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DETERMINATE SYSTEMS

Controllability of Nonlinear Algebraic Differential Systems[1]

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Abstract—We consider a control system of nonlinear ordinary differential equations unsolved for the derivative of the desired vector-function, the system having arbitrarily high index of unsolvability. For such systems the null-controllability by linear approximation is investigated. Conditions of complete controllability are obtained for the linear system with smooth coefficients. It is shown that the complete controllability implies the local null-controllability in the linear case.

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1. INTRODUCTION

Consider a control system of nonlinear ordinary differential equations

$$F(t, x(t), x'(t), u(t)) = 0, \quad t \in I = (a_0 - \varepsilon, a_0 + \varepsilon), \tag{1}$$

where the n-variate vector-function $F(t, x, y, u)$ is defined on the domain

$$\mathcal{D} = \{(t, x, y, u) : \ t \in I; \ \|x\|, \ \|y\|, \|u\| < K_0\} \subset \mathbf{R}^{2n+l+1};$$

$x(t)$ is the desired n-variate vector-function; $u(t)$ is the l-variate function of control; K_0, ε are positive constants. We use hereafter the following notation: $\| * \|$ stands for a norm in a Euclidean space; $\phi'(t) = \dfrac{d}{dt}\phi(t)$, $\phi^{(i)}(t) = \left(\dfrac{d}{dt}\right)^i \phi(t) \ \forall \phi(t) \in \mathbf{C}^i(I)$.

The function $F(t, x, y, u)$ is assumed to have sufficient number of continuous partial derivatives on $\mathcal{D}$; in addition, $\det \dfrac{\partial F(t, x, y, u)}{\partial y} \equiv 0$ over $\mathcal{D}$. Such systems are called the algebraic differential systems (ADS). An integer r: $0 \leqslant r \leqslant n$ is used as the measure of unsolvability with respect to the desired vector-function. r is called the index of unsolvability.

The function F is supposed to satisfy

$$F(t, 0, 0, 0) = 0 \quad \forall t \in I. \tag{2}$$

Paper [1] has the detailed controllability and observability analysis of linear ADS on the base of the reduction to the canonical Kronecker form. Different types of controllability and observability for linear autonomous ADS (including controllability at infinity, impulsive controllability and observability) are considered in [2–4].

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A criterion of R-controllability for linear ADS with infinitely differentiable coefficients is obtained in [5]. The definition of R-controllability cannot be given in terms of initial system, since it is associated with a certain structural form. The existance of this form is proven, but constructive methods to find it are not presented.

Conditions of complete controllability for linear time-variant ADS are proven in [6]. The technique of solving for derivatives, which is based on the transformation to a so-called central canonical form [7], allows to study systems with real-analytical coefficients.

The present paper realizes the very first (as far as the author knows) attempt to investigate the local null-controllability of nonlinear system (1) by its linear approximation. Since the study of complete controllability of linear ADS with smooth coefficients is of particular importance, appropriate criteria are obtained. It is shown that the complete controllability implies local null-controllability in the linear case.

2. DEFINITIONS AND NOTATIONS

Definition 1. A system of finite equations

$$\mathcal{F}_r(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) = \begin{pmatrix} F(t,x,y,u) \\ F_1(t,x,y,z_1,u,v_1) \\ \dots \\ F_r(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) \end{pmatrix} = 0, \tag{3}$$

such that $x, y, z_j \in \mathbf{R}^n$; $u, v_j \in \mathbf{R}^l$, and the functions $F_j(t,x,y,z_1,\dots,z_j,u,v_1,\dots,v_j)$ $(j=\overline{1,r})$ satisfy the condition: for all two vector functions $\phi(t) \in \mathbf{C}^{j+1}(I)$, $\psi(t) \in \mathbf{C}^j(I)$ (n- and l-variate, respectively) such that $(t, \phi(t), \phi'(t), \psi(t)) \in \mathcal{D}$ $\forall t \in I$,

$$F_j\left(t,\phi(t),\phi'(t),\phi''(t),\dots,\phi^{(j+1)}(t),\psi(t),\dots,\psi^{(j)}(t)\right) = \left(\frac{d}{dt}\right)^j F(t,\phi(t),\phi'(t),\psi(t)), \quad t \in I,$$

is called the r-extended system with respect to ADS (1).

Let's associate the following matrices with the function $F(t,x,y,u)$: the $n(r+1) \times nr$-matrix

$$\Gamma_{r,z} = \Gamma_{r,z}(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) = \left(\partial\mathcal{F}_r/\partial z_1 \quad \dots \quad \partial\mathcal{F}_r/\partial z_r\right),$$

the square $n(r+1)$-matrix $\Gamma_{r,y} = (\partial\mathcal{F}_r/\partial y \ \ \Gamma_{r,z})$, and the $n(r+1) \times n(r+2)$-matrix $\Gamma_{r,x} = \left(\partial\mathcal{F}_r/\partial x \quad \Gamma_{r,y}\right)$.

According to (2), the point $t = a_0$, $x = y = z_j = 0$, $u = v_j = 0$ $(j = \overline{1,r})$ satisfies the extended system (3). Denote this point by $\alpha_r = (a_0, 0, \dots, 0)$. Then $\mathcal{F}_r(\alpha_r) = 0$. If $\operatorname{rank}\Gamma_{r,x}(\alpha_r) = n(r+1)$, then system (3) satisfies all conditions of the implicit function theorem [8, p. 66]. According to this theorem, $n(r+1)$ components of the vector colon $(x,y,z_1,\dots,z_r)$[2] can be expressed explicitly from (3) (let's denote them by ξ) in terms of $t, u, v_1, \dots, v_r$ and the rest n components of the vector (let's denote them by η):

$$\xi = \xi(t,\eta,u,v_1,\dots,v_r), \quad (t,\eta,u,v_1,\dots,v_r) \in \mathcal{W}, \tag{4}$$

[2] Here colon $(c_1, c_2, \dots, c_n) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

where $\mathcal{W} = I_0 \times \widetilde{\mathcal{W}}$; $I_0 = (a_0 - \varepsilon_0, a_0 + \varepsilon_0) \subseteq I$, $0 < \varepsilon_0 \leqslant \varepsilon$; $\widetilde{\mathcal{W}} \subset \mathbf{R}^{n+l(r+1)}$ is a neighborhood of $\eta = 0$, $u = v_1 = \ldots = v_r = 0$;

$$\operatorname{colon}(\xi, \eta) = P \operatorname{colon}(x, y, z_1, \ldots, z_r), \tag{5}$$

$\xi \in \mathbf{R}^{n(r+1)}$, $\eta \in \mathbf{R}^n$, P is a row permutation matrix.

Since the matrix $\Gamma_{r,x}$ has the dimensions $n(r+1) \times n(r+2)$, a nonsingular $n(r+1)$-minor of the matrix $\Gamma_{r,x}(\alpha_r)$ that determines functions (4) is not unique in the general case. Let's find it. Take $\varrho = \operatorname{rank}\Gamma_{r,y}(\alpha_r)$ ($\varrho \leqslant n(r+1)$) linearly independent columns of the matrix $\Gamma_{r,y}(\alpha_r)$ such that there is as many as possible of first n columns of the matrix. Augment this set of columns by $n(r+1) - \varrho$ linearly independent columns of the matrix $\partial\mathcal{F}_r/\partial x$, computed at α_r (note that the matrix $\partial\mathcal{F}_r/\partial x$ represents first n columns of the matrix $\Gamma_{r,x}(\alpha_r)$). The obtained $n(r+1)$ linearly independent columns constitute the desired minor, and $\varrho \geqslant nr$.

Definition 2. The above constructed $n(r+1)$-minor of the matrix $\Gamma_{r,x}(\alpha_r)$ is called the solving minor.

We assume that functions (4) correspond to the solving minor.

Denote the matrix resulting from the substitution of (4) to $\Gamma_{r,z}(t, x, y, z_1, \ldots, z_r, u, v_1, \ldots, v_r)$ by $\overline{\Gamma}_{r,z}(t, \eta, u, v_1, \ldots, v_r)$.

3. EQUIVALENT FORMS

3.1. Nonlinear ADS

System (3) is equivalent, under some assumptions, to the special system in a neighborhood of the point $\alpha_r = (a_0, 0, \ldots, 0) \in \mathbf{R}^{n(r+2)+l(r+1)+1}$.

Lemma 1. *Let*

(1) $F(t, x, y, u) \in \mathbf{C}^{r+1}(\mathcal{D})$;

(2) $\mathcal{F}_r(\alpha_r) = 0$, $\operatorname{rank}\Gamma_{r,x}(\alpha_r) = n(r+1)$;

(3) $\operatorname{rank}\overline{\Gamma}_{r,z}(t, \eta, u, v_1, \ldots, v_r) = \rho = \text{const}$ *over the domain* $\mathcal{W}$;

(4) *the solving minor of the matrix* $\Gamma_{r,x}(\alpha_r)$ *comprise* ρ *columns of the matrix* $\Gamma_{r,z}(\alpha_r)$ *and* n *first columns of the matrix* $\Gamma_{r,y}(\alpha_r)$.

Then system (3) *is equivalent to the system*

$$y - f(t, x_1, u, v_1, \ldots, v_r) = 0, \tag{6}$$

$$x_2 - f_0(t, x_1, u, v_1, \ldots, v_r) = 0, \tag{7}$$

$$Z_1 - \varphi(t, x_1, Z_2, u, v_1, \ldots, v_r) = 0 \tag{8}$$

in some neighborhood $\mathcal{A}$ *of* α_r. *Functions* f, f_0, φ *are defined on* $\mathcal{W}$ *and have continuous partial derivatives with respect to each variable;*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q_1 x, \quad \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = Q_2 \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}; \tag{9}$$

$x_1 \in \mathbf{R}^{n-d}$, $x_2 \in \mathbf{R}^d$, $d = nr - \rho$; $Z_1 \in \mathbf{R}^\rho$, $Z_2 \in \mathbf{R}^d$; *Q_1 and Q_2 are row permutation matrices.*

The proof of the Lemma is given in the Appendix.

Remark 1. The minimal value of r $(0 \leqslant r \leqslant n)$ such that the conditions of Lemma 1 are satisfied is called the *index* of unsolvability with respect to the derivative of (1). In the general case the solution of ADS (1) depends on r derivatives of the control $u(t)$ [7].

Definition 3. Any vector-function $u(t)\colon I \to \mathbf{R}^l$ is called the admissible control for ADS (1) if $u(t) \in \mathbf{C}^r(I)$ and $\forall t \in I$ $\operatorname{colon}(u(t), u'(t), \ldots, u^{(r)}(t)) \in \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of the origin in the space $\mathbf{R}^{l(r+1)}$.

Definition 4. Let $u_*(t)$ be an admissible control for ADS (1). An n-variate vector-function $x_*(t) \in \mathbf{C}^1(I)$ is called the solution of the system $F(t, x(t), x'(t), u_*(t)) = 0$, $t \in I$ if the system turns into identity over I after substitution of $x_*(t)$.

