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On the stability issues of switched singular time-delay systems with slow switching based on average dwell-time

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SUMMARY

The issue of exponential stability analysis of continuous-time switched singular systems consisting of a family of stable and unstable subsystems with time-varying delay is investigated in this paper. It is very difficult to analyze the stability of such systems because of the existence of time-delay and unstable subsystems. In this regard, on the basis of the free-weighting matrix approach, by constructing the new Lyapunov-like Krasovskii functional, and using the average dwell-time approach, delay-dependent sufficient conditions are derived and formulated in terms of LMIs to check the exponential stability of such systems. This paper also highlights the relationship between the average dwell-time of the switched singular time-delay system, its stability, exponential convergence rate of differential states, and algebraic states. Finally, a numerical example is given to confirm the analytical results and illustrate the effectiveness of the proposed strategy. Copyright © 2012 John Wiley & Sons, Ltd.

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KEY WORDS: switched systems; singular systems; time-varying delay; exponential stability; average dwell-time

1. INTRODUCTION

Ordinary differential equations (ODEs) are the base of the classical state-space model for a proper system whereas many of the systems are presented by ODEs. However, in many models that may result in a singular system model, some of the relationships of variables are dynamic while others are purely static. It should also be noted that a state-space model is generally obtained under the assumption that the plant under consideration is governed by the causality principle, but in certain situations the state of a system in the past may depend on its state in the future. Such anticipatory characteristic of a system leads to a violation of the causality assumption. In this case, a singular system model is necessary for the description of such systems. Because singular systems constitute an important class of systems of both theoretical and practical significance, many approaches have been developed to investigate their stability and stabilization [1–3], solvability [4], optimal control [5, 6], and applications [7].

During the last twdo decades, the stability and stabilization of time-delay systems have been the topics of recurring interest and have been encountered in various engineering and physical systems. For more results of stability and stabilization of time-delay systems, we refer the readers to Ref. [8] and the references cited therein. Many important and interesting results have also been reported on the stability analysis of singular time-delay systems by various methods [9–11]. These methods may be classified into two categories: delay-independent cases and delay-dependent cases.

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Generally speaking, delay-independent cases are likely to be conservative, especially when the time delays are small.

In recent years, the so-called switched systems have been widely studied and many interesting results have been reported in the literature [12–16]. The motivation to study switched systems is mainly twofold. First, many engineering systems can be represented by switched systems, such as networked control systems (NCSs), traffic control, automotive engine control, and aircraft control, see Refs. [15–17] and their bibliography for more applications. Second, the idea of controller switching is introduced to overcome the shortcomings of the global single controller and improve system performance.

For switched systems, because of the complicated behavior caused by the interaction between the continuous dynamics and discrete switching, the problem of time delays is more difficult to study and has a strong engineering background. Therefore, dealing with time-delay has been a hot topic in the stability analysis and special attention has been attracted. Sun *et al*. presented stability analysis of switched system with time-delay in [16], which was a pioneering research and basis of many researches in switched systems and singular switched systems. Sun *et al*. considered the stability problem for a class of time-delay switched systems with derivative of time-varying delay allowed to be bounded with an unknown constant. Thereafter, several useful results have been reported in the literature such as the issues on the stability and stabilization analysis [12], observer design [13, 14], tracking problem [14], output feedback control [15], L_2-gain problem [18], exponential stability analysis subject to controller failure [19, 20], and H_∞ controller design for networked predictive control systems [21]; see Refs. [22, 23] for more details.

As a crucial factor, switching signals determine the dynamic behavior of a switched system in most cases. Some analysis and synthesis results have been presented assuming the switching signals are arbitrary [22, 23] in virtue of general linear or nonlinear system theories. Also, for the controlled switched systems, the switching signal constitutes system design such that the corresponding problem is more complicated in finding a suitable switching signal for improving system performance. Hence, in switched control systems, switching signals are crucial to determine system behavior, which depends on either time or system state, or both, or other supervisory decision procedures (see Ref. [23] and its bibliography). Usually, the switching sequence of subsystems is considered as completely unknown in advance, and the switching points of subsystems are considered either arbitrary or ones with dwell (or average dwell)-time [24]. The average dwell-time switching signal has been recognized to be more flexible and efficient in system stability analysis. Often, the switched systems with dwell (or average dwell)-time are also viewed as slowly switched systems in the literature. Therefore, stability analysis for the slowly switched systems usually needs to specify dwell (or average dwell)-time of the switching signal, which thereby can also be viewed as a design problem of switching laws and are frequently encountered in the switching control practice and some results have been reported including both linear cases and nonlinear cases within the continuous-time context [25–27].

There are some features in switched singular systems with time-delay, which are neither found in singular systems in switched time-delay systems. Therefore, it can be concluded that for switched singular time-delay systems, analysis of the stability is more complicated. Therefore, it is interesting and challenging to investigate the stability problem of switched singular time-delay systems. In [28] stability analysis of discrete-time switched singular time-delay systems is presented, but unfortunately the exponential stability of algebraic states has not been shown well. In this issue, unstable subsystems have not been considered as well. Stability analysis of continuous-time switched singular time-delay systems consisting of only stable subsystems has been considered in [29]. There are some other papers that have presented stability analysis of the switched singular system [30–34], but the simultaneous presence of switching signals with average dwell-time property, time-delay, or both have not been fully investigated.

In the present paper, the problem of finding a switching signal and the exponential stability of switched singular time-delay systems consisting of stable and unstable subsystems are investigated. First, some properties of switched singular time-delay systems are introduced and discussed. Then, with the help of the average dwell-time approach incorporated with a switching signal condition, a class of switching signals is found under which the switched singular time-delay system is

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exponentially stable. Also, an upper bound is given for algebraic states. The LMI-based existence conditions of such a stability analysis are derived by the introduction of free-weighting matrices to eliminate the cross coupling of system matrices. Some additional instrumental matrix variables are introduced, which makes the stability analysis feasible.

The outline of this paper is as follows. Preliminaries and the problem formulation are given in Section 2. Then, the main results are presented in Section 3. In this section, exponential convergence rate for differential variables is derived for both stable and unstable subsystems. Then, to make the switched singular time-delay system regular, impulse-free, and exponentially stable, some sufficient conditions are derived with the average dwell-time approach and the state variable transformation technique. A numerical example is also provided to show the effectiveness of the main result in Section 4. Finally, concluding remarks and future works are drawn in Section 5.

2. PRELIMINARIES

Let the dynamic of a class of switched singular time-delay systems be described by the following:

$$\begin{cases} E\dot{x}(t) = A_p x(t) + A_{dp} x(t-d(t)) \\ x(\theta) = \phi(\theta) \quad \forall \theta \in [-h_2, 0],\, p \in \mathcal{P} \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $rank(E) = r \leqslant n$ may be singular, (A_p, A_{dp}) are known constant matrices with appropriate dimensions, $\phi(\theta)$ is an initial vector valued continuous function and $\|\phi(\theta)\|_c$ is defined as $\|\phi(\theta)\|_c = \sup_{-h_2 \leqslant s \leqslant 0} \|\phi(s)\|$, which stands for the norm of initial condition $\phi(\theta)$, and $d(t)$ is a time-varying continuous function that satisfies $0 < h_1 \leqslant d(t) \leqslant h_2$ and $\dot{d}(t) \leqslant \bar{d} < 1$ in which h_2 and h_1 are scalars representing the upper and lower bound of delay, respectively. The index set $\mathcal{P}$ is finite: $\mathcal{P} = \{1, 2, \cdots, m\}$. The piecewise right continuous (and constant) function $\sigma(t) : [0, \infty) \to \mathcal{P}$ is a switching signal to specify, at each time instant t, the index $\sigma(t) \in \mathcal{P}$ of the active subsystem, that is, $\sigma(t) = p$ means that the pth subsystem is activated. Let $t_0 = 0 < \tau_1 < \tau_2 < \cdots < \tau_k < t, k = 1, 2, \cdots$ denote the switching points within $[0, t)$. Constructing switching sequence $\alpha = \{(\sigma(t_0), t_0), (\sigma(\tau_1), \tau_1), \cdots, (\sigma(\tau_k), \tau_k), \cdots | k = 1, 2, \cdots\}$ means that the $\sigma(\tau_k)$th subsystem is activated during $[\tau_k, \tau_{k+1})$.

Here, $\mathcal{P}$ is divided into $\mathcal{S}$ and $\mathcal{U}$ where $\mathcal{S}$ represents the stable subsystems with exponential stable differential variables and satisfying certain conditions stated below as Lemma 1, and $\mathcal{U}$ represents the other subsystems, whose differential variable behavior can be upper-bounded by a potentially increasing exponential function as stated in Lemma 2. In addition, the index i is used for subsystems that belong to $\mathcal{S}$ and j for subsystems that belong to $\mathcal{U}$.

Assumption 1
We remark that, initially it is assumed that the unstable subsystems run during $[\tau_{2k-1}, \tau_{2k})$ whereas the stable subsystems run during $[\tau_{2k-2}, \tau_{2k-1})$.

The contrary assumption can be made too. Without loss of generality, Assumption 1 is made in this paper. This typical switched system has been shown in Figure 1. In continuation, the results will be extended to a more general set-up with any switching sequence of subsystems.

Remark 1
The stability problem for switched singular systems has not been fully investigated yet, which will be challenging because of the difficult extension of the existing stability results. Also, there are

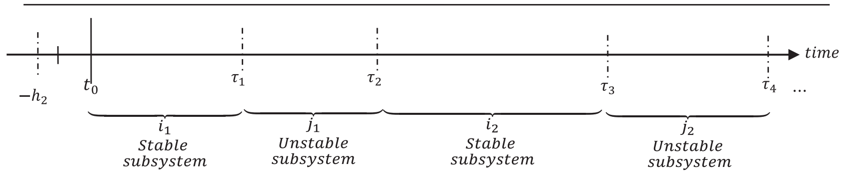

Figure 1. A typical switched system.

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many applications for these systems such as excitation control of the power system connected with On-Load Tap-Changer (OLTC) and constant power load [35], micromachined-switch Micro Electro-Mechanical Systems (MEMS) device as nonlinear descriptor [36], Markovian jump systems with time-delay [37], impulse detection in power electronics systems [38], antiwindup control [39], evaporator vessel [40], rigid bodies [40], and pulse-width modulator boost converter [36] (also see Ref. [41] for more applications). These theoretical and practical significances have motivated us to carry out the present study.

Definition 1

(i) System (1) is impulse-free and regular if (1) has an unique real-valued smooth solution [42].
(ii) The pair (E, A_p) is said to be impulse-free and regular if $\det(sE - A_p)$ is not identically zero and $\deg(\det(sE - A_p)) = \operatorname{rank}(E)$ [1].

It is obvious that in singular time-delay systems, (ii) is a necessary condition for (i). For more details on real-valued smooth solution refer to [42].

Now, choose two nonsingular matrices $M_p (p \in \mathcal{P})$ and N such that $E = M_p \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} N$. Therefore, we can obtain

$$A_p = M_p \begin{pmatrix} \bar{A}_{p_{11}} & \bar{A}_{p_{12}} \\ \bar{A}_{p_{21}} & \bar{A}_{p_{22}} \end{pmatrix} N, A_{dp} = M_p \begin{pmatrix} \bar{A}_{p_{11}} & \bar{A}_{p_{12}} \\ \bar{A}_{p_{21}} & \bar{A}_{p_{22}} \end{pmatrix} N. \tag{2}$$

If (E, A_p) is regular and impulse-free, we can get $\bar{A}_{p_{22}}$ is nonsingular [1], so we let $J_p = \begin{pmatrix} I & -\bar{A}_{p_{12}}\bar{A}_{p_{22}}^{-1} \\ 0 & \bar{A}_{p_{12}}^{-1} \end{pmatrix}$ and $\bar{M}_p = J_p M_p^{-1}$, which imply

$$\bar{E} = \bar{M}_p E N^{-1} = J_p M_p^{-1} M_p \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} N N^{-1} = \begin{pmatrix} I & -\bar{A}_{p_{12}}\bar{A}_{p_{22}}^{-1} \\ 0 & \bar{A}_{p_{22}}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$$\bar{A}_p = \bar{M}_p A_p N^{-1} = J_p M_p^{-1} M_p \begin{pmatrix} \bar{A}_{p_{11}} & -\bar{A}_{p_{12}} \\ \bar{A}_{p_{21}} & \bar{A}_{p_{22}}^{-1} \end{pmatrix} N N^{-1} = \begin{pmatrix} I & -\bar{A}_{p_{12}}\bar{A}_{p_{22}}^{-1} \\ 0 & \bar{A}_{p_{22}}^{-1} \end{pmatrix} \begin{pmatrix} \bar{A}_{p_{11}} & \bar{A}_{p_{12}} \\ \bar{A}_{p_{21}} & \bar{A}_{p_{22}} \end{pmatrix} = \begin{pmatrix} A_{p_{11}} & 0 \\ A_{p_{21}} & I \end{pmatrix}$$

$$\bar{A}_{dp} = \bar{M}_p A_{dp} N^{-1} = \begin{pmatrix} A_{dp_{11}} & A_{dp_{12}} \\ A_{dp_{21}} & A_{p_{22}} \end{pmatrix}, \tag{3}$$

where $A_{p_{11}} = \bar{A}_{p_{11}} - \bar{A}_{p_{12}}\bar{A}_{p_{22}}^{-1}\bar{A}_{p_{21}}$ and $A_{p_{21}} = \bar{A}_{p_{22}}^{-1}\bar{A}_{p_{21}}$. By considering (1)–(2), each subsystem can be rewritten as follows:

$$\dot{\xi}_1(t) = \bar{A}_{p_{11}}\xi_1(t) + \bar{A}_{p_{12}}\xi_2(t) + \bar{A}_{dp_{11}}\xi_1(t - d(t)) + \bar{A}_{dp_{12}}\xi_2(t - d(t)) \tag{4a}$$
$$0 = \bar{A}_{p_{21}}\xi_1(t) + \bar{A}_{p_{22}}\xi_2(t) + \bar{A}_{dp_{21}}\xi_1(t - d(t)) + \bar{A}_{dp_{22}}\xi_2(t - d(t)) \tag{4b}$$

in which $\xi(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix} = N x(t)$. Consider the following initial condition:

$$\begin{cases} \xi_1(t) = \phi_1(t) \\ \xi_2(t) = \phi_2(t) \end{cases}, t \in [-h_2, 0]. \tag{5}$$

By substituting initial condition $\psi(t) = col(\phi_1(t) \quad \phi_2(t))$ into (4b), we have

$$0 = \bar{A}_{\sigma(t_0)_{21}}\phi_1(0) + \bar{A}_{\sigma(t_0)_{22}}\phi_2(0) + \bar{A}_{d\sigma(t_0)_{21}}\phi_1(-d(0)) + \bar{A}_{d\sigma(t_0)_{22}}\phi_2(-d(0)). \tag{6}$$

Equation (6) is a compatible initial condition of switched singular time-delay systems. Consider that $\psi(t) = N\phi(t)$ and $\phi(t)$ was defined in (1). Furthermore, compatibility conditions at switching points (τ_k) can be written as follows:

$$\begin{aligned}
\bar{A}_{\sigma(\tau_k^-)_{21}}\xi_1\left(\tau_k^-\right) &+ \bar{A}_{\sigma(\tau_k^-)_{22}}\xi_2\left(\tau_k^-\right) + \bar{A}_{d\sigma(\tau_k^-)_{21}}\xi_1\left(\tau_k^- - d\left(\tau_k^-\right)\right) \\
&+ \bar{A}_{d\sigma(\tau_k^-)_{22}}\xi_2\left(\tau_k^- - d\left(\tau_k^-\right)\right) = \bar{A}_{\sigma(\tau_k)_{21}}\xi_1\left(\tau_k^+\right) + \bar{A}_{\sigma(\tau_k)_{22}}\xi_2\left(\tau_k^+\right) \\
&+ \bar{A}_{d\sigma(\tau_k)_{21}}\xi_1\left(\tau_k^+ - d\left(\tau_k^+\right)\right) + \bar{A}_{d\sigma(\tau_k)_{22}}\xi_2\left(\tau_k^+ - d\left(\tau_k^+\right)\right),
\end{aligned} \tag{7}$$

which guarantee the continuity of states at switching points. Here, $\sigma\left(\tau_k^+\right) = \sigma(\tau_k)$, $t^- = \lim_{\varepsilon\to 0}(t-\varepsilon)$, and $t^+ = \lim_{\varepsilon\to 0}(t+\varepsilon)$. In singular time-delay systems, if the pair (E, A_p) is regular and impulse-free, the system can still have finite discontinuities because of incompatible initial condition [11, 43]. If the initial condition does not satisfy the (4b) at $t = 0$, the system will have jump discontinuities. Therefore, it can be concluded that for switched singular time-delay systems, analysis of the solution is more complicated. In switched singular time-delay systems, continuity at switching points may not always be satisfied. Therefore, unlike standard switched time-delay systems, discontinuities in switched singular time-delay systems can propagate between different times because of the existence of delayed solution terms and discontinuities of previous switching points. Similar to the discussion in [43] for ordinary singular time-delay systems, the jump can occur only in the algebraic variables or in the derivatives of the differential variables, that is, the differential variables are always continuous. This feature is found in neither singular systems nor switched time-delay systems. Similar to singular time-delay systems, we have the following assumptions to guarantee the continuity of the solution for the first subsystem that is activated in $[\tau_0, \tau_1)$. Thus, the following assumptions are made throughout the paper:

Assumption 2
It is assumed that initial condition $\psi(t) = N\phi(t) = col(\phi_1(t) \quad \phi_2(t))$ satisfies the compatibility condition of the first subsystem that is activated in $[\tau_0, \tau_1)$ as

$$0 = \bar{A}_{\sigma(0)_{21}}\phi_1(0) + \bar{A}_{\sigma(0)_{22}}\phi_2(0) + \bar{A}_{d\sigma(0)_{21}}\phi_1(-d(0)) + \bar{A}_{d\sigma(0)_{22}}\phi_2(-d(0)) \tag{8}$$

Assumption 3
In this paper, it is assumed that Equation (7) is satisfied, which guarantees the continuity of states at switching points.

