Chapter 1

Introduction to Differential Equations

1.1 Definitions and Terminology

1. Second order; linear

2. Third order; nonlinear because of $(dy/dx)^4$

3. Fourth order; linear

4. Second order; nonlinear because of $\cos(r+u)$

5. Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1+(dy/dx)^2}$

6. Second order; nonlinear because of R^2

7. Third order; linear

8. Second order; nonlinear because of $\dot{x}^2$

9. Writing the differential equation in the form $x(dy/dx)+y^2=1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2-1)(dx/dy)+x=0$, we see that it is linear in x.

10. Writing the differential equation in the form $u(dv/du)+(1+u)v=ue^u$ we see that it is linear in v. However, writing it in the form $(v+uv-ue^u)(du/dv)+u=0$, we see that it is nonlinear in u.

11. From $y=e^{-x/2}$ we obtain $y'=-\frac{1}{2}e^{-x/2}$. Then $2y'+y=-e^{-x/2}+e^{-x/2}=0$.

12. From $y=\frac{6}{5}-\frac{6}{5}e^{-20t}$ we obtain $dy/dt=24e^{-20t}$, so that

$$\frac{dy}{dt}+20y=24e^{-20t}+20\left(\frac{6}{5}-\frac{6}{5}e^{-20t}\right)=24.$$

13. From $y=e^{3x}\cos 2x$ we obtain $y'=3e^{3x}\cos 2x-2e^{3x}\sin 2x$ and $y''=5e^{3x}\cos 2x-12e^{3x}\sin 2x$, so that $y''-6y'+13y=0$.

14. From $y = -\cos x \ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x \ln(\sec x + \tan x)$ and $y'' = \tan x + \cos x \ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.

15. The domain of the function, found by solving $x+2 \geq 0$, is $[-2, \infty)$. From $y' = 1+2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+(2(x+2)^{-1/2}]\\ &= y - x + 2(y-x)(x+2)^{-1/2}\\ &= y - x + 2[x + 4(x+2)^{1/2} - x](x+2)^{-1/2}\\ &= y - x + 8(x+2)^{1/2}(x+2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x = -2$.

16. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5\tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$
or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

17. The domain of the function is $\{x \mid 4 - x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ and } x \neq 2\}$. From $y' = 2x/(4-x^2)^2$ we have

$$y' = 2x\left(\frac{1}{4-x^2}\right)^2 = 2xy^2.$$

An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

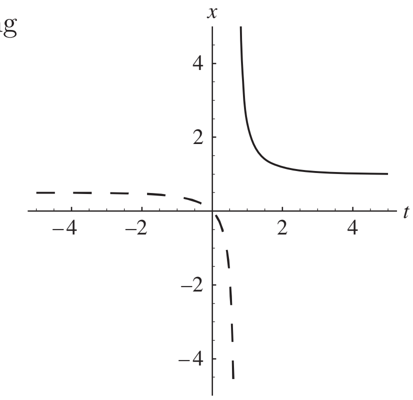
18. The function is $y = 1/\sqrt{1 - \sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1 - \sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1 - \sin x)^{-3/2}\cos x = [(1 - \sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another one is $(5\pi/2, 9\pi/2)$, and so on.

19. Writing $\ln(2X-1)-\ln(X-1)=t$ and differentiating implicitly we obtain

$$\frac{2}{2X-1}\frac{dX}{dt}-\frac{1}{X-1}\frac{dX}{dt}=1$$
$$\left(\frac{2}{2X-1}-\frac{1}{X-1}\right)\frac{dX}{dt}=1$$
$$\frac{2X-2-2X+1}{(2X-1)(X-1)}\frac{dX}{dt}=1$$
$$\frac{dX}{dt}=-(2X-1)(X-1)=(X-1)(1-2X).$$



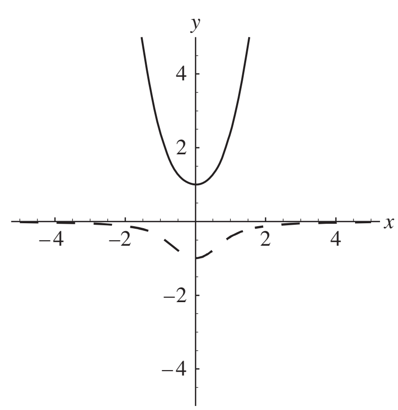
Exponentiating both sides of the implicit solution we obtain

$$\frac{2X-1}{X-1}=e^t$$
$$2X-1=Xe^t-e^t$$
$$(e^t-1)=(e^t-2)X$$
$$X=\frac{e^t-1}{e^t-2}.$$

Solving $e^t-2=0$ we get $t=\ln 2$. Thus, the solution is defined on $(-\infty,\ln 2)$ or on $(\ln 2,\infty)$. The graph of the solution defined on $(-\infty,\ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2,\infty)$ is solid.

20. Implicitly differentiating the solution, we obtain

$$-2x^2\frac{dy}{dx}-4xy+2y\frac{dy}{dx}=0$$
$$-x^2\,dy-2xy\,dx+y\,dy=0$$
$$2xy\,dx+(x^2-y)dy=0.$$



Using the quadratic formula to solve $y^2-2x^2y-1=0$ for y, we get $y=\left(2x^2\pm\sqrt{4x^4+4}\right)/2=x^2\pm\sqrt{x^4+1}$. Thus, two explicit solutions are $y_1=x^2+\sqrt{x^4+1}$ and $y_2=x^2-\sqrt{x^4+1}$. Both solutions are defined on $(-\infty,\infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.

