3

Modeling with First-Order Differential Equations

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In Section 1.3 we saw how a first-order differential equation could be used as a mathematical model in the study of population growth, radioactive decay, cooling of bodies, mixtures, chemical reactions, fluid draining from a tank, velocity of a falling body, and current in a series circuit. Using the methods discussed in Chapter 2, we are now able to solve some of the linear DEs in Section 3.1 and nonlinear DEs in Section 3.2 that frequently appear in applications.

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3.1 Linear Models

INTRODUCTION In this section we solve some of the linear first-order models that were introduced in Section 1.3.

GROWTH AND DECAY The initial-value problem

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0, \tag{1}$$

where k is a constant of proportionality, serves as a model for diverse phenomena involving either growth or decay. We saw in Section 1.3 that in biological applications the rate of growth of certain populations (bacteria, small animals) over short periods of time is proportional to the population present at time t. Knowing the population at some arbitrary initial time t_0, we can then use the solution of (1) to predict the population in the future—that is, at times $t > t_0$. The constant of proportionality k in (1) can be determined from the solution of the initial-value problem, using a subsequent measurement of x at a time $t_1 > t_0$. In physics and chemistry (1) is seen in the form of a *first-order reaction*—that is, a reaction whose rate, or velocity, dx/dt is directly proportional to the amount x of a substance that is unconverted or remaining at time t. The decomposition, or decay, of U-238 (uranium) by radioactivity into Th-234 (thorium) is a first-order reaction.

EXAMPLE 1 Bacterial Growth

A culture initially has P_0 number of bacteria. At $t = 1$ h the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

SOLUTION We first solve the differential equation in (1), with the symbol x replaced by P. With $t_0 = 0$ the initial condition is $P(0) = P_0$. We then use the empirical observation that $P(1) = \frac{3}{2}P_0$ to determine the constant of proportionality k.

Notice that the differential equation $dP/dt = kP$ is both separable and linear. When it is put in the standard form of a linear first-order DE,

$$\frac{dP}{dt} - kP = 0,$$

we can see by inspection that the integrating factor is e^{-kt}. Multiplying both sides of the equation by this term and integrating gives, in turn,

$$\frac{d}{dt}[e^{-kt}P] = 0 \qquad \text{and} \qquad e^{-kt}P = c.$$

Therefore $P(t) = ce^{kt}$. At $t = 0$ it follows that $P_0 = ce^0 = c$, so $P(t) = P_0e^{kt}$. At $t = 1$ we have $\frac{3}{2}P_0 = P_0e^k$ or $e^k = \frac{3}{2}$. From the last equation we get $k = \ln\frac{3}{2} = 0.4055$, so $P(t) = P_0e^{0.4055t}$. To find the time at which the number of bacteria has tripled, we solve $3P_0 = P_0e^{0.4055t}$ for t. It follows that $0.4055t = \ln 3$, or

$$t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ h}.$$

See Figure 3.1.1. ■

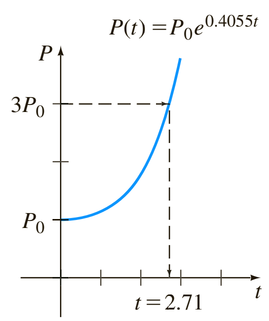

FIGURE 3.1.1 Time in which population triples in Example 1

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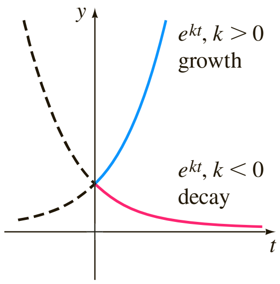

FIGURE 3.1.2 Growth ($k > 0$) and decay ($k < 0$)

Notice in Example 1 that the actual number P_0 of bacteria present at time $t = 0$ played no part in determining the time required for the number in the culture to triple. The time necessary for an initial population of, say, 100 or 1,000,000 bacteria to triple is still approximately 2.71 hours.

As shown in Figure 3.1.2, the exponential function e^{kt} increases as t increases for $k > 0$ and decreases as t increases for $k < 0$. Thus problems describing growth (whether of populations, bacteria, or even capital) are characterized by a positive value of k, whereas problems involving decay (as in radioactive disintegration) yield a negative k value. Accordingly, we say that k is either a **growth constant** ($k > 0$) or a **decay constant** ($k < 0$).

HALF-LIFE In physics the **half-life** is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another element. The longer the half-life of a substance, the more stable it is. For example, the half-life of highly radioactive radium, Ra-226, is about 1700 years. In 1700 years one-half of a given quantity of Ra-226 is transmuted into radon, Rn-222. The most commonly occurring uranium isotope, U-238, has a half-life of approximately 4,500,000,000 years. In about 4.5 billion years, one-half of a quantity of U-238 is transmuted into lead, Pb-206.

EXAMPLE 2 Half-Life of Plutonium

A breeder reactor converts relatively stable uranium-238 into the isotope plutonium-239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

SOLUTION Let $A(t)$ denote the amount of plutonium remaining at time t. As in Example 1 the solution of the initial-value problem

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

is $A(t) = A_0e^{kt}$. If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. To find the decay constant k, we use $0.99957A_0 = A(15)$—that is, $0.99957A_0 = A_0e^{15k}$. Solving for k then gives $k = \frac{1}{15}\ln 0.99957 = -0.00002867$. Hence $A(t) = A_0e^{-0.00002867t}$. Now the half-life is the corresponding value of time at which $A(t) = \frac{1}{2}A_0$. Solving for t gives $\frac{1}{2}A_0 = A_0e^{-0.00002867t}$, or $\frac{1}{2} = e^{-0.00002867t}$. The last equation yields

$$t = \frac{\ln 2}{0.00002867} \approx 24{,}180 \text{ yr.}$$

FIGURE 3.1.3 Willard Libby (1908–1980)

CARBON DATING About 1950, a team of scientists at the University of Chicago led by the chemist **Willard Libby** devised a method using a radioactive isotope of carbon as a means of determining the approximate ages of carbonaceous fossilized matter. See Figure 3.1.3. The theory of **carbon dating** is based on the fact that the radioisotope carbon-14 is produced in the atmosphere by the action of cosmic radiation on nitrogen-14. The ratio of the amount of C-14 to the stable C-12 in the atmosphere appears to be a constant, and as a consequence the proportionate amount of the isotope present in all living organisms is the same as that in the atmosphere. When a living organism dies, the absorption of C-14, by breathing, eating, or photosynthesis, ceases. By comparing the proportionate amount of C-14, say, in a fossil with the constant amount ratio found in the atmosphere, it is possible to obtain a reasonable estimation of its age. The method is based on the knowledge of the

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FIGURE 3.1.4 A page of the Gnostic Gospel of Judas

half-life of C-14. Libby's calculated value of the half-life of C-14 was approximately 5600 years, and is called the **Libby half-life.** Today the commonly accepted value for the half-life of C-14 is the **Cambridge half-life** that is close to 5730 years. For his work, Libby was awarded the Nobel Prize for chemistry in 1960. Libby's method has been used to date wooden furniture found in Egyptian tombs, the woven flax wrappings of the Dead Sea Scrolls, a recently discovered copy of the Gnostic Gospel of Judas written on papyrus, and the cloth of the enigmatic Shroud of Turin. See Figure 3.1.4 and Problem 12 in Exercises 3.1.

EXAMPLE 3 Age of a Fossil

A fossilized bone is found to contain 0.1% of its original amount of C-14. Determine the age of the fossil.

SOLUTION As in Example 2 the starting point is $A(t) = A_0e^{kt}$. To determine the value of the decay constant k we use the fact that $\frac{1}{2}A_0 = A(5730)$ or $\frac{1}{2}A_0 = A_0e^{5730k}$. The last equation implies $5730k = \ln\frac{1}{2} = -\ln 2$ and so we get $k = -(\ln 2)/5730 = -0.00012097$. Therefore $A(t) = A_0e^{-0.00012097t}$. With $A(t) = 0.001A_0$ we have $0.001A_0 = A_0e^{-0.00012097t}$ and $-0.00012097t = \ln(0.001) = -\ln 1000$. Thus

$$t = \frac{\ln 1000}{0.00012097} \approx 57{,}100 \text{ years.}$$

The age found in Example 3 is really at the border of accuracy of this method. The usual carbon-14 technique is limited to about 10 half-lives of the isotope, or roughly 60,000 years. One fundamental reason for this limitation is the relatively short half-life of C-14. There are other problems as well: The chemical analysis needed to obtain an accurate measurement of the remaining C-14 becomes somewhat formidable around the point $0.001A_0$. Moreover, this analysis requires the destruction of a rather large sample of the specimen. If this measurement is accomplished indirectly, based on the actual radioactivity of the specimen, then it is very difficult to distinguish between the radiation from the specimen and the normal background radiation. But recently the use of a particle accelerator has enabled scientists to separate the C-14 from the stable C-12 directly. When the precise value of the ratio of C-14 to C-12 is computed, the accuracy of the dating method can be extended to 70,000–100,000 years. For these reasons and the fact that the C-14 dating is restricted to organic materials, this method is used mainly by archeologists. On the other hand, geologists who are interested in questions about the age of rocks or the age of the Earth use **radiometric-dating** techniques. Radiometric dating, invented by the physicist chemist **Ernest Rutherford** (1871–1937) around 1905, is based on the radioactive decay of a naturally occurring radioactive isotope with a very long half-life and a comparison between a measured quantity of this decaying isotope and one of its decay products using known decay rates. Radiometric methods using potassium-argon, rubidium-strontium, or uranium-lead can give ages of certain kinds of rocks of several billion years. See Problems 5 and 6 in Exercises 3.3 for a brief discussion of the **potassium-argon method of dating.**

The size and location of the sample caused major difficulties when a team of scientists were invited to use carbon-14 dating on the Shroud of Turin in 1988.

The half-life of uranium-238 is about 4.47 billion years.

NEWTON'S LAW OF COOLING/WARMING In equation (3) of Section 1.3 we saw that the mathematical formulation of Newton's empirical law of cooling/warming of an object is given by the linear first-order differential equation

$$\frac{dT}{dt} = k(T - T_m), \tag{2}$$

where k is a constant of proportionality, $T(t)$ is the temperature of the object for $t > 0$, and T_m is the ambient temperature—that is, the temperature of the medium around the object. In Example 4 we assume that T_m is constant.

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EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

SOLUTION In (2) we make the identification $T_m = 70$. We must then solve the initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 300 \tag{3}$$

and determine the value of k so that $T(3) = 200$.

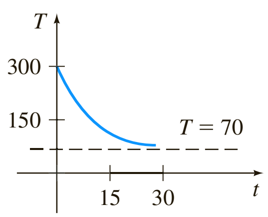
Equation (3) is both linear and separable. If we separate variables,

$$\frac{dT}{T - 70} = k\,dt,$$

yields $\ln|T - 70| = kt + c_1$, and so $T = 70 + c_2e^{kt}$. When $t = 0$, $T = 300$, so $300 = 70 + c_2$ gives $c_2 = 230$; therefore $T = 70 + 230e^{kt}$. Finally, the measurement $T(3) = 200$ leads to $e^{3k} = \frac{13}{23}$, or $k = \frac{1}{3}\ln\frac{13}{23} = -0.19018$. Thus

$$T(t) = 70 + 230e^{-0.19018t}. \tag{4}$$

We note that (4) furnishes no finite solution to $T(t) = 70$, since $\lim_{t\to\infty} T(t) = 70$. Yet we intuitively expect the cake to reach room temperature after a reasonably long period of time. How long is "long"? Of course, we should not be disturbed by the fact that the model (3) does not quite live up to our physical intuition. Parts (a) and (b) of Figure 3.1.5 clearly show that the cake will be approximately at room temperature in about one-half hour. ■



(a)

$T(t)$	t (min)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

(b)

FIGURE 3.1.5 Temperature of cooling cake in Example 4

The ambient temperature in (2) need not be a constant but could be a function $T_m(t)$ of time t. See Problem 18 in Exercises 3.1.

MIXTURES The mixing of two fluids sometimes gives rise to a linear first-order differential equation. When we discussed the mixing of two brine solutions in Section 1.3, we assumed that the rate $A'(t)$ at which the amount of salt in the mixing tank changes was a net rate:

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt}) = R_{in} - R_{out}. \tag{5}$$

In Example 5 we solve equation (8) on page 25 of Section 1.3.

EXAMPLE 5 Mixture of Two Salt Solutions

Recall that the large tank considered in Section 1.3 held 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflow, or solution entering, was 2 lb/gal, so salt was entering the tank at the rate $R_{in} = (2 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = 6$ lb/min and leaving the tank at the rate $R_{out} = (A/300 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = A/100$ lb/min. From this data and (5) we get equation (8) of Section 1.3. Let us pose the question: If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

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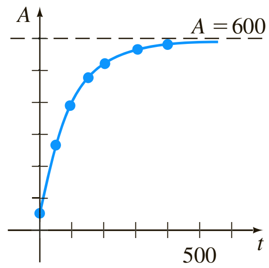

(a)

t (min)	A (lb)
50	266.41
100	397.67
150	477.27
200	525.57
300	572.62
400	589.93

(b)

FIGURE 3.1.6 Pounds of salt in the tank in Example 5

SOLUTION To find the amount of salt $A(t)$ in the tank at time t, we solve the initial-value problem

$$\frac{dA}{dt} + \frac{1}{100}A = 6, \quad A(0) = 50.$$

Note here that the side condition is the initial amount of salt $A(0) = 50$ in the tank and *not* the initial amount of liquid in the tank. Now since the integrating factor of the linear differential equation is $e^{t/100}$, we can write the equation as

$$\frac{d}{dt}[e^{t/100}A] = 6e^{t/100}.$$

Integrating the last equation and solving for A gives the general solution $A(t) = 600 + ce^{-t/100}$. When $t = 0$, $A = 50$, so we find that $c = -550$. Thus the amount of salt in the tank at time t is given by

$$A(t) = 600 - 550e^{-t/100}. \tag{6}$$

The solution (6) was used to construct the table in Figure 3.1.6(b). Also, it can be seen from (6) and Figure 3.1.6(a) that $A(t) \to 600$ as $t \to \infty$. Of course, this is what we would intuitively expect; over a long time the number of pounds of salt in the solution must be $(300 \text{ gal})(2 \text{ lb/gal}) = 600 \text{ lb}$. ■

In Example 5 we assumed that the rate at which the solution was pumped in was the same as the rate at which the solution was pumped out. However, this need not be the case; the mixed brine solution could be pumped out at a rate r_{out} that is faster or slower than the rate r_{in} at which the other brine solution is pumped in. The next example illustrates the case when the mixture is pumped out at rate that is *slower* than the rate at which the brine solution is being pumped into the tank.

