JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 61, 97–112 (1977)

Approximate and Complete Controllability of Nonlinear Systems to a Convex Target Set

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Submitted by J. P. LaSalle

The research deals with complete and approximate controllability of the system (*) $dx/dy = f(t, x, u)$, without control restraints to an arbitrary convex target set. First, some characterizations of complete controllability, to the target of (*) and a special case of (*) namely $\dot{x} = A(t)x + k(t, u)$** are given. As a consequence complete controllability is equivalent to null-controllability. Next certain equations are formulated. These are in the same spirit as J. P. Dauer's "A Controllability Technique for Nonlinear Systems" (*J. Math. Anal. Appl.* oo (1972), 442–451) and are utilized in the main contribution of the paper: Under certain convexity assumption, bounded perturbations of systems which are completely controllable to a fixed target G are completely controllable to G. Without the convexity assumption, but with perturbations satisfying a Lipschitz condition, approximate controllability to G of a perturbed system is equivalent to complete controllability to G of the unperturbed equation.

1. INTRODUCTION

Consider the nonlinear control system

$$dx/dt = f(t, x, u), \tag{1}$$

where f is a continuous function from $E \times E^n \times E^m$ into E^n. Let $I = [t_0, t_1]$ be a compact subset of E. We say that system (1) is G-controllable where $G \subseteq E^n$ if for any $x_0 \in E^n$, there exists a bounded measurable function $u: I \to E^m$ such that the solution $x(t) = x(t, t_0, x_0, u)$ of

$$dx/dt = f(t, x, u(t)), \qquad x(t_0) = x_0, \tag{2}$$

satisfies $x(t_1) \in G$. System (1) is approximately G-controllable if for any $x_0 \in E^n$ and for any $\lambda > 0$ there exists a bounded measurable function $u: I \to E^m$ such that the solution $x(t)$ of (2) satisfies

$$d(x(t_1), G) < \lambda,$$

ISSN 0022-247X

where $d(x, G) = \inf\{| x - p | : p \in G\}$ and $| \cdot |$ denotes a norm in E^n. Note that G can be taken to be a point, and in particular if $G = \{0\}$ the first definition corresponds to the usual one for null-controllability.

This paper deals with G-controllability and approximate G-controllability of systems where G is assumed to be an arbitrary closed and convex target set. First, certain characterizations of G-controllability of (1) and a special case of (1) are given. These are immediately applied to obtain a very curious result that the system

$$dx/dt = A(t)\, x + k(t, u), \tag{3}$$

where A is an $n \times n$ matrix function, is completely controllable if and only if it is null-controllable. Next, certain existence theorems for generalized initial value problems for contigent equations are formulated. These are in the same spirit as [1]. The main contributions of the paper are: Under certain convexity assumptions bounded perturbations of systems which are G-controllable are G-controllable. Whitout the convexity assumption but with the perturbations satisfying a Lipschitz condition, approximate G-controllability of a perturbed system is seen to be equivalent to G-controllability of the unperturbed one.

Recall that system (1) is completely controllable if for any x_0, $x_1 \in E^n$ there exists a bounded measurable function $u: I \to E^m$ such that the solution $x(t) = x(t, t_0, x_0, u)$ of (2) satisfies $x(t_1) = x_1$. It is nullcontrollable if for any $x_0 \in E^n$ there exists a bounded measurable function $u: I \to E^m$ such that the solution $x(t)$ of (2) satisfies $x(t_1) = 0$. Finally, the system (1) is x_1-controllable if for every $x_0 \in E^n$ there exists a bounded measurable function $u: I \to E^m$ such that the solution $x(t)$ of (2) satisfies $x(t_1) = x_1$.

Notation. In what follows $S_N{}^M(0)$ denotes the M-dimensional ball of radius N centered on the origin. The set $h(t, S_\rho{}^M(0))$ is defined by $h(t, S_\rho{}^M(0)) = \{h(t, u): u \in S_\rho{}^M(0)\}$. Aumann's integral of this set is given by

$$\int_I h(t, S_\rho{}^M(0))\, dt = \left\{\int_I h(t, u(t))\, dt \colon u\colon I \to S_\rho{}^M(0), \text{ measurable, } u(t) \in S_\rho{}^M(0)\right\}.$$

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We first present two basic characterizations of G-controllability of the special system

$$dx/dt = A(t)\, x + k(t, u), \tag{3}$$

where A is an $n \times n$-matrix function and both A and k are continuous. The first proposition is distantly to "expanding" characterization of reachable sets in [12, p. 52]. This is equivalent to linear systems being proper in [12, pp. 73, 78] and being completely controllable [12, p. 93]. An immediate important consequence of this is that complete controllability of (3) is equivalent to null controllability when the system's controls are not restrained.

