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Parametric control systems design with applications in missile control

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This paper considers parametric control of high-order descriptor linear systems via proportional plus derivative feedback. By employing general parametric solutions to a type of so-called high-order Sylvester matrix equations, complete parametric control approaches for high-order linear systems are presented. The proposed approaches give simple complete parametric expressions for the feedback gains and the closed-loop eigenvector matrices, and produce all the design degrees of freedom. Furthermore, important special cases are particularly treated. Based on the proposed parametric design approaches, a parametric method for the gain-scheduling controller design of a linear time-varying system is proposed and the design of a BTT missile autopilot is carried out. The simulation results show that the method is superior to the traditional one in sense of either global stability or system performance.

high-order linear systems, parametric approaches, Sylvester matrix equations, BTT missile

1 Introduction

Parametric control system design approaches possess the advantage of providing all the degrees of freedom in the system design and are very effective in dealing with multiple objective design. This paper is concerned with the parametric control of the following high-order dynamical linear system:

$$A_m x^{(m)} + A_{m-1} x^{(m-1)} + \cdots + A_1 \dot{x} + A_0 x = Bu, \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control vector, respectively; $A_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, 2, \dots, m$, and $B \in \mathbb{R}^{n\times r}$ are the system coefficient matrices, which satisfy the following assumptions.

Assumption A1. $\operatorname{rank}(A_m) = n_0, 0 \neq n_0 \leqslant n$.

Assumption A2. $\operatorname{rank}(B) = r$.

The above system (1) clearly contains the first-order system

$$E\dot{x} = Ax + Bu \quad (2)$$

and the second-order dynamical linear system

$$M\ddot{x} + D\dot{x} + Kx = Bu \quad (3)$$

as special cases. Let s denote the differential operator. Then system (1) can be represented in the following operator form:

$$A(s)X(s) = BU(s),$$

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where $X(s)$ and $U(s)$ are the Laplace transforms of $x(t)$ and $u(t)$, respectively, and

$$A(s) = s^m A_m + \cdots + s^2 A_2 + sA_1 + A_0. \tag{4}$$

Like the cases of first- and second-order systems, we call (1) an mth order descriptor linear system when $n_0 < n$. It is called an mth order normal linear system when $n_0 = n$. Furthermore, $\det A(s)$ is called the characteristic polynomial of the system, and the zeros of $\det A(s)$ are called the poles of the system. In this paper we are presenting a unified approach for the control of the above first-, second- and high-order systems. The presentation is given for the general mth order system (1). When $A(s)$ are specified as $A(s) = Ms^2 + Ds + K$ and $A(s) = -Es + A$, corresponding results of the second- and first-order systems can be obtained.

The first-order system (2) has fundamental importance in control systems theory. As a special case of (1), the second-order linear system (3) has found applications in many fields, such as vibration and structural analysis, spacecraft control and robotics control, and hence has attracted much attention[1−11]. However, concerning the control of second-order linear systems, most of the results are focused on stabilization[2,3] and pole assignment[4−7]. Regarding eigenstructure assignment in second-order linear systems, there have been only a few results[8−11]. Ref. [8] considers eigenstructure assignment in a special class of second-order linear systems using inverse eigenvalue methods. Ref. [9] proposes an algorithm for eigenstructure assignment in second-order linear systems, with the system coefficient matrices satisfying certain symmetric positivity condition. This algorithm is attractive because it utilizes only the original system data. Very recently, eigenstructure assignment in second-order linear systems using a proportional plus derivative feedback controller is considered in ref. [10]. Simple general and complete parametric expressions in direct closed forms for both the closed-loop eigenvector matrix and the feedback gains are established. As in ref. [9], the approach utilizes directly the original system data, and involves manipulations on only n-dimensional matrices. However, the approach has the disadvantage that it requires the controllability of the matrix pair (A_1, B), which is not satisfied in some applications.

Control design of the high-order descriptor linear system (1) can be realized by converting the system into the following corresponding extended first-order state-space descriptor system model:

$$E_e \dot{z} = A_e z + B_e u, \tag{5}$$

where

$$\begin{aligned} z^{\mathrm{T}} &= \begin{bmatrix} x^{\mathrm{T}} & \dot{x}^{\mathrm{T}} & \cdots & \left(x^{(m-1)}\right)^{\mathrm{T}} \end{bmatrix}, \\ E_e &= \mathrm{Blockdiag}\left(I_n, \ldots, I_n, A_m\right), \end{aligned} \tag{6}$$

$$\begin{aligned} A_e &= \begin{bmatrix} 0 & I_n & & \\ & & \ddots & \\ & & & I_n \\ -A_0 & -A_1 & \cdots & -A_{m-1} \end{bmatrix}, \\ B_e &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix}. \end{aligned} \tag{7}$$

We point out that such a conversion is not preferable in most practical analysis and design problems because of the following several reasons:

1. it gives additional computation load and makes the treatment more complicated;
2. it is often not numerically reliable, and the error given in this very first conversion step will be carried to the final analysis or design results and makes the final results inaccurate;
3. it destroys the physical meanings of the original system parameters.

