alpha(-t_l^- + s)} x^T(s)\vartheta ds \\ &= x^T(t_l)E^T(\nu_{i,0} - \nu_{j,S}) \\ &\quad + \int_{t_l - d(t_l)}^{t_l} e^{\alpha(-t_l + s)} x^T(s)[\vartheta - \vartheta] ds, \end{aligned} \tag{69}$$

and from (62), we have

$$V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) = V_{\sigma(t_l)}(x(t_l^+)) - V_{\sigma(t_{l-1})}(x(t_l^-)) \le 0, \quad \forall l \in Z_+. \tag{70}$$

Then it follows from (68) and (70) that

$$V_{\sigma(t)}(x(t)) \le e^{-\alpha(t-t_0)} V_{\sigma(t_0)}(x(t_0)). \tag{71}$$

From the other side, denote $\nu_i(t) = \begin{bmatrix} \nu_i^{(1)}(t) \\ \nu_i^{(2)}(t) \end{bmatrix}$ with $\nu_i^{(1)}(t) \in R^r$ and $\nu_i^{(2)}(t) \in R^{n-r}$, then the DLCLKF (64) leads to

$$\begin{aligned} V_{\sigma(t)}(t) &= x^T(t)E^T\nu_i(t) + \int_{t-d(t)}^{t} e^{\alpha(-t+s)} x^T(s)\vartheta ds \\ &= \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \nu_i^{(1)}(t) \\ \nu_i^{(2)}(t) \end{bmatrix} + \int_{t-d(t)}^{t} e^{\alpha(-t+s)} x^T(s)\vartheta ds \\ &= x_1^T(t)\nu_i^{(1)}(t) + \int_{t-d(t)}^{t} e^{\alpha(-t+s)} x^T(s)\vartheta ds. \end{aligned} \tag{72}$$

Based on (64) and (72), it is obvious that

$$V_{\sigma(t)}(x(t)) \ge \beta_1 \|x_1(t)\|_1, \tag{73}$$

$$V_{\sigma(t_0)}(x(t_0)) \le \beta_2 \|\varphi\|_{1c}, \tag{74}$$

where $\beta_1 = \min_{\forall i \in \underline{N}, s \in \underline{S}} \underline{\lambda}(\nu_{i,s}^{(1)})$, $\beta_2 = \max_{\forall i \in \underline{N}} \overline{\lambda}(\nu_{i,0}^{(1)}) + \frac{(1-e^{-\alpha d_u})}{\alpha} \overline{\lambda}(\vartheta)$, $\|\varphi\|_{1c} = \sup\limits_{-du \le \theta \le 0} \|x(t_0 + \theta)\|_1$.

Then, combining (71) and (73)-(74) leads to, $\forall t \geq 0$,

$$\|x_1(t)\|_1 \leq \beta e^{-\alpha(t-t_0)}\|\varphi\|_{1c}, \tag{75}$$

where $\beta = \beta_2/\beta_1 > 1$, and obviously, $x_1(t)$ is ES with DR α. (75) can further yield

$$\|x_1(t)\|_1 \leq \beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)}, \quad \forall t \geq 0. \tag{76}$$

Next, $x_2(t)$ being ES with DR α is to be proved. One has from (75) that

$$\|x_1(t-d(t))\|_1 \leq \beta e^{-\alpha(t-d(t)-t_0)}\|\varphi\|_{1c} \leq \beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)}, \quad \forall t > d(t), \tag{77}$$

$$\|x_1(t-d(t))\|_1 \leq \|\varphi\|_{1c} \leq \|\varphi\|_{1c}e^{-\alpha(t-d(t))} \leq \beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)}, \quad \forall t \in [0, d(t)], \tag{78}$$

and from (77)-(78), it is obvious that

$$\|x_1(t-d(t))\|_1 \leq \beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)}, \quad \forall t \geq 0. \tag{79}$$

The second equation of system (6) yields, $\forall t \geq 0$,

$$\begin{aligned}\|x_2(t)\|_1 &= \|\overline{A}_{i3}x_1(t) + \overline{A}_{di3}x_1(t-d(t)) + \overline{A}_{di4}x_2(t-d(t))\|_1 \\ &\leq \|\overline{A}_{i3}\|_1\|x_1(t)\|_1 + \|\overline{A}_{di3}\|_1\|x_1(t-d(t))\|_1 + \|\overline{A}_{di4}\|_1\|x_2(t-d(t))\|_1.\end{aligned} \tag{80}$$

When $t \in [0, d(t)]$, one can obtain

$$\|x_2(t-d(t))\|_1 \leq \|\varphi\|_{1c} \leq \|\varphi\|_{1c}e^{-\alpha(t-d(t))} \leq e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)}, \tag{81}$$

and from (76), (79)-(81), we get

$$\begin{aligned}\|x_2(t)\|_1 &\leq \|\overline{A}_{i3}\|_1\|x_1(t)\|_1 + \|\overline{A}_{di3}\|_1\|x_1(t-d(t))\|_1 + \|\overline{A}_{di4}\|_1\|x_2(t-d(t))\|_1 \\ &\leq \|\overline{A}_{i3}\|_1\beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} + \|\overline{A}_{di3}\|_1\beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} + \|\overline{A}_{di4}\|_1 e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} \\ &\leq (\|\overline{A}_{i3}\|_1 + \|\overline{A}_{di3}\|_1)\beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} + \|\overline{A}_{di4}\|_1 e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} \\ &\leq (\|\overline{A}_{di4}\|_1\delta + \delta)\|\varphi\|_{1c}e^{-\alpha(t-t_0)},\end{aligned} \tag{82}$$

where $\delta = \max_{\forall i \in \underline{N}}\{(\|\overline{A}_{i3}\|_1 + \|\overline{A}_{di3}\|_1)\beta e^{\alpha d_u}, e^{\alpha d_u}\}$.

When $t \in [d(t), 2d(t)]$, one can obtain from (82) that

$$\begin{aligned}\|x_2(t-d(t))\|_1 &\leq (\|\overline{A}_{di4}\|_1\delta + \delta)\|\varphi\|_{1c}e^{-\alpha(t-d(t)-t_0)} \\ &\leq (\|\overline{A}_{di4}\|_1\delta + \delta)e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} \\ &\leq (\|\overline{A}_{di4}\|_1\delta^2 + \delta^2)\|\varphi\|_{1c}e^{-\alpha(t-t_0)},\end{aligned} \tag{83}$$

and from (76), (79)-(80) and (83), we get

$$\begin{aligned}\|x_2(t)\|_1 &\leq \|\overline{A}_{i3}\|_1\|x_1(t)\|_1 + \|\overline{A}_{di3}\|_1\|x_1(t-d(t))\|_1 + \|\overline{A}_{di4}\|_1\|x_2(t-d(t))\|_1 \\ &\leq (\|\overline{A}_{i3}\|_1 + \|\overline{A}_{di3}\|_1)\beta e^{\alpha d_u}\|\varphi\|_{1c}e^{-\alpha(t-t_0)} + \|\overline{A}_{di4}\|_1(\|\overline{A}_{di4}\|_1\delta^2 + \delta^2)\|\varphi\|_{1c}e^{-\alpha(t-t_0)} \\ &\leq (\|\overline{A}_{di4}\|_1^2\delta^2 + \|\overline{A}_{di4}\|_1\delta^2 + \delta)\|\varphi\|_{1c}e^{-\alpha(t-t_0)}.\end{aligned} \tag{84}$$

