Simulation and High-Performance Computing

Part 2: Time-Stepping Methods for Special Applications

Steffen Börm

Christian-Albrechts-Universität zu Kiel

September 28th, 2020

Leapfrog method

Goal: Accuracy of Runge's method with the computational work of Euler's.

Idea: As in Runge's method, use the central difference quotient

$$\begin{aligned}
\frac{y(t+\delta) - y(t)}{\delta} &\approx y'(t+\tfrac{\delta}{2}) = f(t+\tfrac{\delta}{2}, y(t+\tfrac{\delta}{2})),\\
y(t+\delta) &\approx y(t) + \delta\, f(t+\tfrac{\delta}{2}, y(t+\tfrac{\delta}{2})).
\end{aligned}$$

Leapfrog method

Goal: Accuracy of Runge's method with the computational work of Euler's.

Idea: As in Runge's method, use the central difference quotient

$$\frac{y(t+\delta)-y(t)}{\delta} \approx y'(t+\tfrac{\delta}{2}) = f(t+\tfrac{\delta}{2}, y(t+\tfrac{\delta}{2})),$$

$$y(t+\delta) \approx y(t) + \delta\, f(t+\tfrac{\delta}{2}, y(t+\tfrac{\delta}{2})).$$

Two-step method: Compute midpoint states also with the central difference quotient.

$$y(t+\tfrac{3}{2}\delta) \approx y(t+\tfrac{\delta}{2}) + \delta\, f(t+\delta, y(t+\delta)).$$

Leapfrog method

Goal: Accuracy of Runge's method with the computational work of Euler's.

Idea: As in Runge's method, use the central difference quotient

$$\frac{y(t+\delta)-y(t)}{\delta} \approx y'(t+\tfrac{\delta}{2}) = f(t+\tfrac{\delta}{2}, y(t+\tfrac{\delta}{2})),$$
$$y(t+\delta) \approx y(t) + \delta\, f(t+\tfrac{\delta}{2}, y(t+\tfrac{\delta}{2})).$$

Two-step method: Compute midpoint states also with the central difference quotient.

$$y(t+\tfrac{3}{2}\delta) \approx y(t+\tfrac{\delta}{2}) + \delta\, f(t+\delta, y(t+\delta)).$$

Algorithm: Alternate between states at $t, t+\delta, t+2\delta, \ldots$ and midpoint states $t+\frac{\delta}{2}, t+\frac{3}{2}\delta, t+\frac{5}{2}\delta, \ldots$.

Leapfrog algorithm

Time grid: Starting at time t_0, let

$$t_k := t_0 + \delta\, k, \qquad\qquad t_{k+1/2} := t_0 + \delta(k + \tfrac{1}{2}).$$

Alternating computation of approximated states

$$\tilde{y}(t_0) := y(t_0),$$

$$\begin{aligned}\tilde{y}(t_{k+1}) &:= \tilde{y}(t_k) + \delta\, f(t_{k+1/2}, \tilde{y}(t_{k+1/2})),\\ \tilde{y}(t_{k+3/2}) &:= \tilde{y}(t_{k+1/2}) + \delta\, f(t_{k+1}, \tilde{y}(t_{k+1})).\end{aligned}$$

Leapfrog algorithm

Time grid: Starting at time t_0, let

$$t_k := t_0 + \delta\, k, \qquad\qquad t_{k+1/2} := t_0 + \delta(k + \tfrac{1}{2}).$$

Alternating computation of approximated states

$$\begin{aligned}
\tilde{y}(t_0) &:= y(t_0), \\
\tilde{y}(t_{1/2}) &:= y(t_0) + \tfrac{\delta}{2} f(t_0, y(t_0)), \\
\tilde{y}(t_{k+1}) &:= \tilde{y}(t_k) + \delta\, f(t_{k+1/2}, \tilde{y}(t_{k+1/2})), \\
\tilde{y}(t_{k+3/2}) &:= \tilde{y}(t_{k+1/2}) + \delta\, f(t_{k+1}, \tilde{y}(t_{k+1})).
\end{aligned}$$

The first midpoint state is computed by Euler's method.