Consider Eq. (6). Multiply it from the left by the permutation matrix Q_1 from (9). Introducing the notation $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q_1 y$, $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = Q_1 f(t, x_1, u, v_1, \ldots, v_r)$, we can represent the system as

$$\begin{aligned} y_1 - f_1(t, x_1, u, v_1, \ldots, v_r) &= 0, \\ y_2 - f_2(t, x_1, u, v_1, \ldots, v_r) &= 0. \end{aligned} \tag{10}$$

The vectors y_1 and y_2 have the same dimensions as x_1 and x_2 from (9).

Putting $y_1 = x_1'(t)$, $\operatorname{colon}(x_1,\ x_2) = \operatorname{colon}(x_1(t), x_2(t)) = Q_1 x(t)$, $u = u(t)$, $v_j = u^{(j)}(t)$ $(j = \overline{1, r})$, associate the following ADS with system (10), (7):

$$x_1'(t) = f_1(t, x_1(t), u(t), u'(t), \ldots, u^{(r)}(t)), \tag{11}$$

$$x_2(t) = f_0(t, x_1(t), u(t), u'(t), \ldots, u^{(r)}(t)), \quad t \in I_0. \tag{12}$$

Fix a point $t_0 \in I_0$ and set initial values for ADS (11), (12):

$$x_1(t_0) = x_{1,0}, \quad x_2(t_0) = x_{2,0}. \tag{13}$$

Let's determine the values of the vectors $x_{1,0} \in \mathbf{R}^{n-d}$, $x_{2,0} \in \mathbf{R}^d$ and the boundaries of the control $u(t)$ such that problem (11)–(13) has a solution $x_*(t) = Q_1^{-1} \operatorname{colon}(x_{*,1}(t),\ x_{*,2}(t))$ over an interval $I_\tau = (t_0 - \tau, t_0 + \tau) \subseteq I_0$, $\tau > 0$.

Suppose that there exist vectors $u_0, v_{1,0}, \ldots, v_{r,0} \in \mathbf{R}^l$ such that

$$x_{2,0} = f_0(t_0, x_{1,0}, u_0, v_{1,0}, \ldots, v_{r,0}). \tag{14}$$

Then the control $u_*(t)$ that satisfies the conditions

$$u(t_0) = u_0, \quad u^{(j)}(t_0) = v_{j,0}, \quad j = \overline{1, r}, \tag{15}$$

can be found in the form $u_*(t) = \sum\limits_{i=0}^{r} b_i (t - t_0)^i$. It is easy to prove that the coefficients $b_i \in \mathbf{R}^l$ are unique: $b_0 = u_0$, $b_j = \dfrac{1}{j!} v_{j,0}$. This control is obviously admissible if u_0, $v_{j,0}$ are sufficiently small.

If problem (11)–(13) has a solution $x_*(t)$ over I_0, which corresponds to the admissible control $u_*(t)$, then the inclusion must hold:

$$(x_*(t), x_*'(t), u_*(t)) \in \mathcal{V} \quad \forall t \in I_0, \tag{16}$$

where $\mathcal{V} \subset \mathbf{R}^{2n+l}$ is a neighborhood of the point $x = 0$, $y = 0$, $u = 0$. Therefore, the vectors $x_{1,0}$, $x_{2,0}$, u_0, $v_{j,0}$ $(j = \overline{1,r})$ in (13)–(15) must be sufficiently close to zero.

Note that given the assumptions of Lemma 1, if $x_*(t)$ is the solution of ADS (1) over I_0 that corresponds to the admissible control $u_*(t) \in \mathbf{C}^r(I_0)$, and inclusion (16) holds, then the same solution turns Eqs. (11) and (12) into identity.

The theorem below represents the sufficient conditions for a solution of the Cauchy problem of ADS (1) with

$$x(t_0) = x_0 \tag{17}$$

to exist locally. The proof is based on the fact that a solution of (11), (12), (17) over $I_\tau \subseteq I$ is the solution of (1), (17) over I_τ.

Theorem 1. *Let*

(1) $F(t, x, y, u) \in \mathbf{C}^{r+2}(\mathcal{D})$*;*

(2) $\mathcal{F}_{r+1}(\alpha_{r+1}) = 0$, $\operatorname{rank}\Gamma_{r,x}(\alpha_r) = n(r+1)$*;*

(3) $\operatorname{rank}\overline{\Gamma}_{r,z}(t, \eta, u, v_1, \ldots, v_r) = \rho = \mathrm{const}$ *over the domain* $\mathcal{W}$*;*

(4) *the solving minor of the matrix* $\Gamma_{r,x}(\alpha_r)$ *comprise* ρ *columns of the matrix* $\Gamma_{r,z}(\alpha_r)$ *and* n *first columns of the matrix* $\Gamma_{r,y}(\alpha_r)$*;*

(5) $\operatorname{rank}\Gamma_{r+1,y}(\alpha_{r+1}) = \operatorname{rank}\Gamma_{r,y}(\alpha_r) + n$.

Then $\forall t_0 \in I_0$ *there exist* $\delta > 0$ *and* $\tau = \tau(t_0) > 0$ *such that for all vectors* $x_0 \in \mathbf{R}^n$, u_0, $v_{j,0} \in \mathbf{R}^l$ $(j = \overline{1,r})$ *that satisfy* (14) *and* $\|x_0\|$, $\|u_0\|$, $\|v_{j,0}\| < \delta$ *the solution* $x_*(t) \in \mathbf{C}^2(I_\tau)$ *of the problem* (11), (12), (17) *exists over the interval* $I_\tau = (t_0 - \tau, t_0 + \tau) \subseteq I_0$*. Here* $x_0 = Q_1^{-1}\mathrm{colon}\,(x_{1,0}, x_{2,0})$*. In addition,* $x_*(t)$ *is the solution of* (1), (17) *over* I_τ*.* $u(t) \in \mathbf{C}^r(I_\tau)$ *in* (11), (12), *and* (1) *is an admissible control that satisfies* (15).

The proof of the Theorem is given in the Appendix.

Definition 5. We call ADS (11), (12) the equivalent form of system (1) on the interval I_0.

3.2. First Approximation System

Let ADS (1) satisfy (2). Introduce the matrices

$$A(t) = \frac{\partial F(t,x,y,u)}{\partial y}(t,0,0,0), \quad B(t) = \frac{\partial F(t,x,y,u)}{\partial x}(t,0,0,0), \quad U(t) = \frac{\partial F(t,x,y,u)}{\partial u}(t,0,0,0).$$

Then the linear ADS

$$A(t)x'(t) + B(t)x(t) + U(t)u(t) = 0, \quad t \in I, \tag{18}$$

is the first approximation system for (1).

The matrices

$$D_{r,z}(t) = \begin{pmatrix} C_1^1 A(t) & O & \dots & O \\ C_2^1 A'(t) + C_2^2 B(t) & C_2^2 A(t) & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^1 A^{(r-1)}(t) + C_r^2 B^{(r-2)}(t) & C_r^2 A^{(r-2)}(t) + C_r^3 B^{(r-3)}(t) & \dots & C_r^r A(t) \end{pmatrix},$$

$$D_{r,y}(t) = \begin{pmatrix} C_0^0 A(t) & O \\ \begin{pmatrix} C_1^0 A'(t) + C_1^1 B(t) \\ \vdots \\ C_r^0 A^{(r)}(t) + C_r^1 B^{(r-1)}(t) \end{pmatrix} & D_{r,z}(t) \end{pmatrix},$$

$$D_{r,x}(t) = \begin{pmatrix} \begin{pmatrix} B(t) \\ B'(t) \\ \vdots \\ B^{(r)}(t) \end{pmatrix} & D_{r,y}(t) \end{pmatrix}$$

are analogous to the matrices $\Gamma_{r,z}$, $\Gamma_{r,y}$, and $\Gamma_{r,x}$, respectively, for the linear system (18) (here C_i^j are binomial coefficients).

The system of linear algebraic equations

$$D_{r,x}(t)\mathrm{colon}\,(x, y, z_1, \dots, z_r) + U_r(t)\mathrm{colon}\,(u, v_1, \dots, v_r) = 0, \tag{19}$$

where $x, y, z_j \in \mathbf{R}^n$; $u, v_j \in \mathbf{R}^l$, $j = \overline{1,r}$;

$$U_r(t) = \begin{pmatrix} C_1^1 U(t) & O & \dots & O \\ C_2^1 U'(t) & C_2^2 U(t) & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^1 U^{(r)}(t) & C_r^2 U^{(r-1)}(t) & \dots & C_r^r U(t) \end{pmatrix}, \tag{20}$$

is called the r-*extended system* with respect to (18).

Lemma 2. *Let*

(1) $A(t), B(t) \in \mathbf{C}^r(I)$*;*

(2) $\operatorname{rank} D_{r,z}(t) = \rho = \mathrm{const}\ \forall t \in I$*;*

(3) *there be a nonsingular* $n(r+1)$*-minor of the matrix* $D_{r,x}(t)$ *comprising* ρ *columns of the matrix* $D_{r,z}(t)$ *and* n *first columns of the matrix* $D_{r,y}(t)$*.*

Then the following operator is defined on I*:*

$$R = R_0(t) + R_1(t)\frac{d}{dt} + \dots + R_r(t)\left(\frac{d}{dt}\right)^r. \tag{21}$$

Coefficients $R_j(t)$ $(j = \overline{0,r})$ *are continuous and satisfy the following: for all vector-functions* $\phi(t) \in \mathbf{C}^{r+1}(I)$ *and* $\psi(t) \in \mathbf{C}^r(I)$ *(*n*- and* l*-variate, respectively)*

$$R[A(t)\phi'(t) + B(t)\phi(t) + U(t)\psi(t)] = \begin{pmatrix} \phi_1'(t) - J_1(t)\phi_1(t) - \sum\limits_{j=0}^{r} L_j(t)\psi^{(j)}(t) \\ \phi_2(t) - J_2(t)\phi_1(t) - \sum\limits_{j=0}^{r} G_j(t)\psi^{(j)}(t) \end{pmatrix},$$

where $\phi_1(t)$, $\phi_2(t)$ *are* $n-d$*-variate and* $d = nr - \rho$*-variate vector-functions; matrices* $L_j(t)$ *and* $G_j(t)$ *have the dimensions* $(n-d)\times l$ *and* $d \times l$*;* $\operatorname{colon}(\phi_1(t), \phi_2(t)) = Q_1\phi(t)$, Q_1 *is a row permutation matrix, matrices* $J_1(t)$ *and* $J_2(t)$ *have the dimensions* $(n-d)\times(n-d)$ *and* $d\times(n-d)$, *respectively;*

$$\begin{pmatrix} L_0(t) & L_1(t) & \ldots & L_r(t) \\ G_0(t) & G_1(t) & \ldots & G_r(t) \end{pmatrix} = -\left(R_0(t) \;\; R_1(t) \;\; \ldots \;\; R_r(t)\right) U_r(t). \tag{22}$$

Suppose that conditions (1)–(3) of Lemma 2 hold, and the columns of the matrix $D_{r,x}(t)$ that take part of the minor (3) are known. Delete the columns of the matrix $\operatorname{colon}\left(B(t), B'(t), \ldots, B^{(r)}(t)\right)$ that don't take part in the minor. Those columns are multiplied in (19) by the components of the vector x_1, while the others are multiplied by the components of x_2 ($\operatorname{colon}(x_1, x_2) = Q_1 x$). Permute the columns of $D_{r,y}(t)$ such that

$$D_{r,y}(t)\operatorname{colon}\left(x'(t), \ldots, x^{(r+1)}(t)\right) = \widetilde{C}(t)\operatorname{colon}\left(x_1'(t), x_2'(t), \ldots, x_1^{(r+1)}(t), x_2^{(r+1)}(t)\right)$$

($\widetilde{C}(t)$ is the result of the permutation). So we obtain the matrix $\Theta_r(t)$ from the matrix $D_{r,x}(t)$. $\Theta_r(t)$ can be defined by the formula

$$\Theta_r(t) = D_{r,x}(t)\operatorname{diag}\left(Q_1^{-1}\begin{pmatrix} O \\ E_d \end{pmatrix}, \; Q_1^{-1}, \; \ldots, \; Q_1^{-1}\right).[3]$$

By construction, the matrix $\Theta_r(t)$ has full row rank on I.

