# A Controllability Theory for Nonlinear Systems

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*Abstract*—**A Lyapunov-like approach to the controllability of nonlinear dynamic systems is presented. A theory is developed which yields sufficient conditions for complete controllability for some classes of nonlinear systems; feedback controllers which drive the systems to desired terminal conditions, at a specified final time, are also obtained. Well-known controllability conditions for linear dynamic systems are derived using this general controllability theory. Elliptical regions are found which contain (bound) the trajectories of a class of systems controlled according to these methods. These regions are used in synthesizing controllers for nonlinear systems and for a class of state-variable inequality constrained problems.**

**An uncontrollability theorem, based also upon Lyapunov-like notions, is presented; this yields sufficiency conditions for uncontrollability for some types of nonlinear systems. Relationships of the theories to other nonlinear controllability approaches are indicated.**

## I. Introduction

CONSIDER the system of ordinary differential equations

$$\dot{x} = f(x, u, t). \tag{1}$$

Here, $x$ is an $n$-dimensional state vector of the dynamic system and $u$ is an $m$-dimensional control vector. Given the value of the state $x(t_0) = x_0$ at the initial time $t_0$, a problem of practical interest is to find a control function $u(\cdot) \in U$ (where $U$ is a prescribed set of functions) which transfers the state from its initial value to a given terminal value $x(t_f) = x_f$ at the given terminal time $t_f$.

In this paper a theory is presented which provides:

1) controllability—sufficient conditions to ensure that the system state can be transferred from $x(t_0) = x_0$ to $x_f$ in the alotted time interval $[t_0, t_f]$;
2) methods for synthesizing (feedback) controllers which accomplish the desired transfer;
3) bounds on the trajectories of a class of dynamic systems;
4) uncontrollability—sufficient conditions to ensure that the desired transfer cannot be accomplished.

The case where (1) is a linear system

$$\dot{x} = Ax + Bu \tag{2}$$

where $A$ and $B$ are time-varying matrices, was studied by Kalman [1], [2] and controllability was defined.

*Definition 1* [1]: A state $x_0$ is said to be *controllable at time* $t_0$ if there exists a control function $u(\cdot)$ depending on $x_0$ and $t_0$ and defined over some finite closed interval $[t_0, t_f]$ such that $x(t_f) = 0$. If this is true for every state $x_0$, then we say that the system is *completely controllable at time* $t_0$.

In this definition of controllability, $t_f$ is not specified *a priori*. It is shown in [1], [2] that if a constant coefficient system is controllable for some $\bar{t}_f > t_0$, it is controllable for all $t_f > t_0$, and that if a variable coefficient system is controllable for some $\bar{t}_f > t_0$, it is controllable for all $t_f > \bar{t}_f$.

Lee and Markus [3], [4] applied the concept of controllability to autonomous nonlinear systems. If system (1) is sufficiently smooth in a neighborhood of the origin,[1] then (1) behaves like (2) in a neighborhood of the origin, where

$$A = \frac{\partial f}{\partial x}(0, 0)$$

$$B = \frac{\partial f}{\partial u}(0, 0). \tag{3}$$

Then, according to Definition 1 [4], (1) is controllable if, for some $\tilde{u}(x)$, the system

$$\dot{x} = f(x, \tilde{u}(x)) = g(x) \tag{4}$$

is asymptotically stable and system (2) is completely controllable at time $t_0$. With such a concept of controllability, it is of interest to find conditions under which (4) can be made stable and the appropriate stabilizing controls; this has been studied in [5], [14]–[16].

In this paper the following definition of controllability is used.

*Definition 2*: System (1) is *controllable from* $(x_0, t_0)$ *to* $(0, t_f)$ if, for some control $u(t)$, $t_0 \le t \le t_f$, the solution of (1) with $x(t_0) = x_0$ is such that $x(t_f) = 0$, where $t_f$ is a preassigned terminal time. If the system is controllable for all $x_0$ at $t = t_0$, it will be called *completely controllable from* $t_0$ *to* $(0, t_f)$.

The principal difference between Definitions 1 and 2 is that in the latter the terminal time $t_f$ is specified. Thus controllability according to Definition 2 is a more restrictive

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[1] The point $x_f$ is assumed to be 0 for notational convenience. If $x_f \neq 0$, define $y = x - x_f$, which has as boundary conditions $y(t_0) = x_0 - x_f$ and $y(t_f) = 0$, and is governed by the differential equations $\dot{y} = f(y + x_f, u, t)$.

property. Moreover, the design of suitable controllers may be more difficult.

If (1) is controllable, a control function (of time) can be found by imposing the requirement that (in addition to transferring the state from $(x_0, t_0)$ to $(0, t_f)$) some performance index be minimized. If the minimum exists, the control may be found by the techniques of optimal control theory (see, for example, [6]–[10], [12], [13]).

However, these techniques are, in general, difficult to apply to nonlinear systems: they require considerable computer time, and results are most often obtained for a particular initial condition $x_0$, so that if deviations from the optimal trajectory occur, considerable extra computation must be performed to correct for the deviations (unless these deviations are very small).

By contrast, the (feedback) controllers found in this paper are easy to design and to use. Moreover, when these controllers are in feedback form, large deviations from a nominal initial state $x_0$ can be tolerated.

The controllability (uncontrollability) theory developed in this paper is motivated by Lyapunov stability theory and by optimal control theory. We present three main theorems. Theorems 1 and 2 provide sufficiency conditions for complete controllability; Theorem 3 yields sufficiency conditions for uncontrollability. The theorems are applicable to dynamic systems described by (1), where the control function may be required to belong to some constraint set.

Application of Theorems 1–3 to a number of control problem examples is described. However, all of the examples are of the form

$$\dot{x} = a(x, t) + b(x, t)u \tag{5}$$

and the control is *unrestricted*, i.e. $U$ is the set of all $E^m$-valued functions of $x$ and $t$.

