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A Test Set of Functional Differential Equations

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A Test Set of Functional Differential Equations

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Abstract

This document is a collection of delay, neutral and functional differential equations (mostly with analytical solutions) taken from the mathematical literature. Also included is the original source of the equation, and other useful information. Further test equations are welcomed, and will appear in the **anonymous ftp** version of this document. The reference solutions quoted were obtained using an explicit fifth-order continuous Runge-Kutta method based on the fifth-order Dormand & Prince method with a Hermite interpolant.

Key words. test set, functional differential equations

AMS subject classifications. primary 65Q05

Classification & Specification of Equations

The test equations in this document are categorized as follows:

- Delay differential equations,
- Neutral differential equations,
- Integro-differential equations, and
- Functional differential equations.

Each category consists of four subcategories:

- Constant-delay scalar equations,
- Varying-delay scalar equations,
- State-dependent delay scalar equations, and
- Systems of equations.

The following information is provided with each test equation:

- Source,
- Analytical solution (if available),
- Position of analytical discontinuities,
- Any other interesting/relevant information.

1 Delay Differential Equations

1.1 Constant-delay scalar DDEs

Equation: 1.1.1

$$\begin{aligned} y'(t) &= Ay(t-B) \qquad t \geq 0, \\ Y(t) &= C \qquad\qquad\quad t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = C \sum_{n=0}^{[\frac{t}{B}]+1} A^n \frac{(t-(n-1)B)^n}{n\,!} \quad t \geq 0.$$

Sources: S.B. Norkin & L.E. El'sgol'ts, Intro. to the Theory and Applics. of Diff. Eqns. with Deviating Arguments, Math. in Sci. and Eng. Vol. 105, Academic Press (1973), p. 8.

Discontinuities: An $(n+1)$-st order discontinuity at $t = nB$.

Other information: For $A = B = C = 1$ the solution at $t = 10$ is $y(10) = \frac{14640251}{44800}$. For $A = -1$ and $B = C = 1$ the solution at $t = 10$ is $y(10) = \frac{10493}{518400}$.

Equation: 1.1.2

$$\begin{aligned} y'(t) &= -y(t-\tfrac{\pi}{2}) \qquad t \geq 0, \\ Y(t) &= \sin(t) \qquad\qquad t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \sin(t) \quad t \geq 0.$$

Sources: R. Hügel, Numerischer Vergleich von Programmen zur Lösung von Delay Gleichungen, 5/85 N, Westfalische Wilhelms-Universitat, Münster, West Germany (1985).

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However an $(n+1)$-st order discontinuity may occur in the numerical solution at $t = \frac{n\pi}{2}$.

Equation: 1.1.3

$$\begin{aligned} y'(t) &= -\lambda y(t-1)(1+y(t)) \qquad t \geq 0, \\ Y(t) &= t \qquad\qquad\qquad\qquad\quad t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: S. Katutani & L. Markus, On the Non-Linear Difference Diff. Eqn. $y'(t) = (A - By(t-\tau))y(t)$ *contained in* Contrib. to the Theory of Non-linear Oscillations, Princeton Univ. Press (1958), pp. 1–18.

G.S. Jones, On the Non-Linear Diff.-Difference Eqn. $f'(x) = -cf(x-1)(1+f(x))$, J. Math. Anal. Applic. Vol. 4 (1962), pp. 440–469.

Discontinuities: An $(n+1)$-st order discontinuity at $t = n$.

Other information: The DDE displays a Hopf bifurcation at $\lambda = \frac{\pi}{2}$. For $\lambda > \frac{\pi}{2}$ the solution may tend to a stable limit cycle. For $\lambda = 1.5$ a reference solution is $y(20) = -0.235184625529838$, and for $\lambda = 3$ a reference solution is $y(20) = 4.671437497493366$. [Also see (1.1.4).]

Equation: 1.1.4

$$\begin{aligned} y'(t) &= (\lambda - y(t-1))y(t) & t \geq 0, \\ Y(t) &= 0.01 & t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: E. Hairer, G. Wanner & S.P. Nørsett, Solving Ordinary Diff. Eqns. 1, Springer Series in Comp. Math. Vol. 8 (1980).

Discontinuities: An $(n+1)$-st order discontinuity at $t = n$.

Other information: This DDE is known as the *delay logistic equation.* For $\lambda \in (0, \frac{\pi}{2})$, $y(t) \to 0$ or λ as $t \to \infty$ depending on $Y(t)$. For $\lambda > \frac{\pi}{2}$, $y(t)$ may also tend to a periodic limit cycle. For $\lambda < e^{-1}$, $y(t)$ is monotonic increasing. There is a Hopf bifurcation at $\lambda = \frac{\pi}{2}$. This equation may be transformed into equation (1.1.3) by the substitution $y(t) \to \lambda(y(t)+1)$. For $\lambda = 1.4$ a reference solution is $y(10) = 1.367208017754907$.

Equation: 1.1.5

$$\begin{aligned} y'(t) &= y(t) + y(t-\pi) + 3\cos(t) + 5\sin(t) & t \geq 0, \\ Y(t) &= 3\sin(t) - 5\cos(t) & t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = 3\sin(t) - 5\cos(t) \quad t \geq 0.$$

Sources: R. Hügel, Numerischer Vergleich von Programmen zur Lösung von Delay Gleichungen, 5/85 N, Westfalische Wilhelms-Universitat, Münster, West Germany (1985)

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However an $(n+1)$-st order discontinuity may occur in the numerical solution at $t = n\pi$. Also, neglecting the forcing term, the resulting DDE is unstable – this may explain the poor performance of some codes on this equation.

Equation: 1.1.6

$$\begin{aligned} y'(t) &= -y(t-1)+y(t-2)-y(t-3)y(t-4) && t \geq 0, \\ Y(t) &= 1 && t < 0, \\ Y(0) &= 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} -t & 0 \leq t \leq 1, \\ \frac{1}{2}t^2 - t - \frac{1}{2} & 1 \leq t \leq 2, \\ -\frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{7}{6} & 2 \leq t \leq 3, \\ \frac{1}{24}t^4 - \frac{1}{6}t^3 - \frac{1}{4}t^2 + t - \frac{19}{24} & 3 \leq t \leq 4, \\ -\frac{1}{120}t^5 + \frac{1}{6}t^4 - \frac{5}{3}t^3 + \frac{109}{12}t^2 - 24t + \frac{2689}{120} & 4 \leq t \leq 5. \end{cases}$$

Sources: C.A.H. Paul, Concerning Explicit Runge-Kutta Techniques for Delay Diff. Eqns., M.Sc. thesis, Math. Dept., Manchester Univ., England (1989).

Discontinuities: A zeroth-order discontinuity at $t = 0$ and an n-th order discontinuity at $t = \{4n - 3,\ 4n-2,\ 4n-1,\ 4n\}$ for $n \geq 1$.

Other information: This DDE is linear upto $t = 4$ and non-linear beyond $t = 4$, with $y(5) = -\frac{43}{60}$.

Equation: 1.1.7

$$\begin{aligned} y'(t) &= \frac{\lambda\theta^m y(t-\tau)}{\theta^m + y(t-\tau)^m} + \gamma y(t) && t \geq 0, \\ Y(t) &= \alpha && t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: L. Glass & M.C. Mackey, Pathological Conditions Resulting from Instabilities in Physiological Control Systems, Annals New York Academy of Sci. Vol. 316 (1979), pp. 78–85.

Discontinuities: An $(n+1)$-st order discontinuity at $t = n\tau$.

Other information: For $\alpha = \frac{1}{2}$, $\lambda = 2$, $\theta = 1$, $\tau = 2$ and $\gamma = -1$ phase plots of $y(t) \times y(t-\tau)$ show period doubling and chaotic limit cycles for $m \in [7, 10]$. For $m = 7$ (which has a non-chaotic limit cycle), a reference solution is $y(20) = 1.202617750066138$. [Also see (1.1.17).]

Equation: 1.1.8

$$\begin{aligned} y'(t) &= -\mu y(t) + \mu y(t-\tau)(1 + q(1 - \left(\frac{y(t-\tau)}{K}\right)^z)) && t \geq 0, \\ Y(t) &= \alpha && t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: R.M. May, The Dynamics of Natural and Managed Populations, The Mathematical Theory of the Dynamics of Biological Populations II, *Eds. Hiorns & Cooke*, Academic Press (1981).

Discontinuities: An $(n+1)$-st order discontinuity at $t = n\tau$.

Other information: This DDE is used by the International Whaling Commission to model the baleen whale population. The DDE exhibits period doubling and chaotic behaviour. For $\alpha = 999$, $K = 1000$, $\mu = 1$, $q = 1.5$, $z = 2.5$ and $\tau = 2$ a reference solution is $y(20) = 820.185301965284$.

Equation: 1.1.9

$$\begin{aligned} y'(t) &= 5y(t) + y(t-1) \qquad & t \geq 0, \\ Y(t) &= 5 & t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} 6\exp(5t) - 1 & 0 \leq t \leq 1, \\ 6(e^5 + t - \frac{6}{5})\exp(5t-5) + \frac{1}{5} & 1 \leq t \leq 2. \end{cases}$$

Sources: H.T. Banks & F. Kappel, Spline Approximations for Functional Diff. Eqns., J. Diff. Eqns. Vol. 34 (1979), pp. 496–522.
K. Ito & R. Teglas, Legendre-Tau Approximations for Functional Diff. Eqns., SIAM J. Control Optim. Vol. 24 (1986), pp. 737–759.

Discontinuities: An $(n+1)$-st order discontinuity at $t = n$.

Other information: This equation is a particular example of the linear stability DDE test equation $y'(t) = \lambda y(t) + \mu y(t-\tau)$. For this particular choice of λ and μ the DDE is *unstable*.

Equation: 1.1.10

$$\begin{aligned} y'(t) &= y(t-\pi)y(t) & t \geq 0, \\ Y(t) &= \begin{cases} 0 & t < -\frac{\pi}{2}, \\ -2 & -\frac{\pi}{2} \leq t < 0, \\ -1 & t = 0. \end{cases} \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} -1 & 0 \leq t \leq \frac{\pi}{2}, \\ -\exp(\pi - 2t) & \frac{\pi}{2} \leq t \leq \pi, \\ -e^{-t} & \pi \leq t \leq \frac{3\pi}{2}, \\ -\exp\left(-\frac{3}{2}\pi + \frac{1}{2}(\exp(3\pi - 2t) - 1)\right) & \frac{3\pi}{2} \leq t \leq 6. \end{cases}$$

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: An n-th order discontinuity at $t = \{\frac{(2n-1)\pi}{2}, n\pi\}$.

Other information: This DDE has a discontinuous initial function. [Also see (1.1.11).]

Equation: 1.1.11

$$\begin{aligned} y'(t) &= -y(t-\pi)y(t) & t \geq 0, \\ Y(t) &= \begin{cases} 0 & t < -\frac{\pi}{2}, \\ -2 & -\frac{\pi}{2} \leq t < 0, \\ -1 & t = 0. \end{cases} \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} -1 & 0 \leq t \leq \frac{\pi}{2}, \\ -\exp(2t-\pi) & \frac{\pi}{2} \leq t \leq \pi, \\ -e^t & \pi \leq t \leq \frac{3\pi}{2}, \\ -\exp\left(\frac{3}{2}\pi + \frac{1}{2}(\exp(2t-3\pi)-1)\right) & \frac{3\pi}{2} \leq t \leq 6. \end{cases}$$

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: An n-th order discontinuity at $t = \{\frac{(2n-1)\pi}{2},\, n\pi\}$.

Other information: This DDE has a discontinuous initial function. [Also see (1.1.10).]

Equation: 1.1.12

$$\begin{aligned} y'(t) &= y(t) + y(t-1) & t \geq 0, \\ Y(t) &= \begin{cases} 0 & -1 \leq t < -\frac{1}{3}, \\ 1 & -\frac{1}{3} \leq t \leq 0. \end{cases} \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} e^t & 0 \leq t \leq \frac{2}{3}, \\ c_1 e^t - 1 & \frac{2}{3} \leq t \leq 1, \\ t\exp(t-1) + c_2 e^t & 1 \leq t \leq \frac{5}{3}, \\ 1 + c_1 t \exp(t-1) + c_3 e^t & \frac{5}{3} \leq t \leq 2, \\ (\frac{1}{2}t^2 - t)\exp(t-2) + c_2 t\exp(t-1) + c_4 e^t & 2 \leq t \leq \frac{8}{3}, \end{cases}$$

where $c_1 = 1+\exp(-\frac{2}{3})$, $c_2 = c_1 - 2e^{-1}$, $c_3 = \frac{5}{3}e^{-1}(1-c_1)+c_2-\exp(-\frac{5}{3})$ and $c_4 = e^{-2}+c_3+2(c_1-c_2)e^{-1}$.

Sources: K. Ito, H.T. Tran & A. Manitius, A Fully-Discrete Spectral Method for Delay Diff. Eqns., SIAM J. Num. Anal. Vol. 28 (1991), pp. 1121–1140.

