Maximum Allowable Delay Bound Estimation Using Lambert W Function

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***Abstract*— The widespread of communication networks make them very promising to play a great role in future control systems. The communication networks will be present in the feedback control system which makes it a kind of time delay system. Closing the feedback system through a communication network introduces many challenges for the controller designers. Communication networks induce inherent time delay and some of the data may be lost which can destabilize the control system or result in poor system performance. It is important to identify the maximum time delay that the control system can withstand. In this paper, we report the application of the Lambert W function for calculating the maximum allowable delay bound in linear time delay control systems. The results of the calculation are compared with the most widely used Linear Matrix Inequalities based method.**

***Keywords*— delay; LMI; Lambert W function; MADB; maximum allowable delay bound, networked control system; time delay**

I. INTRODUCTION

The availability of high processing computers and the very fast communication networks have opened the doors for the networked control system (NCS) and the Internet of Things (IoT). A typical NCS is shown in Fig. 1. The communications networks are the backbone of such systems. Comparing with the direct conventional wiring, the communications networks are cheap, flexible and reliable. These advantages are coupled with some drawbacks. The time delay and data loss are unavoidable in such systems which makes them typical time delay systems.

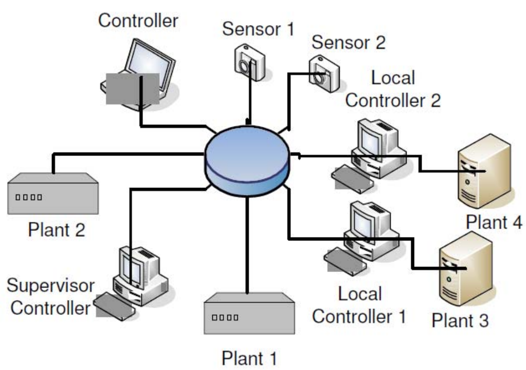

Fig. 1. A typical networked control system

Time delay control systems can be classified into delay dependent and delay-independent systems. In the delay independent systems, the stability of the system is guaranteed for every delay. In the delay dependent system, the system becomes unstable if the network induced delay is larger than the system maximum allowable delay bound (MADB) [1-3]. It is essential to identify the maximum time delay allowed for maintaining the system stability which, in turn, is also associated with the process of controller design. Most of the real-time control systems are delay-dependent control systems.

Time delay systems have gathered a lot of attention in the previous two decades especially with the emergence of the communication networks. The time delay issue has been under research since the 1874. The main focus was on the explicit analytical solution to the problem. With the emergence of the computers, the Linear Matrix Inequality (LMI) formulation of the problem has attracted many researchers. In general, the methods for analyzing the stability of time delay systems can be classified into two categories: Frequency-Domain Methods and Time-Domain Methods. The control theory is rich of the different methods and the different approaches for the stability analysis of time delay systems, the reader can refer to [4] and the references therein.

Lambert *W* function has been recently used for analyzing the stability of time delay systems [5]. Lambert *W* function is one of the frequency domain stability analysis methods. The Lambert *W* function is used to transform the transcendental equation to complex valued equation. Through Lambert *W* function the solution of the time delay system is accurate. Lambert *W* function was introduced by Lambert to solve the trinomial equation $x = q + x^m$ and later developed by Euler. In [6,7] the Lambert *W* function is used to analyze the stability of scalar and multidimensional time delay systems. The authors in [8] use a method based on Lambert *W* function and the eigenvalue assignment for a DC motor controlled with PI controller. In [9] the conditions of the controllability and the observability of time delay systems have been derived using Lambert *W* function. There are some shortcomings prevent the widespread use of Lambert *W* function in control system research. The authors in [10] pointed out the limitations and the deficiencies in the previous approaches and proved that the rightmost root of the original transcendental characteristic equation is not necessarily a principal branch Lambert *W*

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function solution. The applications of the Lambert *W* for controller design purpose were reported in [11], where the PI controller gains for fist-order system are selected to achieve a predefined system performance through eigenvalue assignment. The Lambert *W* function has been applied for the analysis of multidimensional time delay system as reported in [12]. A transformation is used to transform the multidimensional to common canonical form. The two deficiencies in this approach is that it cannot be applied to any time delay system and the authors rely only on the third order time delay system to verify their results.