In this paper we consider the problem of generalized eigenstructure assignment in the high-order descriptor linear system (1) via proportional plus derivative coordinate control. The intension is to provide simple direct methods which utilizes only the original system coefficients A_i, $i = 0, 1, 2, \ldots, m$, and B. Based on a complete parametric solution to a type of high-order generalized Sylvester matrix equation, a complete parametric solution is provided. Very simple, complete parametric expressions for both the closed-loop eigenvector matrices and the feedback gains are established. These expressions contain a matrix parameter which represents the design degrees of freedom, which can be further properly chosen to produce a closed-loop system with some desired system specifications. Furthermore, complete parametric approaches are proposed for three important special cases. The first one employs the right factorization of the system, happening to be a natural generalization of the parametric method proposed for eigenstructure assignment in first-order state-space descriptor linear systems[12,13]. With this method, besides the group of parameter vectors, the closed-loop eigenvalues may also be treated as part of the design degrees of freedom since they appear directly in the expressions of the eigenvector matrix and the feedback gains, and hence are not necessarily chosen *a priori*. The second one depends only on a series of singular value decompositions, hence is numerically very simple and reliable. Finally, in order to realize the BTT missile autopilot, we propose a parametric method for the gain-scheduling controller design of a linear time-varying system based on the parametric design approaches. The simulation results show that the method is superior to the traditional one in sense of either global stability or system performance.

The paper is composed of five sections. Section 2 gives the formulation of the generalized eigenstructure assignment problem for high-order descriptor linear systems. Solutions to the general parametric control problem are presented in sections 3. Three important special cases are treated in section 4. Section 5 proposes a parametric gain-scheduling controller design method for the BTT missile autopilot.

2 Problem statement

For the high-order dynamical system (1), by choosing the following control law:

$$u = K_0 x + K_1 \dot{x} + \cdots + K_{m-1} x^{(m-1)}, \quad K_i \in \mathbb{R}^{r\times n}, \tag{8}$$

we obtain the closed-loop system as follows:

$$A_m x^{(m)} + A_{m-1}^c x^{(m-1)} + \cdots + A_0^c x = 0, \quad A_i^c = A_i - BK_i, \quad i = 0, 1, \ldots, m-1, \tag{9}$$

which can be written in the first-order state-space form

$$E_{ec}\dot{z} = A_{ec} z, \tag{10}$$

where

$$E_{ec} = \mathrm{Blockdiag}(I_n, \ldots, I_n, A_m), \tag{11}$$

$$A_{ec} = \begin{bmatrix} 0 & I_n & & \\ & & \ddots & \\ & & & I_n \\ -A_0^c & -A_1^c & \cdots & -A_{m-1}^c \end{bmatrix}. \tag{12}$$

Following the pole assignment theory for first-order descriptor linear systems, under the R- and I-controllability of system (1), the number of finite eigenvalues that can be assigned to the closed-loop system (10)–(12) is

$$n_e = n(m-1) + n_0. \tag{13}$$

Therefore, the desired Jordan matrix J which can be assigned to the matrix pair (E_{ec}, A_{ec}) must be of order n_e. In this paper, we consider a generalized eigenstructure assignment problem: instead of assigning a Jordan matrix J to the matrix pair (E_{ec}, A_{ec}), we assign an arbitrary square matrix $F \in \mathbb{C}^{n_e\times n_e}$ to the matrix pair (E_{ec}, A_{ec}), that is, find a matrix V_{ec}^f satisfying

$$A_{ec}V_{ec}^f = E_{ec}V_{ec}^f F.$$

In this case, we say F is a core matrix of the matrix pair (E_{ec}, A_{ec}), while V_{ec}^f is the generalized finite eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) associated with the core matrix F.

Lemma 1. Let E_{ec} and A_{ec} be given by (11) and (12), and let $F \in \mathbb{C}^{n_e\times n_e}$ be arbitrarily given. Then

1. the core matrix of the matrix pair (E_{ec}, A_{ec}) is F if and only if there exists a matrix $V \in \mathbb{C}^{n\times n_e}$ satisfying

$$A_m V F^m + A_{m-1}^c V F^{m-1} + \cdots + A_0^c V = 0, \tag{14}$$

and in this case the corresponding generalized finite eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) is given by

$$V_{ec}^f = \begin{bmatrix} V^{\mathrm{T}} & (VF)^{\mathrm{T}} & \cdots & (VF^{m-1})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}};$$

2. the infinite eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) is given by

$$V_{ec}^{\infty} = \begin{bmatrix} 0 & V_{\infty}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

where $V_\infty \in \mathbb{R}^{n\times(n-n_0)}$ is determined by

$$A_m V_\infty = 0, \ \mathrm{rank}(V_\infty) = n - n_0; \tag{15}$$

3. the entire eigenvector matrix of the matrix pair (E_{ec}, A_{ec}) is

$$V_{ec} = \begin{bmatrix} V^{\mathrm{T}} & (VF)^{\mathrm{T}} & \cdots & (VF^{m-1})^{\mathrm{T}} \\ 0 & \cdots & 0 & V_{\infty}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

For the proof of Lemma 1, one can refer to Lemma 1 in ref. [14].

With the above understanding, the problem of generalized eigenstructure assignment in the high-order descriptor dynamical system (1) via the proportional plus derivative feedback law (8) can be stated as follows.

Problem GESA (Generalized eigenstructure assignment)**.** Given system (1) satisfying Assumptions A1 and A2, and an arbitrary matrix $F \in \mathbb{C}^{n_e \times n_e}$, find a general parametric form for the matrices $K_i \in \mathbb{R}^{r\times n}$, $i = 0, 1, 2, \ldots, m-1$, and $V \in \mathbb{C}^{n\times n_e}$ such that the matrix equation (14) and the condition

$$\det V_{ec} \neq 0 \tag{16}$$

are satisfied.