When $t \in [(k-1)d(t), kd(t)]$, assume that

$$\|x_2(t)\|_1 \leq (\|\overline{A}_{di4}\|_1^k \delta^k + \|\overline{A}_{di4}\|_1^{k-1}\delta^k + \cdots + \|\overline{A}_{di4}\|_1 \delta^2 + \delta)\|\varphi\|_{1c} e^{-\alpha(t-t_0)}. \tag{85}$$

then for $t \in [kd(t), (k+1)d(t)]$, one can obtain from (85) via inductive supposition method that

$$\begin{aligned}\|x_2(t-d(t))\|_1 &\leq (\|\overline{A}_{di4}\|_1^k \delta^k + \|\overline{A}_{di4}\|_1^{k-1}\delta^k + \cdots + \|\overline{A}_{di4}\|_1 \delta^2 + \delta)\|\varphi\|_{1c} e^{-\alpha(t-d(t)-t_0)} \\ &\leq (\|\overline{A}_{di4}\|_1^k \delta^k + \|\overline{A}_{di4}\|_1^{k-1}\delta^k + \cdots + \|\overline{A}_{di4}\|_1 \delta^2 + \delta) e^{\alpha d_2} \|\varphi\|_{1c} e^{-\alpha(t-t_0)}, \end{aligned} \tag{86}$$

and from (76), (79)-(80) and (86), we get

$$\begin{aligned}\|x_2(t)\|_1 \leq\; & \|\overline{A}_{i3}\|_1 \|x_1(t)\|_1 + \|\overline{A}_{di3}\|_1 \|x_1(t-d(t))\|_1 + \|\overline{A}_{di4}\|_1 \|x_2(t-d(t))\|_1 \\ \leq\; & (\|\overline{A}_{i3}\|_1 + \|\overline{A}_{di3}\|_1)\beta e^{\alpha d_u} \|\varphi\|_{1c} e^{-\alpha(t-t_0)} \\ & + \|\overline{A}_{di4}\|_1 (\|\overline{A}_{di4}\|_1^k \delta^k + \|\overline{A}_{di4}\|_1^{k-1}\delta^k + \cdots + \|\overline{A}_{di4}\|_1 \delta^2 + \delta) e^{\alpha d_2} \|\varphi\|_{1c} e^{-\alpha(t-t_0)} \\ \leq\; & (\|\overline{A}_{di4}\|_1^{(k+1)} \delta^{(k+1)} + \|\overline{A}_{di4}\|_1^k \delta^{(k+1)} + \cdots + \|\overline{A}_{di4}\|_1 \delta^2 + \delta)\|\varphi\|_{1c} e^{-\alpha(t-t_0)}. \end{aligned} \tag{87}$$

If $0 < \|\overline{A}_{di4}\|_1 \delta < 1$, we have

$$\begin{aligned}\|x_2(t)\|_1 &\leq [1 + \|\overline{A}_{di4}\|_1 \delta + \|\overline{A}_{di4}\|_1^2 \delta^2 + \cdots + \|\overline{A}_{di4}\|_1^k \delta^k + \cdots]\delta \|\varphi\|_{1c} e^{-\alpha(t-t_0)} \\ &\leq \frac{\delta}{1 - \|\overline{A}_{di4}\|_1 \delta} \|\varphi\|_{1c} e^{-\alpha(t-t_0)}. \end{aligned} \tag{88}$$

Finally, equations (75) and (88) yield $\|x(t)\|_1 \leq \varpi e^{-\alpha(t-t_0)} \|\varphi\|_{1c}$, $\varpi = \max\{\beta, \frac{\delta}{1-\|\overline{A}_{di4}\|_1\delta}\}$, $\forall t \geq 0$, that is, in light of Definition 3, SSPS (2) is ES with DR α.

The proof is completed.

□

Remark 8. *For the sake of proving exponential stability, a traditional method based on Razumikhin and Halanay inequality is adopted in literature [35], where DR is computed as fixed values via seeking solutions of complex functions. This causes some certain restrictions in engineering applications. To address this drawback, a new method based on Lyapunov-Krasovskii functional is proposed in Theorem 4 in this paper, where DR can be set as free parameters satisfying different situations through inserting the particular exponential term into constructed functional (64). This reduces the restrictions of the former method and brings great flexibility to the exponential stability analysis for SSPSs.*

When $T_i^* = T_j^*$, $\forall i, j \in \underline{N}$, $i \neq j$, the MDMDT constraint is degenerated into MDT constraint, and exponential stability issue with MDT constraint is presented accordingly.

Corollary 5. *For system (2), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T^* \in R_+$, if there exist vectors $\nu_{i,s} \in R_+^n$, $\vartheta \in R_+^n$, $i \in \underline{N}$,*

$s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,

$$A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S,$$
$$A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S,$$
$$A_{di}^T \nu_{i,s} - (1 - d_d)e^{-\alpha d_u}\vartheta \preceq 0,$$
$$A_i^T \nu_{i,S} + \alpha E_i^T \nu_{i,S} + \vartheta \preceq 0,$$
$$\nu_{i,0} - \nu_{j,S} \preceq 0,$$

then SSPS (2) is ES with MDT constraint meeting

$$T_l \geq T^*, \quad T_l = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z.$$

4.2. MDCDT case

For system (2), on the basis of DLCLKF method, Subsection 4.2 presents an exponential stability result under MDCDT constraint, and further gives a degenerated one under CDT constraint.

Theorem 5. *For system (2), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T_i^* \in R_+$, $i \in \underline{N}$, if there exist vectors $\nu_{i,s} \in R_+^n$, $\vartheta \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,*

$$A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \tag{89}$$
$$A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \tag{90}$$
$$A_{di}^T \nu_{i,s} - (1 - d_d)e^{-\alpha d_u}\vartheta \preceq 0, \tag{91}$$
$$\nu_{i,0} - \nu_{j,S} \preceq 0, \tag{92}$$

then SSPS (2) is ES with MDCDT constraint meeting

$$T_{li} \equiv T_i^*, \quad T_{li} = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z. \tag{93}$$

Proof. Select the DLCLKF candidate (64) where $\nu_{\sigma(t)}(t)$ satisfies conditions (33)-(34).

For $t \in [t_l, t_{l+1})$, from (2), (64) and (89)-(91), we get

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T]\nu_i(t) + x^T(t)E^T\dot{\nu}_i(t) \\
&\quad + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T][(1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}] \\
&\quad + x^T(t)E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq x^T(t)\{(1-\rho(t))[A_i^T\nu_{i,s} + \alpha E^T\nu_{i,s} + E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta] \\
&\quad + \rho(t)[A_i^T\nu_{i,s+1} + \alpha E^T\nu_{i,s+1} + E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta]\} \\
&\quad + x^T(t-d(t))\{(1-\rho(t))[A_{di}^T\nu_{i,s} - (1-d_d)e^{-\alpha d_u}\vartheta] \\
&\quad + \rho(t)[A_{di}^T\nu_{i,s+1} - (1-d_d)e^{-\alpha d_u}\vartheta]\} \\
&\leq 0,
\end{aligned} \tag{94}$$

which yields

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_l)}V_{\sigma(t_l)}(x(t_l)), \quad t \in [t_l, t_{l+1}). \tag{95}$$

Then, taking the switching instant t_l into account, where $\sigma(t) = j$ for $t \in [t_{l-1}, t_l)$ and $\sigma(t) = i$ for $t \in [t_l, t_{l+1})$, $\forall i, j \in \underline{N}$, (2) and (64) lead to

$$\begin{aligned}
V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) &= x^T(t_l^+)E^T\nu_{\sigma(t_l^+)}(t_l^+) + \int_{t_l^+-d(t_l^+)}^{t_l^+} e^{\alpha(-t_l^+ + s)}x^T(s)\vartheta ds \\
&\quad - x^T(t_l^-)E^T\nu_{\sigma(t_l^-)}(t_l^-) - \int_{t_l^- - d(t_l^-)}^{t_l^-} e^{\alpha(-t_l^- + s)}x^T(s)\vartheta ds \\
&= x^T(t_l)E^T(\nu_{i,0} - \nu_{j,S}) \\
&\quad + \int_{t_l - d(t_l)}^{t_l} e^{\alpha(-t_l+s)}x^T(s)[\vartheta - \vartheta]ds,
\end{aligned} \tag{96}$$

and from (92), we have

$$V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) = V_{\sigma(t_l)}(x(t_l^+)) - V_{\sigma(t_{l-1})}(x(t_l^-)) \leq 0, \quad \forall l \in Z_+. \tag{97}$$

Then it follows from (95) and (97) that

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_0)}V_{\sigma(t_0)}(x(t_0)). \tag{98}$$

The remaining proof is similar to that in Theorem 4.

