Appendix B

Appendix for Chapter 4

B.1 Proof Regarding Minimal Energy

Let us define

$$\bar{\mathbf{x}} = \mathbf{x}(t_1) - \mathbf{M}(t_1; 0, \mathbf{g}, \mathbf{x}_0) \tag{B.1}$$

Then the assumption that $\mathbf{u'}$ and $\mathbf{u}$ transfer $(\mathbf{x}_0, 0)$ to $(\mathbf{0}, t_1)$ implies that

$$\bar{\mathbf{x}} = \int_0^{t_1} \mathbf{K}(\xi, t_1)\mathbf{B}\mathbf{u}(\xi)d\xi = \int_0^{t_1} \mathbf{K}(\xi, t_1)\mathbf{B}\mathbf{u}'(\xi)d\xi \tag{B.2}$$

Subtracting both sides, one can obtain

$$\int_0^{t_1} \mathbf{K}(\xi, t_1)\mathbf{B}\{\mathbf{u}(\xi)' - \mathbf{u}(\xi)\}(\xi)d\xi = \mathbf{0} \tag{B.3}$$

which implies that

$$\left\langle \int_0^{t_1} \mathbf{K}(\xi, t_1)\mathbf{B}\{\mathbf{u}(\xi)' - \mathbf{u}(\xi)\}(\xi)d\xi, C_o^{-1}(0, t_1)\bar{\mathbf{x}} \right\rangle = 0 \tag{B.4}$$

where $<\ ,\ >$ indicates the inner product of vectors. By using the following property of the inner product

$$\langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle = \left\langle \mathbf{A}^T\mathbf{x}, \mathbf{y} \right\rangle \tag{B.5}$$

this equation can rewritten as

$$\int_0^{t_1} \left\langle \mathbf{u}(\xi)' - \mathbf{u}(\xi), \{\mathbf{K}(\xi, t_1)\mathbf{B}\}^T C_o^{-1}(0, t_1)\bar{\mathbf{x}} \right\rangle d\xi = 0 \tag{B.6}$$

With the use of (4.5), and then (B.6) becomes

$$\int_0^{t_1} \langle \mathbf{u}(\xi)' - \mathbf{u}(\xi), \mathbf{u}(\xi) \rangle \, d\xi = 0 \tag{B.7}$$

Consider now

$$\int_0^{t_1} \|\mathbf{u}(\xi)'\|^2 \, d\xi \tag{B.8}$$

where $\|\mathbf{x}\| \equiv (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}$. After some manipulation, and using (B.7), one can obtain

$$\begin{aligned}
\int_0^{t_1} \|\mathbf{u}(\xi)'\|^2 \, d\xi &= \int_0^{t_1} \|\mathbf{u}(\xi)' - \mathbf{u}(\xi) + \mathbf{u}(\xi)\|^2 \, d\xi \\
&= \int_0^{t_1} \|\mathbf{u}(\xi)' - \mathbf{u}(\xi)\|^2 \, d\xi + \int_0^{t_1} \|\mathbf{u}(\xi)\|^2 \, d\xi \\
&\qquad + 2 \int_0^{t_1} \langle \mathbf{u}(\xi)' - \mathbf{u}(\xi), \mathbf{u}(\xi) \rangle \, d\xi \\
&= \int_0^{t_1} \|\mathbf{u}(\xi)' - \mathbf{u}(\xi)\|^2 \, d\xi + \int_0^{t_1} \|\mathbf{u}(\xi)\|^2 \, d\xi
\end{aligned} \tag{B.9}$$

Since

$$\int_0^{t_1} \|\mathbf{u}(\xi)' - \mathbf{u}(\xi)\|^2 \, d\xi \tag{B.10}$$

is always nonnegative, it can be concluded that

$$\int_0^{t_1} \|\mathbf{u}(\xi)'\|^2 \, d\xi \geq \int_0^{t_1} \|\mathbf{u}(\xi)\|^2 \, d\xi \tag{B.11}$$

□

B.2 Comparisons with Other Types of Controllability and Observability

Depending on the nature of the problem under consideration, there exist various definitions of controllability and observability for time-delay systems. For example, spectral controllability has been developed to apply Finite Spectrum Assignment, a stabilizing method counteracting the effect of the delay based on prediction of the state. Spectral controllability is a sufficient condition for point-wise controllability used in our paper (sometimes point-wise controllability is also called controllability or fixed time complete controllability). The other definitions of controllability and observability are not related to *linear feedback controller* or *linear observer* as in systems of ODEs. The presented definitions and theorems in Chapter 4 are most similar to those for ODEs among the existing ones. The main purpose of the study in Chapter 4 is to put the controllability and observability Gramians to practical use by approximating them with the Lambert W function approach. Figures B.1 and B.2 show the relationships between various types of controllability and observability.