Discontinuities: An $(n+1)$-st order discontinuity at $t = \{n,\, n+\frac{2}{3}\}$.

Other information: This DDE is a version of the linear stability DDE test equation, but with a discontinuous initial function.

Equation: 1.1.13

$$\begin{aligned} y'(t) &= ry(t)(1 - \tfrac{y(t-\tau)}{M}) & t \geq 0, \\ Y(t) &= 19.001 & t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: An $(n+1)$-st order discontinuity at $t = n\tau$.

Other information: This DDE models the Lemmings population cycle of about 4 years. For $\tau = \frac{37}{50}$, $r = 3.5$ and $M = 19$ a reference solution is $y(40) = 24.7645486398692$.

Equation: 1.1.14

$$\begin{array}{rcll} y'(t) & = & Ay(t) + y(t - \frac{3\pi}{2}) - A\sin(t) & t \geq 0, \\ Y(t) & = & \exp(pt) + \sin(t) & t \leq 0, \end{array}$$

where $A = p - \exp(-\frac{3p\pi}{2})$.

Analytical solution:

$$y(t) = \exp(pt) + \sin(t) \quad t \geq 0.$$

Sources: K. Ito, H.T. Tran & A. Manitius, A Fully-Discrete Spectral Method for Delay Diff. Eqns., SIAM J. Num. Anal. Vol. 28 (1991), pp. 1121–1140.

Discontinuities: No discontinuities.

Other information: The parameter p is a *stiffness* parameter, with the stiffness of the DDE increasing substantially for large negative values of p. The analytical solution is an analytic continuation of the initial function. However an $(n+1)$-st order discontinuity may occur in the numerical solution at $t = \frac{3n\pi}{2}$.

Equation: 1.1.15

$$\begin{array}{rcll} y'(t) & = & -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}) & t \geq 0, \\ Y(t) & = & 1 + \exp(-3t) & t \leq 0. \end{array}$$

Analytical solution:

$$y(t) = 1 + \exp(-3t) \quad t \geq 0.$$

Sources: R. Weiner & K. Strehmel, A Type Insensitive Code for Delay Diff. Eqns. based on Adaptive and Explicit Runge-Kutta Interpolation Methods, Comp. Vol. 40 (1988), pp. 255–265.

Discontinuities: No discontinuities.

Other information: This DDE is derived from the linear stability DDE test equation. The choice of parameters produces a *stiff* DDE. The analytical solution is an analytic continuation of the initial function. However an $(n+1)$-st order discontinuity may occur in the numerical solution at $t = n$.

Equation: 1.1.16

$$
\begin{aligned}
y'(t) &= y(t-1) & t \geq 0, \\
Y(t) &= (-1)^{[-5t]} & t \leq 0.
\end{aligned}
$$

Analytical solution:

$$
y(t) = \begin{cases}
1+t & 0 \leq t \leq \frac{1}{5}, \\
\frac{7}{5} - t & \frac{1}{5} \leq t \leq \frac{2}{5}, \\
\frac{3}{5} + t & \frac{2}{5} \leq t \leq \frac{3}{5}, \\
\frac{9}{5} - t & \frac{3}{5} \leq t \leq \frac{4}{5}, \\
\frac{1}{5} + t & \frac{4}{5} \leq t \leq 1.
\end{cases}
$$

Sources: D.R. Willé & C.T.H. Baker, DELSOL – A Numerical Code for the Solution of Systems of Delay Diff. Eqns., App. Num. Math. Vol. 9 (1992), pp. 223–234.

Discontinuities: An n-th order discontinuity at $t = \{n,\, n-\frac{1}{5},\, n-\frac{2}{5},\, n-\frac{3}{5},\, n-\frac{4}{5}\}$.

Other information: This DDE has a discontinuous initial function.

Equation: 1.1.17

$$
\begin{aligned}
y'(t) &= \tfrac{\beta y(t-\tau)}{1+y(t-\tau)^m} - \gamma y(t) & t \geq 0, \\
Y(t) &= \tfrac{1}{10} & t \leq 0.
\end{aligned}
$$

Analytical solution: Not available.

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: An $(n+1)$-st order discontinuity at $t = n\tau$.

Other information: For $\beta = \frac{1}{5}$, $m = 10$, $\gamma = \frac{1}{10}$ and $\tau = 20$ a reference solution is $y(1000) = 1.12163648455960$. [See also (1.1.7).]

Equation: 1.1.18

$$
\begin{aligned}
y'(t) &= \begin{cases}
-ry(t)I(t) & 0 \leq t \leq \gamma, \\
-ry(t)(I(t) + y(t_0) - e^{\mu} y(t)) & \gamma \leq t \leq \gamma + \sigma, \\
-ry(t)e^{\mu}(y(t-\sigma) - y(t)) & t \geq \gamma + \sigma,
\end{cases} \\
I(t) &= \begin{cases}
\frac{2}{5}(1-t) & 0 \leq t \leq 1, \\
0 & t \geq 1,
\end{cases}
\end{aligned}
$$

where $y(0) = 10$, $r = 0.5$, $\mu = 0.05$, $\sigma = 1$ and $\gamma = 1 - \frac{1}{\sqrt{2}}$.

Analytical solution: Not available.

Sources: H.J. Oberle & H.J. Pesch, Numerical Treatment of Delay Diff. Eqns. by Hermite Interpolation, Numer. Math. Vol. 37 (1981), pp. 235–255.

Discontinuities: A first-order discontinuity at $t = \gamma$. Also n-th order discontinuities at $t = (n-1)\sigma+1$ and $n\sigma + \gamma$ for $n \geq 1$.

Other information: This DDE describes a model of the spread of an infection – $y(t)$ represents the number of susceptible people and $I(t)$ the number of infected people. A reference solution is $y(10) = 0.0630208986890493$.

1.2 Varying-delay scalar DDEs

Equation: 1.2.1

$$\begin{aligned} y'(t) &= ay(bt^c) \qquad 0 \leq t \leq b^{\frac{1}{1-c}}, \\ Y(0) &= d. \end{aligned}$$

Analytical solution:

$$y(t) = d\sum_{n=0}^{\infty}\left(\prod_{m=1}^{n}(c^m-1)^{-1}\right)a^n b^{\frac{c^n+n(1-c)-1}{(c-1)^2}}(c-1)^n t^{\left(\frac{c^n-1}{c-1}\right)} \quad 0 \leq t \leq b^{\frac{1}{1-c}}.$$

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: No discontinuities.

Other information: This equation is an *initial value* DDE and is only defined on the interval $[0,\, b^{\frac{1}{1-c}}]$. For $a = 3$, $b = \frac{19}{20}$, $c = \frac{11}{10}$ and $d = 1$ the solution at $t = (\frac{20}{19})^{10}$ is $y((\frac{20}{19})^{10}) = 91.22490537957470909$.

Equation: 1.2.2

$$\begin{aligned} y'(t) &= y(t-\tfrac{1}{10})y(t^2) \qquad 0 \leq t \leq 1, \\ Y(t) &= 1 \qquad\qquad\qquad\quad t < 0, \\ Y(0) &= 2. \end{aligned}$$

Analytical solution: Not available.

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: A zeroth-order discontinuity at $t = 0$ and an $(n+m)$-th order discontinuity at $t = \{\sqrt[2^m]{\frac{n}{10}}, \sqrt[2^{m-1}]{\frac{n}{10}} + \frac{1}{10}, \sqrt{\sqrt[2^{m-2}]{\frac{n}{10}} + \frac{1}{10}}, ldots, \}$ for $n \geq 1$.

Other information: This DDE has an infinite number of analytical discontinuities in the interval [0, 1]. However, dependent on how discontinuities are treated, further non-existent first-order discontinuities $\{s_i\}$ may be detected when $s_i^2 < \mathcal{E}$, where $\mathcal{E}$ is the tolerance in the root-finding algorithm for tracking discontinuities. The 'propagation' of these non-existent discontinuities leads to further non-existent discontinuities being detected. Thus the smaller the unit-roundoff, and consequently (probably) $\mathcal{E}$, the more non-existent discontinuities detected. A reference solution is $y(1) = 67.981879379380$.

Equation: 1.2.3

$$\begin{aligned} y'(t) &= y(2t-1) \qquad 0 \le t \le 1, \\ Y(t) &= 1 \qquad t \le 0. \end{aligned}$$

Analytical solution:

$$y(t) = \sum_{i=0}^{n=\left[\frac{\ln(1-t)}{-\ln 2}\right]+1} 2^{\frac{-(i+1)i}{2}} \frac{\left(2^i t - 2^i + 1\right)^i}{i\,!}\alpha_{n-i} \quad 0 \le t \le 1,$$

where

$$\begin{aligned} \alpha_n &= \sum_{i=1}^{n} \left(i\left(1-2^{i-n}\right)^{i-1} - \left(\frac{1}{2} - 2^{i-n}\right)^i \right) \frac{2^{\frac{(1-i)i}{2}}}{i\,!}\alpha_{n-i}, \\ \alpha_0 &= 1. \end{aligned}$$

Sources: C.A.H Paul & C.T.H. Baker, Explicit Runge-Kutta Methods for the Numerical Solution of Singular Delay Diff. Eqns., Tech. Rep. No. 212, Math. Dept., Manchester Univ. (1992).

Discontinuities: An $(n+1)$-st order discontinuity at $t = 1 - \frac{1}{2^n}$.

Other information: This DDE is only defined on the interval [0, 1], and has a zero lag at the point $t = 1$. The point $t = 1$ is also a cluster point for discontinuities. The solution at the endpoint is $y(1) = 2.271492555500812$.

Equation: 1.2.4

$$\begin{aligned} y'(t) &= y(t - t^{-10}) \qquad t \ge 1, \\ Y(t) &= t \qquad t \le 1. \end{aligned}$$

Analytical solution: Not available.

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: An n-th order discontinuity at $t = s_n$, where $s_n - s_n^{-10} = s_{n-1}$ and $s_1 = 1$.

Other information: This DDE has a vanishing (but non-singular) lag. At $t = 6$ the lag $\tau(t) \approx 10^{-8}$, and a reference solution is $y(6) = 134.759541416008$.

Equation: 1.2.5

$$\begin{aligned} y'(t) &= -y\left(\frac{3t - \sin(t)}{5}\right) \qquad t \ge 0, \\ Y(0) &= 1. \end{aligned}$$

Analytical solution: Not available

Sources: H.G. Bock & J.P Schlöder, Numerical Solution of Retarded Diff. Eqns. with State Dependent Time Lags, ZAMM Vol. 61 (1981), pp. 269–271.

Discontinuities: No discontinuities.

Other information: This is an *initial value* DDE. A reference solution is $y(8) = 0.282862375579844$.

Equation: 1.2.6

$$\begin{aligned} y'(t) &= y\left(\tfrac{t}{(1+2t)^2}\right)^{(1+2t)^2} & t \geq 0, \\ Y(0) &= 1. & \end{aligned}$$

Analytical solution:

$$y(t) = e^t \quad t \geq 0.$$

Sources: M.A. Feldstein, Discretization Methods for Retarded Ordinary Diff. Eqns., Ph.D. thesis, Math. Dept., UCLA (1964).

Discontinuities: No discontinuities.

Other information: This is an *initial value* DDE.

Equation: 1.2.7

$$\begin{aligned} y'(t) &= 1 - y(\exp(1 - \tfrac{1}{t})) & t \geq \alpha, \\ Y(t) &= \ln(t) & 0 < t \leq \alpha. \end{aligned}$$

Analytical solution:

$$y(t) = \ln(t) \quad t > 0.$$

Sources: K.W. Neves, Automatic Integration of Functional Diff. Eqns.: An Approach, ACM Trans. Math. Soft. Vol. 1 (1975), pp. 357–368.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However discontinuities may occur in the numerical solution, their positions being dependent on the initial point α.

Equation: 1.2.8

$$\begin{aligned} y'(t) &= \lambda\left(\tfrac{t-1}{t}\right) y(t - \ln(t) - 1)y(t) & t \geq 1, \\ Y(t) &= 1 & t \leq 1. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} \exp(\lambda(t - \ln(t) - 1)) & \xi_0 \leq t \leq \xi_1, \\ \exp(\lambda + \lambda \int\limits_{\xi_1}^{t} \frac{x-1}{x} \exp(\lambda(x - \ln(x^2 - x\ln(x) - x) - 2))\, dx) & \xi_1 \leq t \leq \xi_2, \end{cases}$$

where $\xi_0 = 1$, $\xi_1 = 3.1461932206205825852$ and $\xi_2 = 5.9254498245082464926$.

Sources: A. Bellen, One-Step Collocation for Delay Diff. Eqns., J. Comp. Appl. Math. Vol. 10 (1984), pp. 275–283.

Discontinuities: An $(n+1)$-st order discontinuity at $t = \xi_n$.