Remark 1. Let (3) be G-controllable; that is, for any $x_0 \in E^n$ there is a bounded measurable function $u: I \to E^m$ such that

$$x_1 = X(t_1)\left[x_0 + \int_I X^{-1}(t)\, k(t, u(t))\, dt\right]$$

for some $x_1 \in G$ or equivalently

$$x_0 = X^{-1}(t_1)\, x_1 - \int_I X^{-1}(t)\, k(t, u(t))\, dt.$$

Let

$$A_N^{(3)}(G, \delta) = X^{-1}(t_1)\, G - \int_I X^{-1}(t)\, k(t, S_\delta{}^M(0))\, dt$$

and

$$A_N^{(3)}(G) = \bigcup_{\delta \geqslant 0} A_N^{(3)}(G, \delta).$$

It is clear that (3) is G-controllable if and only if

$$A_N^{(3)} = E^n.$$

Moreover nothe that if $\delta \leqslant \delta'$ then

$$A_N^{(3)}(G, \delta) \subseteq A_N^{(3)}(G, \delta').$$

PROPOSITION 1. *A necessary and sufficient condition that* (3) *be G-controllable, where G is a convex subset of E^n is that for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\int_I X^{-1}(t)\, (k(t, S_\delta{}^M(0))\, dt - X^{-1}(t_1)\, G \supseteq S_\epsilon{}^M(0), \tag{4}$$

or equivalently

$$-A_N^{(3)}(G, \delta) \supseteq S_\epsilon{}^N(0). \tag{4'}$$

Proof. Assume (3) is G-controllable. Let $\epsilon > 0$. Choose $2n$ points x_i, $i = 1,..., 2n$, in E^n such that

$$S_\epsilon{}^N(0) \subseteq \operatorname{cvx}\{x_i\,,\, i = 1, 2,..., 2n\},$$

where cvx denotes convex hull. Since (3) is G-controllable there exists a δ_i such that $x_i \in A_N^{(3)}(G, \delta_i)$ for $i = 1,..., 2n$. By [2] the set $A_N^{(3)}(G, \delta_i)$ is convex for each δ_i. Thus if

$$\delta = \max\{\delta_i : i = 1,..., 2n\}$$

$$S_\epsilon{}^n(0) \subseteq A_N^{(3)}(G, \delta).$$

Relation (4') follows from the symmetry of $S_\epsilon{}^n(0)$.

For the converse assume condition (4′), that is, given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$S_\epsilon^n(0) \subseteq -A_N^{(3)}(G, \delta).$$

Since

$$\bigcup_{\epsilon>0} S_\epsilon^n(0) = E^n,$$

and

$$\bigcup_{\delta>0} - A_N^{(3)}(G, \delta) \supseteq -A_N^{(3)}(G, \delta)$$

we have

$$E^n \subseteq \bigcup_{\delta>0} - A_N^{(3)}(G, \delta),$$

from which it follows that

$$A_N(G) = E^n.$$

COROLLARY 1. *System* (3) *is completely controllable if and only if it is null-controllable.*

Proof. Evidently, complete controllability implies null-controllability. For the converse suppose (3) is null-controllable; then by the preceding proposition for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\int_I X^{-1}(t)\, k(t, S_\delta^M(0))\, dt - X^{-1}(t_1)\{0\} \supseteq S_\epsilon^n(0)$$

or equivalently

$$\int_I X^{-1}(t)\, k(t, S_\delta^M(0))\, dt \supseteq S_\epsilon^n(0).$$

By [6, Proposition I] it follows that (3) is completely controllable.

It is important to notice that the lack of restrictions on the range of controls plays a crucial role.

The following example will show that is we assume controls to range in a unit cube the above proposition becomes false.

EXAMPLE. Let $G = \{(0, 0)\}$ and

$$\dot{x} = Ax + Bu,$$

where $x, u \in E^2$ and $| u_i | \leqslant 1$ for $i = 1, 2$. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the inverse of the corresponding fundamental matrix is

$$X^{-1}(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$$

and

$$X^{-1}(t)\, Bu = (e^t u_1, e^t u_2).$$

Thus

$$\left\| \int_{t_0}^{t_1} X^{-1}(t)\, Bu(t)\, dt \right\| \leqslant 2^{1/2}(e^{t_1} - e^{t_0}).$$

Let $\epsilon > 2^{1/2}(e^{t_1} - e^{t_0})$. It is clear that you cannot find a suitable $\delta > 0$ such that $S_\mu^{(2)}(0)$ should be contained in a unit cube and that condition (4) should be satisfied.