In this paper, the Lambert *W* function is used to analyze the stability of the time delay systems with focus on calculating the MADB. The results of the MADB are compared with the most widely used LMI method. In the next section, the mathematical model and the analysis with the Lambert *W* function is briefly described. Then the Lambert *W* function and the LMI method are compared through a number of numerical examples.

II. Mathematical Modeling

The general linear time delay system usually has the following form:

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \tag{1}$$

where A and A_d are constant matrices with appropriate dimensions, τ is the time delay. Taking Laplace transform of (1):

$$(sI - A - A_d e^{-s\tau})x(s) = 0 \tag{2}$$

The necessary and sufficient stability conditions for (1) to be asymptotically stable $f(\lambda) = \det(sI - A - A_d e^{-s\tau})$ for a given time delay. In other words, the system is stable if all the roots lie on the left-hand side or the real parts of the roots are all negative. The main dilemma is that (1) is a transcendental equation and the solution is complex. Furthermore, for uncertain time-delay systems, the solution process becomes very sophisticated. Lambert *W* function was introduced in 1758 by Lambert [5]. Later in 1779, Euler introduced modified version of Lambert transformation [5]. Back to the 1700s, Lambert function was used to solve the trinomial equation. The solution to the time delay is complete in analytical form. The method has been applied successfully for a scalar case and is extended to higher order systems [5]. The main problem in the method is that it is not generalized. For the existence and the uniqueness of the solution. The Lambert *W* function is defined to be any function, $W(s)$, that satisfies:

$$W(s)e^{W(s)} = s \tag{3}$$

where $W(s)$ is known as Lambert *W* function [5]. Lambert *W* function is a complex function with complex arguments. The complex valued Lambert *W* function has infinite branches, $k = 0, \pm 1, \pm 2, .., \pm\infty$. For a single-input-single-output s is a scalar and for the multi-input-multi-output systems it is a matrix. The advantages of the function consist of the fact that it enables us to express the characteristic roots of the systems explicitly. In the next section two cases are considered; the scalar case and the multidimensional case.

A. Lambert W function for Scalar Case

Consider the first-order time delay system [6]:

$$\begin{aligned} &\dot{x}(t) = ax(t) + a_d x(t-\tau) + bu(t), t > 0 \\ &x(0) = x_0, \quad t = 0 \\ &x(t) = g(t),\ t \in [-\tau, 0) \end{aligned} \tag{4}$$

Two initial conditions need to be specified for this time delay system: a preshape function, $g(t)$, for $-\tau \le t < 0$ and the initial point, x_0, at the time, $t = 0$. This permits a discontinuity at $t = 0$, when $x_0 \ne g(t=0)$. The quantity, τ, denotes the time-delay. The Lambert *W* function is applied to solve the transcendental characteristic equation of (4), which can be written as:

$$(s-a)e^{s\tau} = a_d \tag{5}$$

Multiplying both sides of (5) by $\tau e^{-a\tau}$ yields:

$$\tau(s-a)e^{\tau(s-a)} = a_d \tau e^{-a\tau} \tag{6}$$

Based on the definition of the Lambert *W* function in (3) it is clear that:

$$W(a_d \tau e^{-a\tau})e^{W(a_d \tau e^{-a\tau})} = a_d \tau e^{-a\tau} \tag{7}$$

Comparing (6) and (7):

$$\tau(s-a) = W(a_d \tau e^{-a\tau}) \tag{8}$$

Thus, the solution of the characteristic equation in (5) can be written in terms of the Lambert *W* function as:

$$s = \frac{1}{\tau}W(a_d \tau e^{-a\tau}) + a \tag{9}$$

The infinite spectrum of the scalar delay differential equation in (4) is, thus, obtained using the infinite branches of the Lambert *W* function, and is given explicitly in terms of parameters a, a_d and τ of the system. The roots of the characteristic equation (4), for $k = 0, \pm 1, \pm 2, \pm 3, .., \pm\infty$, are:

$$s_k = \frac{1}{\tau}W_k(a_d \tau e^{-a\tau}) + a \tag{10}$$

where k indicates a branch of the Lambert *W* function. The solution of (4) is then:

$$x(t) = \sum_{k=-\infty}^{\infty} e^{S_k t} C_k^I \tag{11}$$

where;

$$C_k^I = \frac{x_0 + a_d e^{-S_k \tau} G(S_k)}{1 + a_d \tau e^{-S_k \tau}} \tag{12}$$