Other information: For $\lambda = 1$ the solution at $t = \xi_2$ is $y(\xi_2) = 76.373472669376805627$, and for $\lambda = -1$ the solution at $t = \xi_2$ is $y(\xi_2) = 0.080847377928312249328$.

Equation: 1.2.9

$$\begin{array}{rcll} y'(t) & = & f(y(\tfrac{1}{2}t)) - y(t) & \quad t \geq 0, \\ Y(0) & = & 1, & \end{array}$$

where $f(s) = 1$ if $s < 0$ and $f(s) = -1$ if $s \geq 0$.

Analytical solution:

$$y(t) = \begin{cases} 2e^{-t} - 1 & 0 \leq t \leq 2\ln(2), \\ 1 - 6e^{-t} & 2\ln(2) < t \leq 2\ln(6), \\ 66e^{-t} - 1 & 2\ln(6) < t \leq 2\ln(66). \end{cases}$$

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: A first-order discontinuity at $t = 2\ln(2)$, $2\ln(6)$ and $2\ln(66)$.

Other information: This DDE has a discontinuous derivative function.

Equation: 1.2.10

$$\begin{array}{rcll} y'(t) & = & y(t - \tfrac{1}{t}) & \quad t \geq 0, \\ Y(t) & = & 1 & \quad t < 0, \\ Y(0) & = & 0. & \end{array}$$

Analytical solution:

$$y(t) = \begin{cases} t & \xi_0 \leq t \leq \xi_1, \\ \frac{1}{2}t^2 - \ln(t) + \frac{1}{2} & \xi_1 \leq t \leq \xi_2, \\ \frac{1}{6}t^3 + \frac{1}{2}t - \frac{1}{2t} - t\ln\left(t - \frac{1}{t}\right) + \ln\left(\frac{t-1}{t+1}\right) + \frac{5}{12} + \frac{\sqrt{5}}{12} - \ln\left(\frac{1}{2}(3 - \sqrt{5})\right) & \xi_2 \leq t \leq \xi_3, \end{cases}$$

where $\xi_0 = 0$, $\xi_1 = 1$, $\xi_2 = \frac{1}{2}(1 + \sqrt{5})$ and $\xi_3 = \frac{1}{4}(1 + \sqrt{5} + \sqrt{22 + 2\sqrt{5}})$.

Sources: C.A.H. Paul, Developing a Delay Diff. Equation Solver, Appl. Num. Math. Vol. 9 (1992), pp. 403–414.

Discontinuities: An n-th order discontinuity at $t = \xi_n$.

Other information:

Equation: 1.2.11

$$\begin{aligned} y'(t) &= \mu y(t)(1-y([t])) & t \geq 0, \\ Y(t) &= \alpha & t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: K.W. Neves & S. Thompson, Software for the Numerical Solution of Systems of Functional Diff. Eqns. with State-Dependent Delays, App. Num. Math. Vol. 9 (1992), pp. 385–401.

Discontinuities: A first-order discontinuity at $t = n$ for $n \geq 0$.

Other information: This DDE has a discontinuous derivative function. For $\mu = 1$ and $\alpha = \frac{1}{2}$ a reference solution is $y(49.5) = 1$.

1.3 State-dependent delay scalar DDEs

Equation: 1.3.1

$$\begin{aligned} y'(t) &= \tfrac{\exp\left(y(y(t)-\ln(2)+1)\right)}{t} & t \geq 1, \\ Y(t) &= 0 & t \leq 1. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} \ln(t) & 1 \leq t \leq 2, \\ \frac{1}{2}t + \ln(2) - 1 & 2 \leq t \leq 4. \end{cases}$$

Sources: K.W. Neves & M.A. Feldstein, High-Order Methods for State-Dependent Delay Diff. Eqns. with Non-Smooth Solutions, SIAM J. Num. Anal. Vol. 21 (1984), pp. 844–863.

Discontinuities: An n-th order discontinuity at $t = 2^{n-1}$ for $n \in \{1, 2, 3\}$.

Other information:

Equation: 1.3.2

$$\begin{aligned} y'(t) &= \tfrac{y(y(t)-\sqrt{2}+1)}{2\sqrt{t}} & t \geq 1, \\ Y(t) &= 1 & t \leq 1. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} \sqrt{t} & \xi_1 \leq t \leq \xi_2, \\ \frac{t}{4} + \frac{1}{2} + (1 - \frac{1}{\sqrt{2}})\sqrt{t} & \xi_2 \leq t \leq \xi_3, \end{cases}$$

where $\xi_1 = 1$, $\xi_2 = 2$ and $\xi_3 = 5.0294372515248$.

Sources: K.W. Neves & M.A. Feldstein, High-order Methods for State-Dependent Delay Diff. Eqns. with Non-Smooth Solutions, SIAM J. Num. Anal. Vol. 21 (1984), pp. 844–863.

Discontinuities: An n-th order discontinuity at $t = \xi_n$.

Other information:

Equation: 1.3.3

$$\begin{aligned} y'(t) &= y(t - y(t - t^2)) && t \geq 0, \\ Y(t) &= t^2 && t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} 0 & 0 \leq t \leq \gamma, \\ f(t) - f(\gamma) & t \geq \gamma, \end{cases}$$

where $f(x) = \frac{1}{9}x^9 - \frac{1}{2}x^8 + \frac{6}{7}x^7 - x^6 + x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3$ and $\gamma \approx 1.755$ is a root of $1 - t + 2t^2 - t^3$.

Sources: C.A.H. Paul & C.T.H Baker, Explicit Runge-Kutta Methods for the Numerical Solution of Singular Delay Diff. Eqns., Tech. Rep. No. 212, Math. Dept., Manchester Univ. (1992).

Discontinuities: A first-order discontinuity at $t = \gamma$.

Other information: The secondary lag vanishes at $t = 0$. From the analytical solution, it is clear that the primary lag vanishes for $t \in [0, \gamma)$. The discontinuity at $t = \gamma$ occurs spontaneously, and as such its position cannot be predicted. Thus derivative discontinuity tracking algorithms can fail to maintain an efficient sequence of stepsizes.

Equation: 1.3.4

$$\begin{aligned} y'(t) &= \tfrac{y(t)y(\ln(y(t)))}{t} && t \geq 1, \\ Y(t) &= 1 && t \leq 1. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} t & 1 \leq t \leq e, \\ \exp(\frac{t}{e}) & e \leq t \leq e^2. \end{cases}$$

Sources: K.W. Neves, Automatic Integration of Functional Diff. Eqns.: An Approach, ACM Trans. Math. Soft. Vol. 1 (1975), pp. 357–368.

Discontinuities: An $(n+1)$-st order discontinuity at $t = e^n$ for $n \in \{0, 1, 2\}$.

Other information:

Equation: 1.3.5

$$\begin{aligned} y'(t) &= -ky(t - 1 - |y(t)|)(1 - y(t)^2) && t \geq 0, \\ Y(t) &= \tfrac{1}{2} && t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: H.L. Smith & Y. Kuang, Slowly Oscillating Periodic Solutions of Autonomous State-Dependent Delay Eqns., IMA Rep. No. 754, Univ. of Minnesota (1990).

Discontinuities: Dependent on the parameter k.

Other information: For $k = 2$ a reference solution is $y(30) = 0.99553206380499$. Also there is an n-th order discontinuity at $t = \xi_n$ where $\xi_1 = 0$, $\xi_2 = 1.8659458287241$, $\xi_3 = 3.4748519444129$, $\xi_4 = 5.4675542263291$, $\xi_5 = 7.2412923812919$, etc.

Equation: 1.3.6

$$\begin{array}{rcll} y'(t) & = & -ky(t - \frac{1}{2}(1 + \exp(-y(t)^2))) & t \geq 0, \\ Y(t) & = & 1 & t \leq 0. \end{array}$$

Analytical solution: Not available.

Sources: H.L. Smith & Y. Kuang, Slowly Oscillating Periodic Solutions of Autonomous State-Dependent Delay Eqns., IMA Rep. No. 754, Univ. of Minnesota (1990).

Discontinuities: Dependent on the parameter k.

Other information: For $k = 10$ a reference solution is $y(20) = 16.295087992773$. Also there is a first-order discontinuity at $t = 0$ and a second-order discontinuity at $t = 0.5000000562673341$.

Equation: 1.3.7

$$\begin{array}{rcll} y'(t) & = & y(y(t)) + (3 + \alpha)t^{(2+\alpha)} - t^{(3+\alpha)^2} & 0 \leq t \leq 1, \\ Y(0) & = & 0. & \end{array}$$

Analytical solution:

$$y(t) = t^{(3+\alpha)} \quad 0 \leq t \leq 1.$$

Sources: L. Tavernini, The Approximate Solution of Volterra Diff. Systems with State-Dependent Time Lags, SIAM J. Num. Anal. Vol. 15 (1978), pp. 1039–1052.

Discontinuities: No discontinuities.

Other information: This is an *initial value* DDE.

Equation: 1.3.8

$$\begin{array}{rcll} y'(t) & = & \omega \cot(g(t))y(t) - \frac{\omega}{\sin(g(t))}y(t - \tau(t, y(t))) & t \geq 0, \\ Y(t) & = & \sin(\omega t) & t \leq 0, \end{array}$$

where $\tau(t, y(t)) = \frac{1}{\omega}(2 + \frac{1}{5}\exp(y(t)))$ and $g(t) = \omega\tau(t, \sin(\omega t))$.

Analytical solution:

$$y(t) = \sin(\omega t) \quad t \geq 0.$$

Sources: H. Arndt, P.J. van der Houwen & B.P. Sommeijer, Numerical Integration of Retarded Diff. Eqns. with Periodic Solutions, ISNM Vol. 74 (1985), pp. 41–51.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However discontinuities may occur in the numerical solution, their positions being dependent on ω.

Equation: 1.3.9

$$\begin{array}{rcll} y'(t) & = & -y(t - \ln(1 + 2|y(t)| \exp(1 - 2e^{-t})))y(t) & t \geq 0, \\ Y(t) & = & 1 & t \leq 0. \end{array}$$

Analytical solution:

$$y(t) = \begin{cases} e^{-t} & 0 \leq t \leq \ln(2), \\ \frac{1}{2} \exp\left(2e^{-t} - 1\right) & \ln(2) \leq t \leq 1. \end{cases}$$

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: An $(n+1)$-st order discontinuity at $t = \xi_n$, where $\xi_0 = 0$, $\xi_1 = \ln(2)$, $\xi_2 = 1.386294361120601$, $\xi_3 = 2.072274257202227$, $\xi_4 = 2.732500205874182$, etc.

Other information:

Equation: 1.3.10

$$\begin{array}{rcll} y'(t) & = & y(y(t)) & t \geq 2, \\ Y(t) & = & \frac{1}{2} & t < 2, \\ Y(2) & = & 1. & \end{array}$$

Analytical solution:

$$y(t) = \begin{cases} \frac{1}{2}t & \xi_0 \leq t \leq \xi_1, \\ 2\exp(\frac{1}{2}t - 2) & \xi_1 \leq t \leq \xi_2, \\ -2\ln\left(e^{-2}(1 + \ln(4e^4) - t)\right) & \xi_2 \leq t \leq \xi_3, \end{cases}$$

where $\xi_0 = 2$, $\xi_1 = 4$, $\xi_2 = \ln(4e^4)$ and $\xi_3 = \frac{1}{2} + \ln(4e^4)$.

Sources: C.A.H. Paul, Developing a Delay Diff. Equation Solver, Appl. Num. Math. Vol. 9 (1992) pp. 403–414.

Discontinuities: An n-th order discontinuity at $t = \xi_n$.

Other information: This equation tests how discontinuities are included in the meshpoints. If $y(4.0) < 2$ or $y(4.0) > 2$ the stepsize may be severely restricted about the point $t = 4$, due to the zeroth-order discontinuity at $t = 2$.

Equation: 1.3.11

$$\begin{array}{rcll} y'(t) & = & y(t - y(t) - 1) + \frac{1}{2} & t \geq 0, \\ Y(t) & = & \begin{cases} 1 & t < -1, \\ 0 & -1 \leq t \leq 0. \end{cases} & \end{array}$$

Analytical solution:

$$y(t) = \frac{3}{2} \quad 0 \leq t \leq 2,$$

OR

$$y(t) = \frac{1}{2} \quad 0 \leq t \leq 2,$$

Sources: S.B. Norkin & L.E. El'sgol'ts, Intro. to the Theory and Applics. of Diff. Eqns. with Deviating Arguments, Math. in Sci. and Eng. Vol. 105, Academic Press (1973), p. 43.

Discontinuities: An n-th order discontinuity at $t = \xi_n$, where $\xi_0 = -1$, $\xi_1 = 0$ and $\xi_2 = 2$.

Other information: The solution of this DDE bifurcates at the point $t = 0$.