Leapfrog method: Implementation

Leapfrog method for the mass-spring system in C:

```
/* Use Euler's method for first midpoint state */
xm = x + 0.5 * delta * v;
vm = v - 0.5 * delta * c / m * x;

for(k=0; k<n; k++) {
  /* Update state */
  x += delta * vm;
  v -= delta * c / m * xm;

  /* Update midpoint state */
  xm += delta * v;
  vm -= delta * c / m * x;
}
```

Leapfrog method: Improved implementation

Idea: For the mass-spring system, $x'(t)$ depends only on $v(t)$ and $v'(t)$ depends only on $x(t)$.
$\rightarrow$ Compute midpoint states only for the velocity.

```
/* Use Euler's method for first midpoint state */
vm = v - 0.5 * delta * c / m * x;

for(k=0; k<n; k++) {
  /* Update state */
  x += delta * vm;

  /* Update midpoint state */
  vm -= delta * c / m * x;
}
```

Result: Accuracy of Runge's method at the "price" of Euler's method.

Experiment: Leapfrog method for the mass-spring system

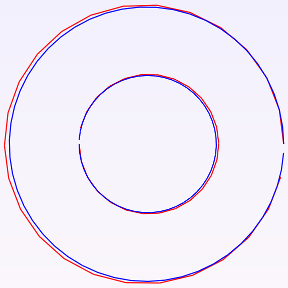
Approach: Start at $t = 0$, perform successive timesteps to reach $t = 20$.

	Euler		Runge		Leapfrog	
δ	error	ratio	error	ratio	error	ratio
1	1.0_{+3}		9.6_{+0}		9.1_{-1}	
1/2	8.2_{+1}	12.2	8.7_{-1}	11.0	2.0_{-1}	4.6
1/4	7.9_{+0}	10.4	1.9_{-1}	4.6	4.8_{-2}	4.2
1/8	1.3_{+0}	6.1	4.6_{-2}	4.1	1.2_{-2}	4.0
1/16	4.0_{-1}	3.3	1.2_{-2}	3.8	3.0_{-3}	4.0
1/32	1.6_{-1}	2.5	2.9_{-3}	4.1	7.4_{-4}	4.1
1/64	7.1_{-2}	2.3	7.4_{-4}	3.9	1.9_{-4}	3.9
1/128	3.4_{-2}	2.1	1.9_{-4}	3.9	4.6_{-5}	4.1
1/256	1.6_{-2}	2.1	4.6_{-5}	4.1	1.2_{-5}	3.8

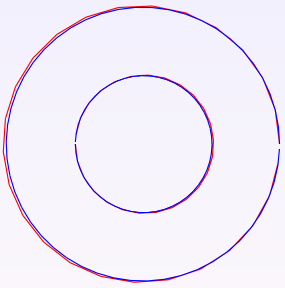
Observation: For this particular problem, the leapfrog method is more efficient and more accurate than Runge's method.

Experiment: Leapfrog for the gravity system

Runge's method
with 25/50 timesteps.

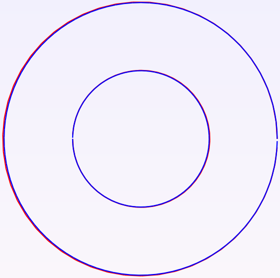


Leapfrog method
with 25/50 timesteps.

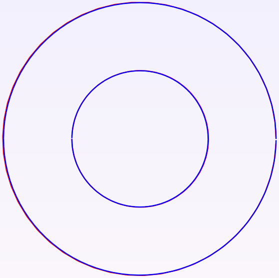


Experiment: Leapfrog for the gravity system

Runge's method
with 50/100 timesteps.

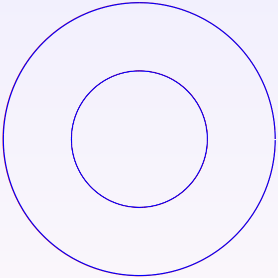


Leapfrog method
with 50/100 timesteps.