Then the coefficients $R_j(t)$ of operator (21) associated with the minor of the matrix $D_{r,x}(t)$ are uniquely determined by

$$(R_0(t)R_1(t)\ldots R_r(t)) = (E_n O \ldots O)\,\Theta_r^{\top}(t)\left(\Theta_r(t)\Theta_r^{\top}(t)\right)^{-1}.[4] \tag{23}$$

Consider the system

$$\begin{aligned} x_1'(t) &= J_1(t)x_1(t) + \sum_{j=0}^{r} L_j(t)u^{(j)}(t), \\ x_2(t) &= J_2(t)x_1(t) + \sum_{j=0}^{r} G_j(t)u^{(j)}(t), \quad t \in I, \end{aligned} \tag{24}$$

where matrices L_j, G_j are determined according to (22), (23), (20),

$$\begin{pmatrix} J_1(t) \\ J_2(t) \end{pmatrix} = \left(R_0(t) \;\; R_1(t) \;\; \ldots \;\; R_r(t)\right)\operatorname{colon}\left(B(t), B'(t), \ldots, B^{(r)}(t)\right) Q_1^{-1}\begin{pmatrix} E_{n-d} \\ O \end{pmatrix}. \tag{25}$$

Theorem 2. *Let*

(1) $A(t), B(t), U(t), u(t) \in \mathbf{C}^{2r+1}(I)$;

(2) *conditions* (2) *and* (3) *of Lemma 2 be satisfied*;

(3) $\operatorname{rank} D_{r+1,y}(t) = \operatorname{rank} D_{r,y}(t) + n \;\; \forall t \in I$.

Than any solution of (18) *is the solution of* (24) *and vice versa.*

[3] The notation $\operatorname{diag}(A_1, \ldots, A_s)$ means a quasidiagonal matrix that has blocks in parentheses on its main diagonal, other entries being zeros.

[4] Here $^{\top}$ means matrix transpose.

The proofs of Lemma 2 and Theorem 2, as well as the justification of (23) are given in [9].

Definition 6. System (24) is called the equivalent form of ADS (18).

Let the conditions of Theorem 1 be satisfied. Property (2) ensures that

$$f_1(t,0,\dots,0)=0, \quad f_0(t,0,\dots,0)=0 \quad \forall t\in I_0$$

for system (11), (12). Construct the first approximation system for (11), (12).

$$\begin{aligned} x_1'(t) &= \widetilde{J}_1(t)x_1(t)+\sum_{j=0}^{r}\widetilde{L}_j(t)u^{(j)}(t),\\ x_2(t) &= \widetilde{J}_2(t)x_1(t)+\sum_{j=0}^{r}\widetilde{G}_j(t)u^{(j)}(t), \quad t\in I_0, \end{aligned} \tag{26}$$

where $\widetilde{J}_1(t)=\frac{\partial f_1}{\partial x_1}(t,0,\dots,0)$, $\widetilde{J}_2(t)=\frac{\partial f_0}{\partial x_1}(t,0,\dots,0)$, $\widetilde{L}_0(t)=\frac{\partial f_1}{\partial u}(t,0,\dots,0)$, $\widetilde{L}_j(t)=\frac{\partial f_1}{\partial v_j}(t,0,\dots,0)$, $\widetilde{G}_0(t)=\frac{\partial f_0}{\partial u}(t,0,\dots,0)$, $\widetilde{G}_j(t)=\frac{\partial f_0}{\partial v_j}(t,0,\dots,0)$, $j=\overline{1,r}$.

Under some assumptions, systems (24) and (26) turn out to be the same on I_0, providing that ADS (24) is taken as the equivalent form of (18), which, in turn, is the linear approximation of ADS (1). In other words, the operation of taking linear approximation and that of taking the equivalent form are commutative.

Theorem 3. *Let $F(t,x,y,u)\in \mathbf{C}^{2r+2}(\mathcal{D})$, $u(t)\in\mathbf{C}^{2r+1}(I)$, $\operatorname{rank}\Gamma_{r,x}(\alpha_r)=n(r+1)$; conditions* (3)–(5) *of Theorem 1 be satisfied, and* (2) *take place. Then systems* (24) *and* (26) *coinside on a sufficiently small interval $I_0=(a_0-\varepsilon_0,a_0+\varepsilon_0)\subseteq I$, $0<\varepsilon_0\leqslant\varepsilon$.*

The proof is given in the Appendix.

4. CONTROLLABILITY CRITERIA

Definition 7. System (1) is called *locally null-controllable* on the segment $T=[t_0,t_1]\subset I$ if there exists a δ-neighborhood of the point $0\in\mathbf{R}^n$ such that for any vector $x_0\in\mathbf{R}^n$: $\|x_0\|<\delta$ there is an admissible control $u_0(t)$ that takes the solution of the system

$$F(t,x(t),x'(t),u_0(t))=0, \quad t\in T,$$

from $x(t_0)=x_0$ to $x(t_1)=0$.

Definition 8. System (18) is called completely controllable on the segment $T=[t_0,t_1]\subset I$ if for any vectors $x_0,x_1\in\mathbf{R}^n$ there is a control $u(t)\in\mathbf{C}^r(T)$ that takes the solution of (18) from $x(t_0)=x_0$ to $x(t_1)=x_1$.

4.1. Linear Systems

In this subsection we get complete controllability and null-controllability criteria for linear ADS (18).

At first, let's derive controllability conditions for the system

$$x_1'(t)=J_1(t)x_1(t)+\sum_{j=0}^{r}L_j(t)u^{(j)}(t), \tag{27}$$

$$x_2(t)=J_2(t)x_1(t)+\sum_{j=0}^{r}G_j(t)u^{(j)}(t), \quad t\in T=[t_0,t_1], \tag{28}$$

where $J_1(t)$ is a $(n-d)\times(n-d)$-matrix, $J_2(t)$ is a $d\times(n-d)$-matrix; matrices $L_j(t)$ and $G_j(t)$ have the dimensions $(n-d)\times l$ and $d\times l$, respectively; $x_1(t), x_2(t)$ are desired $n-d$- and d-variate vector-functions; $u(t)$ is l-variate control function.

Denote by $\Omega(t)$ the matrizant of (27). Then

$$\Omega'(t) = J_1(t)\Omega(t), \quad t\in T; \quad \Omega(t_0) = E_{n-d}. \tag{29}$$

Lemma 3. *Let for the system* (27), (28) $J_1(t)$, $J_2(t)$, $L_j(t)$, $G_j(t)\in \mathbf{C}(T)$, $j=\overline{0,r}$; $u(t)\in\mathbf{C}^r(T)$. *Then* (27), (28) *is completely controlable on the segment* T *iff:*

(1) rank $(G_0(t_0)G_1(t_0)\dots G_r(t_0)) = \text{rank }(G_0(t_1)G_1(t_1)\dots G_r(t_1)) = d$;

(2) $\forall h\in\mathbf{R}^{n-d} : h\neq 0,\ h^\top\Omega^{-1}(t)\,(L_0(t)L_1(t)\dots L_r(t))\not\equiv 0$ *on* T.

The proof is given in the Appendix.

Lemma 4. *Let* $J_2(t)$, $G_j(t)\in\mathbf{C}(T)$; $J_1(t)$, $L_j(t)\in\mathbf{C}^{n-d}(T)$, $j=\overline{0,r}$; $u(t)\in\mathbf{C}^r(T)$. *System* (27), (28) *is completely controllable on the segment* T *if the following conditions are satisfied:*

(1) rank $(G_0(t_0)\ G_1(t_0)\ \dots\ G_r(t_0)) = \text{rank }(G_0(t_1)\ G_1(t_1)\ \dots\ G_r(t_1)) = d$;

(2) $\exists\sigma\in T:\ \text{rank}\, Q(\sigma) = n-d$, *where*

$$\begin{gathered} Q(t) = (Q_0(t)Q_1(t)\dots Q_{n-d-1}(t))\,,\\ Q_0(t) = (L_0(t)L_1(t)\dots L_r(t))\,,\\ Q_i(t) = J_1(t)Q_{i-1}(t) - Q'_{i-1}(t), \quad i=\overline{1,n-d-1}. \end{gathered} \tag{30}$$

This lemma follows from Lemma 3 and the result presented in [10, p. 169].

Lemma 5. *Let* $J_1(t)$, $J_2(t)$, $L_j(t)$, $G_j(t)\in\mathbf{C}(T)$, $j=\overline{0,r}$; $u(t)\in\mathbf{C}^r(T)$. *If system* (27), (28) *is completely controllable on the segment* $T=[t_0,t_1]$ *then it is null-controllable on the same segment.*

The proof is given in the Appendix.

Suppose that all assumptions of Theorem 2 are satisfied on the segment $T=[t_0,t_1]$, and system (27), (28) is the equivalent form of the ADS

$$A(t)x'(t) + B(t)x(t) + U(t)u(t) = 0, \quad t\in T. \tag{31}$$

Then the matrices L_j, G_j $(j=\overline{0,r})$ in (27), (28) are determined by formulae (22), (23), (20), while the matrices $J_1(t)$, $J_2(t)$ are determined by (25).

It follows directly from Theorem 2 that if ADS (31) is completely controllable or locally null-controllable, then system (27), (28) has the same property. And conversely, a controllability property of (27), (28) implies the appropriate controllability property of (31). Therefore, Lemmas 3–5 yield the following theorem.

Theorem 4. *Let*

(1) $A(t), B(t), U(t), u(t)\in\mathbf{C}^{2r+1}(T)$;

(2) rank $D_{r,z}(t) = \rho = \text{const}\ \ \forall t\in T$;

(3) *there be a nonsingular* $n(r+1)$*-minor of the matrix* $D_{r,x}(t)$ $\forall t\in T$ *that comprises* ρ *columns of the matrix* $D_{r,z}(t)$ *and* n *first columns of the matrix* $D_{r,y}(t)$;

4) rank $D_{r+1,y}(t) = \text{rank}\, D_{r,y}(t) + n\ \forall t\in T$.

