INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control (2016)
Published online in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.3711

On l_1 stability of switched positive singular systems with time-varying delay

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SUMMARY

This paper investigates the problem of exponential stability and l_1-gain performance analysis for a class of discrete-time switched positive singular systems with time-varying delay. Firstly, a necessary and sufficient condition of positivity for the system is established by using the singular value decomposition method. Then by constructing an appropriate co-positive Lyapunov functional and using the average dwell time scheme, we develop a sufficient delay-dependent condition and identify a class of switching signals for the switched positive singular system to be exponentially stable and meet a prescribed l_1-gain performance level under the switching signal. Based on this condition, the decay rate of the system can be tuned and the optimal system performance level can be determined by solving a convex optimization problem. All of the criteria obtained in this paper are presented in terms of linear programming, which suggests a good scalability and applicability to high dimensional systems. Finally, a numerical example is presented to demonstrate the effectiveness of the proposed method. Copyright © 2016 John Wiley & Sons, Ltd.

Received 16 September 2015; Revised 16 July 2016; Accepted 17 October 2016

KEY WORDS: switched positive singular systems; l_1-gain performance; exponential stability; time-varying delay; average dwell time; co-positive linear Lyapunov function

1. INTRODUCTION

In this paper, we are interested in the stability and performance analysis of switched positive singular systems (SPSSs). Our motivations come from the following aspects. Firstly, many physical systems in the real world involve variables that have non-negative property, like population level, concentration of substances, and absolute temperature, which are known as positive systems [1, 2] and are frequently encountered in various fields, for instance, pharmacokinetics [3], chemical reaction [4], internet congestion control [5], and system theories [6–10]. Secondly, singular systems provide a natural framework for modeling of any practical dynamic systems, such as power systems, economics systems, robotic systems, biological systems [11–13], and to describe a larger class of systems than state-space models [14]. Thirdly, many practical systems are always subject to abrupt variations in their structures and parameters, such as failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. One effective method of characterizing the abrupt changes is by using the switched system approach [15–17].

Therefore, the SPSS considered here is of practical importance and meanwhile poses theoretical challenges in its formal analysis because of the singularity of system model and the non-negativity of state variables. For positive systems, l_1-norm is usually used as a performance measure because it accounts for the sum of the quantities [18]. Recently, many significant results on positive systems

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have been obtained in literature, particularly with respect to the stability analysis by means of the linear co-positive Lyapunov functional (see [19–21]), and the corresponding existence conditions are formulated in the linear programming (LP) form.

Recently, there are some results on positive singular systems reported [22–24]. As pointed out in [22], the internal positivity property of continuous descriptor systems is studied. In [24], the problems of positivity and stability are addressed for a class of discrete-time positive delayed singular systems. It is widely recognized that few of the previous results are concerned with exponential stability analysis for positive singular systems. The main difficulty is how to estimate the exponential decay rate of the system. Therefore, the first aim of this work is to develop some effective methods to estimate the decay rate for positive singular systems.

There are only a few recent publications investigating SPSSs [25, 26]. In [25], the problems of stability for a class of switched positive descriptor systems with average dwell time switching were investigated. Reference [26] dealt with the problem of state feedback controller design for singular positive Markovian jump systems with partly known transition rates. However, both [25, 26] only focused on the autonomous case without taking time delay and exogenous disturbance into account. As time delay and exogenous disturbance are frequently encountered in practice, it is necessary and significant to further investigate SPSSs with those two kinds of phenomena. When taking time delays and exogenous disturbance into account, the problem of choosing an appropriate mode-dependent co-positive Lyapunov function and how to analyze the stability and system performance will be more complicated and challenging. Furthermore, one may expect to establish a relation between the switching law and the system performance (like the disturbance attenuation performance). Such a relation may provide us some useful guidelines to analyze the stability with a desired system performance. To the best of authors' knowledge, no relevant work that considers this kind of system has been published. Therefore, the main objective of this work is to investigate the exponential stability, the optimal l_1-gain performance for SPSSs with time-varying delay.

The main contributions of this paper are as follows: (i) provide a necessary and sufficient condition of positivity for SPSSs with time-varying delay by using the singular value decomposition method; (ii) identify a class of switching signals specified by average dwell time to guarantee the exponential stability of the considered system; (iii) derive a sufficient delay-dependent criterion for the first time with regard to the considered system to be exponentially stable with a prescribed l_1-gain performance by constructing an appropriate co-positive Lyapunov function and using the average dwell time scheme. Moreover, the decay rate can be tuned according to practical applications; (iv) propose a convex optimization problem for SPSSs with time-varying delay such that the optimal l_1-gain performance level can be determined; (v) establish the relation between the switching law, that is, the minimal average dwell time and the l_1-gain performance level implicitly through the obtained results; and (vi) present all the conditions obtained in this paper in terms of LP.

The remainder of the paper is as follows. The problem formulation and some necessary lemmas are provided in Section 2. In Section 3, the positivity, exponential stability, and l_1-gain performance analysis for the considered system are developed. A numerical example is presented to demonstrate the effectiveness of proposed method in Section 4. In Section 5, concluding remarks are given.

Notations: $A \succeq 0\,(\preceq, \succ, \prec)$ means that all entries of matrix A are non-negative (non-positive, positive, and negative); $A \succ B$ ($A \succeq B$) means that $A - B \succ 0$ ($A - B \succeq 0$). A^T means the transpose of matrix A. det (A) denotes the determinant of matrix A; deg $(\cdot)$ denotes the degree of a polynomial; rank (A) denotes the rank of matrix A. R (R_+) is the set of all real (positive real) numbers; R^n is n-dimensional real vector space; R^n_+ is the set of all n-dimensional positive real vectors; $R^{m \times n}$ is the set of all $m \times n$-dimensional real matrices. Z^+ refers to the set of all positive integers. $1_n \in R^n$ denotes a column vector with each entry equal to 1; $[A]_{pq}$ means the (p, q)-th entry of a matrix A and $[x]_p$ means the p-th entry of a vector x. $\|x\|_1 = \sum_{p=1}^{n} \left|[x]_p\right|$ is the 1-norm of a vector $x \in R^n$; $\|A\|_1 = \max_{1 \leqslant q \leqslant m} \sum_{p=1}^{n} \left|[A]_{pq}\right|$ is the 1-norm of a matrix $A \in R^{n \times m}$. $\|\nu\|_{l_1} = \sum_{k=k_0}^{\infty} \|\nu(k)\|_1$ is the l_1-norm of a sequence of vectors $\nu(k)$, $k \geqslant k_0$, l_1 means the space of all vectors $\nu(k)$ having a finite l_1-norm, that is, $l_1 = \left\{\nu(k), k \geqslant k_0 \middle|\, \|\nu\|_{l_1} < \infty\right\}$. $\bar{\rho}(\nu)$ stands for the maximal element of ν, and similarly, $\underline{\rho}(\nu)$ stands for the minimal element of ν, for any $\nu \in R^n$. All matrices and vectors are assumed to have compatible dimensions for algebraic operations, if their dimensions are not explicitly stated.