EXAMPLE 6 Example 5 Revisited

If the well-stirred solution in Example 5 is pumped out at a slower rate of, say, $r_{out} = 2$ gal/min, then liquid will accumulate in the tank at the rate of

$$r_{in} - r_{out} = (3 - 2) \text{ gal/min} = 1 \text{ gal/min}.$$

After t minutes,

$$(1 \text{ gal/min}) \cdot (t \text{ min}) = t \text{ gal}$$

will accumulate, so the tank will contain $300 + t$ gallons of brine. The concentration of the outflow is then $c(t) = A/(300 + t)$ lb/gal, and the output rate of salt is $R_{out} = c(t) \cdot r_{out}$, or

$$R_{out} = \left(\frac{A}{300 + t} \text{ lb/gal}\right) \cdot (2 \text{ gal/min}) = \frac{2A}{300 + t} \text{ lb/min}.$$

Hence equation (5) becomes

$$\frac{dA}{dt} = 6 - \frac{2A}{300 + t} \quad \text{or} \quad \frac{dA}{dt} + \frac{2}{300 + t}A = 6.$$

The integrating factor for the last equation is

$$e^{\int 2\,dt/(300+t)} = e^{2\ln(300+t)} = e^{\ln(300+t)^2} = (300 + t)^2$$

and so after multiplying by the factor the equation is cast into the form

$$\frac{d}{dt}[(300 + t)^2 A] = 6(300 + t)^2.$$

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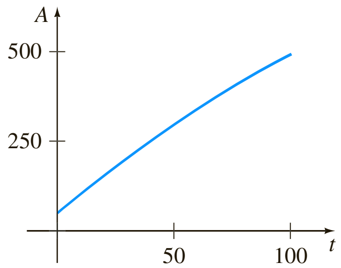

FIGURE 3.1.7 Graph of $A(t)$ in Example 6

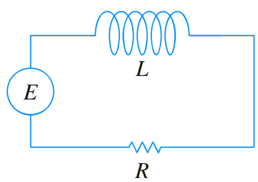

FIGURE 3.1.8 *LR*-series circuit

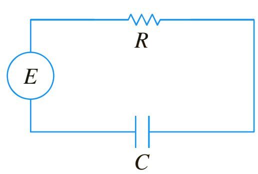

FIGURE 3.1.9 *RC*-series circuit

Integrating the last equation gives $(300 + t)^2A = 2(300 + t)^3 + c$. By applying the initial condition $A(0) = 50$ and solving for A yields the solution $A(t) = 600 + 2t - (4.95 \times 10^7)(300 + t)^{-2}$. As Figure 3.1.7 shows, not unexpectedly, salt builds up in the tank over time, that is, $A \to \infty$ as $t \to \infty$. ■

SERIES CIRCUITS For a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor $(L(di/dt))$ and the voltage drop across the resistor (iR) is the same as the impressed voltage $(E(t))$ on the circuit. See Figure 3.1.8.

Thus we obtain the linear differential equation for the current $i(t)$,

$$L\frac{di}{dt} + Ri = E(t), \tag{7}$$

where L and R are known as the inductance and the resistance, respectively. The current $i(t)$ is also called the **response** of the system.

The voltage drop across a capacitor with capacitance C is given by $q(t)/C$, where q is the charge on the capacitor. Hence, for the series circuit shown in Figure 3.1.9, Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t). \tag{8}$$

But current i and charge q are related by $i = dq/dt$, so (8) becomes the linear differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{9}$$

EXAMPLE 7 *LR*-Series Circuit

A 12-volt battery is connected to a series circuit in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

SOLUTION From (7) we see that we must solve the equation

$$\frac{1}{2}\frac{di}{dt} + 10i = 12,$$

subject to $i(0) = 0$. First, we multiply the differential equation by 2 and read off the integrating factor e^{20t}. We then obtain

$$\frac{d}{dt}[e^{20t}i] = 24e^{20t}.$$

Integrating each side of the last equation and solving for i gives $i(t) = \frac{6}{5} + ce^{-20t}$. Now $i(0) = 0$ implies that $0 = \frac{6}{5} + c$ or $c = -\frac{6}{5}$. Therefore the response is $i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}$. ■

From (4) of Section 2.3 we can write a general solution of (7):

$$i(t) = \frac{e^{-(R/L)t}}{L}\int e^{(R/L)t}E(t)\,dt + ce^{-(R/L)t}. \tag{10}$$

In particular, when $E(t) = E_0$ is a constant, (10) becomes

$$i(t) = \frac{E_0}{R} + ce^{-(R/L)t}. \tag{11}$$

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Note that as $t \to \infty$, the second term in equation (11) approaches zero. Such a term is usually called a **transient term;** any remaining terms are called the **steady-state** part of the solution. In this case E_0/R is also called the **steady-state current;** for large values of time it appears that the current in the circuit is simply governed by Ohm's law ($E = iR$).

REMARKS

The solution $P(t) = P_0e^{0.4055t}$ of the initial-value problem in Example 1 described the population of a colony of bacteria at any time $t \geq 0$. Of course, $P(t)$ is a continuous function that takes on *all* real numbers in the interval $P_0 \leq P < \infty$. But since we are talking about a population, common sense dictates that P can take on only positive integer values. Moreover, we would not expect the population to grow continuously—that is, every second, every microsecond, and so on—as predicted by our solution; there may be intervals of time $[t_1, t_2]$ over which there is no growth at all. Perhaps, then, the graph shown in Figure 3.1.10(a) is a more realistic description of P than is the graph of an exponential function. Using a continuous function to describe a discrete phenomenon is often more a matter of convenience than of accuracy. However, for some purposes we may be satisfied if our model describes the system fairly closely when viewed macroscopically in time, as in Figures 3.1.10(b) and 3.1.10(c), rather than microscopically, as in Figure 3.1.10(a).

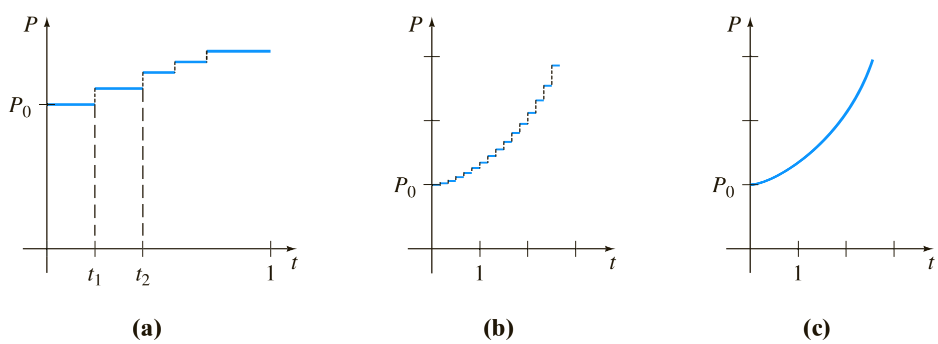

FIGURE 3.1.10 Population growth is a discrete process

EXERCISES 3.1

Answers to selected odd-numbered problems begin on page ANS-3.

Growth and Decay

1. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?

2. Suppose it is known that the population of the community in Problem 1 is 10,000 after 3 years. What was the initial population P_0? What will be the population in 10 years? How fast is the population growing at $t = 10$?

3. The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years? How fast is the population growing at $t = 30$?

4. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time t and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

6. Initially 100 milligrams of a radioactive substance was present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time t, find the amount remaining after 24 hours.

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7. Determine the half-life of the radioactive substance described in Problem 6.

8. **(a)** Consider the initial-value problem $dA/dt = kA$, $A(0) = A_0$ as the model for the decay of a radioactive substance. Show that, in general, the half-life T of the substance is $T = -(\ln 2)/k$.

(b) Show that the solution of the initial-value problem in part (a) can be written $A(t) = A_0 2^{-t/T}$.

(c) If a radioactive substance has the half-life T given in part (a), how long will it take an initial amount A_0 of the substance to decay to $\frac{1}{8}A_0$?

9. When a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in feet). In clear seawater, the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

10. When interest is compounded continuously, the amount of money increases at a rate proportional to the amount S present at time t, that is, $dS/dt = rS$, where r is the annual rate of interest.

(a) Find the amount of money accrued at the end of 5 years when \$5000 is deposited in a savings account drawing $5\frac{3}{4}\%$ annual interest compounded continuously.

(b) In how many years will the initial sum deposited have doubled?

(c) Use a calculator to compare the amount obtained in part (a) with the amount $S = 5000\left(1 + \frac{1}{4}(0.0575)\right)^{5(4)}$ that is accrued when interest is compounded quarterly.

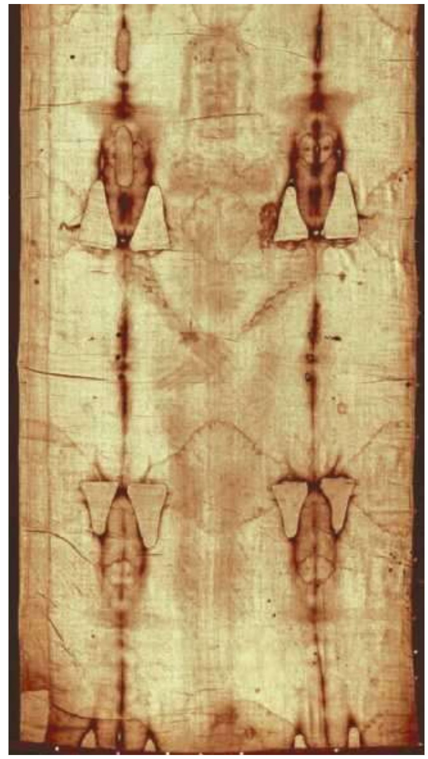
Carbon Dating

11. Archaeologists used pieces of burned wood, or charcoal, found at the site to date prehistoric paintings and drawings on walls and ceilings of a cave in Lascaux, France. See Figure 3.1.11. Use the information on page 87 to determine the approximate age of a piece of burned wood, if it was found that 85.5% of the C-14 found in living trees of the same type had decayed.

Rock painting showing a horse and a cow, c. 17000 BC (cave painting), Prehistoric/Caves of Lascaux, Dordogne, France/Bridgeman Images

FIGURE 3.1.11 Cave wall painting in Problem 11

12. The Shroud of Turin, which shows the negative image of the body of a man who appears to have been crucified, is believed by many to be the burial shroud of Jesus of Nazareth. See Figure 3.1.12. In 1988 the Vatican granted permission to have the shroud carbon-dated. Three independent scientific laboratories analyzed the cloth and concluded that the shroud was approximately 660 years old, an age consistent with its historical appearance. Using this age, determine what percentage of the original amount of C-14 remained in the cloth as of 1988.



Source: Wikipedia.org

FIGURE 3.1.12 Shroud image in Problem 12

Newton's Law of Cooling/Warming

13. A thermometer is removed from a room where the temperature is 70° F and is taken outside, where the air temperature is 10° F. After one-half minute the thermometer reads 50° F. What is the reading of the thermometer at $t = 1$ min? How long will it take for the thermometer to reach 15° F?

14. A thermometer is taken from an inside room to the outside, where the air temperature is 5° F. After 1 minute the thermometer reads 55° F, and after 5 minutes it reads 30° F. What is the initial temperature of the inside room?

15. A small metal bar, whose initial temperature was 20° C, is dropped into a large container of boiling water. How long will it take the bar to reach 90° C if it is known that its temperature increases 2° in 1 second? How long will it take the bar to reach 98° C?

16. Two large containers A and B of the same size are filled with different fluids. The fluids in containers A and B are maintained at 0° C and 100° C, respectively. A small metal bar, whose initial temperature is 100° C, is lowered into container A. After 1 minute the temperature of the bar is 90° C. After 2 minutes the bar is removed and instantly transferred to the other container. After 1 minute in container B the temperature of the bar rises 10°. How long, measured from the start of the entire process, will it take the bar to reach 99.9° C?

17. A thermometer reading 70° F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110° F after $\frac{1}{2}$ minute and 145° F after 1 minute. How hot is the oven?

18. At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is 80° F. The liquid bath has a controlled

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temperature (measured in degrees Fahrenheit) given by $T_m(t) = 100 - 40e^{-0.1t}$, $t \geq 0$, where t is measured in minutes.

(a) Assume that $k = -0.1$ in (2). Before solving the IVP, describe in words what you expect the temperature $T(t)$ of the chemical to be like in the short term. In the long term.

(b) Solve the initial-value problem. Use a graphing utility to plot the graph of $T(t)$ on time intervals of various lengths. Do the graphs agree with your predictions in part (a)?

19. A dead body was found within a closed room of a house where the temperature was a constant 70° F. At the time of discovery the core temperature of the body was determined to be 85° F. One hour later a second measurement showed that the core temperature of the body was 80° F. Assume that the time of death corresponds to $t = 0$ and that the core temperature at that time was 98.6° F. Determine how many hours elapsed before the body was found. [*Hint*: Let $t_1 > 0$ denote the time that the body was discovered.]

20. The rate at which a body cools also depends on its exposed surface area S. If S is a constant, then a modification of (2) is

$$\frac{dT}{dt} = kS(T - T_m),$$

where $k < 0$ and T_m is a constant. Suppose that two cups A and B are filled with coffee at the same time. Initially, the temperature of the coffee is 150° F. The exposed surface area of the coffee in cup B is twice the surface area of the coffee in cup A. After 30 min the temperature of the coffee in cup A is 100° F. If $T_m = 70°$ F, then what is the temperature of the coffee in cup B after 30 min?

Mixtures

21. A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.

22. Solve Problem 21 assuming that pure water is pumped into the tank.

23. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

24. In Problem 23, what is the concentration $c(t)$ of the salt in the tank at time t? At $t = 5$ min? What is the concentration of the salt in the tank after a long time, that is, as $t \to \infty$? At what time is the concentration of the salt in the tank equal to one-half this limiting value?

25. Solve Problem 23 under the assumption that the solution is pumped out at a faster rate of 10 gal/min. When is the tank empty?

26. Determine the amount of salt in the tank at time t in Example 5 if the concentration of salt in the inflow is variable and given by $c_{in}(t) = 2 + \sin(t/4)$ lb/gal. Without actually graphing, conjecture what the solution curve of the IVP should look like. Then use a graphing utility to plot the graph of the solution on the interval [0, 300]. Repeat for the interval [0, 600] and compare your graph with that in Figure 3.1.6(a).

27. A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing $\frac{1}{2}$ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

28. In Example 5 the size of the tank containing the salt mixture was not given. Suppose, as in the discussion following Example 5, that the rate at which brine is pumped into the tank is 3 gal/min but that the well-stirred solution is pumped out at a rate of 2 gal/min. It stands to reason that since brine is accumulating in the tank at the rate of 1 gal/min, any finite tank must eventually overflow. Now suppose that the tank has an open top and has a total capacity of 400 gallons.

(a) When will the tank overflow?

(b) What will be the number of pounds of salt in the tank at the instant it overflows?

(c) Assume that although the tank is overflowing, brine solution continues to be pumped in at a rate of 3 gal/min and the well-stirred solution continues to be pumped out at a rate of 2 gal/min. Devise a method for determining the number of pounds of salt in the tank at $t = 150$ minutes.

(d) Determine the number of pounds of salt in the tank as $t \to \infty$. Does your answer agree with your intuition?

(e) Use a graphing utility to plot the graph of $A(t)$ on the interval [0, 500).

Series Circuits

29. A 30-volt electromotive force is applied to an LR-series circuit in which the inductance is 0.1 henry and the resistance is 50 ohms. Find the current $i(t)$ if $i(0) = 0$. Determine the current as $t \to \infty$.

30. Solve equation (7) under the assumption that $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$.

31. A 100-volt electromotive force is applied to an RC-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current $i(t)$.

32. A 200-volt electromotive force is applied to an RC-series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4$. Determine the charge and current at $t = 0.005$ s. Determine the charge as $t \to \infty$.

33. An electromotive force

$$E(t) = \begin{cases} 120, & 0 \leq t \leq 20 \\ 0, & t > 20 \end{cases}$$

is applied to an LR-series circuit in which the inductance is 20 henries and the resistance is 2 ohms. Find the current $i(t)$ if $i(0) = 0$.

34. An LR-series circuit has a variable inductor with the inductance defined by

$$L(t) = \begin{cases} 1 - \frac{1}{10}t, & 0 \leq t < 10 \\ 0, & t > 10. \end{cases}$$

Find the current $i(t)$ if the resistance is 0.2 ohm, the impressed voltage is $E(t) = 4$, and $i(0) = 0$. Graph $i(t)$.

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Additional Linear Models

35. Air Resistance In (14) of Section 1.3 we saw that a differential equation describing the velocity v of a falling mass subject to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv,$$

where $k > 0$ is a constant of proportionality. The positive direction is downward.

(a) Solve the equation subject to the initial condition $v(0) = v_0$.

(b) Use the solution in part (a) to determine the limiting, or terminal, velocity of the mass. We saw how to determine the terminal velocity without solving the DE in Problem 40 in Exercises 2.1.