The proof is completed. □

When $T_i^* = T_j^*$, $\forall i, j \in \underline{N}$, $i \neq j$, the MDCDT constraint is equal to CDT constraint, and exponential stability issue with CDT constraint is presented accordingly.

Corollary 6. *For system (2), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $T^* \in R_+$, if there exist vectors $\nu_{i,s} \in R_+^n$, $\vartheta \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $s \in \{0, 1, \cdots, S\}$,*

$$\begin{aligned} A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \\ A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{T^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \\ A_{di}^T \nu_{i,s} - (1 - d_d) e^{-\alpha d_u} \vartheta \preceq 0, \\ \nu_{i,0} - \nu_{j,S} \preceq 0, \end{aligned}$$

then SSPS (2) is ES with CDT constraint meeting

$$T_l \equiv T^*, \quad T_l = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z.$$

4.3. MDRDT case

For system (2), on the basis of DLCLKF method, Subsection 4.3 presents an exponential stability result under MDRDT constraint, and further gives a degenerated one under RDT constraint.

Theorem 6. *For system (2), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars $\alpha > 0$, $S \in Z_+$, $\overline{T}_i^* \in R_+$, $\underline{T}_i^* \in R_+$, $i \in \underline{N}$, if there exist vectors $\nu_{i,s} \in R_+^n$, $\vartheta \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, so that, $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $m \in \{\lceil \frac{S\underline{T}_i^*}{\overline{T}_i^*} \rceil, \lceil \frac{S\underline{T}_i^*}{\overline{T}_i^*} \rceil + 1, \lceil \frac{S\underline{T}_i^*}{\overline{T}_i^*} \rceil + 2, \cdots, S\}$, $s \in \{0, 1, \cdots, S\}$,*

$$A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \tag{99}$$

$$A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{\overline{T}_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta \preceq 0, \quad s \neq S, \tag{100}$$

$$A_{di}^T \nu_{i,s} - (1 - d_d) e^{-\alpha d_u} \vartheta \preceq 0, \tag{101}$$

$$\nu_{i,0} - \nu_{j,m} \preceq 0, \tag{102}$$

then SSPS (2) is ES with MDRDT constraint meeting

$$\underline{T}_i^* \leq T_{li} \leq \overline{T}_i^*, \quad T_{li} = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z. \tag{103}$$

Proof. Select the DLCLKF candidate (64) where $\nu_{\sigma(t)}(t)$ meets conditions (44)-(45).

For $t \in [t_l, t_{l+1})$, from (2), (64) and (99)-(101), we have

$$\begin{aligned}
\dot{V}_i(x(t)) + \alpha V_i(x(t)) &\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T]\nu_i(t) + x^T(t)E^T\dot{\nu}_i(t) \\
&\quad + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq [x^T(t)A_i^T + x^T(t-d(t))A_{di}^T + \alpha x^T(t)E^T][(1-\rho(t))\nu_{i,s} + \rho(t)\nu_{i,s+1}] \\
&\quad + x^T(t)E^T\frac{S}{T_i^*}(\nu_{i,s+1} - \nu_{i,s}) \\
&\quad + x^T(t)\vartheta - (1-d_d)e^{-\alpha d_u}x^T(t-d(t))\vartheta \\
&\leq x^T(t)\{(1-\rho(t))[A_i^T\nu_{i,s} + \alpha E^T\nu_{i,s} + E^T\frac{S}{\overline{\overline{T}}_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta] \\
&\quad + \rho(t)[A_i^T\nu_{i,s+1} + \alpha E^T\nu_{i,s+1} + E^T\frac{S}{\overline{\overline{T}}_i^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta]\} \\
&\quad + x^T(t-d(t))\{(1-\rho(t))[A_{di}^T\nu_{i,s} - (1-d_d)e^{-\alpha d_u}\vartheta] \\
&\quad + \rho(t)[A_{di}^T\nu_{i,s+1} - (1-d_d)e^{-\alpha d_u}\vartheta]\} \\
&\leq 0,
\end{aligned} \tag{104}$$

which yields

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_l)}V_{\sigma(t_l)}(x(t_l)), \quad t \in [t_l, t_{l+1}). \tag{105}$$

Then, taking the switching instant t_l into account, where $\sigma(t) = j$ for $t \in [t_{l-1}, t_l)$ and $\sigma(t) = i$ for $t \in [t_l, t_{l+1})$, $\forall i, j \in \underline{N}$, (2) and (64) lead to

$$\begin{aligned}
V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) &= x^T(t_l^+)E^T\nu_{\sigma(t_l^+)}(t_l^+) + \int_{t_l^+-d(t_l^+)}^{t_l^+} e^{\alpha(-t_l^+ + s)}x^T(s)\vartheta ds \\
&\quad - x^T(t_l^-)E^T\nu_{\sigma(t_l^-)}(t_l^-) - \int_{t_l^- - d(t_l^-)}^{t_l^-} e^{\alpha(-t_l^- + s)}x^T(s)\vartheta ds \\
&= x^T(t_l)E^T(\nu_{i,0} - \nu_{j,m}) \\
&\quad + \int_{t_l - d(t_l)}^{t_l} e^{\alpha(-t_l + s)}x^T(s)[\vartheta - \vartheta]ds,
\end{aligned} \tag{106}$$

and from (102), we have

$$V_{\sigma(t_l^+)}(x(t_l^+)) - V_{\sigma(t_l^-)}(x(t_l^-)) = V_{\sigma(t_l)}(x(t_l^+)) - V_{\sigma(t_{l-1})}(x(t_l^-)) \leq 0, \quad \forall l \in Z_+. \tag{107}$$

Then it follows from (105) and (107) that

$$V_{\sigma(t)}(x(t)) \leq e^{-\alpha(t-t_0)}V_{\sigma(t_0)}(x(t_0)). \tag{108}$$

The remaining proof is similar to that in Theorem 4.
The proof is completed.

□

When $\overline{T}_i^* = \overline{T}_j^*$, $\underline{T}_i^* = \underline{T}_j^*$, $\forall i, j \in \underline{N}$, $i \neq j$, the MDRDT constraint is turned to RDT constraint, and exponential stability issue under RDT constraint is presented accordingly.

Corollary 7. *For system (2), suppose that Lemma 1 and Assumption 1 are satisfied. Given scalars* $\alpha > 0$, $S \in Z_+$, $\overline{T}^* \in R_+$, $\underline{T}^* \in R_+$, *if there exist vectors* $\nu_{i,s} \in R_+^n$, $\vartheta \in R_+^n$, $i \in \underline{N}$, $s \in \{0, 1, \cdots, S\}$, *so that,* $\forall (i, j) \in \underline{N} \times \underline{N}$, $i \neq j$, $m \in \{\lceil \frac{S\underline{T}^*}{\overline{T}^*} \rceil, \lceil \frac{S\underline{T}^*}{\overline{T}^*} \rceil + 1, \lceil \frac{S\underline{T}^*}{\overline{T}^*} \rceil + 2, \cdots, S\}$, $s \in \{0, 1, \cdots, S\}$,

$$
\begin{aligned}
A_i^T \nu_{i,s} + \alpha E^T \nu_{i,s} + E^T \frac{S}{\overline{T}^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta &\preceq 0, \quad s \neq S, \\
A_i^T \nu_{i,s+1} + \alpha E^T \nu_{i,s+1} + E^T \frac{S}{\overline{T}^*}(\nu_{i,s+1} - \nu_{i,s}) + \vartheta &\preceq 0, \quad s \neq S, \\
A_{di}^T \nu_{i,s} - (1 - d_d) e^{-\alpha d_u} \vartheta &\preceq 0, \\
\nu_{i,0} - \nu_{j,m} &\preceq 0,
\end{aligned}
$$

then SSPS (2) is ES with RDT constraint meeting

$$
\underline{T}^* \leq T_l \leq \overline{T}^*, \quad T_l = t_{l+1} - t_l, \quad \sigma(t) = i, \quad t \in [t_l, t_{l+1}), \quad l \in Z.
$$

4.4. Discussions

In real life, due to the physical constraints, the system cannot switch arbitrarily fast, and the system has to stay in each subsystem for at least a period of time, so we focus on the switching signals limited by dwell time. Different dwell-time conditions are suitable for different switching situations, and we have presented six theorems and seven corollaries in this paper. What are the differences between these dwell-time results? How to use these results? These will be discussed in detail in the following.