## II. First Controllability Theorem

In this section Theorem 1 is proved and is applied to several examples. Some well-known results for linear systems [1], [2], [6] are rederived, and new results are obtained for a class of nonlinear systems.

It is assumed in Theorem 1 that $u(t), t_0 \le t \le t_f$, is restricted to some constraint set $U$.

*Theorem* 1 (*Controllability*)

If a scalar function $V(x, t)$ exists such that

1) $V_x(x, t)$ and $V_t(x, t)$ exist, for all $x, t, t \neq t_f$
2) for all continuous $c(t)$ ($n$-vector function of $t$),

$$\lim_{t \to t_f} c(t) \neq 0 \implies \lim_{t \to t_f} V(c(t), t) = \infty \tag{6}$$

and if a control function $u^*(\cdot) \in U$ exists such that

3) along trajectories of (1), the full time derivative of $V(x, t)$ satisfies

$$\dot{V} = V_t + V_x f(x, u^*, t) \le M < \infty, \qquad \forall t, \quad t_0 \le t < t_f \tag{7}$$

4) the solution to

$$x(t_0) = x_0 \tag{8}$$

$$\dot{x} = f(x, u^*(x, t), t) \tag{9}$$

exists and is unique

then system (1) is controllable from $(x_0, t_0)$ to $(0, t_f)$, and $u^*(x, t)$ accomplishes this transfer.

*Proof*: For all $t < t_f$,

$$V(x(t), t) = V(x_0, t_0) + \int_{t_0}^{t} \dot{V}\, dt \le V(x_0, t_0) + M(t - t_0).$$

Therefore,

$$\lim_{t \to t_f} V(x(t), t) \le V(x_0, t_0) + M(t_f - t_0) < \infty$$

and so, by condition (6),

$$x(t_f) = \lim_{t \to t_f} x(t) = 0.$$

*Example* 1

Consider the following time-varying system:

$$\dot{x} = F(t)x + G(t)u. \tag{10}$$

Let

$$V = x^T S(t) x \tag{11}$$

where $S(t)$ is a time-varying matrix chosen such that

$$\lim_{t \to t_f} c^T(t) S(t) c(t) = \infty \tag{12}$$

for all $c(t)$ such that $\lim_{t \to t_f} c(t) \neq 0$, and where

$$V(x_0, t_0) = x_0^T S(t_0) x_0 \le B < \infty. \tag{13}$$

Along trajectories of (10)

$$\dot{V} = 2x^T S(Fx + Gu) + x^T \dot{S} x. \tag{14}$$

If the feedback control law

$$u = -\tfrac{1}{2} G^T S x \tag{15}$$

is used, then

$$\dot{V} = x^T(\dot{S} + SF + F^T S - SGG^T S)x. \tag{16}$$

Equation (16) becomes

$$\dot{V} = 0, \qquad \text{for all } x, t$$

if

$$\dot{S} + SF + F^T S - SGG^T S = 0. \tag{17}$$

The conditions of Theorem 1 are satisfied if it is possible to satisfy the Riccati equation (17) and requirements (12) and (13). If these can be satisfied, then $x(t_f) = 0$ when control law (15) is used.

It is clear that $S^{-1}(t)$ satisfies

$$\frac{d}{dt} S^{-1} = FS^{-1} + S^{-1}F^T - GG^T. \tag{18}$$

If boundary condition

$$S^{-1}(t_f) = 0 \tag{19}$$

is used, and if $S(t)$ exists for $t \in [t_0, t_f)$, then the conditions of the theorems are satisfied. It is interesting to note that

$$S^{-1}(t) = \int_t^{t_f} \Phi(t, \tau)G(\tau)G^T(\tau)\Phi^T(t, \tau)\, d\tau \tag{20}$$

where $\Phi$ is the transition (fundamental) matrix associated with $F(t)$. The condition that $S^{-1}(t)$ be invertible (positive definite) for $t \in [t_0, t_f)$ is

$$\int_t^{t_f} \Phi(t, \tau)G(\tau)G^T(\tau)\Phi^T(t, \tau)\, d\tau > 0, \qquad \forall t \in [t_0, t_f) \tag{21}$$

which is precisely Kalman's condition [1] for complete controllability of (10) from the point $(x_0', t_0')$ to $(0, t_f)$, for all $x_0'$ and $t_0' \in [t_0, t_f)$.

*Example* 2

The previous example may be generalized by the addition of a nonlinear vector function $h(x, t)$. Let the system equation be

$$\dot{x} = F(t)x + G(t)u + h(x, t). \tag{22}$$

As before, let

$$V = x^T S x$$

where

$$\lim_{t \to t_f} [S(t)]^{-1} = 0 \tag{23}$$

and

$$V(x_0, t_0) = x_0^T S(t_0) x_0 \leq B < \infty. \tag{24}$$

If $S$ satisfies (17) and the control law (15) is modified to account for $h(x, t)$

$$u = -\tfrac{1}{2} G^T S x + q(x, t) \tag{25}$$

where $q(x, t)$ is yet to be determined, then along trajectories

$$\dot{V} = 2x^T S(Gq + h). \tag{26}$$

If the linear system (10) is controllable from $(x_0, t_0)$ to $(0, t_f)$, the nonlinear system (22) is also controllable from $(x_0, t_0)$ to $(0, t_f)$ if some function $q(x, t)$ exists such that $\dot{V}$ calculated from (26) is bounded and if the trajectory due to (25) exists and is unique. This occurs in the following cases.

*Case* 1 [11] (*Trivial Case*): It is possible to solve

$$G(t)q(x, t) + h(x, t) = 0. \tag{27}$$

In this case, if

$$u = \tilde{u} + q \tag{28}$$

then the system equation becomes

$$\dot{x} = F(t)x + G(t)\tilde{u} \tag{29}$$

because of (27). System (29) has been assumed controllable, so system (22) is also controllable. Thus, $x(t_f) = 0$ if the control is calculated from (28) using any solution $q$ to (27).

