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A. THOWSEN [a]

[a] Department of Applied Mathematics , University of the Witwatersrand , Johannesburg

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On pointwise degeneracy, controllability and minimal time control of linear dynamical systems with delays

A. THOWSEN†

The first part of this paper surveys controllability and degeneracy results for linear time-invariant delay systems and discusses their relationship to minimal time, delay feedback control. In the second part a new construction for augmented delay feedback which achieves function space null controllability is presented. New results on the pointwise degeneracy of dynamic systems with pure delays are also stated.

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1. Introduction

This paper has two objectives. The first objective is to give an introduction to the related areas of pointwise degeneracy, controllability and time optimal control for linear dynamical systems with time delays. In exploring the relationship which exists between these topics, the author draws on the results obtained by several researchers whose individual contributions can be discerned from the comprehensive references. The second objective is to present new results on the pointwise degeneracy of systems with multiple pure delays and to give a new construction for delay feedback control which ensures function space null controllability. The latter result also solves some minimal time control problems.

We will consider the dynamic behaviour of three free linear time-invariant systems, namely the *single delay system* (SDS) :

$$\dot{x}(t) = Ax(t) + Bx(t-1), \quad t > 0 \tag{1}$$

the *multiple delay system* (MDS) :

$$\dot{x}(t) = Ax(t) + B_1x(t-1) + B_2x(t-2) + \dots + B_mx(t-m), \qquad m \geqslant 2, \quad B_m \neq 0, \quad t > 0 \tag{2}$$

and the *pure delay system* (PDS) :

$$\dot{x}(t) = B_1x(t-1) + B_2x(t-2) + \dots + B_mx(t-m), \quad m \geqslant 1, \quad B_m \neq 0, \quad t > 0 \tag{3}$$

Here $x(t) \in R^n$; A, B and B_i, $i = 1, 2, \dots, m$, are constant $n \times n$ matrices, and the unit delay has been set equal to 1 without loss of generality. The corresponding linear control systems are governed by the inhomogeneous equations obtained by adding control terms $Cu(t)$ (or $C_0u(t) + C_1u(t-1) + \dots + C_ru(t-r)$) to the right-hand sides of eqns. (1)–(3). The equations, together with the more general functional equation

$$\dot{x}(t) = f(x(t), x(t+\theta), u(t), t), \quad \theta \in [-h, 0)$$

describe the behaviour of several dynamic systems in which the rate of change depends not only on the present condition of the system but also on its past

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† Department of Applied Mathematics, University of the Witwatersrand, Johannesburg.

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history. Systems (1)–(3) are infinite dimensional. Hence, consistent with the convention that the state represents the minimum amount of information needed to fully determine the future behaviour of a free system, the state at time t is the trajectory segment in R^n given by $x(t)$ on $[t-m, t]$ ($m=1$ for (1)). With initial condition $\varphi \in C([-m, 0]; R^n)$, i.e. in the space of all continuous functions mapping $[-m, 0]$ into R^n, there exists a unique solution to each eqn. (1)–(3) (Bellman and Cooke 1963).

It is well known that the finite dimensional dynamic system

$$\dot{x}(t) = Ax(t), \quad x(t) \in R^n, \quad t > 0$$

has the property that for every pair $(\xi, T) \in R^n \times R^+$ (R^+ is the set of positive real numbers) there exists a vector $x_0 \in R^n$ such the trajectory emanating from $(x_0, 0)$ reaches ξ at time T. Hence the mapping $R^n \to R^n$ from the set of initial conditions to the state space R^n is always surjective. Weiss (1967) studied the single delay system (SDS) and conjectured that the mapping $\varphi \mapsto x(t) : C([-1, 0]; R^n) \to R^n$ is also surjective for all positive t. A system with this property he called pointwise complete. It was subsequently shown independently by Yorke and Kato (see Popov 1972 a) that the conjecture is true for $n=2$. Brooks and Schmitt (1971) verified the conjecture when A and B commute; Popov (1971) for the case of rank $B=1$ and Lee (see Popov 1972 a) when rank $B=n$. First in 1971, by Popov (1972 a) for the SDS and independently by Zverkin (1971) for the multiple delay system (MDS), was it shown that there exist linear constant delay systems with the property that the trajectories associated with admissible initial functions all attain values in a proper subspace of R^n at some $t>0$. This feature is readily observed in Popov's example:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix} \qquad \begin{matrix} (4\,a) \\ (4\,b) \\ (4\,c) \end{matrix}$$