Corollary 1 is nevertheless fascinating. One begins to wonder what is so special about controllability to the origin which would force the system to be completely controllable. We are thus led to the following proposition.

PROPOSITION 2. *Let $x_1 \in E^n$. System (3) is x_1-controllable if and only if it is completely controllable.*

Proof. Since complete controllability implies x_1-contrallability, we need only prove the converse. Assume system (3) is x_1-controllable. Consider the system

$$\dot{y} = A(t)\, y + h(t, u), \tag{5}$$

where $h(t, u) = A(t)\, x_1 + k(t, u)$. System (5) is null controllable. Indeed, let $x_0' \in E^n$ and consider the point $x_0' + x_1$. Since (3) is x_1-controllable there exists a solution of $x(t)$ of (3) such that

$$x(t_0) = x_0' + x_1 \qquad \text{and} \qquad x(t_1) = x_1 .$$

The function $y(t) \equiv x(t) - x_1$ is a solution of (5), since

$$\dot{y}(t) = \dot{x}(t) = A(t)\, x(t) + k(t, u) = A(t)\, y(t) + h(t, u).$$

Also, (5) steers x_0' to the origin. Clearly $y(t_0) = x_0'$, $y(t_1) = 0$. Since $h(t, u)$ is continuous and satisfies the conditions of Proposition 2, and since we have proved null-controllability, Corollary 1 implies that (5) is completely controllable.

Now let $x_0 \in E^n$, and consider a solution $y(t)$ of (5) steering $x_0 - x_1$ to $-x_1$. Set $x(t) = y(t) + x_1$. Clearly $x(t)$ is a solution of (3) that steers x_0 to the origin. Thus we have shown that if (3) is x_1-controllable then it is null-controllable, and by Corollary 1 is completely controllable. This completes the proof.

The next proposition is a useful growth condition criterion for G-controllability of (3).

PROPOSITION 3. *In system* (3) *assume that* $k(t, 0) = 0$. *Then* (3) *is G-controllable if and only if for every* $\epsilon > 0$, *any* $\eta \in E^n$, $\eta \neq 0$, *there exists a bounded measurable* $u: I \to E^m$, *a* $p \in G$ *such that*

$$\eta^T \left\{ \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u(s))\, ds - X^{-1}(t_1)\, p \right\} \geqslant \epsilon. \tag{6}$$

It is assumed that G is convex and contains 0.

Proof. Let $\epsilon > 0$ and $\eta \neq 0$, $\eta \in E^n$, be given. Choose $x_0 \in E^n$ such that

$$\eta^T(-x_0) \geqslant \epsilon.$$

Assume now that (3) is G-controllable. Choose $u: I \to E^m$ such that

$$x(t_1, t_0, x_0, u) = X(t_1) \left[x_0 + \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u(s))\, ds \right] = p$$

for some $p \in G$ or equivalently $-x_0 = \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u(s))\, ds - X^{-1}(t_1)\, p$. Hence

$$\eta^T(-x_0) = \eta^T \left\{ \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u(s))\, ds - X^{-1}(t_1)\, p \right\} \geqslant \epsilon.$$

Conversely, consider the set $S(u) = \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u(s))\, ds - X^{-1}(t_1)\, G$ and set $S = \{S(u): u \text{ measurable, bounded, } u: I \to E^m\}$.

Because $k(s, 0) = 0$ and $0 \in G$, S contains the origin. By the Richter theorem [9] the integral is convex, so that S is convex. Assume now that (6) holds. Then by [Lemma 3, p. 7],

$$\bigcup_u S(u) = E^n.$$

Take any $x_0 \in E^n$, there exists a bounded measurable $u_1: I \to E^m$ such that

$$-x_0 \in S(u_1) = \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u_1(s))\, ds - X^{-1}(t_1)\, G.$$

Thus there exists a $p \in G$ such that

$$-x_0 = \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u_1(s))\, ds - X^{-1}(t_1)\, p.$$

It follows from this that

$$p = X(t_1) \left[x_0 + \int_{t_0}^{t_1} X^{-1}(s)\, k(s, u(s))\, ds \right] - x(t_1, u_1) \in G$$

since $p \in G$. This concludes the proof.