Then

(I) *ADS* (31) *is completely controllable on the segment* T *iff the following conditions are satisfied*:

(a) $\operatorname{rank}\,(G_0(t_0)G_1(t_0)\dots G_r(t_0)) = \operatorname{rank}\,(G_0(t_1)G_1(t_1)\dots G_r(t_1)) = d$;

(b) $\forall h \in \mathbf{R}^{n-d} : h \neq 0,\ h^\top \Omega^{-1}(t)\,(L_0(t)L_1(t)\dots L_r(t)) \not\equiv 0$ *on* T.

(II) *System* (31) *is completely controllable on the segment* T *if condition* (a) *is satisfied, and* $\exists \sigma \in T :\ \operatorname{rank} Q(\sigma) = n - d$, *where the matrix* $Q(t)$ *is determined according to* (30).

(III) *If ADS* (31) *is completely controllable on the segment* T, *it is null-controllable on the same segment.*

4.2. Nonlinear Systems

Before passing to the proof of local null-controllability criterium for a nonlinear ADS (1), consider the system of special form

$$\begin{aligned} x_1'(t) &= f_1(t, x_1(t), u(t), u'(t), \dots, u^{(r)}(t)), \\ x_2(t) &= f_0(t, x_1(t), u(t), u'(t), \dots, u^{(r)}(t)), \quad t \in T = [t_0, t_1]. \end{aligned} \tag{32}$$

Functions f_1 and f_0 are supposed to be defined and have continuous partial derivatives with respect to all their variables on $T \times \mathcal{V}$, where $\mathcal{V}$ is the domain of variation for the variables $x_1, u, u', \dots, u^{(r)}$, $\mathcal{V}$ being a neighborhood of $0 \in \mathbf{R}^{n+l(r+1)}$. In addition, $f_0(t, 0, 0, \dots, 0) = 0$, $f_1(t, 0, 0, \dots, 0) = 0\ \forall t \in T$.

Using the notation

$$\begin{aligned} J_1(t) &= \frac{\partial f_1}{\partial x_1}(t, 0, 0, \dots, 0), \quad J_2(t) = \frac{\partial f_0}{\partial x_1}(t, 0, 0, \dots, 0), \\ L_i(t) &= \frac{\partial f_1}{\partial u^{(i)}}(t, 0, 0, \dots, 0), \quad G_i(t) = \frac{\partial f_0}{\partial u^{(i)}}(t, 0, 0, \dots, 0), \quad i = \overline{0, r}, \end{aligned} \tag{33}$$

we get that ADS (27), (28) is the the first approximation system for (32).

Lemma 6. *Let*

(1) *functions* $f_1(t, x_1, u, u', \dots, u^{(r)})$ *and* $f_0(t, x_1, u, u', \dots, u^{(r)})$ *be defined and have continuous partial derivatives with respect to all their variables in the domain* $T \times \mathcal{V}$;

(2) $f(t, 0, 0, \dots, 0) = 0,\ f_0(t, 0, 0, \dots, 0) = 0\ \forall t \in T$;

(3) *first approximation system* (27), (28) *for ADS* (32) *be completely controllable on the segment* T.

Then (32) *is locally null-controllable on the segment* T.

Lemma 7. *Let conditions* (1) *and* (2) *of Lemma 6 be satisfied, and system* (27), (28) *be locally null-controllable on the segment* T. *Then system* (32) *is also locally null-controllable on* T.

Proofs of the Lemmas are given in the Appendix.

Return to ADS (1). Let all conditions of Theorem 3 be satisfied. Then Theorem 1 holds. This means that ADS (1) is equivalent to (11), (12) on I_0. ADS (26) is the first approximation for (11), (12). Applying Lemmas 6 and 7, we get that if (26) is completely controllable or locally null-controllable on some segment $T = [t_0, t_1] \subset I_1 = (a_0 - \epsilon, a_0 + \epsilon) \subseteq I_0$ ($\epsilon > 0$ is sufficiently small) then system (11), (12) is locally null-controllable on this segment. It follows directly from Theorem 1 that the local null-controllability of (11), (12) on T, system (1) has the same property. Thus, local null-controllability of (1) follows from complete controllability or local null-controllability of (26).

On the other hand, system (24) is the first approximation system for (1). According to Theorem 3, ADS (26) and (24) coinside on $I_0 = (a_0 - \varepsilon_0, a_0 + \varepsilon_0) \subseteq I$. Hence, those systems have controllability properties simultaneously. So, complete controllability or local null-controllability of (24) on the segment T implies local null-controllability of ADS (1). Thus we have proven the following theorem.

Theorem 5. *Let*

(1) $F(t, x, y, u) \in \mathbf{C}^{2r+2}(\mathcal{D})$, $u(t) \in \mathbf{C}^{2r+1}(I)$;

(2) $F(t, 0, 0, 0) = 0 \ \forall t \in I$;

(3) $\operatorname{rank} \Gamma_{r,x}(\alpha_r) = n(r+1)$;

(4) $\operatorname{rank} \overline{\Gamma}_{r,z}(t, \eta, u, v_1, \ldots, v_r) = \rho = \text{const}$ *on the domain* $\mathcal{W}$;

(5) *the solving minor of the matrix* $\Gamma_{r,x}(\alpha_r)$ *comprise* ρ *columns of the matrix* $\Gamma_{r,z}(\alpha_r)$ *and* n *first columns of the matrix* $\Gamma_{r,y}(\alpha_r)$;

(6) $\operatorname{rank} \Gamma_{r+1,y}(\alpha_{r+1}) = \operatorname{rank} \Gamma_{r,y}(\alpha_r) + n$.

If the first approximation system (24) *is completely controllable or locally null-controllable on the segment* $T = [t_0, t_1] \subset I_1 = (a_0 - \epsilon, a_0 + \epsilon)$ ($\epsilon > 0$ *is sufficiently small*) *then ADS* (1) *is locally null-controllable on* T.

5. CONCLUSION

Criteria of complete controllability are obtained for linear ADS with smooth coefficients. The obtained criteria turn into known criteria for systems solved for the derivative of required vector-function [10, p. 159, 357].

The conversion to the equivalent form was used to analyse controllability of both linear and nonlinear systems. In particular, one of complete controllability criteria is stated in terms of equivalent form coefficients. In the nonlinear case the construction of equivalent form is a complex problem, which involves the finding an implicit function satisfying the r-extended system. For a linear ADS this problem, however, is solvable, its complexity being comparable to that of matrix innversion.

It is shown that the operations of linearization and conversion to the equivalent form are commutative for nonlinear systems under certain assumptions. This allows to avoid the construction of equivalent form for nonlinear ADS during controllability analysis. It is proven that the local null-controllability of a nonlinear ADS follows from complete controllability or local null-controllability of its linear approximation.

It should be noted that both linear and nonlinear ADS are studied under general assumptions. In particular, the index of unsolvability is supposed to be arbitrarily high. In the linear case it is not required that the rank of the matrix at the required vector function be constant. For nonlinear ADS the matrix $\Gamma_{r,y}$ can be of variable rank. The requirement of constant rank for the matrix $\overline{\Gamma}_{r,z}$ is close to the condition of regular solution behavior.

APPENDIX

Proof of Lemma 1. Since the solving minor of the implicit function (4), (5) has the structure determined by condition (4),

$$\xi = \text{colon}\,(y, x_2, Z_1), \quad \eta = \text{colon}\,(x_1, Z_2),$$

where the variables x_1, x_2, Z_1, Z_2 are related with $x, z_1, \ldots, z_r$ according to (9). So, function (4) can be presented in the form

$$\begin{gathered} y = f(t, x_1, Z_2, u, v_1, \ldots, v_r), \quad x_2 = f_0(t, x_1, Z_2, u, v_1, \ldots, v_r), \\ Z_1 = \varphi(t, x_1, Z_2, u, v_1, \ldots, v_r). \end{gathered} \tag{A.1}$$

According to condition (1), by the implicit function theorem, the functions f, f_0, φ have continuous partial derivatives with respect to all their variables on their own domains.

Substitute (A.1) to $\Gamma_{r,z}(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r)$ to get the matrix $\overline{\Gamma}_{r,z}(t,\eta,u,v_1,\dots,v_r) = \widehat{\Gamma}_{r,z}(t,x_1,Z_2,u,v_1,\dots,v_r)$.

Find the Jacobi matrix of system (A.1) with respect to the variables Z_1, Z_2:

$$J(t,x_1,Z_2,u,v_1,\dots,v_r) = \begin{pmatrix} O & -\partial f(t,x_1,Z_2,u,v_1,\dots,v_r)/\partial Z_2 \\ O & -\partial f_0(t,x_1,Z_2,u,v_1,\dots,v_r)/\partial Z_2 \\ E_\rho & -\partial \varphi(t,x_1,Z_2,u,v_1,\dots,v_r)/\partial Z_2 \end{pmatrix}. \tag{A.2}$$

By the implicit function derivative theorem [8, p. 73],

$$\begin{pmatrix} \partial y/\partial Z_2 \\ \partial x_2/\partial Z_2 \\ \partial Z_1/\partial Z_2 \end{pmatrix} = -\left(\partial\mathcal{F}_r/\partial y \ \partial\mathcal{F}_r/\partial x_2 \ \partial\mathcal{F}_r/\partial Z_1\right)^{-1} \left(\partial\mathcal{F}_r/\partial Z_2\right). \tag{A.3}$$

Multiply the matrix $\Gamma_{r,z}(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) = (\partial\mathcal{F}_r/\partial Z_1 \ \ \partial\mathcal{F}_r/\partial Z_2)\,Q_2$ from the left by the matrix $(\partial\mathcal{F}_r/\partial y \ \ \partial\mathcal{F}_r/\partial x_2 \ \ \partial\mathcal{F}_r/\partial Z_1)^{-1}$ (Q_2 is the permutation matrix from (9)). Combining with (A.3), we have

$$(\partial\mathcal{F}_r/\partial y \ \ \partial\mathcal{F}_r/\partial x_2 \ \ \partial\mathcal{F}_r/\partial Z_1)^{-1}\,\Gamma_{r,z} = \begin{pmatrix} O & -\partial y/\partial Z_2 \\ O & -\partial x_2/\partial Z_2 \\ E_\rho & -\partial Z_1/\partial Z_2 \end{pmatrix} Q_2.$$

Substitute functions (A.1) to the obtained formula and multiply it from the left by the matrix Q_2^{-1}. This gives the expression for the matrix J:

$$J(t,x_1,Z_2,u,v_1,\dots,v_r) = H(t,x_1,Z_2,u,v_1,\dots,v_r)\widehat{\Gamma}_{r,z}(t,x_1,Z_2,u,v_1,\dots,v_r)Q_2^{-1}, \tag{A.4}$$

where $H(t,x_1,Z_2,u,v_1,\dots,v_r)$ is the matrix $(\partial\mathcal{F}_r/\partial y \ \ \partial\mathcal{F}_r/\partial x_2 \ \ \partial\mathcal{F}_r/\partial Z_1)^{-1}$, after substitution of functions (A.1). Hence, the matrix H is invertible everywhere over its domain $\mathcal{W}$.