If $\tilde{u}$ is calculated according to (15), then

$$u = -\tfrac{1}{2} G^T S(t) x + q(x, t).$$

*Case* 2: If it is possible to write

$$Gq + h = -p(x, t)R(t)x \tag{30}$$

where $p$ is a scalar-valued function and $R(t)$ is a time-varying matrix such that

$$p(x, t) \geq 0, \qquad \text{for all } x, t \tag{31}$$

and

$$SR + R^T S \tag{32}$$

is positive semidefinite, then, from (26),

$$\dot{V} = -p(x, t)x^T(SR + R^T S)x \leq 0 \tag{33}$$

and thus, from the theorem, $x(t_f) = 0$.

*Example* 3

Let system (10) be controllable from $(x_0, t_0)$ to $(0, t_f)$, and let

$$h(x, t) = -p(x, t)x \tag{34}$$

where $p(x, t) \geq 0$. If we set $q = 0$, then

$$Gq + h = h = -p(x, t)x.$$

Thus $R = I$ and $SR + R^T S = 2S$, which is positive definite (see (21)). We have

$$\dot{V} = -2p(x, t)x^T S x \leq 0$$

so that the conditions of Theorem 1 are satisfied.

Thus if system (10) is controllable from $(x_0, t_0)$ to $(0, t_f)$, and $h(x, t)$ is given by (31), system (22) is also controllable from $(x_0, t_0)$ to $(0, t_f)$. A noteworthy consequence of this result is that the linear controller (15), which is designed for the linear system (10), will drive the state of the nonlinear system to zero at $t = t_f$. Thus, in designing the controller the nonlinearity may be ignored, provided that $p(x, t) \geq 0$.

If $p$ is a function of $u$, i.e.,

$$p(x, u, t) \geq 0, \qquad \text{for all } x, u, t$$

the system is controllable using the same linear controller. Indeed it is only necessary that

$$p(x, -\tfrac{1}{2} G^T S x, t) \geq 0, \qquad \text{for all } x, t.$$

*Example* 4

Let system (10) be controllable from $(x_0, t_0)$ to $(0, t_f)$; let

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{35}$$

and let

$$h = \begin{bmatrix} 0 \\ -p(x, t)x_2 \end{bmatrix}, \qquad p(x, t) \geq 0. \tag{36}$$

(Note that this example is not included in Case 1.) If

$$q = -p(x, t)x_1$$

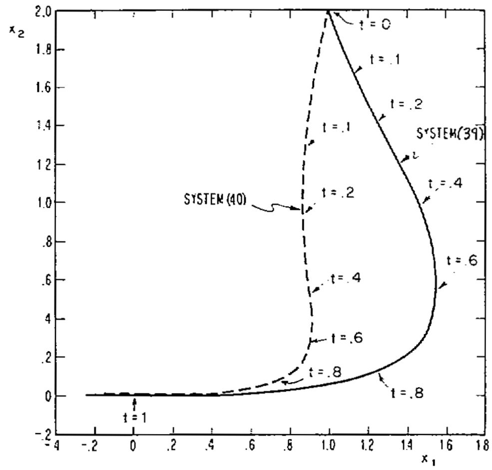


Fig. 1. Trajectories of linear and nonlinear systems.

then

$$Gq + h = -p(x, t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -p(x, t)x.$$

As in the last example,

$$\dot{V} = -2p(x, t)x^T Sx \leq 0$$

so that the conditions of Theorem 1 are satisfied. In this case, however, the control used is not the linear feedback law (15); rather, it is

$$u = -\tfrac{1}{2}G^T Sx - p(x, t)x_1. \tag{37}$$

Fig. 1 illustrates Example 4.

If

$$F = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \tag{38}$$

and $G$ is given by (35), system (10) becomes

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 - x_2 \end{aligned} \tag{39}$$

which is controllable. Also, if $h$ is as in (36) and $p(x, t) = x_2^2$, system (22) is

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 - x_2 - x_2^3. \end{aligned} \tag{40}$$

Let $t_0 = 0$, $t_f = 1$, $x_1(0) = 1$, $x_2(0) = 2$. Control (15), which drives (39) to the origin, is

$$u = -\frac{S_{11}x_1 + S_{12}x_2}{2} \tag{41}$$

where

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} = S$$

which satisfies (17) and (19). Control (37) becomes

$$u = -\frac{S_{11}x_1 + S_{12}x_2}{2} - x_1 x_2^2.$$

Note that both system (39) and system (40) tend to go to $x_2 = 0$ relatively quickly. In both cases (39) and (40), there is "overshoot," but $x_1(t = 1) = x_2(t = 1) = 0$. It should also be noted that there is less overshoot in the case of system (40).

*Example 5*

Let[2]

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 - \operatorname{sat}(x_2)x_2. \end{aligned}$$

Rewrite the system as

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 + x_2 - (\operatorname{sat}(x_2) + 1)x_2. \end{aligned}$$

Here

$$F = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $(F, G)$ is controllable.

Let

$$q = -(\operatorname{sat}(x_2) + 1)x_1.$$

Thus

$$Gq + h = -(\operatorname{sat}(x_2) + 1)x$$

and, since $\operatorname{sat}(x_2) + 1 \geq 0$, the conditions of Theorem 1 are satisfied and the control

$$u = -\tfrac{1}{2}G^T Sx - (\operatorname{sat}(x_2) + 1)x_1$$

drives $x$ to the origin at $t = t_f$.