DOI: 10.1002/rnc

2. PROBLEM FORMULATION

A discrete-time linear switched singular system with time-varying delay can be stated as

$$\begin{cases} Ex(k+1) = A_{\sigma(k)}x(k) + A_{d\sigma(k)}x(k-d(k)) + F_{\sigma(k)}w(k), \\ z(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}w(k), \\ x(\theta) = \varphi(\theta), \quad \theta = -d_2, -d_2+1, \ldots, 0, \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state vector, $w(k) \in R^u$ is the disturbance input which belongs to l_1, and $z(k) \in R^m$ is the controlled output. $d(k)$ is a time-varying delay, which is assumed to satisfy $d(k) \in \{d_1, \ldots, d_2\}$ with $d_1 \geqslant 0$. $\varphi(\theta)$ is a given discrete vector-valued initial condition of the system. $\{\sigma(k), k \geqslant 0\}$ is a switching signal taking values in a finite set $\underline{N} = \{1, 2, \cdots, N\}$ with N being the number of subsystems; moreover, $\sigma(k) = i$ denotes the i-th subsystem is activated, $i \in \underline{N}$. Meanwhile, for switching time sequence $k_0 < k_1 < k_2 < \cdots$ of the switching signal $\sigma(k)$, the holding time between k_l and k_{l+1} is called the dwell time of the currently engaged l-th subsystem, where l is a non-negative integer. A_i, A_{di}, F_i, C_i, and D_i, $i \in \underline{N}$, are known constant matrices with appropriate dimensions. $E \in R^{n\times n}$ is singular and rank$E = r < n$. Without loss of generality, we assume that System (1) is activated at $k_0 = 0$.

Definition 1 ([27]*)*
System (1) is said to be

1. regular if every pair (E, A_i) is regular, that is, $\det(sE - A_i) \neq 0$, $\forall i \in \underline{N}$;
2. causal if every pair (E, A_i) is causal, that is, degree$(\det(sE - A_i)) = $ rank$E$, $\forall i \in \underline{N}$;
3. exponentially stable, if for any initial conditions $\varphi(\theta)$ and the switching signal $\sigma(k)$, the solution $x(k)$ of System (1) with $w(k) = 0$ satisfies $\|x(k)\|_1 \leqslant \iota\chi^k \|\varphi\|_{1c}$, $\forall k \geqslant 0$, where $\|\varphi\|_{1c} = \sup_{-d_2\leqslant\theta\leqslant 0} \|\varphi(\theta)\|_1$, $0 < \chi < 1$ and $\iota > 0$ are called the decay rate and decay efficient, respectively.

Definition 2 ([28]*)*
System (1) is said to be positive if, for any initial conditions $\varphi(\theta)\succeq 0$, $w(k)\succeq 0$ and any switching signals $\sigma(k)$, we have $x(k)\succeq 0$ and $z(k)\succeq 0$ for all $k \geqslant 0$.

For System (1), because rank$E = r < n$, then there exist non-singular matrices P, $Q \in R^{n\times n}$ such that $PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. In this paper, for simplicity, let $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and other system matrices have the following form, $\forall i \in \underline{N}$:

$$A_i = \begin{bmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{bmatrix}, A_{di} = \begin{bmatrix} A_{di1} & A_{di2} \\ A_{di3} & A_{di4} \end{bmatrix}, F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix}, C_i = \begin{bmatrix} C_{i1} & C_{i2} \end{bmatrix}. \tag{2}$$

Lemma 1 ([29]*)*
System (1) with (2) is causal if and only if A_{i4} is non-singular, $\forall i \in \underline{N}$.

In the light of Lemma 1, when System (1) with (2) is regular and causal, under the coordinate transformation $x(k) = [x_1{}^T(k) \quad x_2{}^T(k)]^T$, $x_1(k) \in R^r$, $x_2(k) \in R^{n-r}$, System (1) can be rewritten as

$$\begin{cases} x_1(k+1) = \bar{A}_{\sigma(k)1}x_1(k) + \bar{A}_{d\sigma(k)1}x_1(k-d(k)) + \bar{A}_{d\sigma(k)2}x_2(k-d(k)) + \bar{F}_{\sigma(k)1}w(k), \\ x_2(k) = \bar{A}_{\sigma(k)3}x_1(k) + \bar{A}_{d\sigma(k)3}x_1(k-d(k)) + \bar{A}_{d\sigma(k)4}x_2(k-d(k)) + \bar{F}_{\sigma(k)2}w(k), \\ z(k) = C_{\sigma(k)1}x_1(k) + C_{\sigma(k)2}x_2(k) + D_{\sigma(k)}w(k), \\ x_1(\theta) = \varphi_1(\theta), \quad \theta = -d_2, -d_2+1, \cdots, 0, \\ x_2(\theta) = \varphi_2(\theta), \quad \theta = -d_2, -d_2+1, \cdots, 0, \end{cases} \tag{3}$$

where for each $i \in \underline{N}$ in (3),

$$\bar{A}_{i1} = A_{i1} - A_{i2}A_{i4}^{-1}A_{i3}, \quad \bar{A}_{di1} = A_{di1} - A_{i2}A_{i4}^{-1}A_{di3},$$

DOI: 10.1002/rnc

$$\bar{A}_{di2} = A_{di2} - A_{i2}A_{i4}^{-1}A_{di4}, \quad \bar{F}_{i1} = F_{i1} - A_{i2}A_{i4}^{-1}F_{i2},$$

$$\bar{A}_{i3} = -A_{i4}^{-1}A_{i3}, \quad \bar{A}_{di3} = -A_{i4}^{-1}A_{di3}, \bar{A}_{di4} = -A_{i4}^{-1}A_{di4}, \quad \bar{F}_{i2} = -A_{i4}^{-1}F_{i2}.$$

Lemma 2 ([24]*)*
Assume that (E, A_i), $\forall i \in \underline{N}$, is regular and causal, then System (1) with (2) is positive if and only if System (3) is positive.

Remark 1
By Definition 1, one can see that the regularity, causality, and exponential stability of System (1) are equivalent to those of System (3). Furthermore, Systems (1) and (3) have the same disturbance input $w(k)$, controlled output $z(k)$, and the same relation from $w(k)$ to $z(k)$, so they satisfy the same l_1-gain performance. Thus, the exponential stability and l_1-gain performance for System (1) can be analyzed by investigating the corresponding problem for System (3).

Definition 3 ([20, 30]*)*
For $\lambda > 0$ and $\gamma > 0$, discrete-time System (1) is said to have a prescribed l_1-gain performance level γ if there exists a switching signal $\sigma(k)$ such that the following conditions are satisfied:

(1) System (1) is regular, causal, and exponentially stable when $w(k) = 0$,
(2) Under zero initial condition, that is, $\varphi(\theta) = 0$, $\theta = -d_2, -d_2 + 1, \ldots, 0$, System (1) satisfies

$$\sum_{k=0}^{\infty} \lambda^k \|z(k)\|_1 < \gamma \sum_{k=0}^{\infty} \|w(k)\|_1, \forall w(k) \in l_1. \tag{4}$$

Definition 4 ([16]*)*
For the switching signal $\sigma(k)$ and any $T_2 > T_1 \geqslant 0$, let $N_\sigma(T_1, T_2)$ denotes the number of switches of $\sigma(k)$ over the interval $[T_1, T_2)$. For given $T_a > 0$ and $N_0 \geqslant 0$, if the inequality

$$N_\sigma(T_1, T_2) \leqslant N_0 + \frac{(T_2 - T_1)}{T_a} \tag{5}$$

holds, then the positive constant T_a is called the average dwell time and N_0 is called the chattering bound. As commonly used in the literature, we choose $N_0 = 0$.

The problem to be addressed in this paper can be formulated as follows:

(i) Provide a necessary and sufficient condition of the positivity for System (1);
(ii) Develop a sufficient delay-dependent LP condition and identify a class of switching signals $\sigma(k)$ such that System (1) is exponentially stable and meets a prescribed l_1-gain performance level.
(iii) Propose a convex optimization problem for System (1), such that the optimal l_1-gain performance level can be determined.

3. MAIN RESULTS

In this section, we aim at investigating the l_1 stability of the discrete-time SPSS (1). Firstly, we will present a necessary and sufficient condition for the positivity of System (1).