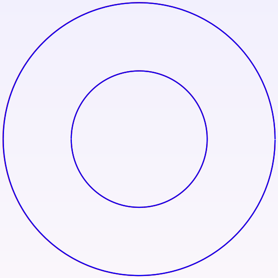


Experiment: Leapfrog for the gravity system

Runge's method
with 100/200 timesteps.



Leapfrog method
with 100/200 timesteps.



Stiff ordinary differential equations

Example: $y'(t) = -\lambda\, y(t)$ with $\lambda > 0$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler method takes the form

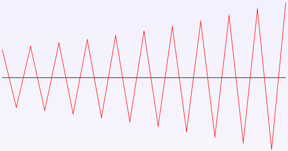
$$y(t+\delta) \approx y(t) - \delta\lambda\, y(t) = (1 - \delta\lambda) y(t).$$

Stiff ordinary differential equations

Example: $y'(t) = -\lambda\, y(t)$ with $\lambda > 0$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler method takes the form

$$y(t+\delta) \approx y(t) - \delta\lambda\, y(t) = (1-\delta\lambda) y(t).$$



Observation:

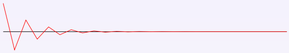
- Completely unrealistic oscillations if $2 < \delta\lambda$.

Stiff ordinary differential equations

Example: $y'(t) = -\lambda\, y(t)$ with $\lambda > 0$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler method takes the form

$$y(t+\delta) \approx y(t) - \delta\lambda\, y(t) = (1-\delta\lambda) y(t).$$



Observation:

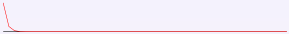
- Completely unrealistic oscillations if $2 < \delta\lambda$.
- Oscillations with exponential decay if $1 < \delta\lambda < 2$.

Stiff ordinary differential equations

Example: $y'(t) = -\lambda\, y(t)$ with $\lambda > 0$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler method takes the form

$$y(t+\delta) \approx y(t) - \delta\lambda\, y(t) = (1-\delta\lambda)y(t).$$



Observation:

- Completely unrealistic oscillations if $2 < \delta\lambda$.
- Oscillations with exponential decay if $1 < \delta\lambda < 2$.
- Proper behaviour only if $\delta\lambda < 1$.

Implicit Euler method

Backward difference quotient: Taylor expansion yields $\eta \in [t, t+\delta]$ with

$$y(t) = y(t+\delta) - \delta\, y'(t+\delta) + \frac{\delta^2}{2} y''(\eta),$$

$$\frac{y(t+\delta) - y(t)}{\delta} = y'(t+\delta) + \frac{\delta}{2} y''(\eta)$$

Implicit Euler method

Backward difference quotient: Taylor expansion yields $\eta \in [t, t+\delta]$ with

$$y(t) = y(t+\delta) - \delta\, y'(t+\delta) + \frac{\delta^2}{2} y''(\eta),$$

$$\frac{y(t+\delta) - y(t)}{\delta} = y'(t+\delta) + \frac{\delta}{2} y''(\eta)$$

Implicit Euler: Drop last term, use $y'(t+\delta) = f(t+\delta, y(t+\delta))$.

$$\tilde{y}(t+\delta) = y(t) + \delta\, f(t+\delta, \tilde{y}(t+\delta)).$$

Problem: Both sides depend on unknown future state $\tilde{y}(t+\delta)$.

Implicit Euler method

Backward difference quotient: Taylor expansion yields $\eta \in [t, t+\delta]$ with

$$y(t) = y(t+\delta) - \delta\, y'(t+\delta) + \frac{\delta^2}{2} y''(\eta),$$
$$\frac{y(t+\delta) - y(t)}{\delta} = y'(t+\delta) + \frac{\delta}{2} y''(\eta)$$

Implicit Euler: Drop last term, use $y'(t+\delta) = f(t+\delta, y(t+\delta))$.

$$\tilde{y}(t+\delta) = y(t) + \delta\, f(t+\delta, \tilde{y}(t+\delta)).$$

Problem: Both sides depend on unknown future state $\tilde{y}(t+\delta)$.