From the definitions of dwell time, in the periodic switching case, it is meant here that the dwell time is periodic, i.e., $t_{l+1} - t_l \equiv T_i^*$, $T_i^* > 0$, the MDCDT condition is desired; In the aperiodic switching case, if the dwell time is limited with a lower-bound, i.e., $t_{l+1} - t_l \geq T_i^*$, $T_i^* > 0$, the MDMDT condition is expected; If the dwell time lies within a certain range with both lower-bound and upper-bound, i.e., $\underline{T}_i^* \leq t_{l+1} - t_l \leq \overline{T}_i^*$, $0 < \underline{T}_i^* < \overline{T}_i^* < +\infty$, the MDRDT condition is preferred.

More concretely, in the case of periodic switching, the MDCDT condition in Theorem 2 is obviously preferred, however, the order of subsystems is not necessarily periodic, and the aperiodic switching case needs to be considered furthermore. In the case of aperiodic switching, if all the subsystems are Hurwitz stable, the MDMDT condition in Theorem 1 is a good choice, however, in the framework under consideration, some unstable subsystems may indeed exsit and the MDMDT condition in Theorem 1 may not be applicable anymore; If the subsystems contain anti-stable ones, the MDRDT condition in Theorem 3 is more desired due to Remark 7, and then can be applied to a wider range of systems.

Noted that, the aforementioned results obtained in Theorems 1-3 are only focused on the delay-free case, and the widespread time-varying delays are not considered. Furthermore, considering the more general time-varying delay case, the MDMDT condition is further studied and obtained in Theorem 4, the MDCDT condition is further discussed and presented in Theorem 5, and the MDRDT condition is further investigated and proposed in Theorem 6.

As mentioned earlier, the results in Theorems 1-6 are all based on mode-dependent dwell time case, which is less conservative than mode-independent dwell time case. When the MDMDT(or MDCDT or MDRDT) case degenerated into the MDT(or CDT or RDT) case, the results in Theorems 1-6 will degenerate into the ones in Corollaries 1, 3-7, respectively. Specifically, the arbitrary switching result in Corollary 2 is a special one of MDT switching result in Corollary 1.

In the next, the difficulties to be overcome in this research work are list one by one.

For the MDMDT condition in Theorem 1, the main difficulties are stated in the following, and the same difficulties are also with the MDCDT condition in Theorem 2 and the MDRDT condition in Theorem 3.

1. Due to the combination of singularity, i.e., the complex structural characteristics of singular systems, positivity and switching compatibility, the study of SSPSs becomes more difficult and challenging than that of standard ones, and how to select some suitable approaches to analyze the system is not easy;
2. Compared with the asymptotical stability analysis, in the exponential stability analysis, it is a major difficulty to determine the DR for systems with singularity constraints;
3. With dwell-time constraints(including MDMDT, MDCDT and MDRDT constraints), it is a huge challenge to explore an appropriate method of stability analysis and manage some specific mathematical difficulties accordingly, and the existing dwell-time theory has not reached the quantitative level.

Furthermore, considering the more general time-varying delay case, in addition to the difficulties mentioned above, new difficulties for the conditions in Theorems 4-6 are:

1. When considering the time-varying delay phenomena, it is extremely difficult to construct a suitable Lyapunov functional to analyze the stability for systems with singularity constraints, because such system is coupled with matrix delay differential equation and matrix difference equation, which greatly increases the complexity and challenge of the research.
2. During the analytical process, it is very challenging to manage the detailed mathematical problems because of the complex form of the time-varying delays.

5. Examples

The validity and significance of the results are illustrated by seven examples.

Example 1. ***(MDMDT)*** *Consider system (2) with two modes, i.e.,* $\sigma(t) \in \{1, 2\}$,

$$A_1 = \begin{bmatrix} -1.5 & 0.9 \\ 1.0 & -0.5 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.01 & 0.05 \\ 0.05 & 0.01 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1.6 & 0.8 \\ 0.9 & -0.4 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.05 & 0.01 \\ 0.01 & 0.05 \end{bmatrix}, \tag{109}$$

and $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *By direct calculation, we have*

$$\begin{array}{l} det(sE - A_1) = 0.5s - 0.15 \neq 0, \quad for \quad s = 0, \\ det(sE - A_2) = 0.4s - 0.08 \neq 0, \quad for \quad s = 0, \\ deg(det(sE - A_1)) = deg(0.5s - 0.15) = rankE = 1, \\ deg(det(sE - A_2)) = deg(0.4s - 0.08) = rankE = 1. \end{array}$$

Then, system (2) is regular and impulse-free, and one get from the definitions in system (6) that

$$\begin{array}{lll} \overline{A}_{11} = 0.300, & \overline{A}_{d11} = 0.100, & \overline{A}_{d12} = 0.068, \\ \overline{A}_{13} = 2.000, & \overline{A}_{d13} = 0.100, & \overline{A}_{d14} = 0.020, \\ \overline{A}_{21} = 0.200, & \overline{A}_{d21} = 0.070, & \overline{A}_{d22} = 0.110, \\ \overline{A}_{23} = 2.250, & \overline{A}_{d23} = 0.025, & \overline{A}_{d24} = 0.125. \end{array}$$

Hence, system (2) with (109) is positive in light of Lemma 1.

Set $\alpha = 0.5$, $d(t) = 0.2 + 0.1sint$, $T_1^* = 1.0$, $T_2^* = 2.0$ *and* $S = 2$. *Solving the conditions in Theorem 4 leads to*

$$\begin{array}{l} \nu_{10} = \begin{bmatrix} 24.8022 \\ 22.5772 \end{bmatrix}, \nu_{11} = \begin{bmatrix} 30.9268 \\ 30.2851 \end{bmatrix}, \nu_{12} = \begin{bmatrix} 35.3016 \\ 40.4045 \end{bmatrix}, \\ \nu_{20} = \begin{bmatrix} 18.5057 \\ 19.6464 \end{bmatrix}, \nu_{21} = \begin{bmatrix} 29.1414 \\ 32.7275 \end{bmatrix}, \nu_{22} = \begin{bmatrix} 38.4287 \\ 46.6817 \end{bmatrix}, \vartheta = \begin{bmatrix} 39.5803 \\ 41.7220 \end{bmatrix}. \end{array}$$

By analysis, we can verify that $\max_{\forall i \in \underline{2}} \|\overline{A}_{di4}\|_1 \delta = 0.5214 < 1$. *According to Theorem 4, system (2) with (109) is positive and ES with MDMDT constraint meeting:*

$$\begin{cases} T_{l1} \geq T_1^* = 1.0, & l \in Z, \\ T_{l2} \geq T_2^* = 2.0, & l \in Z. \end{cases} \tag{110}$$

The simulation results are shown in Figs. 1-2, where $x(0) = [1 \quad 2]^T$, $x(t) = [0 \quad 0]^T$, $t \in [-0.3, 0)$. *Fig. 1 plots the MDMDT switching signal satisfying (110). From Fig. 1, it is obvious that the dwell-time intervals for Subsystem 1 are* $[0,1)$, $[3,4.5)$, $[7,8)$, $[10,11.5)$, $[14,15)$, $[17,18.5)$, $[21,22)$, $[24,25)$, $[27,28)$, *and the dwell-time intervals for Subsystem 2 are* $[1,3)$, $[4.5,7)$, $[8,10)$, $[11.5,14)$, $[15,17)$, $[18.5,21)$, $[22,24)$, $[25,27)$, $[28,30)$, *then it is easy to conclude that the switching rule* $\sigma(t)$ *plotted in Fig. 1 satisfies the MDMDT condition (110). Fig. 2 shows the system's state* $x(t)$, *which indicates the obvious positivity and exponential stability of system (2) with (109). The validity of the results is illustrated.*