Theorem 1
Assume that (E, A_i), $\forall i \in \underline{N}$ is regular and causal. System (1) is positive if and only if $\bar{A}_{i1} \succeq 0$, $\bar{A}_{di1} \succeq 0$, $\bar{A}_{di2} \succeq 0$, $\bar{F}_{i1} \succeq 0$, $\bar{A}_{i3} \succeq 0$, $\bar{A}_{di3} \succeq 0$, $\bar{A}_{di4} \succeq 0$, $\bar{F}_{i2} \succeq 0$, $C_{i1} \succeq 0$, $C_{i2} \succeq 0$, $D_i \succeq 0$, $\forall i \in \underline{N}$, where $\bar{A}_{i1}$, $\bar{A}_{di1}$, $\bar{A}_{di2}$, $\bar{F}_{i1}$, $\bar{A}_{i3}$, $\bar{A}_{di3}$, $\bar{A}_{di4}$, $\bar{F}_{i2}$, C_{i1}, C_{i2} are defined in (3).

Proof
Sufficiency. We have to show that System (1) is positive. For $T = 0$ and $k_0 = 0$, we obviously have from the initial condition $\varphi(\theta) \succeq 0$, $\theta = -d_2, -d_2 + 1, \ldots, 0$ that $x(0) \succeq 0$.

DOI: 10.1002/rnc

For any $T > 0$ and $k_0 = 0$, we denote $k_1, k_2, \ldots, k_l, k_{l+1}, \ldots, k_{N_\sigma(k_0,T)}$ the switching instances on the interval $[k_0, T)$. First, we prove that the solution $x(k) = [x_1^T(k) \quad x_2^T(k)]^T$ is positive on $[k_0, k_1)$. It follows from Lagrange's formula and (3) that, $\forall k \in [k_0, k_1)$,

$$\begin{aligned} x_1(k) &= \bar{A}^k_{\sigma(k_0)1} x_1(k_0) + \sum_{s=k_0}^{k-1} \bar{A}^{k-s-1}_{\sigma(k_0)1} [\bar{A}_{d\sigma(k_0)1} x_1(s-d(s)) \\ &\quad + \bar{A}_{d\sigma(k_0)2} x_2(s-d(s)) + \bar{F}_{\sigma(k_0)1} w(s)], \\ &= \bar{A}^k_{\sigma(k_0)1} x_1(k_0) + \sum_{s=k_0}^{k-1} \bar{A}^{k-s-1}_{\sigma(k_0)1} \bar{A}_{d\sigma(k_0)1} x_1(s-d(s)) \\ &\quad + \sum_{s=k_0}^{k-1} \bar{A}^{k-s-1}_{\sigma(k_0)1} \bar{A}_{d\sigma(k_0)2} x_2(s-d(s)) + \sum_{s=k_0}^{k-1} \bar{A}^{k-s-1}_{\sigma(k_0)1} \bar{F}_{\sigma(k_0)1} w(s), \end{aligned} \tag{6}$$

$$x_2(k) = \bar{A}_{\sigma(k_0)3} x_1(k) + \bar{A}_{d\sigma(k_0)3} x_1(k-d(k)) + \bar{A}_{d\sigma(k_0)4} x_2(k-d(k)) + \bar{F}_{\sigma(k_0)2} w(k), \tag{7}$$

$$z(k) = C_{\sigma(k_0)1} x_1(k) + C_{\sigma(k_0)2} x_2(k) + D_{\sigma(k_0)} w(k). \tag{8}$$

By Theorem 1, if $\bar{A}_{\sigma(k_0)1} \succeq 0$, $\bar{A}_{d\sigma(k_0)1} \succeq 0$, $\bar{A}_{d\sigma(k_0)2} \succeq 0$, $\bar{F}_{\sigma(k_0)1} \succeq 0$, $\bar{A}_{\sigma(k_0)3} \succeq 0$, $\bar{A}_{d\sigma(k_0)3} \succeq 0$, $\bar{A}_{d\sigma(k_0)4} \succeq 0$, $\bar{F}_{\sigma(k_0)2} \succeq 0$, $C_{\sigma(k_0)1} \succeq 0$, $C_{\sigma(k_0)2} \succeq 0$, and $D_{\sigma(k_0)} \succeq 0$, it is straightforward to obtain $x_1(k) \succeq 0$, $x_2(k) \succeq 0$, $z(k) \succeq 0$, $\forall k \in [k_0, k_1)$. Now, similar to the aforementioned process, we have, $\forall k \in [k_1, k_2)$,

$$\begin{aligned} x_1(k) &= \bar{A}^k_{\sigma(k_1)1} x_1(k_1) + \sum_{s=k_1}^{k-1} \bar{A}^{k-s-1}_{\sigma(k_1)1} [\bar{A}_{d\sigma(k_1)1} x_1(s-d(s)) \\ &\quad + \bar{A}_{d\sigma(k_1)2} x_2(s-d(s)) + \bar{F}_{\sigma(k_1)1} w(s)] \succeq 0, \end{aligned}$$

$$\begin{aligned} x_2(k) &= \bar{A}_{\sigma(k_1)3} x_1(k) + \bar{A}_{d\sigma(k_1)3} x_1(k-d(k)) \\ &\quad + \bar{A}_{d\sigma(k_1)4} x_2(k-d(k)) + \bar{F}_{\sigma(k_1)2} w(k) \succeq 0, \end{aligned}$$

$$z(k) = C_{\sigma(k_1)1} x_1(k) + C_{\sigma(k_1)2} x_2(k) + D_{\sigma(k_1)} w(k) \succeq 0.$$

Recursively, if $\bar{A}_{i1} \succeq 0$, $\bar{A}_{di1} \succeq 0$, $\bar{A}_{di2} \succeq 0$, $\bar{F}_{i1} \succeq 0$, $\bar{A}_{i3} \succeq 0$, $\bar{A}_{di3} \succeq 0$, $\bar{A}_{di4} \succeq 0$, $\bar{F}_{i2} \succeq 0$, $C_{i1} \succeq 0$, $C_{i2} \succeq 0$, and $D_i \succeq 0$, $\forall i \in \underline{N}$, we can obtain that $x_1(T) \succeq 0$, $x_2(T) \succeq 0$, $z(T) \succeq 0$, $\forall T \geqslant 0$.

Necessity. Conversely, we first suppose that there exists an element $[\bar{A}_{i1}]_{pq} < 0$, $\forall i \in \underline{N}$, $p \neq q$, where $[\bar{A}_{i1}]_{pq}$ is the (p, q)-th scalar entry of $\bar{A}_{i1}$. From (3), we have

$$\begin{aligned} x_{1p}(k+1) = & \sum_{g=1, g\neq p, g\neq q}^{r} [\bar{A}_{i1}]_{pg} x_{1g}(k) + [\bar{A}_{i1}]_{pp} x_{1p}(k) + [\bar{A}_{i1}]_{pq} x_{1q}(k) \\ & + \sum_{g=1}^{r} [\bar{A}_{di1}]_{pg} x_{1g}(k-d(k)) + \sum_{g=r+1}^{n} [\bar{A}_{di2}]_{pg} x_{2g}(k-d(k)) + \sum_{g=1}^{u} [\bar{F}_{i1}]_{pg} w_g(k), \end{aligned}$$

where $x_{1g}(k)$, $x_{2g}(k)$ and $w_g(k)$ represent the g-th element of $x_1(k)$, $x_2(k)$ and $w(k)$, respectively. Then if $x_{1q}(k) \neq 0$, we can see that $x_{1p}(k+1) < 0$ is possible. It follows that System (1) is not positive. Second, further assume that $\bar{A}_{di1}$ has an element $[\bar{A}_{di1}]_{pq} < 0$, $\forall i \in \underline{N}$, we have

$$\begin{aligned} x_{1p}(k+1) = & \sum_{g=1, g\neq p}^{r} [\bar{A}_{i1}]_{pg} x_{1g}(k) + [\bar{A}_{i1}]_{pp} x_{1p}(k) + \sum_{g=1, g\neq q}^{r} [\bar{A}_{di1}]_{pg} x_{1g}(k-d(k)) \\ & + [\bar{A}_{di1}]_{pq} x_{1q}(k-d(k)) + \sum_{g=1}^{u} [\bar{F}_{i1}]_{pg} w_g(k) + \sum_{g=r+1}^{n} [\bar{A}_{di2}]_{pg} x_{2g}(k-d(k)). \end{aligned}$$

DOI: 10.1002/rnc

Then if $x_{1q}(k-d(k)) \neq 0$, we can see that $x_{1p}(k+1) < 0$ is possible. It follows that System (1) is not positive. Similarly, if $\bar{A}_{di2}$ has an element $[\bar{A}_{di2}]_{pq} < 0$, or $\bar{F}_{i1}$ has an element $[\bar{F}_{i1}]_{pq} < 0$, one can get that System (1) is not positive.