21. Differentiating $P = c_1e^t/\left(1+c_1e^t\right)$ we obtain

$$\frac{dP}{dt} = \frac{\left(1+c_1e^t\right)c_1e^t - c_1e^t\cdot c_1e^t}{\left(1+c_1e^t\right)^2} = \frac{c_1e^t}{1+c_1e^t}\,\frac{\left[\left(1+c_1e^t\right)-c_1e^t\right]}{1+c_1e^t}$$

$$= \frac{c_1e^t}{1+c_1e^t}\left[1-\frac{c_1e^t}{1+c_1e^t}\right] = P(1-P).$$

22. Differentiating $y = 2x^2-1+c_1e^{-2x^2}$ we obtain $\dfrac{dy}{dx} = 4x - 4xc_1e^{-2x^2}$, so that

$$\frac{dy}{dx} + 4xy = 4x - 4xc_1e^{-2x^2} + 8x^3 - 4x + 4c_1xe^{-x^2} = 8x^3$$

23. From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\dfrac{dy}{dx} = (2c_1+c_2)e^{2x} + 2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2} = (4c_1+4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1+4c_2-8c_1-4c_2+4c_1)e^{2x} + (4c_2-8c_2+4c_2)xe^{2x} = 0.$$

24. From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$

$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2},$$

so that

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y &= (-6c_1+4c_1+c_1+c_1)x^{-1} + (-c_3+2c_3-c_2-c_3+c_2)x \\ &\quad + (-c_3+c_3)x\ln x + (16-8+4)x^2 \\ &= 12x^2 \end{aligned}$$

In Problems 25–28, we use the Product Rule and the derivative of an integral ((12) of this section): $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt = g(x)$.

25. Differentiating $y = e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt$ we obtain $\dfrac{dy}{dx} = 3e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt + \frac{e^{-3x}}{x}\cdot e^{3x}$ or $\dfrac{dy}{dx} = 3e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt + \frac{1}{x}$, so that

$$\begin{aligned} x\frac{dy}{dx} - 3xy &= x\left(3e^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt + \frac{1}{x}\right) - 3x\left(e^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt\right) \\ &= 3xe^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt + 1 - 3xe^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt = 1 \end{aligned}$$

26. Differentiating $y = \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt$ we obtain $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + \frac{\cos x}{\sqrt{x}}\cdot\sqrt{x}$ or $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + \cos x$, so that

$$\begin{aligned} 2x\frac{dy}{dx} - y &= 2x\left(\frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + \cos x\right) - \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt \\ &= \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + 2x\cos x - \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt = 2x\cos x \end{aligned}$$

27. Differentiating $y = \frac{5}{x} + \frac{10}{x}\int_1^x \frac{\sin t}{t}\,dt$ we obtain $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{\sin x}{x}\cdot\frac{10}{x}$ or $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{10\sin x}{x^2}$, so that

$$\begin{aligned} x^2\frac{dy}{dx} + xy &= x^2\left(-\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{10\sin x}{x^2}\right) + x\left(\frac{5}{x} + \frac{10}{x}\int_1^x \frac{\sin t}{t}\,dt\right) \\ &= -5 - 10\int_1^x \frac{\sin t}{t}\,dt + 10\sin x + 5 + 10\int_1^x \frac{\sin t}{t}\,dt = 10\sin x \end{aligned}$$

28. Differentiating $y = e^{-x^2} + e^{-x^2}\int_0^x e^{t^2}\,dt$ we obtain $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + e^{x^2}\cdot e^{-x^2}$ or $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1$, so that

$$\begin{aligned} \frac{dy}{dx} + 2xy &= \left(-2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1\right) + 2x\left(e^{-x^2} + e^{-x^2}\int_0^x e^{t^2}\,dt\right) \\ &= -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1 + 2xe^{-x^2} + 2xe^{-x^2}\int_0^x e^{t^2}\,dt = 1 \end{aligned}$$

29. From

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

we obtain

$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

so that $xy' - 2y = 0$.

30. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x\to 0^-} y(x) = 5$ and $\lim_{x\to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

31. Substitute the function $y = e^{mx}$ into the equation $y' + 2y = 0$ to get

$$\begin{aligned} (e^{mx})' + 2(e^{mx}) &= 0 \\ me^{mx} + 2e^{mx} &= 0 \\ e^{mx}(m+2) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of x, we must have $m = -2$ and so $y = e^{-2x}$ is a solution.

32. Substitute the function $y = e^{mx}$ into the equation $5y' - 2y = 0$ to get

$$\begin{aligned} 5(e^{mx})' - 2(e^{mx}) &= 0 \\ 5me^{mx} - 2e^{mx} &= 0 \\ e^{mx}(5m - 2) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of x, we must have $m = 2/5$ and so $y = e^{2x/5}$ is a solution.

33. Substitute the function $y = e^{mx}$ into the equation $y'' - 5y' + 6y = 0$ to get

$$\begin{aligned} (e^{mx})'' - 5(e^{mx})' + 6(e^{mx}) &= 0 \\ m^2e^{mx} - 5me^{mx} + 6e^{mx} &= 0 \\ e^{mx}(m^2 - 5m + 6) &= 0 \\ e^{mx}(m - 2)(m - 3) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of x, we must have $m = 2$ or $m = 3$ therefore $y = e^{2x}$ and $y = e^{3x}$ are solutions.

34. Substitute the function $y = e^{mx}$ into the equation $2y'' + 7y' - 4y = 0$ to get

$$\begin{aligned} 2(e^{mx})'' + 7(e^{mx})' - 4(e^{mx}) &= 0 \\ 2m^2e^{mx} + 7me^{mx} - 4e^{mx} &= 0 \\ e^{mx}(2m^2 + 7m - 4) &= 0 \\ e^{mx}(m + 4)(2m - 1) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = -4$ or $m = 1/2$ therefore $y = e^{-4x}$ and $y = e^{x/2}$ are solutions.