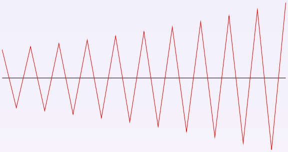
Model problem is linear, therefore easy to handle.

$$\tilde{y}(t+\delta) = y(t) - \delta\lambda\, \tilde{y}(t+\delta) \iff \tilde{y}(t+\delta) = \frac{1}{1+\delta\lambda} y(t)$$

Experiment: Explicit vs implicit Euler

Example: $y'(t) = -\lambda\, y(t)$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler:



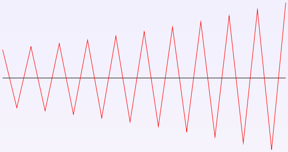
Implicit Euler:

Observation: At the same timestep size, the implicit method yields far more realistic results.

Experiment: Explicit vs implicit Euler

Example: $y'(t) = -\lambda\, y(t)$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler:



Implicit Euler:

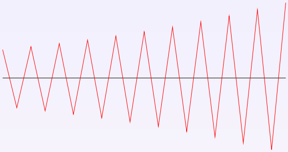
Observation: At the same timestep size, the implicit method yields far more realistic results.

Application: Parabolic and hyberbolic partial differential equations, e.g., heat or wave equations. $\rightarrow$ Courant-Friedrichs-Levy condition.

Experiment: Explicit vs implicit Euler

Example: $y'(t) = -\lambda\, y(t)$, exact solution $y(t) = e^{-\lambda t}$.

Explicit Euler:



Implicit Euler:

Observation: At the same timestep size, the implicit method yields far more realistic results.

Application: Parabolic and hyberbolic partial differential equations, e.g., heat or wave equations. $\rightarrow$ Courant-Friedrichs-Levy condition.

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad\qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy:

$$E'(t) = c\,x(t)x'(t) + m\,v(t)v'(t)$$

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy:

$$\begin{aligned} E'(t) &= c\,x(t)x'(t) + m\,v(t)v'(t) \\ &= c\,x(t)v(t) \end{aligned}$$

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad\qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy:

$$\begin{aligned} E'(t) &= c\,x(t)x'(t) + m\,v(t)v'(t) \\ &= c\,x(t)v(t) + m\,v(t)\left(-\frac{c}{m}\right)x(t) \end{aligned}$$

Conservation of energy

Example: Mass-spring system.

$$x'(t) = v(t), \qquad v'(t) = -\frac{c}{m}x(t).$$

Energy of the system consists of potential and kinetic energy.

$$E(t) := \frac{c}{2}x(t)^2 + \frac{m}{2}v(t)^2.$$

Conservation of energy:

$$\begin{aligned} E'(t) &= c\,x(t)x'(t) + m\,v(t)v'(t) \\ &= c\,x(t)v(t) + m\,v(t)\left(-\frac{c}{m}\right)x(t) = 0. \end{aligned}$$

The total energy of the system is constant.

Energy and Euler's methods

Explicit Euler method:

$$\tilde{E}(t+\delta) = \frac{c}{2}(x(t) + \delta\, v(t))^2 + \frac{m}{2}\left(v(t) - \delta\,\frac{c}{m}x(t)\right)^2$$

Energy and Euler's methods

Explicit Euler method:

$$\begin{aligned}\tilde{E}(t+\delta) &= \frac{c}{2}(x(t)+\delta\, v(t))^2 + \frac{m}{2}\left(v(t) - \delta\,\frac{c}{m}x(t)\right)^2 \\ &= E(t) + \delta^2\frac{c}{m}E(t) \geq E(t).\end{aligned}$$

Energy and Euler's methods

Explicit Euler method: Total energy increased with each timestep.

$$\begin{aligned}\tilde{E}(t+\delta) &= \frac{c}{2}(x(t)+\delta\, v(t))^2 + \frac{m}{2}\left(v(t) - \delta\,\frac{c}{m}x(t)\right)^2 \\ &= E(t) + \delta^2\frac{c}{m}E(t) \geq E(t).\end{aligned}$$

Energy and Euler's methods

Explicit Euler method: Total energy increased with each timestep.

$$\begin{aligned}\tilde{E}(t+\delta) &= \frac{c}{2}(x(t)+\delta\, v(t))^2 + \frac{m}{2}\left(v(t)-\delta\,\frac{c}{m}x(t)\right)^2 \\ &= E(t) + \delta^2\frac{c}{m}E(t) \geq E(t).\end{aligned}$$