*Example 6*

If system (10) is controllable and

$$G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 0 \\ -x_1^3 \end{bmatrix}$$

then choose

$$q = x_1^3 \frac{S_{22}}{S_{12}}$$

where the solution to (17) is

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}.$$

Here

$$Gq + h = x_1^3 \begin{bmatrix} \dfrac{S_{22}}{S_{12}} \\ -1 \end{bmatrix}.$$

[2] $$\operatorname{sat}(S) = \begin{cases} +1, & \text{if } S > +1 \\ S, & \text{if } -1 \leq S \leq +1 \\ -1, & \text{if } S < -1. \end{cases}$$

Let

$$p(x_1, x_2, t) = x_1^2$$

$$R = \begin{bmatrix} \dfrac{-S_{22}}{S_{12}} & 0 \\ 0 & 0 \end{bmatrix}$$

so that

$$SR + R^T S = 2\begin{bmatrix} \dfrac{S_{12}^2 - S_{11}S_{22}}{S_{12}} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\dot V = 2\frac{S_{11}S_{22} - S_{12}^2}{S_{12}} x_1^4.$$

Note that if $S_{12} < 0$, we shall have $\dot V \le 0$, and thus $x(t_f) = 0$. In particular, if

$$F = \begin{bmatrix} 0 & 0 \\ -\epsilon & -1 \end{bmatrix}$$

so that the system equations are

$$\dot x_1 = u$$

$$\dot x_2 = -\epsilon x_1 - x_2 - x_1^3 \tag{42}$$

then (10) is controllable if $\epsilon \ne 0$. It can be shown that the solution to the Riccati equation (17) is such that $S_{12}(t) = -\epsilon k(t)$, where $k(t) \ge 0$. Thus system (42) is controllable from any initial state and time $(x_0, t_0)$ to $(0, t_f)$ if $\epsilon > 0$. A control that will drive the state to the origin is

$$u = -\tfrac{1}{2}S_{11}x_1 - \tfrac{1}{2}S_{12}x_2 + \frac{S_{22}}{S_{12}}x_1^3.$$

Of course, the system may be controllable if $\epsilon < 0$; however, the form of control law assumed above is not appropriate.

The behavior of this example is illustrated by the trajectory shown in Fig. 2, where it is assumed that $\epsilon = 1$, $t_0 = 0$, $t_f = 1$, $x_1(0) = 1$, $x_2(0) = 2$. This should be compared with Fig. 1 because (39) is the linear part of (42).

*Example 7*

The function $h(x, t)$ need not be nonlinear in $x$. If

$$h(x, t) = -C(t)x$$

then (22) is controllable with $u$ given by (15) (i.e., ignoring $h(x, t)$) if

$$SC + C^T S \ge 0. \tag{43}$$

Equation (43) is satisfied if, for example, $C = k(t)I$, where $k(t) \ge 0$ is a scalar function.

*Example 8*

The previous example illustrates that a system can be controlled even though its parameters are not precisely known, or where it is more convenient to use $F = \tilde F(t)$ rather than $F = \tilde F(t) - k(t)I$ to calculate $u(x, t)$. A similar result is available for the $G$ matrix. If the system is

$$\dot x = Fx + \tilde G u \tag{44}$$

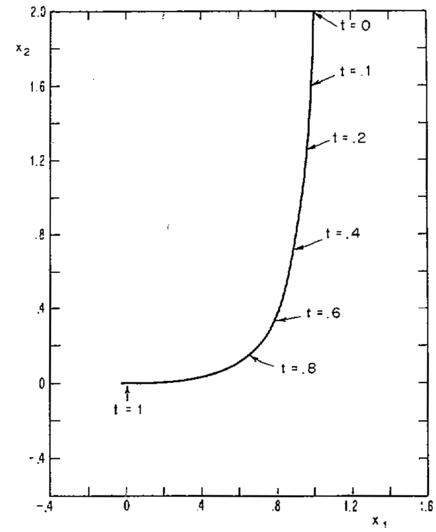


Fig. 2. Trajectory of nonlinear system.

and the feedback law is

$$u = -\tfrac{1}{2}\tilde G^T S x \tag{45}$$

but the $S$ matrix satisfies (17) with $G \ne \tilde G$

$$\dot S + SF + F^T S - SGG^T S = 0 \tag{17}$$

then

$$\dot V = \tfrac{1}{2}x^T S(GG^T - \tilde G\tilde G^T)Sx.$$

The control law (45) drives any initial state to zero (by satisfying the conditions of Theorem 1) if

$$GG^T - \tilde G \tilde G^T \le 0. \tag{46}$$

Note that the true value of $\tilde G$ is needed here to compute the feedback law (45). Inequality (46) is very useful because it allows one to obtain an $S$ matrix which satisfies the conditions of Theorem 1 by using a constant (or at worst time-varying) $G(t)$ matrix. However, $\tilde G(t)$ may be a function of $t$, $x$, and even of $u$.

If (44) describes the system and $S$ satisfies (17) but the control law is given by

$$u = -\tfrac{1}{2}G^T S x \tag{47}$$

(so that $\tilde G$ is not used in calculating the control), then $x(t_f) = 0$ if $G$ and $\tilde G$ satisfy

$$2GG^T - \tilde G G^T - G\tilde G^T \le 0. \tag{48}$$

Finally, if the system is given by (44) but $S$ satisfies

$$\dot S + SF + F^T S - S\tilde G\tilde G S = 0 \tag{49}$$

and the control is given by (47), then $x(t_f) = 0$ if

$$2\tilde G\tilde G^T - \tilde G G^T - G\tilde G^T \le 0.$$

*Ellipsoids Which Contain System Trajectories*

It is possible to extend the applicability of Theorem 1 by means of the following observation. Let $F$ and $G$ be constant matrices and let $S(t)$ satisfy (17) and (19). Define the ellipse

$E(t)$ as

$$E(t) = \{x | x^T S(t) x \le x^T(t) S(t) x(t)\} \tag{50}$$

and let $t_1 < t_2 < t_f$. If $V(x, t) = x^T S(t) x$ and $x(t)$ is a function of $t$ such that

$$\frac{d}{dt} V(x(t), t) \le 0 \tag{51}$$

then

$$x(t_2) \in E(t_1). \tag{52}$$

This is true because (see the Appendix) $S(t) \ge 0$, and so $x^T S(t_1) x \le x^T S(t_2) x$. In particular, we have

$$x^T(t_2) S(t_1) x(t_2) \le x^T(t_2) S(t_2) x(t_2). \tag{53}$$

Inequality (51) implies

$$x^T(t_2) S(t_2) x(t_2) \le x^T(t_1) S(t_1) x(t_1). \tag{54}$$

From (53) and (54)

$$x^T(t_2) S(t_1) x(t_2) \le x^T(t_1) S(t_1) x(t_1) \tag{55}$$

and (55) is equivalent to (52).

