for $k \geqq 6$. Therefore, we have

$$\begin{aligned} f(x) \sim 207 &+ 396(x-2) + 295(x-2)^2 \\ &+ 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5 \end{aligned}$$

In this example, it is not difficult to see that $\sim$ may be replaced by $=$. Simply expand all the terms in the Taylor series and collect them to get the original form for f. Taylor's Theorem, discussed soon, will allow us to draw this conclusion without doing any work! ■

Complete Horner's Algorithm

An application of Horner's algorithm is that of finding the Taylor expansion of a polynomial about any point. Let $p(x)$ be a given polynomial of degree n with coefficients a_k as in Equation (2) in Section 1.1, and suppose that we desire the coefficients c_k in the equation

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \\ &= c_n (x-r)^n + c_{n-1}(x-r)^{n-1} + \cdots + c_1(x-r) + c_0 \end{aligned}$$

Of course, Taylor's Theorem asserts that $c_k = p^{(k)}(r)/k!$, but we seek a more efficient algorithm. Notice that $p(r) = c_0$, so this coefficient is obtained by applying Horner's algorithm to the polynomial p with the point r. The algorithm also yields the polynomial

$$q(x) = \frac{p(x) - p(r)}{x - r} = c_n(x-r)^{n-1} + c_{n-1}(x-r)^{n-2} + \cdots + c_1$$

This shows that the second coefficient, c_1, can be obtained by applying Horner's algorithm to the polynomial q with point r, because $c_1 = q(r)$. (Notice that the first application of Horner's algorithm does not yield q in the form shown but rather as a sum of powers of x. (See Equations (3)–(4) in Section 1.1.) This process is repeated until all coefficients c_k are found.

We call the algorithm just described the **complete Horner's algorithm**. The pseudocode for executing it is arranged so that the coefficients c_k *overwrite* the input coefficients a_k.

```
integer n, k, j;   real r;   real array (a_i)_{0:n}
for k = 0 to n − 1 do
    for j = n − 1 to k do
        a_j ← a_j + r a_{j+1}
    end for
end for
```

This procedure can be used in carrying out Newton's method for finding roots of a polynomial, which we discuss in Chapter 3. Moreover, it can be done in complex arithmetic to handle polynomials with complex roots or coefficients.

EXAMPLE 4 Using the complete Horner's algorithm, find the Taylor expansion of the polynomial

$$p(x) = x^4 - 4x^3 + 7x^2 - 5x + 2$$

about the point $r = 3$.

Solution The work can be arranged as follows:

$$
\begin{array}{rrrrrr}
 & 1 & -4 & 7 & -5 & 2 \\
3\,) & & 3 & -3 & 12 & 21 \\ \cline{2-6}
 & 1 & -1 & 4 & 7 & 23 \\
 & & 3 & 6 & 30 & \\ \cline{2-5}
 & 1 & 2 & 10 & 37 & \\
 & & 3 & 15 & & \\ \cline{2-4}
 & 1 & 5 & 25 & & \\
 & & 3 & & & \\ \cline{2-3}
 & 1 & 8 & & &
\end{array}
$$

The calculation shows that

$$p(x) = (x-3)^4 + 8(x-3)^3 + 25(x-3)^2 + 37(x-3) + 23 \qquad \blacksquare$$

Taylor's Theorem in Terms of $(x - c)$

■ **THEOREM 2**

TAYLOR'S THEOREM FOR $f(x)$

If the function f possesses continuous derivatives of orders $0, 1, 2, \dots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1} \tag{8}$$

where the error term E_{n+1} can be given in the form

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Here ξ is a point that lies between c and x and depends on both.

In practical computations with Taylor series, it is usually necessary to *truncate* the series because it is not possible to carry out an infinite number of additions. A series is said to be **truncated** if we ignore all terms after a certain point. Thus, if we truncate the exponential Series (1) after seven terms, the result is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$$

This no longer represents e^x except when $x = 0$. But the truncated series should *approximate* e^x. Here is where we need Taylor's Theorem. With its help, we can assess the difference between a function f and its truncated Taylor series.

The explicit assumption in this theorem is that $f(x), f'(x), f''(x), \dots, f^{(n+1)}(x)$ are all continuous functions in the interval $I = [a, b]$. The final term E_{n+1} in Equation (8) is the **remainder** or **error term**. The given formula for E_{n+1} is valid when we assume only that $f^{(n+1)}$ exists at each point of the open interval (a, b). The error term is similar to the terms preceding it, but notice that $f^{(n+1)}$ must be evaluated at a point other than c. This point ξ depends on x and is in the open interval (c, x) or (x, c). Other forms of the remainder