Let γ_0, γ_1 and γ_2 be constants of integration. Then (4 a) and (4 c) clearly yield $x_1(t-1) - x_3(t) = \gamma_0$ for $t \geqslant 1$. From (4 b) $x_2(t) = \gamma_0 t + \gamma_1$ and $x_1(t) = \gamma_0 t^2 + 2\gamma_1 t + \gamma_2$ by (4 a). Consequently $x_1(t) - 2x_2(t) - x_3(t) \equiv 0$ for $t \geqslant 2$. Viewed geometrically all trajectories reach the plane P: $x_1(t) - 2x_2(t) - x_3(t) = 0$ no later than $t=2$ and remain on P for all future times. Such a system is called pointwise degenerate (as opposed to pointwise complete). The discovery of degenerate systems sparked a reassessment of earlier controllability results for time-delay systems. Motivated by the purely algebraic controllability conditions developed for finite dimensional linear systems research focused on algebraic conditions for pointwise degeneracy and controllability. In turn several interesting time optimal control problems were solved algebraically using the results on degeneracy. In this paper some of these results are stated and their implications discussed. Also new results for minimal time-delay feedback control problems and for the pointwise degeneracy of systems with multiple pure delays are given.

The organization of the paper is as follows. In § 2 we give formal definitions of pointwise degeneracy and controllability. Section 3 discusses the relationship between degeneracy, controllability and minimal time control.

The methods used and the results obtained in studying these concepts are described. Sections 4 and 5, respectively, present new results by the author on delay feedback control and on pointwise degeneracy of pure delay systems. Throughout the emphasis is on the basic ideas rather than on the details of proofs which can be found in the references.

2. Definitions

Consider the linear constant coefficient delay system

$$\dot{x}(t)=Ax(t)+\sum_{i=1}^{m}B_ix(t-i)+Cu(t), \quad B_m\neq 0, \quad t>0 \tag{5}$$

with initial condition

$$x(t)=\varphi(t), \quad -m\leqslant t\leqslant 0, \quad \varphi\in C([-m,0]\,;\,R^n)$$

The solution of (5) evaluated at time t with initial condition φ and control u on $(0, t]$ will be denoted by $x(t, 0\,;\,\varphi, u)$. $x(.,0\,;\,\varphi,u)$ is called the *trajectory* of (5) in R^n and the *state* of system (5) at time t is the trajectory segment

$$x_t(t, 0\,;\,\varphi, u)=x(s, 0\,;\,\varphi, u), \quad t-m\leqslant s\leqslant t$$

The following are standard definitions :

Definition 1 (Pointwise degeneracy)

The system is *pointwise degenerate with respect to* $\eta\in R^n$ $(\eta\neq 0)$ *at time* t_1 if $\eta^{\mathrm{T}}x(t_1, 0\,;\,\varphi, 0)=0$ for all initial conditions $\varphi\in C([-m,0]\,;\,R^n)$.

The system is *pointwise degenerate at time* t_1 if it is pointwise degenerate with respect to some non-zero $\eta\in R^n$ at t_1.

The system is *pointwise degenerate* if it is pointwise degenerate at some $t_1>0$.

The complementary property is called *pointwise completeness*.

Definition 2 (Controllability)

The system is *R^n null controllable* if for every φ there exist a finite time t_1 and a control u on $(0, t_1]$ such that $x(t_1, 0\,;\,\varphi, u)=0$. The system is *$R^n$ controllable* if for every φ and every $\xi\in R^n$ there exist a finite time t_1 and a control u on $(0, t_1]$ such that $x(t_1, 0\,;\,\varphi, u)=\xi$.

The system is *function space null controllable* if for every φ there exist a finite time t_1 and a control u on $(0, t_1]$ such that $x_t(t_1, 0\,;\,\varphi, u)=0$.

Several other controllability concepts have been investigated. In the sequel we consider only the definitions given above.

Definition 3 (Force free attainable set)

The *force-free attainable set* at time t_1 is the subset of R^n given by

$$\mathscr{A}(t_1)=\{x(t_1, 0\,;\,\varphi, 0)\,|\,\varphi\in C([-m,0]\,;\,R^n)\}$$

Definition 4 (Reachable set)

The *set of reachability* at time t_1 is the subset of R^n given by $\mathscr{R}(t_1)=\{x(t_1, 0\,;\,0, u)\,|\,u$ is admissible$\}$.

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As the class of admissible control we take the class of piecewise continuous functions.

Definition 5 (Degenerate set)

The *set of degeneracy* for a dynamic system is the time set given by $\mathscr{D} = \{t \in R^+ \mid \text{the system is degenerate at } t\}$.

3. Controllability, pointwise degeneracy and time optimal control

Before elucidating the connection between controllability, pointwise degeneracy and time optimal control we discuss the individual concepts in that order and give a fairly up-to-date presentation of published results. First, however, some preliminary results are needed. It is well known that for the finite dimensional system

$$\left.\begin{aligned} \dot{y}(t) &= Ay(t) + Cu(t), \quad t > 0 \\ y(0) &= y_0 \end{aligned}\right\} \tag{6}$$

there exists a non-singular transition matrix $\phi(t)$ uniquely determined by the system matrix A such that the solution to (6) at time t is

$$y(t) = \phi(t)y_0 + \int_0^t \phi(t-\tau)Cu(\tau)\,d\tau$$

Furthermore, $\phi(t)$ satisfies the equation

$$\begin{aligned} \dot{\phi}(t) &= A\phi(t), \quad t > 0 \\ \phi(0) &= I \end{aligned}$$

