Super-Resolution with Total-Variation Minimization

Bachelorarbeit

zur Erlangung des Grades

Bachelor of Science

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vorgelegt von

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Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den 27. März 2018

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Unterschrift

Deutsche Zusammenfassung

Es ist oft von großem Interesse, feine Details von einem Signal in einem Maßstab außerhalb der von verfügbaren Messungen unterstützten Auflösung zu untersuchen. Dies führt zu einem wichtigen Problem, nämlich dem Super-Resolution-Problem, wobei man im Allgemeinen versucht, hochaufgelöste Informationen aus Messungen vom groben Maßstab zu rekonstruieren. Dieses Problem tritt in vielen praktischen Bereichen der Wissenschaft auf, wie z.B. Astronomie, medizinische Bildverarbeitung, Geophysik, Mikroskop usw.

In dieser Bachelorarbeit geht es darum, eine mathematische Einführung zum Super-Resolution-Problem aufzubauen. Dazu wird im ersten Teil der Arbeit der theoretische Grundstein gelegt: der Begriff der komplexen Maße wird vorgestellt und ein ganz kurzer Überblick über konvex Optimierung und Compressed Sensing wird gegeben. Danach fängt der eigentliche Teil der Arbeit an.

Ein natürliches mathematisches Modell zu diesem Problem wird im Kapitel 3 vorgestellt: wir untersuchen ein Signal in der Form

$$x = \sum_j a_j \delta_{t_j},$$

wobei $\{t_j\}$ Stellen in $[0,1]$ sind, δ_t ein Diracmaß ist und $\{a_j\}$ komplexwertige Amplituden sind. Die Informationen, die uns zur Verfügung stehen, sind die niedrigsten $n = 2f_c + 1$ Koeffizienten der Fourierreihe:

$$y(k) = \int_0^1 e^{-i2\pi kt} dx(t) = \sum_j a_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z}, |k| \leq f_c, \tag{*}$$

hier bezeichnet die positive ganze Zahl f_c die Abschneidefrequenz. Diese Frequenz-Abschneide verursacht eine zu $\lambda_c := 1/f_c$ proportionale Resolutionsbegrenzung. Man kann (*) umformulieren als $x = F_n y$, wobei F_n die n niedrigste Fourier-Koeffizienten sammelnde lineare Abbildung ist. Um x aus y zurückzubekommen lösen wir das folgende konvexe Optimierungsproblem:

$$\min_z \|z\|_{TV} \quad \text{sodass} \quad F_n x = y; \tag{P_{TV}}$$

wobei $\|\cdot\|_{TV}$ die Totalvariation eines komplexen Maßes bezeichnet und die Minimierung über die Menge aller endlichen komplexen Maßen auf $[0,1]$ (versehen mit der Borel'schen σ-Algebra) ausgeführt werden soll. Unter der Annahme, dass die Stellen t_j ausreichend getrennt sind, zeigt unser Resultat, dass die Lösung des Konvexprogramms P_{TV} genau das Originalsignal x ist. Um die Trennungsannahme klarer zu machen, führen wir den Begriff Minimumseparation ein:

Definition. *Wir identifizieren die Endpunkte von* $[0,1]$ *um einen Kreis* $\mathbb{T}$ *zu erhalten. Dann wird der* wrap-around Abstand *zwischen zwei Punkte* $t, t' \in \mathbb{T}$ *with* $t < t'$ *definiert durch*

$$|t - t'|_w := \min\{t' - t, 1 + t - t'\}.$$

Für eine Familie T von Punkten in $\mathbb{T}$ definieren wir ihre Minimumseparation *als den kleinsten Abstand zwischen zwei beliebigen Elementen in T,*

$$\Delta(T) := \inf_{\substack{t,t' \in T: \\ t \neq t'}} |t - t'|_w$$

Nun lautet das Hauptresultat wie folgt:

Satz. *Erfüllt der Träger $T = \{t_j\} \subset [0,1]$ des Signals x die folgende sogenannte Minimumseparation-Bedingung,*

$$\Delta(T) \geq 2/f_c = 2\lambda_c,$$

und ist $f_c \geq 128$, so ist x die eindeutige Lösung von P_{TV}. Wenn x bekannt ist als ein reelles Signal, dann kann sich die Minimumseparation auf $1.87\lambda_c$ verringern.

Der Beweis dieses Resultats wird in mehrere Teile aufgeteilt und diese werden in Abschnitt 3.4 vorgestellt. Die Idee besteht darin, dass man zu jedem die Minimumseparation-Bedingung erfüllenden Signal x die Existenz eines low-frequency (d.h. Grad $\leq f_c$) trigonometrischen Polynoms p zeigt, das das Signum von x interpoliert, d.h. $p(t_j) = \operatorname{sgn}(a_j)$. Um ein solches Polynom zu erstellen interpolieren wir das Signum des Signals mit einem low-frequency Kern, in diesem Fall also dem Quadrat des Fejér-Kerns

$$K(t) = \left[\frac{\sin\left(\left(\frac{f_c}{2} + 1\right)\pi t\right)}{\left(\frac{f_c}{2} + 1\right)\sin(\pi t)} \right]^4, \quad 0 < t < 1,$$

and $K(0) = 1$. In Abschnitt 3.5 wird diskutiert, wie man dieses Resultat auch zu Mehrdimensionalen und zu anderen Typen von Signalen erweitern kann.

Es ist nicht schwierig zu erkennen, dass die Totalvariationsnorm gesehen werden kann als die verallgemeinerte stetige Version der ℓ_1-Norm und somit lässt sich unser Hauptresultat auch mit Resultaten in Compressed Sensing vergleichen, wo man bei der Wiederherstellung des Signals die *Basis Pursuit*, ein ℓ_1-Minimierungsproblem, betrachtet. In diesem Zusammenhang hat unser Modell einen großen Vorteil, nämlich die Zeitkontinuierlichkeit. Tatsächlich kann man leicht eine diskrete Version des Resultats ableiten, indem man die Stellen t_j in die Gitter $\{j/N : j = 0, \ldots, N-1\}$ bringt. Das bedeutet, wir betrachten ein Signal $x = \{x_j\}_{j=0}^{N-1} \in \mathbb{C}^N$ mit den low-frequency Daten $y = F_n x$, wobei F_n die zu den niedrigen Koeffizienten gehörige partielle Fouriermatrix ist, d.h.

$$y_k = [F_n x](k) = \sum_{j=0}^{N-1} x_j e^{-i2\pi k \frac{j}{N}}, \quad |k| \leq f_c.$$

Diese Beobachtung ergibt sich das folgende Korollar:

Korollar. *Erfüllt der Träger $T \in \{0, \ldots, N-1\}$ eines Signals $x = \{x_j\}_{j=0}^{N-1} \in \mathbb{C}^N$ die (umgestellte) Minimumseparation - Bedingung,*

$$\min_{\substack{t,t' \in T: \\ t \neq t'}} \left| \frac{t - t'}{N} \right|_w \geq 2\lambda_c,$$

und $f_c \geq 128$, so ist x die eindeutige Lösung zum folgenden Minimierungsproblem

$$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \textit{sodass} \quad F_n z = y. \tag{P_{ℓ_1}}$$

Der wesentliche Unterschied zwischen dem Super-Resolution-Problem und dem typischen Problem von Compressed Sensing (vorgestellt in Abschnitt 2.3) liegt darin, dass die Matrix F_n nicht frei oder zufällig gewählt werden darf, sondern schon festgelegt wird als eine partielle Fouriermatrix, die den niedrigen Frequenzen entspricht. Eine weitere auffällige Tatsache ist, dass unser Resultat in der diskreten Form, also das obige Korollar, die Minimumseparation-Bedingung fordert statt der Sparsity (sparse $\sim$ dünn-besetzt), in vielen Fällen ist die Minimumseparation-Bedingung aber stärker (aus dieser Bedingung folgt, dass das Signal $n/4$-sparse ist). Auf diese Bedingung kann man bei der praktischen Super-Resolution trotzdem nicht verzichten: nur unter der Voraussetzung der Sparsity gibt es Signale mit ℓ_2-Norm gleich 1, die durch den Messprozess fast komplett unterdrückt werden, d.h die zugehörigen Messungsvektoren sind fast gleich 0 (z.B. für $N = 4096$, $n = 1024$ gibt es 48-sparse Vektoren $x \in \mathbb{C}^N$ mit $\|x\|_2 = 1$ sodass $\|y\|_2 \approx 2.3 \times 10^{-67}$). Die Begründung dafür wird im Abschnitt 4.2 geliefert, nachdem wir kurz im Abschnitt 4.1 über die Stabilität unserer Methode diskutiert haben.

Unser zu betrachtendes Optimierungsproblem (P_{TV}) dient nicht nur zur theoretischen Seite, es ist tatsächlich auch numerisch lösbar. Dazu betrachten wir sein Dualproblem, was zu einer äquivalenten endlichdimensionalen Konvexproblem überführt werden kann. Die Lösung dieses Problems liefert ungefähr den Träger des Originalsignals, die Amplituden sind dann durch das Lösen des zugehörigen linearen Gleichungssystems zu finden.

Contents

Chapter 1

Introduction

{ch:Intro}

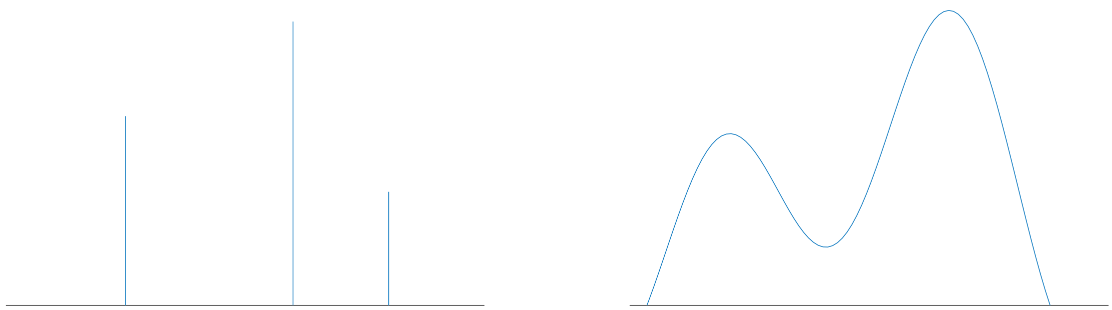
In many applications of practical interest, one needs to recover an object, e.g. a signal or an image, from its partial Fourier samples. Super-resolution is one of those recovery processes, in which the fine scale structure of the object is retrieved from only the coarse scale features. In other words, this includes techniques that enhance the resolution of a sensing system. Super-resolution is motivated by the fact that in a sensing system there usually is a physical limit on the highest possible resolution (for an imaging device the spatial resolution may be measured by how closely lines can be resolved). In electrical imaging limitations come from the lens and the size of the sensors, e.g. pixel sizes. Here, due to photon shot noise which reduces image quality when pixels are made smaller, there is an inflexible limit to the effective resolution of the whole system. For an optical system, diffraction is well known as the crucial reason for limitation on resolution. In microscopy, this is refered to as the Abbe diffraction, which is an essential obstacle to observing sub-wavelength structures. This explains why resolving sub-wavelength features is a crucial challenge in microscopy [18], astronomy [16], medical imaging [11], etc. There are also other fields where it is desirable to extrapolate fine scale details from low-resolution data, or resolve sub-pixel details: spectroscopy [12], radar [19], non-optical medical imaging [15], geophysics [17]. For a survey of super-resolution techniques in imaging, see [1] and the references therein.

Consider the following mathematical model: start with an object $x(t)$ (in two-dimensional case $t = (t_1, t_2)$), which is measured by an optical instrument with the *point-spread function* $h(t)$. This instrument acts as a filter in the sense that we may observe samples from the convolution product $y = x*h$ or equivalently, in the frequency domain, by convolution theorem,

$$\hat{y}(\omega) = \hat{x}(\omega)\hat{h}(\omega).$$

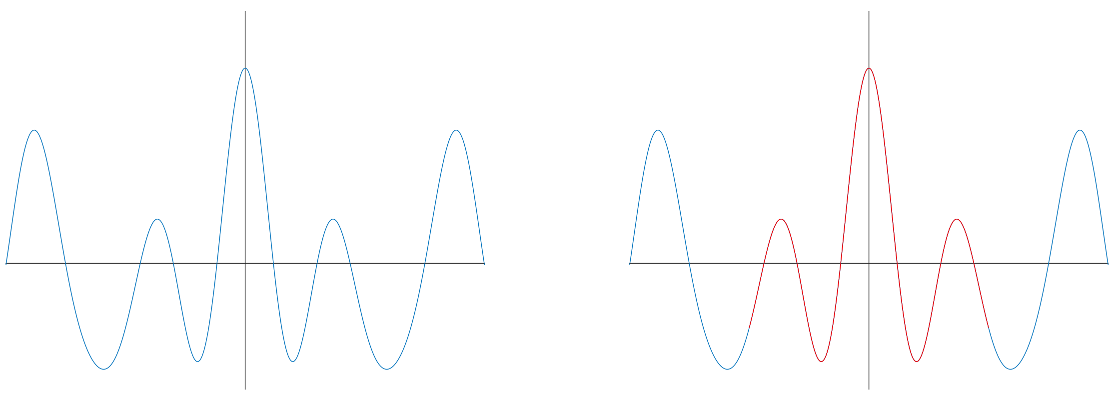
Here, $\hat{x}$ denotes the Fourier transform of x and $\hat{h}$ is the *modulation transfer function* (or simply *transfer function*) of the instrument. Common optical instruments act as low-pass filters, i.e. their transfer function $\hat{h}(\omega)$ vanishes for all $|\omega| > \Omega$, where Ω is a frequency cut-off (one can simply imagine it as the indicator function of the set corresponding to $|\omega| \leq \Omega$). Now the super-resolution problem consists in recovering the fine-scale, or equivalently the high-frequency features of x, which have been killed by the measurement process. This is schematically represented in Figure 1.1: the signal on the left should be recovered from its low-frequency information on the right. Viewed in the frequency domain, the super-resolution problem consists in extrapolating the high-end, or the missing part, of the spectrum from the low-end part, as seen in Figure 1.2. Compared to a typical compressed sensing problem [7], in super-resolution one wishes to extrapolate the spectrum instead of interpolate.

As a simple model, we choose to study a signal in the form of a spike-train supported within $[0, 1]$ as in



(a) Highly resolved signal. **(b)** Low-pass data of the signal.

Figure 1.1: Super-resolution is about recovering the signal in (a) by deconvolving the data in (b). {fig: Super_res



(a) Real part of the Fourier transform of the signal. **(b)** In red, the portion of the observed spectrum.

Figure 1.2: Super-resolution is about extrapolating the red fragment to recover the whole curve. {fig: Super-res

Figure 1.1. Intuitively, if we create a grid $\{j/N\}$ on the unit interval, where $j = 0, \ldots, N-1$ denotes the indexes of the entries and N is the length of the signal vector as in typical compressed sensing problem (e.g. as in Section 2.3), and let $N \to \infty$, then this should be able to approximate the support of the spikes. In this context, our model is an extension of the typical discrete setting, whereby one recovers the signal in the form of a sparse vector by solving the ℓ_1 norm minimization, or equivalently, the basis pursuit. In fact, the sparsity condition may be interpreted in the continuous-time model that there are just a few spikes whose amplitudes are non-zero. Under the additional assumption that the discrete signal is real valued and nonnegative, [6] and [13] show that k spikes can be recovered from $2k+1$ Fourier coefficients by solving the basis pursuit. This result is extended in [3] to the continuous setting by using the total-variation minimization.

However, for many applications, one has to process with complex valued signals. Hence, in this thesis, we intend to construct a model which also allows this kind of signals. As a consequence, we require a different assumption but only about the support of the signals (not their amplitudes), which avoids the clustering of the spikes. This assumption, however, is a necessary and natural assumption that one cannot avoid since the measurements in super-resolution are fixed to be the low-pass filtering, instead of being free to choose as in the ordinary compressed sensing problem. This is also the main reason why the results in typical compressed sensing cannot be applied in our situation and super-resolution problem needs to be considered carefully.

Actually, the assumption about the separation of the spikes is not new, it is already suggested in the work of Donoho [4] regarding the robustness of super-resolution in presence of noise. While super-resolution for general sparse signals is hopelessly ill-posed, the work suggests that there is still hope to super-resolve spread-out signals. To make this become reality, we propose solving a convex optimization problem, which is analogous to the basis pursuit. In other words, among many methods of super-resolution, the thesis will discuss the optimization-based one.

Outline. In this thesis we would like to develope a mathematical theory of super-resolution. Firstly, Chapter 2 will present the theoretical basics that we need to understand the results and their proofs. In Chapter 3, we introduce our mathematical model of super-resolution, namely a superposition of pointwise events (as in Figure 1.1), and the main result, stating that we can recover such a signal exactly from low-frequency samples by convex optimization under the condition that the distance between the spikes is on the order of the resolution limit. This result can also be extended to other models or to higher dimensions. Afterwards, Chapter 4 discusses the stability of the minimization procedure used in the result and Chapter 5 shows how this can be numerically implemented as a semidefinite program.

Chapter 2

Background

{ch:Backgro

In this chapter, we want to introduce the basic knowledge that is necessary for this thesis. First, we extend the concept of measures to signed and complex measures, which are important for the theoretical part of the thesis. Then, we get a brief overview of convex optimization problems, which we need to bring our results to practice. Finally, we have a look at some basic results of compressed sensing, which later will be compared with our results.

2.1 Signed measures and complex measures

{section:

Throughout this section let (X, Σ) be a measurable space. First, we present the concepts of signed measures and complex measures on this measureable space and then, we will collect some results regarding this concepts.

2.1.1 Signed measures and complex measures

For our models later we need the notion of complex measures, i.e. the measures on (X, Σ) with values in $\mathbb{C}$. Broadly speaking, each complex measure is a combination of two signed measures, where a signed measure is in turn a combination of two (classical) measures. First let us introduce the formal definition of each of these concepts:

Definition 2.1. *[2, Chapter 4.1]*

(i) A function $\mu : \Sigma \to \overline{\mathbb{R}} := \{\mathbb{R}, \pm\infty\}$ is called a signed measure *on (X, Σ) if it is σ-additive, i.e. for any sequence $(A_n) \subset \Sigma$ of pairwise disjoint subsets of X we have*

$$\mu\left(\bigcup_n A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

(more precisely: $\mu\left(\bigcup_n A_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) \in \mathbb{R}$ if there is no expressions of the form "$\infty - \infty$" in the sum).

(ii) A function $\mu : \Sigma \to \mathbb{C}$ is called a complex measure *on (X, Σ) if it is σ-additive, i.e. for any sequence $(A_n) \subset \Sigma$ of pairwise disjoint subsets of X we have*

$$\mu\left(\bigcup_n A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

(more precisely: $\mu\left(\bigcup_n A_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) \in \mathbb{C}$ if there is no expressions of the form "$\infty - \infty$" in the sum).

Remark 2.2. *(i) The σ-additivity implies $\mu(\emptyset) = 0$.*

(ii) In the definition of complex measures, we have restricted them to be finite*, i.e. they cannot have infinite values.*

The next theorem and its corollary shows the relation between signed measures and positive measures, as mentioned before a signed measure can be decomposed into two positive measures:

Theorem 2.3 (Hahn Decomposition)**.** *[2, Theorem 4.1.5] Let μ be a signed measure on (X, Σ). Then there exist $X_-, X_+ \in \Sigma$ with the following properties:*

(i) $X_- \cap X_+ = \emptyset$ and $X_- \cup X_+ = X$

(ii) $\mu(A) \leq 0$ for every $A \in \Sigma$ with $A \subset X_-$

(iii) $\mu(A) \geq 0$ for every $A \in \Sigma$ with $A \subset X_+$.

dan decomp} **Corollary 2.4** (Jordan decompostion)**.** *[2, Corollary 4.1.6] Every signed measure μ on (X, Σ) has a unique decomposition into $\mu = \mu_+ - \mu_-$ where μ_+, μ_- are (not negative) measures on (X, Σ) with the following properties:*

(i) At least one of the measures μ_+ and μ_- is finite

(ii) For each Hahn decomposition (X_-, X_+) we have for all $A \in \Sigma$ that

$$\mu_+(A) = \mu(A \cap X_+) \quad \text{and} \quad \mu_-(A) = \mu(A \cap X_-).$$

Now we come to the notions variation and total variation of a signed measure and then of a complex measure. One can define the variation and total variation of a signed measure using its Jordan decomposition as in the proposition below:

Proposition 2.5. *Let μ be a signed measure on (X, Σ) with the Jordan decomposition (μ_-, μ_+). Then $|\mu| := \mu_+ + \mu_-$ is a measure called* variation *of μ. Furthermore, the number $\|\mu\|_{TV} := \mu(X)$ is called the* total variation *of μ. For any $A \in \Sigma$ we have*

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{n} |\mu(A_k)| : A_1, ..., A_n \subset A \text{ mutually disjoint}\right\}. \tag{2.1}$$

As mentioned at the begin of the section, for a set function $\mu : \Sigma \to \mathbb{C}$ we have that μ is a complex measure if and only if $Re(\mu)$ and $Im(\mu)$ are finite signed measures. However, one easily realizes that the definition above of variation and variation using the Jordan decomposition cannot be extended to the complex measures. Fortunately, the equality (2.1) can be used to define the variation of a complex measure, and its total variation respectively.

Definition 2.6 (Variation, total variation). *Let μ be a complex measure on (X,Σ). Then the* variation $|\mu|$ *of μ is defined by*

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{n} |\mu(A_k)| : A_1, ..., A_n \subset A \text{ mutually disjoint}\right\} \qquad \text{for } A \in \Sigma.$$

Moreover, the total variation of $\|\mu\|_{TV}$ of μ is given by $|\mu|(X)$.

Indeed, one can check that the variation $|\mu|$ of a complex measure is a finite measure on (X,Σ) (note that we have restricted complex measures to be finite). Also, the set of complex measures on (X,Σ) is a Banach space as claimed in the following theorem:

Theorem 2.7. *[24, Section I.1] The set of complex measures $\mathcal{B}(X,\Sigma)$ on a measurable space (X,Σ) is a Banach space w.r.t the total variation norm $\|\cdot\|_{TV}$.* {Theo: com

So far we have introduced the concepts signed and complex measures. Of course one can integrate with respect to these new set functions. Let μ be a signed measure with the Jordan decomposition $\mu = \mu^+ - \mu^-$ and f be a numerical measurable function (i.e. the value "∞" is allowed) on (X,Σ). Under the condition

$$\int |f|\, d|\mu| = \int |f|\, d\mu^+ + \int |f|\, d\mu^- < \infty,$$

the integral of f on some arbitrary set $A \in \Sigma$ is defined by

$$\int_A f d\mu := \int_A f d\mu^+ - \int_A f d\mu^-.$$

Furthermore, for a complex measure $\mu = \mu_1 + i\mu_2$, where μ_1 and μ_2 are finite signed measures, and for a function f as before we define

$$\int_A f d\mu := \int_A f d\mu_1 + i \int_A f d\mu_2.$$

2.1.2 Some useful results about complex measures

This section collects some necessary results about complex measures, most of them are just natural extensions from results about (positive) measures. For simplicity, in some of those results we do not consider their most general setting but just the setting, which fits our application later. Let us begin with a result in functional analysis, which characterizes the dual space of $\mathcal{C}(X)$, i.e. the space of continuous complex-valued functions on X, if X is a compact metric space. Indeed, it is isometrically isomorphic to a subspace of $\mathcal{B}(X,\Sigma)$ (i.e. the space of all complex measures on (X,Σ)), namely the subspace of all regular Borel measures.

Definition 2.8. *Let X be a metric space and Σ be the associated Borel σ-algebra. A positive, signed or complex measure on (X,Σ) is called a* Borel measure. *Furthermore, a positive Borel measure μ on X is called* regular *if*

(i) $\mu(C) < \infty$ for every $C \subset X$ compact,

(ii) for every $A \in \Sigma$ it holds

$$\mu(A) = \sup\{\mu(C) : C \subset A,\, C \text{ compact}\} = \inf\{\mu(O) : A \subset O,\, O \text{ open}\}$$

Finally, a signed measure or a complex measure on (X,Σ) is called regular *if its variation is regular. We use the notation $\tilde{\mathcal{B}}(X)$ to denote the space of all complex regular Borel measure on X.*

{example: dirac

Example 2.9. *A Dirac measure δ_x with $x \in X$ is obviously a regular Borel measure. Hence, a linear combination of Dirac measures is also a regular Borel measure.*

Now we have the all the necessary ingredients to understand the following theorem:

{Theo: Werner_D

Theorem 2.10. *[24, Theorem 2.15] Let X be a compact metric space. Then $(\mathcal{C}(X), \|\cdot\|_\infty)' \cong (\mathcal{M}(X), \|\cdot\|_{TV})$ i.e. the spaces are isometrically isomorphic, under the mapping*

$$T : \mathcal{M}(X) \to \mathcal{C}(X)', (T\mu)(f) = \int_X f d\mu.$$

After a beautiful result from functional analysis, let us we come back to measure theory with a very natural extension of the well-known Radon-Nikodym Theorem. But before that, we need to extend the notion of absolute continuity to complex measures. Analogously as for positive measures, one can define the absolute continuity of a signed or complex measure w.r.t. a positive measure as follows:

Definition 2.11 (Absolute continuity)**.** *Let ν be a signed or complex measure and μ a σ-finite (positive) measure on the same measurable space (X, Σ). Then ν is called* absolutely continuous *w.r.t. μ (in notation $\nu \ll \mu$), if its variation $|\nu|$ is absolutely continuous w.r.t. μ (that is $|\nu| \ll \mu$).*

Recall that $|\nu| \ll \mu$ means that for every set $A \in \Sigma$ with $\mu(A) = 0$ we also have $|\nu|(A) = 0$. Observe that this happens if and only if the measures appearing in the Jordan decomposition of ν (i.e. ν_+, ν_- if $\nu = \nu_+ - \nu_-$ is a signed measure, or ν_j, $j = 1, 2, 3, 4$, if $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ is a complex measure) are all absolutely continuous w.r.t. μ. Hence one can easily extend the classical Radon-Nikodym theorem to signed or complex measures as follows:

on-Nikodym}

Theorem 2.12 (Radon-Nikodym)**.** *Let μ be a positive σ-finite measure and ν be a finite signed or complex measure on (X, Σ). Then ν is absolutely continuous w.r.t. μ if and only if there exists some $f \in \mathcal{L}^1(X, \Sigma, \mu)$, that is a measurable function f on (X, Σ) with $\int_X |f| d\mu < \infty$ such that*

$$\nu(A) = \int_A f d\mu \quad \text{for every } A \in \Sigma. \tag{2.2}$$

Moreover, this function is unique up to μ-null sets.

Proof. By the observation above we decompose $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ and use the Radon-Nikodym theorem for each positive measure ν_j, $j \in \{1, 2, 3, 4\}$. For the uniqueness let f, g be two density functions. Plugging $A := \{x \in X : \operatorname{Re} f(x) > \operatorname{Re} g(x)\}$ in (2.2) we get $\operatorname{Re} f(x) \leq \operatorname{Re} g(x)$ for μ-almost every $x \in X$. Analogously $\operatorname{Re} f(x) \geq \operatorname{Re} g(x)$ and hence $\operatorname{Re} f(x) = \operatorname{Re} g(x)$ for μ-almost every $x \in X$. The same holds for $\operatorname{Im} f$ and $\operatorname{Im} g$, therefore $f = g$ μ-almost everywhere on X. $\square$

The next important result in this section is the Lebesgue decomposition theorem, at first we need the notion of singularity of measures.

Definition 2.13. *The signed or complex measures μ, ν on (X, Σ) are called* mutually singular *(in notation $\mu \perp \nu$, one also says ν is singular w.r.t. μ or vice versa), if they concentrated on disjoint measurable subsets, i.e. there exists some set $A \in \Sigma$ with $|\mu|(A^c) = 0$ and $|\nu|(A) = 0$.*

Further to the notion "concentrated", a positive measure μ on (X, Σ) is said to be *concentrated* on a subset $A \in \Sigma$ if $\mu(A^c) = 0$. A signed or complex measure μ is said to be *concentrated* on $A \in \Sigma$ if its variation is concentrated on this set. Now we finish this section with the Lebesgue decomposition theorem which states that each complex measure can be decomposed into an absolute continuous and a singular part w.r.t. a positive measure.

e decomposition} **Theorem 2.14** (Lebesgue decomposition). *[2, Theorem 4.3.2] Let μ be a positive measure and ν be a signed or complex measure on (X, Σ) s.t. ν is σ-finite. Then there exists a unique pair (ν_a, ν_s) of signed measures on (X, Σ) with the following properties:*

(i) $\nu_a \ll \mu$

(ii) $\nu_s \perp \mu$

(iii) $\nu_a + \nu_s = \nu$

2.2 Convex Optimization

This section gives a very brief overview on convex optimization, we focus mostly on the terminology in this area. In this sense, this short section might be seen as a collection of technical terms from convex optimization used in the thesis.