3 Solution to Problem GESA

Let

$$\begin{aligned} W &= K_{m-1}VF^{m-1} + \cdots + K_1VF + K_0V \\ &= \begin{bmatrix} K_0 & K_1 & \cdots & K_{m-1} \end{bmatrix} \begin{bmatrix} V \\ VF \\ \vdots \\ VF^{m-1} \end{bmatrix}. \end{aligned} \tag{17}$$

Then (14) becomes the following high-order Sylvester matrix equation:

$$A_mVF^m + \cdots + A_1VF + A_0V = BW. \tag{18}$$

Introduce the following auxiliary equation

$$W_{\infty} = K_{m-1}V_{\infty}. \tag{19}$$

Then combining (19) with (17), we have

$$[W \quad W_{\infty}] = \begin{bmatrix} K_0 & K_1 & \cdots & K_{m-1} \end{bmatrix} V_{ec}. \tag{20}$$

Therefore, when condition (16) holds, the feedback gain matrices can be obtained as

$$\begin{aligned} &\begin{bmatrix} K_0 & K_1 & \cdots & K_{m-1} \end{bmatrix} \\ &\quad = \begin{bmatrix} W & W_{\infty} \end{bmatrix} V_{ec}^{-1}. \end{aligned} \tag{21}$$

By specifying the general solution to the high-order Sylvester matrix equation (18) for this special case, we can immediately obtain the corresponding solution to the generalized eigenstructure assignment problem.

3.1 Solution to the Sylvester matrix equation

Definition 1. Let $F \in \mathbb{C}^{p\times p}$ be an arbitrary matrix. Then

1. a pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ is said to be F-right coprime if

$$\operatorname{rank}\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = r, \ \forall s \in \sigma(F); \tag{22}$$

2. a pair of polynomial matrices $H(s) \in \mathbb{R}^{m\times n}[s]$ and $L(s) \in \mathbb{R}^{m\times m}[s]$ is said to be F-left coprime if

$$\operatorname{rank}\begin{bmatrix} H(s) & L(s) \end{bmatrix} = m, \ \forall s \in \sigma(F).$$

According to the above definition, $A(s)$ and B are F-left coprime if

$$\operatorname{rank}\begin{bmatrix} A(s) & B \end{bmatrix} = n, \ \forall s \in \sigma(F). \tag{23}$$

Regarding the existence of solutions and the degree of freedom in the solution (V, W) to the high-order Sylvester matrix equation (18), we have the following result[15].

Lemma 2. Let $A(s) \in \mathbb{R}^{n\times n}[s]$ be given in (4), $F \in \mathbb{C}^{n_e\times n_e}$, $B \in \mathbb{R}^{n\times r}$ with $\operatorname{rank}(B) = r$, $A(s)$ and B be F-left coprime, and $N(s)$ and $D(s)$ be given by

$$\begin{cases} N(s) = \sum\limits_{i=0}^{\omega} N_i s^i, \ N_i \in \mathbb{R}^{n\times r}, \\ D(s) = \sum\limits_{i=0}^{\omega} D_i s^i, \ D_i \in \mathbb{R}^{r\times r}, \end{cases} \tag{24}$$

and satisfy

$$A(s)N(s) - BD(s) = 0. \tag{25}$$

1. The matrices $V \in \mathbb{C}^{n\times n_e}, W \in \mathbb{C}^{r\times n_e}$ given by

$$\begin{cases} V = N_0Z + N_1ZF + \cdots + N_{\omega}ZF^{\omega}, \\ W = D_0Z + D_1ZF + \cdots + D_{\omega}ZF^{\omega}, \end{cases} \tag{26}$$

satisfy the high-order Sylvester matrix equation (18) for arbitrary matrix $Z \in \mathbb{C}^{r\times n_e}$.

2. All the matrices $V \in \mathbb{C}^{n\times n_e}$, $W \in \mathbb{C}^{r\times n_e}$ satisfying the matrix equation (18) can be parameterized as (26) if and only if $N(s)$ and $D(s)$ are F-right coprime, i.e., condition (22) holds.

Remark 1. In the case in which $(A(s), B)$ is R-controllable, $A(s)$ and B are F-left coprime for arbitrary F. Therefore, in this case the F matrix can be arbitrarily chosen.

The solution (26) can be immediately written out as soon as the two polynomial matrices $N(s)$ and $D(s)$ determined by (25) are obtained. In many applications, we have $B = I_n$. In this case we can obviously take

$$N(s) = I,\ D(s) = A(s).$$

When the polynomial $\det A(s)$ is not identically zero, we can take

$$N(s) = \mathrm{adj}(A(s))B,\ \ D(s) = \det(A(s))I_r.$$

For more general case, general numerical algorithms[16,17], solving the right factorization $A^{-1}(s)B = N(s)D^{-1}(s)$ can also be readily applied to find the polynomial matrices $N(s)$ and $D(s)$.