35. Substitute the function $y = x^m$ into the equation $xy'' + 2y' = 0$ to get

$$\begin{aligned} x \cdot (x^m)'' + 2(x^m)' &= 0 \\ x \cdot m(m - 1)x^{m-2} + 2mx^{m-1} &= 0 \\ (m^2 - m)x^{m-1} + 2mx^{m-1} &= 0 \\ x^{m-1}[m^2 + m] &= 0 \\ x^{m-1}[m(m + 1)] &= 0 \end{aligned}$$

The last line implies that $m = 0$ or $m = -1$ therefore $y = x^0 = 1$ and $y = x^{-1}$ are solutions.

36. Substitute the function $y = x^m$ into the equation $x^2y'' - 7xy' + 15y = 0$ to get

$$\begin{aligned} x^2 \cdot (x^m)'' - 7x \cdot (x^m)' + 15(x^m) &= 0 \\ x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m &= 0 \\ (m^2 - m)x^m - 7mx^m + 15x^m &= 0 \\ x^m[m^2 - 8m + 15] &= 0 \\ x^m[(m-3)(m-5)] &= 0 \end{aligned}$$

The last line implies that $m = 3$ or $m = 5$ therefore $y = x^3$ and $y = x^5$ are solutions.

In Problems 37–40, we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$

37. Solving $5c = 10$ we see that $y = 2$ is a constant solution.

38. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

39. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

40. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

41. From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}\,.$$

42. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}\,.$$

43. $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all differentiable functions $y = \phi(x)$.

44. The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

45. The first derivative of $f(x) = e^x$ is e^x. The first derivative of $f(x) = e^{kx}$ is ke^{kx}. The differential equations are $y' = y$ and $y' = ky$, respectively.

46. Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c \sin x$ or $y = c \cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.

47. We first note that $\sqrt{1 - y^2} = \sqrt{1 - \sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of x for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\frac{d}{dx}(\sin x)\right|_{x=\pi} = \cos x\big|_{x=\pi} = \cos \pi = -1,$$

but

$$\sqrt{1 - y^2}\,\big|_{x=\pi} = \sqrt{1 - \sin^2 \pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1 - y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

48. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A \sin t + B \cos t$, could be a solution of the differential equation. Using $y' = A \cos t - B \sin t$ and $y'' = -A \sin t - B \cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A \sin t - B \cos t + 2A \cos t - 2B \sin t + 4A \sin t + 4B \cos t \\ &= (3A - 2B) \sin t + (2A + 3B) \cos t = 5 \sin t \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13} \sin t - \frac{10}{13} \cos t$.

49. One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.

50. One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

51. Differentiating $(x^3+y^3)/xy = 3c$ we obtain

$$\frac{xy(3x^2+3y^2y') - (x^3+y^3)(xy'+y)}{x^2y^2} = 0$$

$$3x^3y + 3xy^3y' - x^4y' - x^3y - xy^3y' - y^4 = 0$$

$$(3xy^3 - x^4 - xy^3)y' = -3x^3y + x^3y + y^4$$

$$y' = \frac{y^4 - 2x^3y}{2xy^3 - x^4} = \frac{y(y^3-2x^3)}{x(2y^3-x^3)}\,.$$

52. A tangent line will be vertical where y' is undefined, or in this case, where $x(2y^3-x^3)=0$. This gives $x=0$ or $2y^3=x^3$. Substituting $y^3=x^3/2$ into $x^3+y^3=3xy$ we get

$$x^3 + \frac{1}{2}x^3 = 3x\left(\frac{1}{2^{1/3}}\,x\right)$$

$$\frac{3}{2}x^3 = \frac{3}{2^{1/3}}x^2$$

$$x^3 = 2^{2/3}x^2$$

$$x^2(x-2^{2/3}) = 0.$$

Thus, there are vertical tangent lines at $x=0$ and $x=2^{2/3}$, or at $(0,0)$ and $(2^{2/3}, 2^{1/3})$. Since $2^{2/3}\approx 1.59$, the estimates of the domains in Problem 50 were close.

53. The derivatives of the functions are $\phi_1'(x) = -x/\sqrt{25-x^2}$ and $\phi_2'(x) = x/\sqrt{25-x^2}$, neither of which is defined at $x=\pm 5$.

54. To determine if a solution curve passes through $(0,3)$ we let $t=0$ and $P=3$ in the equation $P = c_1e^t/(1+c_1e^t)$. This gives $3 = c_1/(1+c_1)$ or $c_1 = -\frac{3}{2}$. Thus, the solution curve

$$P = \frac{(-3/2)e^t}{1-(3/2)e^t} = \frac{-3e^t}{2-3e^t}$$

passes through the point $(0,3)$. Similarly, letting $t=0$ and $P=1$ in the equation for the one-parameter family of solutions gives $1 = c_1/(1+c_1)$ or $c_1 = 1 + c_1$. Since this equation has no solution, no solution curve passes through $(0,1)$.

55. For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y = \int(\int f(x)dx)\,dx + c_1x + c_2$.

56. Solving for y' using the quadratic formula we obtain the two differential equations

$$y' = \frac{1}{x}\left(2 + 2\sqrt{1+3x^6}\right) \quad \text{and} \quad y' = \frac{1}{x}\left(2 - 2\sqrt{1+3x^6}\right),$$

so the differential equation cannot be put in the form $dy/dx = f(x,y)$.