For the linear delay system (5) there exists (Bellman and Cooke 1963, Thowsen 1975) analogously an $n \times n$ fundamental matrix $X(t)$ such that

$$x(t, 0\,;\ \varphi, u) = X(t)\varphi(0) + \sum_{i=1}^{m} \int_{-i}^{0} X(t-\tau-i)B_i\varphi(\tau)\,d\tau + \int_0^t X(t-\tau)Cu(\tau)\,d\tau \tag{7}$$

where $X(t)$ satisfies the differential equation

$$\left.\begin{aligned} X(t) &= AX(t) + \sum_{i=1}^{m} BX(t-i), \quad t > 0 \\ X(0) &= I \\ X(t) &= 0, \quad -m \leqslant t < 0 \end{aligned}\right\} \tag{8}$$

Of particular interest is the fact that $X(t)$ can be singular. Indeed, for pointwise degenerate systems $X(t)$ is always singular for large enough t.

The earliest result on the controllability of (5) was obtained by Chyung and Lee (1966). Using the form (7) for the solution they proved as a necessary

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and sufficient condition for R^n controllability that there exists a finite T such that

$$\operatorname{rank} \int_0^T X(T-\tau)B(\tau)B^{\mathrm{T}}(\tau)X^{\mathrm{T}}(T-\tau)\,d\tau = n$$

The condition is similar to Kalman's integral criterion for controllability and the proof is exactly the same. Computation of the fundamental matrix for given A and B_i ($i=1, 2, \ldots, m$) is generally extremely difficult and simpler criteria were sought. Kirillova and Churakova (1967) derived algebraic necessary and sufficient conditions for R^n null controllability for special cases of (5) and gave separate necessity and sufficiency conditions for the general single delay system. Sufficiency conditions for function space null controllability were published by Buckalo (1968) and Tahim (1965) in a partly algebraic form, and for R^n controllability Weiss (1970) derived sufficient, algebraic conditions. It should be noted that while R^n controllability is unaffected by whether the system is pointwise degenerate, R^n null controllability is not. Since the null-vector in R^n lies on the terminal manifold on which all trajectories of the free (undriven) degenerate system end up, the available controls need only effect a transfer to the origin from any point on this manifold to guarantee R^n null controllability. Conditions for R^n controllability are therefore sufficient for R^n null controllability but not vice versa unless the system is pointwise complete. Zmood (1971) developed a representational form for the fundamental matrix to study its singularity properties and to determine algebraically the sets of force-free attainability and reachability.

Let $X(t)$ be the fundamental matrix for the SDS and write $X_k(\tau) = X(\tau+k)$ for $\tau \in [0, 1]$ and $k = 0, 1, \ldots$. Furthermore, introduce the $n \times n(k+1)$ matrices

$$E_k = [0, \ldots, 0, I]$$

$$Z_k^{\mathrm{T}}(\tau) = [X_0^{\mathrm{T}}(\tau), \ldots, X_k^{\mathrm{T}}(\tau)]$$

and the $n(k+1) \times n(k+1)$ matrix

$$A_k = \begin{bmatrix} A & 0 & 0 & \ldots & 0 & 0 \\ B & A & 0 & \ldots & 0 & 0 \\ 0 & B & A & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & B & A \end{bmatrix}$$

Then on the interval $[k, k+1]$,

$$X(k+\tau) = X_k(\tau) = E_k \exp(A_K \tau) Z_k(0)$$

where $Z_k(0)$ can be found recursively from

$$Z_i(0) = \begin{bmatrix} I \\ \exp(A_{k-1}) Z_{i-1}(0) \end{bmatrix}, \quad Z_0(0) = I$$

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The next theorems, all relating to the system $\dot{x}(t)=Ax(t)+Bx(t-1)+Cu(t)$, $x(t)\in R^n$, $u(t)\in R^r$, $t>0$, were proved by Zmood (1971). For notational simplicity let $C_k=Z_k(0)C$ and $F_k=Z_k(0)B$.

Theorem 1

A necessary and sufficient condition for R^n controllability at time $t\in(k, k+1]$, $k=0, 1, \ldots$ is that the matrix

$$Q(t)=[E_0C_0, \ldots, E_0A_0^{n-1}C_0, \ldots, E_kC_k, \ldots, E_kA_k^{n(k+1)-1}C_k]$$

has rank n.

Theorem 2

A necessary and sufficient condition for pointwise completeness at time $t\in(k, k+1)$, $k=0, 1, \ldots$ is that the matrix

$$M(t)=[E_kF_k, \ldots, E_kA_k^{n(k+1)-1}F_k, \ldots, E_{k-1}A_{k-1}^{nk-1}F_{k-1}, E_k \times \exp[A_k(t-k)]Z_k(0)]$$

has rank n.