For the free response $G(S_k)=0$, so:

$$C_k^I = \frac{x_0}{1 + a_d \tau e^{-S_k \tau}} = \sigma_{ck} + j\omega_{ck} \tag{13}$$

Then (11) can be written as:

$$x(t) = \mathrm{Re}[\sum_{k=-\infty}^{\infty} e^{(\sigma_{sk} + \omega_{sk})t}(\sigma_{ck} + j\omega_{ck})] \tag{14}$$

$$x(t) = \mathrm{Re}[\sum_{k=\infty}^{\infty} e^{\sigma_{sk} t}(\cos\omega_{sk} t + j\sin\omega_{sk} t)(\sigma_{ck} + j\omega_{ck})] \tag{15}$$

$$x(t) = \sum_{k=-\infty}^{\infty} e^{\sigma_{sk} t}(\sigma_{ck}\cos\omega_{sk} t - \omega_{ck}\sin\omega_{sk} t) \tag{16}$$

B. Lambert W Function for multi-dimensional System

Lambert *W* function has been extended to deal with multi-dimensional time delay systems [7], however, it is restricted to some specific cases. The time delay can be represented as:

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \quad t \geq 0 \tag{17}$$

$$x(t) = x_0 \qquad t = 0$$

$$x(t) = \Phi(t) \qquad t \in [-\tau, 0] \tag{18}$$

A straight forward complete solution can be found if A and A_d commute. The solution is assumed as follows:

$$x(t) = e^{St} C^I \tag{19}$$

where S is $n\times n$ matrix and C^I is $n\times 1$ vector. Substituting into (17), we get:

$$Se^{St}C^I - Ae^{St}C^I - A_d e^{S(t-\tau)}C^I = 0 \tag{20}$$

$$(S - A - A_d e^{-\tau S})e^{St}C^I = 0 \tag{21}$$

From which;

$$S - A - A_d e^{-\tau S} = 0 \tag{22}$$

The characteristic equation of the system is then $S - A - A_d e^{-\tau S}$, and finding the solution of (22) provides a complete solution to the time delay system. Equation (22) can be written as:

$$S - A = A_d e^{-\tau S} \tag{23}$$

Multiplying both sides of (23) by $\tau e^{\tau S} e^{-\tau A}$ we get;

$$\tau(S-A)e^{S\tau}e^{-A\tau} = \tau A_d e^{-A\tau} \tag{24}$$

In the especial case if A and A_d commute then S and A commute, then;

$$e^{S\tau}e^{-A\tau} = e^{\tau(S-A)} \tag{25}$$

Substituting (25) in (24):

$$\tau(S-A)e^{\tau(S-A)} = \tau A_d e^{-A\tau} \tag{26}$$

Lambert *W* matrix function is given as:

$$W(H)e^{W(H)} = H \tag{27}$$

Comparing (26) with (27) we can write;

$$\tau(S - A) = W(H) \tag{28}$$

As Lambert *W* function has infinite number of branches one can write;

$$\tau(S_k - A) = W_k(H_k) \tag{29}$$

$$S_k = \frac{1}{\tau} W_k(H_k) + A \tag{30}$$

In many cases A and A_d do not commute and there are some methods in the literature for solving this problem. In [7] the authors introduced unknown matrix Q satisfying the following:

$$\tau(S-A)e^{\tau(S-A)} = \tau A_d Q \tag{31}$$

Comparing (27) with (31), yields;

$$\tau(S - A) = W(\tau A_d Q) \tag{32}$$

As the function has infinite branches one can write;