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In this section we present two propositions which represent slight generalizations of [1, Theorems 1 and 2] on the existence of an absolutely continuous function satisfying

$$\dot{x}(t) \in R(t, x(t)) \qquad \text{a.e. on} \qquad I = [t_0, t_1],$$
$$x(t_0) = x_0, \qquad d(x(t_1), G) \leqslant \epsilon,$$

where $\epsilon \geqslant 0$. Here R denotes a set-valued mapping $I \times E^n$ into the set of nonempty closed subsets of E^n which is upper semicontinuous with respect to set inclusion.

To fix notation, let $x_0 \in E^n$ and $G \in E^n$ be fixed; and let B be the set of all functions x which are Lipschitz continuous and are such that $x(t_0) = x_0$ and $x(t_1) \in G$. Define the following two norms in B:

$$| x | = \max_{t \in I} | x(t) |,$$

$$\| x \| = \max_{t \in I} | x(t) | + \inf\{L_x\},$$

where $\{L_x\}$ is the set of all Lipschitz constants for the function x. Let

$$B_p = \{x \in B : \| x - x_0 \| \leqslant p\}$$

where $(x - x_0)(t) = x(t) - x_0$. Define the multifunction Φ on B_p as

$$\Phi(y) = \{z \in B_p : \dot{z}(t) \in R(t, y(t)) \text{ a.e. on } I\}. \tag{7}$$

Recall that for $x \in B_p$ Aumann's integral of $R(t, x(t))$ is given by

$$\int_I R(s, x(s))\, ds = \left\{ \int_I r(s)\, ds : r \text{ measurable}, r(s) \in R(s, x(s)) \text{ a.e. on } I \right\}.$$

The following existence result is crucial in the sequel.

PROPOSITION 4. *Assume that $p > 0$ is so large that $B_p \neq \varnothing$ and, that $R(t, x(t))$ is convex and such that*

$$\bigcup_{\substack{t \in I \\ x \in S_p{}^n(x_0)}} R(t, x) \subseteq S_M{}^n(0),$$

where

$$M = \min\{p/2, p/2(t_1 - t_0)\}.$$

If for every $y \in B_p$ we have

$$-x_0 \in \int_I R(s, y(s))\, ds - G \tag{8}$$

then there exists $x \in B_p$ such that

$$x(t_0) = x_0, \qquad x(t_1) \in G$$

and

$$\dot{x}(t) \in R(t, x(t)) \qquad \textit{a.e. on } I.$$

Proof. Let multifunction Φ be as defined by (7) where $p > 0$ satisfies the conditions of Proposition 4. If Φ has a fixed point $x \in B_p$ then $x \in \Phi(x)$, $x \in B_p$ implies

$$x(t_0) = x_0, \qquad x(t_1) \in G$$

and

$$\dot{x}(t) \in R(t, x(t)) \qquad \text{a.e. on } I,$$

proving the result. We now establish the existence of such a fixed point using Fan's fixed-point theorem [4]. It states that any upper semicontinuous multifunction from a compact, convex subset S of a locally convex linear topological space into the set of nonempty closed convex subsets of S has a fixed point in S. Clearly B_p is a compact and convex subset of E^n. If $y \in B_p$ then $\Phi(y)$ is convex since B_p and $R(t, y(t))$ are convex. We now show that $\Phi(y)$ is nonempty. Indeed from (8) since G is nonempty there exists $p \in G$ such that for some measurable function r with

$$r(t) \in R(t, y(t)) \qquad \text{a.e.} \quad \text{on} \quad I,$$

$$-x_0 = \int_I r(s)\, ds - p;$$

that is

$$p = x_0 + \int_I r(s)\, ds.$$

If we set $z(t) = x_0 + \int_{t_0}^{t} r(s)\, ds$, $t \in I$, we see at once that $z \in B$ since $z(t_0) = x_0$ and $z(t_1) = x_0 + \int_{t_0}^{t_1} r(s)\, ds \in G$, and z is Lipschitz continuous. Furthermore

$$| z(t) - x_0 | \leqslant \int_{t_0}^{t} (p\, ds/2(t_1 - t_0)) = p/2$$

for $t \in I$ and

$$| z(t) - z(\bar{t})| \leqslant \int_{\bar{t}}^{t} (p/2)\, dx = (p/2)\,(t - \bar{t})$$

for all t, $\bar{t} \in I$. Hence

$$\| z(t) - x_0 \| = \max_{t \in I} \| z(t) - x_0 \| + \operatorname{Inf}\{L_x\} \leqslant (p/2) + (p/2) = p.$$