3.2 The complete parametric solution

Theorem 1. Let $A(s) \in \mathbb{R}^{n\times n}[s]$ be given in (4), $F \in \mathbb{C}^{n_e\times n_e}$, $B \in \mathbb{R}^{n\times r}$ with $\mathrm{rank}(B) = r$, and $A(s)$ and B be F-left coprime. Further, let $N(s)$ and $D(s)$ be given by (24) and satisfy (25), and be F-coprime. Then

1. problem GESA has a solution if and only if there exists a matrix $Z \in \mathbb{C}^{r\times n_e}$ satisfying

$$\det V_{ca} \neq 0,$$

where

$$V_{ca} = \begin{bmatrix} N_0Z + \cdots + N_\omega ZF^\omega & 0 \\ (N_0Z + \cdots + N_\omega ZF^\omega)F & \vdots \\ \vdots & 0 \\ (N_0Z + \cdots + N_\omega ZF^\omega)F^{m-1} & V_\infty \end{bmatrix};$$

2. when the above condition is met, all the solutions to the Problem GESA are given by

$$V = N_0Z + N_1ZF + \cdots + N_\omega ZF^\omega,$$

and

$$\begin{bmatrix} K_0 & K_1 & \cdots & K_{m-1} \end{bmatrix} = (D_0Z + D_1ZF + \cdots + D_\omega ZF^\omega)V_{ca}^{-1},$$

where $Z \in \mathbb{C}^{r\times n_e}$ is an arbitrary parameter matrix making the matrix V_{ca} nonsingular, and $W_\infty \in \mathbb{R}^{r\times(n-n_0)}$ is an arbitrary parameter matrix.

Remark 2. The above theorem gives a complete parametric solution to the Problem GESA. The free parameter matrices Z and W_∞ represent the degrees of freedom in the eigenstructure assignment design, and can be sought to meet certain desired system performances. Furthermore, it should be noted that the core matrix F appears explicitly in the general solution, and hence does not need to be prescribed. It may be sought together with the parameter matrices Z and W_∞ to meet certain closed-loop specifications. This is in fact a very important feature of the approach.

Remark 3. When $\mathrm{rank}(A_m) = n_0 < n$, both the open-loop system (1) and the closed-loop system (9) are singular ones. For this case, closed-loop regularity has to be addressed. Note that the control of system (1) via the feedback control law (8) is equivalent to the state feedback control in the first-order descriptor linear system (5)–(7). Therefore, under the controllability condition of system (1), the first-order descriptor linear system (5)–(7) is regularizable via state feedback[18], and hence for “almost all” controllers in the form of (8), the corresponding closed-loop system (10)–(12) is regular.

4 Special cases

In this section, let us consider the important case of

$$F = \mathrm{diag}\left(s_1,\quad s_2,\quad \ldots,\quad s_{n_e}\right). \tag{27}$$

In accordance with the structure of the matrix F, we denote

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_q \end{bmatrix}, \tag{28}$$

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_q \end{bmatrix}. \tag{29}$$

By specifying the general solution to the high-order Sylvester matrix equations for this special case, we can immediately obtain the corresponding solution to the generalized eigenstructure assignment problem.

4.1 Case of undetermined s_i, i=1,2,…,n_e

With the above deduction and the results in section 3, we can now obtain the following corollary regarding the solution to the Problem GESA.

Corollary 1. Let system (1) be R-controllable, and let $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in$

$\mathbb{R}^{r\times r}[s]$ be a pair of polynomial matrices satisfying the right factorization (25).

1. Problem GESA has a solution if and only if there exist a group of parameters $f_i \in \mathbb{C}^r$, $i=1,2,\dots,n_e$, satisfying

Constraint C1: $f_i = \bar{f}_j$ if $s_i = \bar{s}_j$;

Constraint C2$_a$: $\det V_{ca} \neq 0$ with

$$V_{ca} = \begin{bmatrix} N(s_1)f_1 & \cdots & N(s_{n_e})f_{n_e} & 0 \\ s_1 N(s_1)f_1 & \cdots & s_{n_e}N(s_{n_e})f_{n_e} & \vdots \\ \vdots & \vdots & \vdots & 0 \\ s_1^{m-1}N(s_1)f_1 & \cdots & s_{n_e}^{m-1}N(s_{n_e})f_{n_e} & V_\infty \end{bmatrix}.$$

2. When the above condition is met, all the solutions to the Problem GESA are given by

$$V = \begin{bmatrix} N(s_1)f_1 & \cdots & N(s_{n_e})f_{n_e} \end{bmatrix}$$

and

$$\begin{bmatrix} K_0 & K_1 & \cdots & K_{m-1} \end{bmatrix} = \begin{bmatrix} D(s_1)f_1 & \cdots & D(s_{n_e})f_{n_e} & W_\infty \end{bmatrix} V_{ca}^{-1},$$

where $f_i \in \mathbb{C}^r, i=1,2,\dots,n_e$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_a$, $W_\infty \in \mathbb{R}^{r\times(n-n_0)}$ is an arbitrary parameter matrix.

4.2 Case of prescribed s_i, i=1,2,…, n_e

Let

$$\varPi_i = \begin{bmatrix} A(s_i) & -B \end{bmatrix},\ i=1,2,\dots,q. \tag{30}$$

The following algorithm produces two sets of constant matrices N_i and D_i, $i=1,2,\dots,n_e$, to be used in the representation of the solution to the matrix equation (18).

Algorithm ND (Solving N_i and D_i, $i=1,2,\dots,n_e$)

Step 1. Through applying SVD to the matrix $\varPi_i$, $i=1,2,\dots,q$, obtain two sets of unitary matrices $P_i \in \mathbb{R}^{n\times n}$ and $Q_i \in \mathbb{R}^{(n+r)\times(n+r)}$, $i=1,2,\dots,q$, satisfying

$$P_i \varPi_i Q_i = \begin{bmatrix} \operatorname{diag}(\sigma_1,\sigma_2,\dots,\sigma_{n_i}) & 0 \\ 0 & 0 \end{bmatrix}, \tag{31}$$

$$i=1,2,\dots,q,$$

where $\sigma_k > 0, k=1,2,\dots,n_i$, are the singular values of Π_i, and

$$n_i = \operatorname{rank}\begin{bmatrix} A(s_i) & B \end{bmatrix},\ i=1,2,\dots,q. \tag{32}$$

Step 2. Obtain the matrices $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i=1,2,\dots,q$, by partitioning the matrix Q_i as follows:

$$Q_i = \begin{bmatrix} * & N_i \\ * & D_i \end{bmatrix},\ i=1,2,\dots,q. \tag{33}$$

Corresponding to the above Algorithm and Theorem 1, we have the following result on Problem GESA.