$$S_k = \frac{1}{\tau} W_k(\tau A_d Q_k) + A \tag{33}$$

Now the problem is finding the matrix Q. There are different approaches for solving (33). In all the approaches, Q_k is obtained from numerical solution (e.g., using *fsolve* in MATLAB) of:

$$W_k(A_d \tau Q_k)e^{W_k(A_d \tau Q_k) + A\tau} = A_d \tau \tag{34}$$

The other approach is to find Q using the **QPmR** algorithm with **LambertW_DDE** Toolbox [7]. In [13] the authors introduce a matrix D_k instead of Q_k, then (34) becomes:

$$D_k e^{D_k + A\tau} = A_d \tau \tag{35}$$

The problem now is to find the matrix D_k not Q_k. D_k is found using the Gauss–Newton–Powell's Dog-Leg algorithm [13]. For some special cases the problem of finding the matrix Q_k is simpler [14].

C. Time Delay Analysis using Lyapunov-Krasovskii Functional

In order to verify the results of the maximum allowable delay bound obtained through Lambert *W* function, a comparison has been made with the most widely used method for calculating the maximum allowable delay bound. The time-varying delay is $\tau(t)$ and satisfies the following:

$$0 \leq \tau(t) \leq \rho \ , \quad \dot{\tau}(t) \leq \mu \leq 1 \tag{36}$$

where ρ is the upper bound of the time delay, μ is the upper bound on the variable rate of the time delay. For the delay-dependent stability the following Lyapunov-Krasovskii functional is usually used;

$$V(x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(s)Qx(s)ds + \int_{-\rho}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta \tag{37}$$

where $P = P^T > 0$, and $Q = Q^T > 0$, and $P = P^T > 0$ are the variable matrices to be determined. Differentiating $V(x)$ along with (17);

$$\begin{aligned} \dot{V}_1(x) &= x^T(t)P\dot{x}(t) + \dot{x}^T(t)Px(t) + x^T(t)Qx(t) \\ &\quad - (1-\dot{\tau})x^T(t-\tau(t))Qx(t-\tau(t)) \end{aligned} \tag{38}$$

Theorem 1 [4]: Given scalars $\rho > 0$ and $\mu > 0$, the time-delay system (17) is asymptotically stable if there exist symmetric positive-definite matrices $P = P^T > 0$, $Q = Q^T > 0$ and $Z = Z^T > 0$, asymmetric semi-positive-definite matrix $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$, and any appropriate dimensioned matrixes Y and T such that the following LMIs are true:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \rho A^T Z \\ \Phi_{12}^T & \Phi_{22} & \rho A_d^T Z \\ \rho Z A & \rho Z A_d & -\rho Z \end{bmatrix} < 0 \quad \Psi = \begin{bmatrix} X_{11} & X_{12} & Y \\ X_{12}^T & X_{22} & T \\ Y^T & T^T & Z \end{bmatrix} \geq 0$$

where;

$$\Phi_{11} = PA + A^T P + Y + Y^T + Q + \rho X_{11}$$

$$\Phi_{12} = PA_d - Y + T^T + \rho X_{12}$$
$$\Phi_{22} = -T - T^T - (1-\mu)Q + \rho X_{22}$$

The MADB can be found be solving the LMIs in theorem 1 along with the binary iteration algorithm.

III. NUMERICAL RESULTS

Example 1:

Consider the first order system in (4) with the parameters a=1, a_d=-2. One can obtain the values of S_k using (10) for τ = 0.5 s, and the values of C_k^I using (13), where x_0 = 0.1 and $g(t)$ = 0 for - $\tau < t < 0$. The eigenvalues are shown in Table I. Also, the free response of (4) can be obtained using (11) for τ =0.1, 0.3, 0.5, and 0.6 s. The system response, $x(t)$, with the Matlab simulation and the time delay approximation by Lambert W function is shown in Fig. 2. The results show good agreement as more branches are used as can be seen from Fig. 3. As we add more terms (i.e., branches), the error between the free response produced by the Matlab model and approximate model is reduced. Furthermore, the stability is determined by the rightmost eigenvalue in the s-plane, as reported by some researchers [15]. The rightmost eigenvalues can be obtained using only the principal branch (i.e., k = 0) of the Lambert W function as shown in Fig. 4. So the MADB can be estimated for k=0 as 0.60462 seconds.