Statement (52) applies to the following linear-quadratic optimization problem. Minimize over $u(\cdot)$,

$$V(x_0, t_0) = \int_{t_0}^{t_f} u^T u \, dt \tag{56}$$

where

$$\dot{x} = Fx + Gu$$
$$x(t_0) = x_0$$
$$x(t_f) = 0.$$

When $F$ and $G$ are such that this problem has a solution $u(\cdot)$, for all $x_0$ and all $t_0 < t_f$, then the function $V$ defined by (56) satisfies Theorem 1 and (51). Therefore, the trajectory $x(t)$ is such that (52) holds. Statement (52) has the following additional significant consequences.

1) A class of state-variable inequality constrained problems can be solved by means of Theorem 1. Consider the following problem. Let region $\mathscr{R}$ be such that $x_0 \in \mathscr{R}$ and $0 \in \mathscr{R}$. Find a control $u(\cdot) \in U$ for (1) satisfying

$$x(t_0) = x_0$$
$$x(t_f) = 0$$

and satisfying the additional requirement that

$$x(t) \in \mathscr{R}, \qquad t_0 \le t \le t_f.$$

If $E(t_0) \subset \mathscr{R}$, a solution to this state-constrained problem is a control function $u^*(\cdot)$ which satisfies Theorem 1 and (51) (i.e., which satisfies Theorem 1 with $M = 0$).

2) We restate the original problem as treated by Theorem 1. Choose $u(\cdot) \in U$ so that

$$\dot{x} = f(x, u, t)$$
$$x(t_0) = x_0$$
$$x(t_f) = 0.$$

Let $V = x^T S(t) x$, where $S(t)$ satisfies (17) and (19). Thus 1) and 2) of Theorem 1 are satisfied. Let $u^*(\cdot) \in U$ and define

$$W(x, t) = \dot{V}(x, t) = V_t(x, t) + V_x(x, t) f(x, u^*(x, t), t). \tag{57}$$

Let $\mathscr{R}$ be the region in which $W(x, t)$ is bounded by zero, i.e.,

$$\mathscr{R} = \{x | W(x, t) \le 0, t_0 \le t \le t_f\}. \tag{58}$$

If $\mathscr{R} = E^n$, then Theorem 1 guarantees that $u^*(\cdot)$ solves the boundary-value problem. For $\mathscr{R} \subset E^n$, a partial solution of the boundary-value problem can be obtained by imposing the state constraint

$$x(t) \in \mathscr{R}, \qquad t_0 \le t \le t_f$$

and treating the problem as in the previous paragraphs.[3] Clearly $u^*(\cdot)$ is a solution if

$$E(t_0) \subset \mathscr{R}.$$

Note that this is only a partial solution because the latter condition holds only for certain $x_0$.

*Example 9*

Consider the state-constrained problem: find a control $u(\cdot)$ such that

$$\dot{x} = Fx + Gu \tag{59}$$

where

$$F = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x^T(0) = x_0^T = (x_{10}, x_{20})$$
$$x^T(1) = (0, 0)$$
$$x_1(t) \le 1, \qquad 0 \le t \le 1.$$

(It is assumed that $x_{10} \le 1$.) Define

$$\mathscr{R} = \{x | x_1 \le 1\}.$$

A solution for certain $x_0$ is obtained as follows. Define $S(t)$ as in (17) and (19) and let

$$E = E(t_0) = \{x = (x_1, x_2) | x^T S(t_0) x \le x_0^T S(t_0) x_0\}.$$

If

$$E(t_0) \subset \mathscr{R} \tag{60}$$

then the control

$$u = -\tfrac{1}{2} G^T S(t) x \tag{61}$$

solves the problem. This is only a partial solution because of (60). See Fig. 3, which displays $\mathscr{R}$, ellipse $E$, and (in solid lines) the trajectories originating from three values of $x_0$.

*Example 10*

Find a control $u(\cdot)$ such that

$$\dot{x}_1 = u$$
$$\dot{x}_2 = -x_1 - 2x_2 + x_1 x_2 \tag{62}$$

[3] Note that $W(0, t) = 0$ so that $0 \in \mathscr{R}$. It must also be verified that $x_0 \in \mathscr{R}$.