(c) If the distance s, measured from the point where the mass was released above ground, is related to velocity v by $ds/dt = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$.

36. How High?—No Air Resistance Suppose a small cannonball weighing 16 pounds is shot vertically upward, as shown in Figure 3.1.13, with an initial velocity $v_0 = 300$ ft/s. The answer to the question "How high does the cannonball go?" depends on whether we take air resistance into account.

(a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by $d^2s/dt^2 = -g$ (equation (12) of Section 1.3). Since $ds/dt = v(t)$ the last differential equation is the same as $dv/dt = -g$, where we take $g = 32$ ft/s^2. Find the velocity $v(t)$ of the cannonball at time t.

(b) Use the result obtained in part (a) to determine the height $s(t)$ of the cannonball measured from ground level. Find the maximum height attained by the cannonball.

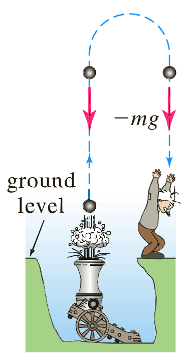

FIGURE 3.1.13 Find the maximum height of the cannonball in Problem 36

37. How High?—Linear Air Resistance Repeat Problem 36, but this time assume that air resistance is proportional to instantaneous velocity. It stands to reason that the maximum height attained by the cannonball must be *less* than that in part (b) of Problem 36. Show this by supposing that the constant of proportionality is $k = 0.0025$. [*Hint*: Slightly modify the differential equation in Problem 35.]

38. Skydiving A skydiver weighs 125 pounds, and her parachute and equipment combined weigh another 35 pounds. After exiting from a plane at an altitude of 15,000 feet, she waits 15 seconds and opens her parachute. Assume that the constant of proportionality in the model in Problem 35 has the value $k = 0.5$ during free fall and $k = 10$ after the parachute is opened. Assume that her initial velocity on leaving the plane is zero. What is her velocity and how far has she traveled 20 seconds after leaving the plane? See Figure 3.1.14. How does her velocity at 20 seconds compare with her terminal velocity? How long does it take her to reach the ground? [*Hint*: Think in terms of two distinct IVPs.]

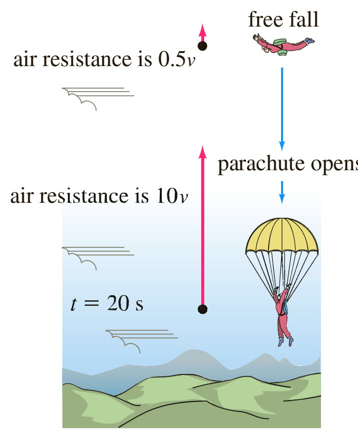

FIGURE 3.1.14 Find the time to reach the ground in Problem 38

39. Rocket Motion Suppose a small single-stage rocket of total mass $m(t)$ is launched vertically, the positive direction is upward, the air resistance is linear, and the rocket consumes its fuel at a constant rate. In Problem 22 of Exercises 1.3 you were asked to use Newton's second law of motion in the form given in (17) of that exercise set to show that a mathematical model for the velocity $v(t)$ of the rocket is given by

$$\frac{dv}{dt} + \frac{k - \lambda}{m_0 - \lambda t}v = -g + \frac{R}{m_0 - \lambda t},$$

where k is the air resistance constant of proportionality, λ is the constant rate at which fuel is consumed, R is the thrust of the rocket, $m(t) = m_0 - \lambda t$, m_0 is the total mass of the rocket at $t = 0$, and g is the acceleration due to gravity.

(a) Find the velocity $v(t)$ of the rocket if $m_0 = 200$ kg, $R = 2000$ N, $\lambda = 1$ kg/s, $g = 9.8$ m/s^2, $k = 3$ kg/s, and $v(0) = 0$.

(b) Use $ds/dt = v$ and the result in part (a) to find the height $s(t)$ of the rocket at time t.

40. Rocket Motion—Continued In Problem 39 suppose of the rocket's initial mass m_0 that 50 kg is the mass of the fuel.

(a) What is the burnout time t_b, or the time at which all the fuel is consumed?

(b) What is the velocity of the rocket at burnout?

(c) What is the height of the rocket at burnout?

(d) Why would you expect the rocket to attain an altitude higher than the number in part (b)?

(e) After burnout what is a mathematical model for the velocity of the rocket?

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41. Evaporating Raindrop As a raindrop falls, it evaporates while retaining its spherical shape. If we make the further assumptions that the rate at which the raindrop evaporates is proportional to its surface area and that air resistance is negligible, then a model for the velocity $v(t)$ of the raindrop is

$$\frac{dv}{dt} + \frac{3(k/\rho)}{(k/\rho)t + r_0}v = g.$$

Here ρ is the density of water, r_0 is the radius of the raindrop at $t = 0$, $k < 0$ is the constant of proportionality, and the downward direction is taken to be the positive direction.

(a) Solve for $v(t)$ if the raindrop falls from rest.

(b) Reread Problem 36 of Exercises 1.3 and then show that the radius of the raindrop at time t is $r(t) = (k/\rho)t + r_0$.

(c) If $r_0 = 0.01$ ft and $r = 0.007$ ft 10 seconds after the raindrop falls from a cloud, determine the time at which the raindrop has evaporated completely.

42. Fluctuating Population The differential equation $dP/dt = (k \cos t)P$, where k is a positive constant, is a mathematical model for a population $P(t)$ that undergoes yearly seasonal fluctuations. Solve the equation subject to $P(0) = P_0$. Use a graphing utility to graph the solution for different choices of P_0.

43. Population Model In one model of the changing population $P(t)$ of a community, it is assumed that

$$\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt},$$

where dB/dt and dD/dt are the birth and death rates, respectively.

(a) Solve for $P(t)$ if $dB/dt = k_1P$ and $dD/dt = k_2P$.

(b) Analyze the cases $k_1 > k_2$, $k_1 = k_2$, and $k_1 < k_2$.

44. Constant-Harvest Model A model that describes the population of a fishery in which harvesting takes place at a constant rate is given by

$$\frac{dP}{dt} = kP - h,$$

where k and h are positive constants.

(a) Solve the DE subject to $P(0) = P_0$.

(b) Describe the behavior of the population $P(t)$ for increasing time in the three cases $P_0 > h/k$, $P_0 = h/k$, and $0 < P_0 < h/k$.

(c) Use the results from part (b) to determine whether the fish population will ever go extinct in finite time, that is, whether there exists a time $T > 0$ such that $P(T) = 0$. If the population goes extinct, then find T.

45. Drug Dissemination A mathematical model for the rate at which a drug disseminates into the bloodstream is given by

$$\frac{dx}{dt} = r - kx,$$

where r and k are positive constants. The function $x(t)$ describes the concentration of the drug in the bloodstream at time t.

(a) Since the DE is autonomous, use the phase portrait concept of Section 2.1 to find the limiting value of $x(t)$ as $t \to \infty$.

(b) Solve the DE subject to $x(0) = 0$. Sketch the graph of $x(t)$ and verify your prediction in part (a). At what time is the concentration one-half this limiting value?

46. Memorization When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2A,$$

where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount memorized in time t, M is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized.

(a) Since the DE is autonomous, use the phase portrait concept of Section 2.1 to find the limiting value of $A(t)$ as $t \to \infty$. Interpret the result.

(b) Solve the DE subject to $A(0) = 0$. Sketch the graph of $A(t)$ and verify your prediction in part (a).

47. Heart Pacemaker A heart pacemaker, shown in Figure 3.1.15, consists of a switch, a battery, a capacitor, and the heart as a resistor. When the switch S is at P, the capacitor charges; when S is at Q, the capacitor discharges, sending an electrical stimulus to the heart. In Problem 58 in Exercises 2.3 we saw that during this time the electrical stimulus is being applied to the heart, the voltage E across the heart satisfies the linear DE

$$\frac{dE}{dt} = -\frac{1}{RC}E.$$

(a) Let us assume that over the time interval of length t_1, $0 < t < t_1$, the switch S is at position P shown in Figure 3.1.15 and the capacitor is being charged. When the switch is moved to position Q at time t_1 the capacitor discharges, sending an impulse to the heart over the time interval of length t_2: $t_1 \le t < t_1 + t_2$. Thus over the initial charging/discharging interval $0 < t < t_1 + t_2$ the voltage to the heart is actually modeled by the piecewise-linear differential equation

$$\frac{dE}{dt} = \begin{cases} 0, & 0 \le t < t_1 \\ -\dfrac{1}{RC}E, & t_1 \le t < t_1 + t_2. \end{cases}$$

By moving S between P and Q, the charging and discharging over time intervals of lengths t_1 and t_2 is

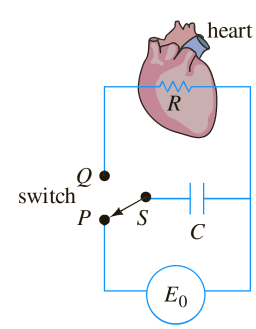

FIGURE 3.1.15 Model of a pacemaker in Problem 47

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repeated indefinitely. Suppose $t_1 = 4$ s, $t_2 = 2$ s, $E_0 = 12$ V, and $E(0) = 0$, $E(4) = 12$, $E(6) = 0$, $E(10) = 12$, $E(12) = 0$, and so on. Solve for $E(t)$ for $0 \le t \le 24$.

(b) Suppose for the sake of illustration that $R = C = 1$. Use a graphing utility to graph the solution for the IVP in part (a) for $0 \le t \le 24$.

48. Sliding Box **(a)** A box of mass m slides down an inclined plane that makes an angle θ with the horizontal as shown in Figure 3.1.16. Find a differential equation for the velocity $v(t)$ of the box at time t in each of the following three cases:

(*i*) No sliding friction and no air resistance

(*ii*) With sliding friction and no air resistance

(*iii*) With sliding friction and air resistance

In cases (*ii*) and (*iii*), use the fact that the force of friction opposing the motion of the box is μN, where μ is the coefficient of sliding friction and N is the normal component of the weight of the box. In case (*iii*) assume that air resistance is proportional to the instantaneous velocity.

(b) In part (a), suppose that the box weighs 96 pounds, that the angle of inclination of the plane is $\theta = 30°$, that the coefficient of sliding friction is $\mu = \sqrt{3}/4$, and that the additional retarding force due to air resistance is numerically equal to $\frac{1}{4}v$. Solve the differential equation in each of the three cases, assuming that the box starts from rest from the highest point 50 ft above ground.

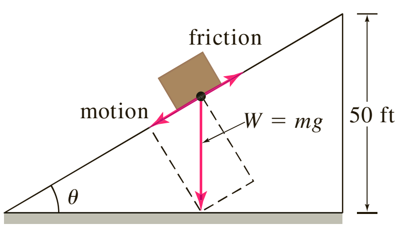

FIGURE 3.1.16 Box sliding down inclined plane in Problem 48

49. Sliding Box—Continued **(a)** In Problem 48 let $s(t)$ be the distance measured down the inclined plane from the highest point. Use $ds/dt = v(t)$ and the solution for each of the three cases in part (b) of Problem 48 to find the time that it takes the box to slide completely down the inclined plane. A root-finding application of a CAS may be useful here.

(b) In the case in which there is friction ($\mu \ne 0$) but no air resistance, explain why the box will not slide down the plane starting from *rest* from the highest point above ground when the inclination angle θ satisfies $\tan\theta \le \mu$.

(c) The box *will* slide downward on the plane when $\tan\theta \le \mu$ if it is given an initial velocity $v(0) = v_0 > 0$. Suppose that $\mu = \sqrt{3}/4$ and $\theta = 23°$. Verify that $\tan\theta \le \mu$. How far will the box slide down the plane if $v_0 = 1$ ft/s?

(d) Using the values $\mu = \sqrt{3}/4$ and $\theta = 23°$, approximate the smallest initial velocity v_0 that can be given to the box so that, starting at the highest point 50 ft above ground, it will slide completely down the inclined plane. Then find the corresponding time it takes to slide down the plane.

50. What Goes Up . . . **(a)** It is well known that the model in which air resistance is ignored, part (a) of Problem 36, predicts that the time t_a it takes the cannonball to attain its maximum height is the same as the time t_d it takes the cannonball to fall from the maximum height to the ground. Moreover, the magnitude of the impact velocity v_i will be the same as the initial velocity v_0 of the cannonball. Verify both of these results.

(b) Then, using the model in Problem 37 that takes air resistance into account, compare the value of t_a with t_d and the value of the magnitude of v_i with v_0. A root-finding application of a CAS (or graphic calculator) may be useful here.

3.2 Nonlinear Models

INTRODUCTION We finish our study of single first-order differential equations with an examination of some nonlinear models.

POPULATION DYNAMICS If $P(t)$ denotes the size of a population at time t, the model for exponential growth begins with the assumption that $dP/dt = kP$ for some $k > 0$. In this model, the **relative,** or **specific, growth rate** defined by

$$\frac{dP/dt}{P} \tag{1}$$

is a constant k. True cases of exponential growth over long periods of time are hard to find because the limited resources of the environment will at some time exert restrictions on the growth of a population. Thus for other models, (1) can be expected to decrease as the population P increases in size.

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The assumption that the rate at which a population grows (or decreases) is dependent only on the number P present and not on any time-dependent mechanisms such as seasonal phenomena (see Problem 33 in Exercises 1.3) can be stated as

$$\frac{dP/dt}{P} = f(P) \quad \text{or} \quad \frac{dP}{dt} = Pf(P). \tag{2}$$

The differential equation in (2), which is widely assumed in models of animal populations, is called the **density-dependent hypothesis.**

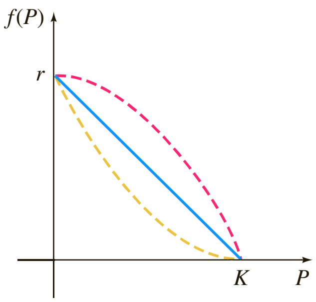

FIGURE 3.2.1 Simplest assumption for $f(P)$ is a straight line (blue color)

LOGISTIC EQUATION Suppose an environment is capable of sustaining no more than a fixed number K of individuals in its population. The quantity K is called the **carrying capacity** of the environment. Hence for the function f in (2) we have $f(K) = 0$, and we simply let $f(0) = r$. Figure 3.2.1 shows three functions f that satisfy these two conditions. The simplest assumption that we can make is that $f(P)$ is linear—that is, $f(P) = c_1P + c_2$. If we use the conditions $f(0) = r$ and $f(K) = 0$, we find, in turn, $c_2 = r$ and $c_1 = -r/K$, and so f takes on the form $f(P) = r - (r/K)P$. Equation (2) becomes

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right). \tag{3}$$

With constants relabeled, the nonlinear equation (3) is the same as

$$\frac{dP}{dt} = P(a - bP). \tag{4}$$

Around 1840 the Belgian mathematician-biologist **P. F. Verhulst** (1804–1849) was concerned with mathematical models for predicting the human populations of various countries. One of the equations he studied was (4), where $a > 0$ and $b > 0$. Equation (4) came to be known as the **logistic equation,** and its solution is called the **logistic function.** The graph of a logistic function is called a **logistic curve.**

The linear differential equation $dP/dt = kP$ does not provide a very accurate model for population when the population itself is very large. Overcrowded conditions, with the resulting detrimental effects on the environment such as pollution and excessive and competitive demands for food and fuel, can have an inhibiting effect on population growth. As we shall now see, the solution of (4) is bounded as $t \to \infty$. If we rewrite (4) as $dP/dt = aP - bP^2$, the nonlinear term $-bP^2$, $b > 0$, can be interpreted as an "inhibition" or "competition" term. Also, in most applications the positive constant a is much larger than the constant b.