Due to (A.4), condition (3) of the Lemma implies

$$\operatorname{rank} J(t,x_1,Z_2,u,v_1,\dots,v_r) = \rho = \text{const} \quad \forall (t,x_1,Z_2,u,v_1,\dots,v_r) \in \mathcal{W}.$$

Then it follows from the representation (A.2) for the matrix J that

$$\frac{\partial f(t,x_1,Z_2,u,v_1,\dots,v_r)}{\partial Z_2} = O, \quad \frac{\partial f_0(t,x_1,Z_2,u,v_1,\dots,v_r)}{\partial Z_2} = O$$

$\forall (t,x_1,Z_2,u,v_1,\dots,v_r) \in \mathcal{W}$. This means that functions f and f_0 in (A.1) do not depend on the variable Z_2. □

Proof of Theorem 1. Given the assumptions, it follows from Lemma 1 that there exists a continuous inversible operator Λ such that

$$\Lambda[\mathcal{F}_r(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r)] = \begin{pmatrix} y - f(t,x_1,u,v_1,\dots,v_r) \\ x_1 - f_0(t,x_1,u,v_1,\dots,v_r) \\ Z_1 - \varphi(t,x_1,Z_2,u,v_1,\dots,v_r) \end{pmatrix}. \tag{A.5}$$

Variables x_1, x_2, Z_1, Z_2 are related with $x, z_1,\dots,z_r$ by (9). It is clear that the operator Λ satisfies

$$\Lambda[0] = 0. \tag{A.6}$$

Let's describe the action of the operator Λ by means of a general structure nonlinear function

$$\Lambda\left[\mathcal{F}_r(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r)\right] = L(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,$$
$$F(t,x,y,u), F_1(t,x,y,z_1,u,v_1),\dots,F_r(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r)), \tag{A.7}$$

which is defined on some neighborhood $\mathcal{A}$ of the point $\alpha_r = (a_0, 0, \dots, 0)$.

Given the assumptions of the theorem, the function L has continuous partial derivatives with respect to its arguments $t, x, y, z_1, \dots, z_r, u, v_1, \dots, v_r, F, F_1, \dots, F_r$ up to the second order. Condition (A.6) is written in terms of the function L as

$$L\left(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,0,\dots,0\right) = 0 \qquad \forall (t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) \in \mathcal{A}.$$

Consider Eq. (6). Multiply it from the left by the row permutation matrix Q_1 from (9). Using the notation $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q_1 y$, $\begin{pmatrix} f_1(t,x_1,u,v_1,\dots,v_r) \\ f_2(t,x_1,u,v_1,\dots,v_r) \end{pmatrix} = Q_1 f(t,x_1,u,v_1,\dots,v_r)$ the obtained system can be represented as

$$\begin{aligned} y_1 &= f_1(t,x_1,u,v_1,\dots,v_r), \\ y_2 &= f_2(t,x_1,u,v_1,\dots,v_r). \end{aligned} \tag{A.8}$$

The vectors y_1 and y_2 are of the same dimensions as the vectors x_1 and x_2 from (9), respectively.

Consider an n-variate vector-function

$$R\left(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,F,F_1,\dots,F_r\right)$$
$$= \begin{pmatrix} O_{nr-\rho} & E_n & O_\rho \end{pmatrix} \begin{pmatrix} Q_1 & E_{nr} \end{pmatrix} L(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,F,F_1,\dots,F_r). \tag{A.9}$$

The subscript of zero matrices means the number of columns.

It follows from (A.5), (A.7), and (A.9) that $\forall (t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) \in \mathcal{A}$

$$R(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,F(t,x,y,u),F_1(t,x,y,z_1,u,v_1),\dots,$$
$$F_r(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r)) = \begin{pmatrix} y_1 - f_1(t,x_1,u,v_1,\dots,v_r) \\ x_2 - f_0(t,x_1,u,v_1,\dots,v_r) \end{pmatrix}. \tag{A.10}$$

As it is the case for the function L, the function R has continuous partial derivatives of the second order with respect to its arguments on its domain $\mathcal{A}$, and

$$R(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,0,\dots,0) = 0 \quad \forall (t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) \in \mathcal{A}. \tag{A.11}$$

It is clear, by definition of the partial derivative, that the partial derivatives of R with respect to the variables $t, x, y, z_1, \dots, z_r$, $u, v_1, \dots, v_r$ have the same property. In particular, the following holds at the points $(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r,0,\dots,0)$ at any $(t,x,y,z_1,\dots,z_r,u,v_1,\dots,v_r) \in \mathcal{A}$:

$$\frac{\partial}{\partial t}R = 0,\ \frac{\partial}{\partial x}R = 0,\ \frac{\partial}{\partial y}R = 0,\ \frac{\partial}{\partial u}R = 0,\ \frac{\partial}{\partial z_j}R = 0,\ \frac{\partial}{\partial v_j}R = 0,\ j = \overline{1,r}. \tag{A.12}$$

Introduce the function

$$R_1(t,x,y,z_1,\dots,z_{r+1},u,v_1,\dots,v_{r+1},F,F_1,\dots,F_{r+1})$$
$$= \frac{\partial R}{\partial t} + \begin{pmatrix} \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} & \dfrac{\partial R}{\partial z_1} & \dots & \dfrac{\partial R}{\partial z_r} \end{pmatrix} \operatorname{colon}\,(y,z_1,\dots,z_{r+1})$$
$$+ \begin{pmatrix} \dfrac{\partial R}{\partial u} & \dfrac{\partial R}{\partial v_1} & \dots & \dfrac{\partial R}{\partial v_r} \end{pmatrix} \operatorname{colon}\,(v_1,\dots,v_{r+1})$$
$$+ \begin{pmatrix} O_n & \dfrac{\partial R}{\partial F} & \dfrac{\partial R}{\partial F_1} & \dots & \dfrac{\partial R}{\partial F_r} \end{pmatrix} \mathcal{F}_{r+1}(t,x,y,z_1,\dots,z_{r+1},u,v_1,\dots,v_{r+1}). \tag{A.13}$$

The function R_1 is determined according to the following rule: for any vector-functions $\phi(t) \in \mathbf{C}^{r+2}(I_0)$, $\psi(t) \in \mathbf{C}^{r+1}(I_0)$, n-variate and l-variate, respectively, such that for all $t \in I_0$ the values $(t, \phi(t), \phi'(t), \dots, \phi^{(r+2)}(t), \psi(t), \psi'(t), \dots, \psi^{(r+1)}(t))$ are in some neighborhood $\overline{\mathcal{A}}$ of the point α_{r+1}, the following holds:

$$R_1\left(t, \phi(t), \phi'(t), \dots, \phi^{(r+2)}(t), \psi(t), \psi'(t), \dots, \psi^{(r+1)}(t),\right.$$
$$\left.F\left(t, \phi(t), \phi'(t), \psi(t)), \dots, F_{r+1}(t, \phi(t), \phi'(t), \dots, \phi^{(r+2)}(t), \psi(t), \psi'(t), \dots, \psi^{(r+1)}(t)\right)\right)$$
$$= \frac{d}{dt} R\left(t, \phi(t), \phi'(t), \dots, \phi^{(r+1)}(t), \psi(t), \psi'(t), \dots, \psi^{(r)}(t),\right.$$
$$\left.F(t, \phi(t), \phi'(t), \psi(t)), \dots, F_r\left(t, \phi(t), \phi'(t), \dots, \phi^{(r+1)}(t), \psi(t), \psi'(t), \dots, \psi^{(r)}(t)\right)\right).$$

Therefore, taking into account (A.10),

$$R_1(t, x, y, z_1, \dots, z_{r+1}, u, v_1, \dots, v_{r+1}, F, F_1, \dots, F_{r+1})$$
$$= \begin{pmatrix} z_{1,1} - \dfrac{\partial f_1}{\partial t} - \dfrac{\partial f_1}{\partial x_1} y_1 - \dfrac{\partial f_1}{\partial u} v_1 - \sum\limits_{i=1}^{r} \dfrac{\partial f_1}{\partial v_i} v_{i+1} \\ y_2 - \dfrac{\partial f_0}{\partial t} - \dfrac{\partial f_0}{\partial x_1} y_1 - \dfrac{\partial f_0}{\partial u} v_1 - \sum\limits_{i=1}^{r} \dfrac{\partial f_0}{\partial v_i} v_{i+1} \end{pmatrix}, \tag{A.14}$$

where colon $(z_{1,1},\ z_{1,2}) = Q_1 z_1$.

Consider the $(r+1)$-extended system

$$\mathcal{F}_{r+1}(t, x, y, z_1, \dots, z_{r+1}, u, v_1, \dots, v_{r+1}) = 0. \tag{A.15}$$

Implicit function (6)–(8) turns into identity its first $n(r+1)$ equations. Let's analyse the other n equations: $F_{r+1}(t, x, y, z_1, \dots, z_{r+1}, u, v_1, \dots, v_{r+1}) = 0$. Substituting function (6)–(8), we get

$$\overline{F}_{r+1}(t, x_1, Z_2, z_{r+1}, u, v_1, \dots, v_{r+1}) = 0. \tag{A.16}$$

Assumption (5) of the Theorem 1 ensures that the row rank of the matrix

$$\begin{pmatrix} \dfrac{\partial \overline{F}_{r+1}}{\partial Z_2} & \dfrac{\partial \overline{F}_{r+1}}{\partial z_{r+1}} \end{pmatrix} (t_0, 0, \dots, 0)$$

is full. By assumption (2) of the theorem, the point $(t_0, 0, \dots, 0)$ satisfies system (A.16). So, all conditions of the implicit function theorem are satisfied, and n components of the vector colon (Z_2, z_{r+1}) (denote them by $Z_{2,1}$)) can be expressed explicitly from (A.16) in terms of the variables $t, x_1, u, v_1, \dots, v_{r+1}$ and other components of the vector (denote them by $Z_{2,2}$)):

$$Z_{2,1} = f_3(t, x_1, Z_{2,2}, u, v_1, \dots, v_{r+1}). \tag{A.17}$$

We have obtained the implicit function (6)–(8), (A.17) that turns (A.15) into identity.