We have shown that $z \in B_p$ since also $\dot{z}(t) = r(t) \in R(t, y(t))$ a.e. on I, clearly $z \in \Phi(y)$, and so $\Phi(y) \neq \varnothing$. By [1, Lemma 1], Φ has a closed graph and hence has closed values. The compactness of B_p now implies that Φ is upper semicontinuous. Now apply Fan's fixed-point theorem to obtain the desired fixed point $x \in B_p$.

We next remove the convexity assumption on $R(t, x(t))$ and impose a Lipschitz condition to obtain an ϵ-approximate existence result. The extended Hausdorff metric on the space of ubsets of E^n is denoted by h.

PROPOSITION 5. *Assume that R is continuous closed valued and satisfies the Lipschitz condition*

$$h(R(t, x), R(t, y)) \leqslant w(t) \mid x - y \mid$$

with $w \in L^1(I)$.

Assume that $p < 0$ is such that $B_p \neq \varnothing$ and that

$$\bigcup_{\substack{t \in I \\ x \in S_p{}^n(x_0)}} R(t, x) \subseteq S_M{}^n(0),$$

where $M = \min\{p/2, p/2(t_1 - t_0)\}$.

If for every $y \in B_p$ we have

$$-x_0 \in \int_I R(s, y(s))\, ds - G,$$

then for every $\epsilon > 0$ there exists an absolutely continuous function x such that

$$\dot{x}(t) \in R(t, x(t)) \qquad a.e. \text{ on } I,$$
$$x(t_0) = x_0 \qquad and \qquad d(x(t_1), G) < \epsilon.$$

Proof. Let $H(t, x)$ denote the closed convex hull of the closed set $R(t, x)$. Evidently H satisfies the hypotheses of Proposition 4. Therefore there exists $y \in B_p$ such that

$$\dot{y}(t) \in H(t, y(t)) \qquad \text{a.e. on } I,$$
$$y(t_0) = x_0 \qquad \text{and} \qquad y(t_1) \in G.$$

By a result due to Filippov [5, Theorem 3], for each $\epsilon > 0$ there exists x such that

$$\dot{x}(t) \in R(t, x(t)) \qquad \text{a.e.} \quad \text{on} \quad I,$$
$$x(t_0) = x_0 \qquad \text{and} \qquad \max_{t \in I} \mid y(t) - x(t) \mid < \epsilon.$$

The last inequality implies that

$$\mid y(t_1) - x(t_1) \mid < \epsilon$$

and since $y(t_1) \in G$ we conclude that

$$d(x(t_1), G) < \epsilon,$$

which proves the proposition.

4

The main results are stated and proved in this section. By applying the results of Section 3, we develop conditions under which perturbations of systems which are G-controllable remain G-controllable, or approximately G-controllable. Our treatment parallels the paper of Dauer [6] on the systems

$$\dot{x} = A(t)\, x + k(t, u) \tag{3}$$

and

$$\dot{x} = A(t)\, x + k(t, u) + g(t, x, u), \tag{9}$$

where A is an n-square continuous matrix function, and k and g are continuous n-vector functions.

THEOREM 1. *Assume that g is bounded on $I \times E^n \times E^m$. Further assume that for sufficiently large μ the set*

$$k(t, S_\mu^m(0)) + g(t, x, S_\mu^m(0))$$

is convex for $(t, x) \in I \times E^n$. Let G be a fixed convex subset of E^n. Then the system (9) *is G-controllable if and only if* (3) *is G-controllable.*

Proof. Suppose (3) is G-controllable. Let $x_0 \in E^n$ be given and select $\epsilon \geqslant | x_0 |$. Let

$$\rho_1 = \sup\{X^{-1}(t)\, g(t, x, u) \colon (t, x, u) \in I \times E^n \times E^m\}.$$

ρ_1 exists since $X^{-1}(t)$ is continuous on I and g is bounded on $I \times E^n \times E^m$. From G-controllability of (3) we have by Proposition 1 that there exists $\delta > 0$ such that if

$$r = 2(\epsilon + \rho_1(t_1 - t_0)),$$

then

$$\int_I X^{-1}(t)\, k(t, S_\delta^M(0))\, dt - X^{-1}(t_1)\, G \supseteq S_r^n(0). \tag{10}$$