Logistic curves have proved to be quite accurate in predicting the growth patterns, in a limited space, of certain types of bacteria, protozoa, water fleas (*Daphnia*), and fruit flies (*Drosophila*).

SOLUTION OF THE LOGISTIC EQUATION One method of solving (4) is separation of variables. Decomposing the left side of $dP/P(a - bP) = dt$ into partial fractions and integrating gives

$$\left(\frac{1/a}{P} + \frac{b/a}{a - bP}\right)dP = dt$$

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a - bP| = t + c$$

$$\ln\left|\frac{P}{a - bP}\right| = at + ac$$

$$\frac{P}{a - bP} = c_1e^{at}.$$

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It follows from the last equation that

$$P(t) = \frac{ac_1 e^{at}}{1 + bc_1 e^{at}} = \frac{ac_1}{bc_1 + e^{-at}}.$$

If $P(0) = P_0$, $P_0 \neq a/b$, we find $c_1 = P_0/(a - bP_0)$, and so after substituting and simplifying, the solution becomes

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}. \tag{5}$$

GRAPHS OF $P(t)$ The basic shape of the graph of the logistic function $P(t)$ can be obtained without too much effort. Although the variable t usually represents time and we are seldom concerned with applications in which $t < 0$, it is nonetheless of some interest to include this interval in displaying the various graphs of P. From (5) we see that

$$P(t) \to \frac{aP_0}{bP_0} = \frac{a}{b} \quad \text{as} \quad t \to \infty \qquad \text{and} \qquad P(t) \to 0 \quad \text{as} \quad t \to -\infty.$$

The dashed line $P = a/2b$ shown in Figure 3.2.2 corresponds to the ordinate of a point of inflection of the logistic curve. To show this, we differentiate (4) by the Product Rule:

$$\begin{aligned} \frac{d^2P}{dt^2} = P\left(-b\frac{dP}{dt}\right) + (a - bP)\frac{dP}{dt} &= \frac{dP}{dt}(a - 2bP) \\ &= P(a - bP)(a - 2bP) \\ &= 2b^2P\left(P - \frac{a}{b}\right)\left(P - \frac{a}{2b}\right). \end{aligned}$$

From calculus recall that the points where $d^2P/dt^2 = 0$ are possible points of inflection, but $P = 0$ and $P = a/b$ can obviously be ruled out. Hence $P = a/2b$ is the only possible ordinate value at which the concavity of the graph can change. For $0 < P < a/2b$ it follows that $P'' > 0$, and $a/2b < P < a/b$ implies that $P'' < 0$. Thus, as we read from left to right, the graph changes from concave up to concave down at the point corresponding to $P = a/2b$. When the initial value satisfies $0 < P_0 < a/2b$, the graph of $P(t)$ assumes the shape of an S, as we see in Figure 3.2.2(b). For $a/2b < P_0 < a/b$ the graph is still S-shaped, but the point of inflection occurs at a negative value of t, as shown in Figure 3.2.2(c).

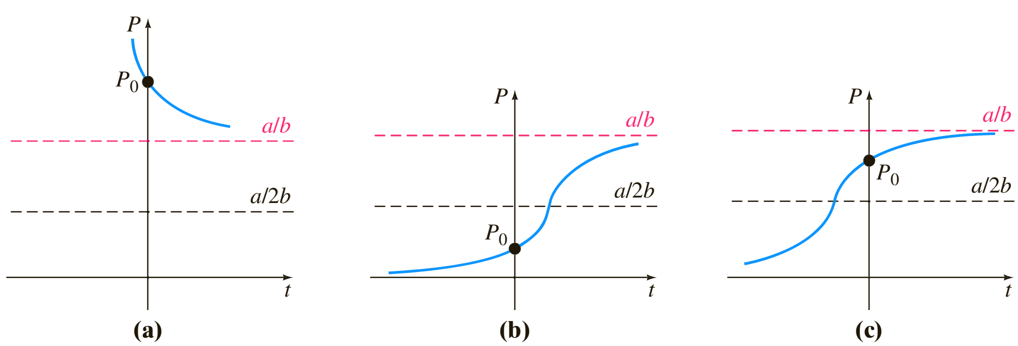

FIGURE 3.2.2 Logistic curves for different initial conditions

We have already seen equation (4) in (5) of Section 1.3 in the form $dx/dt = kx(n + 1 - x)$, $k > 0$. This differential equation provides a reasonable model for describing the spread of an epidemic brought about initially by introducing an infected individual into a static population. The solution $x(t)$ represents the number of individuals infected with the disease at time t.

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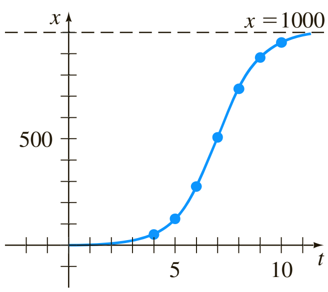
EXAMPLE 1 Logistic Growth

Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected, determine the number of infected students after 6 days if it is further observed that after 4 days $x(4) = 50$.



(a)

t (days)	x (number infected)
4	50 (observed)
5	124
6	276
7	507
8	735
9	882
10	953

(b)

FIGURE 3.2.3 Number of infected students in Example 1

SOLUTION Assuming that no one leaves the campus throughout the duration of the disease, we must solve the initial-value problem

$$\frac{dx}{dt} = kx(1000 - x), \quad x(0) = 1.$$

By making the identification $a = 1000k$ and $b = k$, we have immediately from (5) that

$$x(t) = \frac{1000k}{k + 999ke^{-1000kt}} = \frac{1000}{1 + 999e^{-1000kt}}.$$

Now, using the information $x(4) = 50$, we determine k from

$$50 = \frac{1000}{1 + 999e^{-4000k}}.$$

We find $-1000k = \frac{1}{4} \ln \frac{19}{999} = -0.9906$. Thus

$$x(t) = \frac{1000}{1 + 999e^{-0.9906t}}.$$

Finally, $$x(6) = \frac{1000}{1 + 999e^{-5.9436}} = 276 \text{ students}.$$

Additional calculated values of $x(t)$ are given in the table in Figure 3.2.3(b). Note that the number of infected students $x(t)$ approaches 1000 as t increases. ■

MODIFICATIONS OF THE LOGISTIC EQUATION There are many variations of the logistic equation. For example, the differential equations

$$\frac{dP}{dt} = P(a - bP) - h \quad \text{and} \quad \frac{dP}{dt} = P(a - bP) + h \tag{6}$$

could serve, in turn, as models for the population in a fishery where fish are **harvested** or are **restocked** at rate h. When $h > 0$ is a constant, the DEs in (6) can be readily analyzed qualitatively or solved analytically by separation of variables. The equations in (6) could also serve as models of the human population decreased by **emigration** or increased by immigration, respectively. The rate h in (6) could be a function of time t or could be population dependent; for example, harvesting might be done periodically over time or might be done at a rate proportional to the population P at time t. In the latter instance, the model would look like $P' = P(a - bP) - cP$, $c > 0$. The human population of a community might change because of **immigration** in such a manner that the contribution due to immigration was large when the population P of the community was itself small but small when P was large; a reasonable model for the population of the community would then be $P' = P(a - bP) + ce^{-kP}$, $c > 0$, $k > 0$. See Problem 24 in Exercises 3.2. Another equation of the form given in (2),

$$\frac{dP}{dt} = P(a - b \ln P), \tag{7}$$

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is a modification of the logistic equation known as the **Gompertz differential equation** named after the English mathematician **Benjamin Gompertz** (1779–1865). This DE is sometimes used as a model in the study of the growth or decline of populations, the growth of solid tumors, and certain kinds of actuarial predictions. See Problem 8 in Exercises 3.2.

CHEMICAL REACTIONS Suppose that a grams of chemical A are combined with b grams of chemical B. If there are M parts of A and N parts of B formed in the compound and $X(t)$ is the number of grams of chemical C formed, then the number of grams of chemical A and the number of grams of chemical B remaining at time t are, respectively,

$$a - \frac{M}{M+N}X \quad \text{and} \quad b - \frac{N}{M+N}X.$$

The law of mass action states that when no temperature change is involved, the rate at which the two substances react is proportional to the product of the amounts of A and B that are untransformed (remaining) at time t:

$$\frac{dX}{dt} \propto \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right). \tag{8}$$

If we factor out $M/(M+N)$ from the first factor and $N/(M+N)$ from the second and introduce a constant of proportionality $k > 0$, (8) has the form

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \tag{9}$$

where $\alpha = a(M+N)/M$ and $\beta = b(M+N)/N$. Recall from (6) of Section 1.3 that a chemical reaction governed by the nonlinear differential equation (9) is said to be a **second-order reaction.**

EXAMPLE 2 Second-Order Chemical Reaction

A compound C is formed when two chemicals A and B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. It is observed that 30 grams of the compound C is formed in 10 minutes. Determine the amount of C at time t if the rate of the reaction is proportional to the amounts of A and B remaining and if initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes? Interpret the solution as $t \to \infty$.

SOLUTION Let $X(t)$ denote the number of grams of the compound C present at time t. Clearly, $X(0) = 0$ g and $X(10) = 30$ g.

If, for example, 2 grams of compound C is present, we must have used, say, a grams of A and b grams of B, so $a + b = 2$ and $b = 4a$. Thus we must use $a = \frac{2}{5} = 2\left(\frac{1}{5}\right)$ g of chemical A and $b = \frac{8}{5} = 2\left(\frac{4}{5}\right)$ g of B. In general, for X grams of C we must use

$$\frac{1}{5}X \text{ grams of } A \quad \text{and} \quad \frac{4}{5}X \text{ grams of } B.$$

The amounts of A and B remaining at time t are then

$$50 - \frac{1}{5}X \quad \text{and} \quad 32 - \frac{4}{5}X,$$

respectively.

Now we know that the rate at which compound C is formed satisfies

$$\frac{dX}{dt} \propto \left(50 - \frac{1}{5}X\right)\left(32 - \frac{4}{5}X\right).$$

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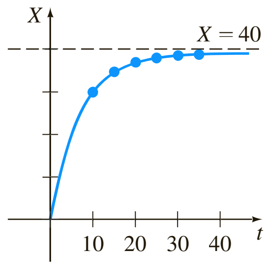

(a)

t (min)	X (g)
10	30 (measured)
15	34.78
20	37.25
25	38.54
30	39.22
35	39.59

(b)

FIGURE 3.2.4 Number of grams of compound C in Example 2

To simplify the subsequent algebra, we factor $\frac{1}{5}$ from the first term and $\frac{4}{5}$ from the second and then introduce the constant of proportionality:

$$\frac{dX}{dt} = k(250 - X)(40 - X).$$

By separation of variables and partial fractions we can write

$$-\frac{\frac{1}{210}}{250 - X}\,dX + \frac{\frac{1}{210}}{40 - X}\,dX = k\,dt.$$

Integrating gives

$$\ln\frac{250 - X}{40 - X} = 210kt + c_1 \quad \text{or} \quad \frac{250 - X}{40 - X} = c_2 e^{210kt}. \tag{10}$$

When $t = 0$, $X = 0$, so it follows at this point that $c_2 = \frac{25}{4}$. Using $X = 30$ g at $t = 10$, we find $210k = \frac{1}{10}\ln\frac{88}{25} = 0.1258$. With this information we solve the last equation in (10) for X:

$$X(t) = 1000\,\frac{1 - e^{-0.1258t}}{25 - 4e^{-0.1258t}}. \tag{11}$$

From (11) we find $X(15) = 34.78$ grams. The behavior of X as a function of time is displayed in Figure 3.2.4. It is clear from the accompanying table and (11) that $X \to 40$ as $t \to \infty$. This means that 40 grams of compound C is formed, leaving

$$50 - \frac{1}{5}(40) = 42 \text{ g of } A \quad \text{and} \quad 32 - \frac{4}{5}(40) = 0 \text{ g of } B.$$

REMARKS

The indefinite integral $\int du/(a^2 - u^2)$ can be evaluated in terms of logarithms, the inverse hyperbolic tangent, or the inverse hyperbolic cotangent. For example, of the two results

$$\int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\frac{u}{a} + c, \quad |u| < a \tag{12}$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right| + c, \quad |u| \neq a, \tag{13}$$

(12) may be convenient in Problems 15 and 26 in Exercises 3.2, whereas (13) may be preferable in Problem 27.

EXERCISES 3.2

Answers to selected odd-numbered problems begin on page ANS-3.

Logistic Equation

1. The number $N(t)$ of supermarkets throughout the country that are using a computerized checkout system is described by the initial-value problem

$$\frac{dN}{dt} = N(1 - 0.0005N), \quad N(0) = 1.$$

(a) Use the phase portrait concept of Section 2.1 to predict how many supermarkets are expected to adopt the new procedure over a long period of time. By hand, sketch a solution curve of the given initial-value problem.

(b) Solve the initial-value problem and then use a graphing utility to verify the solution curve in part (a). How many companies are expected to adopt the new technology when $t = 10$?

2. The number $N(t)$ of people in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially, $N(0) = 500$, and it is observed that $N(1) = 1000$. Solve

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for $N(t)$ if it is predicted that the limiting number of people in the community who will see the advertisement is 50,000.

3. A model for the population $P(t)$ in a suburb of a large city is given by the initial-value problem

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

where t is measured in months. What is the limiting value of the population? At what time will the population be equal to one-half of this limiting value?

4. (a) Census data for the United States between 1790 and 1950 are given in Table 3.2.1. Construct a logistic population model using the data from 1790, 1850, and 1910.

(b) Construct a table comparing actual census population with the population predicted by the model in part (a). Compute the error and the percentage error for each entry pair.

TABLE 3.2.1

Year	Population (in millions)
1790	3.929
1800	5.308
1810	7.240
1820	9.638
1830	12.866
1840	17.069
1850	23.192
1860	31.433
1870	38.558
1880	50.156
1890	62.948
1900	75.996
1910	91.972
1920	105.711
1930	122.775
1940	131.669
1950	150.697

Modifications of the Logistic Model

5. (a) If a constant number h of fish are harvested from a fishery per unit time, then a model for the population $P(t)$ of the fishery at time t is given by

$$\frac{dP}{dt} = P(a - bP) - h, \quad P(0) = P_0,$$

where a, b, h, and P_0 are positive constants. Suppose $a = 5$, $b = 1$, and $h = 4$. Since the DE is autonomous, use the phase portrait concept of Section 2.1 to sketch representative solution curves corresponding to the cases $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$. Determine the long-term behavior of the population in each case.

(b) Solve the IVP in part (a). Verify the results of your phase portrait in part (a) by using a graphing utility to plot the graph of $P(t)$ with an initial condition taken from each of the three intervals given.

(c) Use the information in parts (a) and (b) to determine whether the fishery population becomes extinct in finite time. If so, find that time.

6. Investigate the harvesting model in Problem 5 both qualitatively and analytically in the case $a = 5$, $b = 1$, $h = \frac{25}{4}$. Determine whether the population becomes extinct in finite time. If so, find that time.

7. Repeat Problem 6 in the case $a = 5$, $b = 1$, $h = 7$.

8. (a) Suppose $a = b = 1$ in the Gompertz differential equation (7). Since the DE is autonomous, use the phase portrait concept of Section 2.1 to sketch representative solution curves corresponding to the cases $P_0 > e$ and $0 < P_0 < e$.

(b) Suppose $a = 1$, $b = -1$ in (7). Use a new phase portrait to sketch representative solution curves corresponding to the cases $P_0 > e^{-1}$ and $0 < P_0 < e^{-1}$.

(c) Find an explicit solution of (7) subject to $P(0) = P_0$.

Chemical Reactions

9. Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 40 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 5 minutes. How much is formed in 20 minutes? What is the limiting amount of C after a long time? How much of chemicals A and B remains after a long time?