Since the function R satisfies (A.11), (A.12), it follows from the representation (A.13) that function (6)–(8), (A.17) makes the left-hand sides of (A.10) and (A.14) identical zero over a neighborhood $\overline{\mathcal{A}}$ of the point α_{r+1}. Thus, function (6)–(8), (A.17) satisfies the system

$$y_1 = f_1(t, x_1, u, v_1, \dots, v_r), \quad x_2 = f_0(t, x_1, u, v_1, \dots, v_r), \tag{A.18}$$

$$z_{1,1} = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} y_1 + \frac{\partial f_1}{\partial u} v_1 + \sum_{i=1}^{r} \frac{\partial f_1}{\partial v_i} v_{i+1}, \quad y_2 = \frac{\partial f_0}{\partial t} + \frac{\partial f_0}{\partial x_1} y_1 + \frac{\partial f_0}{\partial u} v_1 + \sum_{i=1}^{r} \frac{\partial f_0}{\partial v_i} v_{i+1}. \tag{A.19}$$

Substituting the expressions for $Z_{2,1}$ and y_1 from (A.17), (A.18), we obtain

$$y_2 = \frac{\partial f_0}{\partial t} + \frac{\partial f_0}{\partial x_1} f_1(t, x_1, u, v_1, \ldots, v_r) + \frac{\partial f_0}{\partial u} v_1 + \sum_{i=1}^{r} \frac{\partial f_0}{\partial v_i} v_{i+1}. \tag{A.20}$$

On the other hand, function (6)–(8) satisfies (A.18). This means that Eqs. (A.18) and (A.20) are the same.

Put

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = Q_1 x'(t), \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = Q_1 x(t), \tag{A.21}$$

and associate the system of differential-algebraic Eqs. (11) and (12) with the system of finite Eqs. (A.18).

Let the vector of initial values x_0 of (17) be such that there are values $u_0, v_{1,0}, \ldots, v_{r,0} \in \mathbf{R}^l$ such that (14) holds at some sufficiently small $\delta > 0$ $\|x_0\|$, $\|u_0\|$, $\|v_{1,0}\|, \ldots, \|v_{r,0}\| < \delta$. In addition, suppose that $u(t) \in \mathbf{C}^r(I_0)$ is an admissible control in (11), (12) that satisfies (15). Then the solution $x_*(t)$ of the problem (11), (12), (17) is uniquely defined on some interval I_τ, and $x_*(t) \in \mathbf{C}^2(I_\tau)$.

It is evident that the substitution of $x_*(t)$ to (A.18), (A.19) turns them into identity over the interval I_τ. Hence, $x_*(t)$ satisfies (A.20) and, therefore, (A.8). The variables of (A.18), (A.19), (A.20), and (A.8) should be interpreted in the sense of (A.21). Since the implicit function (A.18), (A.8) satisfies the equation $F(t, x, y, u) = 0$, this means that the function $x_*(t)$ turns (1) into identity over I_τ. $\square$

Proof of Theorem 3. Consider ADS (1) and its first approximation (18). Construct the r-extended system (3) for (1) and write the matrix $\Gamma_{r,x}$. Substitute the values $x = 0$, $y = 0$, $z_j = 0$, $u = 0$, $v_j = 0$, $j = \overline{1,r}$ to $\Gamma_{r,x}$. By definition of the partial derivative, it can be shown that

$$\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial F(t,x,y,u)}{\partial \chi}(t,0,0,0)\right) = \frac{\partial^{j+1} F(t,x,y,u)}{(\partial t)^j \, \partial \chi}(t,0,0,0), \quad j = \overline{1,r},$$

where χ means either x, or y, or u. Combining this fact and the structure of r-extended system with respect to the variables $z_1, \ldots, z_r, v_1, \ldots, v_r$ we get the formula

$$\Gamma_{r,x}(t, 0, \ldots, 0) = D_{r,x}(t), \quad \left(\frac{\partial \mathcal{F}_r}{\partial u} \frac{\partial \mathcal{F}_r}{\partial v_1} \cdots \frac{\partial \mathcal{F}_r}{\partial v_r}\right)(t, 0, \ldots, 0) = U_r(t) \quad \forall t \in I. \tag{A.22}$$

One can see that, given the assumptions of the Theorem, due to (A.22), all conditions of Theorem 2 are satisfied on a sufficiently small interval I_0.

Recall that ADS (11), (12) corresponds to system (10), (7). The Jacobi matrix with respect to the variables x_1, x_2, y_1, y_2, u, $v_1, \ldots, v_r$ for system (10), (7)

$$\Delta_{r,x} = \begin{pmatrix} -\partial f_1/\partial x_1 & O & E_{n-d} & O_d & -\partial f_1/\partial u & -\partial f_1/\partial v_1 & \ldots & -\partial f_1/\partial v_r \\ -\partial f_0/\partial x_1 & E_d & O & O & -\partial f_0/\partial u & -\partial f_0/\partial v_1 & \ldots & -\partial f_0/\partial v_r \end{pmatrix}$$

can be determined by the formula $\Delta_{r,x} = S\left(\Gamma_{r,x} \; \frac{\partial \mathcal{F}_r}{\partial u} \; \frac{\partial \mathcal{F}_r}{\partial v_1} \; \ldots \; \frac{\partial \mathcal{F}_r}{\partial v_r}\right)$, where

$$S = S(t, x_1, u, v_1, \ldots, v_r) = P_1 \left(\frac{\partial \mathcal{F}_r}{\partial y} \; \frac{\partial \mathcal{F}_r}{\partial x_2} \; \frac{\partial \mathcal{F}_r}{\partial Z_1}\right)^{-1} P_2,$$

P_1, P_2 are permutation matrices.

Substitute the values $x_1 = 0$, $u = 0$, $v_j = 0$ $(j = \overline{1,r})$ to $\Delta_{r,x}$ to get the matrix

$$\begin{pmatrix} -\widetilde{J}_1(t) & O & E_{n-d} & O_d & -\widetilde{L}_0(t) & \ldots & -\widetilde{L}_r(t) \\ -\widetilde{J}_2(t) & E_d & O & O & -\widetilde{G}_0(t) & \ldots & -\widetilde{G}_r(t) \end{pmatrix},$$

which is the Jacobi matrix with respect to the variables x_1, x_2, y_1, y_2, u, $v_1, \ldots, v_r$ for the system

$$\begin{aligned} y_1 - \widetilde{J}_1(t)x_1 - \widetilde{L}_0(t)u - \sum_{j=1}^{r} \widetilde{L}_j(t)v_j &= 0, \\ x_2 - \widetilde{J}_2(t)x_1(t) - \widetilde{G}_0(t)u - \sum_{j=1}^{r} \widetilde{G}_j(t)v_j &= 0. \end{aligned} \tag{A.23}$$

Consider now the system

$$\begin{aligned} y_1 - J_1(t)x_1 - L_0(t)u - \sum_{j=1}^{r} L_j(t)v_j &= 0, \\ x_2 - J_2(t)x_1(t) - G_0(t)u - \sum_{j=1}^{r} G_j(t)v_j &= 0, \end{aligned} \tag{A.24}$$

that corresponds to ADS (24). The following holds for (A.24):

$$\begin{pmatrix} -J_1(t) & O & E_{n-d} & O_d & -L_0(t) & \ldots & -L_r(t) \\ -J_2(t) & E_d & O & O & -G_0(t) & \ldots & -G_r(t) \end{pmatrix} = \overline{S}(t) \left(D_{r,x}(t) \; U_r(t)\right),$$

where $\overline{S}(t)$ is the the matrix inversible for all $t \in I_0$, which has the minor from condition (3) of Lemma 2 as its determinant.

It follows from (A.22) that $S(t, 0, \ldots, 0) = \overline{S}(t)$ $\forall t \in I_0$. Hence, systems (A.23) and (A.24) are the same, and systems (24) and (26) coinside on the interval I_0. □

Proof of Lemma 3. Necessity. Suppose that system (27), (28) is completely controllable.

It is clear that any solution of (27), (28) satisfies

$$\begin{aligned} x_1(t_1) &= \Omega(t_1)x_1(t_0) + \Omega(t_1) \int_{t_0}^{t_1} \Omega^{-1}(\tau) \left(\sum_{j=0}^{r} L_j(\tau)u^{(j)}(\tau) \right) d\tau, \\ x_2(t_1) &= J_2(t_1)x_1(t_1) + \sum_{j=0}^{r} G_j(t_1)u^{(j)}(t_1), \\ x_2(t_0) &= J_2(t_0)x_1(t_0) + \sum_{j=0}^{r} G_j(t_0)u^{(j)}(t_0). \end{aligned} \tag{A.25}$$

Denote

$$\begin{aligned} g_1 &= \Omega^{-1}(t_1)x_1(t_1) - x_1(t_0), \\ g_2 &= x_2(t_1) - J_2(t_1)x_1(t_1), \\ g_3 &= x_2(t_0) - J_2(t_0)x_1(t_0). \end{aligned}$$

Then the following controllability of (27), (28) can be understood as the existence of a control $u(t) \in \mathbf{C}^r(T)$, such that at any g_1, g_2, g_3 equations

$$g_1 = \int\limits_{t_0}^{t_1} \Omega^{-1}(\tau) \left(\sum_{j=0}^{r} L_j(\tau) u^{(j)}(\tau) \right) d\tau, \tag{A.26}$$

$$g_2 = \sum_{j=0}^{r} G_j(t_1) u^{(j)}(t_1), \quad g_3 = \sum_{j=0}^{r} G_j(t_0) u^{(j)}(t_0) \tag{A.27}$$

hold.

According to the assumption made in the beginning of the proof, the system

$$g_2 = (G_0(t_1) G_1(t_1) \ldots G_r(t_1))\, v$$

is solvable for the vector v at all values of g_2. Obviously, this is possible only when the system matrix has full row rank, i.e., rank $(G_0(t_1)G_1(t_1)\ldots G_r(t_1)) = d$. Similarly, the matrix $(G_0(t_0)G_1(t_0)\ldots G_r(t_0))$ has the same property. Thus, condition (1) of the Lemma is satisfied.

Let's show that condition (2) is satisfied as well. Assume the contrary. Then there exists a nonzero vector $\overline{h} \in \mathbf{R}^{n-d}$ such that

$$\overline{h}^{\top} \Omega^{-1}(t) \left(L_0(t) L_1(t) \ldots L_r(t) \right) = 0 \quad \forall t \in T. \tag{A.28}$$

Put $g_1 = \overline{h}$ in (A.26) and multiply (A.26) from the left by $\overline{h}^{\top}$. Taking into account (A.28), we get the contradiction $\overline{h}^{\top}\overline{h} = \overline{h}^{\top} \int\limits_{t_0}^{t} \Omega^{-1}(\tau) \left(\sum\limits_{j=0}^{r} L_j(\tau) u^{(j)}(\tau) \right) d\tau = 0$.

Sufficiency. Let conditions (1) and (2) of the Lemma be satisfied. We have to show that there exists a control $u(t) \in \mathbf{C}^r(T)$, such that (A.26), (A.27) hold at any values of g_1, g_2, g_3.

Let's seek control $u(t)$ in the form

$$u(t) = \sum_{j=0}^{r} \left(\alpha_j (t-t_0)^{r+1+j} + \beta_j (t-t_1)^{r+1+j} + \gamma_j (t-t_0)^{r+1} (t-t_1)^{r+1} (t-c)^{s+j} \right), \tag{A.29}$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbf{R}^l$ are unknown coefficients, $c \notin T$ is a fixed number, $s \geqslant 0$ is a sufficiently large number.