Implicit Euler method:

$$\tilde{E}(t+\delta) = \frac{c}{2}(x(t)+\delta\,\tilde{v}(t+\delta))^2 + \frac{m}{2}\left(v(t)-\delta\,\frac{c}{m}\tilde{x}(t+\delta)\right)^2$$

Energy and Euler's methods

Explicit Euler method: Total energy increased with each timestep.

$$
\begin{aligned}
\tilde{E}(t+\delta) &= \frac{c}{2}(x(t) + \delta\, v(t))^2 + \frac{m}{2}\left(v(t) - \delta\, \frac{c}{m} x(t)\right)^2 \\
&= E(t) + \delta^2 \frac{c}{m} E(t) \geq E(t).
\end{aligned}
$$

Implicit Euler method:

$$
\begin{aligned}
\tilde{E}(t+\delta) &= \frac{c}{2}(x(t) + \delta\, \tilde{v}(t+\delta))^2 + \frac{m}{2}\left(v(t) - \delta\, \frac{c}{m} \tilde{x}(t+\delta)\right)^2 \\
&= E(t) - \delta^2 \frac{c}{m} \tilde{E}(t+\delta) \leq E(t).
\end{aligned}
$$

Energy and Euler's methods

Explicit Euler method: Total energy increased with each timestep.

$$\begin{aligned}\tilde{E}(t+\delta) &= \frac{c}{2}(x(t)+\delta\,v(t))^2 + \frac{m}{2}\left(v(t)-\delta\,\frac{c}{m}x(t)\right)^2\\ &= E(t)+\delta^2\frac{c}{m}E(t) \geq E(t).\end{aligned}$$

Implicit Euler method: Total energy decreased with each timestep.

$$\begin{aligned}\tilde{E}(t+\delta) &= \frac{c}{2}(x(t)+\delta\,\tilde{v}(t+\delta))^2 + \frac{m}{2}\left(v(t)-\delta\,\frac{c}{m}\tilde{x}(t+\delta)\right)^2\\ &= E(t)-\delta^2\frac{c}{m}\tilde{E}(t+\delta) \leq E(t).\end{aligned}$$

Crank-Nicolson method

Idea: Combine the explicit and the implicit Euler method.

$$\tilde{y}_{ex}(t+\delta) := y(t) + \delta f(t, y(t)),$$
$$\tilde{y}_{im}(t+\delta) := y(t) + \delta f(t+\delta, \tilde{y}_{im}(t+\delta)),$$

Crank-Nicolson method

Idea: Combine the explicit and the implicit Euler method.

$$\begin{aligned}
\tilde{y}_{ex}(t+\delta) &:= y(t) + \delta f(t, y(t)),\\
\tilde{y}_{im}(t+\delta) &:= y(t) + \delta f(t+\delta, \tilde{y}_{im}(t+\delta)),\\
\tilde{y}_{cn}(t+\delta) &:= y(t) + \frac{\delta}{2}\Big(f(t, y(t)) + f(t+\delta, \tilde{y}_{cn}(t+\delta))\Big).
\end{aligned}$$

Crank-Nicolson method

Idea: Combine the explicit and the implicit Euler method.

$$
\begin{aligned}
\tilde{y}_{ex}(t+\delta) &:= y(t) + \delta f(t, y(t)),\\
\tilde{y}_{im}(t+\delta) &:= y(t) + \delta f(t+\delta, \tilde{y}_{im}(t+\delta)),\\
\tilde{y}_{cn}(t+\delta) &:= y(t) + \frac{\delta}{2}\Big(f(t, y(t)) + f(t+\delta, \tilde{y}_{cn}(t+\delta))\Big).
\end{aligned}
$$

Mass-spring system: Solve linear equations

$$
\begin{aligned}
\tilde{x}(t+\delta) &= x(t) + \frac{\delta}{2}(v(t) + \tilde{v}(t+\delta)),\\
\tilde{v}(t+\delta) &= v(t) - \frac{\delta}{2}\frac{c}{m}(x(t) + \tilde{x}(t+\delta)),
\end{aligned}
$$