Continuously, suppose that there exists an element $[\bar{A}_{i3}]_{pq} < 0$, $\forall i \in \underline{N}$, $p \neq q$, where $[\bar{A}_{i3}]_{pq}$ is the (p,q)-th scalar entry of $\bar{A}_{i3}$. From System (3), we obtain

$$\begin{aligned} x_{2p}(k) = & \sum_{g=1, g\neq p, g\neq q}^{r} [\bar{A}_{i3}]_{pg} x_{1g}(k) + [\bar{A}_{i3}]_{pp} x_{1p}(k) + [\bar{A}_{i3}]_{pq} x_{1q}(k) \\ & + \sum_{g=1}^{r} [\bar{A}_{di3}]_{pg} x_{1g}(k-d(k)) + \sum_{g=r+1}^{n} [\bar{A}_{di4}]_{pg} x_{2g}(k-d(k)) + \sum_{g=1}^{u} [\bar{F}_{i2}]_{pg} w_g(k). \end{aligned}$$

Then if $x_{1q}(k) \neq 0$, we can see that $x_{2p}(k) < 0$ is possible. It follows that System (1) is not positive. Similarly, if $\bar{A}_{di3}$ has an element $[\bar{A}_{di3}]_{pq} < 0$, or $\bar{A}_{di4}$ has an element $[\bar{A}_{di4}]_{pq} < 0$, or $\bar{F}_{i2}$ has an element $[\bar{F}_{i2}]_{pq} < 0$, $\forall i \in \underline{N}$, one can get that System (1) is not positive.

Finally, suppose that there exists an element $[C_{i1}]_{pq} < 0$, $\forall i \in \underline{N}$, $p \neq q$, where $[C_{i1}]_{pq}$ is the (p,q)-th scalar entry of $[C_{i1}]$. From System (3), we obtain

$$\begin{aligned} z_p(k) = & \sum_{g=1, g\neq p, g\neq q}^{r} [C_{i1}]_{pg} x_{1g}(k) + [C_{i1}]_{pp} x_{1p}(k) + [C_{i1}]_{pq} x_{1q}(k) \\ & + \sum_{g=r+1}^{n} [C_{i2}]_{pg} x_{2g}(k) + \sum_{g=1}^{u} [D_i]_{pg} w_g(k). \end{aligned}$$

Then if $x_{1q}(k) \neq 0$, we can see that $z_p(k) < 0$ is possible. It follows that System (1) is not positive. Similarly, if C_{i2} has an element $[C_{i2}]_{pq} < 0$, or D_i has an element $[D_i]_{pq} < 0$, $\forall i \in \underline{N}$, one can get that System (1) is not positive.

That is to say, System (1) is positive under arbitrary switching signals, if and only if $\bar{A}_{i1} \succeq 0$, $\bar{A}_{di1} \succeq 0$, $\bar{A}_{di2} \succeq 0$, $\bar{F}_{i1} \succeq 0$, $\bar{A}_{i3} \succeq 0$, $\bar{A}_{di3} \succeq 0$, $\bar{A}_{di4} \succeq 0$, $\bar{F}_{i2} \succeq 0$, $C_{i1} \succeq 0$, $C_{i2} \succeq 0$, $D_i \succeq 0$, and $\forall i \in \underline{N}$.

This completes the proof. □

Remark 2

In the case of $E = I$, positivity condition of discrete-time positive switched non-singular system with time-varying delay has been proposed in [18], that is, System (1) with $E = I$ is positive if and only if $A_i \succeq 0$, $A_{di} \succeq 0$, $F_i \succeq 0$, $C_i \succeq 0$, $D_i \succeq 0$, and $\forall i \in \underline{N}$.

In the sequence, by applying the average dwell time approach and the co-positive Lyapunov function technique, a sufficient condition for l_1 stability of System (1) is proposed.

Theorem 2

Suppose that (E, A_i), $\forall i \in \underline{N}$ is regular and causal. Let $0 < \alpha < 1$ and $\gamma > 0$ be prescribed scalars, $\bar{A}_{i1} \succeq 0$, $\bar{A}_{di1} \succeq 0$, $\bar{A}_{di2} \succeq 0$, $\bar{F}_{i1} \succeq 0$, $\bar{A}_{i3} \succeq 0$, $\bar{A}_{di3} \succeq 0$, $\bar{A}_{di4} \succeq 0$, $\bar{F}_{i2} \succeq 0$, $C_{i1} \succeq 0$, $C_{i2} \succeq 0$, $D_i \succeq 0$, if $0 < \left\|\bar{A}_{di4}\right\|_1 < 1$, and there exist vectors ν_i, υ_{1i}, υ_{2i}, $\upsilon_{3i} \in R_+^n$, such that, $\forall i \in \underline{N}$,

$$\left[\left(A_i^T - \alpha E^T\right)\nu_i + \upsilon_{1i} + \upsilon_{2i} + (d_{12}+1)\upsilon_{3i} + C_i^T 1_m\right] \preceq 0, \tag{9}$$

$$A_{di}^T \nu_i - \alpha^{d_2} \upsilon_{3i} \preceq 0, \tag{10}$$

$$F_i^T \nu_i + D_i^T 1_m - \gamma 1_u \preceq 0, \tag{11}$$

where $d_{12} = d_2 - d_1$, then System (1) is positive and exponentially stable with its decay rate estimate $\beta = \alpha\mu^{1/T_a}$ and achieves a prescribed l_1-gain performance level γ for any switching signal $\sigma(k)$ with average dwell time

$$T_a > T_a^* = -\frac{\ln \mu}{\ln \alpha}, \tag{12}$$

where $\mu \geqslant 1$ satisfies

$$\nu_i \preceq \mu \nu_j, \quad \upsilon_{1i} \preceq \mu \upsilon_{1j}, \quad \upsilon_{2i} \preceq \mu \upsilon_{2j}, \quad \upsilon_{3i} \preceq \mu \upsilon_{3j}, \quad \forall i, j \in \underline{N}. \tag{13}$$

Proof

First, we prove that System (1) with $w(k) = 0$ is exponentially stable. Choose the following co-positive Lyapunov functional:

$$V_i(k) = V_{1i}(k) + V_{2i}(k), \quad k \in [k_l, k_{l+1}), \tag{14}$$

where

$$V_{1i}(k) = x^T(k)E^T \nu_i,$$
$$V_{2i}(k) = \sum_{s=k-d_1}^{k-1} \alpha^{k-s-1} x^T(s)\upsilon_{1i} + \sum_{s=k-d_2}^{k-1} \alpha^{k-s-1} x^T(s)\upsilon_{2i} + \sum_{s=-d_2}^{-d_1} \sum_{m=k+s}^{k-1} \alpha^{k-m-1} x^T(m)\upsilon_{3i}.$$