10. Solve Problem 9 if 100 grams of chemical A is present initially. At what time is chemical C half-formed?

Additional Nonlinear Models

11. Leaking Cylindrical Tank A tank in the form of a right-circular cylinder standing on end is leaking water through a circular hole in its bottom. As we saw in (10) of Section 1.3, when friction and contraction of water at the hole are ignored, the height h of water in the tank is described by

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh},$$

where A_w and A_h are the cross-sectional areas of the water and the hole, respectively.

(a) Solve the DE if the initial height of the water is H. By hand, sketch the graph of $h(t)$ and give its interval I of definition in terms of the symbols A_w, A_h, and H. Use $g = 32$ ft/s^2.

(b) Suppose the tank is 10 feet high and has radius 2 feet and the circular hole has radius $\frac{1}{2}$ inch. If the tank is initially full, how long will it take to empty?

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12. Leaking Cylindrical Tank—Continued When friction and contraction of the water at the hole are taken into account, the model in Problem 11 becomes

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh},$$

where $0 < c < 1$. How long will it take the tank in Problem 11(b) to empty if $c = 0.6$? See Problem 13 in Exercises 1.3.

13. Leaking Conical Tank A tank in the form of a right-circular cone standing on end, vertex down, is leaking water through a circular hole in its bottom.

(a) Suppose the tank is 20 feet high and has radius 8 feet and the circular hole has radius 2 inches. In Problem 14 in Exercises 1.3 you were asked to show that the differential equation governing the height h of water leaking from a tank is

$$\frac{dh}{dt} = -\frac{5}{6h^{3/2}}.$$

In this model, friction and contraction of the water at the hole were taken into account with $c = 0.6$, and g was taken to be 32 ft/s^2. See Figure 1.3.12. If the tank is initially full, how long will it take the tank to empty?

(b) Suppose the tank has a vertex angle of 60° and the circular hole has radius 2 inches. Determine the differential equation governing the height h of water. Use $c = 0.6$ and $g = 32$ ft/s^2. If the height of the water is initially 9 feet, how long will it take the tank to empty?

14. Inverted Conical Tank Suppose that the conical tank in Problem 13(a) is inverted, as shown in Figure 3.2.5, and that water leaks out a circular hole of radius 2 inches in the center of its circular base. Is the time it takes to empty a full tank the same as for the tank with vertex down in Problem 13? Take the friction/contraction coefficient to be $c = 0.6$ and $g = 32$ ft/s^2.

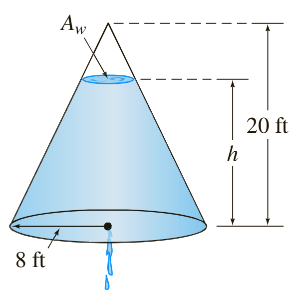

FIGURE 3.2.5 Inverted conical tank in Problem 14

15. Air Resistance A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2,$$

where $k > 0$ is a constant of proportionality. The positive direction is downward.

(a) Solve the equation subject to the initial condition $v(0) = v_0$.

(b) Use the solution in part (a) to determine the limiting, or terminal, velocity of the mass. We saw how to determine the terminal velocity without solving the DE in Problem 41 in Exercises 2.1.

(c) If the distance s, measured from the point where the mass was released above ground, is related to velocity v by $ds/dt = v(t)$, find an explicit expression for $s(t)$ if $s(0) = 0$.

16. How High?—Nonlinear Air Resistance Consider the 16-pound cannonball shot vertically upward in Problems 36 and 37 in Exercises 3.1 with an initial velocity $v_0 = 300$ ft/s. Determine the maximum height attained by the cannonball if air resistance is assumed to be proportional to the square of the instantaneous velocity. Assume that the positive direction is upward and take $k = 0.0003$. [*Hint*: Slightly modify the DE in Problem 15.]

17. That Sinking Feeling **(a)** Determine a differential equation for the velocity $v(t)$ of a mass m sinking in water that imparts a resistance proportional to the square of the instantaneous velocity and also exerts an upward buoyant force whose magnitude is given by Archimedes' principle. See Problem 18 in Exercises 1.3. Assume that the positive direction is downward.

(b) Solve the differential equation in part (a).

(c) Determine the limiting, or terminal, velocity of the sinking mass.

18. Solar Collector The differential equation

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y}$$

describes the shape of a plane curve C that will reflect all incoming light beams to the same point and could be a model for the mirror of a reflecting telescope, a satellite antenna, or a solar collector. See Problem 29 in Exercises 1.3. There are several ways of solving this DE.

(a) Verify that the differential equation is homogeneous (see Section 2.5). Show that the substitution $y = ux$ yields

$$\frac{u\,du}{\sqrt{1 + u^2}\left(1 - \sqrt{1 + u^2}\right)} = \frac{dx}{x}.$$

Use a CAS (or another judicious substitution) to integrate the left-hand side of the equation. Show that the curve C must be a parabola with focus at the origin and is symmetric with respect to the x-axis.

(b) Show that the first differential equation can also be solved by means of the substitution $u = x^2 + y^2$.

19. Tsunami **(a)** A simple model for the shape of a tsunami is given by

$$\frac{dW}{dx} = W\sqrt{4 - 2W},$$

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where $W(x) > 0$ is the height of the wave expressed as a function of its position relative to a point offshore. By inspection, find all constant solutions of the DE.

(b) Solve the differential equation in part (a). A CAS may be useful for integration.

(c) Use a graphing utility to obtain the graphs of all solutions that satisfy the initial condition $W(0) = 2$.

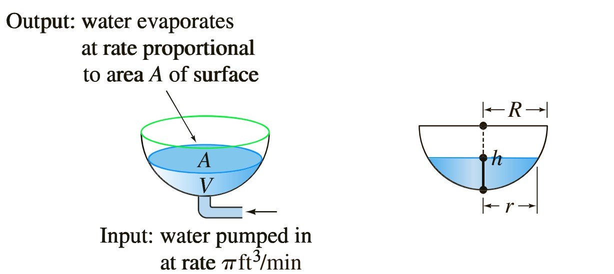
20. Evaporation An outdoor decorative pond in the shape of a hemispherical tank is to be filled with water pumped into the tank through an inlet in its bottom. Suppose that the radius of the tank is $R = 10$ ft, that water is pumped in at a rate of π ft^3/min, and that the tank is initially empty. See Figure 3.2.6. As the tank fills, it loses water through evaporation. Assume that the rate of evaporation is proportional to the area A of the surface of the water and that the constant of proportionality is $k = 0.01$.

(a) The rate of change dV/dt of the volume of the water at time t is a net rate. Use this net rate to determine a differential equation for the height h of the water at time t. The volume of the water shown in the figure is $V = \pi Rh^2 - \frac{1}{3}\pi h^3$, where $R = 10$. Express the area of the surface of the water $A = \pi r^2$ in terms of h.

(b) Solve the differential equation in part (a). Graph the solution.

(c) If there were no evaporation, how long would it take the tank to fill?

(d) With evaporation, what is the depth of the water at the time found in part (c)? Will the tank ever be filled? Prove your assertion.



(a) hemispherical tank **(b)** cross-section of tank

FIGURE 3.2.6 Decorative pond in Problem 20

21. Doomsday Equation Consider the differential equation

$$\frac{dP}{dt} = kP^{1+c},$$

where $k > 0$ and $c \geq 0$. In Section 3.1 we saw that in the case $c = 0$ the linear differential equation $dP/dt = kP$ is a mathematical model of a population $P(t)$ that exhibits unbounded growth over the infinite time interval $[0, \infty)$, that is, $P(t) \to \infty$ as $t \to \infty$. See Example 1 on page 85.

(a) Suppose for $c = 0.01$ that the nonlinear differential equation

$$\frac{dP}{dt} = kP^{1.01}, \quad k > 0,$$

is a mathematical model for a population of small animals, where time t is measured in months. Solve the differential equation subject to the initial condition $P(0) = 10$ and the fact that the animal population has doubled in 5 months.

(b) The differential equation in part (a) is called a **doomsday equation** because the population $P(t)$ exhibits unbounded growth over a finite time interval $(0, T)$, that is, there is some time T such that $P(t) \to \infty$ as $t \to T^-$. Find T.

(c) From part (a), what is $P(50)$? $P(100)$?

22. Doomsday or Extinction Suppose the population model (4) is modified to be

$$\frac{dP}{dt} = P(bP - a).$$

(a) If $a > 0$, $b > 0$ show by means of a phase portrait (see page 40) that, depending on the initial condition $P(0) = P_0$, the mathematical model could include a doomsday scenario ($P(t) \to \infty$) or an extinction scenario ($P(t) \to 0$).

(b) Solve the initial-value problem

$$\frac{dP}{dt} = P(0.0005P - 0.1), \; P(0) = 300.$$

Show that this model predicts a doomsday for the population in a finite time T.

(c) Solve the differential equation in part (b) subject to the initial condition $P(0) = 100$. Show that this model predicts extinction for the population as $t \to \infty$.

Project Problems

23. Regression Line Read the documentation for your CAS on *scatter plots* (or *scatter diagrams*) and *least-squares linear fit*. The straight line that best fits a set of data points is called a **regression line** or a **least squares line.** Your task is to construct a logistic model for the population of the United States, defining $f(P)$ in (2) as an equation of a regression line based on the population data in the table in Problem 4. One way of doing this is to approximate the left-hand side $\dfrac{1}{P}\dfrac{dP}{dt}$ of the first equation in (2), using the forward difference quotient in place of dP/dt:

$$Q(t) = \frac{1}{P(t)}\frac{P(t+h) - P(t)}{h}.$$

(a) Make a table of the values t, $P(t)$, and $Q(t)$ using $t = 0, 10, 20, \ldots, 160$ and $h = 10$. For example, the first line of the table should contain $t = 0$, $P(0)$, and $Q(0)$. With $P(0) = 3.929$ and $P(10) = 5.308$,

$$Q(0) = \frac{1}{P(0)}\frac{P(10) - P(0)}{10} = 0.035.$$

Note that $Q(160)$ depends on the 1960 census population $P(170)$. Look up this value.

(b) Use a CAS to obtain a scatter plot of the data $(P(t), Q(t))$ computed in part (a). Also use a CAS to find an equation of the regression line and to superimpose its graph on the scatter plot.

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(c) Construct a logistic model $dP/dt = Pf(P)$, where $f(P)$ is the equation of the regression line found in part (b).

(d) Solve the model in part (c) using the initial condition $P(0) = 3.929$.

(e) Use a CAS to obtain another scatter plot, this time of the ordered pairs $(t, P(t))$ from your table in part (a). Use your CAS to superimpose the graph of the solution in part (d) on the scatter plot.

(f) Look up the U.S. census data for 1970, 1980, and 1990. What population does the logistic model in part (c) predict for these years? What does the model predict for the U.S. population $P(t)$ as $t \to \infty$?

24. Immigration Model **(a)** In Examples 3 and 4 of Section 2.1 we saw that any solution $P(t)$ of (4) possesses the asymptotic behavior $P(t) \to a/b$ as $t \to \infty$ for $P_0 > a/b$ and for $0 < P_0 < a/b$; as a consequence the equilibrium solution $P = a/b$ is called an attractor. Use a root-finding application of a CAS (or a graphic calculator) to approximate the equilibrium solution of the immigration model

$$\frac{dP}{dt} = P(1 - P) + 0.3e^{-P}.$$

(b) Use a graphing utility to graph the function $F(P) = P(1 - P) + 0.3e^{-P}$. Explain how this graph can be used to determine whether the number found in part (a) is an attractor.

(c) Use a numerical solver to compare the solution curves for the IVPs

$$\frac{dP}{dt} = P(1 - P), \quad P(0) = P_0$$

for $P_0 = 0.2$ and $P_0 = 1.2$ with the solution curves for the IVPs

$$\frac{dP}{dt} = P(1 - P) + 0.3e^{-P}, \quad P(0) = P_0$$

for $P_0 = 0.2$ and $P_0 = 1.2$. Superimpose all curves on the same coordinate axes but, if possible, use a different color for the curves of the second initial-value problem. Over a long period of time, what percentage increase does the immigration model predict in the population compared to the logistic model?

25. What Goes Up . . . In Problem 16 let t_a be the time it takes the cannonball to attain its maximum height and let t_d be the time it takes the cannonball to fall from the maximum height to the ground. Compare the value of t_a with the value of t_d and compare the magnitude of the impact velocity v_i with the initial velocity v_0. See Problem 50 in Exercises 3.1. A root-finding application of a CAS might be useful here. [*Hint*: Use the model in Problem 15 when the cannonball is falling.]

26. Skydiving A skydiver is equipped with a stopwatch and an altimeter. As shown in Figure 3.2.7, he opens his parachute 25 seconds after exiting a plane flying at an altitude of 20,000 feet and observes that his altitude is 14,800 feet. Assume that air resistance is proportional to the square of the instantaneous velocity, his initial velocity on leaving the plane is zero, and $g = 32$ ft/s^2.

(a) Find the distance $s(t)$, measured from the plane, the skydiver has traveled during freefall in time t. [*Hint*: The constant of proportionality k in the model given in Problem 15 is not specified. Use the expression for terminal velocity v_t obtained in part (b) of Problem 15 to eliminate k from the IVP. Then eventually solve for v_t.]

(b) How far does the skydiver fall and what is his velocity at $t = 15$ s?

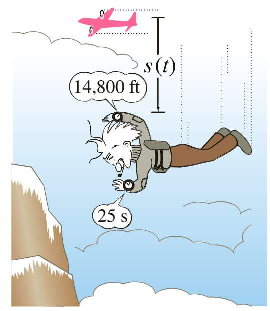

FIGURE 3.2.7 Skydiver in Problem 26

27. Hitting Bottom A helicopter hovers 500 feet above a large open tank full of liquid (not water). A dense compact object weighing 160 pounds is dropped (released from rest) from the helicopter into the liquid. Assume that air resistance is proportional to instantaneous velocity v while the object is in the air and that viscous damping is proportional to v^2 after the object has entered the liquid. For air take $k = \frac{1}{4}$, and for the liquid take $k = 0.1$. Assume that the positive direction is downward. If the tank is 75 feet high, determine the time and the impact velocity when the object hits the bottom of the tank. [*Hint*: Think in terms of two distinct IVPs. If you use (13), be careful in removing the absolute value sign. You might compare the velocity when the object hits the liquid—the initial velocity for the second problem—with the terminal velocity v_t of the object falling through the liquid.]

28. Old Man River . . . In Figure 3.2.8(a) suppose that the y-axis and the dashed vertical line $x = 1$ represent, respectively, the straight west and east beaches of a river that is 1 mile wide. The river flows northward with a velocity $\mathbf{v}_r$, where $|\mathbf{v}_r| = v_r$ mi/h is a constant. A man enters the current at the point (1, 0) on the east shore and swims in a direction and rate relative to the river given by the vector $\mathbf{v}_s$, where the speed $|\mathbf{v}_s| = v_s$ mi/h is a constant. The man wants to reach the west beach exactly at (0, 0) and so swims in such a manner that keeps his velocity vector $\mathbf{v}_s$ always directed toward the point (0, 0). Use Figure 3.2.8(b) as an aid in showing that a mathematical model for the path of the swimmer in the river is

$$\frac{dy}{dx} = \frac{v_s y - v_r\sqrt{x^2 + y^2}}{v_s x}.$$

[*Hint*: The velocity $\mathbf{v}$ of the swimmer along the path or curve shown in Figure 3.2.8 is the resultant $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r$. Resolve $\mathbf{v}_s$ and $\mathbf{v}_r$ into components in the x- and y-directions. If $x = x(t)$, $y = y(t)$ are parametric equations of the swimmer's path, then $\mathbf{v} = (dx/dt, dy/dt)$.]