Example 2. ***(MDCDT)*** *Consider system (2) with (109), and with the same analysis as in Example 1, system (2) with (109) is positive in light of Lemma 1.*

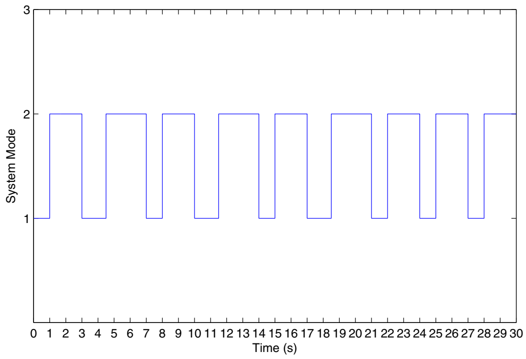

Figure 1: Switching rule $\sigma(t)$ with MDMDT constraint

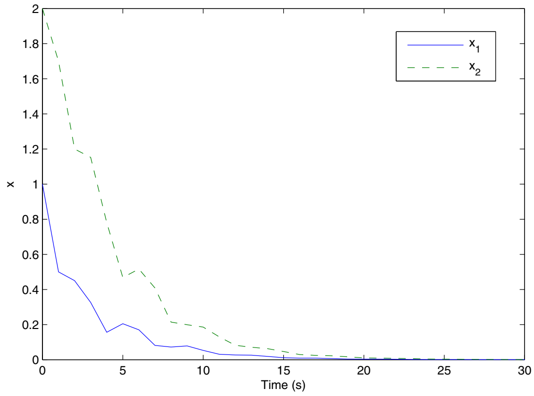

Figure 2: The state $x(t)$ with MDMDT constraint

Set $\alpha = 0.5$, $d(t) = 0.2 + 0.1sint$, $T_1^* = 1.0$, $T_2^* = 2.0$ *and* $S = 2$. *Solving the conditions in Theorem 5 leads to*

$$\nu_{10} = \begin{bmatrix} 26.5378 \\ 23.2512 \end{bmatrix}, \nu_{11} = \begin{bmatrix} 33.5108 \\ 32.2822 \end{bmatrix}, \nu_{12} = \begin{bmatrix} 39.1185 \\ 42.0297 \end{bmatrix},$$
$$\nu_{20} = \begin{bmatrix} 19.5642 \\ 20.5962 \end{bmatrix}, \nu_{21} = \begin{bmatrix} 31.2809 \\ 34.3661 \end{bmatrix}, \nu_{22} = \begin{bmatrix} 44.3411 \\ 47.2247 \end{bmatrix}, \vartheta = \begin{bmatrix} 42.8732 \\ 44.6696 \end{bmatrix}.$$

By analysis, we can verify that $\max_{\forall i \in \underline{2}} \|\overline{A}_{di4}\|_1 \delta = 0.5444 < 1$. *According to Theorem 5, system (2) with (109) is positive and ES with MDCDT constraint meeting:*

$$\begin{cases} T_{l1} \equiv T_1^* = 1.0, & l \in Z, \\ T_{l2} \equiv T_2^* = 2.0, & l \in Z. \end{cases} \tag{111}$$

The simulation results are shown in Figs. 3-4, where $x(0) = [1 \quad 2]^T$, $x(t) = [0 \quad 0]^T$, $t \in [-0.3, 0)$. *Fig. 3 plots the MDCDT switching signal satisfying (111). From Fig. 3, it is obvious that the dwell-time intervals for Subsystem 1 are* $[0, 1)$, $[3, 4)$, $[6, 7)$, $[9, 10)$, $[12, 13)$, $[15, 16)$, $[18, 19)$, $[21, 22)$, $[24, 25)$, $[27, 28)$, *and the dwell-time intervals for Subsystem 2 are* $[1, 3)$, $[4, 6)$, $[7, 9)$, $[10, 12)$, $[13, 15)$, $[16, 18)$, $[19, 21)$, $[22, 24)$, $[25, 27)$, $[28, 30)$, *then it is easy to conclude that the switching rule* $\sigma(t)$ *plotted in Fig. 3 satisfies the MDCDT condition (111). Fig. 4 shows the system's state* $x(t)$, *which indicates the obvious positivity and exponential stability of system (2) with (109). The validity of the results is illustrated.*

Example 3. ***(MDRDT)*** *Consider system (2) with (109), and with the same analysis as in Example 1, system (2) with (109) is positive in light of Lemma 1.*

Set $\alpha = 0.5$, $d(t) = 0.2 + 0.1sint$, $\underline{T}_1^* = 1.0$, $\overline{T}_1^* = 2.0$, $\underline{T}_2^* = 2.0$, $\overline{T}_2^* = 4.0$ *and* $S = 2$. *Solving the conditions in Theorem 6 leads to*

$$\nu_{10} = \begin{bmatrix} 15.6177 \\ 13.3382 \end{bmatrix}, \nu_{11} = \begin{bmatrix} 26.6566 \\ 28.4349 \end{bmatrix}, \nu_{12} = \begin{bmatrix} 29.9199 \\ 31.8190 \end{bmatrix},$$
$$\nu_{20} = \begin{bmatrix} 11.5208 \\ 11.3558 \end{bmatrix}, \nu_{21} = \begin{bmatrix} 27.5640 \\ 33.2218 \end{bmatrix}, \nu_{22} = \begin{bmatrix} 30.9977 \\ 35.2051 \end{bmatrix}, \vartheta = \begin{bmatrix} 30.9251 \\ 33.1622 \end{bmatrix}.$$

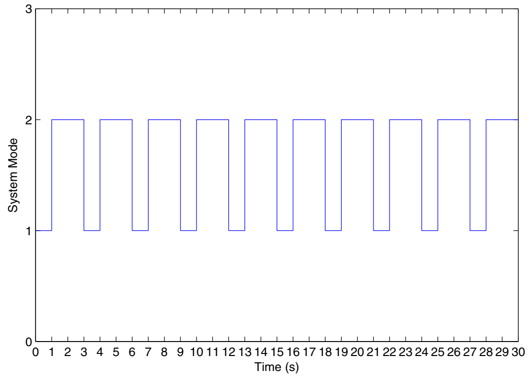

Figure 3: Switching rule $\sigma(t)$ with MDCDT constraint

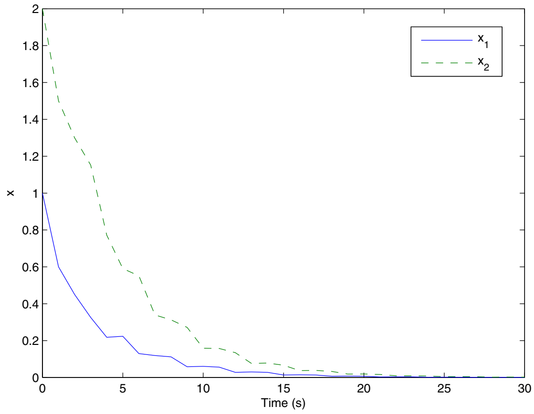

Figure 4: The state $x(t)$ with MDCDT constraint

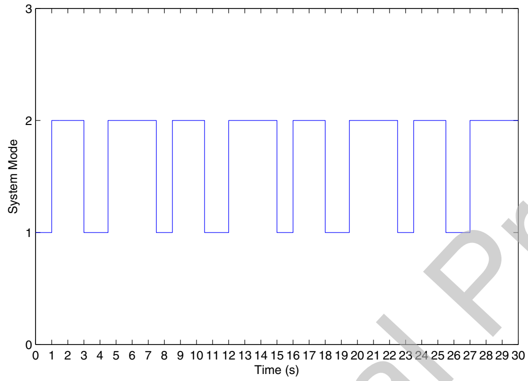

Figure 5: Switching rule $\sigma(t)$ with MDRDT constraint

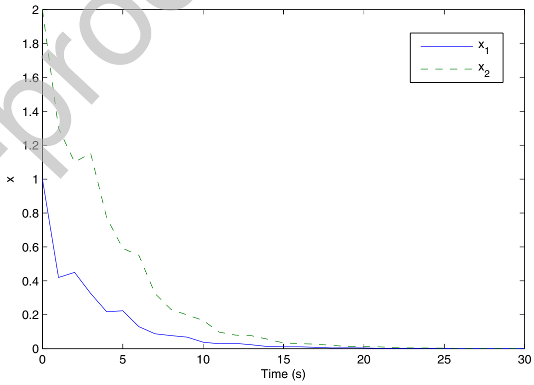

Figure 6: The state $x(t)$ with MDRDT constraint

By analysis, we can verify that $\max_{\forall i \in \underline{2}} \|\overline{A}_{di4}\|_1 \delta = 0.5126 < 1$. *According to Theorem 6, system (2) with (109) is positive and ES with MDRDT constraint meeting:*

$$\begin{cases} 1.0 = \underline{T}_1^* \leq T_{l1} \leq \overline{T}_1^* = 2.0, & l \in Z, \\ 2.0 = \underline{T}_2^* \leq T_{l2} \leq \overline{T}_1^* = 4.0, & l \in Z. \end{cases} \tag{112}$$