Equation: 1.3.12

$$\begin{aligned} y'(t) &= -y(t - y(t)^2 - 2) + 5 & 0 \leq t \leq \tfrac{125}{121}, \\ Y(t) &= \begin{cases} \frac{9}{2} & t \leq -1, \\ -\frac{1}{2} & -1 < t \leq 0. \end{cases} \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} \frac{1}{2}(t-1) & 0 \leq t \leq 1, \\ \frac{11}{2}(t-1) & 1 \leq t \leq \frac{125}{121}. \end{cases}$$

Sources: S.B. Norkin & L.E. El'sgol'ts, Intro. to the Theory and Applics. of Diff. Eqns. with Deviating Arguments, Math. in Sci. and Eng. Vol. 105, Academic Press (1973), p. 50.

Discontinuities: A zeroth-order discontinuity at $t = -1$, and first-order discontinuities at $t = 0$ and $t = 1$.

Other information: This DDE has a lacunary interval, starting at $t = \frac{125}{121}$. Thus the numerical solution cannot proceed beyond this point.

1.4 Systems of DDEs

Equation: 1.4.1

$$\begin{aligned} y_1'(t) &= y_1(t-1) + y_2(t) & t \geq 0, \\ y_2'(t) &= y_1(t) - y_1(t-1) & t \geq 0, \\ Y_1(t) &= e^t & t \leq 0, \\ Y_2(0) &= 1 - e^{-1}. \end{aligned}$$

Analytical solution:

$$\begin{aligned} y_1(t) &= e^t & t \geq 0, \\ y_2(t) &= e^t - \exp(t-1) & t \geq 0. \end{aligned}$$

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: No discontinuities.

Other information: This system of DDEs is equivalent to solving a scalar integro-differential equation [see (4.1.2)]. An $(n+1)$-st order discontinuity may occur at $t = n$ in both numerical solutions.

Equation: 1.4.2

$$
\begin{aligned}
y_1'(t) &= -y_1(t)y_2(t-1) + y_2(t-10) && t \geq 0, \\
y_2'(t) &= y_1(t)y_2(t-1) - y_2(t) && t \geq 0, \\
y_3'(t) &= y_2(t) - y_2(t-10) && t \geq 0, \\
Y(t) &= [5,\ 0.1,\ 1]^T && t \leq 0.
\end{aligned}
$$

Analytical solution: Not available.

Sources: E. Hairer, G. Wanner & S.P. Nørsett, Solving Ordinary Diff. Eqns. 1, Springer Series in Comp. Math. Vol. 8 (1980).

Discontinuities: A first-order discontinuity in all solutions at $t = 0$. An $(n+1)$-st order discontinuity in $y_1(t)$ and $y_2(t)$, and an $(n+2)$-nd order discontinuity in $y_3(t)$ at $t = n$ for $1 \leq n \leq 9$. A $(2m+n)$-th order discontinuity in $y_1(t)$ and $y_3(t)$, and a $(2m+n+1)$-st order discontinuity in $y_2(t)$ at $t = 10m+n$ for $m \geq 1$.

Other information: This system of DDEs describes an S-I-R model. A reference solution is $y_1(40) = 0.0912491205663460$, $y_2(40) = 0.0202995003350707$ and $y_3(40) = 5.98845137909849$.

Equation: 1.4.3

$$
\begin{aligned}
y_1'(t) &= \tfrac{21}{2} - \tfrac{y_1(t)}{1+0.0005y_4(t-4)^3} && t \geq 0, \\
y_2'(t) &= \tfrac{y_1(t)}{1+0.0005y_4(t-4)^3} - y_2(t) && t \geq 0, \\
y_3'(t) &= y_2(t) - y_3(t) && t \geq 0, \\
y_4'(t) &= y_3(t) - \tfrac{1}{2}y_4(t) && t \geq 0, \\
Y(t) &= [60,\ 10,\ 10,\ 20]^T && t \leq 0.
\end{aligned}
$$

Analytical solution: Not available.

Sources: M. Okamoto & K. Hayashi, Frequency Conversion Mechanisms in Enzymatic Feedback Systems, J. Theor. Bio. Vol. 108 (1984), pp. 529–537.

Discontinuities: A first-order discontinuity in all solutions at $t = 0$. A $(3n-1)$-st order discontinuity in $y_1(t)$ and $y_2(t)$ at $t = 4n$ for $n \geq 1$. A $3n$-th order discontinuity in $y_3(t)$ at $t = 4n$ for $n \geq 1$. A $(3n+1)$-st order discontinuity in $y_4(t)$ at $t = 4n$.

Other information: This system of DDEs arises in the study of enzyme kinetics; the delay in the model describes the action of an inhibitor molecule. A reference solution is $y_1(160) = 33.9257947111253$, $y_2(160) = 22.7065371313083$, $y_3(160) = 18.8305767317378$ and $y_4(160) = 22.2659907877122$.

Equation: 1.4.4

$$
\begin{array}{rcll}
y_1'(t) &=& y_5(t-1) + y_3(t-1) & t \geq 0, \\
y_2'(t) &=& y_1(t-1) + y_2(t-\frac{1}{2}) & t \geq 0, \\
y_3'(t) &=& y_3(t-1) + y_1(t-\frac{1}{2}) & t \geq 0, \\
y_4'(t) &=& y_5(t-1)y_4(t-1) & t \geq 0, \\
y_5'(t) &=& y_1(t-1) & t \geq 0, \\
Y_1(t) &=& \exp(t+1) & t \leq 0, \\
Y_2(t) &=& \exp(t+\frac{1}{2}) & t \leq 0, \\
Y_3(t) &=& \sin(t+1) & t \leq 0, \\
Y_4(t) &=& \exp(t+1) & t \leq 0, \\
Y_5(t) &=& \exp(t+1) & t \leq 0.
\end{array}
$$

Analytical solution:

$$
\begin{array}{rcll}
y_1(t) &=& e^t - \cos(t) + e & 0 \leq t \leq 1, \\
y_2(t) &=& \left\{ \begin{array}{l} 2e^t + \exp(\frac{1}{2}) - 2 \\ e^t + 2\exp(t-\frac{1}{2}) + t\exp(\frac{1}{2}) - 2t + \frac{3}{2}\exp(\frac{1}{2}) - 3 \end{array} \right. & \begin{array}{l} 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{2} \leq t \leq 1, \end{array} \\
y_3(t) &=& \left\{ \begin{array}{l} \exp(t+\frac{1}{2}) - \cos(t) + 1 - \exp(\frac{1}{2}) + \sin(1) \\ -\cos(t) + \exp(t-\frac{1}{2}) - \sin(t-\frac{1}{2}) + (t+\frac{1}{2})e - \exp(\frac{1}{2}) + \sin(1) \end{array} \right. & \begin{array}{l} 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{2} \leq t \leq 1, \end{array} \\
y_4(t) &=& \frac{1}{2}\exp(2t) - \frac{1}{2} + e & 0 \leq t \leq 1, \\
y_5(t) &=& e^t + e - 1 & 0 \leq t \leq 1.
\end{array}
$$

Sources: K.W. Neves, Automatic Integration of Functional Diff. Eqns.: An Approach, ACM Trans. Math. Soft. Vol. 1 (1975), pp. 357–368.

Discontinuities: An $(n+1)$-st order discontinuity in all solutions at $t = n$. Also an $(n+1)$-st order discontinuity at $t = n - \frac{1}{2}$ in $y_1(t)$ for $n \geq 2$, in $y_2(t)$ and $y_3(t)$ for $n \geq 1$, in $y_4(t)$ for $n \geq 4$ and in $y_5(t)$ for $n \geq 3$.

Other information:

Equation: 1.4.5

$$
\begin{array}{rcll}
y_1'(t) &=& 2y_2(t) & t \geq 0, \\
y_2'(t) &=& -y_3(t) + y_1(t-1) & t \geq 0, \\
y_3'(t) &=& 2y_2(t-1) & t \geq 0, \\
Y(t) &=& [0,\, 0,\, 0]^T & t < 0, \\
Y(0) &=& [1,\, 1,\, 1]^T. &
\end{array}
$$

Analytical solution:

$$
\begin{aligned}
y_1(t) &= \begin{cases} 1+2t-t^2 & 0 \le t \le 1, \\ 2 & 1 \le t \le 3, \end{cases} \\
y_2(t) &= \begin{cases} 1-t & 0 \le t \le 1, \\ 0 & 1 \le t \le 3, \end{cases} \\
y_3(t) &= \begin{cases} 1 & 0 \le t \le 1, \\ -2+4t-t^2 & 1 \le t \le 2, \\ 2 & 2 \le t \le 3. \end{cases}
\end{aligned}
$$

Sources: H.T. Banks & F. Kappel, Spline Approximations for Functional Diff. Eqns., J. Diff. Eqns. Vol. 34 (1979), pp. 496–522.

Discontinuities: A $2n$-th order discontinuity in $y_1(t)$ at $t = n$. A zeroth-order discontinuity in $y_2(t)$ at $t = 0$ and a $(2n-1)$-st order discontinuity in $y_2(t)$ at $t = n$ for $n \ge 1$. A zeroth-order discontinuity at $t = 0$, a first-order discontinuity at $t = 1$ and a $(2n-2)$-nd order discontinuity at $t = n$ in $y_3(t)$ for $n \ge 2$.

Other information:

Equation: 1.4.6

$$
\begin{aligned}
y_1'(t) &= y_2(t) & t \ge 0, \\
y_2'(t) &= -y_1(t) - y_2(t-1) & t \ge 0, \\
Y(t) &= [0, \sin(2\pi t)]^T & t \le 0.
\end{aligned}
$$

Analytical solution:

$$
\begin{aligned}
y_1(t) &= \begin{cases} \frac{1}{4\pi^2-1}(\sin(2\pi t) - 2\pi\sin(t)) & 0 \le t \le 1, \\ \frac{2\pi}{4\pi^2-1}(\frac{1}{2}(t+1)\sin(t-1) - \sin(t) + \frac{1}{4\pi^2-1}(\cos(2\pi t) - \cos(t-1))) & 1 \le t \le 2, \end{cases} \\
y_2(t) &= \begin{cases} \frac{2\pi}{4\pi^2-1}(2\cos(\pi t)^2 - \cos(t) - 1) & 0 \le t \le 1, \\ \frac{\pi}{4\pi^2-1}(\sin(t-1) + (t+1)\cos(t-1) - 2\cos(t) + \frac{2\sin(t-1) - 4\pi\sin(2\pi t)}{4\pi^2-1}) & 1 \le t \le 2. \end{cases}
\end{aligned}
$$

Sources: H.T. Banks & F. Kappel, Spline Approximations for Functional Diff. Eqns., J. Diff. Eqns. Vol. 34 (1979), pp. 496–522.

Discontinuities: A first-order discontinuity at $t = 0$ and a $(n+2)$-nd order discontinuity at $t = n$ in $y_1(t)$ for $n \ge 1$. A first-order discontinuity at $t = 0$ and a $(n+1)$-st order discontinuity at $t = n$ in $y_2(t)$ for $n \ge 1$.

Other information: This system of DDEs is equivalent to the second-order scalar DDE which appears in H.T. Banks & F. Kappel's paper.

Equation: 1.4.7

$$
\begin{aligned}
y_1'(t) &= y_2(t) & t \ge 0, \\
y_2'(t) &= 1 - y_2(t-1) - y_1(t) & t \ge 0, \\
Y(t) &= [0, 0]^T & t \le 0.
\end{aligned}
$$

Analytical solution:

$$
\begin{aligned}
y_1(t) &= \begin{cases} 1-\cos(t) & 0 \le t \le 1, \\ 1-\cos(t)+\frac{1}{2}(t-1)\cos(t-1)-\frac{1}{2}\sin(t-1) & 1 \le t \le 2, \end{cases} \\
y_2(t) &= \begin{cases} \sin(t) & 0 \le t \le 1, \\ \sin(t)+\frac{1}{2}(1-t)\sin(t-1) & 1 \le t \le 2. \end{cases}
\end{aligned}
$$

Sources: H.T. Banks & F. Kappel, Spline Approximations for Functional Diff. Eqns., J. Diff. Eqns. Vol. 34 (1979), pp. 496–522.

Discontinuities: A first-order discontinuity at $t = 0$ and an $(n+2)$-nd order discontinuity at $t = n$ in $y_1(t)$ for $n \geq 1$. An $(n+1)$-st order discontinuity at $t = n$ in $y_2(t)$.

Other information:

Equation: 1.4.8

$$
\begin{aligned}
y_1'(t) &= y_2(t) & t \ge 2, \\
y_2'(t) &= -\tfrac{1}{2}y_1(t) - \tfrac{1}{2} + y_1(\tfrac{1}{2}t - \tfrac{\pi}{4})^2 & t \ge 2, \\
Y(t) &= [\sin(t),\, \cos(t)]^T & t \le 2.
\end{aligned}
$$

Analytical solution:

$$
\begin{aligned}
y_1(t) &= \sin(t) & t \ge 2, \\
y_2(t) &= \cos(t) & t \ge 2.
\end{aligned}
$$

Sources: J. Oppelstrup, The RKFHB4 Method for Delay Diff. Eqns., Lect. Notes in Math. 631, Springer, New York (1978), pp. 133–146.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However a first-order discontinuity may occur in both solutions at $t = 2$. Also a $(2n+1)$-st order discontinuity may occur in $y_1(t)$ and a $2n$-th order discontinuity may occur in $y_2(t)$ at $t = 2^{n+1} + \frac{(2^n-1)\pi}{2}$ for $n \geq 1$.