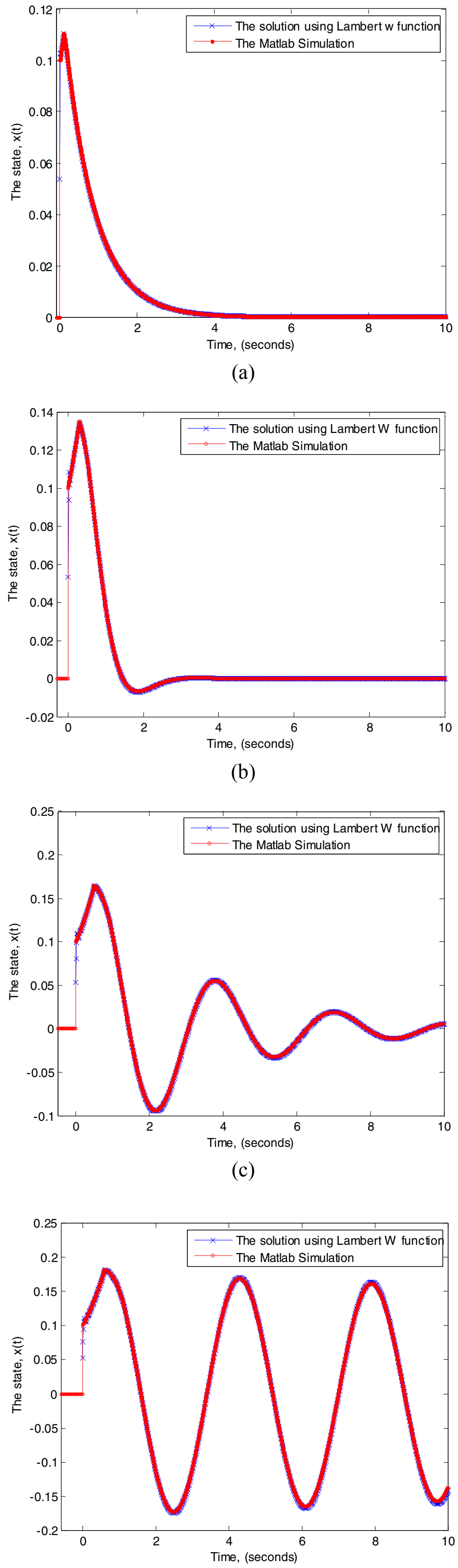
TABLE I. THE EIGENVALUES COEFFICIENTS FOR SCALAR CASE, T=0.5 S.

k	σ_{sk}	ω_{sk}	σ_{ck}	ω_{ck}	IAE
0	-0.3258	1.9450	0.0318	-0.0918	0.1893
+1	-0.3258	-1.9450	0.0318	0.0918	0.0061
-1	-4.1469	15.0489	-0.0027	-0.0127	
+2	-4.1469	-15.0489	-0.0027	0.0127	0.0049
-2	-5.3160	27.8280	-0.0011	-0.0070	
+3	-5.3160	-27.8280	-0.0011	0.0070	0.0045
-3	-6.0459	40.4962	-0.0006	-0.0049	
+4	-6.0459	-40.4962	-0.0006	0.0049	0.0042
-4	-6.5791	53.1236	-0.0004	-0.0037	
+5	-6.5791	-53.1236	-0.0004	0.0037	0.0041
-5	-6.9996	65.7312	-0.0003	-0.0030	
+6	-6.9996	-65.7312	-0.0003	0.0030	0.0040
-6	-7.3468	78.3275	-0.0002	-0.0025	
+7	-7.3468	-78.3275	-0.0002	0.0025	0.0039
-7	-7.6426	90.9166	-0.0002	-0.0022	

Example 2 [1-2]:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad A_d = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix}$$

Using Lambert W function with the principal branch and using the binary iteration algorithm [16-18] the MADB is 1.162 s, and with the simulation the MADB is 1.167 s. The relative error is 0.4284 %. The rightmost eigenvalues using the principal branch are shown in Fig. 5. Solving the LMIs in theorem 1 and using the binary iteration algorithm the MADB is 1.0758 s. In this example the MADB obtained through using the Lambert function is less conservative than the one obtained through the LMI method.