The simulation results are shown in Figs. 5-6, where $x(0) = [1 \quad 2]^T$, $x(t) = [0 \quad 0]^T$, $t \in [-0.3, 0)$. *Fig. 5 plots the MDRDT switching signal satisfying (112). From Fig. 5, it is obvious that the dwell-time intervals for Subsystem 1 are* $[0, 1)$, $[3, 4.5)$, $[7.5, 8.5)$, $[10.5, 12)$, $[15, 16)$, $[18, 19.5)$, $[22.5, 23.5)$, $[25.5, 27)$, *and the dwell-time intervals for Subsystem 2 are* $[1, 3)$, $[4.5, 7.5)$, $[8.5, 10.5)$, $[12, 15)$, $[16, 18)$, $[19.5, 22.5)$, $[23.5, 25.5)$, $[27, 30)$, *then it is easy to conclude that the switching rule* $\sigma(t)$ *plotted in Fig. 5 satisfies the MDRDT condition (112). Fig. 6 shows the system's state* $x(t)$, *which indicates the obvious positivity and exponential stability of system (2) with (109). The validity of the results is illustrated.*

Example 4. *A practical application as the traffic control of cross way is given to verify the validity of the provided results. As described in Fig.7, a "triangular connection" is composed of three main roads, i.e., Road A, Road B and Road C, and is controlled by six traffic lights. Buffer variables x_1, x_2 and x_3 represent the vehicles awaiting at the three traffic lights inside the triangular loop.*

Assume that there exist three symmetrical configurations with respect to the states of six traffic lights. For simplicity, we take one phase of configuration as described in Fig.7 for example, where the four traffic lights corresponding to Buffer x_1, Buffer x_2, Road B and Road C are green, and the other two traffic lights corresponding to Buffer x_3 and Road A are red. Thus, the relation between Buffers x_1, x_2 and x_3 can be generalized in the following:

1. *x_3 increases proportionally with x_2, i.e., $\dot{x}_3 = \psi x_2$, $\psi > 0$;*
2. *x_2 almost remains constant, that is, gaining the inflow from B and x_1, and offering the outflow to A and x_3 simultaneously, i.e., $\dot{x}_2 = 0$;*
3. *x_1 decreases exponentially due to that the inflow from C all flows into x_2 and B, i.e., $\dot{x}_1 = -\chi x_1$, $\chi > 0$,*

which approaches the initial transient of the switching of traffic lights. In addition, through the circular rotation of x_1, x_2 and x_3 as well as A, B and C, other two phases of configurations could be got via the similar steps as previously described. Therefore, the traffic signalling system is described as:

$$\begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t), \\ x(t_0) &= x_0, \end{cases} \tag{113}$$

where $\sigma(t) \in \{1, 2, 3\}$,

$$A_1 = \begin{bmatrix} -\chi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \psi & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & \psi \\ 0 & -\chi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ \psi & 0 & 0 \\ 0 & 0 & -\chi \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

Setting $\chi = 1.5$, $\psi = 1.2$, $\alpha = 0.5$, $T_1^* = T_2^* = T_3^* = 1$ and $S = 2$, the stability of traffic system can be solved through the conditions in Theorem 2, and the feasible solutions are

$$\nu_{10} = \begin{bmatrix} 0.4463 \\ 1.0940 \\ 0.6456 \end{bmatrix}, \nu_{11} = \begin{bmatrix} 0.7114 \\ 1.1182 \\ 0.8042 \end{bmatrix}, \nu_{12} = \begin{bmatrix} 1.3520 \\ 0.8297 \\ 1.1026 \end{bmatrix},$$
$$\nu_{20} = \begin{bmatrix} 0.6456 \\ 0.4463 \\ 1.0940 \end{bmatrix}, \nu_{21} = \begin{bmatrix} 0.8042 \\ 0.7114 \\ 1.1182 \end{bmatrix}, \nu_{22} = \begin{bmatrix} 1.1026 \\ 1.3520 \\ 0.8297 \end{bmatrix},$$
$$\nu_{30} = \begin{bmatrix} 1.0940 \\ 0.6456 \\ 0.4463 \end{bmatrix}, \nu_{31} = \begin{bmatrix} 1.1182 \\ 0.8042 \\ 0.7114 \end{bmatrix}, \nu_{32} = \begin{bmatrix} 0.8297 \\ 1.1026 \\ 1.3520 \end{bmatrix}.$$

Based on Theorem 2, system (113) is positive and ES with MDCDT constraint meeting:

$$\begin{cases} T_{l1} \equiv T_1^* = 1.0, & l \in Z, \\ T_{l2} \equiv T_2^* = 1.0, & l \in Z, \\ T_{l3} \equiv T_3^* = 1.0, & l \in Z. \end{cases} \tag{114}$$

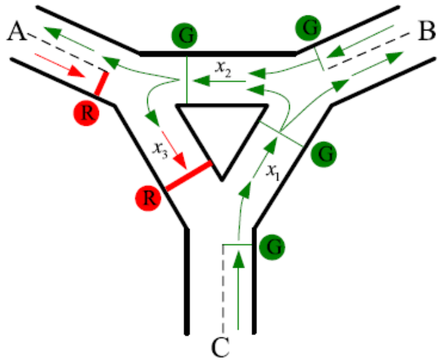

Figure 7: The triangular intersection (Configuration Phase 1)