Corollary 2. Let n_i, $i=1,2,\dots,n_e$, be given by (32), and $N_i \in \mathbb{R}^{r\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{n\times(n+r-n_i)}$, $i=1,2,\dots,n_e$, be given by Algorithm ND.

1. Problem GESA has a solution if and only if there exist a group of parameters $f_i \in \mathbb{C}^{n+r-n_i}$, $i=1,2,\dots,n_e$, satisfying Constraint C1 and

Constraint C2$_b$: $\det V_{cb} \neq 0$ with

$$V_{cb} = \begin{bmatrix} N_1 f_1 & \cdots & N_{n_e} f_{n_e} & 0 \\ s_1 N_1 f_1 & \cdots & s_{n_e} N_{n_e} f_{n_e} & \vdots \\ \vdots & \vdots & \vdots & 0 \\ s_1^{m-1} N_1 f_1 & \cdots & s_{n_e}^{m-1} N_{n_e} f_{n_e} & V_\infty \end{bmatrix}. \tag{34}$$

2. When the above condition is met, all the solutions to the Problem GESA are given by

$$V = \begin{bmatrix} N_1 f_1 & N_2 f_2 & \cdots & N_{n_e} f_{n_e} \end{bmatrix}, \tag{35}$$

and

$$\begin{bmatrix} K_0 & K_1 & \cdots & K_{m-1} \end{bmatrix} = \begin{bmatrix} D_1 f_1 & \cdots & D_{n_e} f_{n_e} & W_\infty \end{bmatrix} V_{cb}^{-1}, \tag{36}$$

where $f_i \in \mathbb{C}^{n+r-n_i}$, $i=1,2,\dots,n_e$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_b$, and $W_\infty \in \mathbb{R}^{r\times(n-n_0)}$ is an arbitrary parameter matrix.

4.3 Case of normal first-order linear system

When $m=1$, and $A_m = I$, system (1) becomes a normal first-order linear system:

$$\dot{x} = Ax + Bu. \tag{37}$$

In this case, with the results in section 3, we can now obtain the following corollary regarding the solution to the Problem GESA.

Corollary 3. Let system (37) be controllable, and let $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ be a pair of polynomial matrices satisfying the right factorization

$$(sI - A)N(s) - BD(s) = 0. \tag{38}$$

1. Problem GESA has a solution if and only if there exist a group of parameters $f_i \in \mathbb{C}^r$, $i = 1, 2, \ldots, n_e$, satisfying Constraint C1 and Constraint C2$_c$: $\det V \neq 0$ with

$$V = \begin{bmatrix} N(s_1)f_1 & N(s_2)f_2 & \cdots & N(s_n)f_n \end{bmatrix}.$$

2. When the above condition is met, all the solutions to the Problem GESA are given by

$$K = \begin{bmatrix} D(s_1)f_1 & \cdots & D(s_n)f_n \end{bmatrix} V^{-1}, \tag{39}$$

where $f_i \in \mathbb{C}^r, i = 1, 2, \ldots, n$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_c$.

In the rest of this section, let us make some remarks on the above results.

Remark 4. The above three corollaries give complete parametric solutions to the Problem GESA. The free parameter matrix W_∞ and the free parameter vectors $f_i, i = 1, 2, \ldots, n_e$, represent the degrees of freedom in the eigenstructure assignment design, and can be sought to meet certain desired system performances. It should be noted that Constraint C1 is not a restriction at all; it only gives a way of selecting these design parameter vectors.

Remark 5. It follows from the well-known pole assignment result that Problem GESA has a solution when system (1) is R- and I-controllable and the closed-loop eigenvalues $s_i, i = 1, 2, \ldots, n_e$, are restricted to be distinct. In this case, there exist parameter vectors $f_i, i = 1, 2, \ldots, n_e$, satisfying Constraints C2$_a$, C2$_b$ or C2$_c$. As a matter of fact, it can be reasoned that, in this case, "almost all" parameter vectors $f_i, i = 1, 2, \ldots, n_e$, satisfy Constraints C2$_a$, C2$_b$ or C2$_c$. Therefore, in such applications Constraints C2$_a$, C2$_b$ or C2$_c$ can often be neglected.

Remark 6. The solution given in Corollary 2 utilizes only a series of singular value decompositions, and hence is numerically very simple and reliable. As for the solution given in Corollary 1, it has the advantage that the closed-loop eigenvalues $s_i, i = 1, 2, \ldots, n_e$, can be set undetermined and used as a part of extra design degrees of freedom to be sought with $f_i, i = 1, 2, \ldots, n_e$, by certain optimization procedures. Furthermore, it happens that the solution given in Corollary 1 is a natural generalization of the parametric solution in refs. [12, 13, 19] proposed for eigenstructure assignment in first-order descriptor state-space systems.

5 Application on autopilot design of a BTT missile

In this section, we propose a new scheduling method for linear time-varying systems based on the parametric approaches of linear invariant systems and apply the approach to the design of a BTT missile autopilot.