2.2.1 Basic terminology

We consider the following *optimization problem*

$$\min \quad f_0(x) \quad \textit{subject to} \begin{cases} f_j(x) \leq 0, & j = 1, ..., m \\ h_j(x) = 0, & j = 1, ..., p \end{cases} . \tag{$\mathcal{P}_{gen}$}$$

The task is to find an x that minimizes $f_0(x)$ among all vectors $x \in \mathbb{R}^n$ that satisfies $f_j(x) \leq 0$, $j = 1, ..., m$ and $h_j(x) = 0$, $j = 1, ..., p$, where $f_j : \mathbb{R}^n \supset \operatorname{dom} f_j \to \mathbb{R}$, $j = 0, ..., m$ and $h_j : \mathbb{R}^n \supset \operatorname{dom} h_j \to \mathbb{R}$, $j = 1, ..., p$ are real valued functions. Now we introduce some basic notions concerning this problem:

Definition 2.15. *Consider the problem ($\mathcal{P}_{gen}$). Then:*

(i) The function f_0 is called the target function *or* cost function.

(ii) The inequalities $f_j(x) \leq 0$, $j = 1, ..., m$ are called inequality constraints, *the equalities $h_j(x) = 0$, $j = 1, ..., p$ are called* equality constraints.

(iii) The set

$$D = \bigcap_{j=0}^{m} \operatorname{dom} f_j \cap \bigcap_{j=1}^{p} \operatorname{dom} h_j$$

is called the domain *of the problem.*

(iv) A point $x \in D$ is called feasible *if it satisfies all given constraints, i.e. $f_j(x) \leq 0$, $\quad j = 1, ..., m$ and $h_j(x) = 0$, $j = 1, ..., p$; the* feasible set *is the set of all feasible points.*

(v) The optimal value *p^* is defined by*

$$p^* = \inf\{f_0(x) : x \textit{ feasible}\},$$

and a feasible point x^* is called an optimal point *of ($\mathcal{P}_{gen}$) if $f_0(x^*) = p^*$. The set of optimal points is called the* optimal set.

Definition 2.16 (Lagrangian-Dual function). *Consider the problem ($\mathcal{P}_{gen}$).*

(i) The associated Lagrangian *is $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \supset D \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$,*

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x).$$

(ii) We call λ_i, $i = 1, ..., m$, *the* Lagrange multiplier *associated with the* i*-th inequality constraint* $f_i(x) \leq 0$, *and similarly* ν_j, $j = 1, ..., p$ *the* Lagrange multiplier *associated with the* j*-th equality constraint* $h_i(x) = 0$.

(iii) The vectors $\lambda \in \mathbb{R}^m$ *and* $\nu \in \mathbb{R}^p$ *are called the* Lagrange multiplier vectors *or* dual variables.

(iv) Finally, we define the (Lagrange) dual function *as* $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$,

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu) = \inf_{x \in \mathbb{R}^n} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right).$$

Easily one realizes that if the condition $\lambda \succeq 0$ (meaning that each coordinate of λ is nonnegative) is satisfied, then $g(\lambda, \nu)$ is a lower bound on the optimal value p^* of the problem $(\mathcal{P}_{gen})$, i.e. $g(\lambda, \nu) \leq p^*$, since then for any feasible point x we have $f_i(x) \leq 0$ while $h_i(x) = 0$, hence $L(x, \lambda, \nu) \leq f_0(x)$. However, when $g(\lambda, \nu) = -\infty$, this estimate still holds but is vacuous. So, we just want to consider the pairs (λ, ν) such that $\lambda \succeq 0$ and $g(\lambda, \nu) > -\infty$.

Definition 2.17 (dual feasible)**.** *Let* $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ *be the dual function associated with the problem* $(\mathcal{P}_{gen})$*. Then the pair* $(\lambda, \nu) \in \mathbb{R}^m \times \mathbb{R}^p$ *is called* dual feasible *if* $\lambda \succeq 0$ *and* $g(\lambda, \nu) > -\infty$.

As discussed above, we know that the Lagrange dual function provides a lower bound on the optimal value p^* of the optimization problem $(\mathcal{P}_{gen})$ but this bound depends on the parameters λ and ν. Thus, the question about the best lower bound is natural and leads to the following problem:

$$\max \quad g(\lambda, \nu) \quad \text{subject to} \quad \lambda \succeq 0 \tag{$\mathcal{P}^*_{gen}$}$$

Definition 2.18 (Primal problem - Dual problem - Dual optimal)**.**

(i) The problem $(\mathcal{P}^*_{gen})$ *is called the* (Lagrange) dual problem *associated with the problem* $(\mathcal{P}_{gen})$*. In this context,* $(\mathcal{P}_{gen})$ *is called the* primal problem.

(ii) A pair (λ^*, ν^*) *is called* dual optimal *if it is an optimal point of the primal problem.*

Remark 2.19.

(i) Note that the dual problem $(\mathcal{P}^*_{gen})$ *remains unchanged if we restrict to the dual feasible pairs* (λ, ν)*. So, a dual feasible pair can be understood as a pair* (λ, ν) *that is feasible w.r.t. dual problem.*

(ii) We denote the optimal value of the dual problem $(\mathcal{P}^*_{gen})$ *by* d^**, then it is the "best" lower bound on* p^* *that can be obtained from the Lagrange dual function. Furthermore, the inequality* $d^* \leq p^*$ *always holds, we call this property* weak duality.

(iii) The difference $p^* - d^*$ *is often refered to as the* optimal duality gap *of the original problem* $(\mathcal{P}_{gen})$.

As we saw in the previous remark, we always have the weak duality, i.e. $d^* \leq p^*$. If this is even an equality, then we get an interesting connection between the dual problem $(\mathcal{P}^*_{gen})$ and the primal problem $(\mathcal{P}_{gen})$.

Definition 2.20 (Strong duality)**.** *Consider the problem* $(\mathcal{P}_{gen})$ *with optimal value* p^* *and the dual problem* $(\mathcal{P}^*_{gen})$ *with optimal value* d^**. We say that the* strong duality *holds, if the following equality holds:*

$$d^* = p^*$$

2.2.2 Some results in convex optimization

Now let us focus on *convex optimization problems*, which are in the form

$$\min \quad f_0(x) \quad \textit{subject to} \quad \begin{cases} f_j(x) \le 0, & j = 1, ..., m \\ a_j^T x = b_j, & j = 1, ..., p \end{cases}, \tag{2.3}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 0, ..., m$ are *convex* functions, $a_j \in \mathbb{R}^n$ are vectors and $b_j \in \mathbb{R}$ are real numbers. Comparing (2.3) to the general problem ($\mathcal{P}_{gen}$) in page 9, the convex optimization problem has some more requirements: the objective function and the functions appearing in the inequality constraints must be convex and the functions in the equality contraints must be affine. In particular, the convex optimization problem (2.3) can also be written in the form

$$\min \quad f_0(x) \quad \textit{subject to} \quad \begin{cases} f_j(x) \le 0, & j = 1, ..., m \\ Ax = b \end{cases}, \tag{$\mathcal{P}_{conv}$}$$

where $A \in \mathbb{R}^{p \times n}$ is a matrix with rows a_j^T, $j = 1, ..., p$ and $b = [b_1, ..., b_p]^T \in \mathbb{R}^p$.

In general, we do not have the strong duality for every optimization problem of the form ($\mathcal{P}_{gen}$), but if the problem is convex, this property usually (but also not always) holds. We refer to *constraint qualifications* as conditions, under which, besides convexity, the strong duality holds. One of the most often used constraint qualifications is the *Slater's condition.* At first, we require some notions.

Definition 2.21. *Let $C \in \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then*

(i) The affine hull *of C is defined by*

$$\operatorname{aff} C = \left\{ \sum_{k=1}^{n} \theta_k x_k \mid x_1, ..., x_n \in C, \sum_{k=1}^{n} \theta_k = 1 \right\}.$$

(ii) The relative interior *of C is defined by*

$$\operatorname{relint} C = \{ x \in C \mid B(x, r) \cap \operatorname{aff} C \subseteq C \textit{ for some } r > 0 \},$$

where $B(x, r) = \{ y \in \mathbb{R}^n \mid \|x - y\| \le r \}$ is the ball of radius r around x w.r.t. the norm $\|\cdot\|$ on $\mathbb{R}^n$.

Remark 2.22. *(i) Note that in the definition of the affine hull of C the numbers θ_k's may be chosen negatively, this is the difference to the so-called* convex hull *of C.*

(ii) The norm used in the definition of the relative interior is an arbitrary norm on $\mathbb{R}^n$. Since all norms on this space are equivalent, they define the same relative interior of C.

Now we introduce a simple constraint qualification that is very often used to obtain strong duality. Recall that the dual problem to (2.3) is given by

$$\max \quad g(\lambda, \nu) \quad \text{subject to} \quad \lambda \succeq 0, \tag{$\mathcal{P}_{conv}^*$}$$

where $g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu) = \inf_{x \in \mathbb{R}^n} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \nu^T (Ax - b) \right)$.

Theorem 2.23 (Slater's Theorem)**.** *[22] Consider the convex optimization problem ($\mathcal{P}_{conv}$) and its dual problem ($\mathcal{P}_{conv}^*$). If there exists some $x \in$* relint D strictly feasible*, i.e.*

$$f_j(x) < 0, \quad j = 1, ..., m, \quad Ax = b,$$

then the strong duality holds.

Note that we have introduced a simple version of convex optimizations, which focus merely on finite dimensional spaces over the field $\mathbb{R}$. The restriction on $\mathbb{R}$ often makes no real assumption more to the problem, because a vector on $\mathbb{C}^n$ can always be identified with a vector on $\mathbb{R}^{2n}$. However, the restriction of finite dimensions is actually a strong assumption. Indeed, for our purpose later on, we need to consider a convex minimization problem over the vector space of all regular complex Borel measures on $[0,1]$ equipped with the total variation norm, which was introduced in Section 2.1. The dual space to this is given in Theorem 2.10 as the space of complex valued continuous functions on $[0,1]$ equipped with the supremum norm. Fortunately, the theory of convex analysis and convex optimization also surrounds this problem (see e.g. [8], [20]). In fact, the theory involves not only general Banach spaces (which we need) but also topological vector spaces. However, it would go beyond the scope of this thesis to develope the entirety of this beautiful theory, hence, we will now omit that, and later just cite a result that we need for our application.

2.3 Compressed Sensing

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2.3.1 Motivation

Let us consider the system of linear equation

$$y = Ax \tag{2.4}$$

for $A \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^n$. In compressed sensing, (2.4) is refered to as a *linear measurement process*, i.e. the vector x is encoded by m linear measureents $y \in \mathbb{C}^m$ using a (known) *measurement matrix* A. The main question is about the *compressibility* of the signal x, namely:

How small can the number of measurements m be such that
we can still (uniquely) determine x from (2.4) in an *efficient* way?

For the case $m < n$, this task is impossible to solve in general as the system is undetermined. Hence we need to impose further assumptions (prior knowledge) on the objective vector x. In many situations, the vector x is not supported everywhere, i.e. has only very few non-zero entries compared to the dimension of the ambient space $\mathbb{C}^n$.

Definition 2.24. *We define*

$$\|x\|_0 := |\mathrm{supp}(x)| := |\{j : x_j \neq 0\}|\,,$$

that is the number of non-zero entries in $x \in \mathbb{C}^n$. A vector $x \in \mathbb{C}^n$ is called k-sparse if $\|x\|_0 \leq k$.

Note that $\|\cdot\|_0$ is neither a norm nor a quasi-norm.

A very natural approach is now to solve the optimization problem

$$\min_{z\in\mathbb{C}^n} \|z\|_0 \quad \text{subject to} \quad Az = y, \tag{$\mathcal{P}_0$}$$

i.e., we are seeking for the sparsest solution of (2.4). Unfortunately, this problem is numerically intractable if m and n are getting larger, as the following theorem shows:

Theorem 2.25. *[9, Ch. 2] The ℓ_0-minimization problem ($\mathcal{P}_0$) with input $A \in \mathbb{C}^{m\times n}$ and $y \in \mathbb{C}^m$ is NP-hard in general.*

Idea: Now the idea to circumvent this problem is to consider the *convex relaxation* of ($\mathcal{P}_0$), which is known as *basis pursuit*:

$$\min_{z\in\mathbb{C}^n} \|z\|_1 \quad \text{subject to} \quad y = Az. \tag{$\mathcal{P}_1$}$$

In the real setting, i.e. $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ and the optimization problem ($\mathcal{P}_1$) is carried out over $\mathbb{R}^n$, it can be solved by linear programming and if the columns of A are linearly independent, the minimizer is sparse [9, Ch. 3].

Goal: Our major goal is now to find a measurement matrix $A \in \mathbb{C}^{m\times n}$ such that we have

(**A**) *Efficient recovery*: Every k-sparse vector $x \in \mathbb{C}^n$ is the unique solution of $(\mathcal{P}_1)$ with input $y = Ax$ (for small $k \ll n$).

(**B**) *Strong compression*: The number of measurements m is as small as possible.

2.3.2 Exact recovery of sparse vectors

{subsection

In this section, we introduce some conditions for the success of the recovery by $(\mathcal{P}_1)$. The first one will be both necessary and sufficient: if the matrix A has the so-called *null space property*, i.e. the energy of any vector in the kernel of A must be split "fairly" onto all of its entries, then $(\mathcal{P}_1)$ with input $y = Ax$ returns exactly the signal x. But at first let us introduce some notations, and then the formal definition of the null space property.

Notation: Let T be a subset of $\{1, ..., n\}$ and $T^c = \{1, ..., n\} \setminus T$ be its complement. For a vector $v \in \mathbb{C}^n$ we denote by v_T either the vector in $\mathbb{C}^{|T|}$, which contains the coordinates of v on T, or the vector in $\mathbb{C}^n$, which equals v on T and is zero on T^c. It will be always clear from the context which notation is used. Moreover, for $A \in \mathbb{C}^{m \times n}$, we denote by A_T the $m \times |T|$ sub-matrix containing the columns of A indexed by T.

Definition 2.26 (Null Space Property)**.** *Let $A \in \mathbb{C}^{m \times n}$ and $k \in \{1, ..., n\}$. Then A is said to have the* null space property (NSP) *of order k if*

$$\|v_T\|_1 < \|v_{T^c}\|_1 \tag{2.5}$$

for all $v \in \ker(A) \setminus \{0\}$ and for all $T \subset \{1, ..., n\}$ with $|T| \le k$.

Remark 2.27. *If $v \in \ker(A)$ is a k-sparse vector with $T = \operatorname{supp}(v)$ and the matrix $A \in \mathbb{C}^{m \times n}$ has the NSP of order k, then $v = 0$ due to (2.5). Consequently, the kernel of A does not contain vectors that are supported on a "small" set (of size at most k).*

The following theorem states exactly what we said at the beginning of the section: the equivalence of the NSP of A and the success of the basis pursuit.

{Theo: NSP

Theorem 2.28. *Let $A \in \mathbb{C}^{m \times n}$ and let $k \in \{1, ..., n\}$. Then the following conditions are equivalent:*

(i) A has the NSP of order k.

(ii) For any k-sparse vector $x \in \mathbb{C}^n$, the unique solution of $(\mathcal{P}_1)$ with input $y = Ax$ is x.

Proof. First assume that A has the NSP of order k. Let $x \in \mathbb{C}^n$ be a k-sparse vector and set $T := \operatorname{supp}(x)$. We have to show $\|x\|_1 < \|z\|_1$ for any $z \in \mathbb{C}^n \setminus \{x\}$ with $Az = Ax$. Since $x - z \in \ker(A) \setminus \{0\}$, it follows from the NSP of A that

$$\begin{aligned} \|x\|_1 &\le \|x - z_T\|_1 + \|z_T\|_1 = \|(x - z)_T\|_1 + \|z_T\|_1 \\ &< \|(x - z)_{T^c}\|_1 + \|z_T\|_1 = \|z_{T^c}\|_1 + \|z_T\|_1 = \|z\|_1 . \end{aligned}$$

For the converse suppose that the second condition is fulfilled. Let $v \in \ker(A) \setminus \{0\}$ and let $T \subset \{1, ..., n\}$ with $|T| \le k$. Then v_T is k-sparse and hence it is the unique solution of

$$\min_{z \in \mathbb{C}^n} \|z\|_1 \quad \text{subject to} \quad Az = Av_T.$$

Since $A(-v_{T^c}) = A(v - v_{T^c}) = Av_T$, it must hold $\|v_T\|_1 < \|v_{T^c}\|_1$. This proves that A has the NSP of order k. $\square$

Although the NSP of the measurement matrix is a sufficient and also necessary condition, under which the basis pursuit $(\mathcal{P}_1)$ recovers exactly the sparse signals, it is difficult to construct a matrix with this property, or to check if this property is satisfied. Hence, we will present the *Restricted Isometry Property*, which also ensures the success of the basis pursuit.

Definition 2.29 (Restricted Isometry Property)**.** *Let* $A \in \mathbb{C}^{m \times n}$ *and* $k \in \{1, ..., n\}$*. Then the* restricted isometry constant $\delta_k = \delta_k(A)$ *of* A *of order* k *is defined as the smallest* $\delta \geq 0$ *such that*

$$(1-\delta)\,\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta)\,\|x\|_2^2 \tag{2.6}$$

for every k*-sparse vector* $x \in \mathbb{C}^n$*. We say that* A *satisfies the* restricted isometry property *(RIP) of order* k *with the constant* δ_k *if* $\delta_k < 1$*.*

Remark 2.30. *The condition (2.6) states that* A *acts almost isometrically when restricted to* k*-sparse vectors, implying that distances and angles are almost preserved on this subset. Note that the inequality* $\delta_1(A) \leq \delta_2(A) \leq ... \leq \delta_k(A)$ *follows trivially.*

The following theorem shows the connection between the RIP and the NSP, namely that if a matrix has the RIP with a sufficiently small RIP-constant then it also satisfies the NSP and hence guarantees the success of the basis pursuit:

Theorem 2.31. *[9, Ch. 9] Let* $A \in \mathbb{C}^{m \times n}$ *and* $k \in \mathbb{N}$ *with* $k \leq n/2$*. If* $\delta_{2k}(A) < 1/3$*, then* A *has the NSP of order* k*. In particular, every* k*-sparse vector* x *is the unique solution of* $(\mathcal{P}_1)$ *with input* $y = Ax$*.*

As mentioned earlier, one wishes to construct matrices that satisfy the RIP. The most simple way to do this is to take the entries to be independent standard normal variables. The following theorem shows that under a certain condition this method in fact creates a matrix that satisfies the RIP with high probability.

o: CS_RIP} **Theorem 2.32.** *[9, Ch. 9] Let* $n, m, k \in \mathbb{N}$ *and* $\varepsilon, \delta \in (0,1)$ *be such that* $n \geq m \geq k \geq 1$ *and*

$$m \geq C\delta^{-2}\Big(k\ln(en/k) + \ln(2/\varepsilon)\Big),$$

where $C > 0$ *is a universal constant. If* $A \in \mathbb{R}^{m \times n}$ *is a Gaussian matrix, i.e.*

$$A = \frac{1}{\sqrt{m}} \begin{bmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{m1} & \cdots & \omega_{mn} \end{bmatrix},$$

where $\omega_{ij} \sim \mathcal{N}(0,1)$*,* $i = 1, ..., m$*,* $j = 1, ..., n$*, are independent and identically distributed standard normal variables. Then*

$$\mathbb{P}(\delta_k(A) \leq \delta) \geq 1 - \varepsilon.$$

Before we move on to another sufficient condition for the NSP, let us derive that **(A)** and **(B)** can be met with only $m \approx C'k\ln(en/2k)$ measurements, which is significantly fewer than the dimension of the ambient space $\mathbb{C}^n$:

asurements}

Corollary 2.33. *Let* $n \geq m \geq m \geq k \geq 1$ *with* $k \leq n/2$ *and*

$$m \geq C'k\ln\left(\frac{en}{k}\right),$$

where $C' > 0$ *is a universal constant. If* $A \in \mathbb{R}^{m \times n}$ *is a Gaussian matrix as in Theorem 2.32, then with probability at least* $1 - 2\exp(-m/(2C'))$ *it holds: Every* k*-sparse vector can be recovered by* $(\mathcal{P}_1)$*.*

Proof. Set $\varepsilon = 2\exp\left(-\delta^2 m/(2C)\right)$. By Theorem 2.32, under the condition

$$m \geq 4C\delta^{-2}k\ln\left(\frac{en}{2k}\right),$$

we have

$$\mathbb{P}(\delta_{2k} \leq \delta) \leq 1 - 2\exp\left(-\delta^2 m/(2C)\right).$$

Applying Theorem 2.28 for $\delta = 1/3$ yields the statement. □

From the theoretical point of view, the NSP is an important condition for the exact recovery by $(\mathcal{P}_1)$. Furthermore, we have shown how to construct matrices satisfying the RIP and hence also this NSP (of lower order) with high probability. However, if a matrix A is given beforehand, it is quite difficult to check if this matrix satisfies the NSP and the RIP, or to calculate the RIP-constant and respectively the order of NSP. Another property of A, which is easily verifiable and also guarantees the exact recovery of $(\mathcal{P}_1)$, is the *mutual coherence* of A.

Definition 2.34 (Mutual Coherence)**.** *Let $A = [a_1, ..., a_n] \in \mathbb{C}^{m\times n}$ (a_i's are the columns of A). Then the* mutual coherence *of A is defined by*

$$\mu(A) := \max_{i\neq j} \frac{|\langle a_i, a_j\rangle|}{\|a_i\|_2 \|a_j\|_2}.$$

The connection between the mutual coherence and the NSP is given by the following corollary.

{Theo: Mut

Corollary 2.35. *Suppose that the matrix $A \in \mathbb{C}^{m\times n}$ has unit-normed columns and let $k \in \{1, ..., n\}$. If*

$$k < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right),$$

then A has the NSP of order k.
In particular, for any $x \in \mathbb{C}^n$ with

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right),$$

the unique solution of $(\mathcal{P}_1)$ with input $y = Ax$ is x.

Proof. Let $v \in \ker(A)\backslash\{0\}$. Then we have $A^*Av = 0$, where $[A^*A]_{j,k} = \langle a_j, a_k\rangle$. Hence, for each $j \in \{1, ..., n\}$ we have $\sum_{k=1}^n \langle a_j, a_k\rangle v_k = 0$ and hence

$$|v_j| = \left|-\sum_{\substack{k=1\\k\neq j}}^n \langle a_j, a_k\rangle v_k\right| \leq \sum_{\substack{k=1\\k\neq j}}^n |\langle a_j, a_k\rangle|\,|v_k| \leq \mu(A)\sum_{\substack{k=1\\k\neq j}}^n |v_k| = \mu(A)(\|v\|_1 - |v_j|).$$

This implies that for each $j \in \{1, ..., n\}$,

$$|v_j| \leq \left(1 + \frac{1}{\mu(A)}\right)^{-1} \|v\|_1.$$

Finally, let $T \subset \{1, ..., n\}$ with $|T| \leq k$, by assumption we have

$$2\|v_T\|_1 = 2\sum_{j\in T} |v_j| \leq 2k\left(1 + \frac{1}{\mu(A)}\right)^{-1} |v|_1 < \|v\|_1 = \|v_T\|_1 + \|v_{T^c}\|_1,$$

which shows that A has the NSP of order k. The particular statement follows directly from Theorem 2.28. □

2.3.3 Stability and robustness

The following two features have to be taken into account when studying the recoverability of sparse signals:

(1) *Stability*: We also intend to recover (or at least approximate) vectors $x \in \mathbb{C}^n$ which are not exactly sparse but *compressible*, meaning that their best k-term approximation error, defined as

$$\sigma_k(x)_p = \min_{\substack{T \subset \{1,\dots,n\}:\\ |T| \le k}} \|x - x_T\|_p , \quad 0 < p < \infty,$$

(in the following we consider $p = 1$) decays rapidly to 0 for $k \to \infty$.

(2) *Robustness*: We would like to recover sparse (or compressible) vectors from noisy measurements, as a basic model, one usually assumes that the measurement vector is given by $y = Ax + e$, where e is small (in a certain sense).

We now focus on the recovery properties of the following slightly modified version of $(\mathcal{P}_1)$:

$$\min_{z \in \mathbb{C}^n} \|z\|_1 \quad \text{subject to} \quad \|Az - y\|_2 \le \eta, \tag{$P_{1,\eta}$}$$

where $\eta \ge 0$ denotes the noise level. Clearly, if $\eta = 0$, then $(P_{1,\eta})$ is equal to $(\mathcal{P}_1)$.

Theorem 2.36. *[9, Ch. 6] If $\delta_{2k} < \frac{4}{\sqrt{41}}$ and $\|e\|_2 \le \eta$, then the solution $\hat{x}$ of $(P_{1,\eta})$ with input $y = Ax + e$ satisfies*

$$\|x - \hat{x}\|_2 \le \frac{C\sigma_k(x)_1}{\sqrt{k}} + D\eta,$$

where $C, D > 0$ are constants depending on δ_{2s}.

Recall that one of our major goals in compressed sensing is to keep the number of measurements m as small as possible. The Corollary 2.33 already gives an upper bound on the optimal number of measurements, indeed if m is of order $k \ln\left(\frac{en}{2k}\right)$, then with high probability, recovery by the basis pursuit succeeds. Now, the following theorem shows that this bound is essentially optimal in the sense that any stable recovery of sparse vectors requires at least $C' k \ln\left(\frac{en}{k}\right)$ measurements, no matter which measurement matrix or recovery algorithm is used:

Theorem 2.37. *[9, Ch. 11] Let $k \le m \le n$ be natural numbers, $q > 1$ fixed, $A \in \mathbb{C}^{m\times n}$ be any measurement matrix and $\Delta : \mathbb{C}^m \to \mathbb{C}^n$ be an arbitrary recovery map such that for some constant $C > 0$*

$$\|x - \Delta(Ax)\|_q \le C\frac{\sigma_k(x)_1}{k^{1-1/q}} \quad \text{for all } x \in \mathbb{C}^n.$$

Then

$$m \ge C' k \ln\left(\frac{en}{k}\right)$$

with some other constant $C' > 0$ depending only on C.

Chapter 3

Noiseless Recovery

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This chapter concentrates on the noiseless super-resolution problem. We will introduce our mathematical model for this problem, and then compare it with the ordinary compressed sensing problem. Of course, the proof of the result will be given and we will end the chapter with some extensions of the result.

3.1 Our model and the main result

{Section:

3.1.1 The basic model

In this section we consider a continuous-time model in which the signal to be analyzed is a weighted superposition of spikes

$$x = \sum_{j=1}^{|T|} a_j \delta_{t_j} \tag{3.1}$$

where δ_t denotes a Dirac measure at t, $T = \{t_j\}_j$ is a subset of $[0,1]$ with cardinality $|T| < \infty$ and $\{a_j\}_j$ contains $\mathbb{C}$-valued amplitudes. As mentioned in Chapter 1, the high-frequency data of this signal is often filtered out in the sense that we may observe the convolution between x and a low-pass point spread function φ,

$$(x * \varphi)(t) = \sum_{j=1}^{|T|} a_j \varphi(t - t_j).$$

Let f_c be the frequency cut-off of φ, meaning that $\widehat{\varphi} = \mathbb{1}_{[-f_c, f_c]}$, i.e. the indicator function of the set $[-f_c, f_c]$, then in Fourier domain we obtain

$$\widehat{x * \varphi} = \widehat{x}\widehat{\varphi} = \widehat{x}\mathbb{1}_{[-f_c, f_c]}.$$

Since the signal $x * \varphi$ is band-limited between $[0,1]$, the Shannon Sampling Theorem states that its spectrum can be entirely determined by the discrete sample

$$y(k) = \widehat{x * \varphi}(k) = \widehat{x}(k) = \int_0^1 e^{-i2\pi kt} dx(t) = \sum_{j=1}^{|T|} a_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z}, |k| \le f_c, \tag{3.2}$$

where we assume for simplicity that f_c is an integer. One can also write the relation (3.2) between the data y and the object x in a compact form as

$$y = F_n x$$

where F_n denotes the linear map collecting the lowest $2f_c + 1$ Fourier series coefficients. Now, the task of super-resolution essentially consists in recovering the signal x from the sample y.