Equation: 1.4.9

$$
\begin{aligned}
y_1'(t) &= y_3(t) & t \ge 0, \\
y_2'(t) &= y_4(t) & t \ge 0, \\
y_3'(t) &= -2my_2(t) + (1+m^2)(-1)^m y_1(t-\pi) & t \ge 0, \\
y_4'(t) &= -2my_1(t) + (1+m^2)(-1)^m y_2(t-\pi) & t \ge 0, \\
Y_1(t) &= \sin(t)\cos(mt) & t \le 0, \\
Y_2(t) &= \cos(t)\sin(mt) & t \le 0, \\
Y_3(t) &= \cos(t)\cos(mt) - m\sin(t)\sin(mt) & t \le 0, \\
Y_4(t) &= m\cos(t)\cos(mt) - \sin(t)\sin(mt) & t \le 0.
\end{aligned}
$$

Analytical solution:

$$
\begin{aligned}
y_1(t) &= \sin(t)\cos(mt) && t \geq 0, \\
y_2(t) &= \cos(t)\sin(mt) && t \geq 0, \\
y_3(t) &= \cos(t)\cos(mt) - m\sin(t)\sin(mt) && t \geq 0, \\
y_4(t) &= m\cos(t)\cos(mt) - \sin(t)\sin(mt) && t \geq 0.
\end{aligned}
$$

Sources: K. Ito, H.T. Tran & A. Manitius, A Fully-Discrete Spectral Method for Delay Diff. Eqns., SIAM J. Num. Anal. Vol. 28 (1991), pp. 1121–1140.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However a $(2n+1)$-st order discontinuity may occur in $y_1(t)$ and $y_2(t)$ at $t = n\pi$, a first-order discontinuity may occur in $y_3(t)$ and $y_4(t)$ at $t = 0$ and a $2n$-th order discontinuity may occur in $y_3(t)$ and $y_4(t)$ at $t = n\pi$ for $n \geq 1$.

Equation: 1.4.10

$$
\begin{aligned}
y_1'(t) &= -y_1(t - \tfrac{\pi}{2}) && t \geq \tfrac{\pi}{2}, \\
y_2'(t) &= -y_2(t - \tfrac{\pi}{2}) && t \geq \tfrac{\pi}{2}, \\
Y(t) &= [\sin(t),\, \cos(t)]^T && t \leq \tfrac{\pi}{2}.
\end{aligned}
$$

Analytical solution:

$$
\begin{aligned}
y_1(t) &= \sin(t) && t \geq \tfrac{\pi}{2}, \\
y_2(t) &= \cos(t) && t \geq \tfrac{\pi}{2}.
\end{aligned}
$$

Sources: R. Weiner & K. Strehmel, A Type Insensitive Code for Delay Diff. Eqns. based on Adaptive and Explicit Runge-Kutta Interpolation Methods, Comp. Vol. 40 (1988), pp. 255–265.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However an n-th order discontinuity may occur in $y_1(t)$ and $y_2(t)$ at $t = \frac{n\pi}{2}$ for $n \geq 1$.

Equation: 1.4.11

$$
\begin{aligned}
y_1'(t) &= \lambda(y_4(t-1) - y_1(t)) && t \geq 0, \\
y_2'(t) &= \lambda(y_1(t-1) - y_2(t)) && t \geq 0, \\
y_3'(t) &= \lambda(y_2(t-1) - y_3(t)) && t \geq 0, \\
y_4'(t) &= \lambda(y_3(t-1) - y_4(t)) && t \geq 0, \\
Y_1(t) &= \begin{cases} 0 & -1 \leq t \leq -\frac{3}{5}, \\ 10(t + \frac{3}{5}) & -\frac{3}{5} \leq t \leq -\frac{1}{2}, \\ -10(t + \frac{2}{5}) & -\frac{1}{2} \leq t \leq -\frac{2}{5}, \\ 0 & -\frac{2}{5} \leq t \leq 0, \end{cases} \\
Y_2(t) &= 0 && t \leq 0, \\
Y_3(t) &= 0 && t \leq 0, \\
Y_4(t) &= 0 && t \leq 0.
\end{aligned}
$$

Analytical solution: Not available.

Sources: D.R. Willé, The Numerical Solution of Delay Diff. Eqns., Ph.D. thesis, Math. Dept., Univ. of Manchester (1990).

Discontinuities: An $(n+1)$-st order discontinuity in all solutions at $t = n$. Also a $(4n+5)$-th order discontinuity in $y_1(t)$ at $t = \{4n+\frac{17}{5},\, 4n+\frac{7}{2},\, 4n+\frac{18}{5}\}$, a $(4n+2)$-nd order discontinuity in $y_2(t)$ at $t = \{4n+\frac{2}{5},\, 4n+\frac{1}{2},\, 4n+\frac{3}{5}\}$, a $(4n+3)$-rd order discontinuity in $y_3(t)$ at $t = \{4n+\frac{7}{5},\, 4n+\frac{3}{2},\, 4n+\frac{8}{5}\}$ and a $(4n+4)$-th order discontinuity in $y_4(t)$ at $t = \{4n+\frac{12}{5},\, 4n+\frac{5}{2},\, 4n+\frac{13}{5}\}$.

Other information: For $\lambda = 15$ a reference solution is $y_1(12) = 0.08812591008676$, $y_2(12) = 3.880069454506 \times 10^{-14}$, $y_3(12) = 3.4873724292788 \times 10^{-8}$ and $y_4(12) = 2.4533504838402 \times 10^{-3}$.

Equation: 1.4.12

$$
\begin{array}{rcll}
y_1'(t) & = & y_2(t) & t \geq 0, \\
y_2'(t) & = & 2y_1(t-\frac{\pi}{2}) + \exp(\sin(t))(\cos(t)^2 - \sin(t)) - 2\exp(-\cos(t)) & t \geq 0, \\
Y(t) & = & [\exp(\sin(t)),\, \cos(t)\exp(\sin(t))]^T & t \leq 0.
\end{array}
$$

Analytical solution:

$$
\begin{array}{rcll}
y_1(t) & = & \exp(\sin(t)) & t \geq 0, \\
y_2(t) & = & \cos(t)\exp(\sin(t)) & t \geq 0.
\end{array}
$$

Sources: A. Bellen, The Collocation Method for the Numerical Approximation of the Periodic Solutions of Functional Diff. Eqns., Comp. Vol. 23 (1979), pp. 55–60.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However a $(2n+1)$-st order discontinuity may occur in $y_1(t)$ at $t = \frac{n\pi}{2}$, a first-order discontinuity may occur in $y_2(t)$ at $t = 0$ and a $2n$-th order discontinuity may occur in $y_2(t)$ at $t = \frac{n\pi}{2}$ for $n \geq 1$.

Equation: 1.4.13

$$
\begin{array}{rcll}
y_1'(t) & = & y_2(t) & t \geq 0, \\
y_2'(t) & = & by_2(t-\tau)^3 - a_1y_2(t) - a_2y_2(t-\tau) - a_3y_1(t) & t \geq 0, \\
Y(t) & = & [\frac{1}{2},\, 2\pi\cos(20\pi t)]^T & t \leq 0.
\end{array}
$$

Analytical solution: Not available.

Sources: H.J. Oberle & H.J. Pesch, Numerical Treatment of Delay Diff. Eqns. by Hermite Interpolation, Numer. Math. Vol. 37 (1981), pp. 235–255.

Discontinuities: A first-order discontinuity in $y_1(t)$ at $t = 0$, an $(n+2)$-nd order discontinuity in $y_1(t)$ at $t = n\tau$ for $n \geq 1$ and an $(n+1)$-st order discontinuity in $y_2(t)$ at $t = n\tau$.

Other information: This system of DDEs represents a second-order DDE which models the propagation of a delayed impulse in an electric circuit. For $a_1 = 10$, $a_2 = 25$, $a_3 = 100$, $b = \frac{1}{20}$ and $\tau = \frac{1}{10}$ a reference solution is $y_1(10) = -0.573584156438595$ and $y_2(10) = 1.119559210717439$.

Equation: 1.4.14

$$
\begin{array}{rcll}
y_1'(t) &=& \frac{1.1}{1+\sqrt{10}(y_1(t-20))^{\frac{5}{4}}} - \frac{10y_1(t)}{1+40y_2(t)} & t \geq 0, \\
y_2'(t) &=& \frac{100y_1(t)}{1+40y_2(t)} - 2.43y_2(t) & t \geq 0, \\
Y(t) &=& \frac{1}{3}[1.05767027,\ 1.030713491]^T & t \leq 0.
\end{array}
$$

Analytical solution: Not available.

Sources: H.J. Oberle & H.J. Pesch, Numerical Treatment of Delay Diff. Eqns. by Hermite Interpolation, Numer. Math. Vol. 37 (1981), pp. 235–255.

Discontinuities: An $(n+1)$-st order discontinuity in $y_1(t)$ at $t = 20n$, a first-order discontinuity in $y_2(t)$ at $t = 0$ and an $(n+2)$-nd order discontinuity in $y_2(t)$ at $t = 20n$.

Other information: This system of DDEs describes a model of chronic granulocytic leukaemia. A reference solution is $y_1(100) = 0.0876801107411822$ and $y_2(100) = 0.293768594326244$.

Equation: 1.4.15

$$
\begin{array}{rcll}
y_1'(t) &=& 2k\exp(-D\tau)y_3(t-\tau)y_1(t-\tau) - (ky_3(t)+D)y_1(t) & t \geq 0, \\
y_2'(t) &=& ky_3(t)y_1(t) - k\exp(-D\tau)y_3(t-\tau)y_1(t-\tau) - Dy_2(t) & t \geq 0, \\
y_3'(t) &=& (s_0 - y_3(t))D - \alpha_0 y_3(t)y_1(t) - \alpha_1 y_2(t), & t \geq 0, \\
Y(t) &=& [1.93,\ 3.87,\ 0.24]^T & t \leq 0,
\end{array}
$$

where $k = 10$, $s_0 = 10$, $\alpha_0 = 10$, $\alpha_1 = 0.1$, $\tau = 1$ and $D = 0.55$.

Analytical solution: Not available.

Sources: R. Weiner & K. Strehmel, A Type Insensitive Code for Delay Diff. Eqns. based on Adaptive and Explicit Runge-Kutta Interpolation Methods, Comp. Vol. 40 (1988), pp. 255–265.

Discontinuities: A first-order discontinuity in all solutions at $t = 0$. Also an $(n+1)$-st order discontinuity in $y_1(t)$ and $y_2(t)$ at $t = n$ and an $(n+2)$-nd order discontinuity in $y_3(t)$ at $t = n$ for $n \geq 1$.

Other information: These DDEs describe a two-state microbial growth model for continuous fermentation. A reference solution is $y_1(100) = 1.37800269802843$, $y_2(100) = 3.78795152200247$ and $y_3(100) = 0.357375799420731$.

Equation: 1.4.16

$$
\begin{array}{rcll}
y_1'(t) &=& 0.1(\exp(-0.02y_3(t-1)) - \exp(-0.02y_3(t))) & t \geq 0, \\
y_2'(t) &=& 3.05 - 0.1\exp(-0.02y_3(t-1)) - y_2(t) & t \geq 0, \\
y_3'(t) &=& 4.2\exp(-0.05y_2(t)) - y_3(t) & t \geq 0, \\
Y(t) &=& [1,\ 0.1,\ 1]^T & t \leq 0.
\end{array}
$$

Analytical solution: Not available.

Sources: H.G. Bock & J.P. Schlöder, Numerical Solution of Retarded Diff. Eqns. with State-Dependent Time Lags, Z. Angew. Math. Mech. Vol. 61 (1981), pp. 269–271.

Discontinuities: A first-order discontinuity in $y_1(t)$ and $y_2(t)$ at $t = 0$, a $2n$-th order discontinuity in $y_1(t)$ and $y_2(t)$ at $t = n$ for $n \geq 1$ and a $(2n + 1)$-st order discontinuity in $y_3(t)$ at $t = n$.

Other information: These DDEs model erythropoesis in the circulation of blood. A reference solution is $y_1(20) = 1.00500911848864$, $y_2(20) = 2.95698924566871$ and $y_3(20) = 3.62275601796740$.