It also follows from this that there exist controls $u_i \colon I \to S_\delta^m(0)$ such that for some $p_i \in G$

$$y_i = \int_I X^{-1}(t)\, k(t, u_i(t))\, dt - X^{-1}(t_1)\, p_i$$

where y_i are $2n$ distinct points on the coordinate axes of E^n satisfying $|y_i| = r$. Let

$$z_i = \int_I X^{-1}(t)\,[k(t, u_i(t)) + g(t, X(t)\,y(t), u_i(t)]\,dt - X^{-1}(t_1)\,p_i\,.$$

It is clear that $z_i \in S^M_{\rho(t_1-t_0)}(y_i)$ for all i. Consider the set

$$S = \int_I X^{-1}(t)\,[k(t, S_\delta{}^M(0)) + g(t, X(t)\,y(t), S_\delta{}^M(0))]\,dt - X^{-1}(t_1)\,G$$

for each continuous $y\colon I \to E^n$.

From the convexity hypotheses S is convex. Thus the convex hull of z_i's, $i = 1,\dots, 2n$, must be contained in S. Consider any n points $z_{i_j} \in S^n_{\rho_1(t_1-t_0)}(y_{i_j})$, $j = 1 \dots n$, with the property that no two of them are on the same axis. The hypersurface determined by these cannot intersect $S_\epsilon{}^n(0)$. Hence $S_\epsilon{}^n(0)$ is contained in the half space containing the origin determined by z_{i_j}'s. It is easy to see that $S_\epsilon{}^n(0)$ is contained in the intersection of all half spaces determined in this fashion. Thus $S_\epsilon{}^n(0)$ is contained in the convex hull of z_i's for all choices of $z_i \in S_{\rho_1(t_1-t_0)}(y_i)$. This in turn shows that $S_\epsilon{}^n(0) \subseteq S$.

Hence

$$-x_0 \in \int_I R(t, y(t))\,dt - X^{-1}(t_1)\,G \tag{11}$$

for each $y(t)$, where

$$R(t, x) = X^{-1}(t)\,[k(t, S_\delta{}^M(0)) + g(t, X(t)\,x(t), S_\delta{}^M(0))]. \tag{12}$$

Since k is continuous on I and g is bounded on $I \times E^n \times E^m$ we can choose M so large that

$$R(t, x) \subseteq S_M{}^n(0) \qquad \text{for all } (t, x) \in I \times E^n.$$

Choose $p > 0$ so that $B_p \neq \varnothing$ and

$$M \leqslant \min\{p/2, p/2(t_1 - t_0)\}.$$

Because R is convex, apply Proposition 4 to deduce the existence of an absolutely continuous function $x\colon I \to E^n$ such that

$$\dot{x}(t) \in R(t, x(t)) \qquad \text{a.e. on } I,$$
$$x(t_0) = x_0 \qquad \text{and} \qquad x(t_1) \in X^{-1}(t_1)\,G.$$

If we now set $z(t) = X(t)\,x(t)$ then

$$\dot{z}(t) \in A(t)\,z(t) + k(t, S_\delta{}^M(0)) + g(t, z(t), S_\delta{}^M(0)) \qquad \text{a.e. on } I,$$
$$z(t_0) = x_0 \qquad \text{and} \qquad z(t_1) \in G.$$

By Filippov's lemma [8, p. 78] there exists $u: I \to E^m$, $u(t) \in S_\delta{}^M(0)$ such that

$$\dot{z}(t) = A(t)\, z(t) + k(t, u(t)) + g(t, z(t), u(t)) \qquad \text{a.e. on } I,$$
$$z(t_0) = x_0 \qquad \text{and} \qquad z(t_1) \in G.$$

Hence (9) is G-controllable.

For the converse, we introduce the following notation. Define the sets

$$A_N^{(3)}(G, \mu) \equiv X^{-1}(t_1)\, G - \int_{t_0}^{t_1} X^{-1}(t)\, k(t, S_\mu{}^m(0))\, dt,$$

$$A_N^{(9)}(G, \mu) \equiv X^{-1}(t_1)\, G - \int_{t_0}^{t_1} X^{-1}(t)\, [k, (t, S_\mu{}^m(0)) + g(t, x, S_\mu{}^m(0))\, dt$$

$$\subseteq A_N^{(3)}(G, \mu) - \int_{t_0}^{t_1} X^{-1}(t)\, g(t, x, S_\mu{}^m(0))\, dt,$$

and now observe that system (3) is G-controllable if and only if

$$A_N^{(3)}(G) = E^n,$$

where

$$A_N^{(3)}(G) = \bigcup_{\mu > 0} A_N^{(3)}(G, \mu).$$

Note that for $\mu \leqslant \mu'$

$$A_N^{(3)}(G, \mu) \subseteq A_N^{(3)}(G, \mu')$$

and that for each μ there is a point common to $A_N^{(3)}(G, \mu)$.