Thus, we have

$$\begin{aligned} V_{1i}(k+1) - \alpha V_{1i}(k) &= x^T(k+1)E^T \nu_i - \alpha x^T(k)E^T \nu_i \\ &= x^T(k)A_i^T \nu_i + x^T(k-d(k))A_{di}^T \nu_i - \alpha x^T(k)E^T \nu_i, \end{aligned} \tag{15}$$

$$\begin{aligned} V_{2i}(k+1) - \alpha V_{2i}(k) = &\sum_{s=k+1-d_1}^{k} \alpha^{k-s} x^T(s)\upsilon_{1i} + \sum_{s=k+1-d_2}^{k} \alpha^{k-s} x^T(s)\upsilon_{2i} \\ &+ \sum_{s=-d_2}^{-d_1} \sum_{m=k+1+s}^{k} \alpha^{k-m} x^T(m)\upsilon_{3i} - \alpha \sum_{s=k-d_1}^{k-1} \alpha^{k-s-1} x^T(s)\upsilon_{1i} \\ &- \alpha \sum_{s=k-d_2}^{k-1} \alpha^{k-s-1} x^T(s)\upsilon_{2i} - \alpha \sum_{s=-d_2}^{-d_1} \sum_{m=k+s}^{k-1} \alpha^{k-m-1} x^T(m)\upsilon_{3i} \\ = &\, x^T(k)\upsilon_{1i} - \alpha^{d_1} x^T(k-d_1)\upsilon_{1i} + x^T(k)\upsilon_{2i} - \alpha^{d_2} x^T(k-d_2)\upsilon_{2i} \\ &+ \sum_{s=-d_2}^{-d_1} \left[x^T(k)\upsilon_{3i} - \alpha^{-s} x^T(k+s)\upsilon_{3i}\right] \\ \leqslant &\, x^T(k)[\upsilon_{1i} + \upsilon_{2i} + (d_{12}+1)\upsilon_{3i}] - \alpha^{d_1} x^T(k-d_1)\upsilon_{1i} \\ &- \alpha^{d_2} x^T(k-d_2)\upsilon_{2i} - \alpha^{d_2} x^T(k-d(k))\upsilon_{3i}. \end{aligned} \tag{16}$$

Add the terms (15)–(16) into $V_i(k+1) - \alpha V_i(k)$ and from (9)–(10), we have

$$\begin{aligned} V_i(k+1) - \alpha V_i(k) \leqslant &\, x^T(k)\left[A_i^T \nu_i - \alpha E^T \nu_i + \upsilon_{1i} + \upsilon_{2i} + (d_{12}+1)\upsilon_{3i}\right] \\ &+ x^T(k-d(k))\left[A_{di}^T \nu_i - \alpha^{d_2}\upsilon_{3i}\right] \\ &- \alpha^{d_1} x^T(k-d_1)\upsilon_{1i} - \alpha^{d_2} x^T(k-d_2)\upsilon_{2i} \leqslant 0. \end{aligned} \tag{17}$$

When $k \in [k_l, k_{l+1})$, from (17), we have

$$V_{\sigma(k)}(k) \leqslant \alpha^{k-k_l} V_{\sigma(k_l)}(k_l). \tag{18}$$

From (13) and (14), at the switching instant k_l, we can easily obtain

$$V_{\sigma(k_l)}(k_l) \leqslant \mu V_{\sigma(k_l^-)}(k_l^-), \quad \forall l = 1, 2, \ldots, \tag{19}$$

DOI: 10.1002/rnc

where k_l^- denotes the left limitation of k_l. Then, it follows from (12), (18), (19), and the relation $k = N_\sigma(k_0, k) \leqslant \frac{k-k_0}{T_a}$ that

$$\begin{aligned} V_{\sigma(k)}(k) &\leqslant \alpha^{k-k_l} V_{\sigma(k_l)}(k_l) \leqslant \alpha^{k-k_l} \mu V_{\sigma(k_l^-)}(k_l^-) \leqslant \alpha^{k-k_l} \mu \alpha^{k_l-k_{l-1}} V_{\sigma(k_{l-1})}(k_{l-1}) \\ &\leqslant \cdots \leqslant \mu^{N_\sigma(k_0,k)} \alpha^{k-k_0} V_{\sigma(k_0)}(k_0) \leqslant \left(\alpha\mu^{\frac{1}{T_a}}\right)^{(k-k_0)} V_{\sigma(k_0)}(k_0). \end{aligned} \tag{20}$$

On the other hand, let $\nu_i = [\nu_{i1}^T \quad \nu_{i2}^T]^T$, with $\nu_{i1} \in R^r$, $\nu_{i2} \in R^{n-r}$, then it follows from the Lyapunov functional (14) that $V_{1i}(k) = x_1^T(k)\nu_{i1}$.

According to (14) and (20), we obtain

$$V_{\sigma(k)}(k) \geqslant \beta_1 \|x_1(k)\|_1 , \tag{21}$$

$$V_{\sigma(k_0)}(k_0) \leqslant \beta_2 \|\varphi\|_{1c} , \tag{22}$$

where $\beta_1 = \min_{\forall i\in \underline{N}} \underline{\rho}(\nu_{i1})$ and $\beta_2 = \max_{\forall i\in \underline{N}} \bar{\rho}(\nu_{i1}) + d_1 \max_{\forall i\in \underline{N}} \bar{\rho}(\upsilon_{1i}) + d_2 \max_{\forall i\in \underline{N}} \bar{\rho}(\upsilon_{2i}) + (d_{12}+1)\frac{1-\alpha^{-d_2}}{1-\alpha} \max_{\forall i\in \underline{N}} \bar{\rho}(\upsilon_{3i})$. Then, combining (20), (21), and (22) yields

$$\|x_1(k)\|_1 \leqslant \frac{\beta_2}{\beta_1}\left(\alpha\mu^{\frac{1}{T_a}}\right)^k \|\varphi\|_{1c} = \tau\beta^k \|\varphi\|_{1c}, \quad \forall k \geqslant 0, \tag{23}$$

where $\tau = \frac{\beta_2}{\beta_1} > 1$ and $\beta = \alpha\mu^{\frac{1}{T_a}}$. Therefore, from the average dwell time condition $T_a > T_a^* = -(\ln\mu/\ln\alpha)$, one can readily obtain $\beta < 1$, which means that $x_1(k)$ is exponentially stable with decay rate β. Moreover, we can easily obtain from (23) that

$$\|x_1(k)\|_1 \leqslant \tau\beta^{-d_2} \|\varphi\|_{1c} \beta^k, \quad \forall k \geqslant 0. \tag{24}$$

Next, we will prove the second component solution $x_2(k)$ of the system is exponentially stable with the same decay rate β. Let us denote $p(k) = \bar{A}_{i3}x_1(k) + \bar{A}_{di3}x_1(k-d(k))$. Observe that, if $k > d(k)$, then

$$\|x_1(k-d(k))\|_1 \leqslant \frac{\beta_2}{\beta_1}\beta^{k-d(k)} \|\varphi\|_{1c} \leqslant \frac{\beta_2}{\beta_1}\beta^k\beta^{-d_2} \|\varphi\|_{1c} = \tau\beta^{-d_2} \|\varphi\|_{1c} \beta^k. \tag{25}$$

For $k \in [0, d(k)]$, we have

$$\begin{aligned} \|x_1(k-d(k))\|_1 &= \|\varphi_1\|_{1c} \leqslant \|\varphi\|_{1c} \leqslant \|\varphi\|_{1c} \beta^{(k-d(k))} \leqslant \beta^{-d_2} \|\varphi\|_{1c} \beta^k \\ &\leqslant \tau\beta^{-d_2} \|\varphi\|_{1c} \beta^k. \end{aligned} \tag{26}$$

From (25) and (26), we have

$$\|x_1(k-d(k))\|_1 \leqslant \tau\beta^{-d_2} \|\varphi\|_{1c} \beta^k, \forall k \geqslant 0. \tag{27}$$

By the notation of vector function $p(k)$, we obtain from (24) and (27) that

$$\|p(k)\|_1 \leqslant \left\|\bar{A}_{i3}\right\|_1 \|x_1(k)\|_1 + \left\|\bar{A}_{di3}\right\|_1 \|x_1(k-d(k))\|_1 \leqslant \tau_1 \|\varphi\|_{1c} \beta^k, \quad \forall k \geqslant 0, \tag{28}$$

where $\tau_1 = \tau\beta^{-d_2}(\left\|\bar{A}_{i3}\right\|_1 + \left\|\bar{A}_{di3}\right\|_1)$. Moreover, from the second equation of (3), we have

$$x_2(k) = \bar{A}_{di4}x_2(k-d(k)) + p(k). \tag{29}$$

Therefore, $\forall k \geqslant 0$,

$$\|x_2(k)\|_1 \leqslant \left\|\bar{A}_{di4}\right\|_1 \|x_2(k-d(k))\|_1 + \|p(k)\|_1 . \tag{30}$$

Set $\delta = \max\{\tau_1; \beta^{-d_2}\}$. If $k \in [0, d(k)]$, then $k - d(k) \in [-d(k), 0]$.