Assumptions 2 and 3 guarantee the continuity of states at switching points. Continuity at initial time $t = t_0$ is considered by Assumption 2 and continuity at switching points $(t = \tau_k)$ is considered by Assumption 3. Assumption 3 implies that switching points occur when the trajectories intersect the compatibility space of the new subsystems.

Without loss of generality, consider the following switched system at time t with the same structure in (2) and (3):

$$\bar{E}\dot{\xi}(t) = M_{\sigma(t)}^{-1}A_{\sigma(t)}N^{-1}\xi(t) + M_{\sigma(t)}^{-1}A_{d\sigma(t)}N^{-1}\xi(t-d(t)). \tag{9}$$

From [43], we can conclude that each subsystem includes a class of natural descriptor system as

$$\begin{bmatrix} \dot{\xi}_1(t) - D_{\sigma(t)}\dot{\xi}_1(t-d(t)) \\ 0 \end{bmatrix} = M_{\sigma(t)}^{-1}A_{\sigma(t)}N^{-1}\xi(t) + M_{\sigma(t)}^{-1}A_{d\sigma(t)}N^{-1}\xi(t-d(t)). \tag{10}$$

System (9) is a functional differential equation as follows:

$$\dot{\xi}_1(t) = \sum_{j=1}^{2} \bar{A}_{\sigma(t)_{1j}}\xi_j(t) + \bar{A}_{d\sigma(t)_{1j}}\xi_j(t-d(t)) \tag{11a}$$

$$0 = \sum_{j=1}^{2} \bar{A}_{\sigma(t)_{2j}}\xi_j(t) + \bar{A}_{d\sigma(t)_{2j}}\xi_j(t-d(t)). \tag{11b}$$

By considering Assumptions 2 and 3, the following proposition is given.

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Proposition 1
For switched system (9) under Assumptions 2 and 3, there exists a unique function $\xi(t)$ defined on $[-h_2, \infty)$ that satisfies (9) on $[0, \infty)$.

Proof
By considering Assumptions 2 and 3, we need only to show the existence of unique $\xi(t)$ for each subsystem. Differentiating (11b) with respect to t and taking into account that $\xi_1(t)$ and $d(t)$ is differentiable, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}(t)}\left(\bar{A}_{\sigma(t)_{22}}\xi_2(t) + \sum_{j=1}^{2}\bar{A}_{d\sigma(t)_{2j}}\xi_j(t-d(t))\right) + \bar{A}_{\sigma(t)_{21}}\left(\sum_{j=1}^{2}\bar{A}_{\sigma(t)_{1j}}\xi_j(t) + \bar{A}_{d\sigma(t)_{1j}}\xi_j(t-d(t))\right) = 0$$

Similar to [43] and [44], and by considering (7), we can conclude that each subsystem of (9) has a unique continuous solution on $[\tau_k, \tau_{k+1})$. Because each subsystem has a unique continuous solution and switching points satisfy (7), it can be concluded that the switched system (9) has unique continuous solution with respect to consistent initial condition (6). □

Now, the following definitions are given to prove the main results.

Definition 2
Switched singular system (1) is said to be exponentially stable if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial condition $\phi(t)$ under Assumption 2, the solution $x(t)$ to the switched singular time-delay system satisfies $\|x(t)\| \leqslant \gamma e^{-\sigma(t-t_0)}\|\phi(t)\|_c$.

Definition 3
Let $N_\sigma(t_0, t)$ and $N_f(t_0, t)$ denote the switching number of the whole system and the unstable subsystems during (t_0, t), respectively. $\kappa = N_f(T_0, t)/N_\sigma(T_0, t)$ is defined as the switching ratio of the unstable subsystem to the whole system.

By considering Definition 3, it can be seen that in Assumption 1, $\kappa = 1/2$. As mentioned before, we will also extend the result without considering Assumption 1.

Definition 4 ([24]*)*
For given scalars T_a and $N_0 \geqslant 0$, we have

$$N_\sigma(t_0, t) \leqslant N_0 + \frac{t - t_0}{T_\mathrm{a}},$$

where $N_\sigma(t_0, t)$ denotes the switching number of $\sigma(t)$ during (t_0, t). Then, T_a is called average dwell-time and N_0 the chattering bound. Here, we assume $N_0 = 0$ for simplicity as commonly used in the literature [45].

By the average dwell-time switching, we mean a class of switching signals such that the average time interval between consecutive switching is at least T_a. The average dwell-time switching strategy may contain signals that occasionally have consecutive discontinuities separated by less than a constant T_a.

Definition 5 ([46]*)*
For a scalar $\alpha^* \in [0, \alpha)$, $T^+(t_0, t)$(resp., $T^-(t_0, t)$) is defined as the total activation time of unstable subsystems that belong to $\mathcal{U}$. (resp., stable subsystems that belong to $\mathcal{S}$) during (t_0, t) and satisfy the condition $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geqslant \frac{\beta+\alpha^*}{\alpha-\alpha^*}$.

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The three objectives to be tackled in the following section are summarized as follows:

- Finding delay-range-dependent LMI conditions that guarantee the exponential stability of system (1) by considering regularity and impulse-free properties.
- Finding the average dwell-time T_a given by Definition 4, which is sufficiently large.
- Finding the total activation time ration $\frac{T^-(t_0,t)}{T^+(t_0,t)}$ in Definition 5 is no less than a specified constant.

3. STABILITY ANALYSIS

In this section, based on the introduced definitions and the following Lemmas, the main results are developed. Initially, sufficient conditions for stability analysis by using the Lyapunov-like Krasovskii functional for stable and unstable subsystems are given. Then, by considering the switching sequence, an upper bound is obtained for algebraic states ($\|\xi_2(t)\|$). Finally, a class of switching signals is found under which the switched singular time-delay system (1) is exponentially stable.

For the ith stable subsystem, ($i \in \mathcal{S}$), the following Lyapunov-like Krasovskii functional is selected to analyze the stability.

$$V_i(t,x(t)) = x(t)^T E^T P_i x(t) + e^{-2\alpha t} \sum_{k=1}^{4} V_{i_k}(t,x(t)), \tag{12}$$

where

$$V_{i_1}(t,x(t)) = \sum_{k=1}^{2} \int_{t-h_k}^{t} x(s)^T e^{2\alpha s} Q_{i_k} x(s)\mathrm{d}s + \int_{t-d(t)}^{t} x(s)^T e^{2\alpha s} Q_{i_3} x(s)\mathrm{d}s$$

$$V_{i_2}(t,x(t)) = \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}(s)^T E^T e^{2\alpha s} Z_{i_1} E\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta,$$

$$V_{i_3}(t,x(t)) = \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}(s)^T E^T e^{2\alpha s} Z_{i_2} E\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta$$

$$V_{i_4}(t,x(t)) = \int_{-h_2}^{-d(t)}\int_{t+\theta}^{t} \dot{x}(s)^T E^T e^{2\alpha s} Z_{i_3} E\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta$$

P_i is a nonsingular matrix in which $E^T P_i = P_i^T E \geqslant 0$, Q_{i_k} and $Z_{i_k} (k = 1,2,3)$ are symmetric positive-definite matrices, and $\alpha > 0$ denotes the decay convergence rate of the differential states of the ith subsystem.

Lemma 1

For the ith subsystem of system (1), which satisfies Assumption 2, given $0 < h_1 \leqslant d(t) \leqslant h_2$, $\dot{d}(t) \leqslant d < 1$, and scalar $\alpha > 0$, if there exists a nonsingular matrix P_i, symmetric and positive-definite matrices Q_{i_k} and $Z_{i_k} (k = 1,2,3)$, and matrices Y_{i_k}, T_{i_k}, L_{i_k} $(k = 1,2)$ such that the following LMI holds:

$$\begin{pmatrix} \Omega_i & Y_i & T_i & L_i & \bar{A}_i^T U_i \\ * & -\ell_1^{-1} Z_{i_1} & 0 & 0 & 0 \\ * & * & -\ell_2^{-1} Z_{i_2} & 0 & 0 \\ * & * & * & -\ell_3^{-1} Z_{i_3} & 0 \\ * & * & * & * & -U_i \end{pmatrix} < 0 \tag{13a}$$

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with the following constraint:

$$E^T P_i = P_i^T E \geqslant 0, \tag{13b}$$

then, the ith subsystem is regular and impulse-free and there exists a scalar χ_i such that $\|A_{di_{22}}{}^{i}\| \leqslant \chi_i e^{-\alpha h_2 i} (i = 1, 2 \cdots)$. Also, we have

$$V_i(t, x(t)) \leqslant e^{-2\alpha(t-t_0)} V_i(t_0, x(t_0)),\ \rho\left(e^{\alpha h_2} A_{di_{22}}\right) < 1, \tag{13c}$$

where

$$\Omega_i = \begin{pmatrix} \Upsilon_{i_{11}} & \Upsilon_{i_{12}} & L_{i_1} E & -T_{i_1} E \\ * & \Upsilon_{i_{22}} & L_{i_2} E & -T_{i_2} E \\ * & * & -e^{-2\alpha h_1} Q_{i_1} & 0 \\ * & * & * & -e^{-2\alpha h_2} Q_{i_2} \end{pmatrix}, Y_i = \begin{pmatrix} Y_{i_1} \\ Y_{i_2} \\ 0 \\ 0 \end{pmatrix}, T_i = \begin{pmatrix} T_{i_1} \\ T_{i_2} \\ 0 \\ 0 \end{pmatrix}, L_i = \begin{pmatrix} L_{i_1} \\ L_{i_2} \\ 0 \\ 0 \end{pmatrix}, \bar{A}_i^T = \begin{pmatrix} A_i^T \\ A_{di}^T \\ 0 \\ 0 \end{pmatrix}$$

$$\Upsilon_{i_{11}} = A_i^T P_i + P_i^T A_i + \sum\nolimits_{k=1}^{3} Q_{i_k} + 2\alpha E^T P_i + Y_{i_1} E + E^T Y_{i_1}^T,$$
$$\Upsilon_{i_{12}} = P_i^T A_{di} + E^T Y_{i_2}^T - Y_{i_1} E + T_{i_1} E - L_{i_1} E$$
$$\Upsilon_{i_{22}} = -(1-d) e^{-2\alpha h_2} Q_{i_3} - Y_{i_2} E - E^T Y_{i2}^T + T_{i_2} E + E^T T_{i_2}^T - L_{i_2} E - E^T L_{i_2}^T$$
$$U_i = h_2 Z_{i_1} + (h_2 - h_1)(Z_{i_2} + Z_{i_3}), \ell_1 = \frac{e^{2\alpha h_2} - 1}{2\alpha}, \ell_2 = \ell_3 \frac{e^{2\alpha h_2} - e^{2\alpha h_1}}{2\alpha}.$$

$\rho(M)$ is the spectral radius of matrix M, and '$*$' denotes the matrix entries implied by the symmetry of a matrix.

Proof

First, we show that the ith subsystem is regular and impulse-free. For this purpose, choose two nonsingular matrices R_i and L such that $\bar{E} = R_i E L = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\begin{aligned}
&\bar{A}_i = R_i A_i L = \begin{pmatrix} A_{i_{11}} & A_{i_{12}} \\ A_{i_{21}} & A_{i_{22}} \end{pmatrix},\ \bar{A}_{di} = R_i A_{di} L = \begin{pmatrix} A_{di_{11}} & A_{di_{12}} \\ A_{di_{21}} & A_{di_{22}} \end{pmatrix}, \\
&\bar{P}_i = R_i^{-T} P_i L = \begin{pmatrix} P_{i_{11}} & P_{i_{12}} \\ P_{i_{21}} & P_{i_{22}} \end{pmatrix} \\
&\bar{Q}_{i_k} = L^T Q_{i_k} L = \begin{pmatrix} Q_{i_{k_{11}}} & Q_{i_{k_{12}}} \\ Q_{i_{k_{21}}} & Q_{i_{k_{22}}} \end{pmatrix},\ \bar{Y}_{i_l} = L^T Y_{i_l} R_i^{-1} = \begin{pmatrix} Y_{i_{l_{11}}} & Y_{i_{l_{12}}} \\ Y_{i_{l_{21}}} & Y_{i_{l_{22}}} \end{pmatrix}, \\
&\bar{T}_{i_l} = L^T T_{i_l} R_i^{-1} = \begin{pmatrix} T_{i_{l_{11}}} & T_{i_{l_{12}}} \\ T_{i_{l_{21}}} & T_{i_{l_{22}}} \end{pmatrix} \\
&\bar{L}_{i_l} = L^T L_{i_l} R_i^{-1} = \begin{pmatrix} L_{i_{l_{11}}} & L_{i_{l_{12}}} \\ L_{i_{l_{21}}} & L_{i_{l_{22}}} \end{pmatrix},\ \bar{Z}_{i_k} = L^T Z_{i_k} R_i^{-1} = \begin{pmatrix} Z_{i_{k_{11}}} & Z_{i_{k_{12}}} \\ Z_{i_{k_{21}}} & Z_{i_{k_{22}}} \end{pmatrix}, \\
&k = 1, 2, 3,\ l = 1, 2.
\end{aligned} \tag{14}$$

From (13b), it can be concluded that $P_{i_{12}} = 0$ and $P_{i_{11}} = P_{i_{11}}^T > 0$. Also, from (13a), $\Upsilon_{i_{11}} < 0$ is obtained. On the basis of (14), pre-multiplying and post-multiplying this inequality by L^T, L, respectively, and noting that $Q_{i_k} > 0$, we have

$$L^T A_i^T P_i L + L^T P_i^T A_i L + L^T \sum\nolimits_{k=1}^{3} Q_{i_k} L + 2\alpha L^T E^T P_i L + L^T Y_{i_1} E L + L^T E^T Y_{i_1}^T L < 0, \tag{15}$$

which implies

$$\begin{pmatrix} \circledast & \circledast \\ \circledast & A_{i_{22}}^T P_{i_{22}} + P_{i_{22}}^T A_{i_{22}} + \sum\limits_{k=1}^{3} Q_{i_{k_{22}}} \end{pmatrix} < 0 \tag{16}$$

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and yields $A_{i_{22}}^T P_{i_{22}} + P_{i_{22}}^T A_{i_{22}} < 0$. Here, $\circledast$ is irrelevant to the results of the following discussion; the real expressions of these variables are omitted here. Thus, $A_{i_{22}}$ is nonsingular, which implies that $\det(sE - A_i)$ is not identically zero and that $\deg(\det(sE - A_i)) = r = \operatorname{rank}(E)$. Then, the pair (E, A_i) is regular and impulse-free [1]. By considering Assumption 2, it can be concluded that the ith subsystem is regular and impulse-free.

In the following, it will be proven that the ith subsystem satisfies (13c). Because the ith subsystem is regular and impulse-free, by setting $R_i = \bar{M}_i$, $L = N^{-1}$, using (13a) and (13b), and the Schur complement Lemma, we get $\begin{pmatrix} \Upsilon_{i_{11}} & \Upsilon_{i_{12}} \\ * & \Upsilon_{i_{22}} \end{pmatrix} < 0$. By pre-multiplying and post-multiplying this inequality by $\operatorname{diag}\{L^T, L^T\}$ and $\operatorname{diag}\{L, L\}$ respectively, and using the Schur complement Lemma, we can conclude that

$$\begin{pmatrix} P_{i_{22}} + P_{i_{22}}^T + \sum_{k=1}^{3} Q_{i_{k_{22}}} & P_{i_{22}}^T A_{di_{22}} \\ * & -(1-d)e^{-2\alpha h_2} Q_{i_{3_{22}}} \end{pmatrix} < 0 . \tag{17}$$

By pre-multiplying and post-multiplying (17) by $\left(-A_{di_{22}}^T, I\right)$ and $\left(-A_{di_{22}}^T, I\right)^T$, respectively, and noting that $Q_{i_k} > 0$, we have $A_{di_{22}}^T Q_{i_{3_{22}}} A_{di_{22}} - e^{-2\alpha h_2} Q_{i_{3_{22}}} < 0$, which implies $\rho\left(e^{\alpha h_2} A_{di_{22}}\right) < 1$. Therefore, there exists a scalar $\chi_i > 0$ such that $\|A_{di_{22}}{}^i\| \leqslant \chi_i e^{-iah_2} (i = 1, 2 \cdots)$.