(d)

Fig. 2. Free response, and comparison between the 7-term (Lambert W function-based method) and the numerical integration of the Matlab/Simulink model, (a) for τ =0.1 s,(b) for τ =0.3 s, (c) for τ =0.5 s, (d) for τ =0.6 s

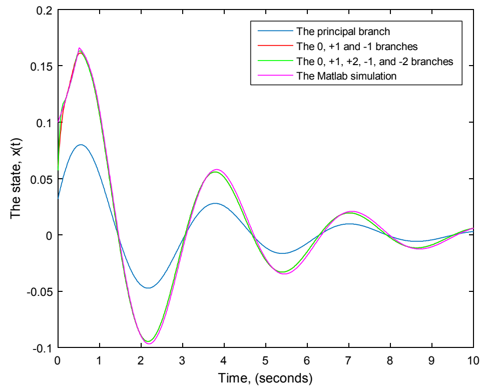


Fig. 3. The comparison between the results of the numerical integration of the Matlab/Simulink and the free response (11) with one, three, and, five terms of Lambert *W* function.

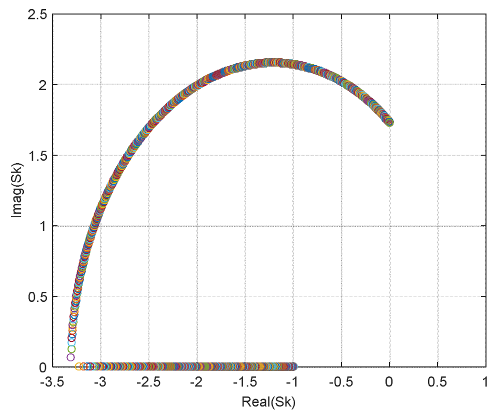


Fig. 4. Rightmost eigenvalues of (4) for k=0

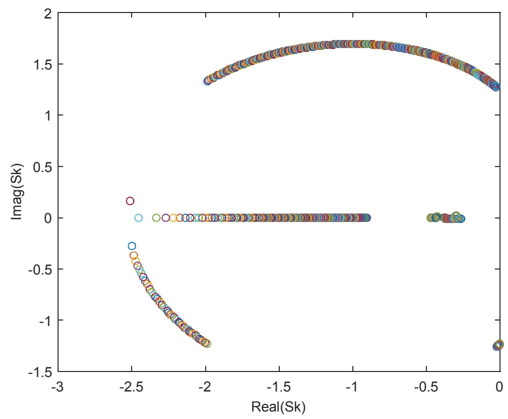


Fig. 5. Rightmost eigenvalues for example 2 for k=0

Example 3 [1]:

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -2 & -3\end{bmatrix}\quad A_d=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -160 & -54 & -11\end{bmatrix}$$

Using Lambert W function with the principal branch and using the binary iteration algorithm the MADB is 0.1172 s, with the simulation the MADB is 0.1178 s. So the relative error is 0.5093%. The rightmost eigenvalues using principal branch are shown in Fig. 6. And the MADB with the LMI is 0.1149 s. As can be seen, the MADB obtained through using the Lambert function is less conservative than the one obtained through the LMI.