So from (30), we have

$$\|x_2(k)\|_1 \leqslant \left\|\bar{A}_{di4}\right\|_1 \|\varphi\|_{1c} + \|p(k)\|_1 \leqslant (\left\|\bar{A}_{di4}\right\|_1 \delta + \delta) \|\varphi\|_{1c} \beta^k. \tag{31}$$

If $k \in [d(k), 2d(k)]$, then $k - d(k) \in [0, d(k)]$. From (30) and (31), we have

$$\|x_2(k)\|_1 \leqslant \left(\left\|\bar{A}_{di4}\right\|_1^2 \delta + \left\|\bar{A}_{di4}\right\|_1 \delta + \delta\right) \|\varphi\|_{1c} \beta^k.$$

Suppose that $\forall k \in [(l-1)d(k), ld(k)]$, then

$$\|x_2(k)\|_1 \leqslant \left(\left\|\bar{A}_{di4}\right\|_1^l \delta + \left\|\bar{A}_{di4}\right\|_1^{l-1} \delta + \cdots + \left\|\bar{A}_{di4}\right\|_1 \delta + \delta\right) \|\varphi\|_{1c} \beta^k.$$

Thus, when $k \in [ld(k), (l+1)d(k)]$, $k - d(k) \in [(l-1)d(k), ld(k)]$, by the inductive supposition and from (30) and (31), we get that

$$\begin{aligned}\|x_2(k)\|_1 &\leqslant \left\|\bar{A}_{di4}\right\|_1 \left(\left\|\bar{A}_{di4}\right\|_1^l \delta + \left\|\bar{A}_{di4}\right\|_1^{l-1} \delta + \cdots + \left\|\bar{A}_{di4}\right\|_1 \delta + \delta\right) \|\varphi\|_{1c} \beta^k + \|p(k)\|_1 \\ &\leqslant \left(\left\|\bar{A}_{di4}\right\|_1^{l+1} \delta + \left\|\bar{A}_{di4}\right\|_1^l \delta + \cdots + \left\|\bar{A}_{di4}\right\|_1^2 \delta + \left\|\bar{A}_{di4}\right\|_1 \delta + \delta\right) \|\varphi\|_{1c} \beta^k.\end{aligned}$$

If $0 < \left\|\bar{A}_{di4}\right\|_1 < 1$, by induction, we obtain

$$\begin{aligned}\|x_2(k)\|_1 &\leqslant \left(\delta + \left\|\bar{A}_{di4}\right\|_1 \delta + \cdots + \left\|\bar{A}_{di4}\right\|_1^l \delta + \cdots\right) \|\varphi\|_{1c} \beta^k \\ &\leqslant \frac{\delta}{1 - \left\|\bar{A}_{di4}\right\|_1} \|\varphi\|_{1c} \beta^k.\end{aligned} \tag{32}$$

From (23) and (32), we finally have $\|x(k)\|_1 \leqslant M \|\varphi\|_{1c} \beta^k$ and $M = \max\{\tau; \frac{\delta}{1-\left\|\bar{A}_{di4}\right\|_1}\}$, $\forall k \geqslant 0$; thus, we conclude that System (3) with $w(k) = 0$ is exponentially stable with decay rate β, which infers System (1) with $w(k) = 0$ is exponentially stable with decay rate β.

We now consider the l_1-gain performance of System (1). Following the similar line as before, we have

$$\begin{aligned}V_i(k+1) - \alpha V_i(k) + \Gamma(k) \leqslant\; & x^T(k)\left[A_i^T \nu_i - \alpha E^T \nu_i + \upsilon_{1i} + \upsilon_{2i} + (d_{12}+1)\upsilon_{3i} + C_i^T 1_m\right] \\ & + x^T(k - d(k))\left[A_{di}^T \nu_i - \alpha^{d_2} \upsilon_{3i}\right] - \alpha^{d_1} x^T(k - d_1)\upsilon_{1i} \\ & - \alpha^{d_2} x^T(k - d_2)\upsilon_{2i} \\ & + w^T(k)\left[F_i^T \nu_i + D_i^T 1_m - \gamma 1_u\right],\end{aligned} \tag{33}$$

where $\Gamma(k) = \|z(k)\|_1 - \gamma \|w(k)\|_1$.

It is seen that (9)–(11) guarantee that

$$V_i(k+1) - \alpha V_i(k) + \Gamma(k) < 0. \tag{34}$$

Applying (34) recursively gives, $\forall k \in [k_l, k_{l+1})$,

$$V_{\sigma(k)}(k) \leqslant \alpha^{k-k_l} V_{\sigma(k_l)}(k_l) - \sum_{k_l}^{k-1} \alpha^{k-s-1} \Gamma(s). \tag{35}$$

It follows from (13) and (35) that

$$
\begin{aligned}
V_{\sigma(k)}(k) &\leqslant \alpha^{k-k_l} V_{\sigma(k_l)}(k_l) - \sum_{k_l}^{k-1} \alpha^{k-s-1}\Gamma(s) \leqslant \alpha^{k-k_l} \mu V_{\sigma(k_l^-)}(k_l^-) - \sum_{k_l}^{k-1} \alpha^{k-s-1}\Gamma(s) \\
&\leqslant \mu\alpha^{k-k_{l-1}} V_{\sigma(k_{l-1})}(k_{l-1}) - \mu \sum_{s=k_{l-1}}^{k_l-1} \alpha^{k-s-1}\Gamma(s) - \sum_{k_l}^{k-1} \alpha^{k-s-1}\Gamma(s) \\
&\leqslant \mu^2\alpha^{k-k_{l-2}} V_{\sigma(k_{l-2})}(k_{l-2}) - \mu^2 \sum_{s=k_{l-2}}^{k_{l-1}-1} \alpha^{k-s-1}\Gamma(s) - \mu \sum_{k_{l-1}}^{k_l-1} \alpha^{k-s-1}\Gamma(s) - \sum_{k_l}^{k-1} \alpha^{k-s-1}\Gamma(s) \\
&\leqslant \cdots \\
&\leqslant \mu^{N_\sigma(k_0,k)}\alpha^{k-k_0} V_{\sigma(k_0)}(k_0) - \mu^{N_\sigma(k_0,k)} \sum_{s=k_0}^{k_1-1} \alpha^{k-s-1}\Gamma(s) - \cdots - \sum_{k_l}^{k-1} \alpha^{k-s-1}\Gamma(s) \\
&= \mu^{N_\sigma(k_0,k)}\alpha^{k-k_0} V_{\sigma(k_0)}(k_0) - \sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)}\alpha^{k-s-1}\Gamma(s).
\end{aligned} \tag{36}
$$

Under the zero initial condition, from (36), we have

$$
0 \leqslant - \sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)}\alpha^{k-s-1}\Gamma(s), \tag{37}
$$

namely,

$$
\sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)}\alpha^{k-s-1} \|z(s)\|_1 < \gamma \sum_{s=k_0}^{k-1} \mu^{N_\sigma(s,k)}\alpha^{k-s-1} \|w(s)\|_1 . \tag{38}
$$