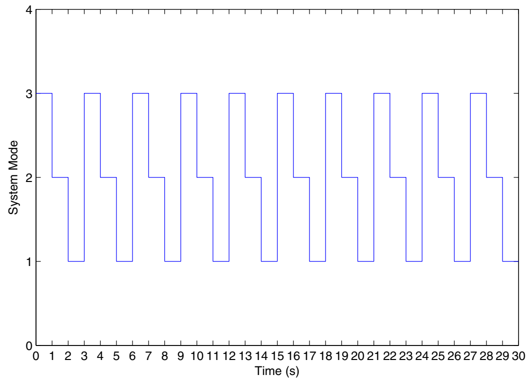

Figure 8: Switching rule $\sigma(t)$

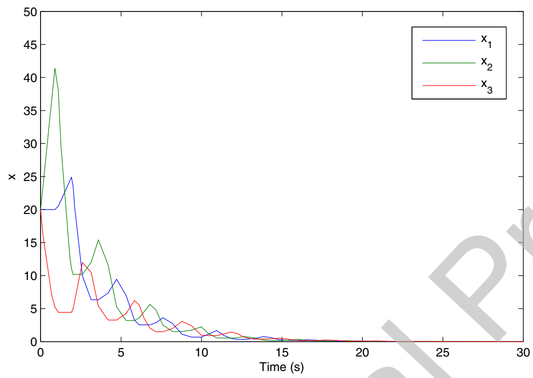

Figure 9: The state $x(t)$

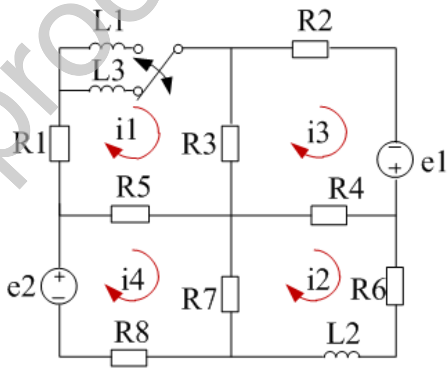

Figure 10: An four-mesh circuit

The simulation results are shown in Figs. 8-9, where $x(0) = [20 \quad 20 \quad 20]^T$. *Fig. 8 plots the MDCDT switching signal satisfying (114). From Fig. 8, it is obvious that the dwell-time intervals for Subsystem 1 are* $[2, 3)$, $[5, 6)$, $[8, 9)$, $[11, 12)$, $[14, 15)$, $[17, 18)$, $[20, 21)$, $[23, 24)$, $[26, 27)$, $[29, 30)$, *the dwell-time intervals for Subsystem 2 are* $[1, 2)$, $[4, 5)$, $[7, 8)$, $[10, 11)$, $[13, 14)$, $[16, 17)$, $[19, 20)$, $[22, 23)$, $[25, 26)$, $[28, 29)$, *and the dwell-time intervals for Subsystem 3 are* $[0, 1)$, $[3, 4)$, $[6, 7)$, $[9, 10)$, $[12, 13)$, $[15, 16)$, $[18, 19)$, $[21, 22)$, $[24, 25)$, $[27, 28)$, *that is to say, the switching signal is a periodic switching signal with a fixed dwell time as* $T_i^* = 1$, $\forall i \in \{1, 2, 3\}$ *and a fixed switching order as* $3 \to 2 \to 1 \to 3 \to 2 \to 1 \cdots$, *then it is easy to conclude that the switching rule* $\sigma(t)$ *plotted in Fig. 8 satisfies the MDCDT condition (114). Fig. 9 show the system's state* $x(t)$, *which indicates the obvious positivity and exponential stability of system (113). It implies that with this kind of signal control strategy, the vehicles of the triangle intersection could be released, and there will be no traffic congestion. The validity of the obtained results is illustrated.*

Example 5. *Another practical application is considered to illustrate the correctness and* *effectiveness of the obtained results. Fig. 10 introduces a practical four-mesh circuit system including resistances, inductances and voltages. In Fig. 10, the switching is among two locations, i.e., inductances L_1 and L_3. In other words, $\sigma(t) = 1$, i.e., subsystem 1 is activated, when inductance L_1 is closed, similarly, $\sigma(t) = 2$, i.e., subsystem 2 is activated, when inductance L_3 is closed. In light of Kirchhoff's law and mesh analysis method, the system model can be expressed as:*

$$\begin{cases} a(\sigma(t))\frac{di_1}{dt} &= -(R_1 + R_3 + R_5)i_1 + R_3 i_3 + R_5 i_4, \\ L_2\frac{di_2}{dt} &= -(R_4 + R_6 + R_7)i_2 + R_4 i_3 + R_7 i_4, \\ 0 &= R_3 i_1 + R_4 i_2 - (R_2 + R_3 + R_4)i_3 + e_1, \\ 0 &= R_5 i_1 + R_7 i_2 - (R_5 + R_7 + R_8)i_4 + e_2, \end{cases}$$

where $a(\sigma(t)) = L_1$ when $\sigma(t) = 1$ and $a(\sigma(t)) = L_3$ when $\sigma(t) = 2$.

Select the state variables $x_p = i_p$, $\forall p \in \{1, 2, 3, 4\}$, the above system model is transformed into

$$E\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),$$

where

$$A_{\sigma(t)} = \begin{bmatrix} -\frac{(R_1+R_3+R_5)}{a(\sigma(t))} & 0 & \frac{R_3}{a(\sigma(t))} & \frac{R_5}{a(\sigma(t))} \\ 0 & -\frac{(R_4+R_6+R_7)}{L_2} & \frac{R_4}{L_2} & \frac{R_7}{L_2} \\ R_3 & R_4 & -(R_2 + R_3 + R_4) & 0 \\ R_5 & R_7 & 0 & -(R_5 + R_7 + R_8) \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{\sigma(t)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x(t) = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}, \quad u(t) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Let $R_q = 1$, $\forall q \in \{1, 2, \cdots 8\}$, $L_1 = 1$ and $L_2 = L_3 = 2$. Given a time-varying delay $d(t) = 0.2 + 0.1sint$, the system turns into

$$E\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t - h(t)) + B_{\sigma(t)}u(t), \tag{115}$$

where

$$A_1 = \begin{bmatrix} -3 & 0 & 1 & 1 \\ 0 & -1.5 & 0.5 & 0.5 \\ 1 & 1 & -3 & 0 \\ 1 & 1 & 0 & -3 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.5 & 0 & 0.5 & 0.5 \\ 0 & -1.5 & 0.5 & 0.5 \\ 1 & 1 & -3 & 0 \\ 1 & 1 & 0 & -3 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Set $u(t) = 0$ because the goal is determining the system's stability. By direct calculation, one has

$$\begin{aligned}
&det(sE - A_1) = 9s^2 + 45s + 63 \neq 0, \quad for \quad s = 0,\\
&det(sE - A_2) = 9s^2 + 48s + 67.5 \neq 0, \quad for \quad s = 0,\\
&deg(det(sE - A_1)) = deg(9s^2 + 45s + 63) = rankE = 2,\\
&deg(det(sE - A_2)) = deg(9s^2 + 48s + 67.5) = rankE = 2.
\end{aligned}$$

Then, the system is regular and impulse-free, and one gets from the definitions in system (6) that

$$\begin{aligned}
&\overline{A}_{11} = \begin{bmatrix} -2.3333 & 0.6667 \\ 0.3333 & -1.1667 \end{bmatrix}, \quad \overline{A}_{d11} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \overline{A}_{d12} = \begin{bmatrix} 0.0333 & 0.0667 \\ 0.0167 & 0.0333 \end{bmatrix},\\
&\overline{A}_{13} = \begin{bmatrix} 0.3333 & 0.3333 \\ 0.3333 & 0.3333 \end{bmatrix}, \quad \overline{A}_{d13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_{d14} = \begin{bmatrix} 0.0333 & 0 \\ 0 & 0.0667 \end{bmatrix},\\
&\overline{A}_{21} = \begin{bmatrix} -1.1667 & 0.3333 \\ 0.3333 & -1.1667 \end{bmatrix}, \quad \overline{A}_{d21} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \overline{A}_{d22} = \begin{bmatrix} 0.0333 & 0.0167 \\ 0.0333 & 0.0167 \end{bmatrix},\\
&\overline{A}_{23} = \begin{bmatrix} 0.3333 & 0.3333 \\ 0.3333 & 0.3333 \end{bmatrix}, \quad \overline{A}_{d23} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_{d24} = \begin{bmatrix} 0.0667 & 0 \\ 0 & 0.0333 \end{bmatrix}.
\end{aligned}$$

Hence, system (115) is positive in light of Lemma 1.