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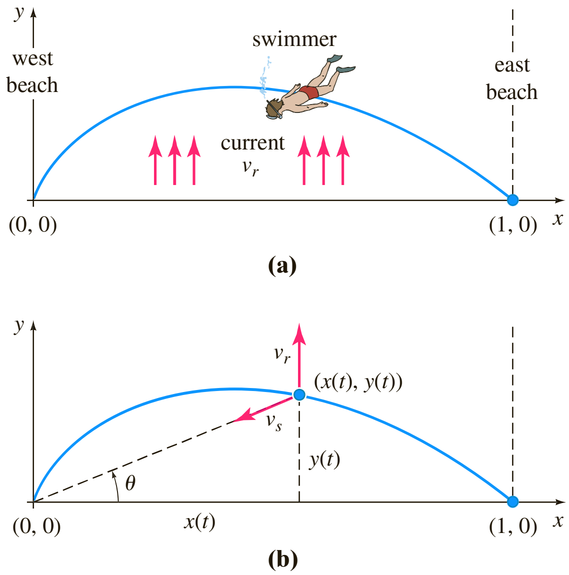

FIGURE 3.2.8 Path of swimmer in Problem 28

29. **(a)** Solve the DE in Problem 28 subject to $y(1) = 0$. For convenience let $k = v_r/v_s$.

(b) Determine the values of v_s for which the swimmer will reach the point (0, 0) by examining $\lim_{x\to 0^+} y(x)$ in the cases $k = 1$, $k > 1$, and $0 < k < 1$.

30. **Old Man River Keeps Moving . . .** Suppose the man in Problem 28 again enters the current at (1, 0) but this time decides to swim so that his velocity vector $\mathbf{v}_s$ is always directed toward the west beach. Assume that the speed $|\mathbf{v}_s| = v_s$ mi/h is a constant. Show that a mathematical model for the path of the swimmer in the river is now

$$\frac{dy}{dx} = -\frac{v_r}{v_s}.$$

31. The current speed v_r of a straight river such as that in Problem 28 is usually not a constant. Rather, an approximation to the current speed (measured in miles per hour) could be a function such as $v_r(x) = 30x(1 - x)$, $0 \le x \le 1$, whose values are small at the shores (in this case, $v_r(0) = 0$ and $v_r(1) = 0$) and largest in the middle of the river. Solve the DE in Problem 30 subject to $y(1) = 0$, where $v_s = 2$ mi/h and $v_r(x)$ is as given. When the swimmer makes it across the river, how far will he have to walk along the beach to reach the point (0, 0)?

32. **Raindrops Keep Falling . . .** When a bottle of liquid refreshment was opened recently, the following factoid was found inside the bottle cap:

The average velocity of a falling raindrop is 7 miles/hour.

A quick search of the Internet found that meteorologist Jeff Haby offers the additional information that an "average" spherical raindrop has a radius of 0.04 in. and an approximate volume of 0.000000155 ft^3. Use this data and, if need be, dig up other data and make other reasonable assumptions to determine whether *"average velocity of . . . 7 mi/h"* is consistent with the models in Problems 35 and 36 in Exercises 3.1 and Problem 15 in this exercise set. Also see Problem 36 in Exercises 1.3.

33. **Time Drips By** The **clepsydra,** or water clock, was a device that the ancient Egyptians, Greeks, Romans, and Chinese used to measure the passage of time by observing the change in the height of water that was permitted to flow out of a small hole in the bottom of a container or tank.

(a) Suppose a tank is made of glass and has the shape of a right-circular cylinder of radius 1 ft. Assume that $h(0) = 2$ ft corresponds to water filled to the top of the tank, a hole in the bottom is circular with radius $\frac{1}{32}$ in., $g = 32$ ft/s^2, and $c = 0.6$. Use the differential equation in Problem 12 to find the height $h(t)$ of the water.

(b) For the tank in part (a), how far up from its bottom should a mark be made on its side, as shown in Figure 3.2.9, that corresponds to the passage of one hour? Next determine where to place the marks corresponding to the passage of 2 hr, 3 hr, . . . , 12 hr. Explain why these marks are not evenly spaced.

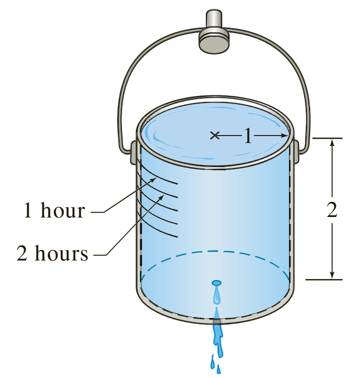

FIGURE 3.2.9 Clepsydra in Problem 33

34. **(a)** Suppose that a glass tank has the shape of a cone with circular cross section as shown in Figure 3.2.10. As in part (a) of Problem 33, assume that $h(0) = 2$ ft corresponds to water filled to the top of the tank, a hole in the bottom is circular with radius $\frac{1}{32}$ in., $g = 32$ ft/s^2, and $c = 0.6$. Use the differential equation in Problem 12 to find the height $h(t)$ of the water.

(b) Can this water clock measure 12 time intervals of length equal to 1 hour? Explain using sound mathematics.

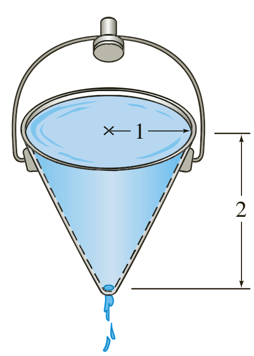

FIGURE 3.2.10 Clepsydra in Problem 34

35. Suppose that $r = f(h)$ defines the shape of a water clock for which the time marks are equally spaced. Use the differential equation in Problem 12 to find $f(h)$ and sketch a typical graph of h as a function of r. Assume that the cross-sectional area A_h of the hole is constant. [*Hint*: In this situation $dh/dt = -a$, where $a > 0$ is a constant.]

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3.3 Modeling with Systems of First-Order DEs

INTRODUCTION This section is similar to Section 1.3 in that we are just going to discuss certain mathematical models, but instead of a single differential equation the models will be systems of first-order differential equations. Although some of the models will be based on topics that we explored in the preceding two sections, we are not going to develop any general methods for solving these systems. There are reasons for this: First, we do not possess the necessary mathematical tools for solving systems at this point. Second, some of the systems that we discuss—notably the systems of *nonlinear* first-order DEs—simply cannot be solved analytically. We shall examine solution methods for systems of *linear* DEs in Chapters 4, 7, and 8.

LINEAR/NONLINEAR SYSTEMS We have seen that a single differential equation can serve as a mathematical model for a single population in an environment. But if there are, say, two interacting and perhaps competing species living in the same environment (for example, rabbits and foxes), then a model for their populations $x(t)$ and $y(t)$ might be a system of two first-order differential equations such as

$$\begin{aligned}\frac{dx}{dt} &= g_1(t, x, y)\\ \frac{dy}{dt} &= g_2(t, x, y).\end{aligned} \tag{1}$$

When g_1 and g_2 are linear in the variables x and y—that is, g_1 and g_2 have the forms

$$g_1(t, x, y) = c_1x + c_2y + f_1(t) \qquad \text{and} \qquad g_2(t, x, y) = c_3x + c_4y + f_2(t),$$

where the coefficients c_i could depend on t—then (1) is said to be a **linear system.** A system of differential equations that is not linear is said to be **nonlinear.**

RADIOACTIVE SERIES In the discussion of radioactive decay in Sections 1.3 and 3.1 we assumed that the rate of decay was proportional to the number $A(t)$ of nuclei of the substance present at time t. When a substance decays by radioactivity, it usually doesn't just transmute in one step into a stable substance; rather, the first substance decays into another radioactive substance, which in turn decays into a third substance, and so on. This process, called a **radioactive decay series,** continues until a stable element is reached. For example, the uranium decay series is U-238 $\rightarrow$ Th-234 $\rightarrow \cdots \rightarrow$ Pb-206, where Pb-206 is a stable isotope of lead. The half-lives of the various elements in a radioactive series can range from billions of years (4.5×10^9 years for U-238) to a fraction of a second. Suppose a radioactive series is described schematically by $X \xrightarrow{-\lambda_1} Y \xrightarrow{-\lambda_2} Z$, where $k_1 = -\lambda_1 < 0$ and $k_2 = -\lambda_2 < 0$ are the decay constants for substances X and Y, respectively, and Z is a stable element. Suppose, too, that $x(t)$, $y(t)$, and $z(t)$ denote amounts of substances X, Y, and Z, respectively, remaining at time t. The decay of element X is described by

$$\frac{dx}{dt} = -\lambda_1 x,$$

whereas the rate at which the second element Y decays is the net rate

$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y,$$

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since Y is *gaining* atoms from the decay of X and at the same time *losing* atoms because of its own decay. Since Z is a stable element, it is simply gaining atoms from the decay of element Y:

$$\frac{dz}{dt} = \lambda_2 y.$$

In other words, a model of the radioactive decay series for three elements is the linear system of three first-order differential equations

$$\begin{aligned} \frac{dx}{dt} &= -\lambda_1 x \\ \frac{dy}{dt} &= \lambda_1 x - \lambda_2 y \\ \frac{dz}{dt} &= \lambda_2 y. \end{aligned} \qquad (2)$$

MIXTURES Consider the two tanks shown in Figure 3.3.1. Let us suppose for the sake of discussion that tank A contains 50 gallons of water in which 25 pounds of salt is dissolved. Suppose tank B contains 50 gallons of pure water. Liquid is pumped into and out of the tanks as indicated in the figure; the mixture exchanged between the two tanks and the liquid pumped out of tank B are assumed to be well stirred. We wish to construct a mathematical model that describes the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t.

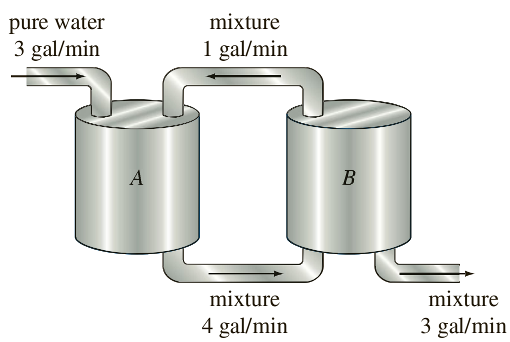

FIGURE 3.3.1 Connected mixing tanks

By an analysis similar to that on page 25 in Section 1.3 and Example 5 of Section 3.1 we see that the net rate of change of $x_1(t)$ for tank A is

$$\begin{aligned} \frac{dx_1}{dt} &= \overbrace{(3\text{ gal/min})\cdot(0\text{ lb/gal}) + (1\text{ gal/min})\cdot\left(\frac{x_2}{50}\text{ lb/gal}\right)}^{\text{input rate of salt}} - \overbrace{(4\text{ gal/min})\cdot\left(\frac{x_1}{50}\text{ lb/gal}\right)}^{\text{output rate of salt}} \\ &= -\frac{2}{25}x_1 + \frac{1}{50}x_2. \end{aligned}$$

Similarly, for tank B the net rate of change of $x_2(t)$ is

$$\begin{aligned} \frac{dx_2}{dt} &= 4\cdot\frac{x_1}{50} - 3\cdot\frac{x_2}{50} - 1\cdot\frac{x_2}{50} \\ &= \frac{2}{25}x_1 - \frac{2}{25}x_2. \end{aligned}$$

Thus we obtain the linear system

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{2}{25}x_1 + \frac{1}{50}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{25}x_1 - \frac{2}{25}x_2. \end{aligned} \qquad (3)$$

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Observe that the foregoing system is accompanied by the initial conditions $x_1(0) = 25$, $x_2(0) = 0$.

A PREDATOR-PREY MODEL Suppose that two different species of animals interact within the same environment or ecosystem, and suppose further that the first species eats only vegetation and the second eats only the first species. In other words, one species is a predator, and the other is a prey. For example, wolves hunt grass-eating caribou, sharks devour little fish, and the snowy owl pursues an arctic rodent called the lemming. For the sake of discussion, let us imagine that the predators are foxes and the prey are rabbits.

Let $x(t)$ and $y(t)$ denote the fox and rabbit populations, respectively, at time t. If there were no rabbits, then one might expect that the foxes, lacking an adequate food supply, would decline in number according to

$$\frac{dx}{dt} = -ax, \qquad a > 0. \tag{4}$$

When rabbits are present in the environment, however, it seems reasonable that the number of encounters or interactions between these two species per unit time is jointly proportional to their populations x and y—that is, proportional to the product xy. Thus when rabbits are present, there is a supply of food, so foxes are added to the system at a rate bxy, $b > 0$. Adding this last rate to (4) gives a model for the fox population:

$$\frac{dx}{dt} = -ax + bxy. \tag{5}$$

On the other hand, if there were no foxes, then the rabbits would, with an added assumption of unlimited food supply, grow at a rate that is proportional to the number of rabbits present at time t:

$$\frac{dy}{dt} = dy, \qquad d > 0. \tag{6}$$

But when foxes are present, a model for the rabbit population is (6) decreased by cxy, $c > 0$—that is, decreased by the rate at which the rabbits are eaten during their encounters with the foxes:

$$\frac{dy}{dt} = dy - cxy. \tag{7}$$

Equations (5) and (7) constitute a system of nonlinear differential equations

$$\begin{aligned} \frac{dx}{dt} &= -ax + bxy = x(-a + by) \\ \frac{dy}{dt} &= dy - cxy = y(d - cx), \end{aligned} \tag{8}$$

where a, b, c, and d are positive constants. This famous system of equations is known as the **Lotka-Volterra predator-prey model.**

Except for two constant solutions, $x(t) = 0$, $y(t) = 0$ and $x(t) = d/c$, $y(t) = a/b$, the nonlinear system (8) cannot be solved in terms of elementary functions. However, we can analyze such systems quantitatively and qualitatively. See Chapter 9, "Numerical Solutions of Ordinary Differential Equations," and Chapter 10, "Systems of Nonlinear First-Order Differential Equations."*

EXAMPLE 1 Predator-Prey Model

Suppose

$$\begin{aligned} \frac{dx}{dt} &= -0.16x + 0.08xy \\ \frac{dy}{dt} &= 4.5y - 0.9xy \end{aligned}$$

*Chapters 10–15 are in the expanded version of this text, *Differential Equations with Boundary-Value Problems*.

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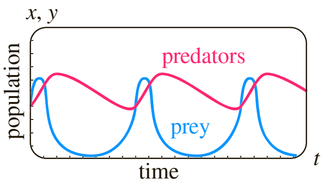

FIGURE 3.3.2 Populations of predators (red) and prey (blue) in Example 1

represents a predator-prey model. Because we are dealing with populations, we have $x(t) \geq 0$, $y(t) \geq 0$. Figure 3.3.2, obtained with the aid of a numerical solver, shows typical population curves of the predators and prey for this model superimposed on the same coordinate axes. The initial conditions used were $x(0) = 4$, $y(0) = 4$. The curve in red represents the population $x(t)$ of the predators (foxes), and the blue curve is the population $y(t)$ of the prey (rabbits). Observe that the model seems to predict that both populations $x(t)$ and $y(t)$ are periodic in time. This makes intuitive sense because as the number of prey decreases, the predator population eventually decreases because of a diminished food supply; but attendant to a decrease in the number of predators is an increase in the number of prey; this in turn gives rise to an increased number of predators, which ultimately brings about another decrease in the number of prey. ■

COMPETITION MODELS Now suppose two different species of animals occupy the same ecosystem, not as predator and prey but rather as competitors for the same resources (such as food and living space) in the system. In the absence of the other, let us assume that the rate at which each population grows is given by

$$\frac{dx}{dt} = ax \qquad \text{and} \qquad \frac{dy}{dt} = cy, \tag{9}$$

respectively.

Since the two species compete, another assumption might be that each of these rates is diminished simply by the influence, or existence, of the other population. Thus a model for the two populations is given by the linear system

$$\begin{aligned} \frac{dx}{dt} &= ax - by \\ \frac{dy}{dt} &= cy - dx, \end{aligned} \tag{10}$$

where a, b, c, and d are positive constants.