57. The differential equation $yy' - xy = 0$ has normal form $dy/dx = x$. These are not equivalent because $y = 0$ is a solution of the first differential equation but not a solution of the second.

58. Differentiating we get $y' = c_1 + 2c_2x$ and $y'' = 2c_2$. Then $c_2 = y''/2$ and $c_1 = y' - xy''$, so

$$y = (y' - xy'')\,x + \left(\frac{y''}{2}\right)x^2 = xy' - \frac{1}{2}x^2y''$$

and the differential equation is $x^2y'' - 2xy' + 2y = 0$.

59. **(a)** Since e^{-x^2} is positive for all values of x, $dy/dx > 0$ for all x, and a solution, $y(x)$, of the differential equation must be increasing on any interval.

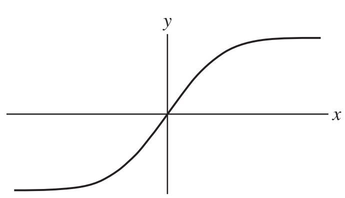
(b) $\displaystyle\lim_{x\to-\infty}\frac{dy}{dx} = \lim_{x\to-\infty} e^{-x^2} = 0$ and $\displaystyle\lim_{x\to\infty}\frac{dy}{dx} = \lim_{x\to\infty} e^{-x^2} = 0$. Since dy/dx approaches 0 as x approaches $-\infty$ and ∞, the solution curve has horizontal asymptotes to the left and to the right.

(c) To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

(d)



60. **(a)** The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

(b) A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

61. **(a)** The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

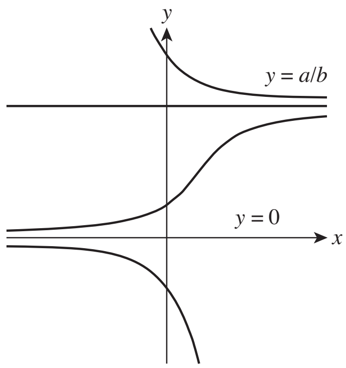
(b) A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

(c) Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Thus $d^2y/dx^2 = 0$ when $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

(d)

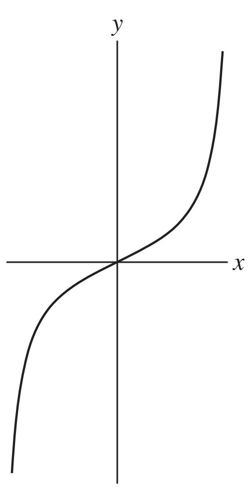


62. **(a)** If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of c.

(b) Since $y' = y^2 + 4 > 0$ for all x where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

(c) Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2 + 4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

(d)



63. In *Mathematica* use

```
Clear[y]
y[x_]:= x Exp[5x] Cos[2x]
y[x]
y''''[x] – 20y'''[x] + 158y''[x] – 580y'[x] + 841y[x]//Simplify
```

The output will show $y(x) = e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

64. In *Mathematica* use

```
Clear[y]
y[x_]:= 20Cos[5Log[x]]/x – 3Sin[5Log[x]]/x
y[x]
x^3 y'''[x] + 2x^2 y''[x] + 20x y'[x] – 78y[x]//Simplify
```

The output will show $y(x) = 20\cos(5\ln x)/x - 3\sin(5\ln x)/x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

1.2 Initial-Value Problems

1. Solving $-1/3 = 1/(1+c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1-4e^{-x})$.

2. Solving $2 = 1/(1+c_1 e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2-e^{-(x+1)})$.

3. Letting $x = 2$ and solving $1/3 = 1/(4+c)$ we get $c = -1$. The solution is $y = 1/(x^2-1)$. This solution is defined on the interval $(1, \infty)$.

4. Letting $x = -2$ and solving $1/2 = 1/(4+c)$ we get $c = -2$. The solution is $y = 1/(x^2-2)$. This solution is defined on the interval $(-\infty, -\sqrt{2})$.

5. Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2+1)$. This solution is defined on the interval $(-\infty, \infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4+c)$ we get $c = -1/2$. The solution is $y = 1/(x^2-1/2) = 2/(2x^2-1)$. This solution is defined on the interval $(-1/\sqrt{2}, 1/\sqrt{2})$.

In Problems 7–10, we use $x = c_1\cos t + c_2\sin t$ and $x' = -c_1\sin t + c_2\cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

7. From the initial conditions we obtain the system

$$c_1 = -1 c_2 = 8$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

8. From the initial conditions we obtain the system

$$c_2 = 0 - c_1 = 1$$

The solution of the initial-value problem is $x = -\cos t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2} = 0$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

10. From the initial conditions we obtain

$$\frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = \sqrt{2}$$
$$[6pt] - \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = 2\sqrt{2}.$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is $x = -\cos t + 3\sin t$.

In Problems 11–14, we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$.

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$

$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^2e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

13. From the initial conditions we obtain

$$e^{-1}c_1 + ec_2 = 5$$
$$e^{-1}c_1 - ec_2 = -5.$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is $y = 5e^{-1}e^{-x} = 5e^{-1-x}$.

14. From the initial conditions we obtain

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

15. Two solutions are $y = 0$ and $y = x^3$.

16. Two solutions are $y = 0$ and $y = x^2$. (Also, any constant multiple of x^2 is a solution.)

17. For $f(x, y) = y^{2/3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}y^{-1/3}$. Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

19. For $f(x, y) = \dfrac{y}{x}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x \neq 0$.

20. For $f(x, y) = x + y$ we have $\dfrac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

21. For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f/\partial y = 2x^2y/(4 - y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

22. For $f(x, y) = \dfrac{x^2}{1 + y^3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2}{(1 + y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

23. For $f(x, y) = \dfrac{y^2}{x^2 + y^2}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2x^2y}{(x^2 + y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0, 0)$.