Now, taking the time derivative of (12), we have

$$\begin{aligned}
\dot{V}_i(t, x(t)) = {} & 2\dot{x}(t)^T E^T P_i x(t) + \sum_{k=1}^{2} \left(x(t)^T Q_{i_k} x(t) - x(t-h_k)^T e^{-2\alpha h_k} Q_{i_k} x(t-h_k) \right) \\
& + x(t)^T Q_{i_3} x(t) - \left(1 - \dot{d}(t)\right) x(t-d(t))^T e^{-2\alpha d(t)} Q_{i_3} x(t-d(t)) \\
& + h_2 \dot{x}(t)^T E^T Z_{i_1} E\dot{x}(t) + (h_2 - h_1)\dot{x}(t)^T E^T Z_{i_2} E\dot{x}(t) \\
& + (h_2 - d(t))\dot{x}(t)^T E^T Z_{i_3} E\dot{x}(t) + 2\alpha x(t)^T E^T P_i x(t) - 2\alpha V_i(t, x(t)) \\
& - e^{-2\alpha t} \int_{t-h_2}^{t} \dot{x}(t)^T E^T e^{2\alpha s} Z_{i_1} E\dot{x}(t) \mathrm{d}s - e^{-2\alpha t} \int_{t-h_2}^{t-h_1} \dot{x}(t)^T E^T e^{2\alpha s} Z_{i_2} E\dot{x}(t) \mathrm{d}s \\
& - e^{-2\alpha t} \int_{t-h_2}^{t-d(t)} \dot{x}(t)^T E^T e^{2\alpha s} Z_{i_3} E\dot{x}(t) \mathrm{d}s .
\end{aligned} \tag{18}$$

Noting that

$$\begin{aligned}
& h_2 \dot{x}(t)^T E^T Z_{i_1} E\dot{x}(t) + (h_2 - h_1)\dot{x}(t)^T E^T Z_{i_2} E\dot{x}(t) + (h_2 - h_1)\dot{x}(t)^T E^T Z_{i_3} E\dot{x}(t) \\
& = \zeta(t)^T \bar{A}_i^T U_i \bar{A}_i \zeta(t)
\end{aligned} \tag{19}$$

and

$$-\left(1 - \dot{d}(t)\right) x(t-d(t))^T e^{-2\alpha d(t)} Q_{i_3} x(t-d(t)) \leqslant -(1-d)x(t-d(t))^T e^{-2\alpha h_2} Q_{i_3} x(t-d(t)), \tag{20}$$

where $\bar{A}_i = (A_i, A_{di}, 0, 0)$, $U_i = h_2 Z_{i_1} + (h_2 - h_1)\left(Z_{i_2} + Z_{i_3}\right)$ and $\zeta(t)^T = \left(x(t)^T, x(t-d(t))^T, x(t-h_1)^T, x(t-h_2)^T\right)$, and adding the following Leibniz–Newton formulas:

$$2\left(x(t)^T Y_{i_1} - x(t-d(t))^T Y_{i_2}\right)\left\{Ex(t) - Ex(t-d(t)) - \int\limits_{t-d(t)}^{t} E\dot{x}(s)\mathrm{d}s\right\} = 0 \tag{21a}$$

$$2\left(x(t)^T T_{i_1} - x(t-d(t))^T T_{i_2}\right)\left\{Ex(t-d(t)) - Ex(t-h_2) - \int\limits_{t-h_2}^{t-d(t)} E\dot{x}(s)\mathrm{d}s\right\} = 0 \tag{21b}$$

$$2\left(x(t)^T L_{i_1} - x(t-d(t))^T L_{i_2}\right)\left\{Ex(t-h_1) - Ex(t-d(t)) - \int\limits_{t-d(t)}^{t-h_1} E\dot{x}(s)\mathrm{d}s\right\} = 0 \tag{21c}$$

to (18) gives

$$\begin{aligned}\dot{V}_i(t,x(t)) + 2\alpha V_i(t,x(t)) \leqslant{}& \zeta(t)^T\left(\Omega_i + \bar{A}_i^T U_i \bar{A}_i\right)\zeta(t) - e^{-2\alpha t}\int\limits_{t-h_2}^{t} \dot{x}(t)^T E^T e^{2\alpha s} Z_{i_1} E\dot{x}(t)\mathrm{d}s \\
&- e^{-2\alpha t}\int\limits_{t-h_2}^{t-h_1} \dot{x}(t)^T E^T e^{2\alpha s} Z_{i_2} E\dot{x}(t)\mathrm{d}s - e^{-2\alpha t}\int\limits_{t-h_2}^{t-d(t)} \dot{x}(t)^T E^T e^{2\alpha s} Z_{i_3} E\dot{x}(t)\mathrm{d}s \\
&- 2\left(x(t)^T Y_{i_1} - x(t-d(t))^T Y_{i_2}\right)\left\{Ex(t) - Ex(t-d(t)) - \int\limits_{t-d(t)}^{t} E\dot{x}(s)\mathrm{d}s\right\} \\
&- 2\left(x(t)^T T_{i_1} - x(t-d(t))^T T_{i_2}\right)\left\{Ex(t-d(t)) - Ex(t-h_2) - \int\limits_{t-h_2}^{t-d(t)} E\dot{x}(s)\mathrm{d}s\right\} \\
&- 2\left(x(t)^T L_{i_1} - x(t-d(t))^T L_{i_2}\right)\left\{Ex(t-h_1) - Ex(t-d(t)) - \int\limits_{t-d(t)}^{t-h1} E\dot{x}(s)\mathrm{d}s\right\},\end{aligned} \tag{22}$$

where Y_{i_k}, T_{i_k}, L_{i_k} $(k = 1, 2)$ are matrices with appropriate dimensions, called the free-weighting matrices [8]. From (22), we can obtain

$$\begin{aligned}\dot{V}_i(t,x(t)) + 2\alpha V_i(t,x(t)) \leqslant{}& \zeta(t)^T\left(\Omega_i + \bar{A}_i^T U_i \bar{A}_i\right)\zeta(t) - \int\limits_{t-d(t)}^{t} Y_{i_3}^T\left(e^{2\alpha(s-t)} Z_{i_1}\right)^{-1} Y_{i_3}\mathrm{d}s \\
&+ \int\limits_{t-d(t)}^{t} \zeta(t)^T Y_i^T\left(e^{2\alpha(s-t)} Z_{i_1}\right)^{-1} Y_i\zeta(t)\mathrm{d}s \\
&- \int\limits_{t-h_2}^{t-d(t)} T_{i_3}^T\left(e^{2\alpha(s-t)} Z_{i_2}\right)^{-1} T_{i_3}\mathrm{d}s \\
&+ \int\limits_{t-h_2}^{t-d(t)} \zeta(t)^T T_i^T\left(e^{2\alpha(s-t)} Z_{i_2}\right)^{-1} T_i\zeta(t)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&-\int_{t-d(t)}^{t-h_1} L_{i_3}^T \left(e^{2\alpha(s-t)} Z_{i_3}\right)^{-1} L_{i_3} \mathrm{d}s \\
&+\int_{t-d(t)}^{t-h_1} \zeta(t)^T L_i^T \left(e^{2\alpha(s-t)} Z_{i_3}\right)^{-1} L_i \zeta(t) \mathrm{d}s,
\end{aligned} \tag{23}
$$

which deduces

$$
\begin{aligned}
\dot{V}_i(t,x(t)) + 2\alpha V_i(t,x(t)) \leqslant \zeta(t)^T \Theta_i \zeta(t) &- \int_{t-d(t)}^{t} Y_{i_3}^T \left(e^{2\alpha(s-t)} Z_{i_1}\right)^{-1} Y_{i_3} \mathrm{d}s \\
&- \int_{t-h_2}^{t-d(t)} T_{i_3}^T \left(e^{2\alpha(s-t)} Z_{i_2}\right)^{-1} T_{i_3} \mathrm{d}s - \int_{t-d(t)}^{t-h_1} L_{i_3}^T \left(e^{2\alpha(s-t)} Z_{i_3}\right)^{-1} L_{i_3} \mathrm{d}s
\end{aligned} \tag{24}
$$

in which

$$
\begin{aligned}
&\Theta_i = \Omega_i + \bar{A}_i^T U_i \bar{A}_i + \ell_1 Y_i Z_{i_1}^{-1} Y_i^T + \ell_2 T_i Z_{i_2}^{-1} T_i^T + \ell_3 L_i Z_{i_3}^{-1} L_i^T, \\
&Y_{i_3}^T = \zeta(t)^T Y_i + \dot{x}(t)^T E^T e^{2\alpha(s-t)} Z_{i_1} \\
&T_{i_3}^T = \zeta(t)^T T_i + \dot{x}(t)^T E^T e^{2\alpha(s-t)} Z_{i_2}, L_{i_3}^T = \zeta(t)^T L_i + \dot{x}(t)^T E^T e^{2\alpha(s-t)} Z_{i_3} \\
&\ell_1 = \frac{e^{2\alpha h_2} - 1}{2\alpha}, \ell_2 = \ell_3 = \frac{e^{2\alpha h_2} - e^{2\alpha h_1}}{2\alpha}, Y_i = \begin{pmatrix} Y_{i_1} \\ Y_{i_2} \\ 0 \\ 0 \end{pmatrix}, T_i = \begin{pmatrix} T_{i_1} \\ T_{i_2} \\ 0 \\ 0 \end{pmatrix}, L_i = \begin{pmatrix} L_{i_1} \\ L_{i_2} \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

Because $\left(e^{2\alpha(s-t)} Z_{i_k}\right) > 0$, $k = 1,2,3$, the last terms in (24) are less than zero. Thus, it follows from the Schur complement Lemma that $\Theta_i < 0$ is equivalent to (13a). Now, by integrating this last equation it can be concluded that $V_i(t,x(t)) \leqslant e^{-2\alpha(t-t_0)} V_i(t_0, x(t_0))$. Because $V_i(t,x(t))$ is a bounded quadratic function, we can find scalar λ_{i_1} and a sufficiently large scalar λ_{i_2} [11, 27] such that $\lambda_{i_1} \|\xi_1(t)\|^2 \leqslant V_i(t,x(t))$ and $V_i(t_0,x(t_0)) \leqslant \lambda_{i_2} \|\phi(t)\|_c^2$, which leads to $\|\xi_1(t)\| \leqslant \sqrt{\frac{\lambda_{i_2}}{\lambda_{i_1}}} e^{-\alpha(t-t_0)} \|\phi(t)\|_c$ in which $\lambda_{i_1} = \lambda_{\min}(P_{i_{11}})$. Thus, the proof is completed. □

For the jth unstable subsystem, ($j \in \mathcal{U}$), the following Lyapunov-like Krasovskii functional is selected to analyze the behavior of unstable subsystems.

$$
V_j(t,x(t)) = x(t)^T E^T P_j x(t) + e^{2\beta t} \sum_{k=1}^{4} V_{j_k}(t,x(t)), \tag{25}
$$

where

$$
\begin{aligned}
&V_{j_1}(t,x(t)) = \sum_{k=1}^{2} \int_{t-h_k}^{t} x(s)^T e^{-2\beta s} Q_{j_k} x(s) \mathrm{d}s + \int_{t-d(t)}^{t} x(s)^T e^{-2\beta s} Q_{j_3} x(s) \mathrm{d}s \\
&V_{j_2}(t,x(t)) = \int_{-h_2}^{0} \int_{t+\theta}^{t} \dot{x}(s)^T E^T e^{-2\beta s} Z_{j_1} E \dot{x}(s) \mathrm{d}s \, \mathrm{d}\theta,
\end{aligned}
$$

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$$V_{j3}(t,x(t)) = \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}(s)^T E^T e^{-2\beta s} Z_{j_2} E\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta$$

$$V_{j4}(t,x(t)) = \int_{-h_2}^{-d(t)}\int_{t+\theta}^{t} \dot{x}(s)^T E^T e^{-2\beta s} Z_{j_3} E\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta.$$

P_j is a nonsingular matrix in which $E^T P_j = P_j^T E \geqslant 0$, Q_{j_k} and Z_{j_k} $(k = 1,2,3)$ are symmetric positive-definite matrices, and $\beta > 0$ denotes the increasing divergence rate of the differential states of the jth subsystem.

Lemma 2
For the jth subsystem of system (1), which satisfies Assumption 2, given $0 < h_1 \leqslant d(t) \leqslant h_2$, $\dot{d}(t) \leqslant d < 1$, and scalar $\beta > 0$, if there exists a nonsingular matrix P_j, symmetric and positive-definite matrices Q_{j_k}, Z_{j_k} $(k = 1,2,3)$, and matrices Y_{j_k}, T_{j_k}, L_{j_k} $(k = 1,2)$ such that the following LMI holds:

$$\begin{pmatrix} \Omega_j & Y_j & T_j & L_j & \bar{A}_j^T U_j \\ * & -g_1^{-1} Z_{j_1} & 0 & 0 & 0 \\ * & * & -g_2^{-1} Z_{j_2} & 0 & 0 \\ * & * & * & -g_3^{-1} Z_{j_3} & 0 \\ * & * & * & * & -U_j \end{pmatrix} < 0 \tag{26a}$$

with the following constraint:

$$E^T P_j = P_j^T E \geqslant 0 \tag{26b}$$

then the jth subsystem is regular and impulse-free and there exists a scalar χ_j such that $\| A_{dj_{22}}{}^{\jmath} \| \leqslant \chi_j e^{\beta h_1 \jmath}$ $(\jmath = 1,2,\cdots)$. Also, we have

$$V_j(t,x(t)) \leqslant e^{2\beta(t-t_0)} V_j(t_0, x(t_0)),\ \rho\left(e^{-\beta h_1} A_{dj_{22}}\right) < 1\,, \tag{26c}$$

where

$$\Omega_j = \begin{pmatrix} \Upsilon_{j_{11}} & \Upsilon_{j_{12}} & L_{j_1}E & -T_{j_1}E \\ * & \Upsilon_{j_{22}} & L_{j_2}E & -T_{j_2}E \\ * & * & -e^{2\beta h_1} Q_{j_1} & 0 \\ * & * & * & -e^{2\beta h_2} Q_{j_2} \end{pmatrix}, Y_j = \begin{pmatrix} Y_{j_1} \\ Y_{j_2} \\ 0 \\ 0 \end{pmatrix}, T_j = \begin{pmatrix} T_{j_1} \\ T_{j_2} \\ 0 \\ 0 \end{pmatrix}, L_j = \begin{pmatrix} L_{j_1} \\ L_{j_2} \\ 0 \\ 0 \end{pmatrix},$$

$$\bar{A}_j^T = \begin{pmatrix} A_j^T \\ A_{dj}^T \\ 0 \\ 0 \end{pmatrix}$$

$$\Upsilon_{j_{11}} = A_j^T P_j + P_j^T A_j + \sum_{k=1}^{3} Q_{j_k} - 2\beta E^T P_j + Y_{j_1}E + E^T Y_{j_1}^T,$$
$$\Upsilon_{j_{12}} = P_j^T A_{dj} + E^T Y_{j_2}^T - Y_{j_1}E + T_{j_1}E - L_{j_1}E$$
$$\Upsilon_{j_{22}} = -(1-d)e^{2\beta h_1} Q_{j_3} - Y_{j_2}E - E^T Y_{j_2}^T + T_{j_2}E + E^T T_{j_2}^T - L_{j_2}E - E^T L_{j_2}^T$$
$$U_j = h_2 Z_{j_1} + (h_2 - h_1)(Z_{j_2} + Z_{j_3}),\ g_1 = \frac{1 - e^{-2\beta h_2}}{2\beta},\ g_2 = g_3 \frac{e^{-2\beta h_1} - e^{-2\beta h_2}}{2\beta}.$$

Proof
Similar to the proof of Lemma 1, it can be proven that $P_{j_{11}} = P_{j_{11}}^T$, $P_{j_{12}} = 0$, and the pair (E, A_j) is regular and impulse-free. Now, by letting $R_j = \bar{M}_j$, $L = N^{-1}$, we get $\begin{pmatrix} \Upsilon_{j_{11}} & \Upsilon_{j_{12}} \\ * & \Upsilon_{j_{22}} \end{pmatrix} < 0.$