Then

$$\begin{pmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(r)}(t) \end{pmatrix} = \mathcal{E}_{r+1}(t-t_0) \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} + \mathcal{E}_{r+1}(t-t_1) \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_r \end{pmatrix} + \Phi(t-t_0, t-t_1)\, \mathcal{E}_s(t-c) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix}, \tag{A.30}$$

where

$$\mathcal{E}_k(t) = \begin{pmatrix} \frac{k!}{k!}t^k E_l & \frac{(k+1)!}{(k+1)!}t^{k+1}E_l & \cdots & \frac{(k+r)!}{(k+r)!}t^{k+r}E_l \\ \frac{k!}{(k-1)!}t^{k-1}E_l & \frac{(k+1)!}{k!}t^k E_l & \cdots & \frac{(k+r)!}{(k+r-1)!}t^{k+r-1}E_l \\ \vdots & \vdots & \ddots & \vdots \\ \frac{k!}{(k-r)!}t^{k-r}E_l & \frac{(k+1)!}{(k+1-r)!}t^{k+1-r}E_l & \cdots & \frac{(k+r)!}{k!}t^k E_l \end{pmatrix}, \quad k = r+1, s;$$

$$\Phi(t,\tau) = \begin{pmatrix} C_0^0 \frac{((r+1)!)^2 (t\tau)^{r+1}}{((r+1)!)^2} E_l & O & \cdots & O \\ C_1^0 \left(\sum\limits_{i=0}^{1} C_1^i \frac{((r+1)!)^2 t^{r+i} \tau^{r+1-i}}{(r+i)!(r+1-i)!} \right) E_l & C_1^1 \frac{((r+1)!)^2 (t\tau)^{r+1}}{((r+1)!)^2} E_l & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^0 \left(\sum\limits_{i=0}^{r} C_r^i \frac{((r+1)!)^2 t^{1+i} \tau^{r+1-i}}{(1+i)!(r+1-i)!} \right) E_l & C_r^1 \left(\sum\limits_{i=0}^{r-1} C_{r-1}^i \frac{((r+1)!)^2 t^{2+i} \tau^{r+1-i}}{(2+i)!(r+1-i)!} \right) E_l & \cdots & C_r^r \frac{((r+1)!)^2 (t\tau)^{r+1}}{((r+1)!)^2} E_l \end{pmatrix}.$$

Substitute (A.30) to (A.27) to get the systems of equations with respect to coefficients α_j and β_j:

$$\begin{gathered} (G_0(t_1)\ G_1(t_1)\ \ldots\ G_r(t_1))\, \mathcal{E}_{r+1}(t_1 - t_0) \text{ colon } (\alpha_0, \alpha_1, \ldots, \alpha_r) = g_2, \\ (G_0(t_0)\ G_1(t_0)\ \ldots\ G_r(t_0))\, \mathcal{E}_{r+1}(t_0 - t_1) \text{ colon } (\beta_0, \beta_1, \ldots, \beta_r) = g_3. \end{gathered} \tag{A.31}$$

It is clear that $\mathcal{E}_k(0) = O$ $(k = r+1, s)$, and the matrix $\mathcal{E}_k(t)$ is not inversible at $t \neq 0$. So, condition (1) from (A.31) ensures that the coefficients α_j and β_j are determined uniquely:

$$\begin{gathered} \text{colon } (\alpha_0, \alpha_1, \ldots, \alpha_r) = \mathcal{E}_{r+1}^{-1}(t_1 - t_0) \begin{pmatrix} G_0^\top(t_1) \\ \cdots \\ G_r^\top(t_1) \end{pmatrix} \left(\sum_{j=0}^{r} G_j(t_1) G_j^\top(t_1) \right)^{-1} g_2, \\ \text{colon } (\beta_0, \beta_1, \ldots, \beta_r) = \mathcal{E}_{r+1}^{-1}(t_0 - t_1) \begin{pmatrix} G_0^\top(t_0) \\ \cdots \\ G_r^\top(t_0) \end{pmatrix} \left(\sum_{j=0}^{r} G_j(t_0) G_j^\top(t_0) \right)^{-1} g_3. \end{gathered} \tag{A.32}$$

Substitution of (A.30), (A.32) to (A.26) yields the system with respect to coefficients γ_j:

$$\widetilde{g}_1 = \int\limits_{t_0}^{t_1} \Omega^{-1}(\tau)\, (L_0(\tau) L_1(\tau) \ldots L_r(\tau))\, \Phi(\tau - t_0, \tau - t_1) \mathcal{E}_s(\tau - c) d\tau \text{ colon} \left(\gamma_0, \gamma_1, \ldots, \gamma_r\right), \tag{A.33}$$

where

$$\begin{aligned} \widetilde{g}_1 = g_1 - \int\limits_{t_0}^{t_1} \Omega^{-1}(\tau)\, (L_0(\tau)\ L_1(\tau) \ldots L_r(\tau))\, (&\mathcal{E}_{r+1}(\tau - t_0) \text{ colon } (\alpha_0, \alpha_1, \ldots, \alpha_r) \\ &+ \mathcal{E}_{r+1}(\tau - t_1) \text{ colon } (\beta_0, \beta_1, \ldots, \beta_r))\, d\tau. \end{aligned} \tag{A.34}$$

As it was mentioned above, the matrix $\mathcal{E}_s(t-c)$ is inversible at $\forall t \in T$. In addition, if t is equal to t_0 or t_1 then $\Phi(t-t_0, t-t_1) = O$, and the matrix $\Phi(t-t_0, t-t_1)$ is also inversible at $t \in (t_0, t_1)$. Combining this fact, condition (2) of the Lemma, and the continuity of matrix coefficients of (27), (28), we get that for any nonzero vector $h \in \mathbf{R}^{n-d}$ $h^\top \Omega^{-1}(\tau)(L_0(\tau)L_1(\tau)\dots L_r(\tau))\Phi(\tau - t_0, \tau - t_1) \not\equiv 0$ on T. In this case, when s is sufficiently large, for all nonzero $h \in \mathbf{R}^{n-d}$

$$h^\top \int\limits_{t_0}^{t_1} \Omega^{-1}(\tau)(L_0(\tau)L_1(\tau)\dots L_r(\tau))\Phi(\tau - t_0, \tau - t_1)\mathcal{E}_s(\tau - c)d\tau \neq 0,$$

which means that the matrix

$$N = \int\limits_{t_0}^{t_1} \Omega^{-1}(\tau)\left(L_0(\tau)L_1(\tau)\dots L_r(\tau)\right)\Phi(\tau - t_0, \tau - t_1)\mathcal{E}_s(\tau - c)d\tau$$

has full row rank. So, at sufficiently large s, system (A.33) is solvable with respect to $\gamma_0, \gamma_1, \dots, \gamma_r$ at any vector $\widetilde{g}_1$:

$$\operatorname{colon}(\gamma_0, \gamma_1, \dots, \gamma_r) = N^\top \left(NN^\top\right)^{-1} \widetilde{g}_1. \tag{A.35}$$

We constructed the control of the form (A.29) that ensures (A.26), (A.27) at any g_1, g_2, g_3. □

Proof of Lemma 5. In (A.25) put

$$x_1(t_0) = x_{1,0}, \quad x_2(t_0) = x_{2,0}, \quad x_1(t_1) = 0, \quad x_2(t_1) = 0,$$

where $x_{1,0} \in \mathbf{R}^{n-d}$, $x_{2,0} \in \mathbf{R}^d$ are some vectors.

The complete controlability of system (27), (28) guarantees the existence of an l-variate vector-function $u(t) \in \mathbf{C}^r(T)$ such that

$$0 = \Omega(t_1)x_{1,0} + \Omega(t_1)\int\limits_{t_0}^{t_1} \Omega^{-1}(\tau)\sum_{j=0}^{r} L_j(\tau)u^{(i)}(\tau)d\tau,$$

$$0 = \sum_{j=0}^{r} G_j(t_1)u^{(j)}(t_1), \quad x_{2,0} = J_2(t_0)x_{1,0} + \sum_{j=0}^{r} G_j(t_0)u^{(j)}(t_0)$$

at any $x_{1,0}$ and $x_{2,0}$ of appropriate dimension. According to the proof of Lemma 3, this control can be sought in the form (A.29), where coefficients α_j, β_j,γ_j ($j = \overline{0,r}$) are determined by formulae (A.32), (A.35), (A.34) with $g_1 = -x_{1,0}$, $g_2 = 0$, $g_3 = x_{2,0} - J_2(t_0)x_{1,0}$.

All matrices of the above formulae are either constant, or continuous on T and, therefore, bounded on T. So the control $u(t)$ can be made admissible by choosing $x_{1,0}$, $x_{2,0}$ such that $\|x_{1,0}\|$, $\|x_{2,0}\| < \delta$ with a sufficiently small $\delta > 0$. □

Proof of Lemma 6. Choose n-variate vectors $p_i = \operatorname{colon}(0, \dots, 0, \sigma_i, 0, \dots, 0)$, where $\sigma_i \neq 0$—ith component. Then the vectors p_i, $i = \overline{1,n}$ are linearly independent in $\mathbf{R}^n$. According to condition (3), for all $i = \overline{1,n}$ there is a control $v_i(t) \in \mathbf{C}^r(T)$ that guarantees the unique existence of solution for the system

$$x_1'(t) = J_1(t)x_1(t) + \sum_{j=0}^{r} L_j(t)v_i^{(j)}(t), \tag{A.36}$$

$$x_2(t) = J_2(t)x_1(t) + \sum_{j=0}^{r} G_j(t)v_i^{(j)}(t), \quad t \in T, \tag{A.37}$$

the solution satisfying the conditions

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = p_i, \quad \begin{pmatrix} x_1(t_1) \\ x_2(t_1) \end{pmatrix} = 0. \tag{A.38}$$

Denote q_k = colon $(0, \ldots, 0, g_k, 0, \ldots, 0)$, where $g_k \neq 0$ is the kth component of the d-variate vector q_k, $k = \overline{1, d}$. These vectors are linearly independent in the space $\mathbf{R}^d$. Due to complete controllability of (27), (28), for any k there exists a control $w_k(t) \in \mathbf{C}^r(T)$ such that solution of the system

$$x_1'(t) = J_1(t)x_1(t) + \sum_{j=0}^{r} L_j(t)w_k^{(j)}(t), \tag{A.39}$$

$$x_2(t) = J_2(t)x_1(t) + \sum_{j=0}^{r} G_j(t)w_k^{(j)}(t), \quad t \in T \tag{A.40}$$

exists, the solution satisfying the conditions

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = 0, \quad x_2(t_1) = q_k. \tag{A.41}$$

The structure of system (A.39), (A.40) yields the uniqueness of this solution.

Determine the control as follows:

$$u(t, \mu, \lambda) = \sum_{i=1}^{n} \mu_i v_i(t) + \sum_{k=1}^{d} \lambda_k w_k(t), \tag{A.42}$$

where $\mu = (\mu_1, \ldots, \mu_n)$, $\lambda = (\lambda_1, \ldots, \lambda_d)$ are unknown parameters. Note that this control is admissible at μ_i and λ_k sufficiently close to zero.