24. For $f(x, y) = (y + x)/(y - x)$ we have $\partial f/\partial y = -2x/(y - x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28, we identify $f(x, y) = \sqrt{y^2 - 9}$ and $\partial f/\partial y = y/\sqrt{y^2 - 9}$. We see that f and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on x.

25. Since $4 > 3$, $(1, 4)$ is in the region defined by $y > 3$ and the differential equation has a unique solution through $(1, 4)$.

26. Since $(5, 3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(5, 3)$.

27. Since $(2, -3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2, -3)$.

28. Since $(-1, 1)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(-1, 1)$.

29. **(a)** A one-parameter family of solutions is $y = cx$. Since $y' = c$, $xy' = xc = y$ and $y(0) = c \cdot 0 = 0$.

(b) Writing the equation in the form $y' = y/x$, we see that R cannot contain any point on the y-axis. Thus, any rectangular region disjoint from the y-axis and containing (x_0, y_0) will determine an interval around x_0 and a unique solution through (x_0, y_0). Since $x_0 = 0$ in part (a), we are not guaranteed a unique solution through $(0, 0)$.

(c) The piecewise-defined function which satisfies $y(0) = 0$ is not a solution since it is not differentiable at $x = 0$.

30. **(a)** Since $\dfrac{d}{dx}\tan(x + c) = \sec^2(x + c) = 1 + \tan^2(x + c)$, we see that $y = \tan(x + c)$ satisfies the differential equation.

(b) Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2, 2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

31. **(a)** Since $\dfrac{d}{dx}\left(-\dfrac{1}{x + c}\right) = \dfrac{1}{(x + c)^2} = y^2$, we see that $y = -\dfrac{1}{x + c}$ is a solution of the differential equation.

(b) Solving $y(0) = -1/c = 1$ we obtain $c = -1$ and $y = 1/(1-x)$. Solving $y(0) = -1/c = -1$ we obtain $c = 1$ and $y = -1/(1+x)$. Being sure to include $x = 0$, we see that the interval of existence of $y = 1/(1-x)$ is $(-\infty, 1)$, while the interval of existence of $y = -1/(1+x)$ is $(-1, \infty)$.

(c) By inspection we see that $y = 0$ is a solution on $(-\infty, \infty)$.

32. (a) Applying $y(1) = 1$ to $y = -1/(x + c)$ gives

$$1 = -\frac{1}{1+c} \quad \text{or} \quad 1 + c = -1$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

(b) Applying $y(3) = -1$ to $y = -1/(x + c)$ gives

$$-1 = -\frac{1}{3+c} \quad \text{or} \quad 3 + c = 1.$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

(c) No, they are not the same solution. The interval I of definition for the solution in part (a) is $(-\infty, 2)$; whereas the interval I of definition for the solution in part (b) is $(2, \infty)$. See the figure.

33. (a) Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

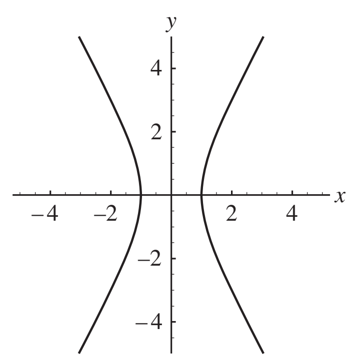
(b) Solving $3x^2 - y^2 = 3$ for y we get

$$y = \phi_1(x) = \sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$

$$y = \phi_2(x) = -\sqrt{3(x^2-1)}, \qquad 1 < x < \infty,$$

$$y = \phi_3(x) = \sqrt{3(x^2-1)}, \qquad -\infty < x < -1,$$

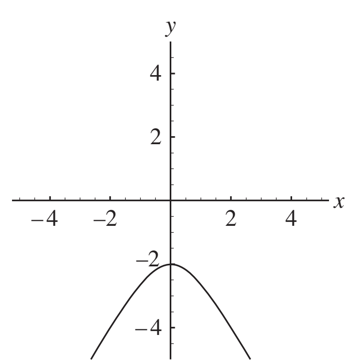
$$y = \phi_4(x) = -\sqrt{3(x^2-1)}, \qquad -\infty < x < -1.$$



(c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

34. (a) Setting $x = 2$ and $y = -4$ in $3x^2 - y^2 = c$ we get $12 - 16 = -4 = c$, so the explicit solution is

$$y = -\sqrt{3x^2 + 4}, \quad -\infty < x < \infty.$$



(b) Setting $c = 0$ we have $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, both defined on $(-\infty, \infty)$.

In Problems 35–38, we consider the points on the graphs with x-coordinates $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).

35. The graph satisfies the conditions in (b) and (f).

36. The graph satisfies the conditions in (e).

37. The graph satisfies the conditions in (c) and (d).

38. The graph satisfies the conditions in (a).

In Problems 39–44 $y = c_1 \cos 2x + c_2 \sin 2x$ is a two parameter family of solutions of the second-order differential equation $y'' + 4y = 0$. In some of the problems we will use the fact that $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$.

39. From the boundary conditions $y(0) = 0$ and $y\left(\frac{\pi}{4}\right) = 3$ we obtain

$$y(0) = c_1 = 0$$

$$y\left(\frac{\pi}{4}\right) = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = c_2 = 3.$$

Thus, $c_1 = 0$, $c_2 = 3$, and the solution of the boundary-value problem is $y = 3 \sin 2x$.