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By pre-multiplying and post-multiplying this inequality by diag $\left\{L^T, L^T\right\}$ and diag$\{L, L\}$, respectively, and using the Schur complement Lemma, we can conclude that

$$\begin{pmatrix} P_{j_{22}} + P_{j_{22}}^T + \sum_{k=1}^{3} Q_{j_{k_{22}}} & P_{j_{22}}^T A_{dj_{22}} \\ * & -(1-d)e^{2\beta h_1} Q_{j_{3_{22}}} \end{pmatrix} < 0. \tag{27}$$

By pre-multiplying and post-multiplying (27) by $\left(-A_{dj_{22}}^T, I\right)$ and $\left(-A_{dj_{22}}^T, I\right)^T$, respectively, and noting that $Q_{j_k} > 0$, we have $A_{dj_{22}}^T Q_{j_{3_{22}}} A_{dj_{22}} - e^{2\beta h_1} Q_{j_{3_{22}}} < 0$, which implies $\rho\left(e^{-\beta h_1} A_{dj_{22}}\right) < 1$. Therefore, there exists scalar $\chi_j > 0$ such that $\|A_{dj_{22}}{}^{\jmath}\| \leqslant \chi_j e^{\jmath \beta h_1} (\jmath = 1, 2, \cdots)$. Now, similar to (18)–(24), by taking the derivative of (25), we can get

$$\begin{aligned} \dot{V}(t, x(t)) - 2\beta V_j(t, x(t)) \leqslant \zeta(t)^T \Theta_j \zeta(t) - &\int_{t-d(t)}^{t} Y_{j_3}^T \left(e^{-2\beta(s-t)} Z_{j_1}\right)^{-1} Y_{j_3} \mathrm{d}s \\ - &\int_{t-h_2}^{t-d(t)} T_{j_3}^T \left(e^{-2\beta(s-t)} Z_{j_2}\right)^{-1} T_{j_3} \mathrm{d}s \\ - &\int_{t-d(t)}^{t-h_1} L_{j_3}^T \left(e^{-2\beta(s-t)} Z_{j_3}\right)^{-1} L_{j_3} \mathrm{d}s \end{aligned} \tag{28}$$

in which

$$\begin{aligned} &\Theta_j = \Omega_j + \bar{A}_j^T U_j \bar{A}_j + g_1 Y_j Z_{j1}^{-1} Y_j^T + g_2 T_j Z_{j2}^{-1} T_j^T + g_3 L_j Z_{j3}^{-1} L_j^T, \\ &Y_{j_3}^T = \zeta(t)^T Y_j + \dot{x}(t)^T E^T e^{-2\beta(s-t)} Z_{j_1} \\ &T_{j_3}^T = \zeta(t)^T T_j + \dot{x}(t)^T E^T e^{-2\beta(s-t)} Z_{j_2}, L_{j_3}^T = \zeta(t)^T L_j + \dot{x}(t)^T E^T e^{-2\beta(s-t)} Z_{j_3} \end{aligned}$$

Because $\left(e^{-2\beta(s-t)} Z_{j_k}\right) > 0$, $k = 1, 2, 3$, the last terms in (28) are less than zero. Thus, it follows from the Schur complement Lemma that $\Theta_j < 0$ is equivalent to (26a). Now, by integrating this last equation, (26c) is obtained. Because $V_j(t, x(t))$ is a bounded quadratic function, we can find scalar λ_{j_1} and a sufficiently large scalar λ_{j_2} such that $\lambda_{j_1} \|\xi_1(t)\|^2 \leqslant V_j(t, x(t))$ and $V_j(t_0, x(t_0)) \leqslant \lambda_{j_2} \|\phi(t)\|_c^2$, which leads to $\|\xi_1(t)\| \leqslant \sqrt{\frac{\lambda_{j_2}}{\lambda_{j_1}}} e^{\beta(t-t_0)} \|\phi(t)\|_c$ in which $\lambda_{j_1} = \lambda_{\min}\left(P_{j_{11}}\right)$. Thus, the proof is completed. □

Remark 2

Conditions (13a) and (13b) in Lemma 1 imply that $\|\xi_1(t)\| \leqslant \sqrt{\frac{\lambda_{i_2}}{\lambda_{i_1}}} e^{-\alpha(t-t_0)} \|\phi(t)\|_c$, which means that differential variables behavior can be upper-bounded by a potentially decreasing exponential function. Also, conditions (26a) and (26b) in Lemma 2 imply $\|\xi_1(t)\| \leqslant \sqrt{\frac{\lambda_{j_2}}{\lambda_{j_1}}} e^{\beta(t-t_0)} \|\phi(t)\|_c$, which means that differential variable behavior can be upper-bounded by a potentially increasing exponential function.

By Lemmas 1 and 2, we characterize both types of subsystems. Now, we want to define an average dwell-time switching signal such that the exponential stability of the switched singular time-delay system under Assumptions 1–3 is guaranteed. The main result is given in the following Theorem 1. To prove Theorem 1, the following discussion and Lemma 3 is necessary to mention, which gives an exponential bound for the algebraic states while switching sequence is considered. For simplicity, Assumption 1 is considered. The results will be extended to the general set-up in Theorem 3.

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Consider a typical switched singular time-delay system under Assumptions 1–3 as shown in Figure 2(a). To prove the exponential stability of the algebraic states, state transformation $\xi(t) = L^{-1}x(t)$ and (3) are used, which yield (for a stable subsystem)

$$0 = A_{i_{21}}\xi_1(t) + \xi_2(t) + A_{d i_{21}}\xi_1(t - d(t)) + A_{d i_{22}}\xi_2(t - d(t)). \tag{29}$$

For time-varying delay, motivated from [11], some new variables are defined to model the dependency of $\xi_2(t)$ on the past instants. Consider the first subsystem in Figure 2(b) and assume $t_i = t_{i-1} - d(t_{i-1})$, $t_0 = t_{\mathcal{K}_{i_1}(t)}$, $i = 1, 2, \cdots, \mathcal{K}_{i_0}(t)$, it can be shown that $\xi_2(t)$ at time $t = t_i$ depends on the value of $\xi(t)$ at $t = t_{i+1}$ and can be found by an iterative method.

In general, because $0 < d(t)$, it can be concluded that for subsystem i_k (resp., j_k) there exists a scalar $\mathcal{K}_{i_k}(t) > 0$ (resp., $\mathcal{K}_{j_k}(t) > 0$) such that $\xi_2(t)$ depends on $\mathcal{K}_{i_k}(t)$ (resp., $\mathcal{K}_{j_k}(t)$) times before t in each subsystem. Here, $\mathcal{K}_{j_k}(t)$ and $\mathcal{K}_{i_k}(t)$ are the number of times in which $\xi(t)$ depends on them in each subsystem. Therefore, there are $\Sigma_{i,j}\left(\mathcal{K}_{j_k}(t) + \mathcal{K}_{i_k}(t)\right)$ points that $\xi_2(t)$ depends on. More details are given in the Appendix as proof of Lemma 3. Consider that existence of h_1 guarantees that $\mathcal{K}_{i_k}(t)$ (resp., $\mathcal{K}_{j_k}(t)$) always exists. As an example, if $(t - ((2^s - 1)/2^s)) = (1/2^{s+1})$ [47], then we cannot find a h_1 such that $h_1 < d(t)$ when $s \to \infty$ and $d(t) = \frac{1}{2}, d(t - ((2^s - 1)/2^s)) = \left(1/2^{s+1}\right), s = 1, 2, \cdots$ cannot be considered as admissible delay for (1).

Lemma 3

For the switched singular time-delay system described by (1), which satisfies Assumptions 1–3, an upper bound for $\|\xi_2(t)\|$ can be obtained by the following:

$$\|\xi_2(t)\| \leqslant \gamma\sigma^{N_\sigma(t_0,t)}\left(\mu_1{}^{\frac{1}{2}}\right)^{N_f(t_0,t)}\left\{\frac{\sigma\left(\mu_1{}^{\frac{1}{2}}\right)}{\gamma} + N_\sigma(t_0,t)\right\} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c, \tag{30a}$$

where γ is a sufficiently large scalar, $\sigma = \max_{i,j\in\mathcal{P}}\left\{\chi_i, \chi_j, \mu^{\frac{1}{2}}\right\}, \mu_1 = e^{2h_2(\alpha+\beta)}$, scalar $\mu \geqslant 1$ that satisfies

$$P_{s_{11}} \leqslant \mu P_{l_{11}}, Q_{s_n} \leqslant \mu Q_{l_n}, Z_{s_n} \leqslant \mu Z_{l_n}, n = 1, 2, 3\ \forall s,\ l \in \mathcal{P} \tag{30b}$$

and P_p, Q_{p_k}, Z_{p_k} $(k = 1, 2, 3)$ are symmetric and positive-definite matrices were defined in (12) and (25). Here, χ_i and χ_j are obtained from Lemmas 1 and 2, and $\begin{pmatrix} P_{p_{11}} & P_{p_{12}} \\ P_{p_{21}} & P_{p_{22}} \end{pmatrix} = R_p^{-T} P_p L$.

Proof

Because the proof is lengthy, it is given in the Appendix instead. □

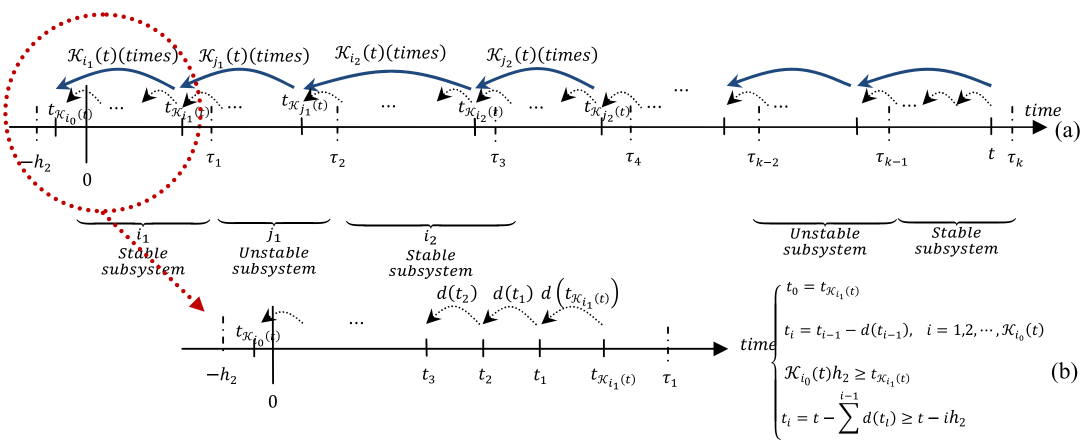

Figure 2. Analysis of algebraic equations based on iterative equations: (a) a typical switched singular system with stable and unstable subsystems and (b) iterative equations for a typical subsystem.

Theorem 1

For given $0 < h_1 \leqslant d(t) \leqslant h_2$, $\dot{d}(t) \leqslant d < 1$, and scalars $\alpha > 0$, $\beta > 0$, if there exists nonsingular matrices P_p, symmetric and positive-definite matrices Q_{p_k}, Z_{p_k} $(k = 1, 2, 3)$, and scalar $\mu \geqslant 1$, which satisfy the following inequalities:

$$P_{s_{11}} \leqslant \mu P_{l_{11}}, Q_{s_n} \leqslant \mu Q_{l_n}, Z_{s_n} \leqslant \mu Z_{l_n}, n = 1, 2, 3 \ \forall s,\ l \in \mathcal{P} \tag{31a}$$

and matrices Y_{p_k}, T_{p_k}, L_{p_k} $(k = 1, 2)$ such that inequalities (13a)-(13b) and (26a)-(26b) hold, the switching signal satisfies the switching condition $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geqslant \frac{\beta+\alpha^*}{\alpha-\alpha^*}$, and the average dwell-time

$$T_a \geqslant \left(ln\left(\sigma \varpi \mu_1{}^{\frac{1}{4}}\right) \Big/ \alpha_0 \right) \tag{31b}$$

then, the switched singular time-delay system (1) under Assumptions 1–3 is regular, impulse-free, and exponentially stable and the states convergence can be estimated as

$$\|x(t)\| \leqslant \mathcal{D} e^{-(\alpha^*-\alpha_0)(t-t_0)} \|\phi(t)\|_c \quad , \tag{31c}$$

where $\mathcal{D}$ is a sufficiently large scalar, $\alpha_0 \in (0, \alpha^*)$, ϖ is obtained by solving the following problem:

$$\begin{cases} \min \varpi \quad \text{s.t.} \\ \dfrac{\sigma\mu_1}{\gamma} + z \geqslant \varpi^z \varpi^{\frac{\sigma\mu_1}{\gamma}} \\ \forall z \geqslant 1 \end{cases} \tag{31d}$$

γ is a sufficiently large scalar, $\sigma = \max_{i,j\in\mathcal{P}} \left\{\chi_i, \chi_j, \mu^{\frac{1}{2}}\right\}$, $\mu_1 = e^{2h_2(\alpha+\beta)}$, and $\begin{pmatrix} P_{p_{11}} & P_{p_{12}} \\ P_{p_{21}} & P_{p_{22}} \end{pmatrix} = R_p^{-T} P_p L$.

Proof

From (12) and (25), the following piecewise Lyapunov functional candidate is chosen:

$$V_{\sigma(t)}(t) = \begin{cases} V_k(t) & \forall k \in \{\text{the set of all stable subsystems}\} \\ V_l(t) & \forall l \in \{\text{the set of all unstable subsystems}\} \end{cases}, \tag{32}$$

where $V_k(t)$ and $V_l(t)$ are defined in (12) and (25), respectively. From (12), (25), and (32) it is obtained on the switching point τ_k that $V_k\left(\tau_k^+\right) \leqslant \mu V_l\left(\tau_k^-\right)$ or $V_l\left(\tau_k^+\right) \leqslant \mu\mu_1 V_k\left(\tau_k^-\right)$. On the basis of (31a), Lemmas 1 and 2, we obtain by induction that

$$\begin{aligned} V_{\sigma(\tau_{k-1})}(t) &\leqslant e^{-2\alpha(t-\tau_{k-1})} V_{\sigma(\tau_{k-1})}(\tau_{k-1}) \\ &\leqslant \mu e^{-2\alpha(t-\tau_{k-1})} V_{\sigma(\tau_{k-2})}\left(\tau_{k-1}^-\right) \\ &\leqslant \mu e^{-2\alpha(t-\tau_{k-1})} e^{2\beta(\tau_{k-1}-\tau_{k-2})} V_{\sigma(\tau_{k-2})}(\tau_{k-2}) \\ &\vdots \\ &\leqslant \mu^{N_\sigma(t_0,t)} \mu_1^{N_f(t_0,t)} e^{-2\alpha T^-(t_0,t)+2\beta T^+(t_0,t)} V_{\sigma(t_0)}(t_0). \end{aligned} \tag{33}$$

From the proof of Lemmas 1 and 2, we have $\|\xi_1(t)\| \leqslant \lambda_{i_2}{}^{0.5} \lambda_{i_1}{}^{-0.5} e^{-\alpha(t-t_0)} \|\phi(t)\|_c$ (for stable subsystem) and $\|\xi_1(t)\| \leqslant \lambda_{j_2}{}^{0.5} \lambda_{j_1}{}^{-0.5} e^{\beta(t-t_0)} \|\phi(t)\|_c$ (for unstable subsystem), which yields

$$\|\xi_1(t)\| \leqslant \sqrt{\frac{b_{\sigma(t_0)}}{a_{\sigma(t_0)}}} \mu^{\frac{N_\sigma(t_0,t)}{2}} \mu_1^{\frac{N_f(t_0,t)}{2}} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)} \|\phi(t)\|_c, \tag{34}$$

where $b_{\sigma(t_0)} = \max_{i,j} \left\{\lambda_{i_2}, \lambda_{j_2}\right\}$, $a_{\sigma(t_0)} = \min_{i,j} \left\{\lambda_{i_1}, \lambda_{j_1}\right\}$. From Lemma 3, we have

$$\|\xi_2(t)\| \leqslant \gamma\sigma^{N_\sigma(t_0,t)} \left(\mu_1{}^{\frac{1}{2}}\right)^{N_f(t_0,t)} \left\{ \frac{\sigma\left(\mu_1{}^{\frac{1}{2}}\right)}{\gamma} + N_\sigma(t_0,t) \right\} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)} \|\phi(t)\|_c, \tag{35}$$

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To use the concept of the average dwell-time, we get

$$\frac{\sigma\left(\mu_1{}^{\frac{1}{2}}\right)}{\gamma} + N_\sigma(t_0,t) \leqslant \varpi^{\frac{\sigma\mu_1}{\gamma}+N_\sigma(t_0,t)} \tag{36}$$