The solution $x_1(t, \mu, \lambda)$, $x_2(t, \mu, \lambda)$ of the problem

$$x_1(t_1) = 0 \tag{A.43}$$

for the system

$$x_1'(t) = f_1(t, x_1(t), u(t, \mu, \lambda), u'(t, \mu, \lambda), \ldots, u^{(r)}(t, \mu, \lambda)), \tag{A.44}$$

$$x_2(t) = f_0(t, x_1(t), u(t, \mu, \lambda), u'(t, \mu, \lambda), \ldots, u^{(r)}(t, \mu, \lambda)), \quad t \in T \tag{A.45}$$

exists and is unique for all parameter values sufficiently close to zero. In particular, $x_1(t, 0, 0) \equiv 0$, $x_2(t, 0, 0) \equiv 0$ on T at $\mu = 0$ and $\lambda = 0$. In addition, due to condition (1) of the Lemma, the functions $x_1(t, \mu, \lambda)$, $x_2(t, \mu, \lambda)$ have continuous partial derivatives with respect to $\mu_1, \ldots, \mu_n$ and $\lambda_1, \ldots, \lambda_d$.

Let's show that the parameters μ and λ can be chosen such that the solution of (A.43)–(A.45) satisfies

$$\begin{pmatrix} x_1(t_0, \mu, \lambda) \\ x_2(t_0, \mu, \lambda) \end{pmatrix} = x_0, \quad x_2(t_1, \mu, \lambda) = 0 \tag{A.46}$$

at any vector $x_0 \in \mathbf{R}^n$ with the norm sufficiently close to zero. This would complete the proof of the Lemma.

Differentiate the identities

$$x_1'(t,\mu,\lambda) = f_1(t, x_1(t,\mu,\lambda), u(t,\mu,\lambda), u'(t,\mu,\lambda), \ldots, u^{(r)}(t,\mu,\lambda)),$$
$$x_2(t,\mu,\lambda) = f_0(t, x_1(t,\mu,\lambda), u(t,\mu,\lambda), u'(t,\mu,\lambda), \ldots, u^{(r)}(t,\mu,\lambda)), \quad t \in T,$$

with respect to μ and λ to get

$$\begin{aligned}
\frac{d}{dt}\frac{\partial x_1(t,\mu,\lambda)}{\partial \mu} &= \frac{\partial f_1}{\partial x_1}\frac{\partial x_1(t,\mu,\lambda)}{\partial \mu} + \sum_{j=0}^{r}\frac{\partial f_1}{\partial u^{(j)}}\frac{\partial u^{(j)}(t,\mu,\lambda)}{\partial \mu},\\
\frac{d}{dt}\frac{\partial x_1(t,\mu,\lambda)}{\partial \lambda} &= \frac{\partial f_1}{\partial x_1}\frac{\partial x_1(t,\mu,\lambda)}{\partial \lambda} + \sum_{j=0}^{r}\frac{\partial f_1}{\partial u^{(j)}}\frac{\partial u^{(j)}(t,\mu,\lambda)}{\partial \lambda},\\
\frac{\partial x_2(t,\mu,\lambda)}{\partial \mu} &= \frac{\partial f_0}{\partial x_1}\frac{\partial x_1(t,\mu,\lambda)}{\partial \mu} + \sum_{j=0}^{r}\frac{\partial f_0}{\partial u^{(j)}}\frac{\partial u^{(j)}(t,\mu,\lambda)}{\partial \mu},\\
\frac{\partial x_2(t,\mu,\lambda)}{\partial \lambda} &= \frac{\partial f_0}{\partial x_1}\frac{\partial x_1(t,\mu,\lambda)}{\partial \lambda} + \sum_{j=0}^{r}\frac{\partial f_0}{\partial u^{(j)}}\frac{\partial u^{(j)}(t,\mu,\lambda)}{\partial \lambda}.
\end{aligned} \tag{A.47}$$

Substitute $\mu = 0,\ \lambda = 0$ to (A.47). Taking into account equations

$$\frac{\partial u^{(j)}}{\partial \mu}(t,0,0) = \left(v_1^{(j)}(t)\ v_2^{(j)}(t)\ \ldots\ v_n^{(j)}(t)\right),$$
$$\frac{\partial u^{(j)}}{\partial \lambda}(t,0,0) = \left(w_1^{(j)}(t)\ w_2^{(j)}(t)\ \ldots\ w_d^{(j)}(t)\right),$$

which follow from (A.42), and notation (33), we obtain

$$\begin{aligned}
\frac{d}{dt}\frac{\partial x_1}{\partial \mu}(t,0,0) &= J_1(t)\frac{\partial x_1}{\partial \mu}(t,0,0) + \sum_{j=0}^{r} L_j(t)\left(v_1^{(j)}(t)\ v_2^{(j)}(t)\ \ldots\ v_n^{(j)}(t)\right),\\
\frac{d}{dt}\frac{\partial x_1}{\partial \lambda}(t,0,0) &= J_1(t)\frac{\partial x_1}{\partial \lambda}(t,0,0) + \sum_{j=0}^{r} L_j(t)\left(w_1^{(j)}(t)\ w_2^{(j)}(t)\ \ldots\ w_d^{(j)}(t)\right),\\
\frac{\partial x_2}{\partial \mu}(t,0,0) &= J_2(t)\frac{\partial x_1}{\partial \mu}(t,0,0) + \sum_{j=0}^{r} G_j(t)\left(v_1^{(j)}(t)\ v_2^{(j)}(t)\ \ldots\ v_n^{(j)}(t)\right),\\
\frac{\partial x_2}{\partial \lambda}(t,0,0) &= J_2(t)\frac{\partial x_1}{\partial \lambda}(t,0,0) + \sum_{j=0}^{r} G_j(t)\left(w_1^{(j)}(t)\ w_2^{(j)}(t)\ \ldots\ w_d^{(j)}(t)\right).
\end{aligned} \tag{A.48}$$

It follows from (A.48) that the columns ξ_i and χ_i of the matrices

$$\frac{\partial x_1}{\partial \mu}(t,0,0) = \left(\xi_1(t)\ \xi_2(t)\ \ldots\ \xi_n(t)\right),$$
$$\frac{\partial x_2}{\partial \mu}(t,0,0) = \left(\chi_1(t)\ \chi_2(t)\ \ldots\ \chi_n(t)\right)$$

are solutions of system (A.36), (A.37), while the columns η_k and ν_k of the matrices

$$\frac{\partial x_1}{\partial \lambda}(t,0,0) = \left(\eta_1(t)\ \eta_2(t)\ \ldots\ \eta_d(t)\right),$$
$$\frac{\partial x_2}{\partial \lambda}(t,0,0) = \left(\nu_1(t)\ \nu_2(t)\ \ldots\ \nu_d(t)\right)$$

turn system (A.39), (A.40) into identity. Due to the choice of controls $v_i(t)$ and $w_k(t)$, the solutions have to satisfy conditions analogous to (A.38) and (A.41), respectively, i.e.,

$$\begin{pmatrix} \xi_i(t_0) \\ \chi_i(t_0) \end{pmatrix} = p_i, \quad \begin{pmatrix} \xi_i(t_1) \\ \chi_i(t_1) \end{pmatrix} = 0, \quad i = \overline{1,n};$$
$$\begin{pmatrix} \eta_k(t_0) \\ \nu_k(t_0) \end{pmatrix} = 0, \quad \nu_k(t_1) = q_k, \quad k = \overline{1,d}.$$

So the matrix

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial \mu}(t_0,0,0) & \dfrac{\partial x_1}{\partial \lambda}(t_0,0,0) \\ \dfrac{\partial x_2}{\partial \mu}(t_0,0,0) & \dfrac{\partial x_2}{\partial \lambda}(t_0,0,0) \\ \dfrac{\partial x_2}{\partial \mu}(t_1,0,0) & \dfrac{\partial x_2}{\partial \lambda}(t_1,0,0) \end{pmatrix} = \begin{pmatrix} \mathrm{diag}\,(\delta_1,\delta_2,\ldots,\delta_n) & O \\ O & \mathrm{diag}\,(g_1,g_2,\ldots,g_d) \end{pmatrix} \tag{A.49}$$

is nonsingular.

Consider Eqs. (A.46). We will treat them as a system of $n+d$ equations with respect to $n+d$ unknowns μ and λ. As it was mentioned above, $x_0 = 0$, $\mu = 0$, $\lambda = 0$ satisfy this system. Since the matrix (A.49) is nonsingular, all conditions of the implicit function theorem are satisfied for system (A.46). According to this theorem, there exists a solution $\mu_* = \phi_1(x_0)$, $\lambda_* = \phi_2(x_0)$ of this systems for any x_0 sufficiently close to zero. □

Proof of Lemma 7. Take a set of n linearly independent n-variate vectors $p_i = \mathrm{colon}\,(0,\ldots,0,\sigma_i, 0,\ldots,0)$, where $\sigma_i \neq 0$ is the ith component of the vector p_i; $\sigma_i < \delta$, $\delta > 0$ is a sufficiently small number. Then $\|p_i\| < \delta$, $i = \overline{1,n}$.

Given these assumptions, there exists at any $i = 1,2,\ldots,n$ an admissible control $v_i(t)$ that guarantees the existence of solution for (A.36) such that (A.38) hold.

Let's demonstrate that $\forall k = \overline{1,d}$ there exists an admissible control $w_k(t)$ such that a solution of (A.39), (A.40) exists and satisfies (A.41) and the equation $x_1(t_1) = x_1$, where $q_k \in \mathbf{R}^d$ $(k = \overline{1,d})$ is a set of linearly independent vectors, $x_1 \in \mathbf{R}^{n-d}$ is some vector. q_k and x_1 are chosen such that their norms are sufficiently close to zero.

The solution of the Cauchy problem $x_1(t_0) = 0$ for (A.39) exists at any sufficiently smooth control $w_k(t)$. So an admissible control $w_k(t)$ has to ensure the following conditions:

$$\begin{gathered} \Omega^{-1}(t_1)x_1 = \int\limits_{t_0}^{t_1} \Omega^{-1}(\tau) \sum_{j=0}^{r} L_j(\tau) w_k^{(j)}(\tau)d\tau, \\ 0 = \sum_{j=0}^{r} G_j(t_0) w_k^{(j)}(t_0), \quad q_k = J_2(t_1)x_1 + \sum_{j=0}^{r} G_j(t_1) w_k^{(j)}(t_1), \end{gathered} \tag{A.50}$$

where $\Omega(t)$ is the matrizant of system (A.39) (v. (29)). As it was done in the proof of Lemma 3, the control $w_k(t) \in \mathbf{C}^r(T)$ that guarantees (A.50) can be constructed in the form of polynomials (A.29), where s is sufficiently large. The admissibility of this control can be provided by the choice of the vectors x_1 and q_k with the norm sufficiently close to zero.

Repetition of the reasoning used in the proof of Lemma 6, beginning with the construction of control of the form (A.42), completes the proof. □

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