3.1.2 Minimum separation condition

An apparent similarity between the super-resolution and the compressed sensing problem is that they are both inverse problems, where we intend to recover signals from their partial spectrum. However, while the RIP-matrices in compressed sensing preserve the energy of arbitrary sparse signals (cf. Section 2.3.2), the low-pass filtering in our super-resolution problem may almost completely suppress the measurement vector of a signal if its support is too clustered together (see Section 4.2 for further details). This means that in practice, super-resolution for general sparse signals is impossible. Therefore, we need to impose conditions on the support of the signals to prevent it to be too clustered. A natural idea is to ensure that consecutive spikes are separated by a minimum distance. Note that due to the periodicity of $t \mapsto e^{-i2\pi kt}$, it is obvious that we need to consider the distance between any two locations of the spikes in a "wrap-around" sense.

Definition 3.1 (Minimum separation). *Let $\mathbb{T}$ be a circle obtained by identifying the endpoints on $[0,1]$. Then the* wrap-around distance *between two points $t, t' \in \mathbb{T}$ with $t < t'$ is defined by*

$$|t - t'|_w := \min\{t' - t, 1 + t - t'\}.$$

Furthermore, for a family of points $T \subset \mathbb{T}^d$, the minimum separation $\Delta(T)$ *of T is defined as the closest (wrap-around) distance between any two elements of T, that is*

$$\Delta(T) = \inf_{t,t' \in T : t \neq t'} |t - t'|_w \,.$$

3.1.3 The main result

In Section 2.3, we have seen that the ℓ_1-minimization recovers exactly sparse discrete signals. On the other hand, the total variation of a complex measure can be interpreted as a continuous counterpart of the ℓ_1 norm. In fact, for signals x as in (3.1) we have $\|x\|_{TV} = \sum_j |a_j|$, or in words, it equals to the ℓ_1 norm of the amplitudes $a = (a_j)_j$. Hence, one might hope that under certain conditions about minimum separation, solving the following total-variation minimization subject to data constraints would recover or at least estimate the continuous original signals:

$$\min_z \|z\|_{TV} \quad \text{subject to} \quad F_n z = y, \tag{$\mathcal{P}_{TV}$}$$

here the minimization is carried out over the set of all (finite) complex Borel measures z supported on $[0,1]$.

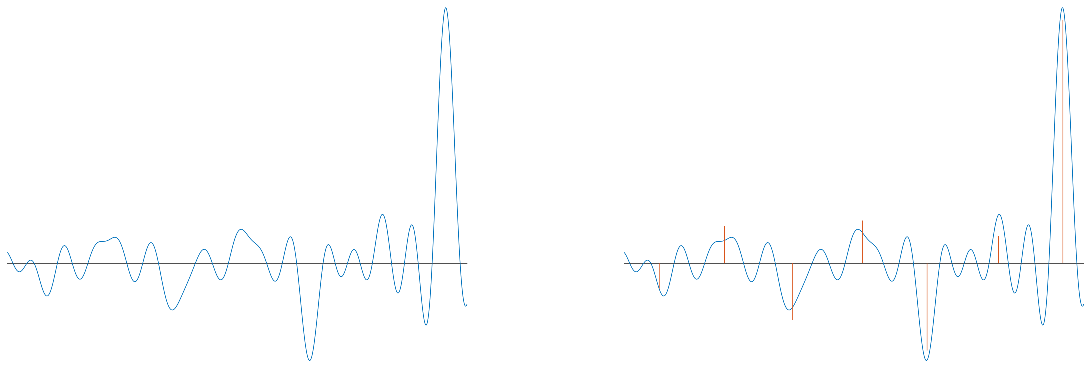
This hope is justified by our first result, which shows that if the spikes are separated by at least $2\lambda_c = 2/f_c$ can be recovered by $(\mathcal{P}_{TV})$ with infinite precision.

ainTheorem}

Theorem 3.2. *Let $T = \{t_j\}$ be the support of x with $f_c \geq 128$ and*

$$\Delta(T) \geq 2/f_c := 2\lambda_c. \tag{3.3}$$

Then x is the unique solution to $(\mathcal{P}_{TV})$ with input $y = F_n x$. If x is known to be real-valued, then the minimum gap can be lowered to $1.87\lambda_c$.



(a) Low-pass data of a superposition of spikes. This is the information we have available to recover the signal.

(b) The red ticks represent the spike-train. The locations of the spikes obey the minimum separation condition (3.3).

Figure 3.1: Actually, by looking at the blue curve, it seems a priori impossible to determine the number of the spikes or their locations.

ency Projection}

Later on, we will refer to (3.3) as the minimum separation condition. Note that under this condition, the total-variation minimization locates the support of the signal with infinite precision and this does not depend on the amplitudes of the spikes. One may find this unexpected because the total-variation norm does not tell much about the structure of the signal. Also, observe that the constraint in $(\mathcal{P}_{TV})$ can be seen also as $P_n z = P_n x = x * \varphi$, where $P_n = \frac{1}{N} F_n^* F_n$, i.e. we only look at the projection onto the first n Fourier modes, or equivalently the convolution of the signal with the Dirichlet kernel

$$\varphi(t) = \sum_{k=-f_c}^{f_c} e^{i2\pi kt} = \frac{\sin{(2f_c+1)\pi t}}{\sin{(\pi t)}},$$

which has slowly decaying side lobes. In this context, the theorem claims that $(\mathcal{P}_{TV})$ will extract the small spikes even though they may be totally buried in the side lobes of the large spikes. This phenomenon is illustrated in Figure 3.1.

On the other hand, unlike the basis pursuit, the convex program $(\mathcal{P}_{TV})$ may not return a spike-train but a more general and complicated measure. And even when it does give us a spike-train, some error would be probably expected. But this is not the case and we obtain hier exactly the position of the original spikes regardless their amplitude.

As an interesting aspect, all the discussion above regarding the infinite precision of the recovery by $(\mathcal{P}_{TV})$ cannot be really tested numerically because for that one would need a numerical solver for $(\mathcal{P}_{TV})$ with infinite precision. Such a solver is of course not available! However in Chapter 5 we will show that $(\mathcal{P}_{TV})$ is actually tractable. Indeed, by solving the dual problem to $(\mathcal{P}_{TV})$, which can be cast as a finite-dimensional semidefinite program, one can determine the sign pattern of the signal and then solve a corresponding system of linear equations to also recover the amplitudes.

Finally, one may wonder whether the minimum separation given in (3.3) is the optimal distance required for exact recovery by $(\mathcal{P}_{TV})$. Quite similarly, the question about the maximum number of the spikes that can be recovered from $n = 2f_c + 1$ low-frequency samples might be asked. Theorem 3.2 gives a lower bound for this maximum value, in fact, if the condition (3.3) is satisfied, by $(\mathcal{P}_{TV})$ we can recover at least $f_c/2$, or equivalently, about $n/4$ spikes. This outcome may be compared with the work in [4] and [13], which show that f_c spikes can be recovered provided that their amplitudes are nonnegative real-valued.

3.2 Discrete super-resolution

{section: discr

Remember that the recovery problem of compressed sensing is often considered in discrete-time settings, i.e. a sparse signal $x \in \mathbb{C}^N$ should be recovered from its spectrum. We can imagine drawing a lattice $\{j/N\}_{j=0}^{N-1}$ on the unit interval and locating the spikes exactly on the grid to obtain a signal in form of a vector of length N, where the amplitudes become the entries of this vector. To be concrete, we intend to recover $x = (x_j)_{j=0}^{N-1} \in \mathbb{C}^N$ from its low-pass data, i.e. the low-frequency samples of the form

$$y_k = \sum_{j=0}^{N-1} x_j e^{-i2\pi k \frac{j}{N}}, \quad k \in \mathbb{Z}, \, |k| \le f_c,$$

or in matrix form

$$y = F_n x,$$

where F_n is the partial Fourier matrix corresponding to the lowest $n = 2f_c + 1$ frequency coefficients. As mentioned before, this discrete-time problem can be seen as a direct corollary of our continuous theory. In fact, one can consider the discrete signal $x = (x_j)_{j=0}^{N-1}$ as a linear combination of Dirac measures that is only supported on the grid $\{j/N\}$, or more precisely, on $\frac{1}{N}T$ where T is the support of the vector x, namely $\sum_{j=0}^{N-1} x_j \delta_{j/N}$.

One may also note that if the resolution of the data, defined as $1/f_c$, is fixed, i.e. the number of samples remains constant, then the continuous-time setting is the limit of infinite resolution in which N tends to infinity. Instead, we choose to fix the ratio between the actual resolution of the signal $1/N$ and the resolution of the data $1/f_c$. In this context we adapt the minimum separation condition (3.3) to the following condition regarding the support T of a vector $x \in \mathbb{C}^N$:

$$\min_{\substack{j,j' \in T: \\ j \neq j'}} \left| \frac{j - j'}{N} \right|_w \ge 2\lambda_c = \frac{2}{f_c}. \tag{3.4}$$

Another aspect that should be transfered into the discrete setting is the total-variation minimization. However, this seems to be simple. As mentioned before, the total variation can be interpreted as the continuous analog to the ℓ_1 norm, and in the discrete setting, it reduces to be exactly the ℓ_1 norm. This means that the convex optimization problem $(\mathcal{P}_{TV})$ now becomes the basis pursuit, i.e.

$$\min_{z \in \mathbb{C}^N} \|z\|_1 \quad \text{subject to} \quad F_n z = y. \tag{P_{ℓ_1}}$$

Not surprisingly, the convex program (P_{ℓ_1}) actually recovers the vector x exactly, as stated in the following corollary.

eorem(cor)}

Corollary 3.3. *Let $T \subset \{0, 1, ..., N-1\}$ be the support of $x = (x_j)_{j=0}^{N-1}$ satisfying the discrete minimum separation, i.e. (3.4). Then x is the unique solution to (P_{ℓ_1}) with input $y = F_n x$.*

Proof. Note that for $z = (z_j)_{j=0}^{N-1}$ the ℓ_1-norm of z is equal to the total variation of the signal $\sum_{j=0}^{N-1} z_j \delta_{j/N}$. Hence the corollary follows directly from Theorem 3.2. □

As seen above, one wishes to resolve a signal on a fine grid with spacing $1/N$ while observing only the lowest $n = 2f_c + 1$ Fourier coefficients. In principle, with these data it is only possible to recover the signal on a coarser grid with spacing $1/(2fc) \approx 1/n$. Hence, the factor N/n can be interpreted as a super-resolution factor, in fact, this is the ratio between the spacings in the coarse and fine grids, as shown in Figure 3.2. From now on, we set

$$\text{SRF} := \frac{N}{n} (\approx \frac{N}{2f_c}).$$

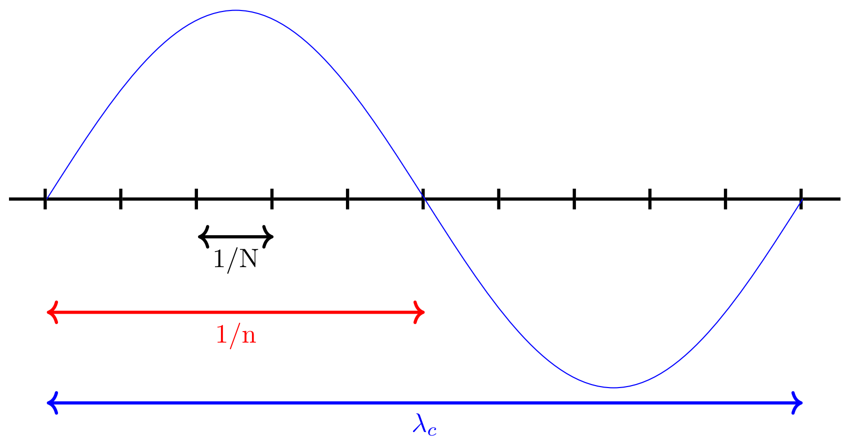


Figure 3.2: Fine grid with spacing $1/N$. Only frequencies between $-f_c$ and f_c are observed, so that the highest frequency sine wave available has wavelength $\lambda_c = 1/f_c$. These data only allow a Nyquist sampling rate of $\lambda_c/2 \approx 1/n$. The ratio N/n between these two resolutions can be interpreted as the super-resolution factor.

{Fig: SRF}

Remark 3.4. *Corollary 3.3 can be reformulated as follows: If the minimum separation of the support T of the signal x is at least $4 \times \mathrm{SRF}$ then the the ℓ_1-minimization recovers x exactly.*

Intuitively, if SRF increases and the length N of the signal is fixed, the spectrum of the signal becomes smaller in the sense that there are fewer Fourier coefficients, from which we seek to recover the signal. In other words, when the SRF gets larger, we get stronger compression but the recovery problem becomes certainly more difficult.

Hence, it is quite natural to guess that the precision of our process in presence of noise would drop if SRF is enlarged. Indeed, in Chapter 4 it will be proved that the super-resolution error is proportional to the noise level and the square of the SRF. A related result can be found in [4], which studies the modulus of continuity of the recovery of a signed measure on a discrete lattice from its spectrum on the interval $[-f_c, f_c]$, a setting which is also equivalent to that of Corollary 3.3 when N tends to infinity. Under the condition that the support of the measure contains at most ℓ elements in any interval of length $2/(\ell f_c)$ (for $\ell = 1$ this is our minimum separation condition, cf. (3.3)), then the modulus of continuity is of order $O(\mathrm{SRF}^{4\ell+1})$ as SRF tends to infinity. This means that for two different signals, if the difference between their measurements in the ℓ_2 norm is at most δ, then the difference between them also in the ℓ_2 norm is of order $O(\mathrm{SRF}^{4\ell+1}\,\delta)$ (the exponent lies between $2\ell - 1$ and $4\ell + 1$). The modulus of continuity gives an upper bound for the smallest error made by a super-resolution method, so the result suggests that robust super-resolution of well-separated signals is in principle not hopelessly ill-conditioned. However, the paper does not suggest any practical recovery algorithm (a brute-force search for sparse measures obeying the low-frequency constraints would be computationally intractable). We will show that such an algorithm actually exists, namely the recovery by $(\mathcal{P}_{TV})$ in continuous and by (P_{ℓ_1}) in discrete setting. The proof for the success of this method is introduced in Section 3.4, while in Chapter 5 we will show that the optimization problem is indeed numerically solvable.

3.3 Connections to compressed sensing

One may realize that the discrete setting discussed in Section 3.2 is very similar to the typical problem of compressed sensing (cf. Section 2.3). Indeed, in both of these inverse problems, we aim to recover exactly sparse signals from their linear measurements by the basis pursuit and the number of measurements should be as small as possible. The main difference is that super-resolution problem focuses on recovery from low-pass measurements, i.e. the measurement matrix is a partial Fourier matrix which maps a signal onto its low-frequency Fourier coefficients, while the compressed sensing problem in Section 2.3 considers general cases, where one does not include any properties on the linear measurements. In fact, the measurement matrix in compressed sensing can be selected freely, and indeed, in Section 2.3 we chose it to be a Gaussian matrix, which, with high probability, preserves almost all the energy of any sparse vector due to the restricted isometry property. Meanwhile, the partial Fourier matrix corresponding to the lowest Fourier coefficients may suppress the signal, meaning that its measurement vector is essentially zeroed out (see Section 4.2 for further details). This is the reason why the super-resolution problem should not be seen as a special case of the compressed sensing problem. In other words, we cannot hope that the conditions discussed in Section 2.3, which guarantees the perfect recovery by the basis pursuit, also work in our situation. In fact:

1. The normalized Fourier matrix $\tilde{F} \in \mathbb{C}^{N\times n}$, i.e. the matrix with columns $f_j = \frac{1}{\sqrt{n}}(e^{-i2\pi kj/N})_{k=0}^{n-1}$, $j = 0, ..., N-1$, does not have the restricted isometry property (RIP), since a submatrix combined from a very small number of consecutive columns is already very close to singular [21]. For example, with $N = 512$ and $\mathrm{SRF} = 4$, the smallest singular value of submatrices composed of eight contiguous columns is 3.32×10^{-5}.

2. By Corollary 2.35, another condition for unique sparse recovery via the basis pursuit (i.e. ℓ_1 minimization) based on the mutual coherence of the matrix $\tilde{F}$ is

$$|T| < \frac{1}{2}\left(1 + \frac{1}{\mu(\tilde{F})}\right). \tag{3.5}$$

   Recall that the mutual coherence of $\tilde{F}$ is defined by $\mu(\tilde{F}) = \max_{i\neq j} |\langle f_i, f_j\rangle|$ (as the columns of $\tilde{F}$ are already normalized). When $N = 1024$ and $\mathrm{SRF} = 4$, one can easily compute that $\mu(\tilde{F}) \approx 0.9003$. In this case, (3.5) yields that for exact recovery one would need $|T| \leq 1.055$, i.e. the signal must be 1-sparse.

Finally, we want to mention the situation in [5, Section 6.3], in which one also considers partial Fourier measurements (but not necessarily only corresponding to low-frequency coefficients). Applying the discrete uncertainty principle proved in this work yields that recovery by ℓ_1 minimization succeeds as long as $2\,|T|\,(N-n) < N$. If $n < N/2$ or equivalently, $\mathrm{SRF} > 2$, this becomes $|T| = 0$. In other words, to recover 1-sparse signals one would need $\mathrm{SRF} \leq 2$, i.e. at least half of the Fourier samples.

3.4 Proof of the main result

ain_result}

In this section we want to introduce the proof to the main result in this thesis, Theorem 3.2. For simplicity, we identify the interval $[0, 1)$ with the circle $\mathbb{T}$. Also, we make use of the notations $\Delta := \Delta(T)$ and $\Delta_{\min} := 2/f_c = 2\lambda_c$. Recall that our minimum separation condition (3.3) states that $\Delta \geq \Delta_{\min}$.

The proof is divided into several steps. Firstly, section 3.4.1 introduces a so-called dual certificate, which guarantees the success of the recovery by $(\mathcal{P}_{TV})$. Given an arbitrary spike-train satisfying the minimum separation condition (3.3), the certificate turns out to be the existence of a low-frequency trigonometric polynomial that interpolates the sign pattern of the spike-train. This also explains the need of the assumption about minimum separation of the spikes. Secondly,

3.4.1 Dual Polynomials

ertificate}

In the discrete setting, [7, Lemma 2.1] shows that the existence of a *dual certificate* ensures that the ℓ_1 minimization solution is unique and equals the original signal. In our continuous setting, an analogous (sufficient) condition for the success of the total-variation solution is as follows:

{lemma:dual_pol}

Lemma 3.5 (Dual certificate). *Suppose that for every $v \in \mathbb{C}^{|T|}$ with $|v_j| = 1$ for each $j \in \{1, ..., |T|\}$, there exists a low-frequency trigonometric polynomial*

$$q(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt} \tag{3.6}$$

satisfying the following properties

$$\begin{cases} q(t_j) = v_j, & t_j \in T \\ |q(t)| < 1, & t \in \mathbb{T} \setminus T. \end{cases} \tag{3.7}$$

Then x is the unique solution of $(\mathcal{P}_{TV})$ with input $y = F_n x$.

Proof. Let $\tilde{x}$ be a (finite) complex measure on $\mathbb{T}$ that solves $(\mathcal{P}_{TV})$ and set $h := \tilde{x} - x$. Towards a contradiction assume that $h \neq 0$. Theorem 2.14 implies the existence of a unique pair (h_T, h_{T^c}) of complex measures on $[0, 1]$ with the following properties:

(i) $h_T \ll |x|$

(ii) $h_{T^c} \perp |x|$.

We aim to prove the following claim

$$\|h_T\|_{TV} < \|h_{T^c}\|_{TV} ,$$

because once we justify it, we will obtain a contradiction as follows:

$$\|x\|_{TV} < \|x\|_{TV} - \|h_T\|_{TV} + \|h_{T^c}\|_{TV} \leq \|x + h_T\|_{TV} + \|h_{T^c}\|_{TV} = \|x + h\|_{TV} = \|\tilde{x}\|_{TV} .$$

It follows from (i) that h_T is concentrated on T, which in turn implies that $h_T = \sum_{j=1}^{|T|} b_j \delta_{t_j}$ for some $b = (b_1, ..., b_{|T|})^T \in \mathbb{C}^{|T|}$. By assumption, there exists a trigonometric polynomial $q(t) = \sum_{k=-f_c}^{f_c} a_k e^{i2\pi kt}$ interpolating the sign pattern of h_T, i.e. satisfying the following properties

$$\begin{cases} q(t_j) = \operatorname{sgn}(b_j), & t_j \in T \\ |q(t)| < 1, & t \in \mathbb{T} \setminus T. \end{cases}$$

Observe that $\int_{\mathbb{T}} q dh = \int_{\mathbb{T}} q dx - \int_{\mathbb{T}} q d\tilde{x} = 0$ since q is a trigonometric polynomial of degree f_c. Also, we have $|h_T| = \sum_{j=1}^{|T|} |b_j| \, \delta_{t_j}$ and $|q|$ is bounded by 1. These observations gives

$$\|h_T\|_{TV} = \sum_{j=1}^{|T|} |b_j| = \sum_{j=1}^{|T|} b_j \operatorname{sgn}(b_j) = \int_{\mathbb{T}} q dh_T = -\int_{\mathbb{T}} q dh_{T^c} \leq \|h_{T^c}\|_{TV} .$$

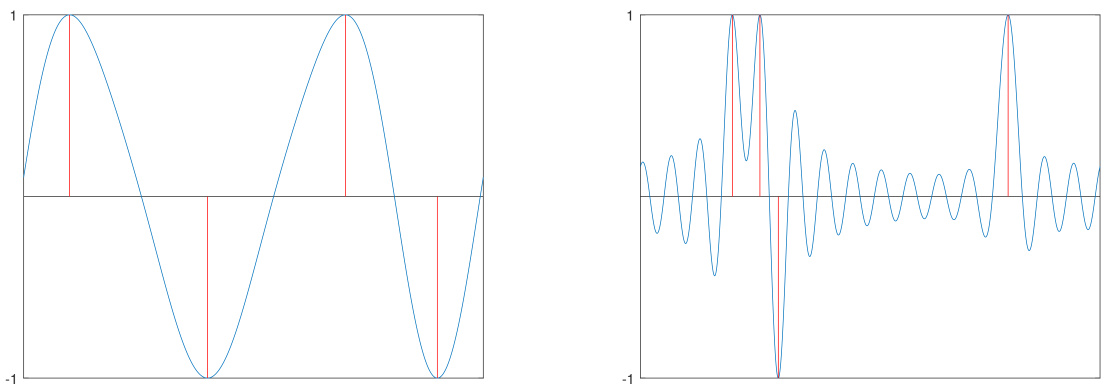
The inequality becomes strict if $h_{T^c} \neq 0$. Consider the opposite case, i.e. $h_{T^c} = 0$, by (ii) this means that $\hat{x}$ is also an atomic complex measure supported on T. Then, the constraint in $(\mathcal{P}_{TV})$, i.e. $F_n x = F_n \tilde{x}$, would lead to an over-determined system of linear equations where the coefficient matrix

$$[e^{i2\pi kt_j}]_{-f_c \leq k \leq f_c,\, 1 \leq j \leq s}$$

is a Vandermonde matrix with entries $e^{i2\pi kt_j}$, $k \in \{-f_c, ..., f_c\}$, $j \in \{1, ..., s\}$, which in turn implies $x = \hat{x}$. This is a contradiction to the assumption that $h \neq 0$, hence the claim follows. $\square$

This dual polynomial not only gives us a theoretical certification for the success of the recovery by $(\mathcal{P}_{TV})$ as seen above, in Chapter 5 we will also see that the dual polynomial corresponding to the original signal actually brings us the support of the spikes.

The vector $v \in \mathbb{C}^{|T|}$ in the Lemma 3.5 can also be seen as the sign pattern corresponding to a spike-train. In this sense, if the support of the spike-train become very near, we would need a rapidly (high-frequency) interpolating polynomial in order to achieve (3.8). The phenomenon is illustrated in Figure 3.1. This also explains the need of the minimum separation condition in the proof of Theorem 3.2.



(a) Low-frequency polynomial interpolating a sign pattern whose support is well-separated and obeying the off-support condition (3.8).

(b) If some spikes lie very near to each other, the polynomial satisfying (3.8) become a high-frequency one.

Figure 3.3: The necessary of the separation between consecutive spikes in the proof. {fig: Dual_poly

Back to the proof of Theorem 3.2, it remains now to prove the existence of such a dual polynomial, provided that the sign pattern satisfies the minimum separation condition. This is stated in the next proposition:

istence_1D}

Proposition 3.6. *Let $v \in \mathbb{C}^{|T|}$ with $|v_j| = 1$ for each $j \in \{1, ..., |T|\}$. If T satisfies the minimum separation condition (3.3), then there exists a low-frequency polynomial as in (3.6) satisfying (3.7).*

With the aid of Lemma 3.5, our main result will follow immediately from this proposition. In other words, to finish to proof of Theorem 3.2, we seek to construct a low-frequency trigonometric polynomial that interpolates the sign pattern v in t_j's and is strictly smaller than 1 on any other point of the unit interval.

The idea for this consists in interpolating v on T with a low-frequency kernel K such that the outcome trigonometric polynomial q has its global maxima 1 (in magnitude) only on T. A necessary condition for an extremum point of a smooth function is that the first derivative vanishes at that point. This suggests to require the derivative of q to vanish on T. Note that K' is also a low-frequency trigonometric polynomial, so that we can interpolate v with both K and K' as

$$q(t) = \sum_{j=1}^{|T|} \alpha_j K(t - t_j) + \beta_j K'(t - t_j)$$

where $\alpha, \beta \in \mathbb{C}^{|T|}$ are coefficient vectors. The required properties mentioned above now lead to

$$\begin{cases} q(t_k) = \sum_{j=1}^{|T|} \alpha_j K(t_k - t_j) + \beta_j K'(t_k - t_j) = v_k, \\ q'(t_k) = \sum_{j=1}^{|T|} \alpha_j K'(t_k - t_j) + \beta_j K''(t_k - t_j) = 0, \quad \text{for every } k \in \{1, ..., |T|\}. \end{cases} \tag{3.8}$$

The reason for interpolating with both K and K' (instead of only K) is that one then obtains a system of linear equations

$$\begin{bmatrix} D_0 & D_1 \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} \tag{3.9}$$

with unknowns $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^{2|T|}$ and coefficient matrix $\begin{bmatrix} D_0 & D_1 \\ D_1 & D_2 \end{bmatrix} \in \mathbb{C}^{2|T| \times 2|T|}$, where

$$\big[D_0\big]_{k,j} := K(t_k - t_j), \quad \big[D_1\big]_{k,j} := K'(t_k - t_j) \quad \big[D_2\big]_{k,j} := K''(t_k - t_j)$$

for $k, j \in \{1, \dots, |T|\}$. If K is well-chosen, then one can control the solution $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and prove the desired properties in (3.7).

3.4.2 The Fejer kernel

{section:

The kernel we employ is

$$K(t) = \begin{cases} \left[\dfrac{\sin\left(\left(\frac{f_c}{2}+1\right)\pi t\right)}{\left(\frac{f_c}{2}+1\right)\sin(\pi t)} \right]^4, & \text{for } t \in (0,1) \\ 1, & \text{for } t = 0. \end{cases} \tag{3.10}$$

If f_c is even, or in other words $f := \frac{f_c}{2} + 1$ is a natural number, then K is the squared Fejer kernel. Clearly, the assumption that f_c is an even number essentially does not make any real restriction to our model (cf. Section 3.1). We can rewrite this kernel on $(0,1)$ as

$$K(t) = \left[\frac{\sin\left(f\pi t\right)}{f \sin(\pi t)} \right]^4,$$

Actually, one might expect interpolating with the Dirichlet kernel, but it would be difficult to control the absolute value of the dual polynomial q on the off-support $\mathbb{T} \setminus T$, due to the slow decay of the tail of the Dirichlet kernel. The kernel K defined above is the fourth power of the Dirichlet kernel, indeed is a good candidate kernel because it is a low-frequency polynomial (with degree at most f_c), attains the value of 1 at its peak and rapidly decays to 0.