Therefore, by solving the optimization problem (31d), we can get

$$\|\xi_2(t)\| \leqslant \mathcal{N}(\sigma\varpi)^{N_\sigma(t_0,t)}\left(\mu_1{}^{\frac{1}{2}}\right)^{N_f(t_0,t)} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c, \tag{37}$$

where $\mathcal{N}^2 = \gamma\varpi^{\left(\sigma\mu_1{}^{0.5}/\gamma\right)}$. Now, it can be concluded that

$$\begin{aligned}\|\xi(t)\| &\leqslant \|\xi_1(t)\| + \|\xi_2(t)\| \\ &\leqslant \mathcal{M}\mu^{\frac{N_\sigma(t_0,t)}{2}}\mu_1^{\frac{N_f(t_0,t)}{2}} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c \\ &\quad + \mathcal{N}(\sigma\varpi)^{N_\sigma(t_0,t)}\mu_1^{\frac{N_f(t_0,t)}{2}} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c \\ &\leqslant \{\mathcal{M}+\mathcal{N}\}(\sigma\varpi)^{N_\sigma(t_0,t)}\mu_1^{\frac{N_f(t_0,t)}{2}} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c,\end{aligned} \tag{38}$$

where $\mathcal{M} = \sqrt{b_{\sigma(t_0)}/a_{\sigma(t_0)}}$. From Definition 4 and (31b), it is obtained that $(\sigma\varpi)^{N_\sigma(t_0,t)}\mu_1^{\frac{N_f(t_0,t)}{2}} \leqslant e^{\alpha_0(t-t_0)}$. Also, from Definition 5, it holds that $-\alpha T^-(t_0,t)+\beta T^+(t_0,t) \leqslant -\alpha^*(t-t_0)$. Noting that $\xi(t) = L^{-1}x(t)$ and substituting into (38) yields $\|x(t)\| \leqslant \{\|L\|(\mathcal{M}+\mathcal{N})\}e^{-(\alpha^*-\alpha_0)(t-t_0)}\|\phi(t)\|_c$. This completes the proof. □

Remark 3

In Theorem 1 'sufficiently large' means that $\mathcal{D} \geqslant \|L\|(\mathcal{M}+\mathcal{N})$ in which $\mathcal{M}$ and $\mathcal{N}$ were defined in (37) and (38). Also, from (35), to obtain the average dwell-time, (35) must be rewritten in an exponential form. The terms $\sigma^{N_\sigma(t_0,t)}$, $\mu_1^{\frac{N_f(t_0,t)}{2}}$, and $e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}$ having exponential form may depend on the number of switching points in (t_0,t). Because the remaining term $\left(\text{i.,e., } {}^{\sigma\mu_1{}^{0.5}}\big/\gamma + N_\sigma(t_0,t)\right)$ depends on $N_\sigma(t_0,t)$, it should be rewritten in exponential form, too. As a solution, we can find a scalar ϖ such that

$$\frac{\sigma\left(\mu_1{}^{\frac{1}{2}}\right)}{\gamma} + N_\sigma(t_0,t) \leqslant \varpi^{\frac{\sigma\mu_1}{\gamma}+N_\sigma(t_0,t)} = \varpi^{\frac{\sigma\mu_1}{\gamma}}\varpi^{N_\sigma(t_0,t)}. \tag{39}$$

By solving (39), we can use the concept of average dwell-time. It is obvious that, for having the minimum average-dwell-time, we should obtain the minimum amount of ϖ by solving the optimization problem (31d). In this problem, obtaining γ is time-consuming (the amount of γ has been given in Appendix, formula (A.19)) and then the optimization problem (31d) seems to be computational. By trial and error, we can see that there are many numbers that satisfy the problem $\left({}^{\sigma\mu_1}\big/_{\gamma} + z \leqslant \varpi^z\varpi^{{}^{\sigma\mu_1}/\gamma}, \forall\, z \geqslant 1\right)$. By letting $X = \left({}^{\sigma\mu_1}\big/\gamma + z\right)$ in which $X > 1$, one can obtain that $\ln\varpi \geqslant \frac{\ln X}{X}$, which yields $\varpi \geqslant e^{\frac{1}{e}}$ where e is Napier's constant. For having a more accurate amount of T_a, we can consider two cases.

Case a: If $\max_{i,j\in\mathcal{P}}\{\chi_i,\chi_j\} \neq \mu^{\frac{1}{2}}$, from (A.21) (see Appendix), we have

$$\begin{aligned}\|\xi_2(t)\| \leqslant \mu_1^{\frac{N_f(t_0,t)}{2}}&\left\{\sigma^{N_\sigma(t_0,t)+1}\left(\mu_1{}^{\frac{1}{2}}\right) + \gamma\sigma^{N_\sigma(t_0,t)}\sum_{l=0}^{N_\sigma(t_0,t)}\left(\min_{i,j\in\mathcal{P}}\left\{\chi_i,\chi_j,\mu^{\frac{1}{2}}\right\}\Big/\sigma\right)^l\right\} \\ &\times e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c,\end{aligned}$$

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where $\sigma = \max_{i,j\in\mathcal{P}}\left\{\chi_i, \chi_j, \mu^{\frac{1}{2}}\right\}$. Then

$$\|\xi_2(t)\| \leqslant \mathcal{N}\sigma^{N_\sigma(t_0,t)}\mu_1^{\frac{N_f(t_0,t)}{2}}e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c$$

in which $\mathcal{N}^2 = \left\{1+\gamma\mathcal{L}_{i_0}\left(\min_{i,j\in\mathcal{P}}\left\{\chi_i,\chi_j,\mu^{\frac{1}{2}}\right\}\Big/\sigma\right)\right\}$ and $\mathcal{L}_{i_s}(z) = \sum_{l=1}^{\infty}\left(z^l\Big/l^s\right)$ is a geometric series with $\mathcal{L}_{i_0}\left(\min_{i,j\in\mathcal{P}}\left\{\chi_i,\chi_j,\mu^{\frac{1}{2}}\right\}\Big/\sigma\right) < \infty$. Similar to the proof of Theorem 1, one can obtain $T_{\mathrm{a}} \geqslant \left(\ln\left(\sigma\mu_1^{\frac{1}{4}}\right)\Big/\alpha_0\right)$.

Case b: If $\max_{i,j\in\mathcal{P}}\left\{\chi_i,\chi_j\right\} = \mu^{\frac{1}{2}}$, from (31a) and similar to (36)–(38), we can set, $\varpi = e^{\frac{1}{e}}$ and then it can be concluded that $T_{\mathrm{a}} \geqslant \left(\ln\left(\sigma\varpi\mu_1^{\frac{1}{2}}\right)\Big/\alpha_0\right)$.

Remark 4

It should be noted that at least one of the subsystems in (1) belongs to $\mathcal{S}$ and its differential states are upper-bounded by an exponentially decreasing function. This is a necessary condition to guarantee the exponential stability of the switched system and switching condition $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geqslant \frac{\beta+\alpha^*}{\alpha-\alpha^*}$.

Remark 5

It is noted that conditions (13a)–(13b) and (26a)–(26b) are nonstrict LMIs, which contain equality constraints. Considering that (13a) (resp., (26a)) and (13b) (resp., (26b)) can be combined into a single strict LMI. Let $P_p > 0$ and $S_p \in R^{n\times(n-r)}$ be any matrix with a full column rank and satisfies $E^T S_p = 0$ [1]. Changing P_p to $(P_p E + S_p \mathcal{Q}_p$ in (13a) (resp., (26a)) yields the strict LMI.

Theorem 2

For given $0 < h_1 \leqslant d(t) \leqslant h_2$, $\dot{d}(t) \leqslant d < 1$, and scalars $\alpha > 0$, $\beta > 0$, if there exist matrices $\mathcal{Q}_p$, full column rank matrices S_p, symmetric positive-definite matrices P_p, Q_{p_k}, Z_{p_k} $(k = 1,2,3)$, and scalar $\mu \geqslant 1$, which satisfy

$$P_{s_{11}} \leqslant \mu P_{l_{11}}, Q_{s_n} \leqslant \mu Q_{l_n}, Z_{s_n} \leqslant \mu Z_{l_n}, n = 1,2,3\ \forall s,\ l\in\mathcal{P} \tag{40a}$$

$$E^T S_p = 0 \tag{40b}$$

and matrices $Y_{p_k}, T_{p_k}, L_{p_k}$ $(k = 1,2)$ such that the following LMIs hold:

$$\begin{pmatrix} \Omega_p & Y_p & T_p & L_p & \bar{A}_p^T U_p \\ * & -k_{p_1}^{-1}Z_{p_1} & 0 & 0 & 0 \\ * & * & -k_{p_2}^{-1}Z_{p_2} & 0 & 0 \\ * & * & * & -k_{p_3}^{-1}Z_{p_3} & 0 \\ * & * & * & * & -U_j \end{pmatrix} < 0 \tag{40c}$$

while the switching signal satisfies the switching condition $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geqslant \frac{\beta+\alpha^*}{\alpha-\alpha^*}$, and average dwell-time (31b), then the switched singular time-delay system (1) under Assumptions 1–3 is regular, impulse-free, and exponentially stable and the state convergence can be estimated as

$$\|x(t)\| \leqslant \mathcal{D}e^{-(\alpha^*-\alpha_0)(t-t_0)}\|\phi(t)\|_c, \tag{40d}$$

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where $\mathcal{D} \geqslant \|L\|(\mathcal{M}+\mathcal{N})$, ϖ is a scalar that satisfies $\varpi \geqslant e^{\frac{1}{e}}$, and

$$\Omega_p = \begin{pmatrix} \Upsilon_{p_{11}} & \Upsilon_{p_{12}} & L_{p_1}E & -T_{p_1}E \\ * & \Upsilon_{p_{22}} & L_{p_2}E & -T_{p_2}E \\ * & * & -\mathscr{k}_{p_4}Q_{p_1} & 0 \\ * & * & * & -\mathscr{k}_{p_5}Q_{p_2} \end{pmatrix}, Y_p = \begin{pmatrix} Y_{p_1} \\ Y_{p_2} \\ 0 \\ 0 \end{pmatrix}, T_p = \begin{pmatrix} T_{p_1} \\ T_{p_2} \\ 0 \\ 0 \end{pmatrix}, L_p = \begin{pmatrix} L_{p_1} \\ L_{p_2} \\ 0 \\ 0 \end{pmatrix},$$

$$\bar{A}_p^T = \begin{pmatrix} A_p^T \\ A_{dp}^T \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \Upsilon_{p_{11}} &= A_p^T\left(P_pE + S_pQ_p\right) + \left(P_pE + S_pQ_p\right)^T A_p + \sum_{k=1}^{3} Q_{p_k} + 2\mathscr{k}_{p_6}E^T\left(P_pE + S_pQ_p\right) \\ &\quad + Y_{p_1}E + E^TY_{p_1}^T \\ \Upsilon_{p_{22}} &= -(1-d)\mathscr{k}_{p_7}Q_{p_3} - Y_{p_2}E - E^TY_{p_2}^T + T_{p_2}E + E^TT_{p_2}^T - L_{p_2}E - E^TL_{p_2}^T \\ \Upsilon_{p_{12}} &= \left(P_pE + S_pQ_p\right)^T A_{dp} + E^TY_{p_2}^T - Y_{p_1}E + T_{p_1}E - L_{p_1}E, U_p = h_2Z_{p_1} \\ &\quad + (h_2-h_1)(Z_{p_2}+Z_{p_3}) \end{aligned}$$

If $p \in \mathcal{S}$ then $\mathscr{k}_{p_k} = \mathscr{f}_k(k=1,2,3), \mathscr{k}_{p_4} = e^{-2\alpha h_1}, \mathscr{k}_{p_5} = e^{-2\alpha h_2}, \mathscr{k}_{p_6} = \alpha, \mathscr{k}_{p_7} = e^{-2\alpha h_2}$
If $p \in \mathcal{U}$ then $\mathscr{k}_{p_k} = \mathscr{g}_k(k=1,2,3), \mathscr{k}_{p_4} = e^{2\beta h_1}, \mathscr{k}_{p_5} = e^{2\beta h_2}, \mathscr{k}_{p_6} = -\beta, \mathscr{k}_{p_7} = e^{2\beta h_1}$.

Note that all notations used here are similar to Theorem 1.

Remark 6
We first choose a, α if LMI (40c) ($\forall p \in \mathcal{S}$) has no solution, a smaller α will be selected to yield a solution. Otherwise, a larger α will be chosen. Hence, a proper α could be obtained. Using a similar method, β is obtained by LMI (40c) ($\forall p \in \mathcal{U}$) and μ is determined by LMI (40a).

Remark 7
It should be noted that Equations (30a) and (38) have been obtained in general set-up of the switching sequence. It means that (30a) and (38) can be still applicable to a switching signal with $\kappa \neq 0.5$.

The results have been extended in the following Theorem 3.

Theorem 3
For given $0 < h_1 \leqslant d(t) \leqslant h_2$, $\dot{d}(t) \leqslant d < 1$, and scalars $\alpha > 0$, $\beta > 0$, if there exist matrices Q_p, full column rank matrices S_p, symmetric positive-definite matrices P_p, Q_{p_k}, Z_{p_k} $(k = 1,2,3)$, and scalar $\mu \geqslant 1$, which satisfy (40a), (40b), (40c), and the switching signal satisfies the switching condition $\frac{T^-(t_0,t)}{T^+(t_0,t)} \geqslant \frac{\beta+\alpha^*}{\alpha-\alpha^*}$, and the following average dwell-time:

$$T_a \geqslant \left(\ln\left(\sigma\varpi\mu_1^{k/4}\right) \Big/ \alpha_0 \right) \tag{41}$$

then, the switched singular time-delay system (1) under Assumptions 2 and 3 is regular, impulse-free, and exponentially stable and the state convergence can be estimated as (40d).

Note that all notations used here are similar to Theorem 2.

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Proof
In case of $\kappa \neq 0.5$, from (A.19) and (30a), and considering (38), we have

$$\|\xi(t)\| \leqslant \{\mathcal{M}+\mathcal{N}\}(\sigma\varpi)^{N_\sigma(t_0,t)}\left(\mu_1^{\frac{1}{2}}\right)^{N_f(t_0,t)} e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c.$$

It is quite clear that this inequality is independent on amount of κ and has a general form. Similar to the proof of Theorem 1, switching condition (41) is obtained easily. Therefore, it is omitted here. □

Remark 8
By supposing that all subsystems are stable (i.e., $T^+(t_0,t)=0$ and $N_f(t_0,t)=0$), from Theorem 1 we obtain $T_a \geqslant \left({\ln(\sigma\varpi)}/{\alpha_0}\right)$. Also, it is easy to find μ such that $\max_{i\in\mathcal{P}}\{\chi_i\} < \mu^{\frac{1}{2}}$, which yields $T_{\mathrm{a}} \geqslant \left({\ln\mu}/{\alpha}\right)$ and $0<\lambda\leqslant\alpha-\left({\ln\mu}/{T_a}\right)$.

Remark 9
A singular system model is formulated as a set of coupled differential and algebraic equations, which includes information on the static and dynamic constraints of a real plant. As mentioned before, the stability analysis based on the average dwell-time of switched singular time-delay systems are closely related to the exponential stability of algebraic equations and there are some features in switched singular systems with time-delay that are found in neither singular systems nor switched time-delay systems.

One of the pioneering researches in nonsingular switched systems is Ref. [16]. In [16], the stability problem for a class of switched systems with time-varying delay is considered with derivative of time-varying delay allowed to be bounded with an unknown constant. Also, in [18], stability and L_2-gain problems for switched systems with time-varying delays were studied. Compared with the other results on switched delay systems, the results in [18] give a design of a class of switching laws, while the existing works aimed at arbitrary switching present delay-dependent criteria, which contain the existing delay-independent ones as a special case. Also, because it is important to establish under what conditions the system is still exponentially stable subject to controller failure and the time varying-delay, Ref. [19] focused on this problem. We can also cite Ref. [20] as an important research in the field of controller temporary failure with stable and unstable subsystems with application to NCSs.

In contrast to the previous method presented in [16, 18–20], in which the stability analysis is considered for nonsingular systems, this paper investigates stability analysis of switched singular time delay systems as an important problem. Therefore, based on existing results for switched nonsingular time-delay systems [16, 18–20], there are many new topics in switched singular time-delay systems that can be considered as future works such as: L_2-gain problems, controller failure analysis, stability analysis of NCSs by using switched system approach, and filter design.