A necessary and sufficient condition for pointwise completeness at time $t=k$, $k=1, 2, \ldots$ is that the matrix

$$M(k)=[E_{k-1}F_{k-1}, \ldots, E_{k-1}A_{k-1}^{nk-1}F_{k-1}, E_kZ_k(0)]$$

has rank n.

Since the force-free attainable set at time t_1 equals the range of $M(t_1)\ M^{\mathrm{T}}(t_1)$ and the reachable set at t_1 equals the range of $Q(t_1)Q^{\mathrm{T}}(t_1)$, conditions for R^n null controllability which are both necessary and sufficient are easily obtained.

Theorem 3

A necessary and sufficient condition for R^n null controllability at time $t\in(k, k+1)$ is that

$$\operatorname{rank}[M(t)M^{\mathrm{T}}(t), Q(t)Q^{\mathrm{T}}(t)]=\operatorname{rank}[Q(t)Q^{\mathrm{T}}(t)]$$

Similarly, the system is R^n null controllable at time $t=k$ if and only if

$$\operatorname{rank}[M(k)M^{\mathrm{T}}(k), Q(k)Q^{\mathrm{T}}(k)]=\operatorname{rank}[Q(k)Q^{\mathrm{T}}(k)]$$

A similar representational form as used by Zmood was developed independently by Popov (1972 a) for studying pointwise degeneracy of the SDS and extended by Asner and Halanay (1973 d) to the MDS. In the latter paper the matrices A_k, C_k and Q_k are defined as follows:

$$A_k=\begin{bmatrix} A & & & & & & \\ B_1 & A & & & & & \\ B_2 & B_1 & A & & & & \\ \cdot & \cdot & \cdot & & & 0 & \\ B_m & B_{m-1} & & & & & \\ 0 & B_m & & & & & \\ 0 & 0 & B_m & B_{m-1} & \cdots & B_2 & B_1 & A \end{bmatrix}_{nk\times nk}$$

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$$C_k = \begin{bmatrix} B_1 & B_2 & \dots & B_m \\ B_2 & B_3 & & \\ \vdots & & & \\ B_m & & & \\ 0 & & & 0 \\ 0 & & & \end{bmatrix}_{nk \times nm}$$

$$Q_k^{\mathrm{T}} = [0 \ \dots \ 0 \ Q^{\mathrm{T}}]_{p \times nk}$$

Also, define the $nk \times nm$ polynomial matrix $S_k(s)$ by

$$(sI - A_k)S_k(s) = C_k \det (sI - A_k)$$

Then $S_k(s)$ can be written as

$$S_k(s) = P_0{}^k + P_1{}^k s + \dots + P_{nk-1}{}^k s^{nk-1}$$

where

$$P_i{}^k = (\alpha_{i+1} I + \alpha_{i+2} A_k + \dots + A_k{}^{nk-1-i}) C_k, \quad i = 0, 1, \dots, nk-1$$

and α_j are the coefficients of the characteristic polynomial of A_k, i.e.

$$\det (sI - A_k) = \sum_{j=0}^{nk} \alpha_j s^j$$

Using this notation the next theorem was proved in Asner and Halanay (1973 d).

Theorem 4

Assume the MDS is pointwise degenerate with respect to the matrix Q at time t_1. Then the degenerate set is of the form $[k-1, \infty)$ where k is the smallest integer with the property

$$Q_{k-1}{}^{\mathrm{T}} S_{k-1}(s) \not\equiv 0, \quad Q_k{}^{\mathrm{T}} S_k(s) \equiv 0$$

Moreover, $k \geqslant 3$.

Popov (1972 a) was the first to show that the set of degeneracy has the form $[p, \infty)$, where p is a positive integer $\geqslant 2$. That part of the proof which shows that a system which is pointwise degenerate at t_1 is also degenerate at any $t_2 \geqslant t_1$ is simple. Assume pointwise degeneracy with respect to η at t_1. But $\eta^{\mathrm{T}} x(t_2, 0 ; \varphi, 0) = \eta^{\mathrm{T}} x(t_1, 0 ; \tilde{\varphi}, 0)$, where $\tilde{\varphi}(t) = x(t_2 - t_1 + t, 0 ; \varphi, 0)$ on $[-m, 0]$ and since $\tilde{\varphi} \in C([-m, 0] ; R^n)$, it follows that $\eta^{\mathrm{T}} x(t_2, 0 ; \varphi, 0) = 0$ for every $\varphi \in C([-m, 0] ; R^n)$. This property of the degenerate set is fundamental for the formulation of the minimal time problems discussed later in this section.

From Theorem 4, Asner and Halanay (1973 d) derived a complete algebraic characterization of pointwise degeneracy for the MDS.