5.1 Gain-scheduled controller design of time-varying systems

Consider a linear time-varying system represented by

$$\dot{x} = A(t)x + B(t)u,\ t \in [t_0, t_e]. \tag{40}$$

Here $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are respectively the state vector and the input vector, $A(t)$ and $B(t)$ are system coefficient matrices of appropriate dimensions which are continuously time-varying. The problem is how to find a controller for system (40) which guarantees both the global stability and desired performance of the resulting closed-loop system. In the following, state feedback controllers are designed for the local subsystems, and new justified technique for scheduling is presented to construct the global controller. Suppose $t_i,\ i = 1, 2, \ldots, r_0$, are the characteristic time points selected in the operating range. Then the local subsystems are

$$\Sigma_i : \dot{x} = A_i x + B_i u,\ i = 1, 2, \ldots, r_0, \tag{41}$$

where $A_i = A(t_i)$, $B_i = B(t_i)$. Linear system theory can be used to derive local controllers for each subsystem Σ_i. State feedback controllers in the form

$$u = K_i x,\ i = 1, 2, \ldots, r_0 \tag{42}$$

are employed to stabilize the subsystems. In order to construct the global controller, a certain interpolation method should be adopted to link K_i, $i = 1, 2, \ldots, r_0$, together. The problem here is how to construct the global state feedback controller $K(t)$ such that the resulted closed-loop system

$$\dot{x} = A_c(t)x,\ A_c(t) = A(t) + B(t)K(t),\ t > 0 \quad (43)$$

is stable and at the same time desired performance is guaranteed. $K(t)$ is a time-varying controller, which can be obtained by combining the local controllers using certain gain-scheduled method. If system (40) is controllable for all $t > 0$, according to the proposed parametric approaches we can design the local stabilizing controllers K_i, $i = 1, 2, \ldots, r_0$, as

$$K = \begin{bmatrix} D(s_1)f_1 & D(s_2)f_2 & \cdots & D(s_n)f_n \end{bmatrix} V^{-1},$$
$$V = \begin{bmatrix} N(s_1)f_1 & N(s_2)f_2 & \cdots & N(s_n)f_n \end{bmatrix},$$

where $f_j \in \mathbb{C}^r$, $j = 1, 2, \ldots, n$, are arbitrary vectors satisfing Constraints C1 and C2$_c$; $N(s)$ and $D(s)$ are right comprime polynomial matrices satisfying (38). Then we can derive the local controller gain matrices K_i, $i = 1, 2, \ldots, r_0$, by properly assigning the closed-loop eigenvalues s_j^i and the free vectors f_j^i, $j = 1, 2, \ldots, n$. The eigenvalues can be determined by desired system performance such as overshoot, and additional requirements can also be fulfilled by extra freedoms proposed by f_j^i. In order to maintain the desired performance of the closed-loop system between adjacent characteristic points, the eigenvalues of the closed-loop system for all fixed time points should stay in a small neighborhood of the desired area. The local controllers of two adjacent characteristic points t_i and t_{i+1} are denoted respectively by $K(s_j^i, f_j^i, j = 1, 2, \ldots, n)$ and $K(s_j^{i+1}, f_j^{i+1}, j = 1, 2, \ldots, n)$. Denote by $s_j(t)$, $j = 1, 2, \ldots, n$, the point-wise eigenvalues of the closed-loop system. Since the performance requirements on the whole operating range are usually similar, we can simply choose the same set of desired eigenvalues between characteristic points, that is,

$$\begin{aligned} s_j(t) &= s_j^i = s_j^{i+1} = s_j, \\ &j = 1, 2, \ldots, n,\ t \in [t_i; t_{i+1}), \end{aligned} \quad (44)$$

where $\mathrm{Re}(s_j) < 0$. Then the controller between characteristic points can be obtained by the following scheduling procedure:

$$\begin{aligned} K(t) &= K(s_j(t), f_j(t); j = 1, 2, \ldots, n) \\ &= \begin{bmatrix} D(s_1)f_1 & \cdots & D(s_n)f_n \end{bmatrix} \\ &\quad \begin{bmatrix} N(s_1)f_1 & \cdots & N(s_n)f_n \end{bmatrix}^{-1}, \end{aligned} \quad (45)$$

where

$$f_j(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} f_j^i + \frac{t - t_i}{t_{i+1} - t_i} f_j^{i+1}. \quad (46)$$

5.2 Stability criterion

According to the above design procedure the system matrix $A_c(t)$ of the resulted closed-loop system is point-wise Hurwitz for all t; thus, the matrix equation

$$A_c^{\mathrm{T}}(t_i)P_i + P_iA_c(t_i) = -I \quad (47)$$

has a unique positive definite solution P_i for $i \in \{0, 1, \ldots, r_0\}$. The following lemma gives the parametric expression for P_i based on the eigenstructure of the system.

Lemma 3[20]**.** The solution to eq. (47) has the following parametric representation:

$$P_i = V^{-\mathrm{T}}Q_iV_i^{-1},$$

where

$$V_i = \begin{bmatrix} v_{i1} & v_{i2} & \cdots & v_{in} \end{bmatrix},$$
$$Q_i = [q_{kj}^i],$$
$$q_{kj}^i = \frac{-(v_k^i)^{\mathrm{T}}v_j^i}{s_k^i + s_j^i},$$

s_j^i, v_j^i $j = 1, 2, \ldots, n$, are respectively the eigenvalues and corresponding eigenvectors of $A_c(t_i)$.

