Short Papers

An Algebraic Criterion for Controllability of Linear Systems with Time Delay

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Abstract—**A new algebraic sufficient condition for controllability of linear time-varying delay-differential systems is established, which reduces to the conventional one for ordinary differential systems when the delay term is absent.**

I. Problem Formulation

We consider the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) + C(t)u(t), \quad t > t_0 \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ are real continuous matrix-valued functions and h is a positive constant. The initial function space is assumed to be $\mathcal{B} = \mathcal{C}([t_0 - h, t_0], R^n)$, the space of continuous functions mapping $[t_0 - h, t_0]$ into R^n, and the R^m-valued control function $u(\cdot)$ is measurable and bounded on every finite time interval. Under the given assumptions, solutions to (1) exist and are unique for given initial data. Let $x(t,t_0,\phi,u)$ denote a solution to (1) at time t corresponding to initial time t_0, initial function $\phi \in \mathcal{B}$ and input u. It is easy to show [1] that such a solution can be represented as

$$x(t,t_0,\phi,u) = x(t,t_0,\phi,0) + \int_{t_0}^{t} K(t,\tau)C(\tau)u(\tau)\, d\tau \tag{2}$$

where $K(t,\tau)$ is the $n \times n$ fundamental matrix for (1) and satisfies the equation

$$\begin{aligned}\partial K(t,\tau)/\partial\tau &= -K(t,\tau)A(\tau) - K(t,\tau+h)B(\tau+h), \quad t_0 \le \tau \le t-h\\ K(t,t) &= I\\ K(t,\tau) &= 0, \quad \text{for } \tau > t.\end{aligned} \tag{3}$$

Now consider the following definition.

Definition 1: The system (1) is R^n controllable to the origin from time t_0 if, for each $\phi \in \mathcal{B}$, there exists a finite time $t_1 > t_0$ and an admissible input u defined on $[t_0,t_1]$ such that $x(t_1,t_0,\phi,u) = 0$.

The following lemma is due to Chyung and Lee [2].

Lemma 1: The system (1) is R^n controllable to the origin from time t_0 if there exists a finite value of time $t_1 > t_0$ such that

$$\operatorname{rank} \int_{t_0}^{t_1} K(t_1,\tau)C(\tau)C'(\tau)K'(t_1,\tau)\, d\tau = n \tag{4}$$

where a prime indicates transpose.

It was shown by Weiss [3] that the condition (4) is necessary if the system (1) is *pointwise complete*, i.e., if the solutions to the homogeneous equation corresponding to all possible initial functions span Euclidian n space at each point in time.

The main difficulty with (4) is its computational intractibility due to the difficulty of computing $K(t,\tau)$. In the case of time-invariant systems, criteria for R^n controllability of (1) are available which depend only on the coefficients of the differential equation (see Kirillova and Čurakova [4]). Our objective is to give an algebraic criterion for R^n controllability of time-varying systems (1) which reduces to the sufficiency results of Kirillova and Čurakova in the time-invariant case. Our result greatly improves upon and includes, as a special case, the only previously reported algebraic controllability result in the literature for such time-varying systems (see Buckalo [5]).

II. Main Result

We begin by defining the matrix

$$Q(t) = [Q_1^1(t),\cdots,Q_1^n(t),Q_2^2(t-h),\cdots,Q_2^n(t-h),\cdots,\\ Q_n^n\,(t-(n-1)h)] \tag{5}$$

where

$$\begin{aligned}Q_1^1(t) &= C(t)\\ Q_j^{k+1}(t) &= (d/dt)Q_j^k(t) - A(t+(j-1)h)Q_j^k(t)\\ &\quad - B(t+(j-1)h)Q_{j-1}^k(t),\\ &\qquad j = 1,\cdots,k, \quad k = 1,\cdots,n\end{aligned}$$

and $Q_j^k = 0$, for $j = 0$ or $j > k$.