Theorem 5

System (2) is pointwise degenerate with respect to the $n \times p$ matrix Q at $t_1 > 0$ if and only if there exist integers $l > 0$, $k > 0$ $(t_1 \geqslant k)$, an $l \times nk$ matrix

$R=(R_1, R_2, \ldots, R_k)$ of rank l, and $l\times l$ matrix V and a $p\times l$ matrix T^{T} such that

$$RC_k=0$$

$$RA_k=VR$$

$$T^{\mathrm{T}}(\exp(V)R_1, \exp(V)R_2-R_1, \ldots, \exp(V)R_k-R_{k-1}, -R_k)=-Q_{k+1}{}^{\mathrm{T}}$$

where A_k, C_k and Q_{k+1} are as defined earlier.

New representational forms for the fundamental matrix for the SDS and the MDS were recently obtained by Tsoi (1975) and Thowsen (1975), respectively. Motivated by the series expansion of the matrix exponential the fundamental matrix was expressed by a series in which the coefficients are $n\times n$ matrices which can be computed recursively. The main disadvantage of all proposed representations of solutions to delay systems is their complexity, which often leads to such a complexity in subsequent theorems as to render them almost useless. As an illustration it is submitted that the matrix $Q(t)$ in Theorem 1 is of dimensions $n\times\frac{1}{2}(k+1)(k+2)nr$. For a fifth-order system with $r=4$ R^n controllability at $t=8$ is determined by checking the rank of a matrix with 720 columns. Similarly, in Theorem 5 no upper bound for l exists. Hence, one may have to search through all positive integers. Clearly we are far from a readily applicable theory. Much interest has therefore focused on systems of low order for which a definitive characterization of pointwise degeneracy can be simply stated. In particular several results are known for the cases $n=2$ and $n=3$. For $n=2$ we have :

(A 1) The SDS is pointwise complete.

(A 2) If the MDS with $m=2$ is pointwise degenerate with respect to some η, then it is pointwise degenerate at $t=2$. Furthermore, (η^{T}, A) is completely observable ; (A, B) with $B=(B_1, B_2)$ is completely controllable ; and these exists a 2×2 matrix $Z=r\eta^{\mathrm{T}}\exp(A)$, where r is defined by $\eta^{\mathrm{T}}r=1$ and $\eta^{\mathrm{T}}\exp(A)r=0$, such that $B_1=AZ-ZA$, $B_2=-ZB_1$ (Asner and Halanay 1973 c). (The pair (η^{T}, A) is said to be completely observable if and only if rank $[\eta, A^{\mathrm{T}}\eta, A^{2\mathrm{T}}\eta, \ldots, A^{n-1\mathrm{T}}\eta]=n$, and the pair (A, B) is completely controllable if and only if rank $[B, AB, A^2B, \ldots, A^{n-1}B]=n$.)

(A 3) With $m=3$ degeneracy cannot occur before $t=3$ (Asner and Halanay 1973 c).

For $n=3$ Popov (1972 a) has shown that :

(B 1) If the SDS is pointwise degenerate with respect to some vector η, then (A, B) is completely controllable, (η^{T}, A) is completely observable, rank $B=2$ and degeneracy occurs at $t=2$.

(B 2) Any pointwise degenerate SDS can be written as $\dot{x}(t)=Ax(t)+(AZ-ZA)x(t-1)$, where $Z=r\eta^{\mathrm{T}}\exp(A)$ and $\eta\in R^3$, $r\in R^3$ satisfy $\eta^{\mathrm{T}}r=1$, $\eta^{\mathrm{T}}\exp(A)r=0$, $\eta^{\mathrm{T}}\exp(A)Ar=0$.

Some interesting results—valid for any order n—are :

(C 1) If A and B in the SDS commute, the system is pointwise complete (Brooks and Schmitt 1971).

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(C 2) If $B_i = bc_i^{\mathrm{T}}$, $i = 1, 2, \ldots, m$, the MDS is pointwise complete (Asner and Halanay 1973 b).

The fact that the trajectory of a system which is pointwise degenerate with respect to η at $t = t_1$ remains on the hyperplane perpendicular to η for all $t \geqslant t_1$ gives rise to formulations of time optimal control problems where the objective is to force the trajectory to reach a given hyperplane in minimal time and to remain on the hyperplane for all future times. Specifically, let the control system be the n-dimensional linear system

$$\dot{x}(t) = Ax(t) + u(t), \quad t > 0 \tag{9}$$

The problem is to determine a delay feedback control law of the form

$$u(t) = Bx(t-1) \tag{10}$$

or

$$u(t) = B_1 x(t-1) + B_2 x(t-2) + \ldots + B_m x(t-m) \tag{11}$$

such that for any initial condition the state $x(t)$ of (9) is brought to the hyperplane perpendicular to η in minimum time. The problem was first studied by Popov (1972 b) who showed that the minimal time was $t = 2$ and determined how B in (10) must be chosen. For $n = 3$ the solution in unique ; for higher order there is some freedom in the choice of B. An interesting variation of this problem was solved by Asner and Halanay (1974). For the linear finite dimensional system (9) they choose the matrices $B = (B_1, B_2, \ldots, B_m)$ and $C = (C_1, C_2, \ldots, C_m)$ in the delay feedback law