Remark 10
In [28] stability analysis of discrete-time switched singular time-delay systems was presented, but unfortunately, exponential stability of algebraic states has not been given correctly and the authors could not investigate a proper discussion on exponential stability. In addition, unstable subsystems have not been considered. Stability analysis of continuous-time switched time-delay singular systems consisting only stable subsystems has been considered in [29]. In this paper, dependency of algebraic states on the switching number in time duration (t_0,t) is not shown well. There are some other papers that have presented stability analysis of the switched singular system [30–34], but average dwell-time switching signal and time-delay have not been fully investigated. As an example, Refs. [32–34] have investigated some issues for switched singular systems without considering switching signal and time-delay or continuous case. However, to the best of our knowledge, delay range-dependent stability problem for switched singular time-delay systems consisting stable and unstable subsystems have not been fully investigated yet, which will be challenging because of the

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difficult extension of the existing stability results. Therefore, it is interesting and challenging to investigate the stability problem of switched singular time-delay systems and practical significances have motivated us to carry out the present study.

Remark 11
When the considered system is nonsingular time delay system, our results can be reduced as those in [45,48] as special cases. Of course, L_2 gain is also considered in stability analysis in [48]. However, sufficient conditions for the exponential estimation of Lyapunov-like Krasovskii functional for all subsystems are presented in Lemmas 1 and 2. Our results are delay dependent, which is less conservative than the delay-dependent results when time delay is small. Also, our results can be applicable to two cases: one is interval time varying and the other is constant time delay. Because some additional matrix variables have been added in stability analysis, this method can be also applicable to continuous and bounded time-varying delay without considering a bound on derivative of time delay. Thus, our results are less conservative.

Remark 12
Notice that our approach can be applicable to address the stability problem of singular and nonsingular NCSs [21,49] with the packets dropout. The main reason for this is that the system may become unstable when the packets are lost while the system is in stable state. Therefore, our proposed approach is of great significance.

4. NUMERICAL EXAMPLE

Consider the following switched singular time-delay system, composed of three subsystems described by:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_1 = \begin{pmatrix} -0.65 & 0.1 & 0 \\ -0.02 & -2 & 0 \\ -0.05 & 1 & 1 \end{pmatrix}, \; A_{d1} = \begin{pmatrix} -0.8 & -0.14 & 0.6 \\ -0.5 & 0.2 & 0.3 \\ -0.1 & 0.3 & 0.45 \end{pmatrix}$$
$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1.2 & -0.01 & 0 \\ 0.2 & 0.6 & 0 \\ -0.05 & 1 & 1 \end{pmatrix}, \; A_{d2} = \begin{pmatrix} -0.76 & 0.63 & 0.19 \\ 0.35 & 0.12 & 0.31 \\ -0.1 & 0.3 & 0.45 \end{pmatrix}$$
$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} -0.85 & 0.3 & 0 \\ -0.12 & -2.5 & 0 \\ -0.05 & 1 & 1 \end{pmatrix}, \; A_{d3} = \begin{pmatrix} -0.7 & -0.24 & 0.16 \\ 0.15 & 0.12 & 0.53 \\ -0.1 & 0.3 & 0.45 \end{pmatrix},$$

where $d(t) = 0.1 + 0.01\sin(t)$. It is found that the subsystems denoted by (E_1, A_1, A_{d1}) and (E_3, A_3, A_{d3}) are stable for $\alpha = 1.45$, whereas the subsystem represented by (E_2, A_2, A_{d2}) is unstable and satisfies Lemma 2 for $\beta = 0.10$. Hence, given $\mu = 10.65$, $\alpha_0 = 0.9$ and $\alpha^* = 1.1$, it follows from Theorem 3 that the proposed system is exponentially stable with average dwell-time $T_{\mathrm{a}} = 1.9855$ s and total activation time proportion $T^- \geqslant 3.4286T^+$. Furthermore, the corresponding matrices can be obtained by using MATLAB software. For the subsystem denoted by (E_1, A_1, A_{d1}), we have

$$P_1 = \begin{pmatrix} 14.068 & -4.1509 & 0.0000 \\ -4.1509 & 75.9926 & 0.0000 \\ 0.0000 & 0.0000 & 55.2770 \end{pmatrix} \qquad Q_{1_1} = \begin{pmatrix} 23.0862 & 1.5773 & -1.2017 \\ 1.5773 & 45.1385 & 17.4961 \\ -1.2017 & 17.4961 & 25.4791 \end{pmatrix}$$
$$Q_{1_2} = \begin{pmatrix} 21.9954 & 1.8737 & -1.1658 \\ 1.8737 & 44.8375 & 17.9425 \\ -1.1658 & 17.9425 & 25.8719 \end{pmatrix} \qquad Q_{1_3} = \begin{pmatrix} 15.0866 & -0.6938 & -2.7031 \\ -0.6938 & 83.0094 & 61.8195 \\ 2.7031 & 61.8195 & 63.7122 \end{pmatrix}$$
$$Z_{1_1} = \begin{pmatrix} 27.9545 & -0.8081 & -1.4581 \\ -0.8081 & 19.5453 & 0.3717 \\ -1.4581 & 0.3717 & 15.1069 \end{pmatrix} \qquad Z_{1_2} = \begin{pmatrix} 4.4794 & -0.7805 & -0.0228 \\ -0.7805 & 2.7992 & 0.0235 \\ -0.0228 & 0.0235 & 2.1477 \end{pmatrix}$$

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$$Z_{1_3} = \begin{pmatrix} 2.1922 & -0.0112 & -0.0032 \\ -0.0112 & 2.1981 & -0.0030 \\ -0.0032 & -0.0030 & 2.1645 \end{pmatrix} \quad Y_{1_1} = \begin{pmatrix} -63.3626 & -0.9572 & 9.8111 \\ -0.9572 & -26.6024 & 7.5090 \\ 9.8111 & 7.5090 & -0.0030 \end{pmatrix}$$

$$Y_{1_2} = \begin{pmatrix} -38.4788 & 13.4583 & -0.2125 \\ 13.4583 & -23.9758 & -15.1097 \\ -0.2125 & -15.1097 & -4.0368 \end{pmatrix} \quad L_{1_1} = \begin{pmatrix} 5.8585 & 0.0142 & 1.4639 \\ 0.0142 & 3.7985 & -4.5634 \\ 1.4639 & -4.5634 & -5.3354 \end{pmatrix}$$

$$L_{1_2} = \begin{pmatrix} -0.8065 & 0.9299 & 0.6449 \\ 0.9299 & -1.1092 & -3.4435 \\ 0.6449 & -3.4435 & -3.4703 \end{pmatrix} \quad T_{1_1} = \begin{pmatrix} 2.9310 & -0.6563 & 0.9321 \\ -0.6563 & 0.8079 & -3.1090 \\ 0.9321 & -3.1090 & -2.5098 \end{pmatrix}$$

$$T_{1_2} = \begin{pmatrix} -84.6966 & 22.0335 & -0.6067 \\ 22.0335 & -37.1781 & -1.3977 \\ -0.6067 & -1.3977 & -1.7241 \end{pmatrix} \quad \mathcal{Q}_1 = [3.4599 \ -44.4544 \ -84.3568] \quad S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For the subsystem denoted by (E_2, A_2, A_{d2}), we have

$$P_2 = \begin{pmatrix} 11.0888 & -5.5498 & 0.0000 \\ -5.5498 & 13.7314 & -0.0000 \\ 0.0000 & -0.0000 & 49.8539 \end{pmatrix} \quad Q_{2_1} = \begin{pmatrix} 20.5198 & -1.0536 & -1.2283 \\ -1.0536 & 31.2779 & 15.3414 \\ -1.2283 & 15.3414 & 22.6178 \end{pmatrix}$$

$$Q_{2_2} = \begin{pmatrix} 19.3353 & -1.1664 & -1.2152 \\ -1.1664 & 30.0326 & 15.292 \\ -1.2152 & 15.292 & 22.660 \end{pmatrix} \quad Q_{2_3} = \begin{pmatrix} 15.1613 & -2.9548 & -2.9984 \\ -2.9548 & -49.6536 & 37.3667 \\ -2.9984 & 37.3667 & 39.8940 \end{pmatrix}$$

$$Z_{2_1} = \begin{pmatrix} 22.2534 & -0.7362 & 0.0361 \\ -0.7362 & 22.8963 & -0.4390 \\ 0.0361 & -0.4390 & 11.2752 \end{pmatrix} \quad Z_{2_2} = \begin{pmatrix} 2.5385 & 0.1438 & 0.0220 \\ 0.1438 & 2.8224 & -0.1399 \\ 0.0220 & -0.1399 & 1.3442 \end{pmatrix}$$

$$Z_{2_3} = \begin{pmatrix} 1.3798 & 0.0049 & 0.0025 \\ 0.0049 & 1.3896 & 0.0000 \\ 0.0025 & 0.0000 & 1.3176 \end{pmatrix} \quad Y_{2_1} = \begin{pmatrix} -57.6136 & 4.0232 & -0.0172 \\ 4.0232 & -51.3221 & 7.1439 \\ -0.0172 & 7.1439 & 0.9127 \end{pmatrix}$$

$$Y_{2_2} = \begin{pmatrix} -31.6507 & -2.8829 & -1.4205 \\ -2.8829 & -43.3237 & -7.5717 \\ -1.4205 & -7.5717 & 0.2156 \end{pmatrix} \quad L_{2_1} = \begin{pmatrix} 4.8349 & 0.3257 & 0.3409 \\ 0.3257 & 2.8884 & -2.7785 \\ 0.3409 & -2.7785 & -2.6756 \end{pmatrix}$$

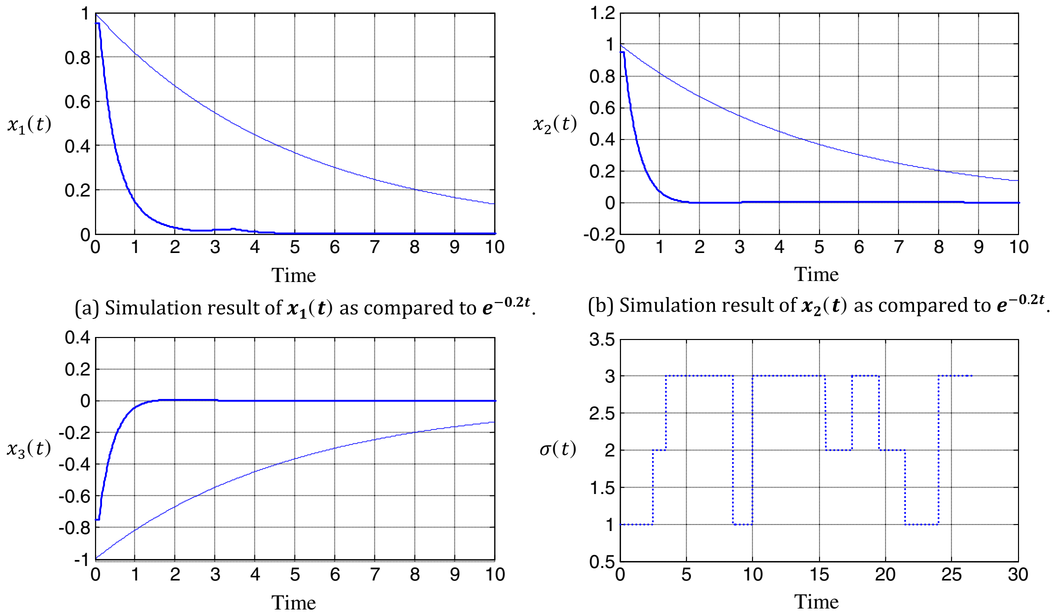
$$L_{2_2} = \begin{pmatrix} 4.0768 & 0.0461 & 0.1387 \\ 0.0461 & 3.4213 & -1.1973 \\ 0.1387 & -1.1973 & -1.8799 \end{pmatrix} \quad T_{2_1} = \begin{pmatrix} -5.2736 & -0.0057 & -0.1158 \\ -0.0057 & -3.3807 & 2.3550 \\ -0.1158 & 2.3550 & 4.6835 \end{pmatrix}$$

$$T_{2_2} = \begin{pmatrix} -54.4124 & -3.4471 & 0.6098 \\ -3.4471 & -74.2510 & 10.4007 \\ 0.6098 & 10.4007 & 3.5804 \end{pmatrix} \quad \mathcal{Q}_2 = [0.9570 \ \ -11.7554 \ \ -20.8289] \quad S_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

For the subsystem denoted by (E_3, A_3, A_{d3}), we have

$$P_3 = \begin{pmatrix} 3.8620 & -0.5314 & 0.0000 \\ -0.5314 & 15.6321 & -0.0000 \\ 0.0000 & -0.0000 & 11.5961 \end{pmatrix} \quad Q_{3_1} = \begin{pmatrix} 4.4326 & -0.2485 & -0.3524 \\ -0.2485 & 10.7722 & 3.3882 \\ -0.3524 & 3.3882 & 5.2124 \end{pmatrix}$$

$$Q_{3_2} = \begin{pmatrix} 4.2592 & -0.2521 & -0.3581 \\ -0.2521 & 10.8883 & 3.4929 \\ -0.3581 & 3.4929 & 5.2721 \end{pmatrix} \quad Q_{3_3} = \begin{pmatrix} 2.9749 & -0.6189 & -0.7039 \\ -0.6189 & 17.4201 & 12.6604 \\ -0.7039 & 12.6604 & 13.7783 \end{pmatrix}$$



(a) Simulation result of $\boldsymbol{x_1(t)}$ as compared to $\boldsymbol{e^{-0.2t}}$. (b) Simulation result of $\boldsymbol{x_2(t)}$ as compared to $\boldsymbol{e^{-0.2t}}$.

(c) Simulation result of $\boldsymbol{x_3(t)}$ as compared to $-\boldsymbol{e^{-0.2t}}$. (d) Switching signal with average dwell-time property.

Figure 3. Responses of the switched singular time-delay system (a) Response $\boldsymbol{x_1(t)}$ (b) Response $\boldsymbol{x_2(t)}$ (c) Response $\boldsymbol{x_3(t)}$ (d) Switching signal with average dwell-time property.

$$Z_{3_1} = \begin{pmatrix} 5.3710 & 0.1400 & 0.0319 \\ 0.1400 & 3.3226 & 0.1031 \\ 0.0319 & 0.1031 & 2.8331 \end{pmatrix} \qquad Z_{3_2} = \begin{pmatrix} 0.6295 & 0.0116 & 0.0083 \\ 0.0116 & 0.3751 & 0.0057 \\ 0.0083 & 0.0057 & 0.3639 \end{pmatrix}$$

$$Z_{3_3} = \begin{pmatrix} 0.3779 & -0.0000 & 0.0013 \\ -0.0000 & 0.3681 & -0.0015 \\ 0.0013 & -0.0015 & 0.3659 \end{pmatrix} \qquad Y_{3_1} = \begin{pmatrix} -11.1520 & -0.1164 & -0.2557 \\ -0.1164 & -2.8836 & 0.1287 \\ -0.2557 & 0.1287 & -1.7122 \end{pmatrix}$$

$$Y_{3_2} = \begin{pmatrix} -5.9592 & -0.5348 & -0.0823 \\ -0.5348 & -1.7918 & -5.3907 \\ -0.0823 & -5.3907 & 1.5117 \end{pmatrix} \qquad L_{3_1} = \begin{pmatrix} 1.0097 & 0.0528 & 0.1167 \\ 0.0528 & 0.5653 & -0.9013 \\ 0.1167 & -0.9013 & -0.4670 \end{pmatrix}$$

$$L_{3_2} = \begin{pmatrix} 0.2350 & 0.1060 & 0.0345 \\ 0.1060 & 0.0073 & -0.5909 \\ 0.0345 & -0.5909 & -0.1424 \end{pmatrix} \qquad T_{3_1} = \begin{pmatrix} 0.7382 & 0.1603 & 0.0951 \\ 0.1603 & 0.2320 & -0.4185 \\ 0.0951 & -0.4185 & -0.3978 \end{pmatrix}$$

$$T_{3_2} = \begin{pmatrix} -12.4237 & -0.3128 & -0.2226 \\ -0.3128 & -2.4206 & -0.9014 \\ -0.2226 & -0.9014 & -0.1229 \end{pmatrix} \qquad \mathcal{Q}_3 = [-0.7897 \quad 5.1525 \quad 16.7308] \qquad S_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

By choosing $\alpha_0 = 0.9$, $\alpha^* = 1.1$ and according to (40d), the state decay of the switched singular time-delay system can be estimated as $\|x(t)\| \leqslant \mathcal{D}e^{-0.2(t-t_0)}\|\phi\|_c$. The complete simulation results with compatible initial conditions are shown in Figures 3(a)–(c), respectively. In addition, the switching signal with average dwell-time property is shown in Figure 3(d).

5. CONCLUSION

In this paper, delay-dependent exponential stability conditions were developed for both stable and unstable singular time-delay subsystems. Then, stability analysis of a class of continuous-time switched singular time-delay systems consisting of a family of stable and unstable subsystems with time-varying delay was investigated. With the help of the average dwell-time approach incorporated

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with a switching signal condition, a class of switching signals is found under which the switched singular time-delay system was exponentially stable. An example was put forth to demonstrate the applicability and effectiveness of the proposed approach. The foregoing results have the potential to be useful for the study of switched singular systems.There are many new topics in switched singular systems that can be considered as future works such as: L_2-gain problems, controller failure analysis, stability analysis of NCSs by using switched system approach, and filter design.