Because a ball $S_p{}^n(0)$ is symmetric it is immediate from Proposition 1 that (3) is G-controllable if and only if for each $\epsilon > 0$ there exists $\mu > 0$ such that

$$X^{-1}(t_1)\, G - \int_{t_0}^{t_1} X^{-1}(t)\, k(t, S_\mu{}^m(0))\, dt \supseteq S_\epsilon{}^n(0).$$

Now, assume that (3) is not G-controllable. Then there exists $\epsilon > 0$ and there exists $s \in S_\epsilon{}^n(0)$ such that

$$s \notin A_N^{(3)}(G).$$

Because the set $A_N^{(3)}(G)$ is a union of nested convex sets with a common point, $A_N^{(3)}(G)$ is convex. Hence there exists a hyperplane π through the point s such that for every $x \in A_N(G)$, $x \cdot n \leqslant 0$, where n is a vector perpendicular to π.

It is clear that

$$A_N^{(9)}(G) = A_N^{(3)}(G) - \int_{t_0}^{t_1} X^{-1}(t)\, g(t, x, E^n)\, dt.$$

From the controllability of (9) it follows that in the above statement we actually have equality. Obviously, if a point $y_0 \in A_N^{(9)}(G)$ then

$$d(y_0, A_N^{(3)}(G)) \leqslant \rho_1(t_1 - t_0),$$

where ρ_1 is as defined in the beginning of the proof. Let

$$\rho_2 = \inf\{d(x, A_N^{(3)}(G)) \colon x \in \pi\}$$

and let $x_0 \in \pi'$, where π' is a hyperplane parallel to π with

$$h(\pi, \pi') > \rho_1(t_1 - t_0)$$

and such that $x_0 \cdot n > 0$. Since

$$d(x_0, A_N^{(3)}(G)) \geqslant h(\pi_0, \pi) + d(\pi, A_N^{(3)}(G)) > \rho_1(t_1 - t_0) + \rho_2 .$$

Thus $x_0 \notin A_N^{(9)}(G)$ and (9) is not G-controllable. Theorem 1 is completely proved.

Next we remove the convexity assumption and then deduce an approximate G-controllability result.

THEOREM 2. *Assume that g is bounded on $I \times E^n \times E^m$ and satisfies a Lipschitz condition*

$$| g(t, x, u) - g(t, y, u) | \leqslant \omega(t)\, | x - y |$$

with $\omega \in L^1(I)$. Let G be a fixed convex subset of E^n. Then the system (3) *is G-controllable if and only if* (9) *is approximately G-controllable.*

Proof. Assume that (3) is G-controllable. Let $x_0 \in E^n$ and $\epsilon > 0$ be given, so that $\epsilon \geqslant | x_0 |$. Just as in the proof of Theorem 1 we have from (11) that

$$-x_0 \in \int_I R(t, y(t))\, dt - X(t_1)\, G$$

for each $y(t)$, where R is given in (12). The required convexity of the integral in (11) follows from Richter's theorem [9]. We now note that R satisfies all the conditions of Proposition 5. Hence if $\epsilon_1 = \epsilon / | X(t_1) |$ there exists x such that

$$\dot{x}(t) \in R(t, x(t)) \qquad \text{a.e. on } I$$
$$x(t_0) = x_0 \qquad \text{and} \qquad d(x(t_1), X^{-1}(t_1)\, G) < \epsilon_1 .$$

Now set $z(t) = X(t)\, x(t)$ and then

$$\dot{z}(t) \in A(t)\, z(t) + k(t, S_\mu{}^M(0)) + g(t, z(t), S_\mu{}^M(0)),$$
$$z(t_0) = x_0\,, \qquad d(X^{-1}(t_1)\, z(t_1), X^{-1}(t_1)\, G) < \epsilon_1\,,$$

which implies $d(z(t_1), G) < \epsilon$.

Apply Filippov's lemma [8] to conclude that (9) is approximately G-controllable.