Equation: 1.4.17

$$\begin{array}{rcll} y_1'(t) & = & y_2(t) & t \geq 1, \\ y_2'(t) & = & (\exp(1 - y_1(t)) - t)y_2(t - \exp(1 - y_1(t)))y_2(t)^2 & t \geq 1, \\ Y(t) & = & [\ln(t),\, \frac{1}{t}]^T & 0 < t \leq 1. \end{array}$$

Analytical solution:

$$\begin{array}{rcll} y_1(t) & = & \ln(t) & t \geq 1, \\ y_2(t) & = & \frac{1}{t} & t \geq 1. \end{array}$$

Sources: A.N. Al-Mutib, An Explicit One-Step Method of Runge-Kutta Type for Solving Delay Diff. Eqns., Utilitas Math. Vol. 31 (1987), pp. 67–80.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However a first-order discontinuity may occur in both solutions at $t = 1$. Also an $(n + 2)$-nd order discontinuity may occur in $y_1(t)$ and an $(n + 1)$-st order discontinuity may occur in $y_2(t)$ at $t = \xi_n$, where $\xi_1 = 2.222870229721045$, $\xi_2 = 3.09979369494714$, $\xi_3 = 3.81274064815172$, $\xi_4 = 4.42679283646861$, etc.

Equation: 1.4.18

$$\begin{array}{rcll} y_1'(t) & = & -r_1 y_1(t) y_2(t) + r_2 y_3(t) & t \geq 0, \\ y_2'(t) & = & -r_1 y_1(t) y_2(t) + \alpha r_1 y_1(t - y_4(t)) y_2(t - y_4(t)) & t \geq 0, \\ y_3'(t) & = & r_1 y_1(t) y_2(t) - r_2 y_3(t) & t \geq 0, \\ y_4'(t) & = & 1 + \frac{3\delta - y_1(t) y_2(t) - y_3(t)}{y_1(t - y_4(t)) y_2(t - y_4(t)) + y_3(t - y_4(t))} \exp(\delta y_4(t)) & t \geq 0, \\ Y(t) & = & [5,\, 0.1,\, 0,\, 0]^T & t \leq 0, \end{array}$$

where $\delta = 0.01$, $\alpha = 3$, $r_1 = 0.02$ and $r_2 = 0.005$.

Analytical solution: Not available.

Sources: J.A. Gatica & P. Waltman, A Threshold Model of Antigen Antibody Dynamics with Fading Memory, Proc. Intl. Conf. Non-Linear Phenomena in Math. Sci. Arlington, Texas (June 16–20 1980), pp. 425–439.

Discontinuities: First-order discontinuities in all solutions at $t = 0$.

Other information: A reference solution is $y_1(40) = 0.259619473196552$, $y_2(40) = 10.4032122566433$, $y_3(40) = 4.74038052680345$ and $y_4(40) = 0.132695506937674$.

Equation: 1.4.19

$$
\begin{array}{rcll}
y_1'(t) &=& (\lambda_1-\sigma_1)(1-r(t))y_1(t)-m_1(1+r(t))y_1(t)+\beta(y_2(t-3)+y_3(t-3))(1-r(t)) & t\geq 0,\\
y_2'(t) &=& (\lambda_2-\sigma_2)(1-r(t))y_2(t)-m_2(1+r(t))y_2(t)+\sigma_1(1-r(t))y_1(t) & t\geq 0,\\
y_3'(t) &=& \lambda_3(1-r(t))y_3(t)-m_3(1+r(t))y_3(t)+\sigma_2(1-r(t))y_2(t) & t\geq 0,\\
y_4'(t) &=& m_1(1+r(t))y_1(t)-(\gamma_1+\mu_1)y_4(t) & t\geq 0,\\
y_5'(t) &=& m_2(1+r(t))y_2(t)-(\gamma_2+\mu_2)y_5(t) & t\geq 0,\\
y_6'(t) &=& m_3(1+r(t))y_3(t)-(\gamma_3+\mu_3)y_6(t) & t\geq 0,\\
y_7'(t) &=& \epsilon(\lambda_1y_1(t)+\lambda_2y_2(t)+\lambda_3y_3(t)+\beta(y_2(t-3)+y_3(t-3))(1-r(t)))- & \\
 & & \quad (m_1y_1(t)+m_2y_2(t)+m_3y_3(t))(1+r(t))(\frac{y_7(t)}{y_1(t)+y_2(t)+y_3(t)})-\eta y_7(t) & t\geq 0,\\
Y(t) &=& [10000,\ 3000,\ 600,\ 250,\ 200,\ 200,\ 2000]^T & t\leq 0,
\end{array}
$$

where $r(t)=\frac{y_1(t)+y_2(t)+y_3(t)}{x_{max}}$, $\lambda_1=0.12$, $\lambda_2=0.05$, $\lambda_3=0.03$, $\sigma_1=0.1$, $\sigma_2=0.025$, $m_1=0.02$, $m_2=0.01$, $m_3=0.005$, $\beta=0.001$, $\mu_1=\mu_2=\mu_3=0.2$, $\gamma_1=\gamma_2=\gamma_3=0.0001$, $\eta=0.266$, $x_{max}=400000$ and $\epsilon=2$.

Analytical solution: Not available.

Sources: L. Tavernini, CTMS User Guide, Math. Div., Comp. Sci. and Systems Design, Univ. of Texas at San Antonio, Texas (1987).

Discontinuities: A first-order discontinuity in all solutions at $t=0$. Also a $2n$-th order discontinuity in $y_1(t)$ and $y_7(t)$ at $t=3n$ and a $(2n+1)$-st order discontinuity in $y_2(t)$, $y_3(t)$, $y_4(t)$, $y_5(t)$ and $y_6(t)$ at $t=3n$.

Other information: This system models the tree biomass dynamics of a closed-canopy forested watershed. $y_1(t)$, $y_2(t)$ and $y_3(t)$ are the total live above-ground biomass density (hg/ha) in years for trees of diameter at breast height of $1.3cm$–$8.8cm$, $8.8cm$–$23.8cm$ and greater than $23.8cm$ respectively. The corresponding standing dead biomass density is represented by $y_4(t)$, $y_5(t)$ and $y_6(t)$. $y_7(t)$ represents the total live root biomass density for all classes. A reference solution is $y_1(500)=2788.63486750721$, $y_2(500)=9804.32821098480$, $y_3(500)=274949.254200888$, $y_4(500)=479.191976738297$, $y_5(500)=851.736632586355$, $y_6(500)=11803.0989856188$ and $y_7(500)=66545.3113112938$.

Equation: 1.4.20

$$
\begin{array}{rcll}
y_1'(t) &=& -ky_1(t-y_2(t)) & t\geq 0,\\
y_2'(t) &=& \frac{|y_1(t-y_2(t))|-|y_1(t)|}{1+|y_1(t-y_2(t))|} & t\geq 0,\\
Y(t) &=& [1,\ \frac{1}{2}]^T & t\leq 0.
\end{array}
$$

Analytical solution: Not available.

Sources: Z. Jackiewicz & E. Lo, The Numerical Integration of Neutral Functional Diff. Eqns. by Fully Implicit One-Step Methods, Tech. Rep. No. 129, Mathematics Dept., Arizona State Univ. (1991).

Discontinuities: Dependent of k.

Other information: The equation for $y_2(t)$ arises from the restriction $\int_{t-\tau}^{t} 1 + |y_1(s)|\, ds = 1$ on the equation for $y_1(t)$. For $k = 2$ a reference solution is $y_1(40) = 0.33905768692880$ and $y_2(40) = 0.64289666786425$. Also for $k = 2$ an n-th order discontinuity occurs in both solutions at $t = \xi_n$, where $\xi_1 = 0$, $\xi_2 = \frac{1}{\sqrt{2}}$, $\xi_3 = 1.2713416574493$, $\xi_4 = 1.9879194328735$, $\xi_5 = 2.5876919714583$, etc.

Equation: 1.4.21

$$
\begin{aligned}
y_1'(t) &= -y_1(t) - H && 0 \le t \le 0.2,\\
y_2'(t) &= 0.9u_2(t-0.02) + 0.1u_2(t) - y_2(t) && 0 \le t \le 0.2,\\
y_3'(t) &= -2y_3(t) - u_1(t)y_3(t-0.015)(y_3(t)+0.125) + 0.25H && 0 \le t \le 0.2,\\
Y(t) &= [0.49,\ 0.002,\ -0.02]T && t \le 0,
\end{aligned}
$$

where $H = (1 + y_1(t))(1 + y_2(t)) \exp\left(\frac{25y_3(t)}{1+y_3(t)}\right) - 1$, $u_i(t) = C_{i,\,[50t+1]}$ and

i/j	0	1	2	3	4	5	6	7	8	9	10
1	0	493	207	-493	157	25	-221	-50	400	293	443
2	0.306	0.847	0.270	0.541	0.180	-0.126	-0.234	-0.414	0	-0.990	-0.036

Analytical solution: Not available.

Sources: M.A. Soliman & W.H. Ray, Optimal control of multivariable systems with pure time delays, Autom. Vol. 7 (1971), pp. 681–691.
H.G. Bock & J.P Schlöder, Numerical Solution of Retarded Diff. Eqns. with State Dependent Time Lags, ZAMM Vol. 61 (1981), pp. 269–271.

Discontinuities: A first-order discontinuity in all solutions at $t = 0$. A first-order discontinuity in $y_2(t)$ at $t = 0.02n$ for $n \ge 0$. An $(n + 2)$-nd order discontinuity in $y_1(t)$ and an $(n + 1)$-st order discontinuity in $y_3(t)$ at $t = 0.015n + 0.02m$. Although the highest order discontinuities that occur in $y_1(t)$ and $y_3(t)$ are orders 5 and 4 respectively.

Other information: This system of DDEs arises from the optimal control of a chemical reactor. The control variables $u_1(t)$ and $u_2(t)$ are clearly discontinuous functions. A reference solution is $y_1(0.2) = 0.0249319667538623$, $y_2(0.2) = 0.00364497681238928$ and $y_3(0.2) = -0.00209576912706439$.

2 Neutral Differential Equations

2.1 Constant-delay scalar NDEs

Equation: 2.1.1

$$\begin{aligned} y'(t) &= y(t) + y(t-1) - \tfrac{1}{4}y'(t-1) & t \geq 0, \\ Y(t) &= -t & t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} -\frac{1}{4} + t + \frac{1}{4}e^t & 0 \leq t \leq 1, \\ \frac{1}{2} - t + \frac{1}{4}e^t + \frac{17}{16}\exp(t-1) + \frac{3}{16}t\exp(t-1) & 1 \leq t \leq 2. \end{cases}$$

Sources: F. Kappel & K. Kunisch, Spline Approximations for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 18 (1981), pp. 1058–1080.

Discontinuities: A first-order discontinuity at $t = n$ for $n \geq 0$.

Other information: Also see (2.1.2), (2.1.3) and (2.1.4).

Equation: 2.1.2

$$\begin{aligned} y'(t) &= y(t) + y(t-1) - 2y'(t-1) & t \geq 0, \\ Y(t) &= -t & t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} -2 + t + 2e^t & 0 \leq t \leq 1, \\ 4 - t + 2e^t - 2(t+1)\exp(t-1) & 1 \leq t \leq 2. \end{cases}$$

Sources: F. Kappel & K. Kunisch, Spline Approximations for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 18 (1981), pp. 1058–1080.

Discontinuities: A first-order discontinuity at $t = n$ for $n \geq 0$.

Other information: Also see (2.1.1), (2.1.3) and (2.1.4).

Equation: 2.1.3

$$\begin{aligned} y'(t) &= y(t) + y(t-1) - \tfrac{1}{4}y'(t-1) + \sin(t) & t \geq 0, \\ Y(t) &= -t & t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} -\frac{1}{4} + t + \frac{3}{4}e^t - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) & 0 \leq t \leq 1, \\ \frac{1}{2} - t + \frac{3}{4}e^t + \frac{3}{16}(3t+1)e^{t-1} + \frac{1}{2}(\cos(t-1) - \cos(t) - \sin(t)) + \frac{1}{8}\sin(t-1) & 1 \leq t \leq 2. \end{cases}$$

Sources: F. Kappel & K. Kunisch, Spline Approximations for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 18 (1981), pp. 1058–1080.

Discontinuities: A first-order discontinuity at $t = n$ for $n \geq 0$.

Other information: Also see (2.1.1), (2.1.2) and (2.1.4).

Equation: 2.1.4

$$\begin{aligned} y'(t) &= y(t) + y'(t-1) && t \geq 0, \\ Y(t) &= 1 && t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \begin{cases} e^t & 0 \leq t \leq 1, \\ (t-1)\exp(t-1) + e^t & 1 \leq t \leq 2, \\ \frac{1}{2}(t^2 - 2t)\exp(t-2) + (t-1)\exp(t-1) + e^t & 2 \leq t \leq 3, \\ \frac{1}{6}(t^3 - 3t^2 - 3t + 9)\exp(t-3) + \frac{1}{2}(t^2 - 2t)\exp(t-2) + (t-1)\exp(t-1) + e^t & 3 \leq t \leq 4. \end{cases}$$

Sources: Z. Jackiewicz & E. Lo, The Numerical Integration of Neutral Functional-Diff. Eqns. by Fully Implicit One-Step Methods, Tech. Rep. No. 129, Mathematics Dept., Arizona State Univ. (1991).

Discontinuities: A first-order discontinuity at $t = n$ for $n \geq 0$.