40. From the boundary conditions $y(0) = 0$ and $y(\pi) = 0$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 0.$$

Thus, $c_1 = 0$, c_2 is unrestricted, and the solution of the boundary-value problem is $y = c_2 \sin 2x$, where c_2 is any real number.

41. From the boundary conditions $y'(0) = 0$ and $y'\left(\frac{\pi}{6}\right) = 0$ we obtain

$$y'(0) = 2c_2 = 0$$

$$y'\left(\frac{\pi}{6}\right) = -2c_1 \sin\left(\frac{\pi}{3}\right) = -\sqrt{3}\, c_1 = 0.$$

Thus, $c_2 = 0$, $c_1 = 0$, and the solution of the boundary-value problem is $y = 0$.

42. From the boundary conditions $y(0) = 1$ and $y'(\pi) = 5$ we obtain

$$y(0) = c_1 = 1$$

$$y'(\pi) = 2c_2 = 5.$$

Thus, $c_1 = 1$, $c_2 = \dfrac{5}{2}$, and the solution of the boundary-value problem is $y = \cos 2x + \dfrac{5}{2} \sin 2x$.

43. From the boundary conditions $y(0) = 0$ and $y(\pi) = 2$ we obtain

$$y(0) = c_1 = 0$$
$$y(\pi) = c_1 = 2.$$

Since $0 \neq 2$, this is not possible and there is no solution.

44. From the boundary conditions $y'\left(\frac{\pi}{2}\right) = 1$ and $y'(\pi) = 0$ we obtain

$$y'\left(\frac{\pi}{2}\right) = 2c_2 = -1$$
$$y'(\pi) = 2c_2 = 0.$$

Since $0 \neq -1$, this is not possible and there is no solution.

45. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)\,dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

46. Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2)\,dx = 6x^2 - 2x + c_1.$$

Then, integrating y' we obtain

$$y = \int (6x^2 - 2x + c_1)\,dx = 2x^3 - x^2 + c_1x + c_2.$$

At $x = 1$ the y-coordinate of the point of tangency is $y = -1 + 5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \qquad \text{or} \qquad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \qquad \text{or} \qquad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

47. When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the red curve.

48. We note that the initial condition $y(0) = 0$,

$$0 = \int_0^y \frac{1}{\sqrt{t^3+1}}\,dt$$

is satisfied only when $y = 0$. For any $y > 0$, necessarily

$$\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt > 0$$

because the integrand is positive on the interval of integration. Then from (12) of Section 1.1 and the Chain Rule we have:

$$\frac{d}{dx}x = \frac{d}{dx}\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt \qquad\qquad \frac{dy}{dx} = \sqrt{y^3+1}$$

and

$$1 = \frac{1}{\sqrt{y^3+1}}\frac{dy}{dx} \qquad\qquad y'(0) = \left.\frac{dy}{dx}\right|_{x=0} = \sqrt{(y(0))^3+1} = \sqrt{0+1} = 1.$$

Computing the second derivative, we see that:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sqrt{y^3+1} = \frac{3y^2}{2\sqrt{y^3+1}}\frac{dy}{dx} = \frac{3y^2}{2\sqrt{y^3+1}}\cdot\sqrt{y^3+1} = \frac{3}{2}y^2$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}y^2.$$

This is equivalent to $2\dfrac{d^2y}{dx^2} - 3y^2 = 0$.

49. If the solution is tangent to the x-axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.

50. The theorem guarantees a unique (meaning single) solution through any point. Thus, there cannot be two distinct solutions through any point.

51. When $y = \frac{1}{16}x^4$, $y' = \frac{1}{4}x^3 = x(\frac{1}{4}x^2) = xy^{1/2}$, and $y(2) = \frac{1}{16}(16) = 1$. When

$$y = \begin{cases} 0, & x < 0 \\ \dfrac{1}{16}x^4, & x \ge 0 \end{cases}$$

we have

$$y' = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^3, & x \ge 0 \end{cases} = x\begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & x \ge 0 \end{cases} = xy^{1/2},$$

and $y(2) = \frac{1}{16}(16) = 1$. The two different solutions are the same on the interval $(0, \infty)$, which is all that is required by Theorem 1.2.1.

1.3 Differential Equations as Mathematical Models

1. $\dfrac{dP}{dt} = kP + r; \qquad \dfrac{dP}{dt} = kP - r$

2. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P$.

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1 P$ and $d = k_2 P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1 P - k_2 P^2$.

4. $\dfrac{dP}{dt} = k_1 P - k_2 P^2 - h, \;\; h > 0$

5. From the graph in the text we estimate $T_0 = 180°$ and $T_m = 75°$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30\cos(\pi t/12)$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\left(\frac{\pi}{12}t\right)\right)\right], \quad t > 0.$$

7. The number of students with the flu is x and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.

8. By analogy, with the differential equation modeling the spread of a disease, we assume that the rate at which the technological innovation is adopted is proportional to the number of people who have adopted the innovation and also to the number of people, $y(t)$, who have not yet adopted it. If one person who has adopted the innovation is introduced into the population, then $x + y = n + 1$ and

$$\frac{dx}{dt} = kx(n + 1 - x), \quad x(0) = 1.$$

9. The rate at which salt is leaving the tank is

$$R_{out}\ (10\ \text{L/min}) \cdot \left(\frac{A}{1000}\ \text{kg/L}\right) = \frac{A}{100}\ \text{kg/min}.$$

Thus $dA/dt = A/100$. The initial amount is $A(0) = 25$.