{lemma: K

Lemma 3.7. *The interpolation kernel K is a trigonometric polynomial of degree f_c.*

Proof. To see that K is actually a low-frequency trigonometric polynomial, let us begin with the k-th Dirichlet kernel, defined by $D_k = \sum_{j=-k}^{k} e^{i2\pi jt}$ for $k \in \mathbb{N}$. With simple calculations, it follows that

$$D_k = \frac{e^{-i2k\pi t} - e^{i2\pi(k+1)t}}{1 - e^{i2\pi t}} = \frac{e^{-i2\pi(k+\frac{1}{2})t} - e^{i2\pi(k+\frac{1}{2})t}}{e^{-i\pi t} - e^{i\pi t}} = \frac{\sin\left((2k+1)\pi t\right)}{\sin\left(\pi t\right)} = \frac{\cos\left(k2\pi t\right) - \cos\left((k+1)2\pi t\right)}{2\sin^2\left(\pi t\right)}.$$

Hence,

$$\left[\frac{\sin(f\pi t)}{f\sin(\pi t)}\right]^2 = \frac{1-\cos(f2\pi t)}{2f^2\sin^2(\pi t)} = \frac{1}{f^2}\sum_{k=0}^{f-1} D_k = \frac{1}{f^2}\sum_{k=0}^{f-1}\sum_{j=-k}^{k} e^{i2\pi jt}$$

and thus, $K(t) = \left[\frac{\sin(f\pi t)}{f\sin(\pi t)}\right]^4$ is a trigonometric polynomial with degree $2(f-1) = f_c$. □

As mentioned before, our interpolation kernel K decays rapidly to 0. In fact, not only the kernel itself, but also its first derivatives have this property. This is an important property of K that will be used later to control the magnitude of the polynomial q. Further details are given in the following technical lemma.

Fejer_kern}

Lemma 3.8. *For $\ell \in \{0,1,2,3\}$ let $K^{(\ell)}$ be the ℓ-th derivative of K. For $\frac{1}{2f_c} = \frac{\lambda_c}{2} \leq t \leq \frac{1}{2}$ we have*

$$\left|K^{(\ell)}(t)\right| \leq B_\ell(t) = \begin{cases} \tilde{B}_\ell(t) = \frac{\pi^\ell H_\ell(t)}{(f_c+2)^{4-l}t^4}, & \frac{\lambda_c}{2} \leq t \leq \sqrt{2}/\pi \\ \frac{\pi^\ell H_\ell^\infty}{(f_c+2)^{4-\ell}t^4}, & \sqrt{2}/\pi < t \leq \frac{1}{2} \end{cases}$$

where $H_0^\infty = 1, H_1^\infty = 4, H_2^\infty = 15, H_3^\infty = 77$ and

$$\begin{aligned} H_0(t) &= a^4(t), \\ H_1(t) &= a^4(t)\Big(2 + 2b(t)\Big), \\ H_2(t) &= a^4(t)\Big(3 + 7b(t) + 5b^2(t)\Big), \\ H_3(t) &= a^4(t)\Big(8 + 24b(t) + 30b^2(t) + 15b^3(t)\Big), \end{aligned}$$

with

$$a(t) = \frac{2}{\pi\left(1 - \frac{\pi^2 t^2}{6}\right)}, \quad b(t) = \frac{1}{f_c}\frac{a(t)}{t}.$$

Proof. Let us divide the proof into steps.

Step 1: First we introduce some bounds on the sine function. An upper bound is the constant 1 and for the lower bound, we will make use of the following estimates

$$\sin(\pi t) \geq 2t, \quad \text{for } t \in [0, 1/2], \tag{3.11}$$

$$\sin(\pi t) \geq \pi t - \frac{\pi^3 t^3}{6} = \frac{2t}{a(t)}, \quad \text{for } t \geq 0, \tag{3.12}$$

The inequality (3.11) holds because of the concavity of the function $t \mapsto \sin(\pi t) - 2t$ while it is 0 when evaluated at 0 and 1/2. and (3.12) can be easily obtained by a Taylor expansion around 0.

Step 2: We calculate $K^{(\ell)}$ for $\ell \in \{1,2,3\}$. For $t \neq 0$ we have

$$K'(t) = 4\pi\left(\frac{\sin(f\pi t)}{f\sin(\pi t)}\right)^3\left(\frac{\cos(f\pi t)}{\sin(\pi t)} - \frac{\sin(f\pi t)\cos(\pi t)}{f\sin^2(\pi t)}\right), \tag{3.13}$$

$$K''(t) = \frac{4\pi^2 \sin^2(f\pi t)}{f^2 \sin^4(\pi t)} \Bigg[3 \left(\cos(f\pi t) - \frac{\sin(f\pi t)\cos(\pi t)}{f \sin(\pi t)} \right)^2 - \sin^2(f\pi t) - \frac{\sin(2f\pi t)}{f \tan \pi t} + \frac{\sin^2(f\pi t)}{f^2 \tan^2(\pi t)} + \frac{\sin^2(f\pi t)}{f^2 \sin^2(\pi t)} \Bigg], \tag{3.14}$$

$$K'''(t) = \frac{4\pi^3 \sin(f\pi t)}{f \sin^4(\pi t)} \Big[6L_1(t) + 9L_2(t) + L_3(t) \sin^2(f\pi t) \Big] \tag{3.15}$$

where

$$\begin{aligned}
L_1(t) &= \left(\cos(f\pi t) - \frac{\sin(f\pi t)\cos(\pi t)}{f \sin(\pi t)} \right)^3, \\
L_2(t) &= \left(\cos(f\pi t) - \frac{\sin(f\pi t)\cos(\pi t)}{f \sin(\pi t)} \right) \left(-\sin^2(f\pi t) - \frac{\sin(2f\pi t)}{f \tan(\pi t)} + \frac{\sin^2(f\pi t)}{f^2 \tan^2(\pi t)} + \frac{\sin^2(f\pi t)}{f^2 \sin^2(\pi t)} \right), \\
L_3(t) &= \left(\frac{3\cos(f\pi t)\left(1 + \cos^2(\pi t)\right)}{f^2 \sin^2(\pi t)} - \cos(f\pi t) + \frac{3\sin(f\pi t)}{f \tan(\pi t)} - \frac{\sin(f\pi t)(1 + 5\cos(\pi t))}{f^3 \sin^3(\pi t)} \right).
\end{aligned}$$

For the derivatives at 0 we consider the Fejer kernel $F(t) = \frac{1}{f^2} \sum_{k=0}^{f-1} \sum_{j=-k}^{k} e^{i2\pi jt}$ as in Lemma 3.7. We get

$$\begin{aligned}
F(0) &= \frac{1}{f^2} \sum_{k=0}^{f-1} \sum_{j=-k}^{k} 1 = 1, \\
F'(0) &= \frac{1}{f^2} \sum_{k=0}^{f-1} \sum_{j=-k}^{k} i2\pi j = 0, \\
F''(0) &= -\frac{1}{f^2} \sum_{k=0}^{f-1} \sum_{j=-k}^{k} 4\pi^2 j^2 = -\frac{2\pi^2(f^2-1)}{3}, \\
F'''(0) &= -\frac{1}{f^2} \sum_{k=0}^{f-1} \sum_{j=-k}^{k} i8\pi^3 j^3 = 0.
\end{aligned}$$

As $K = F^2$ we obtain $K'(0) = 2F(0)F'(0) = 0$, $K''(0) = 2F(0)F''(0) + 2F'(0)^2 = -\frac{4\pi^2(f^2-1)}{3}$ and $K'''(0) = 2F'(0)F''(0) + 2F(0)F'''(0) + 4F'(0)F''(0) = 0$.

Step 3: Now, we use the bounds gathered in Step 1 to prove the desired bounds on K and its derivatives:

- For $\ell = 0$:
  It follows from (3.11) that $|K(t)| \leq \frac{1}{f^4 \sin^4(\pi t)} \leq \frac{1}{2^4 f^4 t^4}$ on $[\sqrt{2}/\pi, 1/2)$, while (3.12) implies that $|K(t)| \leq \frac{1}{f^4 \sin^4 \pi t} \leq \frac{a(t)^4}{2^4 f^4 t^4}$ on $[\lambda_c/2, \sqrt{2}/\pi]$.
- For $\ell = 1$:
  On $[\sqrt{2}/\pi, 1/2)$ plugging (3.11) and $2ft > 1$ in (3.13) gives

$$|K'(t)| \leq \frac{4\pi}{(2ft)^3} \left[\frac{1}{2t} + \frac{1}{f(2t)^2} \right] \leq \frac{4\pi}{(2f)^3 t^4}.$$

On $[\lambda_c/2, \sqrt{2}/\pi]$ plugging (3.12) in (3.13) gives

$$|K'(t)| \leq \frac{4\pi a^3(t)}{f^3(2t)^3}\left[\frac{a(t)}{2t} + \frac{a^2(t)}{f(2t)^2}\right] = \frac{\pi a^4(t)}{(2f)^3 t^4}\left(2 + \frac{2a(t)}{2ft}\right).$$

- <u>For $\ell = 2$:</u>
  On $[\sqrt{2}/\pi, 1/2)$ plugging (3.11) and $2ft > 1$ in (3.14) gives

$$|K''(t)| \leq \frac{4\pi^2}{f^2(2t)^4}\left[3\left(1 + \frac{1}{2ft}\right)^2 + \frac{1}{2ft} + \frac{2}{(2ft)^2}\right] \leq \frac{15\pi^2}{(2f)^2 t^4}.$$

  On $[\lambda_c/2, \sqrt{2}/\pi]$ plugging (3.12) in (3.14) gives

$$|K''(t)| \leq \frac{4\pi^2 a^4(t)}{f^2(2t)^4}\left[3\left(1 + \frac{a(t)}{f2t}\right)^2 + \frac{a(t)}{2ft} + \frac{2a^2(t)}{(2ft)^2}\right] = \frac{\pi^2 a^4(t)}{(2f)^2 t^4}\left[3 + \frac{7a(t)}{2ft} + \frac{5a^2(t)}{(2f)^2 t^2}\right].$$

- <u>For $\ell = 3$:</u>
  On $[\sqrt{2}/\pi, 1/2)$ plugging (3.11) and $2ft > 1$ in (3.15) gives

$$\begin{aligned}|K'''(t)| \leq \frac{4\pi^3}{f(2t)^4}&\left[6\left(1 + \frac{1}{2ft}\right)^3 + 9\left(1 + \frac{1}{2ft}\right)\left(1 + \frac{2}{2ft} + \frac{2}{(2ft)^2}\right)\right.\\ &\left.\quad + \frac{6}{(2ft)^2} + 1 + \frac{3}{2ft} + \frac{6}{(2ft)^3}\right]\\ \leq \frac{77\pi^3}{2ft^4}&.\end{aligned}$$

  On $[\lambda_c/2, \sqrt{2}/\pi]$ plugging (3.12) in (3.15) gives

$$\begin{aligned}|K'''(t)| \leq \frac{4\pi^3}{f(2t)^4}&\left[6\left(1 + \frac{a(t)}{2ft}\right)^3 + 9\left(1 + \frac{a(t)}{2ft}\right)\left(1 + \frac{2a(t)}{2ft} + \frac{2a^2(t)}{(2ft)^3}\right)\right.\\ &\left.\quad + \left(\frac{6a^2(t)}{(2ft)^2} + 1 + \frac{3a(t)}{2ft} + \frac{6a^3(t)}{(2ft)^3}\right)\right]\\ \leq \frac{\pi^3 a^4(t)}{2ft^4}&\left(8 + 24\frac{a(t)}{2ft} + 30\frac{a^2(t)}{(2ft)^2)} + 15\frac{a^3(t)}{(2ft)^3}\right).\end{aligned}$$

<u>Step 4:</u> Having proved the bounds on $\left|K^{(\ell)}(t)\right|$, now we come to the statements about monotonicity. It is clear that $t \mapsto \frac{1}{t^4}$ is monotonically decreasing on $(\sqrt{2}/\pi, \frac{1}{2})$, hence so is B_ℓ. Furthermore, we have for $c(t) := \frac{2}{\pi f_c b(t)} = \left(1 - \frac{\pi^2 t^2}{6}\right) t$ that

$$c'(t) = 1 - \frac{\pi^2 t^2}{2} > 0, \qquad \text{for any } t \in (0, \sqrt{2}/\pi).$$

This means that b is monotonically decreasing on $(0, \sqrt{2}/\pi]$, and hence so is B_ℓ. Finally, in the discontinuity point $t = \sqrt{2}/\pi$ it is easy to check that $H_\ell(\sqrt{2}/\pi) < H_\ell^\infty$ for each $\ell \in \{0, 1, 2, 3\}$. These together show that B_ℓ monotonically decreasing on $(0, \frac{1}{2})$.

For $k \in \mathbb{N}$ and any positive t we have that

$$(b^k)''(t) = k(k-1)b^{k-2}(t)(b')^2(t) + kb^{k-1}(t)b''(t)$$

is positive, since

$$b''(t) = -\frac{c''(t)c^2(t) - 2(c')^2(t)c(t)}{c^4(t)}$$

is positive as $c''(t) = -\pi^2 t < 0$. This shows that b^k is strictly convex on $(0, \infty)$. Hence $b^k(\Delta - t) + b^k(\Delta + t) > 2b^k(\Delta)$ holds for every positve t, and thus the derivative (w.r.t t) of $b^k(\Delta - t) + b^k(\Delta + t)$ is positive for $t \in (0, \Delta/2)$. Therefore $b^k(\Delta - t) + b^k(\Delta + t)$ is monotonically increasing in t. Note that for $\Delta + t \leq \sqrt{2}/\pi$ (just so that $\tilde{B}_\ell(\Delta + t)$ can be defined) we have that $\tilde{B}_\ell(\Delta - t) + \tilde{B}_\ell(\Delta + t)$ is a linear combination of $b^k(\Delta - t) + b^k(\Delta + t)$ with positive coefficients, therefore it is increasing in t. This completes the proof.

□

We continue this section about the interpolation kernel K with a technical lemma, which gives upper bounds for the quantities of the form $\sum_{t_j \in T\setminus\{\tau\}} \left|K^{(\ell)}(t - t_j)\right|$ for some $\tau \in T$, where t lies very near τ. Since $|K|$ is both even and 1-periodic function, we can without loss of generality map the unit circle $\mathbb{T}$ to the interval $[-\frac{1}{2} + t, \frac{1}{2} + t]$ and assume that $0 \in T$. Note that the upper bounds in lemma 3.8 and in lemma 3.9 below will be computed numerically and will be used in the next sections to show the existence of interpolating coefficients (i.e. α, β) and then to control the constructed interpolation polynomial.

{lemma: Fe

Lemma 3.9. *Suppose* $0 \in T$. *Then for all* $t \in [0, \min\{\Delta/2, \Delta_{\min}\}]$ *it holds*

$$\sum_{t_j \in T\setminus\{0\}} \left|K^{(\ell)}(t - t_j)\right| \leq F_\ell(\Delta, t) = F_\ell^+(\Delta, t) + F_\ell^-(\Delta, t) + F_\ell^\infty(\Delta_{\min})$$

where

$$F_\ell^+(\Delta, t) = \max\left\{\max_{\Delta \leq t_+ \leq 3\Delta_{\min}} \left|K^{(\ell)}(t - t_+)\right|, B_\ell(3\Delta_{\min} - t)\right\} + \sum_{j=2}^{20} \tilde{B}_\ell(j\Delta_{\min} - t),$$

$$F_\ell^-(\Delta, t) = \max\left\{\max_{\Delta \leq t_- \leq 3\Delta_{\min}} \left|K^{(\ell)}(t_-)\right|, B_\ell(3\Delta_{\min})\right\} + \sum_{j=2}^{20} \tilde{B}_\ell(j\Delta_{\min} + t),$$

$$F_\ell^\infty(\Delta_{\min}) = \frac{\kappa \pi^\ell H_\ell^\infty}{(f_c + 2)^{4-\ell}\Delta_{\min}^4}, \qquad \textit{where } \kappa = \frac{\pi^4}{45} - 2\sum_{j=1}^{19} \frac{1}{j^4} \leq 8.98 \times 10^{-5}.$$

Moreover, $F_\ell(\Delta, t)$ *is monotonically decreasing in* Δ *for all* t*, and* $F_\ell(\Delta_{\min}, t)$ *is monotonically increasing in* t.

Proof. Let us divide the sum $\sum_{t_j \in T\setminus\{0\}} \left|K^{(\ell)}(t - t_j)\right|$ into the sum over all t_j's that are larger than t and the sum over the other t_j's, i.e. that are smaller than t.

(1) We start we the sum over the t_j's that are larger than t. Without loss of generality let us assume that the t_j's on the right of t are $t_1, t_2, \ldots$ and they are numbered in a way such that $0 < t_1 < t_2 < \ldots$ Then the sum over them is equal to

$$\sum_{\substack{t_j \in T: \\ t < t_j \leq 1/2 + t}} \left|K^{(\ell)}(t - t_j)\right| = \left|K^{(\ell)}(t_1 - t)\right| + \sum_{\substack{t_j \in T\setminus\{t_1\}: \\ t < t_j \leq 1/2 + t}} \left|K^{(\ell)}(t_j - t)\right|. \tag{3.16}$$

a) The second term on the right hand side of (3.16) can be estimated using Lemma 3.8 as follows:

$$\begin{aligned}\sum_{\substack{t_j\in T\setminus\{t_1\}:\\ t<t_j\le 1/2+t}}\left|K^{(\ell)}(t-t_j)\right| &\le \sum_{\substack{t_j\in T\setminus\{t_1\}:\\ t<t_j\le 1/2+t}} B_\ell(t_j-t)\\ &\le \sum_{j=2}^{20} B_\ell(t_j-t)+\sum_{j=21}^{\infty} B_\ell(t_j-t)\\ &\le \sum_{j=2}^{20} B_\ell(j\Delta_{\min}-t)+\sum_{j=21}^{\infty}\frac{\pi^l H_\ell^\infty}{(f_c+2)^{4-l}(t_j-t)^4}\\ &\le \sum_{j=2}^{20} \tilde{B}_\ell(j\Delta_{\min}-t)+\frac{\pi^\ell}{(f_c+2)^{4-\ell}}\sum_{j=21}^{\infty}\frac{H_\ell^\infty}{(j\Delta_{\min}-t)^4};\end{aligned} \tag{3.17}$$

the third inequality holds because $j\Delta_{\min}-t\le t_j-t$ while B_ℓ is decreasing in t, and the last inequality holds because $j\Delta_{\min}-t\le 21\Delta_{\min}\le 21.\frac{2}{f_c}<0.33<\sqrt{2}/\pi$ due to the assumption that $f_c\ge 128$. Now the second term in (3.17) can be upper bounded using the following estimate

$$\sum_{j=21}^{\infty}\frac{H_\ell^\infty}{(j\Delta_{\min}-t)^4}\le\sum_{j=20}^{\infty}\frac{H_\ell^\infty}{(j\Delta_{\min})^4}=\frac{H_\ell^\infty}{\Delta_{\min}^4}\left(\sum_{j=1}^{\infty}\frac{1}{j^4}-\sum_{j=1}^{19}\frac{1}{j^4}\right)=\frac{H_\ell^\infty}{\Delta_{\min}^4}\left(\frac{\pi^4}{90}-\sum_{j=1}^{19}\frac{1}{j^4}\right)=\frac{\kappa H_\ell^\infty}{2\Delta_{\min}^4}.$$

Here we have used that $t\le\Delta_{\min}$ and that the Riemann zeta function at 4, i.e. $\sum_{j=1}^\infty\frac{1}{j^4}$, is equal to $\frac{\pi^4}{90}$. To summarize, we have

$$\sum_{\substack{t_j\in T\setminus\{t_1\}:\\ t<t_j\le 1/2+t}}\left|K^{(\ell)}(t-t_j)\right|\le\sum_{j=2}^{20}\tilde{B}_\ell(j\Delta_{\min}-t)+\frac{\kappa\pi^\ell H_\ell^\infty}{2(f_c+2)^{4-\ell}\Delta_{\min}^4}.$$

b) We come back to the first term in (3.16), it holds

$$\left|K^{(\ell)}(t_1-t)\right|\le\begin{cases}\max_{\Delta\le t_+\le 3\Delta_{\min}}\left|K^{(\ell)}(t-t_+)\right|, & t_+\le 3\Delta_{\min},\\ B_\ell(3\Delta_{\min}-t), & t_+>3\Delta_{\min}.\end{cases}$$

Together we obtain an upper bound for the sum over all t_j's that lie on the right half of $[-\frac{1}{2}+t,\frac{1}{2}+t]$, that is

$$\sum_{\substack{t_j\in T:\\ t<t_j\le 1/2+t}}\left|K^{(\ell)}(t-t_j)\right|\le F_\ell^+(\Delta,t)+F_\ell^\infty(\Delta_{\min})/2.$$

(2) Similarly, the sum over t_j's that lie on the left of t can be estimated by the same process. This actually gives

$$\sum_{\substack{t_j\in T:\\ -1/2+t<t_j<t}}\left|K^{(\ell)}(t-t_j)\right|\le F_\ell^-(\Delta_{\min},t)+F_\ell^\infty(\Delta_{\min})/2.$$

One may note that there is a small difference in the first term of the definition of F_ℓ^- and F_ℓ^+, this is as a result of the positivity of t and the monotonicity of B_ℓ. Now putting the estimates gathered in (1) and (2) together yields

$$\sum_{t_j\in T\setminus\{0\}}\left|K^{(\ell)}(t-t_j)\right|\le F_\ell(\Delta,t)=F_\ell^+(\Delta,t)+F_\ell^-(\Delta,t)+F_\ell^\infty(\Delta_{\min}).$$

(3) Note that for every t the terms

$$F_\ell^+(\Delta,t) = \max\left\{\max_{\Delta\le t_+\le 3\Delta_{\min}}\left|K^{(\ell)}(t-t_+)\right|, B_l(3\Delta_{\min}-t)\right\} + \sum_{j=2}^{20}\tilde{B}_\ell(j\Delta_{\min}-t)$$

and

$$F_\ell^-(\Delta,t) = \max\left\{\max_{\Delta\le t_-\le 3\Delta_{\min}}\left|K^{(\ell)}(t_-)\right|, B_l(3\Delta_{\min})\right\} + \sum_{j=2}^{20}\tilde{B}_\ell(j\Delta_{\min}+t)$$

are both decreasing in Δ, hence $F_\ell(\Delta,t)$ is decreasing in Δ.

(4) Finally we turn to the monotonicity of $F(\Delta_{\min},t)$ w.r.t. t. Fix $\Delta=\Delta_{\min}$. Recall that $j\Delta_{\min}+t\le 21\Delta_{\min} < \sqrt{2}/\pi$ for $j\le 20$, hence by Lemma 3.8 we obtain that $\tilde{B}_\ell(j\Delta_{\min}-t)+\tilde{B}_\ell(j\Delta_{\min}+t)$ is monotonically increasing in t. It remains to check that the first term in the expression for $F_\ell^+(\Delta_{\min},t)$, i.e.

$$\max\left\{\max_{\Delta\le t_+\le 3\Delta_{\min}}\left|K^{(\ell)}(t-t_+)\right|, B_\ell(3\Delta_{\min}-t)\right\}$$

is also increasing in t (because then it will be analogous for the first term in the expression for $F_\ell^-(\Delta_{\min},t)$). This term can be rewritten as

$$\max\left\{\max_{\Delta_{\min}-t\le u\le 3\Delta_{\min}-t}\left|K^{(\ell)}(u)\right|, B_\ell(3\Delta_{\min}-t)\right\}.$$

Let $t'\ge t$ be arbitrary on the intervall $[0,\Delta/2]$. Then we get that

$$\max_{\Delta_{\min}-t'\le u\le 3\Delta_{\min}-t'}\left|K^{(\ell)}(u)\right| \ge \max_{\Delta_{\min}-t\le u\le 3\Delta_{\min}-t'}\left|K^{(\ell)}(u)\right|$$

Also, by Lemma 3.8 we obtain

$$B_\ell(3\Delta_{\min}-t') \ge \begin{cases} B_\ell(3\Delta_{\min}-t), & \\ \left|K^{(\ell)}(u)\right|, & u\ge 3\Delta_{\min}-t'. \end{cases}$$

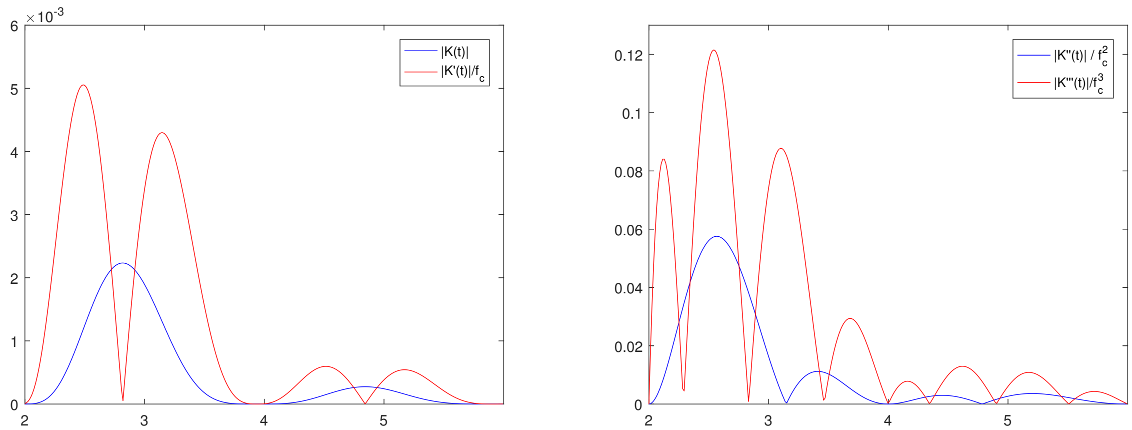
Combining those estimates yields

$$\max\left\{\max_{\Delta_{\min}-t'\le u\le 3\Delta_{\min}-t'}\left|K^{(\ell)}(u)\right|, B_\ell(3\Delta_{\min}-t')\right\} \ge \max\left\{\max_{\Delta_{\min}-t\le u\le 3\Delta_{\min}-t}\left|K^{(\ell)}(u)\right|, B_\ell(3\Delta_{\min}-t)\right\},$$

which shows exactly the desired monotonicity, and hence finishes the proof.

□

As mentioned before, Section 3.4.3 and Section 3.4.4 will make use of the bounds in lemma 3.9 to prove the existence of the interpolating coefficients α, β and then to control them as well as to bound the absolute value of the constructed interpolating trigonometric polynomial on the off-support. It will turn out that we need numerical upper bounds on $F_\ell(\Delta_{\min},t)$ at $t\in\{0, 0.1649\lambda_c, 0.4269\lambda_c, 0.7559\lambda_c\}$ (for the last two quantities we just need bounds for $\ell=0,1$). For a fixed t, the maximum of $\left|K^{(\ell)}(t-t_+)\right|$ where t_+ ranges over $[\Delta_{\min},3\Delta_{\min}]$ can be computed numerically, e.g. by trisection method (after restricting on an interval that contains only one maximum point). For reference, these functions are plotted in Figure 3.4. The upper bounds on $F_\ell(\Delta_{\min},t)$ for t as discussed above are collected in Table 3.1.



(a) **(b)**

bsK_(t-t+)}

Figure 3.4: $\left|K^{(\ell)}(t)\right|$ for $t \in [\Delta_{\min}, 3\Delta_{\min}]$. The scaling of the x-axis is in units of λ_c.

t/λ_c	$F_0(1.98\lambda_c, t)$	$F_1(1.98\lambda_c, t)$	$F_2(1.98\lambda_c, t)$	$F_3(1.98\lambda_c, t)$
0	6.253×10^{-3}	$7.639 \times 10^{-2} f_c$	$1.053 f_c^2$	$8.078 f_c^3$
0.1649	6.279×10^{-3}	$7.659 \times 10^{-2} f_c$	$1.055 f_c^2$	$18.56 f_c^3$
0.4269	8.029×10^{-2}	$0.3042 f_c$		
0.7559	5.565×10^{-2}	$1.918 f_c$		

main_table}

Table 3.1: Numerical upper bounds on $F_\ell(\Delta_{\min}, t)$ with $\Delta_{\min} = 1.98\lambda_c$.

3.4.3 Existence and estimate of the solution

{section:

In this section, we prove the existence of a solution of (3.9) and furthermore gives an upper bound (in infinity norm) for this solution, which will be later used in the proof of the estimate $|q(t)| < 1$ for $t \in \mathbb{T} \setminus T$. First we introduce a sufficient condition for the solvability of (3.9).