Crank-Nicolson method

Idea: Combine the explicit and the implicit Euler method.

$$\begin{aligned}
\tilde{y}_{ex}(t+\delta) &:= y(t) + \delta f(t, y(t)),\\
\tilde{y}_{im}(t+\delta) &:= y(t) + \delta f(t+\delta, \tilde{y}_{im}(t+\delta)),\\
\tilde{y}_{cn}(t+\delta) &:= y(t) + \frac{\delta}{2}\Big(f(t, y(t)) + f(t+\delta, \tilde{y}_{cn}(t+\delta))\Big).
\end{aligned}$$

Mass-spring system: Solve linear equations

$$\begin{aligned}
\tilde{x}(t+\delta) &= x(t) + \frac{\delta}{2}(v(t) + \tilde{v}(t+\delta)),\\
\tilde{v}(t+\delta) &= v(t) - \frac{\delta}{2}\frac{c}{m}(x(t) + \tilde{x}(t+\delta)),\\
\left(1 + \delta^2 \frac{c}{4m}\right)\tilde{x}(t+\delta) &= \left(1 - \delta^2 \frac{c}{4m}\right)x(t) + \delta v(t),
\end{aligned}$$

Crank-Nicolson method

Idea: Combine the explicit and the implicit Euler method.

$$\begin{aligned}
\tilde{y}_{ex}(t+\delta) &:= y(t) + \delta f(t, y(t)),\\
\tilde{y}_{im}(t+\delta) &:= y(t) + \delta f(t+\delta, \tilde{y}_{im}(t+\delta)),\\
\tilde{y}_{cn}(t+\delta) &:= y(t) + \frac{\delta}{2}\Big(f(t, y(t)) + f(t+\delta, \tilde{y}_{cn}(t+\delta))\Big).
\end{aligned}$$

Mass-spring system: Solve linear equations

$$\begin{aligned}
\tilde{x}(t+\delta) &= x(t) + \frac{\delta}{2}(v(t) + \tilde{v}(t+\delta)),\\
\tilde{v}(t+\delta) &= v(t) - \frac{\delta}{2}\frac{c}{m}(x(t) + \tilde{x}(t+\delta)),\\
\left(1 + \delta^2 \frac{c}{4m}\right)\tilde{x}(t+\delta) &= \left(1 - \delta^2 \frac{c}{4m}\right) x(t) + \delta v(t),\\
\left(1 + \delta^2 \frac{c}{4m}\right)\tilde{v}(t+\delta) &= \left(1 - \delta^2 \frac{c}{4m}\right) v(t) - \delta \frac{c}{m} x(t).
\end{aligned}$$

Crank-Nicolson method

Idea: Combine the explicit and the implicit Euler method.

$$\begin{aligned}
\tilde{y}_{ex}(t+\delta) &:= y(t) + \delta f(t, y(t)),\\
\tilde{y}_{im}(t+\delta) &:= y(t) + \delta f(t+\delta, \tilde{y}_{im}(t+\delta)),\\
\tilde{y}_{cn}(t+\delta) &:= y(t) + \frac{\delta}{2}\Big(f(t, y(t)) + f(t+\delta, \tilde{y}_{cn}(t+\delta))\Big).
\end{aligned}$$

Mass-spring system: Solve linear equations to obtain explicit algorithm.

$$\begin{aligned}
\tilde{x}(t+\delta) &= x(t) + \frac{\delta}{2}(v(t) + \tilde{v}(t+\delta)),\\
\tilde{v}(t+\delta) &= v(t) - \frac{\delta}{2}\frac{c}{m}(x(t) + \tilde{x}(t+\delta)),\\
\left(1 + \delta^2 \frac{c}{4m}\right)\tilde{x}(t+\delta) &= \left(1 - \delta^2 \frac{c}{4m}\right)x(t) + \delta v(t),\\
\left(1 + \delta^2 \frac{c}{4m}\right)\tilde{v}(t+\delta) &= \left(1 - \delta^2 \frac{c}{4m}\right)v(t) - \delta \frac{c}{m} x(t).
\end{aligned}$$

Experiment: Euler vs Crank-Nicolson

Approach: Start at $t = 0$, perform successive timesteps to reach $t = 10$.