10. The rate at which salt is entering the tank is

$$R_{in} = (10\ \text{L/min}) \cdot (0.25\ \text{kg/L}) = 2.5\ \text{kg/min}.$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(10 - 7.5)\text{L/min} = 2.5$ L/min. After t minutes there are $1000 + 2.5t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (7.5 \text{ L/min}) \cdot \left(\frac{A}{1000 + 2.5t} \text{ kg/L}\right) = \frac{7.5A}{1000 + 2.5t} \text{ kg/min}.$$

The differential equation is

$$\frac{dA}{dt} = 2.5 - \frac{7.5A}{1000 + 2.5t}.$$

11. The rate at which salt is entering the tank is

$$R_{in} = (10 \text{ L/min}) \cdot (0.25 \text{ kg/L}) = 2.5 \text{ kg/min}.$$

Since the tank loses liquid at the net rate of

$$10 \text{ L/min} - 12 \text{ L/min} = -2 \text{ L/min},$$

after t minutes the number of liters of brine in the tank is $1000 - 2t$ liters. Thus the rate at which salt is leaving is

$$R_{out} = \left(\frac{A}{1000 - 2t} \text{ kg/L}\right) \cdot (12 \text{ L/min}) = \frac{12A}{1000 - 2t} \text{ kg/min} = \frac{6A}{500 - t} \text{ kg/min}.$$

The differential equation is

$$\frac{dA}{dt} = 2.5 - \frac{6A}{500 - t} \quad \text{or} \quad \frac{dA}{dt} + \frac{6}{500 - t}A = 2.5.$$

12. The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ kg/L}) \cdot (r_{in} \text{ L/min}) = c_{in} r_{in} \text{ kg/min}.$$

Now let $A(t)$ denote the number of kilograms of salt and $N(t)$ the number of liters of brine in the tank at time t. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of liters of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of liters of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ kg/L}\right) \cdot (r_{out} \text{ L/min}) = r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \text{ kg/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in} r_{in} - r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t} A = c_{in} r_{in}.$$

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}\,.$$

Using $A_h = \pi\left(\dfrac{50}{1000}\right)^2 = 0.0025\,\pi\,,\ A_w = 4^2 = 16$, and $g = 9.8$, this becomes

$$\frac{dh}{dt} = -0.00217\,c\sqrt{h}\,.$$

14. The volume of water in the tank at time t is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the tank at height h. From the figure in the text we see that $r/h = 3/6$ so that $r = 0.5h$ and $V = \frac{1}{3}\pi\,(0.5h)^2\,h = \frac{1}{12}\pi h^3$. Differentiating with respect to t we have $dV/dt = \frac{1}{4}\pi h^2\,dh/dt$ or

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}\,.$$

From Problem 13 we have $dV/dt = -cA_h\sqrt{2gh}$ where $c = 0.6$, $A_h = \pi\left(\frac{50}{1000}\right)^2$, and $g = 9.8$. Thus $dV/dt = -0.00664\pi\sqrt{h}$ and

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\left(-0.00664\sqrt{h}\right) = -\frac{0.0266}{h^{3/2}}\,.$$

15. Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

16. By Kirchhoff's second law we obtain $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$.

17. From Newton's second law we obtain $m\dfrac{dv}{dt} = -kv^2 + mg$.

18. Since the barrel in Figure 1.3.17(b) in the text is submerged an additional y feet below its equilibrium position the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi\times(\text{radius})^2\times\text{height}$ or $\pi(s/2)^2y$. Then we have from Archimedes' principle

$$\begin{aligned}\text{upward force of water on barrel} &= \text{weight of water displaced}\\ &= (9810)\times(\text{volume of water displaced})\\ &= (9810)\pi(s/2)^2y = 2452.5\pi s^2 y.\end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2} = -2452.5\pi s^2 y \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \frac{2452.5\pi s^2 g}{w}y = 0,$$

where $g = 9.8$ and w is the weight of the barrel in pounds.

19. The net force acting on the mass is

$$F = ma = m\,\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m\,\frac{d^2x}{dt^2} = -kx.$$

20. From Problem 19, without a damping force, the differential equation is $m\,d^2x/dt^2 = -kx$. With a damping force proportional to velocity, the differential equation becomes

$$m\,\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

21. As the rocket climbs (in the positive direction), it spends its amount of fuel and therefore the mass of the fuel changes with time. The air resistance acts in the opposite direction of the motion and the upward thrust R works in the same direction. Using Newton's second law we get

$$\frac{d}{dt}(mv) = -mg - kv + R$$

Now because the mass is variable, we must use the product rule to expand the left side of the equation. Doing so gives us the following:

$$\frac{d}{dt}(mv) = -mg - kv + R$$

$$v \times \frac{dm}{dt} + m \times \frac{dv}{dt} = -mg - kv + R$$

The last line is the differential equation we wanted to find.

22. **(a)** Since the mass of the rocket is $m(t) = m_p + m_v + m_f(t)$, take the time rate-of-change and get, by straight-forward calculation,

$$\frac{d}{dt}m(t) = \frac{d}{dt}(m_p + m_v + m_f(t)) = 0 + 0 + m_f'(t) = \frac{d}{dt}m_f(t)$$

Therefore the rate of change of the mass of the rocket is the same as the rate of change of the mass of the fuel which is what we wanted to show.