On the other hand, we might assume, as we did in (5), that each growth rate in (9) should be reduced by a rate proportional to the number of interactions between the two species:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= cy - dxy. \end{aligned} \tag{11}$$

Inspection shows that this nonlinear system is similar to the Lotka-Volterra predator-prey model. Finally, it might be more realistic to replace the rates in (9), which indicate that the population of each species in isolation grows exponentially, with rates indicating that each population grows logistically (that is, over a long time the population is bounded):

$$\frac{dx}{dt} = a_1x - b_1x^2 \qquad \text{and} \qquad \frac{dy}{dt} = a_2y - b_2y^2. \tag{12}$$

When these new rates are decreased by rates proportional to the number of interactions, we obtain another nonlinear model:

$$\begin{aligned} \frac{dx}{dt} &= a_1x - b_1x^2 - c_1xy = x(a_1 - b_1x - c_1y) \\ \frac{dy}{dt} &= a_2y - b_2y^2 - c_2xy = y(a_2 - b_2y - c_2x), \end{aligned} \tag{13}$$

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where all coefficients are positive. The linear system (10) and the nonlinear systems (11) and (13) are, of course, called **competition models.**

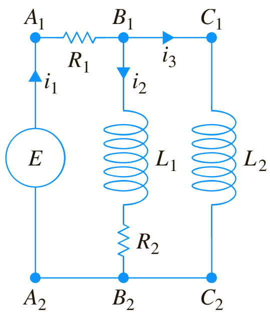

FIGURE 3.3.3 Network whose model is given in (17)

NETWORKS An electrical network having more than one loop also gives rise to simultaneous differential equations. As shown in Figure 3.3.3, the current $i_1(t)$ splits in the directions shown at point B_1, called a *branch point* of the network. By **Kirchhoff's first law** we can write

$$i_1(t) = i_2(t) + i_3(t). \tag{14}$$

We can also apply **Kirchhoff's second law** to each loop. For loop $A_1B_1B_2A_2A_1$, summing the voltage drops across each part of the loop gives

$$E(t) = i_1R_1 + L_1\frac{di_2}{dt} + i_2R_2. \tag{15}$$

Similarly, for loop $A_1B_1C_1C_2B_2A_2A_1$ we find

$$E(t) = i_1R_1 + L_2\frac{di_3}{dt}. \tag{16}$$

Using (14) to eliminate i_1 in (15) and (16) yields two linear first-order equations for the currents $i_2(t)$ and $i_3(t)$:

$$\begin{aligned} L_1\frac{di_2}{dt} + (R_1 + R_2)i_2 + R_1i_3 &= E(t) \\ L_2\frac{di_3}{dt} + \qquad\qquad R_1i_2 + R_1i_3 &= E(t). \end{aligned} \tag{17}$$

We leave it as an exercise (see Problem 16 in Exercises 3.3) to show that the system of differential equations describing the currents $i_1(t)$ and $i_2(t)$ in the network containing a resistor, an inductor, and a capacitor shown in Figure 3.3.4 is

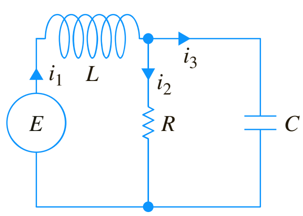

FIGURE 3.3.4 Network whose model is given in (18)

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 \qquad &= E(t) \\ RC\frac{di_2}{dt} + i_2 - i_1 &= 0. \end{aligned} \tag{18}$$

EXERCISES 3.3

Answers to selected odd-numbered problems begin on page ANS-4.

Radioactive Series

1. We have not discussed methods by which systems of first-order differential equations can be solved. Nevertheless, systems such as (2) can be solved with no knowledge other than how to solve a single linear first-order equation. Find a solution of (2) subject to the initial conditions $x(0) = x_0$, $y(0) = 0$, $z(0) = 0$.

2. In Problem 1 suppose that time is measured in days, that the decay constants are $k_1 = -0.138629$ and $k_2 = -0.004951$, and that $x_0 = 20$. Use a graphing utility to obtain the graphs of the solutions $x(t)$, $y(t)$, and $z(t)$ on the same set of coordinate axes. Use the graphs to approximate the half-lives of substances X and Y.

3. Use the graphs in Problem 2 to approximate the times when the amounts $x(t)$ and $y(t)$ are the same, the times when the amounts $x(t)$ and $z(t)$ are the same, and the times when the amounts $y(t)$ and $z(t)$ are the same. Why does the time that is determined when the amounts $y(t)$ and $z(t)$ are the same make intuitive sense?

4. Construct a mathematical model for a radioactive series of four elements W, X, Y, and Z, where Z is a stable element.

5. **Potassium-40 Decay** The chemical element potassium is a soft metal that can be found extensively throughout the Earth's crust and oceans. Although potassium occurs naturally in the form of three isotopes, only the isotope potassium-40 (K-40) is radioactive. This isotope is also unusual in that it decays by two different nuclear reactions. Over time, by emitting beta particles a great percentage of an initial amount K_0 of K-40 decays into the stable isotope calcium-40 (Ca-40), whereas by electron capture a smaller percentage of K_0 decays into the stable isotope argon-40 (Ar-40). Because the rates at which the amounts $C(t)$ of Ca-40 and $A(t)$ of Ar-40 increase are proportional to the amount $K(t)$ of potassium present, and the rate at which $K(t)$ decays is also proportional to $K(t)$, we obtain the system of linear first-order equations

$$\begin{aligned} \frac{dC}{dt} &= \lambda_1K \\ \frac{dA}{dt} &= \lambda_2K \\ \frac{dK}{dt} &= -(\lambda_1 + \lambda_2)K, \end{aligned}$$

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where λ_1 and λ_2 are positive constants of proportionality. By proceeding as in Problem 1 we can solve the foregoing mathematical model.

(a) From the last equation in the given system of differential equations find $K(t)$ if $K(0) = K_0$. Then use $K(t)$ to find $C(t)$ and $A(t)$ from the first and second equations. Assume that $C(0) = 0$ and $A(0) = 0$.

(b) It is known that $\lambda_1 = 4.7526 \times 10^{-10}$ and $\lambda_2 = 0.5874 \times 10^{-10}$. Find the half-life of K-40.

(c) Use $C(t)$ and $A(t)$ found in part (a) to determine the percentage of an initial amount K_0 of K-40 that decays into Ca-40 and the percentage that decays into Ar-40 over a very long period of time.

6. Potassium-Argon Dating The knowledge of how K-40 decays can be used to determine the age of very old igneous rocks. See Figure 3.3.5.

(a) Use the solutions obtained in part (a) of Problem 5 to find the ratio $A(t)/K(t)$.

(b) Use $A(t)/K(t)$ found in part (a) to show that

$$t = \frac{1}{\lambda_1 + \lambda_2} \ln\left[1 + \frac{\lambda_1 + \lambda_2}{\lambda_2} \frac{A(t)}{K(t)}\right].$$

(c) Suppose it is determined that each gram of an igneous rock sample contains 8.5×10^{-7} grams of Ar-40 and 5.4×10^{-6} grams of K-40. Use the result in part (b) to find the approximate age of the rock.

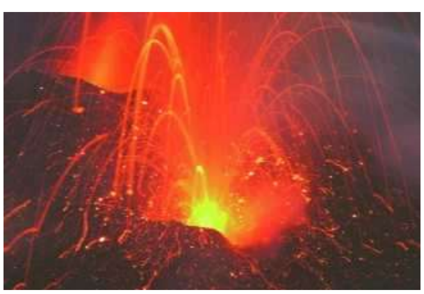

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FIGURE 3.3.5 Igneous rocks are formed through solidification of volcanic lava

Mixtures

7. Consider two tanks A and B, with liquid being pumped in and out at the same rates, as described by the system of equations (3). What is the system of differential equations if, instead of pure water, a brine solution containing 2 pounds of salt per gallon is pumped into tank A?

8. Use the information given in Figure 3.3.6 to construct a mathematical model for the number of pounds of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ at time t in tanks A, B, and C, respectively.

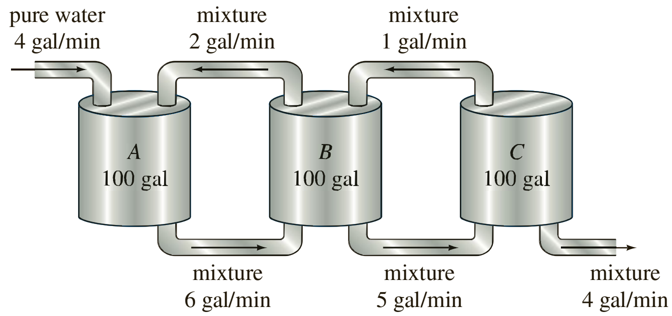

FIGURE 3.3.6 Mixing tanks in Problem 8

9. Two very large tanks A and B are each partially filled with 100 gallons of brine. Initially, 100 pounds of salt is dissolved in the solution in tank A and 50 pounds of salt is dissolved in the solution in tank B. The system is closed in that the well-stirred liquid is pumped only between the tanks, as shown in Figure 3.3.7.

(a) Use the information given in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$ and $x_2(t)$ at time t in tanks A and B, respectively.

(b) Find a relationship between the variables $x_1(t)$ and $x_2(t)$ that holds at time t. Explain why this relationship makes intuitive sense. Use this relationship to help find the amount of salt in tank B at $t = 30$ min.

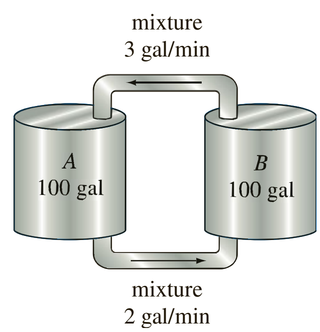

FIGURE 3.3.7 Mixing tanks in Problem 9

10. Three large tanks contain brine, as shown in Figure 3.3.8. Use the information in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ at time t in tanks A, B, and C, respectively. Without solving the system, predict limiting values of $x_1(t)$, $x_2(t)$, and $x_3(t)$ as $t \to \infty$.

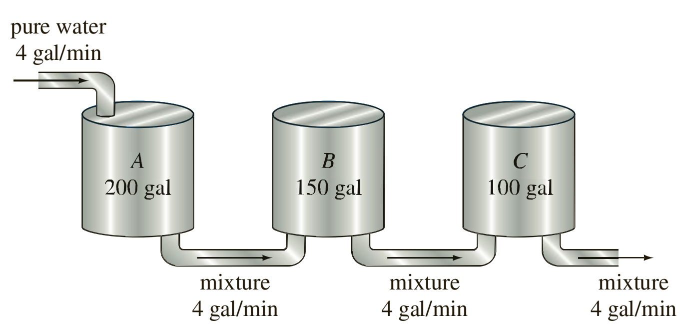

FIGURE 3.3.8 Mixing tanks in Problem 10

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Predator-Prey Models

11. Consider the Lotka-Volterra predator-prey model defined by

$$\frac{dx}{dt} = -0.1x + 0.02xy$$
$$\frac{dy}{dt} = 0.2y - 0.025xy,$$

where the populations $x(t)$ (predators) and $y(t)$ (prey) are measured in thousands. Suppose $x(0) = 6$ and $y(0) = 6$. Use a numerical solver to graph $x(t)$ and $y(t)$. Use the graphs to approximate the time $t > 0$ when the two populations are first equal. Use the graphs to approximate the period of each population.

Competition Models

12. Consider the competition model defined by

$$\frac{dx}{dt} = x(2 - 0.4x - 0.3y)$$
$$\frac{dy}{dt} = y(1 - 0.1y - 0.3x),$$

where the populations $x(t)$ and $y(t)$ are measured in thousands and t in years. Use a numerical solver to analyze the populations over a long period of time for each of the following cases:

(a) $x(0) = 1.5, \quad y(0) = 3.5$

(b) $x(0) = 1, \quad y(0) = 1$

(c) $x(0) = 2, \quad y(0) = 7$

(d) $x(0) = 4.5, \quad y(0) = 0.5$

13. Consider the competition model defined by

$$\frac{dx}{dt} = x(1 - 0.1x - 0.05y)$$
$$\frac{dy}{dt} = y(1.7 - 0.1y - 0.15x),$$

where the populations $x(t)$ and $y(t)$ are measured in thousands and t in years. Use a numerical solver to analyze the populations over a long period of time for each of the following cases:

(a) $x(0) = 1, \quad y(0) = 1$

(b) $x(0) = 4, \quad y(0) = 10$

(c) $x(0) = 9, \quad y(0) = 4$

(d) $x(0) = 5.5, \quad y(0) = 3.5$

Networks

14. Show that a system of differential equations that describes the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in Figure 3.3.9 is

$$L\frac{di_2}{dt} + L\frac{di_3}{dt} + R_1 i_2 = E(t)$$
$$-R_1\frac{di_2}{dt} + R_2\frac{di_3}{dt} + \frac{1}{C}i_3 = 0.$$

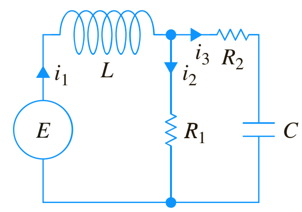

FIGURE 3.3.9 Network in Problem 14

15. Determine a system of first-order differential equations that describes the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in Figure 3.3.10.

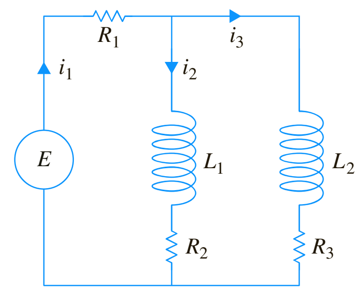

FIGURE 3.3.10 Network in Problem 15

16. Show that the linear system given in (18) describes the currents $i_1(t)$ and $i_2(t)$ in the network shown in Figure 3.3.4. [*Hint*: $dq/dt = i_3$.]

Additional Nonlinear Models

17. SIR Model A communicable disease is spread throughout a small community, with a fixed population of n people, by contact between infected individuals and people who are susceptible to the disease. Suppose that everyone is initially susceptible to the disease and that no one leaves the community while the epidemic is spreading. At time t, let $s(t)$, $i(t)$, and $r(t)$ denote, in turn, the number of people in the community (measured in hundreds) who are *susceptible* to the disease but not yet infected with it, the number of people who are *infected* with the disease, and the number of people who have *recovered* from the disease. Explain why the system of differential equations

$$\frac{ds}{dt} = -k_1 si$$
$$\frac{di}{dt} = -k_2 i + k_1 si$$
$$\frac{dr}{dt} = k_2 i,$$

where k_1 (called the *infection rate*) and k_2 (called the *removal rate*) are positive constants, is a reasonable mathematical model, commonly called a **SIR model,** for the spread of the epidemic throughout the community. Give plausible initial conditions associated with this system of equations.

18. (a) In Problem 17, explain why it is sufficient to analyze only

$$\frac{ds}{dt} = -k_1 si$$
$$\frac{di}{dt} = -k_2 i + k_1 si.$$

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(b) Suppose $k_1 = 0.2$, $k_2 = 0.7$, and $n = 10$. Choose various values of $i(0) = i_0$, $0 < i_0 < 10$. Use a numerical solver to determine what the model predicts about the epidemic in the two cases $s_0 > k_2/k_1$ and $s_0 \leq k_2/k_1$. In the case of an epidemic, estimate the number of people who are eventually infected.