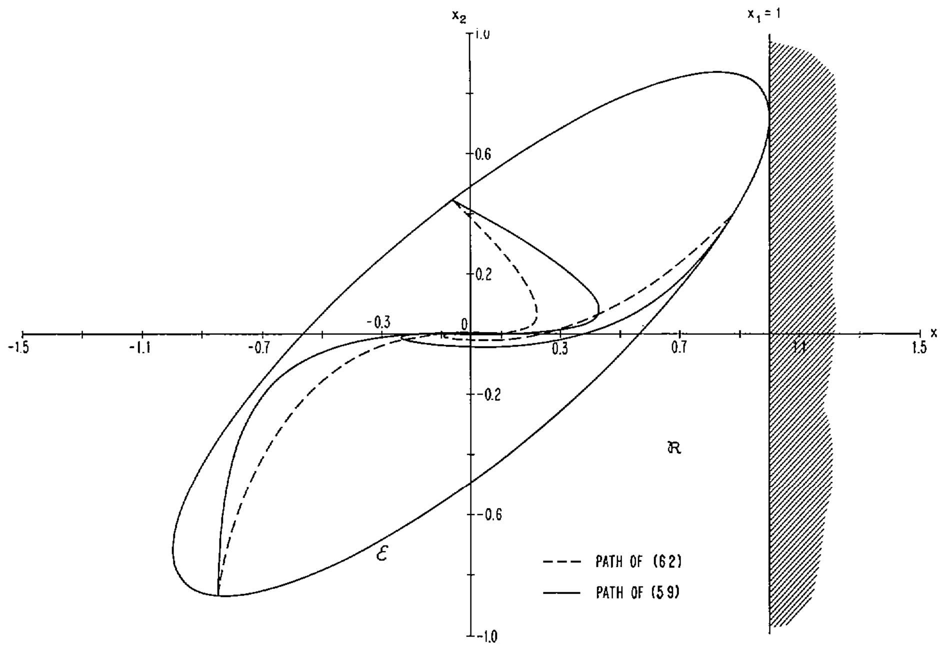


Fig. 3. Bounded trajectories of linear and nonlinear systems.

and

$$x^T(0) = x_0^T = (x_{10}, x_{20})$$
$$x^T(1) = (0, 0).$$

System (62) can be written

$$\dot{x} = Fx + Gu + h$$

where $F$ and $G$ are as in the previous example and

$$h = \begin{pmatrix} 0 \\ -px_2 \end{pmatrix}, \qquad p = 1 - x_1.$$

(Compare this with Example 4, where $p \geq 0$ for all $x$.)

Consider the control

$$u = -\tfrac{1}{2}G^T S(t)x - px_1. \tag{63}$$

If $V = x^T S(t)x$, then

$$\dot{V} = W = -2px^T Sx.$$

Clearly $W \leq 0$ in the region $\mathscr{R}$ defined in Example 9. If $\mathrm{E}(t_0) \subset \mathscr{R}$, then control (63) solves the problem. See Fig. 3, where the dashed lines are trajectories of system (62), driven by control (63), originating at three points on ellipse $E$.

## III. Second Controllability Theorem

In this section, Theorem 2 is proved and applied to systems of the form

$$\dot{x} = f(x, t) + \sum_{i=1}^{m} B_i(t)xu_i$$

where each $B_i(t)$ is an $n \times n$ matrix and $u$ is an $m$ vector. It is found that if the $B_i(t)$ matrices satisfy certain conditions, the system is completely controllable for all intervals $[t_0, t_f]$ regardless of the form of $f(x, t)$ (provided that $f$ is such that a solution to the differential equation exists).

*Theorem 2 (Controllability)*

If a scalar function $V(x, t)$ exists such that

1) $V_x$ and $V_t$ exist for all $x, t$
2) $$V(x, t_f) = 0 \implies x = 0 \tag{64}$$

and a control function $u^*(\cdot) \in U$ and a scalar function $K(t)$ exist such that

3) $$K(t_0) = V(x_0, t_0)$$
$$K(t_f) = 0$$
4) $$\dot{V} = V_t + V_x f(x, u^*, t) = \dot{K}, \qquad \forall t \in [t_0, t_f] \tag{65}$$
5) the solution to
$$x(t_0) = x_0 \tag{66}$$
$$\dot{x} = f(x, u^*(x, t), t) \tag{67}$$
exists and is unique

then $x(t_f) = 0$.

*Proof*: Since $V(x(t_0), t) = K(t_0)$ and $\dot{V} = \dot{K}$, $\forall t \in [t_0, t_f]$, then $V(x(t), t) = K(t), \forall t \in [t_0, t_f]$. Therefore, $V(x(t_f), t_f) = 0$, and by (64), $x(t_f) = 0$.

*Remark*: This theorem is almost a definition of controllability; however, it is useful.

*Corollary*

If a scalar function $V(x, t)$ exists that satisfies 1) and 2) of Theorem 2, and if a function $\psi(a, b)$ exists such that

1) $\psi(a, b)$ satisfies a Lipschitz condition in $a$, for every $b \neq t_f$, and the solution $S(t)$ to the initial-value problem
$$\dot{S} = \psi(S, t) \tag{68}$$
$$S(t_0) = V(x_0, t_0) \tag{69}$$
satisfies $S(t_f) = 0$

2) $u^*(\cdot) \in U$ exists such that

$$\dot{V} = \psi(V, t) \tag{70}$$

and $u^*(x, t)$ satisfies 5) of Theorem 2

then $x(t_f) = 0$.

*Proof*: Let $K$ be the solution to (68) and (69). Then $K(t_f) = 0$. Furthermore, $K(t) = V(x(t), t)$, for all $t \in [t_0, t_f]$. Therefore, $V(x(t_f), t_f) = 0$, and thus $x(t_f) = 0$.

This corollary is particularly useful if it is possible to choose $\psi$ such that all solutions to (68) are such that $S(t_f) = 0$ regardless of the value of $S(t_0)$. (In that case, (68) has a singular point at $t = t_f$.) Then $u^*(x, t)$ is truly a feedback control because even if the state is perturbed (at time $t$, $t_0 \leq t \leq t_f - \epsilon, \epsilon > 0$), $u^*$ will drive the state to the origin. This is not true of the control function of Theorem 2 because if $x(t_1)$ is perturbed, in general $V(t_1)$ will be perturbed. But then $V(t_1^+) \neq V(t_1) = K(t_1)$, and for $t > t_1$, $V(t) \neq K(t)$, so $V(t_f) \neq 0$ and therefore $x(t_f) \neq 0$.