$$u(t) = \sum_{i=1}^{m} [B_i x(t-i) + C_i \dot{x}(t-i)]$$

such that the system is pointwise degenerate with respect to the columns of the $n \times p$ matrix Q ($p < n$, rank $Q = p$) in minimal time. For this problem the minimal time is 1 and B and C satisfy the equations

$$Q^{\mathrm{T}} \exp{(A)} C_{j-1} = Q^{\mathrm{T}} C_j \quad (j = 1, \ldots, m+1)$$

$$C_0 = -I, \quad C_{m+1} = 0$$

$$B_j + AC_j = D_j \quad (j = 1, \ldots, m)$$

$$Q^{\mathrm{T}} \exp{(At)} D_j \equiv 0$$

One limitation of the degeneracy achieved by delay feedback is that only some, and never all, components of $x(t)$ are controlled to and maintained at zero value. This is frequently satisfactory for output control problems where

$$y(t) = Dx(t), \quad y(t) \in R^r$$

and r is strictly less than n. For other problems this limitation caused by the form of solutions to (1)–(3) must be removed. In the next section a simple and seemingly practical way of achieving this is suggested.

4. Function space null controllability with augmented delay feedback control

Consider the linear n-dimensional control system

$$\dot{x}_1(t) = A_1 x_1(t) + u_1(t), \quad t > 0 \tag{12}$$

with complete information about the state $x_1(t)$ available only after a finite time lag. The problem is to construct a control signal u_1 using delayed information only so that $x_1(t) = 0$ for all $t \geqslant t_1 > 0$. Its solution rests on augmenting (11) by another n-dimensional control system

$$\dot{x}_2(t) = A_4 x_2(t) + u_2(t), \quad t > 0 \tag{12}$$

(called the 'synthetic' system) to enrich the class of admissible controls. Since access to information about the state of (12) is instantaneous, the information available to the controllers u_1 and u_2 at time t is $\{x_1(\tau),\ \tau \leqslant t-1\}$ and $\{x_2(\tau),\ \tau \leqslant t\}$. We therefore look for feedback laws of the form

$$u_1(t) = A_2 x_2(t) = \sum_{i=1}^{m} B_{i1} x_1(t-i) + \sum_{i=1}^{m} B_{i2} x_2(t-i)$$

and

$$u_2(t) = \sum_{i=1}^{m} B_{i3} x_1(t-i) + \sum_{i=1}^{m} B_{i4} x_2(t-i)$$

The augmented closed-loop system can then be written as

$$\dot{x}(t) = A x(t) + \sum_{i=1}^{m} B_i x(t-i), \quad t > 0 \tag{13}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} B_{i1} & B_{i2} \\ B_{i3} & B_{i4} \end{bmatrix}, \quad i = 1, 2, \ldots, m$$

We next prove the following theorem.

Theorem 6

System (11) is function space null controllable, i.e. $x_1(t) \equiv 0$ for all $t \geqslant t_1 > 0$, for the following choice of feedback parameters: $m = 2$, $A_2 = \exp(-A_1)$, $A_4 = A_1$

$$B_1 = \begin{bmatrix} -\exp(A_1) & -2I_n \\ 0 & \exp(A_1) \end{bmatrix}, \quad B_2 = \begin{bmatrix} \exp(2A_1) & \exp(A_1) \\ -\exp(3A_1) & -\exp(2A_1) \end{bmatrix}$$

Furthermore, the minimum value of t_1 is 2.

Proof

Define

$$Z = \begin{bmatrix} \exp(A_1) & I_n \\ -\exp(2A_1) & -\exp(A_1) \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

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and note that
$$Z^2=0 \tag{14}$$
$$ZB_2=0 \tag{15}$$
$$ZB_1=-B_2 \tag{16}$$
$$B_1+ZA=AZ \tag{17}$$

Define $y(t)=x(t)+Zx(t-1)$ for $t\geqslant 1$ and obtain from (13)

$$\dot{y}(t)=\dot{x}(t)+Z\dot{x}(t-1)=Ax(t)+(B_1+ZA)x(t-1)+(B_2+ZB_1)x(t-2)+ZB_2x(t-3)$$

By (14)–(17)

$$\dot{y}(t)=Ax(t)+AZx(t-1)=Ay(t) \quad \text{for } t\geqslant 1$$

Hence,

$$y(t)=\exp(A)y(t-1) \quad \text{for } t\geqslant 2 \tag{18}$$

Substituting for $y(t)$ and premultiplying (18) by Q^{T} yields

$$Q^{\mathrm{T}}x(t)+Q^{\mathrm{T}}Zx(t-1)=Q^{\mathrm{T}}\exp(A)x(t-1)+Q^{\mathrm{T}}\exp(A)Zx(t-2)$$

But, since it is easily verified that $Q^{\mathrm{T}}Z=Q^{\mathrm{T}}\exp(A)$ (hence $Q^{\mathrm{T}}\exp(A)Z=0$ by (14)), we obtain

$$Q^{\mathrm{T}}x(t)=x_1(t)=0 \quad \text{for } t\geqslant 2$$

The minimum time being 2 follows from the fact that pointwise degeneracy of (13) cannot occur before $t=2$.