APPENDIX

Proof of Lemma 3

Consider Figure 2(b), assume $t = t_{\mathcal{K}_{i_1}(t)}$. It can be seen that the value of $\xi_2(t)$ depends on the value of $\xi(t)$ at time $t = t_{i+1}$. By noting that

$$\begin{cases} t_0 = t_{\mathcal{K}_{i_1}(t)}, t_i = t_{i-1} - d(t_{i-1}), i = 1, 2, \cdots, \mathcal{K}_{i_0}(t) \\ \mathcal{K}_{i_0}(t)h_2 \geqslant t_{\mathcal{K}_{i_1}(t)}, t_i = t - \sum_{l=1}^{i-1} d(t_l) \geqslant t - ih_2 \end{cases} \tag{A.1}$$

$\xi_2\left(t_{\mathcal{K}_{i_1}(t)}\right)$ is obtained as follows:

$$\begin{aligned} \xi_2\left(t_{\mathcal{K}_{i_1}(t)}\right) &= -A_{i_{1_{21}}}\xi_1\left(t_{\mathcal{K}_{i_1}(t)}\right) - A_{di_{1_{21}}}\xi_1\left(t_{\mathcal{K}_{i_1}(t)} - d\left(t_{\mathcal{K}_{i_1}(t)}\right)\right) \\ &\quad - A_{di_{1_{22}}}\xi_2\left(t_{\mathcal{K}_{i_1}(t)} - d\left(t_{\mathcal{K}_{i_1}(t)}\right)\right). \end{aligned} \tag{A.2}$$

Now, by setting $t_1 = t_{\mathcal{K}_{i_1}(t)} - d\left(t_{\mathcal{K}_{i_1}(t)}\right) = t_0 - d(t_0)$, $\xi_2(t_1)$ can be computed from (A.1) as follows:

$$\begin{aligned} \xi_2(t_1) &= -A_{i_{1_{21}}}\xi_1(t_1) - A_{di_{1_{21}}}\xi_1(t_0 - d(t_0) - d(t_0 - d(t_0))) - A_{di_{1_{22}}}\xi_2(t_0 - d(t_0) \\ &\quad - d(t_0 - d(t_0))) = -A_{i_{1_{21}}}\xi_1(t_1) - A_{di_{1_{21}}}\xi_1(t_1 - d(t_1)) - A_{di_{1_{22}}}\xi_2(t_1 - d(t_1)), \end{aligned} \tag{A.3}$$

which yields

$$\xi_2(t_1) = -A_{i_{1_{21}}}\xi_1(t_1) - A_{di_{1_{21}}}\xi_1(t_2) - A_{di_{1_{22}}}\xi_2(t_2) \tag{A.4}$$

$\xi_2(t_2)$ can also be obtained as follows:

$$\xi_2(t_2) = -A_{i_{1_{21}}}\xi_1(t_2) - A_{di_{1_{21}}}\xi_1(t_3) - A_{di_{1_{22}}}\xi_2(t_3) \tag{A.5}$$

Substituting (A.5) in (A.4), we get

$$\begin{aligned} \xi_2(t_1) &= -A_{i_{1_{21}}}\xi_1(t_1) - A_{di_{1_{21}}}\xi_1(t_2) - A_{di_{1_{22}}}\left\{-A_{i_{1_{21}}}\xi_1(t_2) - A_{di_{1_{21}}}\xi_1(t_3) - A_{di_{1_{22}}}\xi_2(t_3)\right\} \\ &= \left(-A_{i_{1_{21}}}\right)\left\{\xi_1(t_1) + \left(-A_{di_{1_{22}}}\right)\xi_1(t_2)\right\} + \left(-A_{di_{1_{21}}}\right)\left\{\xi_1(t_2) + \left(-A_{di_{1_{22}}}\right)\xi_1(t_3)\right\} \\ &\quad + \left(-A_{di_{1_{22}}}\right)^2\xi_2(t_3) \end{aligned} \tag{A.6}$$

and so on

$$\begin{aligned} \xi_2\left(t_{\mathcal{K}_{i_1}(t)}\right) &= -A_{i_{1_{21}}}\xi_1\left(t_{\mathcal{K}_{i_1}(t)}\right) - A_{di_{1_{21}}}\xi_1(t_1) - A_{di_{1_{22}}}\left\{\left(-A_{i_{1_{21}}}\right)\left\{\xi_1(t_1) + \left(-A_{di_{1_{22}}}\right)\xi_1(t_2)\right\}\right. \\ &\quad \left. + \left(-A_{di_{1_{21}}}\right)\left\{\xi_1(t_2) + \left(-A_{di_{1_{22}}}\right)\xi_1(t_3)\right\} + \left(-A_{di_{1_{22}}}\right)^2\xi_2(t_3)\right\} \\ &= \left(-A_{i_{1_{21}}}\right)\left\{\xi_1\left(t_{\mathcal{K}_{i_1}(t)}\right) + \left(-A_{di_{1_{22}}}\right)\xi_1(t_1) + \left(-A_{di_{1_{22}}}\right)^2\xi_1(t_2)\right\} \\ &\quad + \left(-A_{di_{1_{21}}}\right)\left\{\xi_1(t_1) + \left(-A_{di_{1_{22}}}\right)\xi_1(t_2) + \left(-A_{di_{1_{22}}}\right)^2\xi_1(t_3)\right\} + \left(-A_{di_{1_{22}}}\right)^3\xi_2(t_3). \end{aligned} \tag{A.7}$$

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Note that $t_i = t - \sum_{l=1}^{i-1} d(t_l) \geqslant t - ih_2$, therefore there exists a positive finite integer $\mathcal{K}_{i_0}(t)$ such that (see Figure 2(b))

$$\xi_2\left(t_{\mathcal{K}_{i_1}(t)}\right) = \left(-A_{di_{1_{22}}}\right)^{\mathcal{K}_{i_1}(t)} \xi_2\left(t_{\mathcal{K}_{i_0}(t)}\right) - \sum_{\substack{s_{i_1}=0 \\ \left(t_0=t_{\mathcal{K}_{i_1}(t)}, t_{s_{i_1}}=t_{s_{i_1}-1}-d\left(t_{s_{i_1}-1}\right)\right)}}^{\mathcal{K}_{i_1}(t)-1} \left\{\left(-A_{di_{1_{22}}}\right)^{s_{i_1}}\right\} \times \left\{A_{di_{1_{21}}}\xi_1\left(t_{s_{i_1}+1}\right) + A_{i_{1_{21}}}\xi_1\left(t_{s_{i_1}}\right)\right\} \tag{A.8}$$

For abbreviation, we show $\left(t_0 = t_{\mathcal{K}_p(t)}, t_{s_p} = t_{s_p-1} - d\left(t_{s_p-1}\right)\right)$ by '*Cond.p*'. Now, assume $t = \left(t_{\mathcal{K}_{i_2}(t)}\right)$. Similar to (A.1)–(A.8), we can show that $\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)$ depends on a time $\tau \in [\tau_1, \tau_2)$ such as $t_{\mathcal{K}_{j_1}(t)}$ as follows:

$$\begin{aligned}\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right) = &\left(-A_{di_{2_{22}}}\right)^{\mathcal{K}_{i_2}(t)} \xi_2\left(t_{\mathcal{K}_{j_1}(t)}\right) \\ &- \sum_{\substack{s_{i_2}=0 \\ (Cond.i_2)}}^{\mathcal{K}_{i_2}(t)-1} \left\{\left(-A_{di_{2_{22}}}\right)^{s_{i_2}} \left\{A_{di_{2_{21}}}\xi_1\left(t_{s_{i_2}+1}\right) + A_{di_{2_{21}}}\xi_1\left(t_{s_{i_2}}\right)\right\}\right\}.\end{aligned} \tag{A.9}$$

Now, we have to find $\xi_2\left(t_{\mathcal{K}_{j_1}(t)}\right)$ depending on the past times. Similar to (A.1)–(A.8), we can get

$$\begin{aligned}\xi_2\left(t_{\mathcal{K}_{j_1}(t)}\right) = &\left(-A_{di_{1_{22}}}\right)^{\mathcal{K}_{i_1}(t)} \left(-A_{dj_{1_{22}}}\right)^{\mathcal{K}_{j_1}(t)} \xi_2\left(t_{\mathcal{K}_{j_1}(t)}\right) \\ &- \left(-A_{dj_{2_{22}}}\right)^{\mathcal{K}_{j_2}(t)} \sum_{\substack{s_{i_1}=0 \\ (Cond.i_1)}}^{\mathcal{K}_{i_1}(t)-1} \left\{\left(-A_{di_{1_{22}}}\right)^{s_{i_1}} \left\{A_{di_{1_{21}}}\xi_1\left(t_{s_{i_1}+1}\right) + A_{i_{1_{21}}}\xi_1\left(t_{s_{i_1}}\right)\right\}\right\} \\ &- \sum_{\substack{s_{j_1}=0 \\ (Cond.j_1)}}^{\mathcal{K}s_{j_1}(t)-1} \left\{\left(-A_{dj_{1_{22}}}\right)^{S_{j_1}} \left\{A_{dj_{1_{21}}}\xi_1\left(t_{s_{j_1}+1}\right) + A_{j_{1_{21}}}\xi_1\left(t_{s_{j_1}}\right)\right\}\right\}.\end{aligned} \tag{A.10}$$

By substituting (A.10) and (A.8) in (A.9), we get

$$\begin{aligned}\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right) = &\left(-A_{di_{2_{22}}}\right)^{\mathcal{K}_{i_2}(t)} \left(-A_{dj_{1_{22}}}\right)^{\mathcal{K}_{j_1}(t)} \left(-A_{di_{1_{22}}}\right)^{\mathcal{K}_{i_1}(t)} \xi_2\left(t_{\mathcal{K}_{i_0}(t)}\right) \\ &- \left(-A_{di_{2_{22}}}\right)^{\mathcal{K}_{i_2}(t)} \left(-A_{dj_{1_{22}}}\right)^{\mathcal{K}_{j_1}(t)} \sum_{\substack{s_{i_1}=0 \\ (Cond.i_1)}}^{\mathcal{K}_{i_1}(t)-1} \left\{\left(-A_{di_{1_{22}}}\right)^{s_{i_1}} \left\{A_{di_{1_{21}}}\xi_1\left(t_{s_{i_1}+1}\right)\right.\right. \\ &\left.\left. + A_{i_{1_{21}}}\xi_1\left(t_{s_{i_1}}\right)\right\}\right\} \\ &- \left(-A_{di_{2_{22}}}\right)^{\mathcal{K}_{i_2}(t)} \sum_{\substack{s_{j_1}=0 \\ (Cond.i_1)}}^{\mathcal{K}_{j_1}(t)-1} \left\{\left(-A_{dj_{1_{22}}}\right)^{s_{j_1}} \left\{A_{dj_{1_{21}}}\xi_1\left(t_{s_{j_1}+1}\right) + A_{j_{1_{21}}}\xi_1\left(t_{s_{j_1}}\right)\right\}\right\} \\ &- \sum_{\substack{s_{i_2}=0 \\ (Cond.i_2)}}^{\mathcal{K}_{i_2}(t)-1} \left\{\left(-A_{di_{2_{22}}}\right)^{s_{i_2}} \left\{A_{di_{2_{21}}}\xi_1\left(t_{s_{i_2}+1}\right) + A_{i_{2_{21}}}\xi_1\left(t_{s_{i_2}}\right)\right\}\right\}\end{aligned} \tag{A.11}$$

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According to (33) and (34), we get

$$
\begin{aligned}
&\text{If } t \in [t_0, \tau_1) \Rightarrow \|\xi_1(t)\| \leqslant \mathcal{M} e^{-\alpha(t-t_0)} \|\phi(t)\|_c \\
&\text{If } t \in [\tau_1, \tau_2) \Rightarrow \|\xi_1(t)\| \leqslant \mathcal{M} \left(\mu^{\frac{1}{2}}\right) \left(\mu_1{}^{\frac{1}{2}}\right) e^{\beta(t-\tau_1)} e^{-\alpha(\tau_1-t_0)} \|\phi(t)\|_c \\
&\text{If } t \in [\tau_2, \tau_3) \Rightarrow \|\xi_1(t)\| \leqslant \mathcal{M} \left(\mu^{\frac{1}{2}}\right)^2 \left(\mu_1{}^{\frac{1}{2}}\right) e^{-\alpha(t-\tau_2)} e^{\beta(\tau_2-\tau_1)} e^{-\alpha(\tau_1-t_0)} \|\phi(t)\|_c \\
&\vdots
\end{aligned}
\tag{A.12}
$$

Also, we have $\rho\left(e^{\alpha h_2} A_{di_{22}}\right) \leqslant 1$, $\|A_{di_{22}}{}^{\imath}\| \leqslant \chi_i e^{-\alpha h_2 \imath}$, $\imath = 1, 2, \cdots (i \in \mathcal{S})$ and $\rho\left(e^{-\beta h_1} A_{dj_{22}}\right) \leqslant 1$, $\|A_{dj_{22}}{}^{\jmath}\| \leqslant \chi_i e^{-\beta h_2 \jmath}$, $\jmath = 1, 2, \cdots (j \in \mathcal{U})$, in which $\mathcal{M} = \frac{\max_{i,j}\left(\lambda_{i_2}, \lambda_{j_2}\right)}{\min_{i,j}\left(\lambda_{i_1}, \lambda_{j_1}\right)}$ and $\left(\delta_{\mathcal{K}_{i_k}(t)}, \eta_{i_k}\right)$, $\left(\delta_{\mathcal{K}_{j_k}(t)}, \eta_{j_k}\right)$ are defined as follows:

$$
\begin{aligned}
&\delta_{\mathcal{K}_{i_k}(t)} = \begin{cases} 1 & \text{if } t = t_{\mathcal{K}_{i_k}(t)} \\ 0 & \text{if } t \neq t_{\mathcal{K}_{i_k}(t)} \end{cases}, \delta_{\mathcal{K}_{j_k}(t)} = \begin{cases} 1 & \textit{if } t = t_{\mathcal{K}_{j_k}(t)} \\ 0 & \textit{if } t \neq t_{\mathcal{K}_{j_k}(t)} \end{cases} \\
&\eta_{i_k} = \left(\delta_{\mathcal{K}_{i_k}(t)} \left\| \left(A_{dj_{k_{22}}}{}^{-1}\right) A_{j_{k_{21}}} \right\| + \left\| A_{i_{k_{21}}} \right\|\right), \\
&\eta_{j_k} = \left(\delta_{\mathcal{K}_{j_k}(t)} \left\| \left(A_{di_{k+1_{22}}}{}^{-1}\right) A_{i_{k+1_{21}}} \right\| + \left\| A_{j_{k_{21}}} \right\|\right).
\end{aligned}
\tag{A.13}
$$