Conversely assume that (3) is not G-controllable. Then by Proposition 1 there exists $\epsilon > 0$ and $s \in S_\epsilon{}^n(0)$ with $s \notin A_N^{(3)}(G)$. Just as in the proof of the converse of Theorem 1, there exists a hyperplane π through the point s such that for any $a \in A_N(G)$ $a \cdot n \leqslant 0$, where n is a vector perpendicular to π. We now observe from the definitions that (9) is approximately G-controllable if and only if for every $\lambda > 0$ there exists a subset S of E^n such that $S \subseteq S_\lambda(0)$ and such that

$$X^{-1}(t_1)\,[G + S] - \int_{t_0}^{t_1} X^{-1}(t)\,[k(t, E^n) + g(t, x(t), E^n]\, dt = E^n.$$

Clearly the left-hand side of this equality is contained in the set

$$X^{-1}(t_1)\, S + A_N^{(9)}(G),$$

so that a point $y_0 \in E^n$ can be steered approximately to G if and only if it is a member of this set.

Obviously if $y_0 \in X^{-1}(t_1)\, S + A_N^{(9)}(G)$, then

$$d(y_0\,, A_N^{(3)}(G)) \leqslant |\, X^{-1}(t_1)|\, \lambda + \rho_1(t_1 - t_0)$$

since

$$X^{-1}(t_1)\, S + A_N^{(9)}(G) \subseteq X^{-1}(t_1)\, S + A_N^{(3)}(G) - \int_{t_0}^{t_1} g(t, x(t), E^n)\, dt.$$

Here ρ_1 is as defined in the proof of Theorem 1. Suppose

$$\rho_2 = \inf\{d(x, A_N^{(3)}(G)) : x \in \pi\}.$$

Consider a point $x_0 \in \pi'$, where π' is a hyperplane parallel to π with

$$h(\pi, \pi') > \rho_1(t_1 - t_0) + |\, X^{-1}(t_1)|\, \lambda - \rho_2$$

and such that $x_0 \cdot n > 0$. Obviously

$$d(x_0\,, A_N^{(3)}(G) \geqslant h(\pi', \pi) + d(\pi, A_N^{(3)}(G) > \rho_1(t_1 - t_0) + |\, X^{-1}(t_1)|\, \lambda - \rho_2 + \rho_2\,.$$

Hence $x_0 \notin X^{-1}(t_1)\, S + A_N^{(9)}(G)$. This means that (9) is not approximately G-controllable, proving Theorem 2.

5

Example 1. Consider the system

$$\dot{x} = x + u,$$
$$\dot{y} = y + e^{t-y^2}\sin u^2$$

on $I = [0, 1]$, $G = \operatorname{span}\{[0, T]^{\mathrm{T}}\}$, where T denotes the transpose. Here

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad k(t, u) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$g(t, \underset{\sim}{x}, u) = [0, e^{t-y^2}\sin u^2]^{\mathrm{T}}.$$

It is a consequence of [10, Corollary 2.2] that $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + k(t, u)$ is G-controllable. Obviously g is bounded on $I \times E^2 \times E^2$. The convexity assumption is clearly satisfied. By Theorem 1,

$$\underset{\sim}{x} = A\underset{\sim}{x} + k(t, u) + g(t, \underset{\sim}{x}, u)$$

is G-controllable.

Example 2. Consider the system

$$\dot{x} = x + y + u + e^t((\sin^2 y + \cos^2 u)/(t + 1)),$$
$$\dot{y} = (t^2 e^{-u^2}/(2 + \cos(y - x)).$$

Here

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad k(t, u) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$g(t, x, y, u) = \begin{pmatrix} (e^t(\sin^2 y + \cos^2 u))/(t + 1) \\ t^2 e^{-u^2}/(2 + \cos(y - x)) \end{pmatrix}.$$

Let $G = \operatorname{span}\{[-1, 1]^T\}$.

Because the controllability space $\{A \mid B\} = \operatorname{span}\{[1, 0]^T\}$ and

$$e^{-At}G = \operatorname{span}\{[-1, 1]^T\}$$

it follows from [10, Corollary 2.2] and the fact that

$$E^2 = \{A \mid B\} + \bigcup_{t \geqslant 0} e^{-At}G$$

that the base system is G-controllable even though it is not null-controllable. Since g is bounded and satisfies the Lipschitz condition in x and y, Theorem 2 implies that the system is approximately G-controllable.

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