In the following examples, it is assumed that $U$ is the nonempty set of functions for which condition 5) of Theorem 2 holds.

*Example* 11

If

$$\dot{x} = f(x, t) + G(t)xu$$

where $f(x, t)$ is a function bounded at $x = 0$, $G(t)$ is an $n \times n$ matrix function of time, and $u$ is a scalar, then let

$$V = x^T P(t) x$$

where $P(t)$ is an $n \times n$ positive-definite matrix.

Along trajectories,

$$\dot{V} = 2x^T P f + x^T(PG + G^T P)xu + x^T \dot{P} x. \tag{71}$$

Therefore, $\dot{V} = \dot{K}$ can be satisfied whenever $x^T(PG + G^TP)x$ is nonzero. If $PG + G^TP$ is positive definite or negative definite, then the coefficient of $u$ in (71) is nonzero (except at $x = 0$). According to Lyapunov's theorem, if $G(t)$ is a stability matrix (a real square matrix whose characteristic roots all have negative real parts [17]), $P$ can be found which is positive definite such that $PG + G^TP$ is negative definite. If $-G(t)$ is a stability matrix, $PG + G^TP$ can be made positive definite.

Note that these same considerations apply to finding $u$ such that $\dot{V} = \psi(V, t)$. Appropriate controls are therefore given by

$$u(x, t) = \frac{\dot{K}(t) - x^T \dot{P}(t) x - 2x^T P(t) f(x, t)}{x^T (P(t)G(t) + G(t)^T P(t))x}$$

or

$$u(x, t) = \frac{\psi(x^T P(t) x, t) - x^T \dot{P}(t) x - 2x^T P(t) f(x, t)}{x^T (P(t)G(t) + G^T(t)P(t))x}.$$

If $u$ is an $m$ vector, the system may be written

$$\dot{x} = f(x, t) + \sum_{i=1}^{m} [G_i(t)x]u_i. \tag{72}$$

Then if $V = x^T P x$,

$$\dot{V} = 2x^T P f + \sum_{i=1}^{m} x^T(PG_i + G_i^T P)xu_i = \dot{K}. \tag{73}$$

Equation (73) can be solved at time $t$ if, for some $i$,

$$x^T(t)(P(t)G_i(t) + G_i^T(t)P(t))x(t) \neq 0. \tag{74}$$

Thus system (72) can be controlled to $x = 0$ if for every $x \neq 0$ and for all $t$, (74) holds for some $i$. This is guaranteed if $P(t)G_i(t) + G_i^T(t)P(t)$ can be made to be positive (negative) definite for some $i$.

*Remark*: The methods of Lee and Markus [4] are not applicable to systems of the form considered here. In the neighborhood of $x = 0$, $u = 0$, the linearized version of (72) is (2), where, according to (3),

$$A = \frac{\partial}{\partial x}\left( f(x, t) + \sum_{i=1}^{m} G_i(t) x u_i \right)\Bigg|_{\substack{x=0 \\ u=0}} = f_x(0, t)$$

$$B = \frac{\partial}{\partial u}\left( f(x, t) + \sum_{i=1}^{m} G_i(t) x u_i \right)\Bigg|_{\substack{x=0 \\ u=0}} = 0$$

so that (2) is not controllable; this violates one of the assumptions of Lee and Markus.

In [18] is a sufficient condition for complete controllability of more general systems than (72) (where $f(x, t)$ is linear):

$$\dot{x} = \left( A + \sum_{k=1}^{m} u_k B_k \right) x + Cu \tag{75}$$

and bounds appear on $u_k$. However, a definition of controllability in which $t_f$ is not specified *a priori* is used (see Definition 1).

Equations (72) and (75) have application in nuclear reactor kinetics, biological populations, physiological processes [18], and chemical reaction dynamics [21].

## IV. Uncontrollability Theorem

In this section Theorem 3, which states sufficient conditions for a given system to be controllable, is proved.

*Theorem* 3 (*Uncontrollability*)

If a scalar function $V(x, t)$ exists such that

1) $V_x$ and $V_t$ exist for all $x, t$

2)
$$V(0, t_f) = 0$$
$$V(x, t_f) \neq 0 \implies x \neq 0$$

3) there exists a function $\psi(S, t)$ such that the solution to

$$S(t_0) = V(x_0, t_0) \tag{76}$$

$$\dot{S} = \psi(S, t) \tag{77}$$

exists, is unique, and

$$S(t_f) \neq 0$$

4) for all control laws $u(\cdot) \in U$,

$$\dot{V} = V_t + V_x f(x, u(x, t), t) = \psi(V, t)$$

then the system is not controllable from $(x_0, t_0)$ to $(0, t_f)$.

*Proof*: For all $t \in [t_0, t_f]$, $V(x(t), t) = S(t)$ because the solution to (76) and (77) is unique. Therefore, $V(x(t_f), t_f) \neq 0$, which implies that $x(t_f) \neq 0$.

*Example* 12

If

$$\dot{x}_1 = -x_1 + (2x_1x_2 + 1)u$$
$$\dot{x}_2 = x_2 - x_2^2 u$$

let

$$V = x_1x_2^2 + x_2.$$

Then $\dot{V} = V$. Therefore, $V = V(x(t_0))e^{(t-t_0)}$. Therefore, $x(t_f) \neq 0$ implies

$$V(x(t_0)) = x_{10}x_{20}^2 + x_{20} = 0.$$

Thus the system is not completely controllable.