{lemma: Sc

Lemma 3.10. *The matrix* $\begin{bmatrix} D_0 & D_1 \\ D_1 & D_2 \end{bmatrix}$ *is invertible if* D_2 *and its so-called Schur complement* $D_0 - D_1 D_2^{-1} D_1$ *are invertible. In this case, the unique solution to (3.9) is* $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} I \\ -D_2^{-1} D_1 \end{bmatrix} C^{-1} v$ *where* I *denotes the identity matrix in* $\mathbb{C}^{|T| \times |T|}$ *and* $C := D_0 - D_1 D_2^{-1} D_1$.

Proof. If D_2 is invertible, then analogously to the LDU-Decomposition we have

$$\begin{bmatrix} D_0 & D_1 \\ D_1 & D_2 \end{bmatrix} = \begin{bmatrix} I & D_1 D_2^{-1} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} D_0 - D_1 D_2^{-1} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ D_2^{-1} D_1 & I \end{bmatrix}.$$

The second statement follows by simple matrix multiplications. □

In order to apply this lemma, we have to prove that D_2 and its Schur complement $D_0 - D_1 D_2^{-1} D_1$ are indeed invertible. To gain the invertibility of a matrix and also get an estimate for the solution, we will make use of the following result:

{lemma: Ne

Lemma 3.11. *A matrix* $M \in \mathbb{C}^{|T| \times |T|}$ *is invertible if* $\|I - M\| < 1$, *where* $\|\cdot\|$ *is an arbitrary operator norm. In particular,*

$$\left\| M^{-1} \right\| \leq \frac{1}{1 - \|I - M\|}$$

.

Proof. Set $H := I - M$, then $\|H\| < 1$. We get that

$$\left\| \sum_{k=m+1}^{n} H^k \right\| \leq \sum_{k=m+1}^{n} \|H\|^k$$

for any $m, n \in \mathbb{N}$. This shows that the so-called Neumann series $\sum_{k=0}^{\infty} H^k$ converges in the space of linear bounded operators from $\mathbb{C}^{|T|}$ to $\mathbb{C}^{|T|}$. Thus,

$$\sum_{k=0}^{\infty} H^k M = \lim_{n \to \infty} \sum_{k=0}^{n} H^k (I - H) = \lim_{n \to \infty} \left(I - H^{n+1} \right) = I$$

and also,

$$M \sum_{k=0}^{\infty} H^k = \lim_{n \to \infty} \sum_{k=0}^{n} (I - H) H^k = \lim_{n \to \infty} \left(I - H^{n+1} \right) = I.$$

In other words, $M^{-1} = \sum_{k=0}^{\infty} H^k$.
For the last estimate, we use the triangle inequality

$$\left\| M^{-1} \right\| \leq \|I\| + \left\| I - M^{-1} \right\| \leq 1 + \left\| M^{-1} \right\| \|I - M\|.$$

□

Now we are able to prove the existence of a solution of Proposition 3.6, i.e. an interpolating coefficient vector for our interpolation problem:

{Lemma: dual_po

Lemma 3.12. *Under the hypotheses of Proposition 3.6 there exist coefficient vectors $\alpha, \beta \in \mathbb{C}^{|T|}$ such that*

$$\begin{aligned} \|\alpha\|_\infty &\leq 1 + 8.824 \times 10^{-3} =: \alpha^\infty, \\ \|\beta\|_\infty &\leq 3.294 \times 10^{-2} \lambda_c =: \beta^\infty. \end{aligned} \tag{3.18}$$

Moreover, if $v_1 = 1$ then

$$\begin{aligned} \mathrm{Re}\,(\alpha_j) &\geq 1 - 8.824 \times 10^{-3}, \\ |\mathrm{Im}\,(\alpha_j)| &\leq 8.824 \times 10^{-3}. \end{aligned} \tag{3.19}$$

Proof. From Lemma 3.8 we obtain

$$\|I - D_0\|_\infty \leq F_0(\Delta_{min}, 0) \leq 6.253 \times 10^{-3}, \tag{3.20}$$

$$\|D_1\|_\infty \leq F_1(\Delta_{min}, 0) \leq 7.639 \times 10^{-2} f_c, \tag{3.21}$$

$$\|K''(0)I - D_2\|_\infty \leq F_2(\Delta_{min}, 0) \leq 1.053 f_c^2. \tag{3.22}$$

The bound (3.22) and the equality $K''(0) = -\pi^2 f_c(f_c + 4)/3$ (computed in the proof of Lemma 3.8) give

$$\left\| I - \frac{D_2}{K''(0)} \right\|_\infty < 1.$$

By Lemma 3.11, D_2 is invertible and

$$\left\|D_2^{-1}\right\|_\infty \leq \frac{1}{|K''(0)| - \|K''(0)I - D_2\|_\infty} \leq \frac{0.4275}{f_c^2}. \tag{3.23}$$

Combining this with (3.20) and (3.21) yields

$$\left\|I - (D_0 - D_1 D_2^{-1} D_1)\right\|_\infty \leq \|I - D_0\|_\infty + \|D_1\|_\infty^2 \left\|D_2^{-1}\right\|_\infty \leq 8.747 \times 10^{-3} < 1. \tag{3.24}$$

This estimate combined with Lemma 3.11 implies that $C := D_0 - D_1 D_2^{-1} D_1$ is invertible and hence, by Lemma 3.10 the matrix $\begin{bmatrix} D_0 & D_1 \\ D_1 & D_2 \end{bmatrix}$ is also invertible. Moreover, we obtain the solution of the system of linear equations (3.9) as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} I \\ -D_2^{-1} D_1 \end{bmatrix} C^{-1} v.$$

To bound this vector w.r.t. the infinity norm we make use of Lemma 3.11 and the estimates (3.21), (3.23) and (3.24). Indeed we get

$$\|\alpha\|_\infty = \left\|C^{-1} v\right\|_\infty \leq \left\|C^{-1}\right\|_\infty \leq \frac{1}{1 - \|I - C\|_\infty} \leq 1 + 8.824 \times 10^{-3},$$

and

$$\|\beta\|_\infty \leq \left\|D_2^{-1} D_1 C^{-1}\right\|_\infty \leq \left\|D_2^{-1}\right\|_\infty \|D_1\|_\infty \left\|C^{-1}\right\|_\infty \leq 3.294 \times 10^{-2} \lambda_c.$$

Finally, let $v_1 = 1$, we get $\alpha_1 = [C^{-1} v]_1 = 1 - \gamma_1$ with $\gamma_1 = [(I - C^{-1})v]_1$. Also by Lemma 3.11 and (3.24) we have

$$|\gamma_1| \leq \left\|I - C^{-1}\right\|_\infty = \left\|C^{-1}(I - C)\right\|_\infty \leq \left\|C^{-1}\right\| \|I - C\|_\infty \leq \frac{\|I - C\|_\infty}{1 - \|I - C\|_\infty} \leq 8.824 \times 10^{-3}.$$

From this we obtain $\mathrm{Re}\,(\alpha_1) = 1 - \mathrm{Re}\,(\gamma_1) \geq 1 - 8.824 \times 10^{-3}$ and $|\mathrm{Im}\,(\alpha_1)| = |\mathrm{Im}\,(\gamma_1)| \leq 8.824 \times 10^{-3}$. $\square$

3.4.4 Estimate of the constructed dual polynomial

est_dual_poly}

In the last section, we have shown the existence of a solution of (3.9) and also proved some bounds on this solution. Now, we continue the proof of Proposition 3.6 by showing that the constructed dual polynomial actually satisfies not only the desired interpolating properties but also the inequality condition, i.e. showing that $|q(t)| < 1$ for $t \in \mathbb{T} \setminus T$. The next lemma proves this statement for any t whose distance to some $\tau \in T$ is less than $0.1649\lambda_c$.

{lemma: du

Lemma 3.13. *Fix $\tau \in T$, then under the hypothesis of Theorem 3.2 it holds $|q(t)| < 1$ for $0 < |t - \tau| \leq 0.1649\lambda_c$.*

Proof. As mentioned before, we assume without loss of generality that $\tau = 0 = t_1$, and also that $q(0) = 1$ (otherwise consider $\tilde{q} = \overline{q(0)}q$, $\tilde{v_j} = \overline{q(0)}v_j$ and recall that $|q(0)| = 1$). By symmetry, it is enough to show the claim for $t \in (0, 0.1649\lambda_c]$. Let q_R be the real part of q and q_I the imaginary part. The first and second derivative of $|q|$ are given by

$$\frac{d\,|q|}{dt}(t) = \frac{q_R(t)q_R'(t) + q_I(t)q_I'(t)}{|q|},$$
$$\frac{d^2\,|q|}{dt^2}(t) = -\frac{\Big(q_R(t)q_R'(t) + q_I(t)q_I'(t)\Big)^2}{|q(t)|^3} + \frac{|q'(t)|^2 + q_R(t)q_R''(t) + q_I(t)q_I''(t)}{|q(t)|}.$$

Since $q'(0) = 0$, meaning that $q_R(0) = q_I(0) = 0$, we obtain that $|q|'(0) = 0$. We will show that the second derivative of $|q|$ is strictly negative in the interval $(0, 0.1649\lambda_c)$, because then the first derivative is also strictly negative, i.e. $|q|$ is monotonically decreasing and hence smaller than $|q(0)| = 1$ on this interval. It is enough to prove for $t \in (0, 0.1649\lambda_c)$ that

$$q_R(t)q_R''(t) + |q'(t)|^2 + |q_I(t)|\,|q_I''(t)| < 0, \tag{3.25}$$

as long as $|q(t)| > 0$. We use the Taylor expansion of K and its derivatives (see proof of Lemma 3.8, and note that $K^{(4)}(0) > 0$) around the origin to gain the following estimates for $t \in [-1/2, 1/2]$, which will be applied to bound the terms in (3.25):

$$K(t) \geq 1 - \frac{\pi^2}{6} f_c(f_c + 4)t^2, \tag{3.26}$$
$$|K'(t)| \leq \frac{\pi^2}{3} f_c(f_c + 4)t, \tag{3.27}$$
$$K''(t) \leq -\frac{\pi^2}{3} f_c(f_c + 4) + \frac{\pi^4}{6}(f_c + 2)^4 t^2, \tag{3.28}$$
$$|K''(t)| \leq \frac{\pi^2}{3} f_c(f_c + 4), \tag{3.29}$$
$$|K'''(t)| \leq \frac{\pi^4}{3}(f_c + 2)^4 t. \tag{3.30}$$

Here, the upper bounds are monotonically increasing in t while the lower bounds are monotonically decreasing in t, hence one can evaluate them at $t = 0.1649\lambda_c$ (and remember the assumption that $f_c \geq 128$) to obtain the following estimates for every $t \in [0, 0.1649\lambda_c]$:

$$\begin{aligned} K(t) &\geq 0.9539, & |K''(t)| &\leq -2.923 f_c^2, & & \\ |K'(t)| &\leq 0.5595 f_c, & |K''(t)| &\leq 3.393 f_c^2, & |K'''(t)| &\leq 5.697 f_c^3, \end{aligned}$$

Now we use these estimates combined with Lemma 3.9 and Lemma 3.12 to control the terms in (3.25). Recall from Lemma 3.9 that $F_\ell(\Delta, t)$ is monotonically decreasing in Δ while $F_\ell(\Delta_{\min}, t)$ is increasing in t, meaning that $F_\ell(\Delta, t) \leq F_\ell(\Delta_{\min}, 0.1649\lambda_c)$ for any $t \in [0, 0.1649\lambda_c]$. Thus, for such t we have

$$\begin{aligned} q_R(t) &= \sum_{t_j \in T} \mathrm{Re}\,(\alpha_j)\, K(t - t_j) + \mathrm{Re}\,(\beta_j)\, K'(t - t_j) \\ &\geq \mathrm{Re}\,(\alpha_1)\, K(t) - \|\alpha\|_\infty \sum_{t_j \in T \setminus \{0\}} |K(t - t_j| - \|\beta\|_\infty \sum_{t_j \in T} |K'(t - t_j)| \\ &\geq \mathrm{Re}\,(\alpha_1)\, K(t) - \alpha^\infty F_0(\Delta_{\min}, t) - \beta^\infty \left(|K'(t)| + F_1(\Delta_{\min}, t) \right) \\ &\geq 0.9182, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} |q_I(t)| &= \left| \sum_{t_j \in T} \mathrm{Im}\,(\alpha_j)\, K(t - t_j) + \mathrm{Im}\,(\beta_j)\, K'(t - t_j) \right| \\ &\leq |\mathrm{Im}\,(\alpha_1)| + \|\alpha\|_\infty \sum_{t_j \in T \setminus \{0\}} |K(t - t_j)| + \|\beta\|_\infty \sum_{t_j \in T} |K'(t - t_j)| \\ &\leq |\mathrm{Im}\,(\alpha_1)| + \alpha^\infty F_0(\Delta_{\min}, t) + \beta^\infty \left(|K'(t)| + F_1(\Delta_{\min}, t) \right) \\ &\leq 3.611 \times 10^{-2}, \end{aligned}$$

and

$$\begin{aligned} q_R''(t) &= \sum_{t_j \in T} \mathrm{Re}\,(\alpha_j)\, K''(t - t_j) + \mathrm{Re}\,(\beta_j)\, K'''(t - t_j) \\ &\leq \mathrm{Re}\,(\alpha_1)\, K''(t) + \|\alpha\|_\infty \sum_{t_j \in T \setminus \{0\}} |K''(t - t_j)| + \|\beta\|_\infty \sum_{t_j \in T} |K'''(t - t_j)| \\ &\leq \mathrm{Re}\,(\alpha_1)\, K''(t) + \alpha^\infty F_2(\Delta_{\min}, t) + \beta^\infty \left(|K'''(t)| + F_3(\Delta_{\min}, t) \right) \\ &\leq -1.034 f_c^2, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} |q_I''(t)| &= \left| \sum_{t_j \in T} \mathrm{Im}\,(\alpha_j)\, K''(t - t_j) + \mathrm{Im}\,(\beta_j)\, K'''(t - t_j) \right| \\ &\leq |\mathrm{Im}\,(\alpha_1)|\, |K''(t)| + \|\alpha\|_\infty \sum_{t_j \in T \setminus \{0\}} |K''(t - t_j)| + \|\beta\|_\infty \sum_{t_j \in T} |K'''(t - t_j)| \\ &\leq |\mathrm{Im}\,(\alpha_1)|\, |K''(t)| + \alpha^\infty F_2(\Delta_{\min}, t) + \beta^\infty \left(|K'''(t)| + F_3(\Delta_{\min}, t) \right) \\ &\leq 1.893 f_c^2, \end{aligned}$$

and finally

$$
\begin{aligned}
|q'(t)| &= \left| \sum_{t_j \in T} \alpha_j K'(t - t_j) + \beta_j K''(t - t_j) \right| \\
&\leq \|\alpha\|_\infty \sum_{t_j \in T} |K'(t - t_j)| + \|\beta\|_\infty \sum_{t_j \in T} |K''(t - t_j)| \\
&\leq \alpha^\infty |K'(t)| + \alpha^\infty F_1(\Delta_{\min,t}) + \beta^\infty \Big(|K''(t)| + F_2(\Delta_{\min}, t) \Big) \\
&\leq 0.7882 f_c.
\end{aligned}
$$

Therefore, for any $t \in (0, 0.1649\lambda_c]$, by putting all these bounds together, we have that

$$
q_R(t) q_R''(t) + |q'(t)|^2 + |q_I(t)| \, |q_I''(t)| \leq -9.291 \times 10^{-2} f_c^2 < 0,
$$

while the estimate (3.31) implies that $|q(t)| \geq q_R(t) > 0$. This shows that (3.25) holds and hence completes the proof. □

Now we need to consider the case when $|t - \tau| > 0.1649\lambda_c$ where $\tau \in T$ is fixed. This is discussed in the following lemma:

{lemma:dual

Lemma 3.14. *Fix $\tau \in T$ then under the hypothesis of Theorem 3.2 we have $|q(t)| < 1$ for $0.1649\lambda_c \leq |t - \tau| \leq \Delta/2$. This also holds for any t that is closer to $0 \in T$ than to any other element in T.*

Proof. As before we assume without loss of generality that $\tau = 0$ and $q(0) = 1$. At first we consider the statement for $0.1649\lambda_c \leq |t| \leq \Delta/2$, with symmetry argument we only need to consider the case when $0.1649\lambda_c \leq t \leq \Delta/2$. Using Lemma 3.9 yields a bound on the absolute value of the dual polynomial on $[0.1649\lambda_c, \Delta/2]$:

$$
\begin{aligned}
|q(t)| &= \left| \sum_{t_j \in T} \alpha_j K(t - t_j) + \beta_j K'(t - t_j) \right| \\
&\leq \|\alpha\|_\infty \left(|K(t)| + \sum_{t_j \in T \setminus \{0\}} |K(t - t_j)| \right) + \|\beta\|_\infty \left(|K'(t)| + \sum_{t_j \in T \setminus \{0\}} |K'(t - t_j)| \right) \\
&\leq \alpha^\infty \Big(|K(t)| + F_0(\Delta_{\min}, t) \Big) + \beta^\infty \Big(|K'(t)| + F_1(\Delta_{\min}, t) \Big).
\end{aligned}
$$

Similarly to the proof of Lemma 3.13 we use the Taylor expansion of K and K' around the origin to obtain the following inequalities, which hold on $(0.1649\lambda_c, 0.7559\lambda_c) \subset [-1/2, 1/2]$:

$$
\begin{aligned}
|K(t)| &\leq 1 - \frac{\pi^2}{6} f_c(f_c + 4) t^2 + \frac{\pi^4}{72} (f_c + 2)^4 t^4, \\
|K'(t)| &\leq \frac{\pi^2}{3} f_c (f_c + 4) t.
\end{aligned}
\tag{3.33}
$$

We define

$$
\begin{aligned}
L_1(t) &= \alpha^\infty \left(1 - \frac{\pi^2}{6} f_c (f_c + 4) t^2 + \frac{\pi^4}{72} (f_c + 2)^4 t^4 \right) + \beta^\infty \frac{\pi^2}{3} f_c (f_c + 4) t, \\
L_2(t) &= \alpha^\infty F_0(\Delta_{\min}, t) + \beta^\infty F_1(\Delta_{\min}, t).
\end{aligned}
$$

so that we can rewrite the bound on $|q(t)|$ from above for $t \in [0.1649, \Delta/2]$ as

$$|q(t)| \leq L_1(t) + L_2(t).$$

Now, L_1 is decreasing on $[0.1649\lambda_c, 0.7559\lambda_c]$ because its derivative is strictly negative on this interval:

$$\begin{aligned}
L_1'(t) &= -\alpha^\infty \left(\frac{\pi^2}{3} f_c(f_c+4)t - \frac{\pi^4}{18}(f_c+2)^4 t^3 \right) + \beta^\infty \frac{\pi^2}{3} f_c(f_c+4) \\
&\leq -\alpha^\infty \frac{\pi^2}{3} f_c^2 t + \alpha^\infty \frac{\pi^4}{18} \left(\frac{3f_c}{2} \right)^4 t^3 + \beta^\infty \frac{\pi^2}{3} \frac{3f_c^2}{2} \\
&< -\alpha^\infty \frac{\pi^2 f_c^2 t}{6} + \beta^\infty \frac{\pi^2 f_c^2}{2} < 0.
\end{aligned}$$

Meanwhile, L_2 is increasing on $(0.1649\lambda_c, 0.7559\lambda_c)$ by Lemma 3.9. These together show that

$$|q(t)| \leq L_1(t_1) + L_2(t_2) \qquad \text{for } t \in [t_1, t_2], \tag{3.34}$$

where $[t_1, t_2]$ is a subinterval of $[0.1649\lambda_c, 0.7559\lambda_c]$. Now we take $[t_1, t_2] = [0.1649\lambda_c, 0.4269\lambda_c]$ to get $|q(t)| \leq 0.99992 < 1$ on this interval, and then take $[t_1, t_2] = [0.4269\lambda_c, 0.7559\lambda_c]$ to obtain $|q(t)| \leq 0.9122 < 1$ on this interval. The necessary numerical upper bounds for $L_1(t_1)$ and $L_2(t_2)$ used to prove these estimates are collected in Table 3.2 below:

t_1/λ_c	t_2/λ_c	$L_1(t_1)$	$L_2(t_2)$
0.1649	0.4269	0.9818	1.812×10^{-2}
0.4269	0.7559	0.7929	0.1193

ound_L1_L2}

Table 3.2: Numerical upper bounds on $L_1(t_1), L_2(t_2)$ in (3.34).

Now, for $0.7559\lambda_c \leq t \leq \Delta/2$ we use the same technique as in the proof of Lemma 3.9 (i.e. divide the sum into positive and negative parts) and apply Lemma 3.8 to obtain:

$$\begin{aligned}
|q(t)| &\leq \alpha^\infty \left[|K(t)| + \sum_{t_j \in T\setminus\{0\}} |K(t-t_j)| \right] + \beta^\infty \left[|K'(t)| + \sum_{t_j \in T\setminus\{0\}} |K'(t-t_j)| \right] \\
&\leq \alpha^\infty \left[B_0(t) + \sum_{j=0}^{\infty} B_0(j\Delta_{\min} + \Delta - t) + \sum_{j=1}^{\infty} B_0(j\Delta_{\min} + t) \right] \\
&\quad + \beta^\infty \left[B_1(t) + \sum_{j=0}^{\infty} B_1(j\Delta_{\min} + \Delta - t) + \sum_{j=1}^{\infty} B_1(j\Delta_{\min} + t) \right] \\
&\leq \alpha^\infty \left[B_0(0.7559\lambda_c) + \sum_{j=1}^{\infty} B_0(j\Delta_{\min} - 0.7559\lambda_c) + \sum_{j=1}^{\infty} B_0(j\Delta_{\min} + 0.7559\lambda_c) \right] \\
&\quad + \beta^\infty \left[B_1(0.7559\lambda_c) + \sum_{j=1}^{\infty} B_1(j\Delta_{\min} - 0.7559\lambda_c) + \sum_{j=1}^{\infty} B_1(j\Delta_{\min} + 0.7559\lambda_c) \right] \\
&\leq 0.758.
\end{aligned}$$

Here we have used the monotonicity of B_0 and B_1 at the second and third inequality.

Finally, for the case when $t > \Delta/2$, let τ_+ be the closest spike of $\tau = 0$ to the right, then we have $t \leq \tau_+/2$ if t is closer to 0 than to τ_+ and the last estimate above holds as well (with minor changes). $\square$

Now, combining Lemma 3.13 and Lemma 3.14 gives directly that the desired inequality condition for our constructed dual polynomial holds, $|q(t)| < 1$ for any $t \in \mathbb{T} \setminus T$. In other words, Proposition 3.6 is proved and this actually almost completes the proof of our main result, Theorem 3.2. The last thing we need to prove in this theorem is the improvement for real-valued signals, which will be discussed in Section 3.4.5. But before that we want to introduce a lemma that is similar to Lemma 3.13 and Lemma 3.14 and will be used to derive stability results in Chapter 4. This lemma will also close the section.

{lemma: du

Lemma 3.15. *Suppose further to (3.3) that $\Delta = \Delta(T) \geq 2.5\lambda_c$. Then for any $\tau \in T$ and for all $t \in \mathbb{T}$ with $|t - \tau| \leq 0.1649\lambda_c$, we have*

$$|q(t)| \leq 1 - 0.3353 f_c^2 (t - \tau)^2.$$

Moreover, if $\min_{\tau \in T} |t - \tau| > 0.1649\lambda_c$, an upper bound on $|q(t)|$ is given by the right hand side above evaluated at $|t - \tau| = 0.1649\lambda_c$.

Proof. Again assume without loss of generality that $\tau = 0$. Then we process exactly as in the proof of Lemma 3.12 and Lemma 3.13 but for $\Delta = 2.5\lambda_c$ to obtain that for any t with $0 \leq |t| \leq 0.1649\lambda_c$ it holds

$$\frac{d^2 |q|}{dt^2}(t) \leq -0.6706 f_c^2.$$

Note that also α^∞ and β^∞ from Lemma 3.12 are enhanced. The necessary numerical upper bounds on $F_\ell(2.5\lambda_c, t)$ at $t \in \{0, 0.1649\lambda_c\}$ are collected in Table 3.3 below:

t/λ_c	$F_0(2.5\lambda_c, t)$	$F_1(2.5\lambda_c, t)$	$F_2(2.5\lambda_c, t)$	$F_3(2.5\lambda_c, t)$
0	5.175×10^{-3}	$6.839 \times 10^{-2} f_c$	$0.8946 f_c^2$	$7.644 f_c^3$
0.1649	5.182×10^{-3}	$6.849 \times 10^{-2} f_c$	$0.9459 f_c^2$	$7.647 f_c^3$

Table 3.3: Numerical upper bounds on $F_\ell(\Delta_{\min}, t)$ with $\Delta_{\min} = 2.5\lambda_c$.

{table: nu

Recall that $|q(0)| = 1$ and $q'(0) = 0$, hence for $t \in (0, 0.1649\lambda_c]$ we have

$$|q(t)| \leq |q(0)| - \frac{1}{2} 0.6706 f_c^2 t^2 = 1 - 0.3353 f_c^2 t^2.$$

Now, for $t = 0.1649\lambda_c$, the right-hand side becomes 0.9909. The calculations in the proof of Lemma 3.14 (but with $\Delta = 2.5\lambda_c$) give that for $t \geq 0.1649\lambda_c$ it holds $|q(t)| \leq 0.9843 < 0.9909$. This completes the proof. ☐

3.4.5 Improvement for real-valued signals

{section:

In the previous parts of Section 3.4, we have introduced the proof of Theorem 3.2 for complex-valued signals where $\Delta_{\min} = 2\lambda_c$. Recall that Theorem 3.2 also states that for real-valued signals, the constant can be enhanced, indeed the minimum separation now only needs to satisfy $\Delta(T) \geq 1.87\lambda_c$. The proof for this statement is almost the same as that for complex-valued signal, which we have discussed so far in Section 3.4. The modifications are similar as in Lemma 3.15 and are listed as below:

In Lemma 3.12, the minimum separation of T is reduced to $1.87\lambda_c$, the statement about existence of a solution remains but the bounds on this solution have to be slightly changed. Upper bounds on $F_\ell(\Delta_{\min}, t)$ for $t \in \{0, 0.17\lambda_c\}$ are provided in Table 3.4.

Compared to Lemma 3.13, we now bound $|q|$ on the interval $[0, 0.17\lambda_c]$ (instead of $[0, 0.1649\lambda_c]$ as before). For this, we only need to show that the second derivative of q is negative and that $q > -1$ on this interval. Note that we can use the estimate for q_R'' and q_R in the proof of Lemma 3.13 for q'' and q respectively, i.e. (3.32) and (3.31). Computing these at $\Delta_{\min} = 1.87\lambda_c$ for $t \in [0, 0.17\lambda_c]$ yields $q'' < -0.1181 f_c^2$ and $q > 0.9113$.

Finally, in Lemma 3.14, the statement is proved for the interval $(0, 0.17\lambda_c]$ instead of $(0, 0.1649\lambda_c]$.

t/λ_c	$F_0(1.87\lambda_c, t)$	$F_1(1.87\lambda_c, t)$	$F_2(1.87\lambda_c, t)$	$F_3(1.87\lambda_c, t)$
0	6.708×10^{-3}	$7.978 \times 10^{-2} f_c$	$1.078 f_c^2$	$16.01 f_c^3$
0.17	6.747×10^{-3}	$0.1053 f_c$	$1.081 f_c^2$	$41.74 f_c^3$

Table 3.4: Numerical upper bounds on $F_\ell(\Delta_{\min}, t)$ with $\Delta_{\min} = 1.87\lambda_c$. {table: num_upp

3.5 Extensions

{section: Exten

In the previous section, we have introduced and proven the main result in this thesis, the Theorem 3.2, which ensures the exact recovery of an atomic measure supported on $[0, 1]$ from its lowest Fourier series coefficients under the minimum separation condition of the support. Certainly one can ask if this result can be extended to higher dimensions, or to other types of signals. This section will give an answer to these natural questions.