19. Concentration of a Nutrient Suppose compartments A and B shown in Figure 3.3.11 are filled with fluids and are separated by a permeable membrane. The figure is a compartmental representation of the exterior and interior of a cell. Suppose, too, that a nutrient necessary for cell growth passes through the membrane. A model for the concentrations $x(t)$ and $y(t)$ of the nutrient in compartments A and B, respectively, at time t is given by the linear system of differential equations

$$\frac{dx}{dt} = \frac{\kappa}{V_A}(y - x)$$

$$\frac{dy}{dt} = \frac{\kappa}{V_B}(x - y),$$

where V_A and V_B are the volumes of the compartments, and $\kappa > 0$ is a permeability factor. Let $x(0) = x_0$ and $y(0) = y_0$ denote the initial concentrations of the nutrient. Solely on the basis of the equations in the system and the assumption $x_0 > y_0 > 0$, sketch, on the same set of coordinate axes, possible solution curves of the system. Explain your reasoning. Discuss the behavior of the solutions over a long period of time.

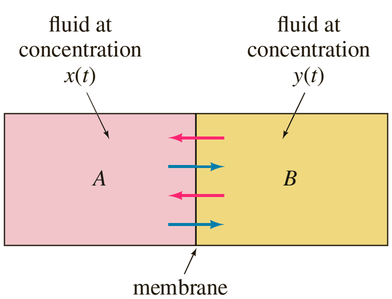

FIGURE 3.3.11 Nutrient flow through a membrane in Problem 19

20. The system in Problem 19, like the system in (2), can be solved with no advanced knowledge. Solve for $x(t)$ and $y(t)$ and compare their graphs with your sketches in Problem 19. Determine the limiting values of $x(t)$ and $y(t)$ as $t \to \infty$. Explain why the answer to the last question makes intuitive sense.

21. Mixtures Solely on the basis of the physical description of the mixture problem on page 108 and in Figure 3.3.1, discuss the nature of the functions $x_1(t)$ and $x_2(t)$. What is the behavior of each function over a long period of time? Sketch possible graphs of $x_1(t)$ and $x_2(t)$. Check your conjectures by using a numerical solver to obtain numerical solution curves of (3) subject to the initial conditions $x_1(0) = 25$, $x_2(0) = 0$.

22. Newton's Law of Cooling/Warming As shown in Figure 3.3.12, a small metal bar is placed inside container A, and container A then is placed within a much larger container B. As the metal bar cools, the ambient temperature $T_A(t)$ of the medium within container A changes according to Newton's law of cooling. As container A cools, the temperature of the medium inside container B does not change significantly and can be considered to be a constant T_B. Construct a mathematical model for the temperatures $T(t)$ and $T_A(t)$, where $T(t)$ is the temperature of the metal bar inside container A. As in Problems 1, 5, and 20, this model can be solved by using prior knowledge. Find a solution of the system subject to the initial conditions $T(0) = T_0$, $T_A(0) = T_1$.

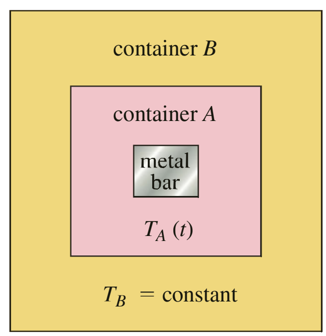

FIGURE 3.3.12 Container within a container in Problem 22

Chapter 3 In Review

Answers to selected odd-numbered problems begin on page ANS-4.

Answer Problems 1 and 2 without referring back to the text. Fill in the blank or answer true or false.

1. If $P(t) = P_0e^{0.15t}$ gives the population in an environment at time t, then a differential equation satisfied by $P(t)$ is ________.

2. If the rate of decay of a radioactive substance is proportional to the amount $A(t)$ remaining at time t, then the half-life of the substance is necessarily $T = -(\ln 2)/k$. The rate of decay of the substance at time $t = T$ is one-half the rate of decay at $t = 0$.

____

3. In March 1976 the world population reached 4 billion. At that time, a popular news magazine predicted that with an average yearly growth rate of 1.8%, the world population would be 8 billion in 45 years. How does this value compare with the value predicted by the model that assumes that the rate of increase in population is proportional to the population present at time t?

4. Air containing 0.06% carbon dioxide is pumped into a room whose volume is 8000 ft^3. The air is pumped in at a rate of 2000 ft^3/min, and the circulated air is then pumped out at the same rate. If there is an initial concentration of 0.2% carbon dioxide in the room, determine the subsequent amount in the room at time t. What is the concentration of carbon dioxide at 10 minutes? What is the steady-state, or equilibrium, concentration of carbon dioxide?

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5. Ötzi the Iceman In September of 1991 two German tourists found the well-preserved body of a man partially frozen in a glacier in the Ötztal Alps on the border between Austria and Italy. See Figure 3.R.1. Through the carbon-dating technique it was found that the body of Ötzi the iceman—as he came to be called—contained 53% as much C-14 as found in a living person. Assume that the iceman was carbon dated in 1991. Use the method illustrated in Example 3 of Section 3.1 to find the approximate date of his death.

FIGURE 3.R.1 Ötzi the iceman in Problem 5

6. In the treatment of cancer of the thyroid, the radioactive liquid Iodine-131 is often used. Suppose that after one day in storage, analysis shows that an initial amount A_0 of iodine-131 in a sample has decreased by 8.3%.

(a) Find the amount of iodine-131 remaining in the sample after 8 days.

(b) Explain the significance of the result in part (a).

7. Solve the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}$$

of the tractrix. See Problem 28 in Exercises 1.3. Assume that the initial point on the y-axis in (0, 10) and that the length of the rope is $x = 10$ ft.

8. Suppose a cell is suspended in a solution containing a solute of constant concentration C_s. Suppose further that the cell has constant volume V and that the area of its permeable membrane is the constant A. By **Fick's law** the rate of change of its mass m is directly proportional to the area A and the difference $C_s - C(t)$, where $C(t)$ is the concentration of the solute inside the cell at time t. Find $C(t)$ if $m = V \cdot C(t)$ and $C(0) = C_0$. See Figure 3.R.2.

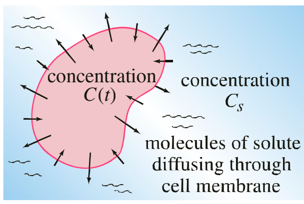

FIGURE 3.R.2 Cell in Problem 8

9. Suppose that as a body cools, the temperature of the surrounding medium increases because it completely absorbs the heat being lost by the body. Let $T(t)$ and $T_m(t)$ be the temperatures of the body and the medium at time t, respectively. If the initial temperature of the body is T_1 and the initial temperature of the medium is T_2, then it can be shown in this case that Newton's law of cooling is $dT/dt = k(T - T_m)$, $k < 0$, where $T_m = T_2 + B(T_1 - T)$, $B > 0$ is a constant.

(a) The foregoing DE is autonomous. Use the phase portrait concept of Section 2.1 to determine the limiting value of the temperature $T(t)$ as $t \to \infty$. What is the limiting value of $T_m(t)$ as $t \to \infty$?

(b) Verify your answers in part (a) by actually solving the differential equation.

(c) Discuss a physical interpretation of your answers in part (a).

10. According to **Stefan's law of radiation** the absolute temperature T of a body cooling in a medium at constant absolute temperature T_m is given by

$$\frac{dT}{dt} = k(T^4 - T_m^4),$$

where k is a constant. Stefan's law can be used over a greater temperature range than Newton's law of cooling.

(a) Solve the differential equation.

(b) Show that when $T - T_m$ is small in comparison to T_m then Newton's law of cooling approximates Stefan's law. [*Hint*: Think binomial series of the right-hand side of the DE.]

11. Suppose an *RC*-series circuit has a variable resistor. If the resistance at time t is defined by $R(t) = k_1 + k_2 t$, where k_1 and k_2 are known positive constants, then the differential equation in (9) of Section 3.1 becomes

$$(k_1 + k_2 t)\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

where C is a constant. If $E(t) = E_0$ and $q(0) = q_0$, where E_0 and q_0 are constants, then show that

$$q(t) = E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}.$$

12. A classical problem in the calculus of variations is to find the shape of a curve $\mathscr{C}$ such that a bead, under the influence of gravity, will slide from point $A(0, 0)$ to point $B(x_1, y_1)$ in the least time. See Figure 3.R.3. It can be shown that a nonlinear differential for the shape $y(x)$ of the path is $y[1 + (y')^2] = k$, where k is a constant. First solve for dx in terms of y and dy, and then use the substitution $y = k \sin^2\theta$ to obtain a parametric form of the solution. The curve $\mathscr{C}$ turns out to be a **cycloid.**

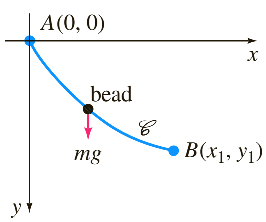

FIGURE 3.R.3 Sliding bead in Problem 12

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13. A model for the populations of two interacting species of animals is

$$\frac{dx}{dt} = k_1x(\alpha - x)$$

$$\frac{dy}{dt} = k_2xy.$$

Solve for x and y in terms of t.

14. Initially, two large tanks A and B each hold 100 gallons of brine. The well-stirred liquid is pumped between the tanks as shown in Figure 3.R.4. Use the information given in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$ and $x_2(t)$ at time t in tanks A and B, respectively.

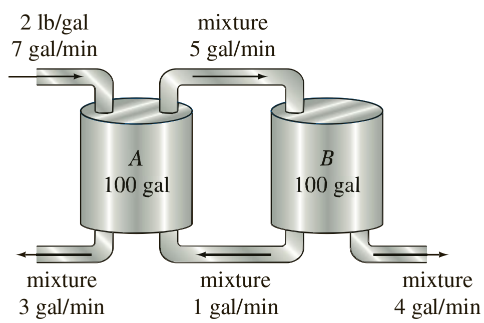

FIGURE 3.R.4 Mixing tanks in Problem 14

When all the curves in a family $G(x, y, c_1) = 0$ intersect orthogonally all the curves in another family $H(x, y, c_2) = 0$, the families are said to be **orthogonal trajectories** of each other. See Figure 3.R.5. If $dy/dx = f(x, y)$ is the differential equation of one family, then the differential equation for the orthogonal trajectories of this family is $dy/dx = -1/f(x, y)$. In Problems 15–18 find the differential equation of the given family by computing dy/dx and eliminating c_1 from this equation. Then find the orthogonal trajectories of the family. Use a graphing utility to graph both families on the same set of coordinate axes.

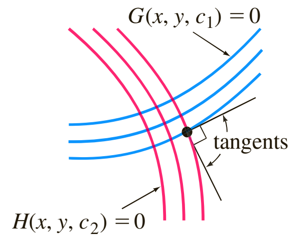

FIGURE 3.R.5 Orthogonal families

15. $y = c_1x$

16. $x^2 - 2y^2 = c_1$

17. $y = c_1e^x$

18. $y = \dfrac{1}{x + c_1}$

19. With the identifications $a = r$, $b = r/K$, and $a/b = K$, Figures 2.1.7 and 3.2.2 show that the logistic population model, (3) of Section 3.2, predicts that for an initial population P_0, $0 < P_0 < K$, regardless of how small P_0 is, the population increases over time but does not surpass the carrying capacity K. Also, for $P_0 > K$ the same model predicts that a population cannot sustain itself over time, so it decreases but yet never falls below the carrying capacity K of the ecosystem. The American ecologist **Warder Clyde Allee** (1885–1955) showed that by depleting certain fisheries beyond a certain level, the fish population never recovers. How would you modify the differential equation (3) to describe a population P that has these same two characteristics of (3) but additionally has a **threshold level** A, $0 < A < K$, below which the population cannot sustain itself and approaches extinction over time. [*Hint*: Construct a phase portrait of what you want and then form a differential equation.]

20. Sawing Wood A long *uniform* piece of wood (cross sections are the same) is cut through perpendicular to its length by a vertical saw blade. See Figure 3.R.6. If the friction between the sides of the saw blade and the wood through which the blade passes is ignored, then it can be assumed that the rate at which the saw blade moves through the piece of wood is

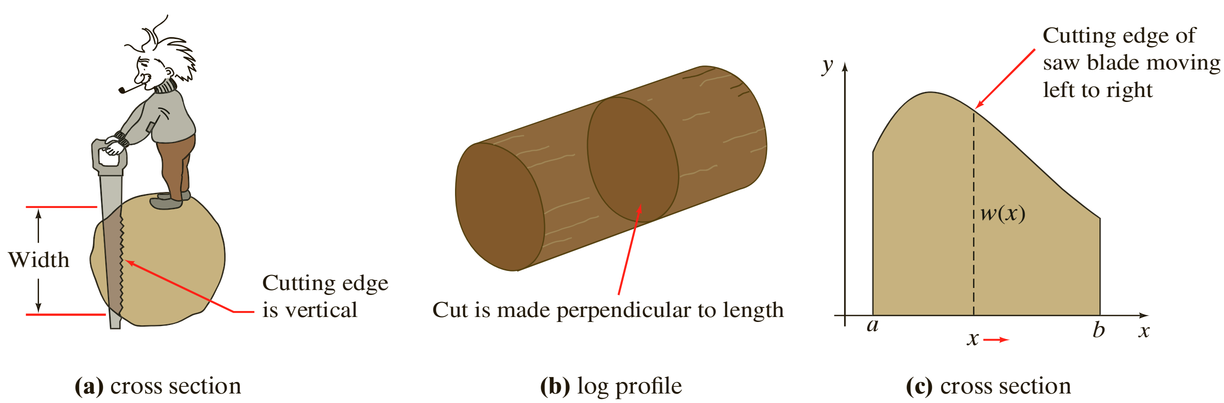

FIGURE 3.R.6 Sawing a log in Problem 20

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inversely proportional to the width of the wood in contact with its cutting edge. As the blade advances through the wood (moving, say, left to right) the width of a cross section changes as a nonnegative continuous function w. If a cross section of the wood is described as a region in the xy-plane defined over an interval $[a, b]$ then, as shown in Figure 3.R.6(c), the position x of the saw blade is a function of time t and the vertical cut made by the blade can be represented by a vertical line segment. The length of this vertical line is the width $w(x)$ of the wood at that point. Thus the position $x(t)$ of the saw blade and the rate dx/dt at which it moves to the right are related to $w(x)$ by

$$w(x)\frac{dx}{dt} = k, \;\; x(0) = a.$$

Here k represents the number of square units of the material removed by the saw blade per unit time.

(a) Suppose the saw is computerized and can be programmed so that $k = 1$. Find an *implicit* solution of the foregoing initial-value problem when the piece of wood is a circular log. Assume a cross section is a circle of radius 2 centered at (0, 0). [*Hint*: To save time see formula 41 in the table of integrals given on the right inside page of the front cover.]

(b) Solve the implicit solution obtained in part (b) for time t as a function of x. Graph the function $t(x)$. With the aid of the graph, approximate the time that it takes the saw to cut through this piece of wood. Then find the exact value of this time.

21. Solve the initial-value problem in Problem 20 when a cross section of a uniform piece of wood is the triangular region given in Figure 3.R.7. Assume again that $k = 1$. How long does it take to cut through this piece of wood?

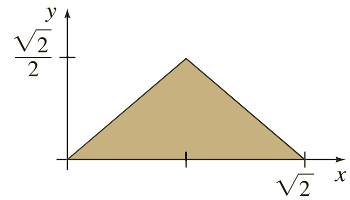

FIGURE 3.R.7 Triangular cross section in Problem 21

22. Chemical Kinetics Suppose a gas consists of molecules of type A. When the gas is heated a second substance B is formed by molecular collision. Let $A(t)$ and $B(t)$ denote, in turn, the number of molecules of types A and B present at time $t \geq 0$. A mathematical model for the rate at which the number of molecules of type A decreases is

$$\frac{dA}{dt} = -kA^2, \;\; k > 0.$$

(a) Determine $A(t)$ if $A(0) = A_0$.

(b) Determine the number of molecules of substance B present at time t if it is assumed that $A(t) + B(t) = A_0$.

(c) By hand, sketch rough graphs of $A(t)$ and $B(t)$ for $t \geq 0$.

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