Theorem 1

If there exists t_1 such that rank $Q(t_1) = n$, then (1) is R^n controllable to the origin from some finite $t_0 < t_1$.

Proof: We shall show that rank $Q(t_1) = n$ implies (4). Fix t_1 and suppose (4) does not hold for any $t_0 < t_1$. Then there exists a nonzero vector $z \in R^n$ such that $z'K(t_1,\tau)C(\tau) = 0$, for all $\tau \le t_1$. In particular we have the set of equations

$$z'K(t_1,\tau)C(\tau) = 0, \quad \text{for all } \tau \in [t_1 - (k+1)h, t_1 - kh]\\ k = 0,1,\cdots,n. \tag{6}$$

Differentiating $(n-1)$ times we obtain, for $k = 0$,

$$z'[k(t_1,\tau)Q_1^i(\tau) + K(t_1,\tau+h)Q_2^i(\tau)] = 0, \quad \tau \in [t_1 - h, t_1],\\ i = 1,\cdots,n. \tag{7}$$

As $\tau \to t_1^-$, (7) becomes $z'Q_1^i(t_1) = 0$, $i = 1,\cdots,n$. As $\tau \to (t_1 - h)^+$, (7) becomes $z'K(t_1,t_1-h)Q_1^i(t_1-h) = 0$, $i = 1,\cdots,n$. Setting $k = 1$ in (6) and differentiating $(n-1)$ times again yields

$$z'[K(t_1,\tau)Q_1^i(\tau) + K(t_1,\tau+h)Q_2^i(t) + K(t_1,\tau+2h)Q_3^i(\tau)] = 0,\\ i = 1,\cdots,n. \tag{8}$$

As $\tau \to (t_1 - h)^-$, (8) becomes

$$z'Q_2^i(t_1 - h) = 0.$$

As $\tau \to (t_1 - 2h)^+$, (8) becomes

$$z'[K(t_1,t_1-2h)Q_1^i(t_1-2h) + K(t_1,t_1-h)Q_2^i(t_1-h)] = 0.$$

Continuing in this manner, for $k = n$, we obtain

$$z'Q_n^n(t_1 - (n-1)h) = 0.$$

That is,

$$z'Q_j^k(t_1 - (j-1)h) = 0, \quad j = 1,\cdots,k, \quad k = 1,\cdots,n.$$

From this it follows that rank $Q(t_1) \ne n$, which proves the theorem.

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Remark 1: For time-invariant systems with $A = 0$ the matrix Q becomes

$$Q = [C, BC, \cdots, B^{n-1}C].$$

Remark 2: For time-varying systems with $B(\cdot) \equiv 0$, the matrix $Q(t)$ becomes

$$Q(t) = [P_0(t), P_1(t), \cdots, P_{n-1}(t)]$$

where

$$P_0(t) = C(t), \quad P_k(t) = (d/dt)P_{k-1}(t) - A(t)P_{k-1}(t). \tag{9}$$

The reader is referred to [6] and [7] for controllability results for nonlinear delay-differential systems, obtained using the concept of Pfaffian forms.

III. Conclusions

A new sufficient condition for R^n controllability of linear time-varying delay-differential systems has been given which depends only on the coefficients of the differential equation. This is a significant advance since the fundamental matrix of a delay equation is considerably harder to compute than its counterpart in ordinary differential equations.