3.5.1 To higher dimensions

We now discuss the 2-dimensional setting and emphasize that the situation in d dimensions is analogous. As before consider an atomic measure

$$x = \sum_j a_j \delta_{t_j},$$

but now, the support $T = \{t_j\}$ should be a subset of $[0, 1]^2$. The information we have available about x is in the form of low-frequency samples

$$y(k) = \int_{[0,1]^2} e^{-i2\pi\langle k,t\rangle} dx(t) = \sum_j a_j e^{-i2\pi\langle k,t_j\rangle}, \quad k = (k_1, k_2) \in \mathbb{Z}^2, |k_1|, |k_2| \leq f_c.$$

We can describe our problem as imaging point sources in 2D plane with an optical device with resolution about $\lambda_c = 1/f_c$ (e.g. imaging idealized stars in the sky with a diffraction limited telescope). An extended result of Theorem 3.2 states that it is possible to locate the point sources with infinite precision if they are "well-separated", and again by minimizing the total variation, i.e. solving the following problem

$$\min_z \|z\|_{TV} \quad \text{subject to } F_n z = y \tag{P_{TV}^2}$$

where F_n is the linear map that collects the n^2 lowest frequency coefficients, with $n = 2f_c + 1$, i.e. the coefficients corresponding to $k = (k_1, k_2) \in \mathbb{Z}^2$ with $-f_c \leq k_1, k_2 \leq f_c$.

Recall that the minimum separation of a family of points T on $[0, 1]$ was defined as the closest (wrap-around) distance between any two elements from T. Now, for $T \subset [0, 1]^d$ we can analogously define it as the closest distance between any two points in T, but now the distance (in d dimensions) means the maximum deviation in any coordinate, i.e. the ℓ_∞ distance.

Definition 3.16 (Minimum Separation in higher dimensions)**.** *Let $\mathbb{T}$ be the circle obtained by identifying the endpoints on $[0, 1]$ and $\mathbb{T}^d$ the d-dimensional torus. Then the distance between two points $t = [t_1, ..., t_d]^T, t' = [t'_1, ..., t'_d]^T \in \mathbb{T}$ is defined as*

$$|t - t'|_\infty := \max_{1\leq j\leq d} |t_j - t'_j|_w .$$

Furthermore, the minimum separation *of a family of points $T \subset \mathbb{T}$ is defined as*

$$\Delta(T) := \inf_{t,t'\in T : t\neq t'} |t - t'|_\infty . \tag{3.35}$$

Not surprisingly, the extended version of Theorem 3.2 to two dimensions states almost the same as the original version, only the constant at the minimum separation condition has to be slightly changed:

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Theorem 3.17. *Let $T = \{t_j\}_{j=1}^{|T|} \subset [0,1]^2$ (identified with $\mathbb{T}^2$) be the support of x. If the minimum separation condition*

$$\Delta(T) \geq 2.38/f_c = 2.38\lambda_c, \tag{3.36}$$

is fulfilled and x is real-valued, then x is the unique minimization solution of (P_{TV}^2) with input $y = F_n x$ restricted on all finite signed (real) measures supported on $[0,1]^2$. For complex-valued signals, the same statement holds but with a slightly different constant.

The proof for Theorem 3.17 for real-valued signals is given in Appendix B. For higher dimensions and for complex measures one can almost directly apply the same proof techniques. In details, suppose that we observe the low frequencies of a d-dimensional signal, i.e. its discrete Fourier coefficients at $k = (k_1, ..., k_d) \in \mathbb{Z}^d$ with $|k|_\infty = \max\{|k_1|, ..., |k_d|\} \leq f_c$. Then the total-variation minimization recovers the signal exactly provided that the minimum separation satisfies $\Delta(T) \geq c_d\lambda_c$, where $c_d > 0$ is some positive numerical constant depending only on the dimension d. Since all norms on a finite dimensional space are equivalent, extensions to other settings, in which low frequencies are defined with respect to other norms, e.g. $|k|_2 \leq f_c$ (instead of ℓ_∞-norm) are straightforward.

3.5.2 To other types of signals

In our main result we have considered a continuous-time model, in which the signal is a linear combination of Dirac measures supported on a subset $T \subset [0,1]$. We have also applied this to gain an analogous result in discrete-time, and in the last section, we have seen that this model can also be extended in higher dimensions. Now in this section, we will give an example for other types of signals, which can be super-resolved using our discussed results and techniques. Again, let $T = \{t_1, t_2, ...\} \subset [0,1]$ and for convenience let $t_0 := 0$ and $t_{|T|+1} := 1$.

We consider the signal $x : [0,1] \to \mathbb{C}$ of the form

$$x(t) = \sum_{j=0}^{|T|} \mathbb{1}_{[t_j, t_{j+1})} p_j(t),$$

where p_j is polynomials of degree $\ell \in \mathbb{N}$ and $\mathbb{1}_A$ denotes the indicator function of some set A. So, x should be a piecewise smooth function with periode 1, or more precisely, on each time interval $[t_j, t_{j+1})$ it is polynomial of degree ℓ. Furthermore, as for splines, we suppose that x is globally $\ell - 1$ times continuous differentiable.

Again let $f_c \in \mathbb{N}$ be the frequency cut-off, the information we have about the signal x is

$$y_k = \int_0^1 x(t) e^{-i2\pi kt} dt, \quad k \in \mathbb{Z}, |k| \leq f_c.$$

One can easily show that the ℓ-th weak derivative of x is $x^{(\ell)} = \sum_j \mathbb{1}_{[t_{j-1}, t_j]} p_j^{(\ell)}$. Note that $p_j^{(\ell)}$ is constant for each j. Hence, for any $\varphi \in C_c^\infty(0,1)$ we have

$$\int_0^1 x^{(\ell)} \varphi' dt = \sum_{j=0}^{|T|} \int_{t_j}^{t_{j+1}} p_j^{(\ell)} \varphi' dt = \sum_{j=0}^{|T|} p_j^{(\ell)} \big(\varphi(t_{j+1}) - \varphi(t_j)\big) = \sum_{j=1}^{|T|} (p_{j-1}^{(\ell)} - p_j^{(\ell)}) \varphi(t_j).$$

This shows that the $(\ell+1)$-th derivative of x (in distributional sense) is an atomic measure support on T given by

$$x^{(\ell+1)} = \sum_{j=1}^{|T|} a_j \delta_{t_j}, \quad a_j = p_j^{(\ell)} - p_{j-1}^{(\ell)}.$$

In order to apply our main result Theorem 3.2 here, we need the k-th Fourier coefficient of this measure, which can be calculated by standard Fourier analysis as

$$y_k^{(\ell+1)} = (i2\pi k)^{\ell+1} y_k, \quad k \neq 0.$$

Also, the Fourier coefficient of $x^{(\ell+1)}$ corresponding to $k = 0$ must vanish, since the periodicity implies for $1 \leq j \leq \ell$ that $\int_0^1 dx^{(\ell+1)}(t) = 0 = \int_0^1 x^{(j)} dt$. Hence, by Theorem 3.2 the convex program

$$\min \left\| z^{(\ell+1)} \right\|_{TV} \quad \text{subject to} \quad F_n z = y \tag{3.37}$$

recovers $x^{(\ell+1)}$ exactly provided that the discontinuity points satisfy the minimum separation condition (3.3). Since x is $(\ell - 1)$ -times continuously differentiable and periodic, $x^{(\ell+1)}$ determines x up to a shift in function value. This shift is equal to the mean value of x, which can be read by $y_0 = \int_0^1 x(t)dt$. Therefore, x can be recovered exactly and we obtain the following corollary:

Corollary 3.18. *If $T = \{t_1, t_2, ...\} \in [0, 1]$ satisfies the minimum separation condition (3.3), then the signal x can be recovered exactly from its low-frequency Fourier coefficients y by solving (3.37).*

Chapter 4

Robustness to noise

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4.1 Robustness to noise

{section:

In chapter 3 we have discussed the noiseless super-resolution based on the total-variation minimization and showed that this method actually return the correct signal with infinite precision. However, noise is unavoidable in practical situations and if the data are comtaminated with noise, we certainly cannot hope to still achieve perfect recovery. In this chapter, we intend to understand the effect of the noise on our super-resolution methods.

For simplicity, we consider here the discrete setting (cf. Section 3.2). Intuitively, suppose that the noise level and the resolution of the data are fixed, then it would become more difficult to recover the fine details of the signal, if the scale of this features become finer. Hence, we may guess about a connection between the super-resolution factor (SRF) and the super-resolution error. The main theorem in this section will justify this intuition, in fact it will turn out that the precision of our super-resolution process is inversely proportional to the noise level and to the square of SRF. But, at first, let us introduce our noise model.

Certainly, one could imagine studying a variety deterministic and stochastic noise models, and a variety of metrics in which to measure the size of the error. As a simple deterministic model, we assume that the measurement y is given by $F_n x + w$, where the projection of the noise w onto the signal space has bounded ℓ_1-norm but is otherwise arbitrary and can be adversarial, that is

$$y = F_n x + w, \quad \frac{1}{N} \|F_n^* w\|_1 \leq \delta \tag{4.1}$$

for some $\delta \geq 0$. Let $P_n := \frac{1}{N} F_n^* F_n$ be the orthogonal projection of a signal onto its first n Fourier modes, meaning that $P_n x$ is the n-degree Fourier series approximation of x. We know that the Discrete Fourier Transformation is unitary up to the constant $\sqrt{N}$, hence F_n is surjective, and $\frac{1}{N} F_n F_n^* = I_n$. Thus there exists $u \in \mathbb{C}^{\mathbb{N}}$ such that $u = P_n u$ and $w = F_n u$. We can rewrite (4.1) as

$$y = F_n(x + u), \quad \|u\|_1 \leq \delta, u = P_n u,$$

Since the high-frequency part of u is filtered out by the measurement process, one can also write this model with arbitrary input noise $u \in \mathbb{C}^n$ as

$$y = F_n(x + u), \quad \|P_n u\|_1 \leq \delta.$$

Finally, we see that F_n^* is injective, thus letting $s := \frac{1}{N} F_n^* y$ we obtain an equivalent form of (4.1)

$$s = P_n x + P_n u, \quad \|P_n u\|_1 \leq \delta. \tag{4.2}$$

Here, the low-pass version of the signal is corrupted with an additive low-pass error whose ℓ_1-norm is at most δ. For example, when $n = N$ we have $P_n = I$, (4.2) becomes

$$s = x + u, \quad \|u\|_1 \leq \delta.$$

In this case, it is impossible to obtain a super-resolved signal $\tilde{x}$ with an error in ℓ_1-norm less than the noise level δ, i.e. $\|x - \tilde{x}\|_1 < \delta$.

We now focus on the recovery properties of the following slightly modified version of (P_{ℓ_1}):

$$\min_z \|z\|_1 \quad \text{subject to } \|P_n z - s\|_1 \leq \delta. \tag{$P_{\ell_1,\delta}$}$$

Our aim is to estimate the recovery error in dependence of the noise level and the super-resolution factor SRF. The main result in this section will be introduced later in Theorem 4.2, which states that the error (measured in ℓ_1-norm) is proportional to δ and to SRF^2. But at first, let us introduce and prove a property of the matrix F_n, namely a strong form of the null space property (NSP) (cf. Section 2.3). Recall that for an arbitrary vector $z = (z_0, z_1, ..., z_{N-1})^T \in \mathbb{C}^N$ and $T \subset \{0, ..., N-1\}$, z_T denotes its orthogonal projection onto the linear space of vectors supported on T, i.e. $(z_T)_j = z_j$ if $j \in T$ and is zero otherwise.

strong_NSP}

Lemma 4.1. *Let $h \in \mathbb{C}^N$ be supported in $T \subset \{0, 1, ..., N-1\}$ and assume furthermore that $F_n h = 0$. Then, it holds*

$$\|h_T\|_1 \leq \rho \|h_{T^c}\|_1 , \tag{4.3}$$

for some numerical constant $\rho \in (0, 1)$. This constant is of the form $1 - \rho = \alpha / SRF^2$ where α is a positive constant. If $\Delta(T) \geq 2.5\lambda_c$ and $SRF > 3.0321$, we can take $\alpha = 0.0838$.

Proof. Applying Proposition 3.6 yields the existence of a low-frequency trigonometric polynomial $q(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt}$ with

$$\begin{cases} q(\frac{j}{N}) = \overline{\mathrm{sgn}\,(h_j)}, & j \in T \\ |q(t)| < 1, & t \in [0,1] \setminus \frac{T}{N}. \end{cases}$$

We shall abuse notations and set $q = (q_j)_{j=0}^{N-1}$ with $q_j = q(j/N)$. Now we have that $|q_j| < 1$ for each $j \in T^c$, hence there exists $\rho \in (0, 1)$ such that $\|q_{T^c}\|_1 \leq \rho < 1$.
Recall that $P_n = \frac{1}{N} F_N^* F_N$ is the orthogonal projection of a signal onto its n lowest Fourier modes. Since the vector q contains only low frequencies, we obtain $P_n q = q$. Hence,

$$\langle q, h \rangle = \langle q, P_n h \rangle = 0.$$

Moreover, by construction of q we have

$$\langle q_T, h_T \rangle = \sum_{j \in T} h_j \overline{\mathrm{sgn}\,(h_j)} = \|h_T\|_1 .$$

Putting everything together, we get

$$\begin{aligned} 0 = \langle q, h \rangle &= \langle q_T, h_T \rangle + \langle q_{T^c}, h_{T^c} \rangle \\ &\geq \|h_T\|_1 - \|q_{T^c}\|_\infty \|h_{T^c}\|_1 \\ &\geq \|h_T\|_1 - \rho \|h_{T^c}\|_1 , \end{aligned}$$

which implies the first statement. Note that as the proofs of lemma 3.13 and lemma 3.15 make clear, ρ has the form $1 - \alpha / \mathrm{SRF}^2$, where $\alpha > 0$ is a constant.

Now, suppose that SRF ≥ 3.0321. Let $j \in T^c$, we need to show that $|q_j| \leq \rho := 1 - 0.0883/\mathrm{SRF}^2$. If the distance from j to T is small, i.e. there exists $j' \in T$ with $\left|\frac{j-j'}{N}\right| \leq 0.1649\lambda_c$, then by Lemma 3.15 we obtain

$$|q_j| = |q(j/N)| \leq 1 - 0.3353\frac{(j-j')^2 f_c^2}{N^2} \leq 1 - 0.0838\frac{4f_c^2}{N^2} \leq 1 - 0.0838/\,\mathrm{SRF}^2 = \rho.$$

In the other case, i.e. the distance from j to T is larger than $0.1649\lambda_c N$, Lemma 3.15 also implies that

$$|q_j| = |q(j/N)| \leq 1 - 0.3353 f_c^2 (0.1649\lambda_c)^2 \approx 1 - 0.0838/3.0321^2 \leq \rho.$$

$\square$

Now we are able to prove the main theorem of this section, which discusses the behavior of the error of our super-resolution using $(P_{\ell_1,\delta})$ in presence of noise (in the form (4.2) discussed above).

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Theorem 4.2. *Assume that the support $T \subset \{0, 1, ..., N-1\}$ of a signal $x \in \mathbb{C}^N$ satisfies the minimum separation condition in discrete setting (3.4). We consider the noise model (4.2) and let $\hat{x}$ be the solution to $(P_{\ell_1,\delta})$. Then it holds*

$$\|x - \hat{x}\|_1 \leq C_0 SRF^2 \delta, \tag{4.4}$$

where C_0 is a positive constant. If further to the minimum separation condition also $\Delta(T) \geq 2.5\lambda_c$ holds and SRF > 3.0321 *then we can take $C_0 \approx 47.733$.*

Proof. We decompose the error into its low- and high-pass components $\hat{x} - x = h + \ell$ where $\ell := P_n(\hat{x} - x)$ and $h := (I - P_n)(\hat{x} - x)$. For the low-pass component we have

$$\|\ell\|_1 = \|P_n(\hat{x} - x)\|_1 \leq \|P_n(\hat{x} - s)\|_1 + \|P_n(s - x)\| \leq 2\delta. \tag{4.5}$$

On the other hand, since $\hat{x}$ is the solution to the minimization problem $(P_{\ell_1,\delta})$ we obtain

$$\begin{aligned} 0 \leq \|x\|_1 - \|\hat{x}\|_1 &= \|x\|_1 - \|x + h_T + h_{T^c}\|_1 + \|\ell\|_1 \\ &\leq \|x\|_1 - \|x + h_{T^c}\|_1 + \|h_T\|_1 + \|\ell\|_1 \\ &= -\|h_{T^c}\|_1 + \|h_T\|_1 + \|\ell\|_1 . \end{aligned}$$

As $P_n h = 0$ or equivalently, $F_n h = 0$, Lemma 4.1 implies that

$$\|h_T\|_1 \leq \rho \|h_{T^c}\|_1 .$$

Therefore, we obtain $\|h_{T^c}\|_1 \leq \frac{1}{1-\rho}\|\ell\|_1$ and $\|h_T\|_1 \leq \frac{\rho}{1-\rho}\|\ell\|_1$. These and (4.5) give

$$\|\hat{x} - x\|_1 \leq \|h_T\|_1 + \|h_{T^c}\|_1 + \|\ell\|_1 \leq \frac{2}{1-\rho}\|\ell\|_1 \leq \frac{4\delta}{1-\rho}.$$

Also from Lemma 4.1 we have $1 - \rho = \alpha/\mathrm{SRF}^2$ for some numerical constant $\alpha > 0$, it follows that $\|\hat{x} - x\|_1 \leq 4\alpha^{-1}\mathrm{SRF}^2\delta$. Finally, if $\Delta(T) \geq 2.5\lambda_c$ and SRF > 3.0321, then one can take $C_0 := 4\alpha^{-1} \approx 4/0.0883 = 47.733$. $\square$

4.2 Sparsity is not enough

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As discussed in the previous chapter, our results do not require the sparsity of the signals, but the minimum separation condition (which, in the discrete setting in Section 3.2, however implies, that the signals should be about $n/4$-sparse). In this section, we explain why robust super-resolution under sparsity constraints alone is hopelessly ill-posed.

Let us remove the minimum distance condition, then the support of sparse signals may be very clustered so that they can be nearly completely annihilated by the low-pass sensing process. This can be explained for large N using the results from [21] as follows. To be concrete, we consider the 'analog version' of (4.2) in which we observe

$$s = \mathcal{P}_W(x+u);$$

here $\mathcal{P}_W(x)$ is computed by taking the discrete-time Fourier transform $y(\omega) = \sum_{j\in\mathbb{Z}} x_j e^{-i2\pi\omega j}$, $\omega \in [-1/2, 1/2]$ and discarding all 'analog' frequencies outside the band $[-W, W]$, $0 < W < 1/2$. Note that by definition, P_n is the convolution with the Dirichlet kernel while $\mathcal{P}_W$ is the convolution with sinc kernel, hence if we put

$$2W = n/N = 1/\operatorname{SRF},$$

then we essentially have $\mathcal{P}_W = P_n$ in the limit $N \to \infty$. Now, let $\mathcal{T}_k$ be the operator that sets the j-th entry of an infinite sequence to itself if $j \in T = \{0, ..., k-1\}$ and to 0 if $j \geq k$. Then, the discrete prolate spheroidal sequences $\{s_j\}_{j=1}^k$ introduced in [21] are the eigenvectors of the operator $\mathcal{P}_W\mathcal{T}_k$ associated with the eigenvalues $\{\lambda_j\}_{j=1}^k$, which lie on $(0,1)$. In short, we have

$$\mathcal{P}_W\mathcal{T}_k s_j = \lambda_j s_j, \quad 1 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0.$$

Set $v_j = \mathcal{T}_k s_j$, then according to [21, Equation (22)] the vectors v_j's form a orthonormal basis of $\mathbb{C}^k$, i.e. the space of sparse vectors with support in $\{0, ..., k-1\}$, and also, the same holds for s_j's up to the constant $\sqrt{\lambda_j}$. In particular, we have

$$\|v_j\|_2 = 1, \quad \|\mathcal{P}_W v_j\|_2 = \|\lambda_j s_j\|_2 = \sqrt{\lambda_j}. \tag{4.6}$$

Moreover, our observation holds not only for the support $T = \{0, \ldots, k-1\}$ but also for any support T with length k.

On the other hand, combining Equation (13) and (58) in [21] yields the approximated value of λ_j for values of j near k, namely

$$\lambda_j \approx A_j e^{-\gamma(k+1)}, \qquad A_j = \frac{\sqrt{\pi}2^{\frac{14(k-j)+9}{4}}\alpha^{\frac{2(k-j)+1}{4}}(k+1)^{k-j+0.5}}{(k-j)!(2-\alpha)^{k-j+0.5}},$$

where

$$\alpha = 1 + \cos(2\pi W), \quad \gamma = \ln\left(1 + \frac{2\sqrt{\alpha}}{\sqrt{2}-\sqrt{\alpha}}\right).$$

For a fixed value of $\operatorname{SRF} = 1/(2W)$, if k is large, these eigenvalues are equal to 0 for all practical purposes. For example, we have:

- If $\operatorname{SRF} = 1.05$ (i.e. we only seek to extend the spectrum by 5%),

  $$\lambda_k \approx 1.94\sqrt{k+1}e^{-0.15(k+1)},$$

  for $k = 256$ this becomes

  $$\lambda_k \approx 6 \times 10^{-16}.$$

- If $\operatorname{SRF} = 4$,

  $$\lambda_k \approx 17.81\sqrt{k+1}e^{-3.02(k+1)},$$

  for $k = 48$ this becomes

  $$\lambda_k = 2.3 \times 10^{-67}.$$

- If SRF increases, using Taylor expansion for cosine function we get that λ_k is of order $O(\mathrm{SRF}^{-2k})$.

In other words, (4.6) implies the existence of unit-normed sparse signals that are almost completely suppressed by the low pass filtering, even if SRF is small. Yet knowing the support ahead of time, it seems to be impossible to recover such signals from noisy measurements.

However, this is not a worst case analysis, certainly, when the super-resolution factor increases, the ill-posedness gets worse. Indeed, [21] also asserts that if k is large, for any $0 < \epsilon < 1/(2W) - 1$ and $j \geq 2kW(1+\epsilon)$ there exist positive constants C_0 and γ_0 (depending on ϵ and W) such that

$$\lambda_j \leq C_0 e^{-\gamma_0 k}.$$

This means that only about the first $\lfloor 2kW \rfloor$ eigenvalues of $\mathcal{P}_W \mathcal{T}_k$ lie near 1 while the other $(1-2W)k$ decay abruptly towards 0. Hence, for any interval T of length k, there is a subspace of signals supported on T with dimension approximately equal to $(1 - 1/\,\mathrm{SRF})k$, which is essentially zeroed out by the measurement process. This implies that if the super-resolution factor is large, e.g. $\mathrm{SRF} \geq 2$, then most of the information from clustered sparse signals is lost by low-pass filtering. For instance, consider a random sparse vector x supported on T with i.i.d entries and its projection onto a fixed subspace of dimension about $(1 - 1/\,\mathrm{SRF})k$ corresponding to the negligible eigenvalues. Although this component contains most of the energy of the signal with high probability, it is practically suppressed by the measurement process.

To conclude, super-resolution for any tightly clustered sparse signal in presence of noise is hopeless. There exist always unit-normed sparse signals (w.r.t. $\ell_2 - norm$) that are irretrievable by any recovery method. If the super-resolution factor gets large, there is even a large-dimensional subspace of sparse signals whose information is almost completely destroyed by the low-pass filtering. This is the reason why we need the minimum separation condition for the support of the signals.

Chapter 5

Minimization via Semi Definite Programming

{ch: SDP}

In Chapter 3, we have shown that solving the convex optimization problem $(\mathcal{P}_{TV})$ recovers exactly the original signal. Recall that $(\mathcal{P}_{TV})$ is given as follows:

$$\min_{z\in\mathcal{B}} \|z\|_{TV} \quad \text{subject to} \quad F_n z = y, \tag{P_{TV}}$$

where $\mathcal{B}$ denotes the vector space of all (finite) Borel complex measures on $[0,1]$. Now, a natural question is how to solve it. This might seem quite challenging because it requires solving an optimization problem over an complex infinite dimensional space. A natural approach is to approximate the solution by discretizing the support of the signal, but this could lead to an increase in complexity if the discretization step is reduced to improve precision. Here we take a different route: we show that the dual program of $(\mathcal{P}_{TV})$ is equivalent to a semidefinite program with just $(n+1)^2/2$, and from that one can find highly accurate solutions to the primal problem.

We begin with the observation that we can restrict the set $\mathcal{B}$ of all Borel complex measures on $[0,1]$ to the set $\tilde{\mathcal{B}}$ of all regular Borel complex measures on $[0,1]$. In fact, this restriction does not change our primal problem $(\mathcal{P}_{TV})$ because its unique solution, namely a linear combination of Dirac measures, is a regular Borel measure as observed in Example 2.9.

The second observation is that the primal problem can be seen as an optimization problem on field $\mathbb{R}$. Indeed, we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $c = c_R + ic_I \mapsto \begin{bmatrix} c_R \\ c_I \end{bmatrix}$ and obtain that

$$\left\langle \begin{bmatrix} y_R \\ y_I \end{bmatrix}, \begin{bmatrix} c_R \\ c_I \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}} = \langle y_R, c_R\rangle_{\mathbb{R}^n} + \langle y_I, c_I\rangle_{\mathbb{R}^n} = \operatorname{Re}\langle y_R + iy_I, c_R + ic_I\rangle_{\mathbb{C}^n} = \operatorname{Re}\langle y, c\rangle_{\mathbb{C}^n}.$$

The same can be done for the signal z, i.e. we identify $\tilde{\mathcal{B}}$ with $\mathcal{M}^2$, where $\mathcal{M}$ denotes the space of all Borel regular signed measures on $[0,1]$, by the mapping $z = z_R + iz_I \mapsto (z_R, z_I)$. Furthermore, define the linear mapping $A : \mathcal{M}^2 \to \mathbb{R}^{2n}$ by $A\begin{bmatrix} z_R \\ z_I \end{bmatrix} = \begin{bmatrix} \operatorname{Re}(F_n z) \\ \operatorname{Im}(F_n z) \end{bmatrix}$. Then our restricted primal problem $(\mathcal{P}_{TV})$ becomes

$$\min_{(z_R,z_I)\in\mathcal{M}^2} \|z_R + iz_I\|_{TV} \quad \text{subject to} \quad A\begin{bmatrix} z_R \\ z_I \end{bmatrix} = \begin{bmatrix} y_R \\ y_I \end{bmatrix}. \tag{5.1}$$

The associated Lagrange dual function is now given by

$$\begin{aligned}
g(c_R, c_I) &= \inf_{(z_R,z_I)\in\mathcal{M}^2} \left(\|z_R + iz_I\|_{TV} + \left\langle A \begin{bmatrix} z_R \\ z_I \end{bmatrix} - \begin{bmatrix} y_R \\ y_I \end{bmatrix}, \begin{bmatrix} c_R \\ c_I \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}} \right) \\
&= -\left\langle \begin{bmatrix} y_R \\ y_I \end{bmatrix}, \begin{bmatrix} c_R \\ c_I \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}} + \inf_{(z_R,z_I)\in\mathcal{M}^2} \left(\|z_R + iz_I\|_{TV} + \left\langle A \begin{bmatrix} z_R \\ z_I \end{bmatrix}, \begin{bmatrix} c_R \\ c_I \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}} \right) \\
&= -\operatorname{Re}\langle y, c\rangle_{\mathbb{C}^n} + \inf_{z\in\tilde{\mathcal{B}}} \left(\|z\|_{TV} + \left\langle \begin{bmatrix} \operatorname{Re}(F_n z) \\ \operatorname{Im}(F_n z) \end{bmatrix}, \begin{bmatrix} c_R \\ c_I \end{bmatrix} \right\rangle_{\mathbb{R}^{2n}} \right) \\
&= -\operatorname{Re}\langle y, c\rangle_{\mathbb{C}^n} + \inf_{z\in\tilde{\mathcal{B}}} \left(\|z\|_{TV} + \operatorname{Re}\langle F_n z, c\rangle_{\mathbb{C}^n} \right) \\
&= -\operatorname{Re}\langle y, c\rangle_{\mathbb{C}^n} + \inf_{z\in\tilde{\mathcal{B}}} \left(\|z\|_{TV} + \operatorname{Re}\langle z, F_n^* c\rangle_{\tilde{\mathcal{B}}\times\mathcal{C}} \right)
\end{aligned}$$

at the last equation, we apply Theorem 2.10 to obtain $\tilde{\mathcal{B}} \cong \mathcal{C}$ and hence by a very common consequence of Hahn-Banach extension theorem we have

$$\|z\|_{TV} = \sup\left\{ \operatorname{Re}\langle z, f\rangle_{\tilde{\mathcal{B}}\times\mathcal{C}} : f \in \mathcal{C},\ \|f\|_\infty \le 1 \right\},$$

where $\langle\cdot,\cdot\rangle_{\tilde{\mathcal{B}}\times\mathcal{C}}$ denotes the dual pairing, $\langle z, f\rangle_{\tilde{\mathcal{B}}\times\mathcal{C}} = \int_0^1 \overline{f(t)} dz(t)$. Thus, we get

$$\inf_{z\in\tilde{\mathcal{B}}} \left(\|z\|_{TV} + \operatorname{Re}\langle z, F_n^* c\rangle_{\tilde{\mathcal{B}}\times\mathcal{C}} \right) = \begin{cases} 0, & \|F_n^* c\|_\infty \le 1 \\ -\infty, & \text{otherwise,} \end{cases}$$

and finally obtain the dual function as

$$g(c) = \begin{cases} -\operatorname{Re}\langle y, c\rangle_{\mathbb{C}^n}, & \|F_n^* c\|_\infty \le 1 \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore, the Lagrange dual problem associated with $(\mathcal{P}_{TV})$ is

$$\max_{c\in\mathbb{C}^n} \operatorname{Re}\langle y, c\rangle \quad \text{subject to} \quad \|F_n^* c\|_\infty \le 1; \tag{P_{TV}^*}$$

here, to ease notation we write $\langle\cdot,\cdot\rangle$ for the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ on $\mathbb{C}^n$. Also note that one can easily determine $F_n^* : \mathbb{C}^n \to \mathcal{C}$ using the dual pairing introduced in Theorem 2.10. Indeed, F_n^* is given by $(F_n^* c)(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt}$, $c \in \mathbb{C}^n$, $t \in [0,1]$. Here and below, we number the entries of a vector $c \in \mathbb{C}^n$ by $c = [c_{-f_c}, \ldots, c_{f_c}]^T$ (remember that $n = 2f_c + 1$). Hence, the constraint inequality means that $|(F_n^* c)(t)| = \left|\sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt}\right| \le 1$ for every $t \in [0,1]$.