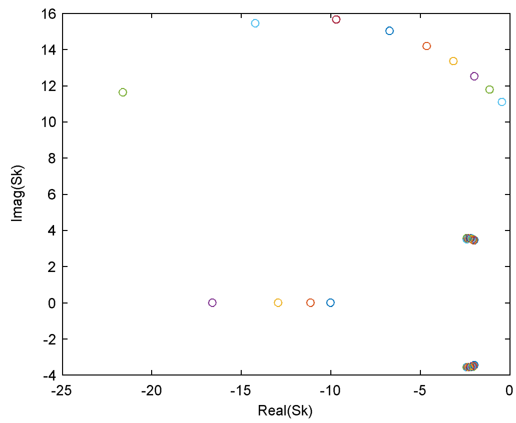


Fig. 6. Rightmost eigenvalues for example 3 for k=0

Example 4 [17]:

$$A=\begin{bmatrix}0 & 0 & K_i\\ 0 & -1/T_g & 0\\ 0 & 1/M & -D/M\end{bmatrix},\ A_d=\begin{bmatrix}0 & 0 & 0\\ 1/T_g & 0 & K_p/T_g\\ 0 & 0 & 0\end{bmatrix}$$

This is the model of the load frequency control system in the Microgrid [17]. The parameters are: K_i=-0.18, K_p=-0.12, M=0.012, D=0.2, and T_g=2. Using Lambert W function the MADB is 1.3 s, while the LMI gives 1.6123 s as the MADB. In this example, the MADB based on the Lambert W function gives more conservative results since the matrices A and A_d are not simultaneously triangularizable.

Example 5 [19,20]:
The forth-order model for single-area load frequency control system is given by [19]:

$$A=\begin{bmatrix}-\dfrac{D}{M} & \dfrac{1}{M} & 0 & 0\\ 0 & -\dfrac{1}{T_{ch}} & \dfrac{1}{T_{ch}} & 0\\ -\dfrac{1}{RT_g} & 0 & -\dfrac{1}{T_g} & 0\\ \beta & 0 & 0 & 0\end{bmatrix},\ A_d=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\dfrac{K_p\beta}{T_g} & 0 & \dfrac{K_i}{T_g} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

where T_{ch}=0.3; T_g=0.1; R=0.05; D=1; β=21; M=10; K_i=0.6; K_p=0.6. Using Lambert W function the MADB is 1.1037 s while the LMI gives 2.0027 s as the MADB. The matrices A and A_d are not simultaneously triangularizable, so the MADB based on the Lambert W function gives more conservative results.

Example 6 [21]:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix} A_d=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -52.1238 & -11.585 & -1 & -2.7252 \\ 0 & 0 & 0 & 0 \\ 26.0619 & 5.7925 & 0.5 & 1.3626 \end{bmatrix}$$

Using Lambert *W* function the MADB is 0.1129 s, while the LMI gives 0.1100 s as the MADB. From the previous examples the proposed method gives good results comparing with the LMI based method, however for higher order systems such as the parallel DC/DC converters in [22] the proposed method gives very conservative results.

IV. CONCLUSION

In this paper, the maximum allowable delay bound in the linear time delay control system has been calculated using the Lambert *W* function. The results have been compared with the most widely used Linear Matrix Inequalities method. The two methods are compared through different examples. For scalar Lambert *W* function the free response is identical to the Matlab simulation and the number of branches must be chosen carefully, with more branches the results show better agreement. Also, the system is stable unless the time delay of the system doesn't exceed the MADB. The rightmost eigenvalues are shown for each example which is the advantages of the Lambert *W* function. For multi-dimensional system when the matrices A and A_d are simultaneously triangularizable the MADB obtained through the method based on the Lambert *W* function gives less conservative results. When A and A_d are not triangularizable the MADB based on the Lambert *W* function is more conservative than the MADB obtained from the method based on LMIs.

REFERENCES

[1] A. F. Khalil, J. Wang, “Stability and Time Delay Tolerance Analysis Approach for Networked Control Systems”, Journal of Mathematical Problems in Engineering, Hindawi Publishing, 2015, pp. 1-9.

[2] A. F. Khalil and J. Wang, "A New Stability Analysis and Time-Delay Tolerance Estimation Approach for Output Feedback Networked Control Systems," UKACC International Conference on Control 2010, Coventry, 2010 pp. 1-6.

[3] A. F. Khalil and J. Wang, "A New Method for Estimating the Maximum Allowable Delay in Networked Control of bounded nonlinear systems," The 17th International Conference on Automation and Computing, Huddersfield, 2011, pp. 80-85.