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Suboptimal Control of a Class of Discrete-Continuous Time Systems

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Abstract—**The synthesis of nonperiodic piecewise-constant feedback gains for a class of nonlinear systems with nonquadratic performance indices is presented in this paper. A mean-square error type performance index measuring the difference between the closed-loop and the open-loop control is minimized. A gradient algorithm is then used to synthesize the feedback gains and switching times for the suboptimal control law. The linear suboptimal control law so obtained is then compared to other recently published results.**

Introduction

The synthesis of suboptimal or near optimal feedback controls has received considerable attention recently [1]–[3]. In a paper by Nahi and Bettwy [1], it was shown that near optimal feedback controls of nonlinear systems with nonquadratic performance indices can be achieved by a suitably chosen measure of "closeness" to the optimal open-loop controls. The authors synthesize a feedback control law which is a nonlinear function of the observable states of the system. It is the purpose of this paper to show that in many cases (especially high-order systems) a simpler control structure can be used to achieve results similar to those of Nahi and Bettwy for a finite control interval $[t_0, t_f]$. In addition, the results of this paper are compared to those of Kleinman *et al.* [2] for linear systems with quadratic performance indices.

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I. Statement of the Problem and Solution

Consider a nonlinear dynamical system described by the vector differential equation

$$\dot{x} = f(x,u,t), \quad x(t_0) = x_0. \tag{1}$$

The state $x(t)$ is an n-dimensional vector in E^n, the n-dimensional Euclidean space; the control $u(t)$ is an r-dimensional vector belonging to the admissible control set Ω; and f is an n-dimensional vector in $C^{(1)}$ on $E^n \times \Omega$. The optimal open-loop control $u^*(t)$ transfers the initial state x_0 to some desired final state $x(t_f)$ in such a manner as to minimize a performance index which is not necessarily quadratic. The performance index is given by

$$J = \int_{t_0}^{t_f} L(x,u,t)\, dt \tag{2}$$

with $L(x,u,t)$ in $C^{(1)}$ on $E^n \times \Omega$. Now assume that a closed-loop solution $u(x)$ is desired for the same problem previously stated. Computational and implementational difficulties in synthesizing optimal closed-loop control laws suggest the synthesis of suboptimal closed-loop controls based upon open-loop controls. Suppose that a nominal or optimal open-loop control $u^*(t)$ generating a solution $x^*(t)$ of (1) is given or can be computed. The difference between $u^*(t)$ and a suboptimal closed-loop control $u(x)$ generating $x(t)$ will be defined as

$$\delta u(t) = u(x) - u^*(t). \tag{3}$$

Similarly, the difference in the solutions of (1) due to $u(x)$ and $u^*(t)$ will be

$$\delta x(t) = x(t) - x^*(t) \tag{4}$$

when

$$x(t_0) = x^*(t_0).$$

The difference in the states at the final time t_f due to the two controls can be approximated by a Taylor series expansion about $x^*(t)$ up to terms of second order such that

$$\delta x(t_f) \cong \int_{t_0}^{t_f} \phi(t,s) \left[\frac{\partial f}{\partial u}\right]_{x=x*} \delta u(s)\, ds \tag{5}$$

where $\phi(t,s)$ is the fundamental matrix solution of

$$\delta \dot{x} = \left[\frac{\partial f}{\partial x}\right]_{x=x*} \delta x, \quad \delta x(t_0) = 0 \tag{6}$$

with $\partial f_i/\partial x_j$ the i,jth element of the $n \times n$ matrix $[\partial f/\partial x]_{x=x*}$ evaluated along the nominal or optimal open-loop trajectory. Taking the norm of both sides of (5) and applying Schwarz's inequality yields

$$\| \delta x(t_f) \|^2 \le \left[\int_{t_0}^{t_f} \left\| \phi(t,s) \left[\frac{\partial f}{\partial u}\right]_{x=x*} \right\|^2 ds\right]^{1/2} \left[\int_{t_0}^{t_f} \| \delta u(s) \|^2 ds\right]^{1/2}. \tag{7}$$

The right-hand side of (7) represents an upper bound on the difference in the terminal states when driven by closed-loop and optimal open-loop controls. In addition, only the second bracket on the right-hand side of (7) is a function of $u(x)$. Therefore, a new performance index I can be defined

$$I \triangleq \int_{t_0}^{t_f} \| \delta u(s) \|^2 ds. \tag{8}$$