Our dual problem (P_{TV}^*) seems to be a little simpler than the primal one, since we minimize over finite dimensional vectors, namely over all $c \in \mathbb{C}^n$, which can be identified with vectors in $\mathbb{R}^{2n}$ as in the observation above. On the other hand, the connection between the primal and dual problem is established by a generalized Slater condition [20]: since the interior of the feasible set contains 0 and hence nonempty, strong duality holds.

However, (P_{TV}^*) is still infinite dimensional due to the constraint. Fortunately, the Fejér-Riesz Factorization Theorem allows to interpret the contraint $\|F_n^* c\|_\infty \le 1$ as the intersection between the cone of positive semidefinite matrices $\{Q \in \mathbb{C}^{n\times n} : Q \succeq 0\}$ and an affine hyperplane. Here and below, for a quadratic matrix Q we use the notation $Q \succeq 0$ to denote that it is positive semidefinite. First let us introduce the Fejér-Riesz Theorem and its quite elementary proof.

n: Fejer-Riesz} **Lemma 5.1.** *(Fejér-Riesz Theorem) Suppose that* $f(z) = \sum_{-n}^{n} c_k z^k$, $z \in \mathbb{C}$ *is nonnegative on the complex unit circle, i.e.* $f(e^{it})$ *has real and nonnegative values for every real* t. *Then we have*

$$f(e^{it}) = \left|h(e^{it})\right|^2$$

for some polynomial $h(z) = \sum_0^m a_k z^k$. *This polynomial* h *can be chosen to have no root* z *with* $|z| < 1$.

Proof. By assumption we have $c_j = \overline{c_{-j}}$ and hence $\overline{f(z)} = f(\frac{1}{\bar{z}})$ for any $z \in \mathbb{C}$. Consequently, the roots of f occur always in pair z_j and $1/\overline{z_j}$, where one can assume that $|z_j| \geq 1$. This means

$$f(z) = c \prod_j (z - z_j)(z^{-1} - \overline{z_j})$$

for some constant c. The polynomial h is now the product of $(z - z_j)$ adjusted by a suitable multiplicative constant. □

We now come to the desired result, which interprets the constraint of (P_{TV}^*) in a more efficient way and thereby brings this convex optimization problem into an equivalent but finite-dimensional problem. {Cor: feje

Corollary 5.2. *Let* $c = [c_{-f_c}, \ldots, c_{f_c}]^T \in \mathbb{C}^n$ *be arbitrary. Then the following conditions are equivalent:*

(i) It holds $\|F_n^* c\|_\infty \leq 1$, *i.e.* $\left|\sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt}\right| \leq 1$ *for every* $t \in [0,1]$.

(ii) There exists a Hermitian matrix $Q \in \mathbb{C}^{n\times n}$ *such that*

$$\begin{bmatrix} Q & c \\ c* & 1 \end{bmatrix} \succeq 0, \quad \sum_{k=1}^{n-j} Q_{k,k+j} = \begin{cases} 1, & j = 0 \\ 0, & j = 1, 2, .., n-1. \end{cases} \tag{5.2}$$

Proof. We first show that (ii) implies (i), for this let $Q \in \mathbb{C}^{n\times n}$ be a Hermitian matrix satisfying the property (5.2). The positive semidefiniteness constraint in (5.2) means that $Q \succeq 0$ and for every $z \in \mathbb{C}^n$ we have $z^*Qz - c^*z - z^*c + 1 \geq 0$, which is equivalent to

$$z^*(Q - cc^*)z + (z^*c - 1)\overline{(z^*c - 1)} \geq 0.$$

This equation holds for every $z \in \mathbb{C}^n$, hence the first part must be nonnegative, meaning that $Q - cc^*$ is positive semidefinite, because otherwise there exists z such that the first part is negative, one can scale this z with a scalar such that $z^*c = 1$, i.e. the second part becomes 0 but the first part remains negative. It follows that $z^*Qz \geq |c^*z|^2$.

For $t \in [0,1]$ we define $z(t) := [e^{i2\pi f_c t}, e^{i2\pi(f_c-1)t}, \ldots, e^{-i2\pi f_c t}]^T$ (remember the numbering $-f_c, \ldots, f_c$). By the equality contraint in (5.2) we get

$$z(t)^*Qz(t) = \sum_{k=1}^{n}\sum_{j=1}^{n} Q_{k,j}\overline{z(t)_{k-1-f_c}}z(t)_{j-1-f_c} = \sum_{k=1}^{n}\sum_{j=1}^{n} Q_{k,j} e^{i2\pi(k-j)t} = 1.$$

Therefore,

$$\left|\sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt}\right|^2 = |c^*z(t)|^2 \leq z(t)^*Qz(t) = 1.$$

Now, we show the other direction. From (i) it follows that $1 - |c^*z(t)|^2$, with $z(t)$ as defined above, is a nonnegative trigonometric polynomial. By the Fejér-Riesz Theorem there exists a polynomial $h(e^{i2\pi t})$ such that $1 - |c^* z(t)|^2 = |h(e^{i2\pi t})|^2$. Comparing the degrees of h and $1 - |c^* z(t)|^2$ gives that $h(e^{i2\pi t}) = \tilde{c}^* z(t)^2$ for some $\tilde{c} \in \mathbb{C}^n$. Set $Q = cc^* + \tilde{c}\tilde{c}^*$, this is obviously a positive semidefinite matrix. We show that it indeed also satisfied the property (5.2).

Since $\tilde{c}\tilde{c}^* = Q - cc^*$ positive semidefinite, it follows easily that the first equation in the first direction holds. Together with the positive semidefiniteness of Q it shows

$$\begin{bmatrix} Q & c \\ c* & 1 \end{bmatrix} \succeq 0.$$

Furthermore, the relation $z(t)^* Q z(t) = 1$ implies that the equality in (5.2) is also satisfied (cf. the second-last equation in the first direction). This completes the proof. $\square$

As said before corollary 5.2 helps us convert the dual problem (P^*_{TV}) into the following equivalent problem:

$$\max_{c,Q} \operatorname{Re}\langle y, c\rangle \qquad \text{subject to (5.2);} \tag{P^*_{SDP}}$$

here we maximize over all Hermitian matrices $Q \in \mathbb{C}^{n\times n}$ and all coefficient vectors $c \in \mathbb{C}^n$. This is a finite dimensional semidefinite program, and hence can be solved by off-the-shelf convex programming software, e.g. CVX [10].

Nevertheless, solving (P^*_{SDP}) just yields the optimal value of the primal problem $(\mathcal{P}_{TV})$ but not a primal solution, which we actually need. In the following, let us assume that (P^*_{SDP}) has been solved and $c \in \mathbb{C}^n$ is a solution to it, we aim to find the primal solution of $(\mathcal{P}_{TV})$. First, we have

$$\begin{aligned}
1 - |(F_n^* c)(t)|^2 &= 1 - \sum_{k=-f_c}^{k=f_c} \sum_{j=-f_c}^{f_c} c_k \overline{c_j} e^{i2\pi(k-j)t} = 1 - \sum_{k=-f_c}^{k=f_c} \sum_{j=k-f_c}^{k+f_c} c_k \overline{c}_{k-j} e^{i2\pi jt} \\
&= 1 - \sum_{j=-2f_c}^{2f_c} \sum_{k=\min\{j-f_c,-f_c\}}^{\max\{f_c, j+f_c\}} c_k \overline{c}_{k-j} e^{i2\pi jt} = 1 - \sum_{j=-2f_c}^{2f_c} d_j e^{i2\pi jt} \\
&= p_{2n-2}(e^{i2\pi t})
\end{aligned} \tag{5.3}$$

where $d_j := \sum_k c_k \overline{c}_{k-j}$ and p_{2n-2} is defined by $p_{2n-2}(z) = 1 - \sum_{j=-2f_c}^{2f_c} u_j z^j$, $z \in \mathbb{C}$. Note that $z^{2f_c} p_{2n-2}(z)$ with $z \in \mathbb{C}$ defines a polynomial of degree at most $4f_c = 2n-2$, and hence is either equal to 0 everywhere or possesses at most $2n-2$ roots. Moreover, if this polynomial is not the zero polynomial, it has, besides the trivial root $z = 0$, the same roots as p_{2n-2}. Therefore, p_{2n-2} has at most $2n-2$ roots or it is the zero polynomial. By construction (5.3) and boundedness of $F_n^* c$, $p_{2n-2}(z)$ is real-valued and nonnegative if z lies on the complex unit circle. Hence p_{2n-2} cannot have single roots on the unit circle, because otherwise its value in a point near the single root is negative. Thus, p_{2n-2} is either the zero polynomial or has at most $n-1$ roots on the unit circle.

On the other hand, the strong duality discussed before implies that any solution $\hat{x}$ to the primal problem $(\mathcal{P}_{TV})$ obeys

$$\|\hat{x}\|_{TV} = \operatorname{Re}\langle y, c\rangle = \operatorname{Re}\langle \hat{x}, F_n^* c\rangle_{\tilde{\mathcal{B}}\times\mathcal{C}} = \operatorname{Re}\left[\int_0^1 \overline{(F_n^* c)(t)} d\hat{x}(t)\right].$$

Since $\hat{x} = \sum_j a_j \delta_{t_j}$ is a linear combination of Dirac measures, this relation implies that

$$\sum_j |a_j| = \operatorname{Re}\left(\sum_j a_j \overline{(F_n^* c)(t_j)}\right) = \sum_j \operatorname{Re}\left(a_j \overline{(F_n^* c)(t_j)}\right).$$

Thus, we obtain $(F_n^* c)(t_j) = \operatorname{sgn}(a_j)$, meaning that the trigonometric polynomial $F_n^* c$ interpolates exactly the sign of $\hat{x}$ when $\hat{x}$ is not vanishing. This phenomenon is illustrated in Figure 5.1. As a consequence,

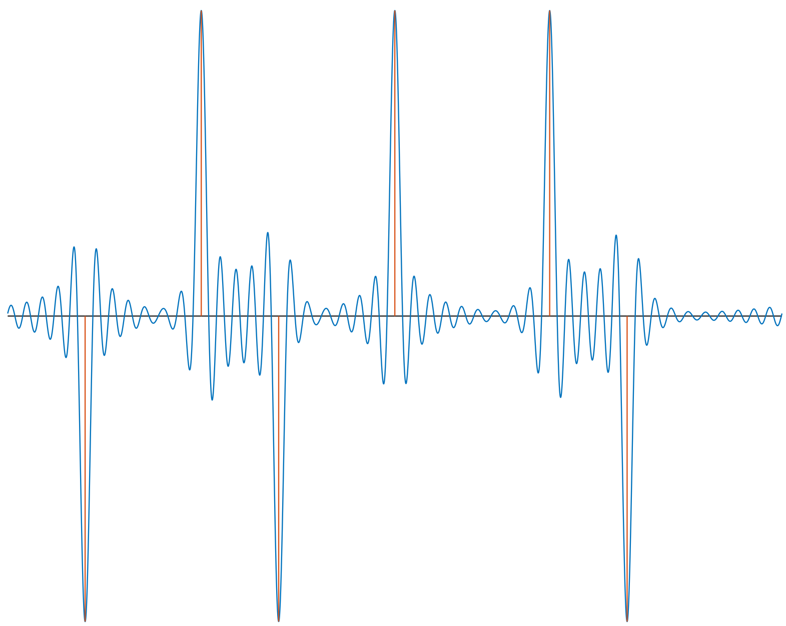


Figure 5.1: The sign of a real atomic measure x is plotted in red. The trigonometric polynomial $F_n^* c$ where c is solution to (P_{SDP}^*) is plotted in blue. Note that $F_n^* c$ interpolates the sign of x. This phenomenon can be used to recover the support of x.

n of the signal}

the support of $\hat{x}$ is a subset of the set of roots of p_{2n-2} that lie on the unit circle. This always holds true regardless of whether p_{2n-2} has at most $n-1$ roots on the unit circle or it is equal to 0 everywhere, in the following, we will have a deeper look at both cases.

We first consider the 'good' case where p_{2n-2} does not vanish everywhere. To calculate its roots one can e.g. use the eigenvalue method mentioned in [23, Section 9.5], i.e. compute the eigenvalues of the companion matrix of $z^{2f_c} p_{2n-2}(z)$. Then one gets an estimated superset $\hat{T}$ of the support T of the original signal $\hat{x}$ with $\left|\hat{T}\right| \leq n-1$, as p_{2n-2} has at most $n-1$ roots on the unit circle. Now, solving the linear system of equation $\sum_{t \in \hat{T}} e^{-i2\pi kt} a_t = y_k$, $|k| \leq f_c$ by using the method of least squares gives the actual support T of the original signal and also its amplitudes. The solution to this system is unique because the coefficient matrix, which is a Vandermonde matrix, has at most $n-1$ columns and they are linearly independent. Here, if $\hat{T}$ is a strict superset of T, then some amplitudes a_t, $t \in \hat{T}$ must vanish. To summarize, in the usual case when p_{2n-2} has less than n roots on the unit circle, we have explained how to retrieve the primal solution of the total-variation minimization problem $(\mathcal{P}_{TV})$, i.e. the original signal, with very high precision. Figure 5.2 illustrates the accuracy of this procedure, while Table 5.1 shows the numerical errors in estimating the support locations. Note that in this step, we have required a root finding procedure, which is not stable in general. Hence, to vindicate the approach to $(\mathcal{P}_{TV})$ by the semidefinite programming (P_{SDP}^*) one would need to argue that the root finding algorithm can be made stable, at least under the conditions of our main result, i.e. Theorem 3.2.

Now we turn to the other case when $p_{2n-2} \equiv 0$. This may happen, e.g. when x is a positive measure obeying the minimum separation condition (3.3) (meaning that it is the unique solution to $(\mathcal{P}_{TV})$) and

f_c	25	50	75	100
Maximum error	2.44×10^{-7}	7.07×10^{-8}	1.29×10^{-7}	1.06×10^{-7}
Average error	5.95×10^{-8}	1.18×10^{-8}	2.19×10^{-8}	8.68×10^{-9}

Table 5.1: Numerical recovery of the support of sparse signals x obtained by solving (P^*_{SDP}) via CVX [10]. The signals were generated with random complex amplitudes situated at approximatedly $f_c/4$ random locations in $[0, 1]$ satisfying the minimum separation condition. The table shows the errors in estimating the support locations. {table: Recover

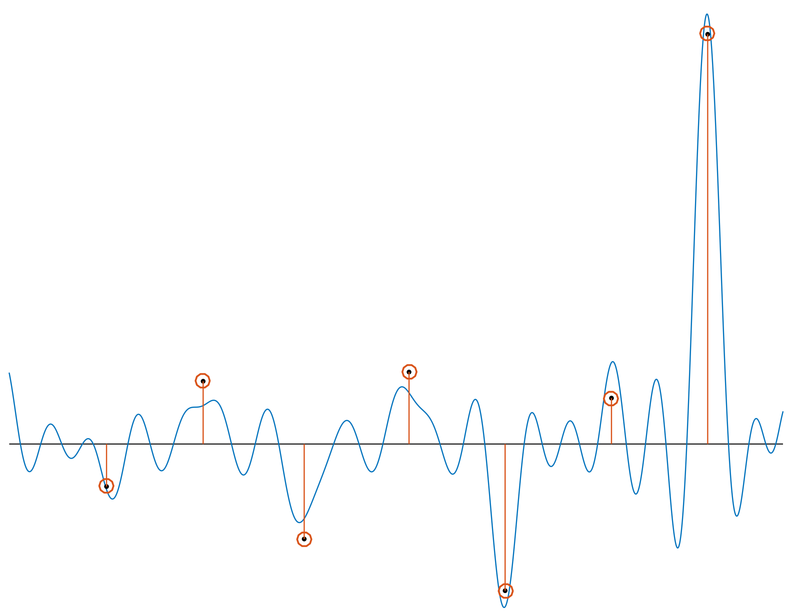


Figure 5.2: There are 21 spikes located at random locations separated by at least $2/f_c$ where $f_c = 50$. The plot shows seven of the original spikes (black dots) and the corresponding low resolution data (blue line) as well as the estimated signal (red line). {fig: original

$c \in \mathbb{C}^n$ with $F_n^* c \equiv 1$, because then we have

$$\operatorname{Re}\langle y, c\rangle = \operatorname{Re}\langle F_n x, c\rangle = \operatorname{Re}\langle x, F_n^* c\rangle = \|x\|_{TV},$$

which shows that c is a solution to the semidefinite program (P_{SDP}^*), but does not carry any information about the support of x. Fortunately, this situation does not happen as long as $F_n^* c$ is not constantly 1 for some solution c of (P_{SDP}^*) or equivalently

$$\text{there exists a solution } \tilde{c} \text{ to } (P_{SDP}^*) \text{ with } |(F_n^* \tilde{c})(t)| < 1 \text{ for some } t \in [0,1], \tag{5.4}$$

and interior point methods as in SDPT3 [14] are applied to solve the semidefinite program (P_{SDP}^*). To explain this phenomenon we assume that the condition (5.4) holds and note that interior point methods approach solutions from the interior of the feasible set by solving the following sequence of optimization problems

$$\max_{c,Q} \operatorname{Re}(y^* c) + t \log\det\left(\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix}\right) \quad \text{subject to (5.2);} \tag{5.5}$$

here an extra term, a scaled *barrier function* is added to the objective function of (P_{SDP}^*) and t is a positive parameter which is gradually reduced towards 0 in order to approach a solution to (P_{SDP}^*) [22]. Let λ_k, $k \in \{1, ..., n\}$ be eigenvalues of $Q - cc^*$. Observe that

$$\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix} = \begin{bmatrix} I & c \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Q - cc^* & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ c^* & 1 \end{bmatrix},$$

and as an easy result of linear algebra also that $\det(Q - cc^*) = \prod_{k=1}^n \lambda_k$. Hence we get

$$\log\det\left(\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix}\right) = \log\det(Q - cc^*) = \sum_{k=1}^{n} \log \lambda_k.$$

Now, by the assumption that Condition (5.4) holds, there is a solution $\tilde{c}$ to (P_{SDP}^*) obeying $|(F_n^* \tilde{c})(t)| < 1$ for some $t \in [0,1]$. Analogously as in the proof of Corollary 5.2, one can choose

$$z := [e^{i2\pi f_c t}, e^{i2\pi (f_c - 1)t}, \ldots, e^{-i2\pi f_c t}]^T$$

to obtain

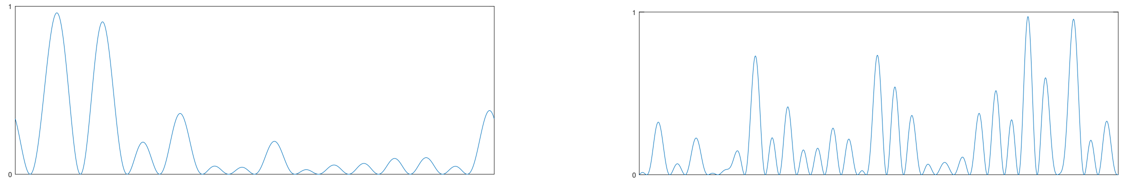
$$|\tilde{c}^* z|^2 = |(F_n^* \tilde{c})(t)| < 1 = z^* Q z.$$

This implies $z^*(Q - \tilde{c}\tilde{c}^*)z > 0$ and hence some eigenvalue of $Q - \tilde{c}\tilde{c}^*$ is strictly positive. This explains why in the limit $t \to 0$ the parametric problem (5.5) will construct a non-vanishing polynomial p_{2n-2} (with at most $n-1$ roots on the unit circle as we have seen) rather than the trivial solution $p_{2n-2} \equiv 0$ (in this situation, all eigenvalues of $Q - cc^*$ vanish).

Above we have deduced that an interior point method can solve the primal problem $(\mathcal{P}_{TV})$ if the condition (5.4) holds. And indeed, this condition holds except in very special cases. To illustrate this, suppose that $y \in \mathbb{C}^n$ is a random vector, *not* a measurement vector corresponding to a sparse signal. Then, we have tested for different values of f_c and random data $y \in \mathbb{C}^n$ and in every single case, we obtained as solutions to (P_{SDP}^*), i.e. dual solutions of $(\mathcal{P}_{TV})$, a vector $\tilde{c}$ satisfying Condition (5.4), this means p_{2n-2} is a non-vanishing polynomial with at most $(n-1)$ roots like in Figure 5.3.

In summary, the minimum total-variation problem $(\mathcal{P}_{TV})$ is solvable, for example by the approach that was discussed in this whole chapter. In other words, our noiseless super-resolution problem is solvable by applying the following scheme:

1. Solve the finite-dimensional semidefinite program (P_{SDP}^*).



(a) $f_c = 8, n = 17$, the polynomial has 14 roots.

(b) $f_c = 16, n = 33$, the polynomial has 27 roots.

Figure 5.3: The trigonometric polynomial $p_{2n-2}(e^{i2\pi t})$ with random data $y \in \mathbb{C}^n$ with i.i.d. complex Gaussian entries typically has less than $n-1$ roots. {fig: Plot of p

2. Construct the polynomial p_{2n-2} as defined in (5.3), and determine its roots on the unit circle to obtain an estimate for the support.

3. Solve the corresponding system of linear equations to obtain an estimate for the amplitudes.

It is beyond the scope of the thesis to rigourously assess or justify this approach. Actually, it would be an interesting research direction to develope a stable super-resolution procedure that can also work well in the presence of noise.

Chapter 6

Implementation

Chapter 7

Closing Remarks

{ch:Closing

Future Work

Acknowledgements

First of all I would like to thank my supervisor Gitta Kutyniok and my cosupervisor Axel Flinth. Without them the thesis would not have been possible. Also, I wish to express my gratitude to all other people, who have helped me completing the thesis.

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Appendix A

MATLAB Code Used in Numerical Experiments

{app:Code}

Appendix B

Proof of Theorem 3.17

{ch: proof

Similarly as before, in this chapter we write Δ for $\Delta(T)$, then it should hold $\Delta = \Delta(T) \geq \Delta_{\min} := 2.38\lambda_c$. Also, for $t, t' \in \mathbb{T}$ unless specified otherwise we write $|t - t'|$ for their ℓ_∞ wrap-around distance. Theorem 3.17 follows from Proposition B.1 below, which shows the existence of a dual polynomial.

{propositio

Proposition B.1. *Let $T = \{r_1, r_2, ...\} \subset \mathbb{T}^2$ be a family of points satisfying the minimum separation condition (3.36), i.e.*

$$|r_j - r_k| \geq 2.38\lambda_c, \quad r_j, r_k \in T, r_j \neq r_k.$$

Assume $f_c \geq 512$. Then for any vector $v \in \mathbb{R}^{|T|}$ with $|v_j| = 1$ for $j = 1, ..., |T|$, there exists a trigonometric polynomial q with Fourier series coefficients supported on $\{-f_c, ..., f_c\}^2$ satisfying

$$\begin{cases} q(r_j) = v_j, & t_j \in T, \\ |q(r)| < 1, & t \in \mathbb{T}^2 \setminus T \end{cases} \tag{B.1}$$

To prove this proposition, we use the same technique as in Proposition 3.6. i.e. we will construct the dual polynomial q by interpolation with a low-frequency rapidly decaying kernel. On a natural way, this 2-dimensional kernal should be obtained by tensorizing the square of the Fejer kernel (3.10), hence we consider

$$K^{2D}(r) = K(x)K(y), \quad forr = (x, y) \in \mathbb{T}^2.$$

As before, we use both K^{2D} and its (partial) derivatives, denoted by $K^{2D}_{(1,0)}$ and $K^{2D}_{(0,1)}$ to interpolate the sign pattern:

$$q(r) = \sum_{r_j \in T} \left[\alpha_j K^{2D}(r - r_j) + \beta_{1j} K^{2D}_{(1,0)}(r - r_j) + \beta_{2j} K^{2D}_{(0,1)}(r - r_j)\right]$$

Now the task is to find the coefficients α, β_1, β_2 such that for any $t_j \in T$

$$\begin{cases} q(r_j) = v_j, \\ \nabla q(t_j) = 0. \end{cases} \tag{B.2}$$

We will use similar techniques as in Section 3.4.3 and Section 3.4.4 to show the existence of a solution to (B.2) and therewith a dual polynomial that satisfies (B.1). Note that we can again apply the same estimates for squared Fejer kernel K and its derivatives as in Section 3.4.2, and hence to bring everything to the 2D setting we will make use of the following lemma:

{lemma: technic

Lemma B.2. *Let $f : \mathbb{R}_+^2 \to \mathbb{R}$ be defined by $f(x) = f_1(x_1)f_2(x_2)$ where f_1 and f_2 are monotonically decreasing real functions on $\mathbb{R}_+$, $x = (x_1, x_2) \in \mathbb{R}_+^2$. Assume that the family of points $X := \{x^{(j)}\}_{j\in\mathbb{N}} \subset \mathbb{R}_+^2$ satisfies $\left|x^{(j)} - x^{(k)}\right| \geq 1$. Then*

$$\sum_{j\in\mathbb{N}} f(x^{(j)}) \leq \sum_{j_1\in\mathbb{N}_0} f_1(j_1) \sum_{j_2\in\mathbb{N}_0} f_2(j_2)$$

Proof. The mapping $X \to \mathbb{R}_+^2$, $x \mapsto (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor)$ is injective, because otherwise there exist two different points in X being mapped to the same pair of integers (j_1, j_2), which would implies that they are both contained in the square $[j_1, j_1 + 1) \times [j_2, j_2 + 1)$ and this contradicts the assumption about their distance. By the monotonicity of f_1 and f_2 we obtain

$$\sum_{j\in\mathbb{N}} f(x^{(j)}) \leq \sum_{j\in\mathbb{N}} f_1(\lfloor x_1^{(j)} \rfloor) f_2(\lfloor x_2^{(j)} \rfloor) \leq \sum_{j_1\in\mathbb{N}_0} f_1(j_1) \sum_{j_2\in\mathbb{N}_0} f_2(j_2)$$

□

f_solution}

Lemma B.3. *Under the hypotheses of Proposition B.1 there exist vectors $\alpha, \beta_1, \beta_2 \in \mathbb{C}^{|T|}$ satisfying (B.2) and*

$$\begin{aligned} \|\alpha\|_\infty &\leq 1 + 5.577 \times 10^{-2}, \\ \|\beta\|_\infty &\leq 2.930 \times 10^{-2} \lambda_c, \end{aligned} \tag{B.3}$$

where $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$. Moreover, if $v_1 = 1$, then

$$\alpha_1 \geq 1 - 5.577 \times 10^{-2}. \tag{B.4}$$

Proof. Just as before, we wish to bring the interpolation constraints in matrix form, hence we define

$$[D_{(0,0)}]_{j,k} = K^{2D}(r_j - r_k), \quad [D_{(1,0)}]_(j,k) = K^{(2D)}_{(1,0)}(r_j - r_k), \quad [D_{(1,0)}]_(j,k) = K^{(2D)}_{(1,0)}(r_j - r_k),$$
$$[D_{(1,1)}]_{(j,k)} = K^{(2D)}_{(1,1)}(r_j - r_k), \quad [D_{(2,0)}]_{(j,k)} = K^{(2D)}_{(2,0)}(r_j - r_k), \quad [D_{(0,2)}]_{(j,k)} = K^{(2D)}_{(0,2)}(r_j - r_k).$$

Here, for $(m_1, m_2) \in \mathbb{N}_0^2$, $m_1 + m_2 \leq 2$ we denote by $K^{(2D)}_{(m_1,m_2)}$ the (m_1, m_2)-derivative of K^{2D}. Since K is an even function, K' is also even and K' is odd, thus $D_{(0,0)}, D_{(1,1)}, D_{(2,0)}, D_{(0,2)}$ are symmetric and $D_{(1,0)}, D_{(0,1)}$ are skew-symmetric. The interpolation constraints in (B.2) are now equivalent to

$$\begin{bmatrix} D_{(0,0)} & D_{(1,0)} & D_{(2,0)} \\ D_{(1,0)} & D_{(2,0)} & D_{(1,1)} \\ D_{(0,1)} & D_{(1,1)} & D_{(0,2)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} D_0 & -D_1^T \\ D_1 & D_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}, \tag{B.5}$$

where $D_0 := D_{(0,0)}, \tilde{D}_1 := \begin{bmatrix} D_{(1,0)} \\ D_{(0,1)} \end{bmatrix}$ and $\tilde{D}_2 := \begin{bmatrix} D_{(2,0)} & D_{(1,1)} \\ D_{(1,1)} & D_{(0,2)} \end{bmatrix}$. We consider

$$\|I - D_0\|_\infty = \sum_{r_j \in T\setminus\{0\}} \left|K^{(2D)}(r_j)\right|,$$

and split this sum into the sum over $r_j \in T \setminus \{0\}$ for which $|x_j| < \Delta/2$ or $|y_j| < \Delta/2$ and over $r_j \in T \setminus \{0\}$ that satisfies $\min\{|x_j|, |y_j|\} \geq \Delta/2$. Now we use B_0 from Lemma 3.8 to bound the first sum, recall that the absolute value of K is bounded by 1, B_0 is monotonically decreasing and x_j's are at least Δ apart, hence:

$$\sum_{\substack{r_j \in T\setminus\{0\}: \\ |y_j| < \Delta/2}} \left|K^{2D}(r_j)\right| \leq \sum_{\substack{r_j \in T\setminus\{0\}: \\ |y_j| < \Delta/2}} |K(x_j)| \leq 2 \sum_{j=1}^{\infty} B_0(j\Delta).$$

For the sum over $\{r_j \in T \setminus \{0\} : |x_j| \leq \Delta/2\}$ we just do similarly. Now we consider the last sum, by Lemma 3.8 and Lemma B.2 we get the following estimate for one part of this sum, which can be done analogously for the other three parts

$$\sum_{\substack{r_j \in T\setminus\{0\}:\\ x_j, y_j \geq \Delta/2}} \left| K^{(2D)}(r_j) \right| \leq \sum_{\substack{r_j \in T\setminus\{0\}:\\ x_j, y_j \geq \Delta/2}} B_0(x_j) B_0(y_j) \leq \left[\sum_{j\geq 0} B_0(\Delta/2 + j\Delta) \right]^2 .$$

Putting all these estimatates together we obtain

$$\|I - D_0\|_\infty \leq 4 \sum_{j\geq 0} B_0(j\Delta) + 4 \left[\sum_{j\geq 0} B_0(\Delta/2 + j\Delta) \right]^2 \leq 4.854 \times 10^{-2}, \tag{B.6}$$

where for the last inequality we use the monotonicity of B_0 and the same technique as in (3.17), starting at $j = 0$ and setting $j_0 = 20$.