[4] Yong He, Jin-Hua She Min Wu, *"Stability Analysis and Robust Control of Time-Delay Systems"*, Springer, 2010.

[5] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey and D. E. Knuth, "On the Lambert W function," Advances in Computational Mathematics, vol. 5, p. 329–359, 1996.

[6] F. M. Asl, A. G. Ulsoy, "Analysis of a System of Linear Delay Differential Equations," Journal of Dynamic Systems, Measurement and Control, 125(2), p. 215–223, 2003.

[7] S. Yi, P. W. Nelson and A. G. Ulsoy, "Time-Delay Systems: Analysis and Control Using the Lambert W Function", New Jersey: World Scientific, 2010.

[8] S. Yi, P. W. Nelson, A. G. Ulsoy, "DC Motor Control Using the Lambert W Function Approach", IFAC Proceedings Volumes, Vol. 45, No. 14, pp. 49-54, 2012.

[9] S. Yi, P. W. Nelson and A. G. Ulsoy, "Controllability and Observability of Systems of Linear Delay Differential Equations Via the Matrix Lambert W Function," in *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 854-860, April 2008.

[10] C. Hwang,Yi-Cheng Cheng, "A note on the use of the Lambert W function in the stability analysis of time-delay systems", Automatica, Vol. 41 No.2005, 1979 – 1985, 2005.

[11] S. Yi, P. W. Nelson, and A. G. Ulsoy, "Proportional-Integral Control of First-Order Time-Delay Systems via Eigenvalue Assignment", IEEE Transactions on Control Systems Technology, Vol. 21, No. 5, September 2013, pp. 1586-1594.

[12] N. Choudhary, J. Sivaramakrishnan, I. N. Kar, "Analysis of multi-dimensional time delay systems using Lambert W function", Journal of Dynamic Systems Measurement and Control, 2017.

[13] I. Ivanoviene, J. Rimas, " Complement to method of analysis of time delay systems via the Lambert W function", Automatica, Vol. 54, No. 2015, pp. 25–28, 2015.

[14] R. Cepeda-Gomez and W. Michiels, "Some special cases in the stability analysis of multi-dimensional time-delay systems using the matrix Lambert W function," Automatica, vol. 53, p. 339–345, Mar. 2015.

[15] H. Shinozaki and T. Mori, "Robust stability analysis of linear time-delay systems by Lambert W function:Some extreme point results," Automatica, 42(1), pp. 1791-1799, 2006.

[16] A. Khalil, S. Elkawafi, A. I. Elgaiya, J. Wang, "Delay-Dependent Stability of DC Microgrid with Time-Varying Delay," 2016 22nd International Conference on Automation and Computing (ICAC), Colchester, 2016, pp. 360-365.

[17] S. Elkawafi, A. Khalil, A. I. Elgaiyar, J. Wang, "Delay-dependent stability of LFC in Microgrid with varying time delays," 2016 22nd International Conference on Automation and Computing (ICAC), Colchester, 2016, pp. 354-359.

[18] A. Alfergani, A. Khalil, "Modeling and Control of Master-Slave Microgrid with Communication Delay," *2017 8th International Renewable Energy Congress (IREC)*, Amman, 2017, pp. 1-6.

[19] A. Khalil, J. Wang, O. Mohamed, "Robust Stabilization of Load Frequency Control System Under Networked Environment," International Journal of Automation and Computing , Vol. 14, No. 1, pp. 93-105, 2017.

[20] A. Khalil, J. Wang, "Stabilization of load frequency control system under networked environment," *2015 21st International Conference on Automation and Computing (ICAC)*, Glasgow, 2015, pp. 1-6.

[21] A. Khalil, J. Wang, "Robust Stabilization of Networked Control Systems using the Markovian Jump System Approach," Proceedings of 2012 UKACC International Conference on Control, Cardiff, 2012, pp. 316-321.

[22] A. Khalil, J. Wang, "Estimation of maximum allowable time-delay bound for system stability of network controlled parallel DC/DC buck converters," *18th International Conference on Automation and Computing (ICAC)*, Loughborough, 2012, pp. 1-6.