By noting (A.11)–(A.14) $\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\|$ can be obtained as follows:

$$
\begin{aligned}
\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\| \leqslant{}& \left\|A_{di_{2_{22}}}{}^{\mathcal{K}_{i_2}(t)}\right\| \left\|A_{dj_{1_{22}}}{}^{\mathcal{K}_{j_1}(t)}\right\| \left\|A_{di_{1_{22}}}{}^{\mathcal{K}_{i_1}(t)}\right\| \left\|\xi_2\left(t_{\mathcal{K}_{i_0}(t)}\right)\right\| \\
&+ \left\|A_{di_{2_{22}}}{}^{\mathcal{K}_{i_2}(t)}\right\| \left\|A_{dj_{1_{22}}}{}^{\mathcal{K}_{j_1}(t)}\right\| \times \sum_{\substack{s_{i_1}=0 \\ (Cond.i_1)}}^{\mathcal{K}_{i_1}(t)-1} \left\{\left\|A_{di_{1_{22}}}{}^{s_{i_1}}\right\| \left\{\left\|A_{di_{1_{21}}}\right\| \left\|\xi_1\left(t_{s_{i_1}+1}\right)\right\|\right.\right. \\
&\left.\left.+ \left(\delta_{\mathcal{K}_{i_1}(t)} \left\|\left(A_{dj_{1_{22}}}{}^{-1}\right) A_{j_{1_{21}}}\right\| + \left\|A_{i_{1_{21}}}\right\|\right) \left\|\xi_1\left(t_{s_{i_1}}\right)\right\|\right\}\right\} \\
&+ \left\|A_{di_{2_{22}}}{}^{\mathcal{K}_{i_2}(t)}\right\| \sum_{\substack{s_{j_1}=0 \\ (Cond.j_1)}}^{\mathcal{K}_{j_1}(t)-1} \left\{\left\|A_{dj_{1_{22}}}{}^{s_{j_1}}\right\| \left\{\left\|A_{dj_{1_{21}}}\right\| \left\|\xi_1\left(t_{s_{j_1}+1}\right)\right\|\right.\right. \\
&\left.\left.+ \left(\delta_{\mathcal{K}_{j_1}(t)} \left\|\left(A_{di_{2_{22}}}{}^{-1}\right) A_{i_{2_{21}}}\right\| + \left\|A_{j_{1_{21}}}\right\|\right) \left\|\xi_1\left(t_{s_{j_1}}\right)\right\|\right\}\right\} \\
&+ \sum_{\substack{s_{i_2}=0 \\ (Cond.i_2)}}^{\mathcal{K}_{i_2}(t)-1} \left\{\left\|A_{di_{2_{22}}}{}^{s_{i_2}}\right\| \left\{\left\|A_{di_{2_{21}}}\right\| \left\|\xi_1\left(t_{s_{i_2}+1}\right)\right\| + \left\|A_{i_{2_{21}}}\right\| \left\|\xi_1\left(t_{s_{i_2}}\right)\right\|\right\}\right\}
\end{aligned}
\tag{A.14}
$$

therefore

$$\begin{aligned}
\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\| \leqslant & \left(\chi_{i_2} e^{-\alpha h_2 \mathcal{K}_{i_2}(t)}\right)\left(\chi_{j_1} e^{\beta h_1 \mathcal{K}_{j_1}(t)}\right)\left(\chi_{i_1} e^{-\alpha h_2 \mathcal{K}_{i_1}(t)}\right)\left\|\xi_2\left(t_{\mathcal{K}_{i_0}(t)}\right)\right\| \\
& +\left(\chi_{i_2} e^{-\alpha h_2 \mathcal{K}_{i_2}(t)}\right)\left(\chi_{j_1} e^{\beta h_1 \mathcal{K}_{j_1}(t)}\right) \\
& \sum_{\substack{s_{i_1}=0 \\ (Cond.i_1)}}^{\mathcal{K}_{i_1}(t)-1}\left\{\left\|A_{d i_{1_{22}}}{}^{s_{i_1}}\right\|\left\{\left\|A_{d i_{1_{21}}}\right\| \mathcal{M} e^{-\alpha\left(t_{s_{i_1}+1}-t_0\right)}\|\phi(t)\|_c\right.\right. \\
& \left.\left.\qquad +\eta_{i_1} \mathcal{M} e^{-\alpha\left(t_{s_{i_1}}-t_0\right)}\|\phi(t)\|_c\right\}\right\}+\left(\chi_{i_2} e^{-\alpha h_2 \mathcal{K}_{i_2}(t)}\right) \\
& \sum_{\substack{s_{j_1}=0 \\ (Cond.j_1)}}^{\mathcal{K}_{j_1}(t)-1}\left\{\left\|A_{d j_{1_{22}}}{}^{s_{j_1}}\right\| \mathcal{M}\left(\mu^{\frac{1}{2}}\right)\left(\mu_1{}^{\frac{1}{2}}\right) e^{-\alpha\left(\tau_1-t_0\right)}\|\phi(t)\|_c\right. \\
& \left.\qquad \times\left\{\left\|A_{d j_{1_{21}}}\right\| e^{\beta\left(t_{s_{j_1}+1}-\tau_1\right)}+\eta_{j_1} e^{\beta\left(t_{s_{j_1}}-\tau_1\right)}\right\}\right\} \\
& +\sum_{\substack{s_{i_2}=0 \\ (Cond.i_2)}}^{\mathcal{K}_{i_2}(t)-1}\left\{\left\|A_{d i_{2_{22}}}{}^{s_{i_2}}\right\| \mathcal{M}\left(\mu^{\frac{1}{2}}\right)^2\left(\mu_1{}^{\frac{1}{2}}\right) e^{\beta\left(\tau_2-\tau_1\right)} e^{-\alpha\left(\tau_1-t_0\right)}\|\phi(t)\|_c\right. \\
& \left.\qquad \times\left\{\left\|A_{d i_{2_{21}}}\right\| e^{-\alpha\left(t_{s_{i_2}+1}-\tau_2\right)}+\left\|A_{i_{2_{21}}}\right\| e^{-\alpha\left(t_{s_{i_2}}-\tau_2\right)}\right\}\right\}.
\end{aligned} \tag{A.15}$$

From (A.1), the following inequalities are given

$$\begin{cases} h_2 \mathcal{K}_{i_m}(t)+t_{\mathcal{K}_{j_{m-1}}(t)} \geqslant t_{\mathcal{K}_{i_m}(t)} \\ h_1 \mathcal{K}_{i_m}(t)+t_{\mathcal{K}_{i_m}(t)} \leqslant t_{\mathcal{K}_{j_m}(t)} \end{cases} \qquad t_{\mathcal{K}_{j_0}(t)}=t_{\mathcal{K}_{i_0}(t)}, m=1,2,\cdots \tag{A.16}$$

Thus,

$$\begin{aligned}
\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\| \leqslant & \chi_{i_2} \chi_{j_1} \chi_{i_1} e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-t_{\mathcal{K}_{j_1}(t)}\right)} e^{\beta\left(t_{\mathcal{K}_{j_1}(t)}-t_{\mathcal{K}_{i_1}(t)}\right)} e^{-\alpha\left(t_{\mathcal{K}_{i_1}(t)}-t_{\mathcal{K}_{i_0}(t)}\right)}\|\phi(t)\|_c \\
& +\chi_{i_2} \chi_{j_1} e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-t_{\mathcal{K}_{j_1}(t)}\right)} e^{\beta\left(t_{\mathcal{K}_{j_1}(t)}-t_{\mathcal{K}_{i_1}(t)}\right)} \\
& \sum_{\substack{s_{i_1}=0 \\ (Cond.i_1)}}^{\mathcal{K}_{i_1}(t)-1}\left\{\left\|A_{d i_{1_{22}}}{}^{s_{i_1}}\right\| \mathcal{M}\|\phi(t)\|_c\left\{\left\|A_{d i_{1_{21}}}\right\| e^{-\alpha\left(t_{\mathcal{K}_{i_1}(t)}-\left(s_{i_1}+1\right) h_2-t_0\right)}\right.\right. \\
& \left.\left.\qquad +\eta_{i_1} e^{-\alpha\left(t_{\mathcal{K}_{i_1}(t)}-s_{i_1} h_2-t_0\right)}\right\}\right\}+\chi_{i_2} e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-t_{\mathcal{K}_{j_1}(t)}\right)} \\
& \sum_{\substack{s_{j_1}=0 \\ (Cond.j_1)}}^{\mathcal{K}_{j_1}(t)-1}\left\{\left\|A_{d j_{1_{22}}}{}^{s_{j_1}}\right\| \mathcal{M} \mu^{\frac{1}{2}} \mu_1{}^{\frac{1}{2}} e^{-\alpha\left(\tau_1-t_0\right)}\|\phi(t)\|_c\right. \\
& \left.\qquad \times\left\{\left\|A_{d j_{1_{21}}}\right\| e^{\beta\left(t_{\mathcal{K}_{j_1}(t)}-\left(s_{j_1}+1\right) h_1-\tau_1\right)}+\eta_{j_1} e^{\beta\left(t_{\mathcal{K}_{j_1}(t)}-s_{j_1} h_1-\tau_1\right)}\right\}\right\}
\end{aligned}$$

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$$
\begin{aligned}
&+\sum_{\substack{s_{i_2}=0\\(Cond.i_2)}}^{\mathcal{K}_{i_2}(t)-1}\left\{\left\|A_{di_{2_{22}}}{}^{s_{i_2}}\right\|\mathcal{M}\left(\mu^{\frac{1}{2}}\right)^2\left(\mu_1{}^{\frac{1}{2}}\right)e^{\beta(\tau_2-\tau_1)}e^{-\alpha(\tau_1-t_0)}\|\phi(t)\|_c\right.\\
&\qquad\left.\times\left\{\left\|A_{di_{2_{21}}}\right\|e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-\left(s_{i_2}+1\right)h_2-\tau_2\right)}+\left\|A_{i_{2_{21}}}\right\|e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-s_{i_2}h_2-\tau_2\right)}\right\}\right\},
\end{aligned}
\tag{A.17}
$$

which leads to

$$
\begin{aligned}
\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\| \leqslant\; &\chi_{i_2}\chi_{j_1}\chi_{i_1}e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-\tau_2\right)}e^{\beta(\tau_2-\tau_1)}e^{(\alpha+\beta)h_2}e^{-\alpha(\tau_1-t_0)}\|\phi(t)\|_c\\
&+\chi_{i_2}\chi_{j_1}e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-\tau_2\right)}e^{\beta(\tau_2-\tau_1)}e^{(\alpha+\beta)h_2}e^{-\alpha(\tau_1-t_0)}\sum_{\substack{s_{i_1}=0\\(Cond.i_1)}}^{\mathcal{K}_{i_1}(t)-1}\\
&\times\left\{\left\|e^{\alpha h_2}A_{di_{1_{22}}}\right\|^{s_{i_1}}\mathcal{M}\|\phi(t)\|_c\left\{\left\|A_{di_{1_{21}}}\right\|e^{\alpha h_2}+\eta_{i_1}\right\}\right\}\\
&+\chi_{i_2}e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-\tau_2\right)}\left(\mu^{\frac{1}{2}}\right)\left(\mu_1{}^{\frac{1}{2}}\right)e^{\beta(\tau_2-\tau_1)}e^{-\alpha(\tau_1-t_0)}\sum_{\substack{s_{j_1}=0\\(Cond.j_1)}}^{\mathcal{K}_{j_1}(t)-1}\\
&\times\left\{\left\|e^{-\beta h_1}A_{dj_{1_{22}}}\right\|^{s_{j_1}}\mathcal{M}\|\phi(t)\|_c\left\{\left\|A_{dj_{1_{21}}}\right\|e^{-\beta h_1}+\eta_{j_1}\right\}\right\}\\
&+e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-\tau_2\right)}\left(\mu^{\frac{1}{2}}\right)^2\left(\mu_1{}^{\frac{1}{2}}\right)e^{\beta(\tau_2-\tau_1)}e^{-\alpha(\tau_1-t_0)}\\
&\times\sum_{\substack{s_{i_2}=0\\(Cond.i_2)}}^{\mathcal{K}_{i_2}(t)-1}\left\{\left\|e^{\alpha h_2}A_{di_{1_{22}}}\right\|^{s_{i_2}}\mathcal{M}\|\phi(t)\|_c\left\{\left\|A_{di_{2_{21}}}\right\|e^{\alpha h_2}+\left\|A_{i_{2_{21}}}\right\|\eta_{i_2}\right\}\right\}.
\end{aligned}
\tag{A.18}
$$

Therefore, it can be concluded that

$$
\begin{aligned}
\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\| \leqslant &\left\{\chi_{i_2}\chi_{j_1}\chi_{i_1}e^{(\alpha+\beta)h_2}+\chi_{i_2}\chi_{j_1}e^{(\alpha+\beta)h_2}\gamma+\chi_{i_2}\left(\mu^{\frac{1}{2}}\right)\left(\mu_1{}^{\frac{1}{2}}\right)\gamma\right.\\
&\left.+\left(\mu^{\frac{1}{2}}\right)^2\left(\mu_1{}^{\frac{1}{2}}\right)\gamma\right\}e^{-\alpha\left(t_{\mathcal{K}_{i_2}(t)}-\tau_2\right)}e^{\beta(\tau_2-\tau_1)}e^{-\alpha(\tau_1-t_0)}\|\phi(t)\|_c,
\end{aligned}
\tag{A.19}
$$

where $e^{(\alpha+\beta)h_2}=\left(\mu_1{}^{\frac{1}{2}}\right)$, $\sigma=\max_{i,j\in\mathcal{P}}\left\{\chi_i,\chi_j,\left(\mu^{\frac{1}{2}}\right)\right\}$, and

$$
\begin{aligned}
&\gamma=\max_{i,j\in\mathcal{P}}\left\{\sum_{s_i=0}^{\mathcal{K}_i(t)-1}\left\{\mathcal{M}\Delta_1\left\|e^{\alpha h_2}A_{di_{22}}\right\|^{s_i}\right\},\quad\sum_{s_j=0}^{\mathcal{K}_j(t)-1}\left\{\mathcal{M}\Delta_2\left\|e^{-\beta h_1}A_{dj_{22}}\right\|^{s_j}\right\}\right\}\\
&\Delta_1=\max_{i\in\mathcal{S}}\left\{\left\|A_{di_{21}}\right\|e^{\alpha h_2}+\eta_i\right\},\ \Delta_2=\max_{j\in\mathcal{U}}\left\{\left\|A_{dj_{21}}\right\|e^{-\beta h_1}+\eta_j\right\}
\end{aligned}
$$

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Therefore, (A.19) can be rewritten as follows:

$$\begin{aligned}\left\|\xi_2\left(t_{\mathcal{K}_{i_2}(t)}\right)\right\| &\leqslant \left\{\sigma^3\left(\mu_1{}^{\frac{1}{2}}\right)+\sigma^2\left(\mu_1{}^{\frac{1}{2}}\right)\gamma+\sigma\left(\mu^{\frac{1}{2}}\right)\left(\mu_1{}^{\frac{1}{2}}\right)\gamma+\left(\mu^{\frac{1}{2}}\right)^2\left(\mu_1{}^{\frac{1}{2}}\right)\gamma\right\}\\ &\quad\times e^{-\alpha T^-\left(t_0,t_{\mathcal{K}_{i_2}(t)}\right)+\beta T^+\left(t_0,t_{\mathcal{K}_{i_2}(t)}\right)}\|\phi(t)\|_c\\ &\leqslant \sigma^3\left(\mu_1{}^{\frac{1}{2}}\right)\{1+3\gamma\}e^{-\alpha T^-\left(t_0,t_{\mathcal{K}_{i_2}(t)}\right)+\beta T^+\left(t_0,t_{\mathcal{K}_{i_2}(t)}\right)}\|\phi(t)\|_c\ . \end{aligned}\tag{A.20}$$

Similar to (A.1)–(A.20), we can also find an upper bound for $\left\|\xi_2\left(t_{\mathcal{K}_{j_2}(t)}\right)\right\|$ as follows:

$$\begin{aligned}\left\|\xi_2\left(t_{\mathcal{K}_{j_2}(t)}\right)\right\| &\leqslant \left\{\chi_{j_2}\chi_{i_2}\chi_{j_1}\chi_{i_1}\left(\mu_1{}^{\frac{1}{2}}\right)^2+\chi_{j_2}\chi_{i_2}\chi_{j_1}\left(\mu_1{}^{\frac{1}{2}}\right)^2\gamma+\chi_{j_2}\chi_{i_2}\left(\mu^{\frac{1}{2}}\right)\left(\mu_1{}^{\frac{1}{2}}\right)^2\gamma\right.\\ &\quad\left.+\chi_{j_2}\left(\mu^{\frac{1}{2}}\right)^2\left(\mu_1{}^{\frac{1}{2}}\right)^2\gamma+\left(\mu^{\frac{1}{2}}\right)^3\left(\mu_1{}^{\frac{1}{2}}\right)^2\gamma\right\}\\ &\quad\times e^{-\alpha T^-\left(t_0,t_{\mathcal{K}_{j_2}(t)}\right)+\beta T^+\left(t_0,t_{\mathcal{K}_{j_2}(t)}\right)}\|\phi(t)\|_c\leqslant\left\{\sigma^4\left(\mu_1{}^{\frac{1}{2}}\right)^2+3\sigma^3\left(\mu_1{}^{\frac{1}{2}}\right)^2\gamma\right\}\\ &\quad\times e^{-\alpha T^-\left(t_0,t_{\mathcal{K}_{j_2}(t)}\right)+\beta T^+\left(t_0,t_{\mathcal{K}_{j_2}(t)}\right)}\|\phi(t)\|_c\ . \end{aligned}\tag{A.21}$$

By induction, we can deduce that if $t\in[\tau_k,\tau_{k+1})$ then

$$\begin{aligned}\|\xi_2(t)\| &\leqslant \left\{\sigma^{N_\sigma(t_0,t)+1}\left(\mu_1{}^{\frac{1}{2}}\right)^{N_f(t_0,t)+1}+N_\sigma(t_0,t)\sigma^{N_\sigma(t_0,t)}\left(\mu_1{}^{\frac{1}{2}}\right)^{N_f(t_0,t)}\gamma\right\}\\ &\quad\times e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c\\ &=\gamma\sigma^{N_\sigma(t_0,t)}\left(\mu_1{}^{\frac{1}{2}}\right)^{N_f(t_0,t)}\left\{\frac{\sigma\left(\mu_1{}^{\frac{1}{2}}\right)}{\gamma}+N_\sigma(t_0,t)\right\}e^{-\alpha T^-(t_0,t)+\beta T^+(t_0,t)}\|\phi(t)\|_c\ . \end{aligned}\tag{A.22}$$

This completes the proof. □

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