Multiplying both sides of (38) by $\mu^{-N_\sigma(k_0,k)}$ yields

$$
\sum_{s=k_0}^{k-1} \mu^{-N_\sigma(k_0,s)}\alpha^{k-s-1} \|z(s)\|_1 < \gamma \sum_{s=k_0}^{k-1} \mu^{-N_\sigma(k_0,s)}\alpha^{k-s-1} \|w(s)\|_1 . \tag{39}
$$

Notice that $N_\sigma(k_0, s) \leqslant (s-k_0)/T_a$ and $T_a > T_a^* = -\ln\mu/\ln\alpha$, we have

$$
\mu^{-N_\sigma(k_0,s)} \geqslant \alpha^{(s-k_0)}. \tag{40}
$$

Combining (39) and (40) leads to

$$
\sum_{s=k_0}^{k-1} \alpha^{k-k_0-1} \|z(s)\|_1 \leqslant \gamma \sum_{s=k_0}^{k-1} \alpha^{k-s-1} \|w(s)\|_1 . \tag{41}
$$

Summing both sides of (39) from $k=0$ to ∞ leads to inequality:

$$
\sum_{k=0}^{\infty} \alpha^{k-1} \|z(k)\|_1 \leqslant \gamma \sum_{k=0}^{\infty} \|w(k)\|_1 .
$$

This completes the proof. □

Remark 3

One can see from Theorem 1 that the minimal average dwell time T_a^* is determined by the scalars α and μ. One can choose different α and μ to have different T_a^*, if (9)–(13) hold. In addition, when

DOI: 10.1002/rnc

$\mu \to 1$, $T_a^* \to 0$, which leads to $\nu_i = \mu\nu_j$, $\upsilon_{1i} = \mu\upsilon_{1j}$, $\upsilon_{2i} = \mu\upsilon_{2j}$, $\upsilon_{3i} = \mu\upsilon_{3j}$, $\forall i, j \in \underline{N}$, then System (1) possesses a common Lyapunov function, and the switching signals can be arbitrary. Hence, the obtained results are quite general.

Remark 4
Unlike the Halanay inequality based approach and the conventional Razumikhin approach, a novel Lyapunov–Krasovskii functional based approach is introduced to prove the exponential stability. Special exponential terms are embedded appropriately into the constructed Lyapunov–Krasovskii functional. The obtained decay rate characterization is a free parameter, which can be selected according to different practical applications, rather than a fixed value computed by solving a transcendental equation or a complicated function as in [31]. The decay rate can be obtained through the solutions of LMIs, which can be checked easily by effective algorithms. In addition, the Lyapunov-functional based approach could be extended to other types of dynamic systems in a straightforward manner.

Remark 5
Based on the Lyapunov–Krasovskii functional and the corresponding technique used in the proof of Theorem 2, it is easy to extend the result to SPSSs with multiple time-varying delays. For the case with multiple time-varying delays, it can be treated by adding the corresponding functionals for different time-varying delays.

Remark 6
Unlike other exponential stability conditions, the decay rate in Theorem 2 is a free value related to α and μ, which can be selected according to different practical applications. This will introduce more flexibility in analysis and synthesis of systems. More importantly, time-varying delay and exogenous disturbance in systems are taken into account in our condition, and thus, the result can be applied to a more general class of systems. It should be emphasized that the left side of (11) is monotonic decreasing functions with respect to γ. This allows us to compute the minimal l_1-gain performance level γ by convex optimization algorithms.

In the case of $E = I$, the results on l_1 stability analysis for discrete-time positive switched nonsingular system with time-varying delay can be easily obtained from Theorem 2. The corresponding results are now concluded in the following Corollary.

Corollary 1
Let $0 < \alpha < 1$ and $\gamma > 0$ be prescribed scalars, $A_i \succeq 0$, $A_{di} \succeq 0$, $F_i \succeq 0$, $C_i \succeq 0$, $D_i \succeq 0$, if there exist vectors $\nu_i, \upsilon_{1i}, \upsilon_{2i}, \upsilon_{3i} \in R_+^n$, such that, $\forall i \in \underline{N}$,

$$\begin{aligned} \left[A_i^T \nu_i - \alpha\nu_i + \upsilon_{1i} + \upsilon_{2i} + (d_{12}+1)\upsilon_{3i} + C_i^T 1_m\right] &\preceq 0, \\ A_{di}^T \nu_i - \alpha^{d_2}\upsilon_{3i} &\preceq 0, \\ F_i^T \nu_i + D_i^T 1_m - \gamma 1_u &\preceq 0, \end{aligned}$$

where $d_{12} = d_2 - d_1$, then System (1) with $E = I$ is positive and exponentially stable with its decay rate estimate $\beta = \alpha\mu^{1/T_a}$ and achieves a prescribed l_1-gain performance level γ for any switching signal $\sigma(k)$ with average dwell time (12).

Problem 1
The l_1 stability can be analyzed by solving feasibility problem of LPs (9)–(13) in Theorem 2. It is also noted that the LP conditions in Theorem 2 are also convex in the scalar γ, thus, given α, d_1, and d_2, the optimal l_1-gain performance level can be determined by solving the following convex optimization problem:

$$\min_{\nu_i, \upsilon_{1i}, \upsilon_{2i}, \upsilon_{3i}} \gamma$$

$$s.t. \quad (9)-(13), \quad \forall i \in \underline{N}.$$

DOI: 10.1002/rnc

4. NUMERICAL RESULTS

In this section, we give a example to show the validity of our theoretical results. The discrete-time system is given.

Example 1

Consider System (1) with two modes, that is, $\underline{N} = \{1, 2\}$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

$$A_1 = \begin{bmatrix} 0.2 & -4 \\ 0.1 & -6 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.01 & 0.1 \\ 0.0 & 0.1 \end{bmatrix}, F_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, D_1 = 0.1,$$
$$A_2 = \begin{bmatrix} 0.1 & -3 \\ 0.2 & -12 \end{bmatrix}, A_{d2} = \begin{bmatrix} 0.01 & 0.1 \\ 0.0 & 0.2 \end{bmatrix}, F_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, C_2 = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix}, D_2 = 0.1.$$

Direct calculation shows that

$$\begin{aligned} \det(sE - A_1) &= 6s - 0.8 \neq 0, \quad \text{for} \quad s = 0, \\ \deg(\det(sE - A_1)) &= \deg(6s - 0.8) = \operatorname{rank} E = 1, \\ \det(sE - A_2) &= 12s - 0.6 \neq 0, \quad \text{for} \quad s = 0, \\ \deg(\det(sE - A_2)) &= \deg(12s - 0.6) = \operatorname{rank} E = 1. \end{aligned}$$

Thus, the switched singular system (1) is said to be regular and causal. Furthermore, by simple computation, we obtain

$$\begin{array}{llll} \bar{A}_{11} = 0.1333, & \bar{A}_{d11} = 0.0100, & \bar{A}_{d12} = 0.0333, & \bar{F}_{11} = 0.0333, \\ \bar{A}_{13} = 0.0167, & \bar{A}_{d13} = 0.0000, & \bar{A}_{d14} = 0.0167, & \bar{F}_{12} = 0.0167, \\ \bar{A}_{21} = 0.0500, & \bar{A}_{d21} = 0.0100, & \bar{A}_{d22} = 0.0500, & \bar{F}_{21} = 0.1750, \\ \bar{A}_{23} = 0.0167, & \bar{A}_{d23} = 0.0000, & \bar{A}_{d24} = 0.0167, & \bar{F}_{22} = 0.0083. \end{array}$$

Therefore, by Theorem 1, System (1) is positive.