Other information: Also see (2.1.1), (2.1.2) and (2.1.3).

Equation: 2.1.5

$$\begin{aligned} y'(t) &= ry(t)(1 - y(t-1) - cy'(t-1)) && t \geq 0, \\ Y(t) &= 2 + t && t \leq 0. \end{aligned}$$

Analytical solution: Not available.

Sources: M.A. Feldstein & Y. Kuang, Boundedness of Solutions of a Non-Linear Non-Autonomous Neutral Delay Eqns., J. Math. Anal. Appl. Vol. 156 (1991), pp. 293–304.

Discontinuities: A first-order discontinuity at $t = n$ for $n \geq 0$.

Other information: This equation models a food-limited population. For $r = \frac{\pi}{\sqrt{3}} + \frac{1}{20}$ and $c = \frac{\sqrt{3}}{2\pi} - \frac{1}{25}$ a reference solution is $y(40) = 0.804413836191349$.

Equation: 2.1.6

$$\begin{aligned} y'(t) &= \frac{1}{c}\left(-y(t)^3 - ky(t-h)^3 + k(\tfrac{1}{z} + g)y(t-h) - (\tfrac{1}{z} - g)y(t)\right) - ky'(t-h) && t \geq 0, \\ Y(t) &= \frac{1}{20} && t \leq 0, \end{aligned}$$

where $c = 10$, $g = \frac{1}{100}$, $h = 2$, $R = \frac{7}{2}$, $z = 25$ and $k = \frac{R-z}{R+z}$.

Analytical solution: Not available.

Sources: R.K. Brayton, Non-Linear Oscillations in a Distributed Network, Quart. Appl. Math. Vol. 24 (1967), pp. 289–301.

Discontinuities: A first-order discontinuity at $t = nh$ for $n \geq 0$.

Other information: This NDE models a circuit consisting of a lossless transmission line terminated by lumped circuits, one of which is non-linear. For the given parameters a reference solution is $y(20) = 0.0323558655290430$.

Equation: 2.1.7

$$\begin{aligned} y'(t) &= y'(t-1) && t \geq 0, \\ Y(t) &= (t+1)^5 && t < 0, \\ Y(0) &= 0. \end{aligned}$$

Analytical solution:

$$y(t) = [t] + (t - [t])^5 \quad t \geq 0.$$

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: A zeroth order discontinuity at $t = 0$ and first-order discontinuities at $t = n$ for $n \geq 0$.

Other information:

2.2 Varying-delay scalar NDEs

Equation: 2.2.1

$$\begin{aligned} y'(t) &= 1 + y(t) - 2y(\tfrac{1}{2}t)^2 - y'(t-\pi) && t \geq 0, \\ Y(t) &= \cos(t) && t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = \cos(t) \quad t \geq 0.$$

Sources: Z. Jackiewicz, Adams Methods for Neutral Functional Diff. Eqns., Numer. Math. Vol. 39 (1982), pp. 615–626.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However a first-order discontinuity may occur at $t = n\pi$ for $n \geq 0$.

Equation: 2.2.2

$$\begin{aligned} y'(t) &= \exp(1 - 2t^2) y(t^2) (y'(t - \tfrac{1}{t+1}))^{t+1} && 0 \leq t \leq 1, \\ Y(t) &= e^t && t \leq 0. \end{aligned}$$

Analytical solution:

$$y(t) = e^t \quad 0 \le t \le 1.$$

Sources: Z. Jackiewicz, One-Step Methods of any Order for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 21 (1984), pp. 486–511.
Z. Jackiewicz, Quasi-Linear Multistep Methods and Variable Step Predictor-Corrector for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 23 (1986), pp. 423–452.

Discontinuities: No discontinuities.

Other information: The analytical solution is an analytic continuation of the initial function. However an $(n+1)$-st order discontinuity may occur at $t = \sqrt[2^n]{s}$, where $s = \sqrt{\frac{5}{4}} - \frac{1}{2}$.

Equation: 2.2.3

$$\begin{aligned} y'(t) &= 1 - 2y(\tfrac{1}{2}t)^2 - (1+\cos(t))H(t-\pi)(1-2y(\tfrac{1}{2}t)^2) - (1+\cos(t))y'(t-\pi) && t \ge 0, \\ Y(t) &= 0 && t \le 0, \end{aligned}$$

where $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \ge 0$.

Analytical solution:

$$y(t) = \sin(t) \quad t \ge 0.$$

Sources: Z. Jackiewicz, Adams Methods for Neutral Functional Diff. Eqns., Numer. Math. Vol. 39 (1982), pp. 615–626.
Z. Jackiewicz, Quasi-Linear Multistep Methods and Variable Step Predictor-Corrector for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 23 (1986), pp. 423–452.

Discontinuities: A first-order discontinuity at $t = n\pi$.

Other information: This NDE has a discontinuous derivative function due to $H(t)$.

Equation: 2.2.4

$$\begin{aligned} y'(t) &= y'(2t - \tfrac{1}{2}) && \tfrac{1}{4} \le t \le \tfrac{1}{2}, \\ Y(t) &= \exp(-t^2) && t \le \tfrac{1}{4}. \end{aligned}$$

Analytical solution:

$$y(t) = y_i(t) = \frac{1}{2^i}\exp(-4^i t^2 + B_i t + C_i) + K_i,$$

for $t \in [t_i, t_{i+1}]$ where $t_i = \frac{1}{2}(1 - 2^{-i})$ and

$$\begin{aligned} B_i &= 2(4^{i-1} + B_{i-1}), \\ C_i &= -4^{i-2} - \tfrac{1}{2}B_{i-1} + C_{i-1}, \\ K_i &= -(\tfrac{1}{2^i}\exp(-4^i t_i^2 + B_i t_i + C_i) - y_{i-1}(t_i)), \end{aligned}$$

with $B_0 = C_0 = K_0 = 0$.

Sources: K.W. Neves & S. Thompson, Software for the Numerical Solution of Systems of Functional Diff. Eqns. with State-Dependent Delays, App. Num. Math. Vol. 9 (1992), pp. 385–401.

Discontinuities: A first-order discontinuity at $t = \{t_i\}$ for $i \geq 1$.

Other information: A reference solution is $y(\frac{1}{2}) = 0.8788261256272320$.

2.3 State-dependent delay scalar NDEs

Equation: 2.3.1

$$\begin{array}{rcll} y'(t) & = & -y'(t - \frac{y(t)^2}{4}) & t \geq 0, \\ Y(t) & = & 1 - t & t \leq 0. \end{array}$$

Analytical solution:

$$y(t) = t + 1 \qquad t \geq 0,$$

OR

$$y(t) = \begin{cases} t + 1 & 0 \leq t \leq 1, \\ 3 - t & 1 \leq t \leq 5 - \sqrt{12}. \end{cases}$$

Sources: R.D. Driver, A Functional Diff. System of Neutral Type Arising in a Two-Body Problem of Classical Electrodynamics, Intl. Symp. on Non-Linear Diff. Eqns. and Non-linear Mechanics, Academic Press, New York (1963), pp. 474–484.
S.B. Norkin & L.E. El'sgol'ts, Intro. to the Theory and Applics. of Diff. Eqns. with Deviating Arguments, Math. in Sci. and Eng. Vol. 105, Academic Press (1973), p. 44.

Discontinuities: A first-order discontinuity at $t = 0$, and in the case of the second possible solution a first-order discontinuity at $t = 1$.

Other information: The solution bifurcates at the point $t = 1$. One solution is valid for $t \geq 0$, whereas the other solution terminates at the point $t = 5 - \sqrt{12}$. The factor that determines which branch of the solution is followed is the size of the numerical solution at $t = 1$. If $y(1) \leq 2$ the solution obtained for $t > 1$ is $y(t) = t + 1$, whereas if $y(1) > 2$ the solution obtained is $y(t) = 3 - t$.

Equation: 2.3.2

$$\begin{array}{rcll} y'(t) & = & \frac{-4ty(t)^2}{\ln(\cos(2t))^2+4} + \tan(2t) + \frac{1}{2}\arctan(y'\left(\frac{ty(t)^2}{1+y(t)^2}\right)) & 0 \leq t \leq \frac{\pi}{4}, \\ Y(0) & = & 0, & \\ Y'(0) & = & 0. & \end{array}$$

Analytical solution:

$$y(t) = -\frac{1}{2}\ln(\cos(2t)) \quad 0 \leq t \leq \frac{\pi}{4}.$$

Sources: R.N. Castleton & L.J. Grimm, A First-Order Method for Diff. Eqns. of Neutral Type, Math. Comp. Vol. 27 (1973), pp. 571–577.

Discontinuities: No discontinuities.

Other information: This is an *initial value* NDE.

Equation: 2.3.3

$$\begin{aligned} y'(t) &= \cos(t)(1+y(ty(t)^2)) + y(t)y'(ty(t)^2) - \sin(t+t\sin(t)^2) \qquad 0 \le t \le \tfrac{\pi}{2}, \\ Y(0) &= 0, \\ Y'(0) &= 1. \end{aligned}$$

Analytical solution:

$$y(t) = \sin(t) \quad 0 \le t \le \frac{\pi}{2}.$$

Sources: R.N. Castleton & L.J. Grimm, A First-Order Method for Diff. Eqns. of Neutral Type, Math. Comp. Vol. 27 (1973), pp. 571–577.

Discontinuities: No discontinuities.

Other information: This is an *initial value* NDE.

Equation: 2.3.4

$$\begin{aligned} y'(t) &= -y'(y(t)-2) \qquad & t \ge 0, \\ Y(t) &= 1-t & t \le 0. \end{aligned}$$

Analytical solution:

$$y(t) = t+1 \quad 0 \le t \le 1.$$

Sources: S.B. Norkin & L.E. El'sgol'ts, Intro. to the Theory and Applics. of Diff. Eqns. with Deviating Arguments, Math. in Sci. and Eng. Vol. 105, Academic Press (1973), pp. 44–45.

Discontinuities: A first-order discontinuity at $t = 0$.

Other information: This NDE has a solution which terminates at $t = 1$.

Equation: 2.3.5

$$\begin{aligned} y'(t) &= \tfrac{3}{25}y(t) - \tfrac{11}{10}y'(t-1) \qquad & t \ge 0, \\ Y(t) &= 1 & t \le 0. \end{aligned}$$

Analytical solution: Not available.

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: A first-order discontinuity at $t = n$ for $n \ge 0$.

Other information: This NDE has a (slowly) exponentially increasing oscillatory solution. A reference solution is $y(100) = 603.643886313941$.

2.4 Systems of NDEs

Equation: 2.4.1

$$\begin{array}{rcll} y_1'(t) & = & y_1(t-\frac{\pi}{2}) + y_1'(t-\pi) - y_2'(t-\frac{\pi}{2}) & t \geq 0, \\ y_2'(t) & = & 2y_1(t-\frac{\pi}{2}) + y_2(t-\pi) + y_1'(t-\pi) & t \geq 0, \\ Y(t) & = & [\cos(t),\, \sin(t)]^T & t \leq 0. \end{array}$$

Analytical solution:

$$\begin{array}{rcl} y_1(t) & = & \begin{cases} 2-\cos(t) & 0 \leq t \leq \frac{\pi}{2}, \\ 2t + 2\sin(t) - \pi & \frac{\pi}{2} \leq t \leq \pi, \end{cases} \\ y_2(t) & = & \begin{cases} 2 - 2\cos(t) & 0 \leq t \leq \frac{\pi}{2}, \\ 2(2t + \cos(t) + 1 - \pi) & \frac{\pi}{2} \leq t \leq \pi. \end{cases} \end{array}$$

Sources: Z. Jackiewicz & E. Lo, The Numerical Solution of Neutral Functional Diff. Eqns. by Adams Predictor-Corrector Methods, Tech. Rep. No. 118, Math. Dept., Arizona State Univ. (1988).

Discontinuities: A first-order discontinuity in $y_1(t)$ at $t = \frac{n\pi}{2}$, a first-order discontinuity in $y_2(t)$ at $t = \frac{n\pi}{2}$ but a second-order discontinuity in $y_2(t)$ at $t = \frac{\pi}{2}$.