Analogously we have

$$\left\| D_{(1,0)} \right\|_\infty \leq 2 \sum_{j\geq 1} B_1(j\Delta) + 2 \left\| K' \right\|_\infty \sum_{j\geq 1} B_0(j\Delta) + 4 \left[\sum_{j\geq 0} B_0(\Delta/2 + j\Delta) \right]^2 \leq 7.723 \times 10^{-2} f_c; \tag{B.7}$$

at the last inequality we have implicitly used that $\|K'\|_\infty \leq 2.08(f_c + 2)$, which follows from combining (3.27) and Lemma 3.8. Similarly,

$$\left\| D_{(1,1)} \right\|_\infty \leq 4 \left\| K' \right\|_\infty \sum_{j\geq 1} B_1(j\Delta) + 4 \left[\sum_{j\geq 0} B_1(\Delta/2 + j\Delta) \right]^2 \leq 0.1576 f_c^2, \tag{B.8}$$

and note that $\|K''\|_\infty = \pi^2 f_c(f_c + 4)/3$ since $|K''|$ has global maximum at 0, hence

$$\left\| K^{2D}_{(2,0)}(0) I - D_{(2,0)} \right\|_\infty \leq 2 \sum_{j\geq 1} B_2(j\Delta) + 2 \left\| K'' \right\|_\infty \sum_{j\geq 1} B_0(j\Delta) + 4 \left[\sum_{j\geq 0} B_0(\Delta/2 + j\Delta) \right]^2 \leq 0.3539 f_c^2. \tag{B.9}$$

Similarly to the proof of Proposition 3.6 we apply Lemma 3.10 and Lemma 3.11 combined with these estimates to show that the system of linear equations (B.5) has a solution that is bounded. To ease notation we set

$$\begin{aligned}
S_1 &:= D_{(2,0)} - D_{(1,1)} D_{(0,2)}^{-1} D_{(1,1)}, \\
S_2 &:= D_{(1,0)} - D_{(1,1)} D_{(0,2)}^{-1} D_{(0,1)}, \\
S_3 &:= D_0 + S_2^T S_1^{-1} S_2 - D_{(0,1)} D_{(0,2)}^{-1} D_{(0,1)}.
\end{aligned}$$

Now, the idea is as follows:

Firstly, it follows from Lemma 3.10 that if $D_{(0,2)}$ and its Schur complement S_1 are invertible, then $\tilde{D}_2 = \begin{bmatrix} D_{(2,0)} & D_{(1,1)} \\ D_{(1,1)} & D_{(0,2)} \end{bmatrix}$ is invertible with

$$\tilde{D}_2^{-1} = \begin{bmatrix} S_1^{-1} & -S_1^{-1} D_{(1,1)} D_{(0,2)}^{-1} \\ -D_{(0,2)}^{-1} D_{(1,1)} S^{-1} & D_{(0,2)}^{-1} + D_{(0,2)}^{-1} D_{(1,1)} S_1^{-1} D_{(1,1)} D_{(0,2)}^{-1} \end{bmatrix}.$$

Then, using again Lemma 3.10 for $\tilde{D}_2$ and its Schur complement S_3 yields a solution to (B.5) as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} I \\ -\tilde{D}_2^{-1}\tilde{D}_1\end{bmatrix}(D_0 + \tilde{D}_1^T\tilde{D}_2^{-1}\tilde{D}_1)^{-1}v \Leftrightarrow \begin{bmatrix}\alpha\\ \beta_1\\ \beta_2\end{bmatrix} = \begin{bmatrix} I \\ -S_1^{-1}S_2 \\ D_{(0,2)}^{-1}(D_{(0,2)}^{-1}S_1^{-1}S_2 - D_{(0,1)})\end{bmatrix} S_3^{-1}v$$

and also the desired estimates on this solution.

To obtain the needed invertibility at each step, we process as follows:

By Lemma 3.11 and (B.9) we get that $D_{(0,2)}$ is invertible and

$$\left\|D_{(0,2)}^{-1}\right\|_\infty \leq \frac{1}{\left|K_{(0,2)}^{2D}(0)\right| - \left\|K_{(0,2)}^{2D}(0)I - D_{(0,2)}\right\|_\infty} \leq \frac{0.3399}{f_c^2}, \tag{B.10}$$

which combined with (B.8) and (B.9) yields

$$\left\|K_{(2,0)}^{2D}(0)I - S_1\right\|_\infty \leq \left\|K_{(2,0)}^{2D}(0)I - D_{(2,0)}\right\|_\infty + \left\|D_{(1,1)}\right\|_\infty^2 \left\|D_{(0,2)}^{-1}\right\|_\infty \leq 0.3624 f_c^2.$$

Again by Lemma 3.11, this equation shows that S_1 is invertible and furthermore

$$\left\|S_1^{-1}\right\|_\infty \leq \frac{1}{\left|K_{(2,0)}^{2D}\right| - \left\|K_{(2,0)}^{2D}(0)I - S_1\right\|_\infty} \leq \frac{0.3408}{f_c^2}. \tag{B.11}$$

Next, by (B.7), (B.8) and (B.10) we obtain

$$\|S_2\|_\infty \leq \left\|D_{(1,0)}\right\|_\infty + \left\|D_{(1,1)}\right\|_\infty \left\|D_{(0,2)}^{-1}\right\|_\infty \left\|D_{(0,1)}\right\|_\infty \leq 8.142 \times 10^{-2} f_c,$$

which together with (B.6), (B.7), (B.10) and (B.11) yields

$$\|I - S_3\|\infty \leq \|I - D_0\|_\infty + \|S_2\|_\infty^2 \left\|S_1^{-1}\right\|_\infty + \left\|D_{(0,1)}\right\|_\infty^2 \left\|D_{(0,2)}^{-1}\right\|_\infty \leq 5.283 \times 10^{-2}.$$

Finally, these results combined with Lemma 3.11 give us the desired bounds on the solution:

$$\begin{aligned}
\|\alpha\|_\infty &\leq \left\|S_3^{-1}\right\|_\infty \leq \frac{1}{1 - \|I - S_3\|_\infty} \leq 1 + 5.577 \times 10^{-2},\\
\|\beta_1\|_\infty &\leq \left\|S_1^{-1}S_2S_3^{-1}\right\|_\infty \leq \left\|S_1^{-1}\right\|_\infty \|S_2\|_\infty \left\|S_3^{-1}\right\|_\infty \leq 2.930 \times 10^{-2}\lambda_c,\\
\|\beta_2\|_\infty &\leq \left\|D_{(0,2)}^{-1}\right\|_\infty \left(\left\|D_{(1,1)}\right\|_\infty \left\|S_1^{-1}\right\|_\infty \|S_2\|_\infty + \left\|D_{(0,1)}\right\|_\infty\right) \left\|S_3^{-1}\right\|_\infty \leq 2.930 \times 10^{-2}\lambda_c.
\end{aligned}$$

Also, for the case $v_1 = 1$ we have

$$\alpha_1 = v_1 - [(I - S_3^{-1})v]_1 \geq 1 - \left\|S_3^{-1}\right\|_\infty \|I - S_3\|_\infty \geq 1 - 5.577 \times 10^{-2}.$$

□

Now we will control the absolute value of the constructed dual polynomial, i.e. show that $|q(r)| < 1$ for $r \in [0,1]^2 \setminus T$. The next lemma gives a bound on $|q|$ near a point $r_1 \in T$, without loss of generality we can assume that $r_1 = 0 \in T$ and that $q(r_1) = 1$ (instead of -1; in that case we just need a minor change in the last inequality of Lemma B.3, the proof of the next lemma remains similar).

Lemma B.4. *Suppose $r_1 = 0 \in T$ and $q(0) = 1$. Then under the hypotheses of (B.2) it holds $|q(r)| < 1$ for any $r \in [0,1]^2$ with $0 < |r| < 0.2447\lambda_c$.*

Proof. Since v is real valued, the vectors α and β are real valued and so is the polynomial q. For $|r| \leq 0.2447\lambda_c$, we will show the negative definiteness the Hessian matrix of q, i.e.

$$H = \begin{bmatrix} q_{(2,0)}(r) & q_{(1,1)}(r) \\ q_{(1,1)}(r) & q_{(0,2)}(r) \end{bmatrix}.$$

The Hessian matrix is symmetric, hence it suffices to show that its determinant is positiv and $q_{(2,0)}(r) < 0$ for $|r| < 0.2447\lambda_c$.

From the proof of Lemma 3.13, namely the equations (3.26) - (3.30) we obtain

$$\begin{aligned} K^{2D}(x,y) &\leq \left(1 - \frac{\pi^2 f_c(f_c+4)x^2}{6}\right)\left(1 - \frac{1}{\pi^2 f_c(f_c+4)y^2}\right), \\ K^{2D}_{(2,0)}(x,y) &\leq \left(-\frac{\pi^2 f_c(f_c+4)}{3} + \frac{\pi^4(f_c+2)^4 x^2}{6}\right)\left(1 - \frac{\pi^2 f_c(f_c+4)y^2}{6}\right), \end{aligned}$$

and

$$\begin{aligned} \left|K^{2D}_{(1,0)}(x,y)\right| &\leq \frac{\pi^2 f_c(f_c+4)x}{3}, \quad \left|K^{2D}_{(1,1)}(x,y)\right| \leq \frac{\pi^4 f_c^2(f_c+4)^2 xy}{9}, \\ \left|K^{2D}_{(2,1)}(x,y)\right| &\leq \frac{\pi^4 f_c^2(f_c+4)^2 y}{9}, \quad \left|K^{2D}_{(3,0)}(x,y)\right| \leq \frac{\pi^4(f_c+2)^4 x}{3}. \end{aligned}$$

Since these bounds are monotone in x and y, we can evalutate them at $x = 0.2447\lambda_c$ and $y = 0.2447\lambda_c$ to obtain the bounds for every $|r| \leq 0.2447\lambda_c$, i.e.

$$\begin{aligned} K^{2D}(r) &\geq 0.8113, & \left|K^{2D}_{(1,0)}(r)\right| &\leq 0.8113, & K^{2D}_{(2,0)}(r) &\leq -2.097 f_c^2, \\ \left|K^{2D}_{(1,1)}(r)\right| &\leq 0.6531 f_c, & \left|K^{2D}_{(2,1)}(r)\right| &\leq 2.669 f_c^2, & \left|K^{2D}_{(3,0)}(r)\right| &\leq 8.070 f_c^3. \end{aligned} \tag{B.12}$$

Note that by symmetry, the bounds for $K^{2D}_{(1,0)}$, $K^{2D}_{(2,0)}$, $K^{2D}_{(2,1)}$ and $K^{2D}_{(3,0)}$ also hold for $K^{2D}_{(0,1)}$, $K^{2D}_{(0,2)}$, $K^{2D}_{(1,2)}$ and $K^{2D}_{(0,3)}$ respectively.

Analogously to Lemma 3.9 we need to bound the sums $\sum_{r_j \in T\setminus\{0\}} \left|K^{2D}_{(m_1,m_2)}(r - r_j)\right|$ for $|r| \leq \Delta/2$ and $m_1, m_2 = 0,1,2,3$.

First we consider the case $(m_1, m_2) = (0,0)$: Without loss of generality assume that $r = (x,y) \in \mathbb{R}^2_+$. Lemma B.2 combined with the monotonicity of B_0 gives an upper bound for the sum over r_j's belonging to the three quadrants $\{r_j \in T : |r_j| > \Delta/2\} \setminus \mathbb{R}^2_+$ (note that for such r_j we have $|x - x_j| \geq \Delta/2$ and $|y - y_j| \geq \Delta/2$):

$$\sum_{\substack{|r_j| \geq \Delta/2: \\ r_j \notin \mathbb{R}^2_+}} \left|K^{2D}(r - r_j)\right| \leq 3\left[\sum_{j\geq 0} B_0(\Delta/2 + j\Delta)\right]^2.$$

Similarly for the sum over r_j's with $|x_j| < \Delta/2$ or $|y_j| < \Delta/2$ we have

$$\sum_{|x_j|<\Delta/2 \text{ or } |y_j|<\Delta/2} \left|K^{2D}(r - r_j)\right| \leq \|K\|_\infty \sum_{j\geq 1} B_0(j\Delta - |r|),$$

here, e.g. if $|x_j| < \Delta/2$ then $|y_j| \geq \Delta$ so that $|y - y_j| \geq \Delta - |r|$. It remains to bound the sum over $r_j \in \mathbb{R}^2_+$ obeying $|r_j| > \Delta/2$. For this we apply Lemma B.2 for $f_1 = f_2$, defined as $f_1(t) = 1$ if $|t| \leq \Delta$ and

$f_1(t) = B_0(t\Delta - |r|)$ otherwise, and note that there is at most just one r_j in this quadrant satisfying either $|x_j - x| \le \Delta$ or $|y_j - y| \le \Delta$ due to the minimum separation condition, thus

$$\sum_{\substack{|r_j| \ge \Delta/2: \\ r_j \in \mathbb{R}_+^2}} \left|K^{2D}(r - r_j)\right| \le \sum_{j \ge 1} B_0(j\Delta - |r|) + \left[\sum_{j \ge 1} B_0(j\Delta - |r|)\right]^2$$

Now we look at the similar sums for $K^{2D}_{(m_1,m_2)}$ for other values of m_1, m_2. These sums can be bounded using the same techniques as above for K^{2D}, and we get that if $|r| \le \Delta/2$

$$\sum_{r_j \in T \setminus \{0\}} \left|K^{2D}_{(m_1,m_2)}(r - r_j)\right| \le Z_{(m_1,m_2)}(|r|), \tag{B.13}$$

where $Z_{(m_1,m_2)} : \mathbb{R}_+^2 \to \mathbb{R}$ is defined by

$$\begin{aligned} Z_{(m_1,m_2)}(u) = 3 \left[\sum_{j \ge 0} B_{m_1}(\Delta/2 + j\Delta)\right] \left[\sum_{j \ge 0} B_{m_2}(\Delta/2 + j\Delta)\right] \\ +2 \left\|K^{(m_1)}\right\|_\infty \sum_{j \ge 1} B_{m_2}(j\Delta - u) + 2 \left\|K^{(m_2)}\right\|_\infty \sum_{j \ge 1} B_{m_1}(j\Delta - u) \\ + \left[\sum_{j \ge 1} B_{m_1}(j\Delta - u)\right] \left[\sum_{j \ge 1} B_{m_2}(j\Delta - u)\right]. \end{aligned}$$

Note that $Z_{(m_1,m_2)} = Z_{(m_2,m_1)}$, hence for $m_1 > m_2$ we will replace $Z_{(m_1,m_2)}$ by $Z_{(m_2,m_1)}$. On the other hand, the absolute value of K and K'' reach their global maximum at 0, hence $\left\|K^{(0)}\right\|_\infty = 1$, $\left\|K^{(2)}\right\|_\infty = \pi^2 f_c(f_c + 4)/3$. Furthermore we can combine the bounds on $|K'|$ and $|K'''|$ in Lemma 3.8 with (3.27) and (3.30) to get that $\left\|K^{(1)}\right\|_\infty \le 2.08(f_c + 2)$ and $\left\|K^{(3)}\right\|_\infty \le 25.3(f_c + 2)^3$ if $f_c \ge 512$. Finally, the necessary numerical upper bounds on $Z_{(m_1,m_2)}$ are collected in the table below:

$Z_{(0,0)}(u)$	$Z_{(0,1)}(u)$	$Z_{(1,1)}(u)$	$Z_{(0,2)}(u)$	$Z_{(1,2)}(u)$	$Z_{(0,3)}(u)$
6.405×10^{-2}	$0.1047 f_c$	$0.1642 f_c^2$	$0.4019 f_c$	$0.6751 f_c^3$	$1.574 f_c^3$

r_bounds_Z}

Table B.1: Numerical upper bounds on $Z_{(m_1,m_2)}(u)$ at $u = 0.2447\lambda_c$.

Recall that by our construction of the dual polynomial,

$$q_{(2,0)}(r) = \sum_{r_j \in T} \alpha_j K^{2D}_{(2,0)}(r - r_j) + \beta_{1j} K^{2D}_{(3,0)}(r - r_j) + \beta_{2j} K^{2D}_{(2,1)}(r - r_j),$$

hence, combining the results from Table B.1, Lemma B.3, (B.12) and (B.13) yields that if $|r| \le 0.2447\lambda_c$ it holds

$$\begin{aligned} q_{(2,0)}(r) &\le \alpha_1 K^{2D}_{(2,0)}(r) + \|\alpha\|_\infty \sum_{r_j \in T \setminus \{0\}} \left|K^{2D}_{(2,0)}(r - r_j)\right| \\ &\quad + \|\beta\|_\infty \left[\left|K^{2D}_{(3,0)}(r)\right| + \sum_{r_j \in T \setminus \{0\}} \left|K^{2D}_{(3,0)}(r - r_j)\right| + \left|K^{2D}_{(2,1)}(r)\right| + \sum_{r_j \in T \setminus \{0\}} \left|K^{2D}_{(2,1)}(r - r_j)\right|\right] \\ &\le \alpha_1 K^{2D}_{(2,0)}(r) + \|\alpha\|_\infty Z_{(0,2)}(|r|) + \|\beta\|_\infty \left[\left|K^{2D}_{(3,0)}(r)\right| + Z_{(0,3)}(|r|) + \left|K^{2D}_{(2,1)}(r)\right| + Z_{(1,2)}(|r|)\right] \\ &\le -1.175 f_c^2. \end{aligned}$$

Clearly, by symmetry the same bound holds for $q_{(0,2)}$. Also, recall that

$$q_{(1,1)}(r) = \sum_{r_j \in T} \alpha_j K^{2D}_{(1,1)}(r - r_j) + \beta_{1j} K^{2D}_{(2,1)}(r - r_j) + \beta_{2j} K^{2D}_{(1,2)}(r - r_j),$$

hence we can analogously gain the bound on its absolute value as

$$\begin{aligned}\left|q_{(1,1)}(r)\right| &\leq \|\alpha\|_\infty \left[\left\|K^{2D}_{(1,1)}(r)\right\| + Z_{(1,1)}(|r|)\right] + \|\beta\|_\infty \left[\left|K^{2D}_{(2,1)}(r)\right| + \left|K^{2D}_{(1,2)}(r)\right| + 2Z_{(1,2)}(|r|)\right] \\ &\leq 1.059 f_c^2.\end{aligned}$$

Together these bounds yield that $q_{(2,0)}(t) < 0$ and $\det(H) = q_{(2,0)}(t) q_{(0,2)}(t) - \left|q_{(1,1)}(t)\right|^2 > 0$, hence H is strictly negative definite. A Taylor expansion of q at 0 show that in the square $|r| \leq 0.2447\lambda_c$ the polynomial q reaches its only maximum at 0 with $q(0) = 1$. To conclude that $|q(r)| < 1$ in this square, we have to show that $q(r) > 1$, and indeed,

$$\begin{aligned}q(r) &= \sum_{r_j \in T} \alpha_j K^{2D}(r - r_j) + \beta_{1j} K^{2D}_{(1,0)}(r - r_j) + \beta_{2j} K^{2D}_{0,1}(r - r_j) \\ &\geq \alpha_1 K^{2D}(r) - \|\alpha\|_\infty Z_{(0,0)}(|r|) - \|\beta\|_\infty \left[\left|K^{2D}_{(0,1)}(r)\right| + \left|K^{2D}_{(1,0)}(r)\right| + 2Z_{(0,1)}(|r|)\right] \\ &\geq 0.6447.\end{aligned}$$

$\square$

Lemma B.5. *Suppose* $0 \in T$ *and* $q(0) = 1$*. Then under the hypotheses of (B.2) we have* $|q(r)| < 1$ *for any* $r \in [0,1]^2$ *with* $0.2447\lambda_c \leq |r| \leq \Delta/2$*. This also holds for all* $r \in [0,1]^2$ *that are closer to* $0 \in T$ *than to any other element of* T *(w.r.t.* ℓ_∞*-distance).*

Proof. We apply (3.33) to obtain that for $t_1 \leq |r| \leq t_2$ with $0 \leq t_1 \leq t_2 \leq 1/2$ the following estimates hold

$$\begin{aligned}\left|K^{2D}(r)\right| &\leq \left(1 - \frac{\pi^2 f_c(f_c+4)x^2}{6} + \frac{\pi^4 (f_c+2)^4 x^4}{72}\right)\left(1 - \frac{\pi^2 f_c(f_c+4)y^2}{6} + \frac{\pi^4 (f_c+2)^4 y^4}{72}\right) \\ &\leq \left[1 - \frac{\pi^2 (f_c+2)^2 t_1^2}{6}\left(1 - \frac{\pi^2 (f_c+2)^2 t_2^2}{12}\right)\right]^2, \\ \left|K^{2D}_{(1,0)}(r)\right| &\leq \frac{\pi^2 f_c(f_c+4) t_2}{3}, \\ \left|K^{2D}_{(0,1)}(r)\right| &\leq \frac{\pi^2 f_c(f_c+4) t_2}{3}.\end{aligned}$$

Now, for $0.2447\lambda_c \leq |r| \leq \Delta/2$ we have

$$\begin{aligned}|q(r)| &= \left|\sum_{r_j \in T} \alpha_j K^{2D}(r - r_j) + \beta_{1j} K^{2D}_{(1,0)}(r - r_j) + \beta_{2j} K^{2D}_{(0,1)}(r - r_j)\right| \\ &\leq \|\alpha\|_\infty \left[\left|K^{(2D)}(r)\right| + Z_{(0,0)}(|r|)\right] + \|\beta\|_\infty \left[\left|K^{2D}_{(1,0)}(r)\right| + \left|K^{2D}_{(0,1)}(r)\right| + 2Z_{(0,1)}(|r|)\right] \\ &= W(r) + \|\alpha\|_\infty Z_{(0,0)}(|r|) + 2\|\beta\|_\infty Z_{(0,1)}(|r|),\end{aligned}$$

where

$$W(r) := \|\alpha\|_\infty \left|K^{2D}(r)\right| + \|\beta\|_\infty \left(\left|K^{2D}_{(1,0)}\right| + \left|K^{2D}_{(1,0)}\right| \right).$$

Note that from lemma B.3 we have upper bounds for $\|\alpha\|_\infty$ and $\|\beta\|_\infty$, these combining with the estimates above give us upper bounds for $W(r)$ for each pair $(t_1, t_2) \in \{(0.2447\lambda_c, 0.27\lambda_c), (0.27\lambda_c, 0.36\lambda_c), (0.36\lambda_c, 0.56\lambda_c), (0.56\lambda_c, 0.84\lambda_c)\}$, which are collected in Table B.2 below:

t/λ_c	t_2/λ_c	$W(r)$	$Z_{(0,0)}(t_2)$	$Z_{(0,1)}(t_2)$
0.2447	0.0.27	0.9203	6.561×10^{-2}	$0.1074 f_c$
0.27	0.36	0.9099	7.196×10^{-2}	$0.1184 f_c$
0.36	0.56	0.8551	0.239×10^{-2}	$0.1540 f_c$
0.56	0.84	0.8118	0.1490	$0.2547 f_c$

r_bounds_W}

Table B.2: Numerical upper bounds on quantities used to bound $|q(r)|$ for $|r| \in [0.2447\lambda_c, 0.84\lambda_c]$.

With the results from Table B.2, we can easily obtain the desired on $|q|$ on each of the intervals $(0.2447\lambda_c, 0.27\lambda_c)$, $(0.27\lambda_c, 0.36\lambda_c)$, $(0.36\lambda_c, 0.56\lambda_c)$, and $(0.56\lambda_c, 0.84\lambda_c)$, namely $|q| < 0.9958$, $|q| < 0.9921$, $|q| < 0.9617$ and $|q| < 0.9841$ respectively. Finally, we consider the case $|r| \in (0.84\lambda_c, \Delta/2)$, for this applying Lemma 3.8 gives

$$\begin{aligned} &W(r) \leq 0.5619, \\ &Z_{(0,0)}(|r|) \leq Z_{(0,0)}(0.84\lambda_c) \leq 0.3646, \\ &Z_{(0,1)}(|r|) \leq Z_{(0,1)}(0.84\lambda_c) \leq 0.6502 f_c \end{aligned}$$

and thus $|q(r)| \leq 0.9850$. Recall that B_0 and B_1 are monotonically decreasing, thus we can also use these last bounds to any location beyond $\Delta/2$ closer to 0 than to any other element of T. This concludes the proof. $\square$

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