	Explicit		Implicit		Crank-Nic	
δ	error	ratio	error	ratio	error	ratio
$1/2$	8.4_{+0}		7.3_{-1}		9.2_{-2}	
$1/4$	2.3_{+0}	3.7	5.6_{-1}	1.3	2.7_{-2}	3.4
$1/8$	7.7_{-1}	3.0	3.7_{-1}	1.5	7.0_{-3}	3.9
$1/16$	3.2_{-1}	2.4	2.2_{-1}	1.7	1.8_{-3}	4.0
$1/32$	1.4_{-1}	2.2	1.2_{-1}	1.8	4.4_{-4}	4.0
$1/64$	6.9_{-2}	2.1	6.3_{-2}	1.9	1.1_{-4}	4.0
$1/128$	3.4_{-2}	2.0	3.2_{-2}	2.0	2.8_{-5}	4.0
$1/256$	1.7_{-2}	2.0	1.6_{-2}	2.0	6.9_{-6}	4.0
$1/512$	8.2_{-3}	2.0	8.1_{-3}	2.0	1.7_{-6}	4.0

Observation: The Crank-Nicolson method is of second order.

Discrete conservation of energy

Crank-Nicolson for the mass-spring system:

$$\tilde{x}(t+\delta) = x(t) + \frac{\delta}{2}(v(t) + \tilde{v}(t+\delta)),$$
$$\tilde{v}(t+\delta) = v(t) - \frac{\delta}{2}\frac{c}{m}(x(t) + \tilde{x}(t+\delta))$$

Energy is preserved, since the third binomial equation yields

$$\tilde{E}(t+\delta) - E(t) = \frac{c}{2}(\tilde{x}(t+\delta)^2 - x(t)^2) + \frac{m}{2}(\tilde{v}(t+\delta)^2 - v(t)^2) = 0.$$

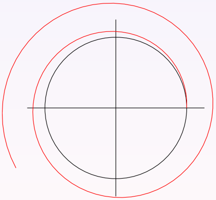
Discrete conservation of energy

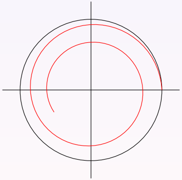
Crank-Nicolson for the mass-spring system:

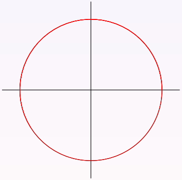
$$\tilde{x}(t+\delta) = x(t) + \frac{\delta}{2}(v(t) + \tilde{v}(t+\delta)),$$
$$\tilde{v}(t+\delta) = v(t) - \frac{\delta}{2}\frac{c}{m}(x(t) + \tilde{x}(t+\delta))$$

Energy is preserved, since the third binomial equation yields

$$\tilde{E}(t+\delta) - E(t) = \frac{c}{2}(\tilde{x}(t+\delta)^2 - x(t)^2) + \frac{m}{2}(\tilde{v}(t+\delta)^2 - v(t)^2) = 0.$$







Summary

Leapfrog method: Uses the central difference quotient, propagates midpoint states.

$$\tilde{y}(t+\delta) := \tilde{y}(t) + \delta\, f(t+\tfrac{\delta}{2}, \tilde{y}(t+\tfrac{\delta}{2})),$$
$$\tilde{y}(t+\tfrac{3}{2}\delta) \approx \tilde{y}(t+\tfrac{\delta}{2}) + \delta\, f(t+\delta, \tilde{y}(t+\delta)).$$

Implicit Euler method: Future state defined via fixed-point equation.

$$\tilde{y}(t+\delta) = y(t) + \delta\, f(t+\delta, \tilde{y}(t+\delta)).$$

Implicit methods are very useful for stiff equations.

Crank-Nicolson method: Provides conservation of energy, at least for the mass-spring system.

$$\tilde{y}(t+\delta) = y(t) + \tfrac{\delta}{2}(f(t, y(t)) + f(t+\delta, \tilde{y}(t+\delta)).$$