Other information:

Equation: 2.4.2

$$\begin{array}{rcll} y_1'(t) & = & y_2(t) + y_1(t-1) - 2y_2(t-1) - 2y_3(t-2) - 4y_2'(t-1) - 4y_3'(t-2) & t \geq 0, \\ y_2'(t) & = & -y_1'(t-1) + 2y_2'(t-1) + 2y_3'(t-2) & t \geq 0, \\ y_3'(t) & = & y_2(t) + y_1(t-1) - 2y_2(t-1) - 2y_3(t-2) - 4y_2'(t-1) - 4y_3'(t-2) & t \geq 0, \\ Y(t) & = & [t,\, \cos(t),\, t]^T & t \leq 0. \end{array}$$

Analytical solution:

$$\begin{array}{rcl} y_1(t) & = & \begin{cases} 4\cos(1) - 2t\cos(1) - 4\cos(t-1) & 0 \leq t \leq 1, \\ 12\cos(1) - 4t - 2t\cos(1) - 8\cos(t-2) & 1 \leq t \leq 2, \end{cases} \\ y_2(t) & = & \begin{cases} 1 - 2\cos(1) + t + 2\cos(t-1) & 0 \leq t \leq 1, \\ -12\cos(1) + 4t + 2t\cos(1) + 8\cos(t-2) & 1 \leq t \leq 2, \end{cases} \\ y_3(t) & = & \begin{cases} 4\cos(1) - 2t\cos(1) - 4\cos(t-1) & 0 \leq t \leq 1, \\ 12\cos(1) - 4t - 2t\cos(1) - 8\cos(t-2) & 1 \leq t \leq 2. \end{cases} \end{array}$$

Sources: F. Kappel & K. Kunisch, Spline Approximations for Neutral Functional Diff. Eqns., SIAM J. Num. Anal. Vol. 18 (1981), pp. 1058–1080.

Discontinuities: A first-order discontinuity in all solutions at $t = n$ for $n \geq 0$.

Other information:

Equation: 2.4.3

$$
\begin{aligned}
y_1'(t) &= y_1(t)(1 - y_1(t-\tau) - \rho y_1'(t-\tau)) - \tfrac{y_2(t)y_1(t)^2}{y_1(t)^2+1} && t \geq 0, \\
y_2'(t) &= y_2(t)(\tfrac{y_1(t)^2}{y_1(t)^2+1} - \alpha) && t \geq 0, \\
Y_1(t) &= \tfrac{33}{100} - \tfrac{1}{10}t && t \leq 0, \\
Y_2(t) &= \tfrac{111}{50} + \tfrac{1}{10}t && t \leq 0.
\end{aligned}
$$

Analytical solution: Not available.

Sources: Y. Kuang, On Neutral Delay Logistic Gauss-Type Predator-Prey Systems, Dynamics and Systems Stability *to appear*.

Discontinuities: A first-order discontinuity in $y_1(t)$ at $t = n\tau$, a first-order discontinuity in $y_2(t)$ at $t = 0$ and a second-order discontinuity in $y_2(t)$ at $t = n\tau$ for $n \geq 1$.

Other information: This system of NDEs is a Gauss-type predator-prey model. For $\alpha = \frac{1}{10}$, $\rho = \frac{29}{10}$ and $\tau = \frac{21}{50}$ a reference solution is $y_1(6) = 0.32932056083118$ and $y_2(6) = 2.2485459883066$.

Equation: 2.4.4

$$
\begin{aligned}
y_1'(t) &= 3.05 - 0.1\exp(-0.02y_3(t)) - (1-\tau'(t))(3.05 - 0.1\exp(-0.02y_3(t-\tau(t)))) && t \geq 0, \\
y_2'(t) &= (1-\tau'(t))(3.05 - 0.1\exp(-0.02y_3(t-\tau(t)))) - y_2(t) && t \geq 0, \\
y_3'(t) &= 4.2\exp(-0.05y_2(t)) - y_3(t) && t \geq 0, \\
Y(t) &= [1,\ 0.1,\ 1]^T && t \leq 0,
\end{aligned}
$$

where $\tau(t) = 0.2 + 1.2\exp(-0.001y_3(t))$.

Analytical solution: Not available.

Sources: H.G. Bock & J.P. Schlöder, Numerical Solution of Retarded Diff. Eqns. with State-Dependent Time Lags, Z. Angew. Math. Mech. Vol. 61 (1981), pp. 269–271.

Discontinuities: A first-order discontinuity in all solutions at $t = 0$. Also a $2n$-th order discontinuity in $y_1(t)$ and $y_2(t)$ and a $(2n+1)$-st order discontinuity in $y_3(t)$ at $t = \xi_n$, where $\xi_1 = 1.3962147007789$, $\xi_2 = 2.7919588463920$, $\xi_3 = 4.1876268693253$, $\xi_4 = 5.5832859953774$, etc.

Other information: A reference solution is $y_1(20) = 0.997721687800035$, $y_2(20) = 2.95698924553131$ and $y_3(20) = 3.62275601804233$.

3 Integro-Differential Equations

3.1 Constant-delay scalar IDEs

Equation: 3.1.1

$$\begin{aligned} y'(t) &= \int_0^t \cos(s)\,ds \qquad t \geq 0, \\ Y(0) &= -1. \end{aligned}$$

Analytical solution:

$$y(t) = -\cos(t) \quad t \geq 0.$$

Sources: H. Brunner & P.J. van der Houwen, The Numerical Solution of Volterra Eqns., North Holland, Amsterdam (1986).

Discontinuities: No discontinuities.

Other information: A simple Volterra equation with no dependence on the solution.

Equation: 3.1.2

$$\begin{aligned} y'(t) &= t\int_0^t s\cos(s^2)\,ds \qquad t \geq 0, \\ Y(0) &= -\tfrac{1}{4}. \end{aligned}$$

Analytical solution:

$$y(t) = -\frac{1}{4}\cos(t^2) \quad t \geq 0.$$

Sources: M.A. Feldstein & J. Sopka, Numerical Methods for Non-Linear Volterra Integro-Diff. Eqns., SIAM J. Num. Anal. Vol. 11 (1974), pp. 826–846.

Discontinuities: No discontinuities.

Other information: A simple Volterra equation with no dependence on the solution.

Equation: 3.1.3

$$\begin{aligned} y'(t) &= 1 - t\exp(-t^2) + y(t) - 2t\int_0^t s\exp(-y(s)^2)\,ds \qquad t \geq 0, \\ Y(0) &= 0. \end{aligned}$$

Analytical solution:

$$y(t) = t \quad t \geq 0.$$

Sources: P. Linz, Linear Multistep Methods for Volterra Integro-Diff. Eqns., J. Assoc. Comput. Mach. Vol. 16 (1969), pp. 295–301.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.4

$$\begin{aligned} y'(t) &= \tfrac{5}{2}t - \tfrac{1}{2}t\exp(t^2) + t\int_0^t s\exp(y(s))\,ds \qquad t \geq 0, \\ Y(0) &= 0. \end{aligned}$$

Analytical solution:

$$y(t) = t^2 \quad t \geq 0.$$

Sources: M.A. Feldstein & J. Sopka, Numerical Methods for Non-Linear Volterra Integro-Diff. Eqns., SIAM J. Num. Anal. Vol. 11 (1974), pp. 826–846.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.5

$$\begin{aligned} y'(t) &= -t - \tfrac{1}{(1+t)^2} + \tfrac{1}{y(t)}\ln\left(\tfrac{1+t}{1+\frac{1}{2}t}\right) + \int_0^t \tfrac{1}{1+(1+t)y(s)}\,ds \qquad t \geq 0, \\ Y(0) &= 1. \end{aligned}$$

Analytical solution:

$$y(t) = \frac{1}{1+t} \quad t \geq 0.$$

Sources: H. Brunner & J.D. Lambert, Stability of Numerical Methods for Volterra Integro-Diff. Eqns., Comp. Vol. 12 (1974), pp. 75–89.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.6

$$\begin{aligned} y'(t) &= 1 + 2t - y(t) + t(1+2t)\int_0^t \exp(st - s^2)y(s)\,ds \qquad t \geq 0, \\ Y(0) &= 1. \end{aligned}$$

Analytical solution:

$$y(t) = \exp(t^2) \quad t \geq 0.$$

Sources: P. Linz, Linear Multistep Methods for Volterra Integro-Diff. Eqns., J. Assoc. Comput. Mach. Vol. 16 (1969), pp. 295–301.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.7

$$\begin{aligned} y'(t) &= 2t - \tfrac{1}{2}\sin(t^4) + t^2 \int_0^t s\cos(t^2 y(s))\, ds \qquad t \geq 0, \\ Y(0) &= 0. \end{aligned}$$

Analytical solution:

$$y(t) = t^2 \quad t \geq 0.$$

Sources: M.A. Feldstein & J. Sopka, Numerical Methods for Non-Linear Volterra Integro-Diff. Eqns., SIAM J. Num. Anal. Vol. 11 (1974), pp. 826–846.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.8

$$\begin{aligned} y'(t) &= (g(t) - ay(t) - b\int_0^t (t+cs)^d y(s)^3\ ds)^3 - 1 \qquad t \geq 0, \\ Y(0) &= 1, \end{aligned}$$

where $g(t) = 1 + a + \frac{bt^{d+1}}{c(d+1)}((1+c)^{d+1} - 1)$, and $a = 40$, $b = 15$, $c = 2$ and $d = \frac{3}{2}$.

Analytical solution:

$$y(t) = 1 \quad t \geq 0.$$

Sources: P.H.M. Wolkenfelt, The Construction of Reducible Quadrature Rules for Volterra Integral and Integro-Diff. Eqns., IMA J. Num. Anal. Vol. 2 (1982), pp. 131–152.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.9

$$\begin{aligned} y'(t) &= ((1+t)e^{-t} - y(t))\cos(t) - (1+2t)e^{-t} + (1+t)(1 - \int_0^t \tfrac{1}{1+s} y(s)\, ds) \qquad t \geq 0, \\ Y(0) &= 1. \end{aligned}$$

Analytical solution:

$$y(t) = (1+t)e^{-t} \quad t \geq 0.$$

Sources: H. Brunner High-Order Methods for the Numerical Solution of Volterra Integro-Diff. Eqns., J. Comp. Appl. Math. Vol. 15 (1986), pp. 301–309.

Discontinuities: No discontinuities.

Other information:

Equation: 3.1.10

$$\begin{aligned} y'(t) &= -\tfrac{1}{4}t^3 + \tfrac{5}{4}\exp(-y(t)) + \tfrac{1}{t}\int_1^t s^2 \exp(y(s))\, ds \qquad t \geq 1, \\ Y(1) &= 0. \end{aligned}$$

Analytical solution:

$$y(t) = \ln(t) \quad t \geq 1.$$

Sources: M.A. Feldstein & J. Sopka, Numerical Methods for Non-Linear Volterra Integro-Diff. Eqns., SIAM J. Num. Anal. Vol. 11 (1974), pp. 826–846.

Discontinuities: No discontinuities.

Other information:

3.2 Varying-delay scalar IDEs

3.3 State-dependent delay scalar IDEs

3.4 Systems of IDEs

4 Functional Differential Equations

4.1 Constant-delay scalar FDEs

Equation: 4.1.1

$$
\begin{aligned}
y'(t) &= y'(t-1) + \int_{t-2}^{t-1} y(s)\,ds && t \geq 0,\\
Y(t) &= t && t < 0,\\
Y(0) &= 1.
\end{aligned}
$$

Analytical solution:

$$
y(t) = \begin{cases}
\frac{1}{2}t^2 - \frac{1}{2}t + 1 & 0 \leq t \leq 1,\\
\frac{1}{24}t^4 - \frac{5}{12}t^3 + \frac{5}{2}t^2 - \frac{59}{12}t + \frac{91}{24} & 1 \leq t \leq 2,\\
\frac{1}{720}t^6 - \frac{7}{240}t^5 + \frac{1}{3}t^4 - \frac{37}{18}t^3 + \frac{47}{6}t^2 - \frac{307}{20}t + \frac{4541}{360} & 2 \leq t \leq 3.
\end{cases}
$$

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: A zeroth-order discontinuity at $t = 0$ and first-order discontinuities at $t = n$ for $n \geq 1$.

Other information: The non-smoothed discontinuity can cause *severe* problems in evaluating the integral term.

Equation: 4.1.2

$$
\begin{aligned}
y'(t) &= y(t-1) + \int_{t-1}^{t} y(s)\,ds && t \geq 0,\\
Y(t) &= e^t && t \leq 0.
\end{aligned}
$$

Analytical solution:

$$y(t) = e^t \quad t \geq 0.$$

Sources: C.A.H Paul & C.T.H. Baker, Explicit Runge-Kutta Methods for the Numerical Solution of Singular Delay Diff. Eqns., Tech. Rep. No. 212, Math. Dept., Manchester Univ. (1992).

Discontinuities: No discontinuities.

Other information: However an $(n+1)$-st order discontinuity may occur in the numerical solution at $t = n$.

4.2 Varying-delay scalar FDEs

4.3 State-dependent delay scalar FDEs

Equation: 4.3.1

$$\begin{aligned} y'(t) &= \frac{1}{t^3}\left(\int_{ty(t)}^{t^2y(t)} s^3 y(s) y'(s)\, ds - 1\right) \qquad t \geq 1, \\ Y(1) &= 1. \end{aligned}$$

Analytical solution:

$$y(t) = \frac{1}{t} \quad t \geq 1.$$

Sources: C.A.H. Paul, Runge-Kutta Methods for Functional Diff. Eqns., Ph.D. thesis, Math. Dept., Manchester Univ. (1992).

Discontinuities: No discontinuities.

Other information: This is an *initial value* FDE. Problems clearly arise if $ty(t) > 1$.

4.4 Systems of FDEs

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