Next consider the same problem when knowledge of the first derivatives, $\dot{x}_1(\tau)$, $\tau\leqslant t-1$, and $\dot{x}_2(\tau)$, $\tau\leqslant t$, are also available to controllers u_1 and u_2 at time t. We seek delay feedback controls of the forms

$$u_1(t)=A_2x_2(t)+\sum_{i=1}^{m}B_{i1}x_1(t-i)+\sum_{i=1}^{m}B_{i2}x_2(t-i)+\sum_{i=1}^{m}C_{i1}\dot{x}_1(t-i)+\sum_{i=1}^{m}C_{i2}\dot{x}_2(t-i)$$

and

$$u_2(t)=\sum_{i=1}^{m}B_{i3}x_1(t-i)+\sum_{i=1}^{m}B_{i4}x_2(t-i)+\sum_{i=1}^{m}C_{i3}\dot{x}_1(t-i)+\sum_{i=1}^{m}C_{i4}\dot{x}_2(t-i)$$

For this problem we show.

Theorem 7

System (11) is function space null controllable for the following choice of feedback parameters :

$$m=1, \quad A_2=I, \quad A_4=A_1$$

$$B_1=\begin{bmatrix}(A_1-I)\exp(A_1) & (A_1-I)\exp(A_1)\\ -A_1\exp(A_1) & -A_1\exp(A_1)\end{bmatrix}$$

$$C_1=\begin{bmatrix}-\exp(A_1) & -\exp(A_1)\\ \exp(A_1) & \exp(A_1)\end{bmatrix}$$

Furthermore, the minimum value of t_1 is 1.

Proof

The augmented system is

$$\dot{x}(t) = Ax(t) + B_1x(t-1) + C_1\dot{x}(t-1), \quad t > 0 \tag{19}$$

Introduce $y(t) = x(t) - C_1x(t-1)$ for $t \geqslant 0$ and observe that $AC_1 = -B_1$. Hence

$$\dot{y}(t) = Ax(t) + B_1x(t-1) = Ay(t), \quad t \geqslant 0$$

and

$$y(t) = \exp(A)y(t-1), \quad t \geqslant 1 \tag{20}$$

Introduce $Q^T = [I_n, 0]$ and note that $Q^T \exp(A)C_1 = 0$ and $Q^TC_1 = -Q^T \exp(A)$. Multiplying (20) by Q^T then yields

$$Q^Tx(t) = Q^TC_1x(t-1) + Q^T \exp(A)x(t-1) - Q^T \exp(A)C_1x(t-2) = 0$$

Consequently

$$x_1(t) = 0 \quad \text{for } t \geqslant 1$$

Finally, $t = 1$ is the minimal value of t_1 since the set of degeneracy of (19) is at most $[1, \infty)$.

The function space null controllability of (11) was obtained irrespective of initial conditions for the synthetic system. They may therefore be set equal to zero for simplicity. Furthermore, it is straightforward to extend the theorems to systems governed by delay differential equations (e.g. (5) with non-singular C matrix). Since the minimal values of t_1 coincide with the lower bounds on the degenerate sets, the theorems also provide solutions to the following minimal time control problems :

Problem 1

Given the $n \times n$ matrix A_1. For the linear system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} B_{i1} & B_{i2} \\ B_{i3} & B_{i4} \end{bmatrix} \begin{bmatrix} x_1(t-i) \\ x_2(t-i) \end{bmatrix}, \quad t > 0$$

determine the matrices A_2, A_3, A_4, $B_{ij} =$, $i = 1, \ldots, m$, $j = 1, \ldots, 4$, which give the minimal value of t_1 such that $x_1(t) \equiv 0$, $t \geqslant t_1$.

Problem 2

Given the $n \times n$ matrix A_1. For the linear system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \sum_{i=1}^{m} \left\{ \begin{bmatrix} B_{i1} & B_{i2} \\ B_{i3} & B_{i4} \end{bmatrix} \begin{bmatrix} x_1(t-i) \\ x_2(t-i) \end{bmatrix} + \begin{bmatrix} C_{i1} & C_{i2} \\ C_{i3} & C_{i4} \end{bmatrix} \begin{bmatrix} \dot{x}_1(t-i) \\ \dot{x}_2(t-i) \end{bmatrix} \right\}, \quad t > 0$$

determine the matrices A_2, A_3, A_4, B_{ij}, C_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, 4$, which give the minimal value of t_1 such that $x_1(t) \equiv 0$, $\forall t \geqslant t_1$.