Set $\alpha = 0.5$, $d(t) = 0.2 + 0.1sint$, $T_1^* = 1.0$, $T_2^* = 2.0$ and $S = 2$. Solving the conditions in Theorem 4 leads to

$$\begin{aligned}
&\nu_{10} = \begin{bmatrix} 3.6417 \\ 8.6531 \\ 2.8349 \\ 2.7961 \end{bmatrix}, \nu_{11} = \begin{bmatrix} 5.6227 \\ 11.5989 \\ 4.4266 \\ 4.3966 \end{bmatrix}, \nu_{12} = \begin{bmatrix} 8.4238 \\ 13.9687 \\ 6.5715 \\ 6.5408 \end{bmatrix},\\
&\nu_{20} = \begin{bmatrix} 6.0991 \\ 6.3337 \\ 2.3012 \\ 2.3224 \end{bmatrix}, \nu_{21} = \begin{bmatrix} 9.9707 \\ 10.1033 \\ 3.8823 \\ 3.8943 \end{bmatrix}, \nu_{22} = \begin{bmatrix} 12.9059 \\ 14.3891 \\ 5.7505 \\ 5.7627 \end{bmatrix}, \vartheta = \begin{bmatrix} 10.3882 \\ 11.9323 \\ 8.9379 \\ 8.9658 \end{bmatrix},
\end{aligned}$$

According to Theorem 4, system (115) is positive and ES with MDMDT constraint meeting:

$$\begin{cases} T_{l1} \geq T_1^* = 1.0, & l \in Z, \\ T_{l2} \geq T_2^* = 2.0, & l \in Z. \end{cases} \tag{116}$$

The simulation results are shown in Figs. 11-12, where $x(0) = [0.2 \quad 0.4 \quad 0.6 \quad 0.8]^T$, $x(t) = [0 \quad 0 \quad 0 \quad 0]^T$, $t = [-0.3, 0)$. Fig. 11 plots the MDMDT switching signal satisfying (116). From Fig. 11, it is obvious that the dwell-time intervals for Subsystem 1 are $[0, 1)$, $[3, 4.5)$, $[7, 8)$, $[10, 11.5)$, $[14, 15)$, $[17, 18.5)$, $[21, 22)$, $[24, 25)$, $[27, 28)$, and the dwell-time intervals for Subsystem 2 are $[1, 3)$, $[4.5, 7)$, $[8, 10)$, $[11.5, 14)$, $[15, 17)$, $[18.5, 21)$, $[22, 24)$, $[25, 27)$, $[28, 30)$, then it is easy to conclude that the switching rule $\sigma(t)$ plotted in Fig. 11 satisfies the MDMDT condition (116). Fig. 12 shows the systems state $x(t)$, which indicates the obvious positivity and exponential stability of system (115). The validity of the results is illustrated.

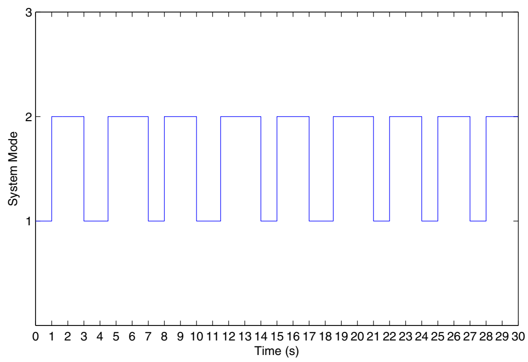

Figure 11: Switching rule $\sigma(t)$

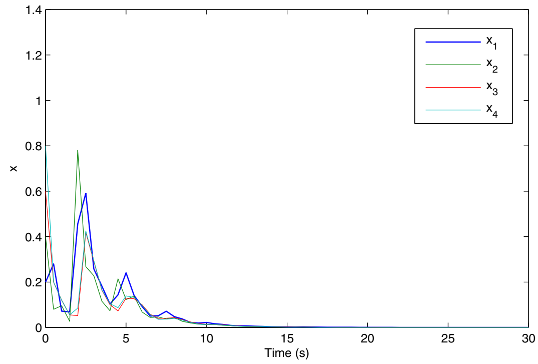

Figure 12: The state $x(t)$

Table 2: The feasibility results for system (117) with $\alpha = 0.5$

	$T^* = 0.2$	$T^* = 0.4$	$T^* = 0.6$	$T^* = 0.8$	$T^* = 1.0$	$T^* = 1.2$	$T^* = 1.4$
$S = 1$					*	*	*
$S = 2$				*	*	*	*
$S = 3$			*	*	*	*	*
$S = 4$			*	*	*	*	*

Example 6. *A comparison is provided to illustrate the affect of various α, S and T_i^* on the feasibility of resulting solutions, and the MDCDT case is taken into account here for simplicity. Consider system (1) with two modes, i.e., $\sigma(t) \in \{1, 2\}$,*

$$A_1 = \begin{bmatrix} -3.5 & 1.6 \\ 1.5 & -1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3.4 & 1.5 \\ 1.6 & -1.6 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{117}$$

The stability issue of system (1) with (117) can be solved via Theorem 2 in this paper.

On the basis of Theorem 2, by selecting various α, S and T_i^*, the corresponding feasibility results are presented in Tables 2-3 below, where the symbol '$*$' implies that one can seek out the feasible solution, and the 'blank' implies the opposite.

Firstly, Table 2 presents the relation among S and the resulting feasibility results when $\alpha = 0.5$, and for simplicity, we set $T_1^* = T_2^* = T^*$ here. As indicated in Table 2, the larger the scalar S is, the wider the range of corresponding feasible solutions is. Moreover, for a fixed S, Table 2 shows that if a larger constant dwell time T^* is selected, a corresponding feasible solution is easier to be found. Furthermore, Table 3 presents the relation among α and the resulting feasibility results when $S = 2$ and $T_1^* = T_2^* = T^* = 2.0$. Table 3 shows that if a smaller exponential DR α is selected, a corresponding feasible solution is easier to be found.

Example 7. *This example is given to illustrate the results in Remark 7. Consider system*

Table 3: The feasibility results for system (117) with $T^* = 2$, $S = 2$

α	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
	*	*	*	*	*			

(1) with two modes, i.e., $\sigma(t) \in \{1, 2\}$,

$$A_1 = \begin{bmatrix} -2 & 1 \\ 5 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{118}$$

where the subsystem 1 is asymptotically stable while the subsystem 2 is anti-stable.

Solve this stability issue via Theorems 1 and 3 in this paper, respectively. Feasible solutions can not be obtained by the MDMDT conditions in Theorem 1, while could be obtained via the MDRDT conditions in Theorem 3 in the case that $\alpha = 0.1$, $d(t) = 0.2 + 0.1sint$, $\underline{T}_1^* = 2$, $\overline{T}_1^* = 4$, $\underline{T}_2^* = 1$, $\overline{T}_2^* = 2$ and $S = 2$. Settling the conditions of Theorem 3 leads to

$$\begin{aligned} \nu_{10} &= \begin{bmatrix} 0.1126 \\ 0.0325 \end{bmatrix}, \nu_{11} = \begin{bmatrix} 0.1944 \\ 0.0653 \end{bmatrix}, \nu_{12} = \begin{bmatrix} 0.2486 \\ 0.0857 \end{bmatrix}, \\ \nu_{20} &= \begin{bmatrix} 0.1932 \\ 0.0066 \end{bmatrix}, \nu_{21} = \begin{bmatrix} 0.1498 \\ 0.0331 \end{bmatrix}, \nu_{22} = \begin{bmatrix} 0.1131 \\ 0.0357 \end{bmatrix}. \end{aligned}$$

Based on Theorem 3, system (1) with (118) is positive and ES with MDRDT constraint meeting:

$$\begin{cases} 2.0 = \underline{T}_1^* \leq T_{l1} \leq \overline{T}_1^* = 4.0, & l \in Z, \\ 1.0 = \underline{T}_2^* \leq T_{l2} \leq \overline{T}_1^* = 2.0, & l \in Z. \end{cases} \tag{119}$$

Therefore, compared with the MDMDT condition, the MDRDT condition is applicable to a wider range for these specific cases.

Remark 9. *Noted that all the conditions obtained in this paper are just sufficient, while the necessity of the obtained conditions is an open problem and is left for future research. Seeking out less conservative conditions via new approaches deserves further investigation.*

6. Conclusions

The paper is focused on the exponential stability analysis for SSPSs without/with time-varying delays under MDMDT, MDCDT and MDRDT constraints, respectively. To analyze the dwell-time stability, a novel DLCLF technique is introduced with the main advantage as its affine dependence on conditions of system matrix, which will be extended and utilized to systems with uncertainties and time-varying parameters quite easily. Meanwhile, the proposed exponential stability conditions under MDMDT(or MDCDT or MDRDT) constraint could be degenerated into the one under MDT(CDT or RDT) case for some specific situations, that is, the considered MDMDT(or MDCDT or MDRDT) constraint is more general and practical. Finally, the validity and significance of the results are illustrated by seven examples.

Based on the discussions in Subsection 3.4, future work is in progress to generalize the obtain theory to uncertain systems, nonlinear systems and stabilization design. From the other side, in the future work, we are also interested in investigating more practical and significant state estimation issues such as observation and filtering issues for SSPSs with dwell-time constraints.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.