*Example* 13

This example was treated by Haynes [19] using techniques involving Pfaffians [20]. The system is

$$\dot{x}_1 = x_1 + x_2u_1$$
$$\dot{x}_2 = x_2 - x_1u_1 + x_3u_2$$
$$\dot{x}_3 = x_3 - x_2u_2 \tag{78}$$

which may be written

$$\dot{x} = x + B_1xu_1 + B_2xu_2 \tag{79}$$

where

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Define

$$V = \tfrac{1}{2}x^Tx. \tag{80}$$

Then

$$\dot{V} = x^T\dot{x} = x^T(x + B_1xu_1 + B_2xu_2)$$
$$= x^Tx + x^TB_1xu_1 + x^TB_2xu_2. \tag{81}$$

But $x^TB_1x = x^TB_2x = 0$ because $B_1$ and $B_2$ are antisymmetric, and (81) becomes

$$\dot{V} = 2V \tag{82}$$

which implies that

$$V = V(t_0)\exp[2(t - t_0)] \tag{83}$$

and thus if $V(t_0) \neq 0$, $V \neq 0$, for all $t$. But $V(t_0) = 0$ if and only if $x(t_0) = 0$. Therefore, system (78) is not completely controllable.

## V. Conclusion

Using notations from Lyapunov stability theory and optimal control theory, two theorems are proved which give sufficient conditions for complete controllability, and a theorem is proved on sufficient conditions for a system to be not completely controllable. The controllability theorems are constructive in that they allow one to design controls which bring the state of the system to the origin at the given final time $t_f$.

The controllability theorems are applied to several classes of systems. It is found that if the linear system

$$\dot{x} = Fx + Gu \tag{84}$$

is controllable and the nonlinear function $h(x, t)$ satisfies certain conditions, then the system

$$\dot{x} = Fx + Gu + h(x, t) \tag{85}$$

is controllable. The system

$$\dot{x} = \tilde{F}(x, u, t)x + \tilde{G}(x, u, t)u \tag{86}$$

is also controllable if a controllable system of the form of (84) can be found and certain inequalities are satisfied between $F$ and $\tilde{F}$, and $G$ and $\tilde{G}$. Conditions are found under which the linear controller which drives the state of (84) to the origin also drives (85) and (86) to the origin. This is useful if $F$, $G$, or $h$ are difficult to calculate or measure in advance, but it is known that they satisfy certain inequalities. The system

$$\dot{x} = f(x, t) + \sum_{i=1}^{m} B_i(t)xu_i \tag{87}$$

is proved to be completely controllable when the matrices $B_1(t), \cdots, B_m(t)$ satisfy certain conditions.

Because the controls that are found can be feedback controls, it is hoped that these methods can be applied to problems where noise enters the state equations, and also that the well-known duality between controllability and observability for linear systems can be extended to the nonlinear case.

## Appendix

### Proof that $\dot{S} \geq 0$

*Lemma*: If $S(t)$ satisfies

$$\dot{S} + SF + F^TS - SGG^TS = 0 \tag{88}$$

$$\lim_{t \to t_f} S^{-1}(t) = 0 \tag{89}$$

and $F$ and $G$ are constant matrices such that the system

$$\dot{x} = Fx + Gu \tag{90}$$

is completely controllable, then $S(t) \geq 0$, for all $t < t_f$.

*Proof*: We write $S(t, t_f)$ to emphasize the dependence of $S$ on $t_f$. Then (88) and (89) become

$$\frac{\partial}{\partial t}S + SF + F^TS - SGG^TS = 0 \tag{91}$$

$$\lim_{t \to t_f} S^{-1}(t, t_f) = 0. \tag{92}$$

Let $Z = S^{-1}$. Then $Z(t, t_f)$ satisfies

$$\frac{\partial}{\partial t}Z - FZ - ZF^T + GG^T = 0 \tag{93}$$

$$Z(t_f, t_f) = 0. \tag{94}$$

The solution to (93), (94) is

$$Z(t, t_f) = \int_t^{t_f} \Phi(t, \tau)GG^T\Phi^T(t, \tau)\, d\tau \tag{95}$$

where $\Phi(t, \tau)$ is the transition matrix of $F$, i.e.,

$$\frac{\partial}{\partial t}\Phi(t, \tau) = F\Phi(t, \tau)$$

$$\Phi(t, t) = I.$$

The assumption of complete controllability of (90) implies that $Z(t, t_f)$ is positive definite, for all $t < t_f$. From (95),

$$\frac{\partial}{\partial t_f} Z = \Phi(t, t_f)GG^T\Phi^T(t, t_f) \geq 0. \tag{96}$$

Also, because $Z = S^{-1}$,

$$\frac{\partial}{\partial t_f} Z = -S^{-1}\left(\frac{\partial}{\partial t_f} S\right)S^{-1}. \tag{97}$$

From (96) and (97)

$$\frac{\partial}{\partial t_f} S \leq 0. \tag{98}$$

Because $F$ is constant, $\Phi(t, \tau)$ depends only on the quantity $\sigma = \tau - t$ and can be written

$$\Phi(t, \tau) = \Psi(\sigma).$$

Then (95) can be written

$$Z(t, t_f) = \int_0^{t_f - t} \Psi(\sigma)GG^T\Psi^T(\sigma)\, d\sigma \tag{99}$$

(where the change of variable $\sigma = \tau - t$ has been performed). From (99), $Z(t, t_f)$ depends only on the quantity $t_f - t$, and so $S(t, t_f)$ also depends only on $t_f - t$. Therefore,

$$\frac{\partial}{\partial t} S = -\frac{\partial}{\partial t_f} S \geq 0. \tag{100}$$

Reverting back to the notation of the statement of the lemma (where $t_f$ is fixed), (100) becomes

$$\dot{S}(t) \geq 0$$

and the lemma is proved.

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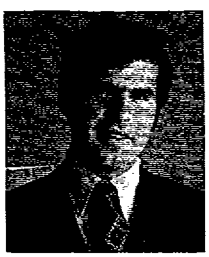


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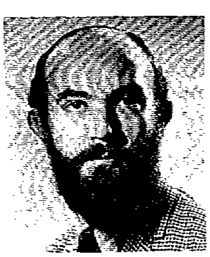


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