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5. Pointwise degeneracy for linear systems with pure delays

In this section we summarize recent results on pointwise degeneracy for the pure delay system (PDS)

$$\dot{x}(t) = B_1 x(t-1) + B_2 x(t-2) + \dots + B_m x(t-m), \quad t > 0 \tag{21}$$

Detailed proofs can be found in Thowsen (1976). The solution to (21) with initial condition φ on $[-m, 0]$ is given by

$$x(t) = X(t)\varphi(0) + \sum_{i=1}^{m} \int_{-i}^{0} X(t-s-i) B_i \varphi(s)\, ds$$

where the fundamental matrix $X(t)$ can be expressed as a finite sum.

Theorem 8

For $t \in [N-1, N]$

$$X(t) = I + \sum_{s=1}^{N-1} \sum_{p=s}^{N-1} \sum_{S(s,p)} B_{j1}\, B_{j2} \dots B_{js} \frac{(t-p)^s}{s!}$$

with

$$1 \leqslant j_1, j_2, \dots, j_s \leqslant m$$

and

$$S(s, p) = \left\{ (j_1, j_2, \dots, j_s) \,\middle|\, \sum_{i=1}^{s} j_i = p \right\}$$

Necessary and sufficient conditions for pointwise degeneracy are given in terms of the fundamental matrix in the next theorem.

Theorem 9

A necessary and sufficient condition for system (21) to be pointwise degenerate at $t = k$ is that there exists a non-zero vector $\eta \in R^n$ such that

$$\eta^{\mathrm{T}} X(k) = 0$$

$$\eta^{\mathrm{T}} \sum_{i=0}^{m-j} X(k-\alpha-i) B_{i+j} = 0, \quad j = 1, 2, \dots, m$$

for all $\alpha \in (0, 1)$.

Using the previous two theorems it is easy to show

Theorem 10

System (21) is pointwise complete at any $t \in (0, 3)$. This result contrasts with that for the MDS which can be degenerated at $t \geqslant 2$.

It is well known that the MDS with m delays has a degenerate set $[k, \infty)$, where k can be greater than m. In Thowsen (1976) the author conjectured that k must be less or equal to m for the PDS and proved the conjecture for some special cases. It is therefore natural to introduce the notion of a proper system.

Definition 6

The PDS (21) with $B_m \neq 0$ is *proper* if there exists a non-zero vector η such that the set of degeneracy with respect to η is $[m, \infty)$. A constructive proof for the existence of proper systems has been developed. The proof was inspired by a recent paper by Asner (1975) in which nilpotent matrices of index r were skilfully used to construct a matrix $B = (B_1, B_2, \ldots, B_m)$ such that the MDS is pointwise degenerate for $t \geqslant r+1$. However, there is no assurance that the constructed delay feedback system is complete for all $t < r+1$. This construction has been suitably modified for the PDS to yield proper systems of order $n \geqslant m-1$. For $n = m-1$ we have

Theorem 11

Let

$$B_1 = H$$
$$B_{j+1} = H^{j+1} + H^j Z - ZB_j, \quad j = 1, 2, \ldots, m-2$$
$$B_m = -ZB_{m-1}$$

where H is the $(m-1)$-dimensional square matrix with unit entries on the first superdiagonal and all other entries equal to zero. Then system (21) is proper.

The construction of proper systems can be used to solve the following problem. Given the scalar dynamic system $\dot{x}_1(t) = u(t)$ at location I and the controller (which transmits signal u to the system) at location II. There is a unit transmission delay between locations I and II. Construct an augmented delay feedback control u such that $x_1(t) \equiv 0$ on $[k, \infty)$ with $k = 4$ as the smallest value of k.

Theorem 11 gives the solution (after coordinate transformation)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & -\frac{3}{4} & \frac{1}{2} \\ -1 & \frac{1}{2} & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-2) \\ x_2(t-2) \\ x_3(t-2) \end{bmatrix}$$
$$+ \begin{bmatrix} -1 & \frac{1}{2} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t-3) \\ x_2(t-3) \\ x_3(t-3) \end{bmatrix} + \begin{bmatrix} 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1(t-4) \\ x_2(t-4) \\ x_3(t-4) \end{bmatrix}$$

which clearly satisfies the requirement that information available to u at time t is $\{x_1(\tau), \tau \leqslant t-2\}$ and $\{x_i(\tau), \tau \leqslant t-1, i = 2, 3\}$.

6. Conclusions

Controllability, pointwise degeneracy and delay feedback controls are surveyed and their relationship discussed. Several recent results are stated and extensive literature references are given.

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A new construction for delay feedback control which yields function space null controllability is proposed. The method is based on augmenting the original system by a 'synthetic' system to enrich the class of feedback signals and overcome the inherent limitations caused by the form of the solution to differential–difference equations.

Finally, new results on the pointwise degeneracy of pure delay systems are given. The set of degeneracy is shown to be $[k, \infty)$ where $k \geqslant 3$ and it is conjectured that the degenerate set for a (degenerate) system with m delays is at least $[m, \infty)$. This leads to the notion of a proper delay system for which a constructive existence proof is given.

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