Let $d_1 = 1$, $d_2 = 3$, and $\alpha = 0.5$. Noted that, suppose $\mu = 1$, by simulation, it can be found that there is no feasible solution to this case, which means that there is no common Lyapunov function for all subsystems (Remark 3). Now, we consider the average dwell time scheme and solve the optimization Problem 1, we have $\gamma^* = 0.6258$, the corresponding solutions are

$$\nu_1 = \begin{bmatrix} 4.9439 \\ 0.2847 \end{bmatrix}, \nu_2 = \begin{bmatrix} 2.3288 \\ 0.3491 \end{bmatrix}, \upsilon_{11} = \begin{bmatrix} 0.0419 \\ 2.3929 \end{bmatrix}, \upsilon_{12} = \begin{bmatrix} 0.0262 \\ 0.9027 \end{bmatrix},$$
$$\upsilon_{21} = \begin{bmatrix} 0.0419 \\ 2.3929 \end{bmatrix}, \upsilon_{22} = \begin{bmatrix} 0.0262 \\ 0.9027 \end{bmatrix}, \upsilon_{31} = \begin{bmatrix} 0.4097 \\ 4.7016 \end{bmatrix}, \upsilon_{32} = \begin{bmatrix} 0.1944 \\ 2.7889 \end{bmatrix}.$$

By analysis, it can be found that the allowable minimum of μ is $\mu = 2.6833$ when $\alpha = 0.5$ is fixed; in this case, $T_a^* = 1.4240$. One can also derive the decay rate of the system as $\beta = \alpha\mu^{1/Ta} = 0.8660$ with $\mu = 3$ and $T_a = 2$. Thus, according to Theorem 2, we can conclude that System (1) is exponentially stable with an optimal l_1-gain performance level $\gamma = 0.6258$ for any switching signal satisfying $T_a > 1.4240$.

The simulation results are shown in Figures 1–3, where the initial condition is $x(0) = [2 \quad 3]^T$, $x(k) = [0 \quad 0]^T$, $k = -3, -2, -1, 0$, and the external disturbance is $w(k) = 0.5e^{-0.5k}$. Figure 1 shows the switching signal $\sigma(k)$ with average dwell time $T_a = 2$; Figures 2 and 3 plot the state x and the controlled output z of the system over 0–40 s with the setting sampling time $T = 1$, respectively. It is easy to see that System (1) is positive and exponentially stable. This demonstrates the proposed results.

In what follows, we will show the relation between the minimal average dwell time T_a^* and the optimal l_1-gain performance level γ^* by selecting different α and μ. For given $\mu = 3$, $d_1 = 1$, and

DOI: 10.1002/rnc

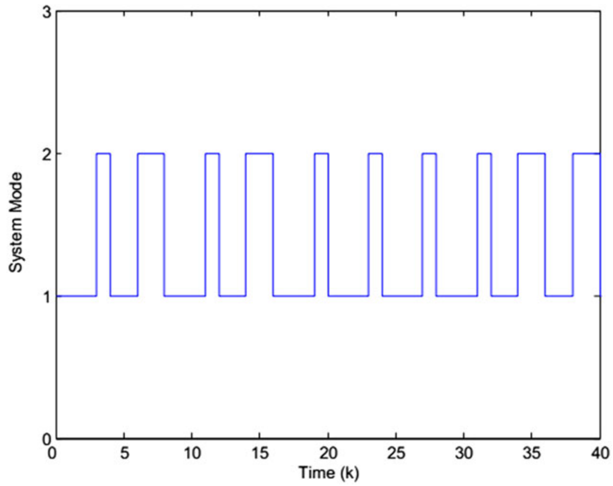

Figure 1. Switching signal $\sigma(k)$.

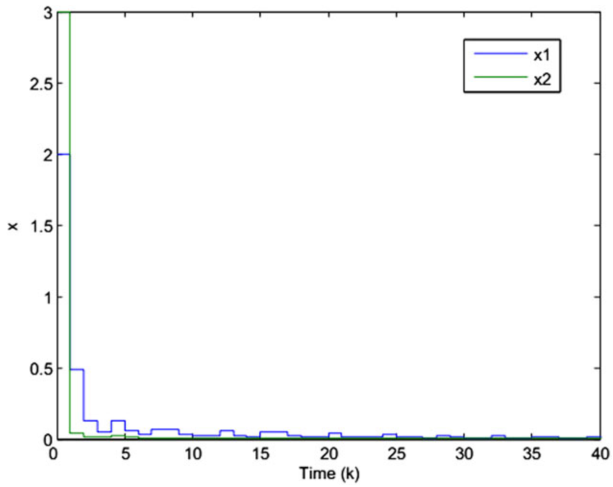

Figure 2. The state $x(k)$ of the given system.

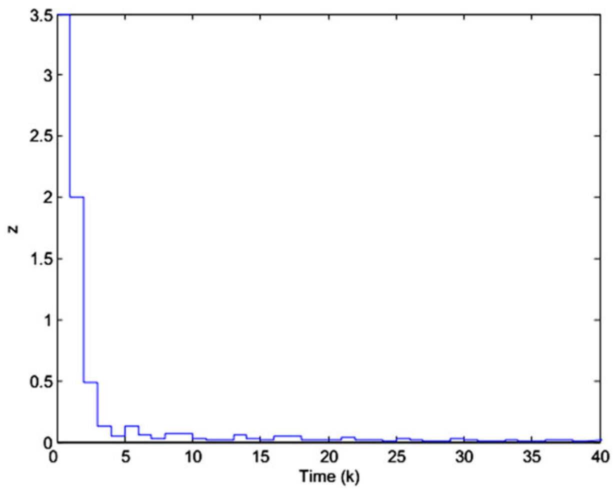

Figure 3. The output $z(k)$ of the given system.

DOI: 10.1002/rnc

Table I. Relation between α and γ^* for $\mu = 3$.

α	0.4	0.5	0.6	0.7	0.8
T_a^*	1.1990	1.5850	2.1507	3.0801	4.9233
γ^*	Infeasible	0.8466	0.2589	0.2272	0.1974

Table II. Relation between μ and γ^* for $\alpha = 0.5$.

μ	2.9	3	3.5	4	4.1
T_a^*	1.5361	1.5850	1.8074	2.0000	2.0356
γ^*	0.8815	0.8466	0.7189	0.6733	0.6180

Table III. Relation between d_2 and γ^* for $\alpha = 0.6$, $\mu = 3$, and $d_1 = 1$.

d_2	1	2	3	4	5
γ^*	0.2461	0.2560	0.2589	0.7771	Infeasible

$d_2 = 3$, the relation between α and γ^* is given in Table I. Table II gives the relation between μ and γ^* for given $\alpha = 0.5$, $d_1 = 1$, and $d_2 = 3$. It can be seen from Tables I and II, the better l_1-gain performance level can be achieved if the system is not switching so fast and frequently. Furthermore, for given $\alpha = 0.6$, $\mu = 3$, and $d_1 = 1$, the relation between the delay bound d_2 and γ^* is given in Table III. Note that the larger delay bound, the worse l_1-gain performance level, and the maximal delay bound is obtained as $d_2 = 4$ for this case.

5. CONCLUSIONS

In this paper, the problem of l_1 stability for a class of discrete-time SPSSs with time-varying delay has been investigated. First, based on the singular value decomposition method, a necessary and sufficient condition for the positivity of this system has been derived. Then, based on an appropriate co-positive Lyapunov functional and the average dwell time scheme, a delay-dependent condition has been derived to solve the problem of l_1 stability. The decay rate estimation has been derived and the optimal system performance level can be determined by solving a convex optimization problem. In addition, the obtained results implicitly established the relation between the switching law and the system performance. The effectiveness of the proposed method has been illustrated by a numerical example.

ACKNOWLEDGEMENT

This work was supported by the Postgraduate Innovation Project of Jiangsu Province under grant no. KYLX-0371.

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DOI: 10.1002/rnc