Without lose of generality, let $A_c(t)$ be bounded from above, i.e. $\exists\zeta > 0$, such that $\|A_c(t)\| < \zeta$, $\forall t > 0$. The Lyapunov function valid on the time interval $[t_i; t_{i+1})$ is defined as

$$L(t) = x^{\mathrm{T}}(t)P_ix(t),\ t \in [t_i; t_{i+1}).$$

Since P_i is positive definite, there exists an invertible matrix D_i such that $D_i^{\mathrm{T}}D_i = P_i$. Denote

$$\begin{aligned} \alpha_i(t) = \max\{\mathrm{eig}(D_i^{-\mathrm{T}}A_c^{\mathrm{T}}(t)D_i^{\mathrm{T}} \\ + D_iA_c(t)D_i^{-1})\}. \end{aligned} \quad (48)$$

Then we have the following result.

Theorem 2. The linear time-varying system (43) is uniformly asymptotically stable in the sense of Lyapunov if the following conditions hold:

$$\int_{t_i}^{t} \alpha_i(\tau)d\tau < 0,\ t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots \quad (49)$$

$$R_i = \frac{\exp\{\int_{t_i}^{t_{i+1}} \alpha_i(\tau)d\tau\}}{\min\{\mathrm{eig}(D_{i+1}^{-\mathrm{T}} P_i D_{i+1}^{-1})\}} < 1. \quad (50)$$

The proof of Theorem 2 can be seen in ref. [21]. Theorem 2 gives a sufficient criterion for asymptotical stability of linear time-varying systems. Accordingly, the global stability of system (43) can be easily guaranteed by applying additional restrictions (49) and (50), which come down to the requirements on the eigenvalues s_j and free vectors f_i^j, $j = 1, 2, \ldots, n$.

5.3 Autopilot design of a BTT missile

BTT steering has been developing rapidly and recently received much attention. Due to the strategic importance of long range and high power capability of BTT missile system, it is a critical research field to establish control theoretical backgrounds for the emerging BTT missile systems. Since in BTT steering, the kinematics and inertial coupling of the roll and yaw systems during combined pitch and roll maneuvers is significant, the conventional three-channel decoupling control is no longer applicable. The mathematic model of the pitch/yaw channel of a BTT missile is given by[22]

$$\begin{cases} \dot{x} = A(t)x + B(t)u, \\ y = C(t)x + D(t)u, \end{cases} \quad (51)$$

where $x = [\omega_z \quad \alpha \quad \omega_y \quad \beta]^{\mathrm{T}}$, $u = [\delta_z \quad \delta_y]^{\mathrm{T}}$, $y = [n_z \quad n_y]^{\mathrm{T}}$ are respectively the state vector, the input vector and the output vector; ω_z, ω_y, α and β are respectively the pitch rate, the yaw rate, the attack angle and the sideslip angle; δ_z, δ_y stand for actuator deflections and n_z, n_y are the overloads in the normal and side direction. The coefficient matrices of the model are set at

$$A(t) = \begin{bmatrix} a_1 & a_2 & \frac{J_x - J_y}{J_z}\omega_x & a_3\omega_x \\ 1 & a_4 & 0 & \frac{-\omega_x}{57.3} \\ \frac{J_z - J_x}{J_y}\omega_x & a_5\omega_x & a_6 & a_7 \\ 0 & \frac{\omega_x}{57.3} & 1 & a_8 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} b_1 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & b_4 \end{bmatrix}^{\mathrm{T}},$$

$$C(t) = \begin{bmatrix} 0 & 0 & 0 & c_1 \\ 0 & c_2 & 0 & 0 \end{bmatrix},$$

$$D(t) = \begin{bmatrix} 0 & d_1 \\ d_2 & 0 \end{bmatrix},$$

where J_x, J_y and J_z are the rotary inertia of the missile corresponding to the body coordinate. During the flight course of the missile, the parameters a_i, $i = 1, \ldots, 8$, b_i, $i = 1, \ldots, 4$, and c_i, d_i, $i = 1, 2$, vary continuously as the height and velocity of the missile change; therefore the system is time-varying. What is more, the changing patterns of these parameters are complicated and it is difficult to give analytical forms. We use the gain-scheduled method proposed above to design a global stabilizing controller for the system. In this paper, the flight at the time interval [4.4 s, 11.9 s] is considered, the case of the whole trajectory can be treated similarly. Choose two characteristic operating points t_1=4.4 s and t_2=11.9 s. Then the global stabilizing controller $K(t)$ can be built according to the above method. Further, in order to realize the tracking of given overload signals, we employ a feed-forward tracking controller based on the model-reference theory[23]

$$G(t) = H(t) - K(t)Z(t), \quad (52)$$

where the coefficient matrices $H(t)$ and $Z(t)$ can be calculated by

$$\begin{bmatrix} Z(t) \\ H(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}. \quad (53)$$

Thus the controller for the original time-varying system (51) is given in the following form:

$$u(t) = K(t)x + G(t)y_r,$$

where y_r is the overload signal to be followed. The closed-loop eigenvalues s_j and the free vectors f_j^1, f_j^2, $j = 1, 2, 3, 4$, give all the design freedoms, determined by solving the following optimization problem:

$$\begin{cases} \min & J \\ \text{s.t.} & \underline{c}_j < \mathrm{Re}(s_j) < \bar{c}_j < 0, \\ & \underline{d}_j < \mathrm{Re}(s_j) < \bar{d}_j < 0, \\ & \text{Constraint C1}, \\ & \int_{t_1}^{t} \alpha_i(\tau) d\tau < 0, \\ & R_1 < 1, \end{cases}$$

where $\underline{c}_j$, $\bar{c}_j$, $\underline{d}_j$ and $\bar{d}_j$ specify the desired areas of the closed-loop eigenvalues and $f_j^{1,2}$ are the free vectors for the local characteristic systems Σ_1 and Σ_2. For the BTT autopilot system, the output of the system should follow the given overload signal accurately and rapidly, and accordingly, a tracking performance index is chosen as

$$J = \sum_{k=1}^{N} \|y_r(k) - y(k)\|_2^2,$$

where $y(k)$ and $y_r(k)$ are respectively the output of the resulting system and the reference signal at every sampling time position and N is the number of sampling points. Suppose the initial state vector is $x_0 = [4 \quad 4 \quad -27 \quad 0]^{\mathrm{T}}$, and the rolling rate of the missile is $\omega_x = 400°/\mathrm{s}$. In order to test the traceability of the system, we give a time-varying overload signal as follows:

$$y_r(t) = \begin{bmatrix} 0 \\ n_{yc}(t) \end{bmatrix},$$

where n_{yc} is a square wave with period 6 s as shown in Figure 1. With the method proposed in this paper, we obtain

$$s_{1,2} = -11.3000 \pm 4.0000i,$$

$$s_{3,4} = -61.0000 \pm 5.0000i,$$

$$f_{1,2}^1 = \begin{bmatrix} -0.0113 & 0.0511 \end{bmatrix}^{\mathrm{T}},$$

$$f_{3,4}^1 = \begin{bmatrix} -0.4617 & -0.8855 \end{bmatrix}^{\mathrm{T}},$$

$$f_{1,2}^2 = \begin{bmatrix} 0.0200 & 0.9998 \end{bmatrix}^{\mathrm{T}},$$

$$f_{3,4}^1 = \begin{bmatrix} -0.0004 & 0.0007 \end{bmatrix}^{\mathrm{T}}.$$

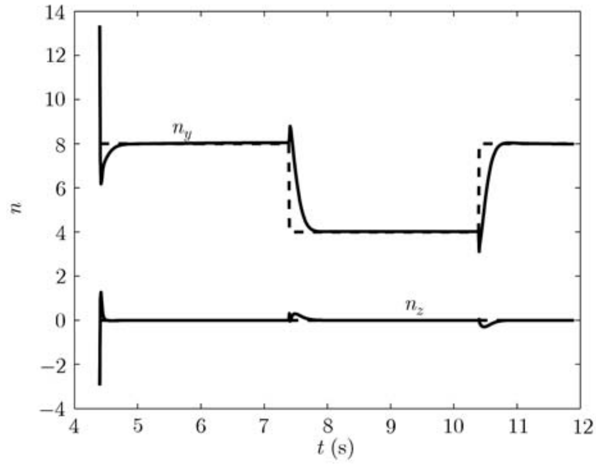

Figure 1 Output responses of the system using eigenstructure interpolation.

Then, the global stabilizing gain-scheduled controller $K(t)$ can be established using (45) and (46). The feed-forward gain matrix $G(t)$ can be calculated by (52) and (53). The simulation results are given in Figure 1. The continuous line and the broken line are, respectively, the output response of the closed-loop system and the given overload signals. For comparison, we also give the output response curves of the system designed using the traditional gain-scheduled method, according to which, the global controller $K(t)$ is constructed by linear interpolation between the feedback gain matrices of local subsystems, that is,

$$K(t) = \frac{t_2 - t}{t_2 - t_1} K_1 + \frac{t - t_1}{t_2 - t_1} K_2, \ t \in (t_1, t_2).$$

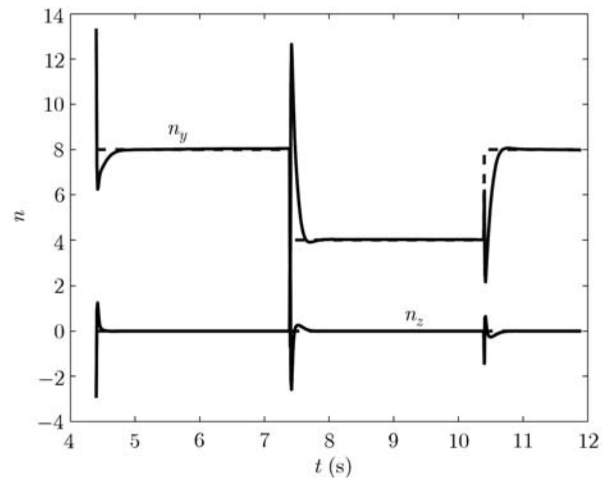

Figure 2 Output responses of the system using gain matrix interpolation.

Obviously, the system outputs in the two cases can both track the given overload signals; however, the system designed using the gain-scheduled method proposed in this paper has much better transient behavior than the traditional one. Since, according to our method, the eigenvalues and eigenvectors of the system are, in some sense, slow-varying, the performance between adjacent characteristic points is guaranteed, as can be seen in Figure 1, while for the one designed using the traditional method, as shown in Figure 2, the performance declines between characteristic points. The superiority of the proposed gain-scheduled method can also be validated from the stability point of view. Stability of the resulted time-varying system can be verified using the sufficient criterion in Theorem 2. For the system designed using the proposed method, we have $R_1 = 2.0328\mathrm{e}-022 < 1$, while, for the traditional one, we have $R_1' = 1.1323\mathrm{e}+003 > 1$. Accordingly, the controller designed in this paper globally asymptotically stabilizes the system. However, stability cannot be guaranteed for the traditional one.

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