(b) The fuel is decreasing at the constant rate of λ and so from part (a) we have

$$\frac{d}{dt}m(t) = \frac{d}{dt}m_f(t) = -\lambda$$

$$m(t) = -\lambda t + c$$

Using the given condition to solve for c, $m(0) = 0 + c = m_0$ and so $m(t) = -\lambda t + m_0$. The differential equation in Problem 21 now becomes

$$\begin{aligned}
v\frac{dm}{dt} + m\frac{dv}{dt} + kv &= -mg + R \\
-\lambda v + (-\lambda t + m_0)\frac{dv}{dt} + kv &= -mg + R \\
(-\lambda t + m_0)\frac{dv}{dt} + (k-\lambda)v &= -mg + R \\
\frac{dv}{dt} + \frac{k-\lambda}{-\lambda t + m_0}v &= \frac{-mg}{-\lambda t + m_0} + \frac{R}{-\lambda t + m_0} \\
\frac{dv}{dt} + \frac{k-\lambda}{-\lambda t + m_0}v &= -g + \frac{R}{-\lambda t + m_0}
\end{aligned}$$

(c) From part (b) we have that $\frac{d}{dt}m_f(t) = -\lambda$ and so by integrating this result we get $m_f(t) = -\lambda t + c$. Now at time $t = 0$, $m_f(0) = 0 + c = c$ therefore $m_f(t) = -\lambda t + m_f(0)$. At some later time t_b we then have $m_f(t_b) = -\lambda t_b + m_f(0) = 0$ and solving this equation for that time we get $t_b = m_f(0)/\lambda$ which is what we wanted to show.

23. From $g = k/R^2$ we find $k = gR^2$. Using $a = d^2r/dt^2$ and the fact that the positive direction is upward we get

$$\frac{d^2r}{dt^2} = -a = -\frac{k}{r^2} = -\frac{gR^2}{r^2} \quad \text{or} \quad \frac{d^2r}{dt^2} + \frac{gR^2}{r^2} = 0.$$

24. The gravitational force on m is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$ we have $M_r = r^3M/R^3$ and

$$F = -k\frac{M_r m}{r^2} = -k\frac{r^3Mm/R^3}{r^2} = -k\frac{mM}{R^3}r.$$

Now from $F = ma = d^2r/dt^2$ we have

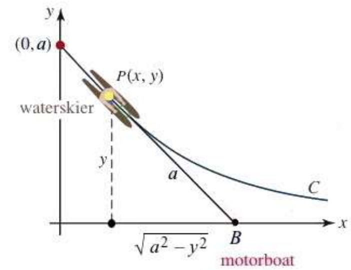
$$m\frac{d^2r}{dt^2} = -k\frac{mM}{R^3}r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}r.$$

25. The differential equation is $\dfrac{dA}{dt} = k(M - A)$.

26. The differential equation is $\dfrac{dA}{dt} = k_1(M - A) - k_2A$.

27. The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.

28. Consider the right triangle fromed by the waterskier (P), the boat (B), and the point on the x-axis directly below the waterskier. Using Pythagorean Theorem we have that the base of the triangle on the x-axis has length $\sqrt{a^2 - y^2}$. Therefore the slope of the line tangent to curve C is

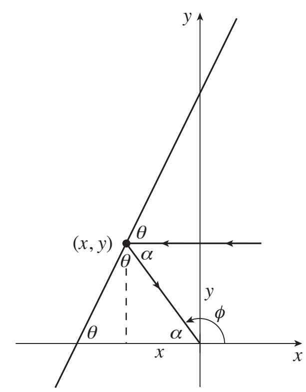


$$y' = -\frac{y}{a^2 - y^2}$$

Notice that the sign of the derivative is negative because as the boat proceeds along the positive x-axis, the y-coordinate decreases.

29. We see from the figure that $2\theta + \alpha = \pi$. Thus

$$\frac{y}{-x} = \tan\alpha = \tan(\pi - 2\theta) = -\tan 2\theta = -\frac{2\tan\theta}{1 - \tan^2\theta}.$$



Since the slope of the tangent line is $y' = \tan\theta$ we have $y/x = 2y'[1-(y')^2]$ or $y - y(y')^2 = 2xy'$, which is the quadratic equation $y(y')^2 + 2xy' - y = 0$ in y'. Using the quadratic formula, we get

$$y' = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}.$$

Since $dy/dx > 0$, the differential equation is

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y} \qquad \text{or} \qquad y\frac{dy}{dx} - \sqrt{x^2 + y^2} + x = 0.$$

30. The differential equation is $dP/dt = kP$, so from Problem 41 in Exercises 1.1, a one-parameter family of solutions is $P = ce^{kt}$.

31. The differential equation in (3) is $dT/dt = k(T - T_m)$. When the body is cooling, $T > T_m$, so $T - T_m > 0$. Since T is decreasing, $dT/dt < 0$ and $k < 0$. When the body is warming, $T < T_m$, so $T - T_m < 0$. Since T is increasing, $dT/dt > 0$ and $k < 0$.

32. The differential equation in (8) is $dA/dt = 2.5 - A/100$. If $A(t)$ attains a maximum, then $dA/dt = 0$ at this time and $A = 250$. If $A(t)$ continues to increase without reaching a maximum, then $A'(t) > 0$ for $t > 0$ and A cannot exceed 250. In this case, if $A'(t)$ approaches 0 as t increases to infinity, we see that $A(t)$ approaches 250 as t increases to infinity.

33. This differential equation could describe a population that undergoes periodic fluctuations.

34. **(a)** As shown in Figure 1.3.23(a) in the text, the resultant of the reaction force of magnitude F and the weight of magnitude mg of the particle is the centripetal force of magnitude $m\omega^2 x$. The centripetal force points to the center of the circle of radius x on which the particle rotates about the y-axis. Comparing parts of similar triangles gives

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = m\omega^2 x.$$

(b) Using the equations in part (a) we find

$$\tan\theta = \frac{F\sin\theta}{F\cos\theta} = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{\omega^2 x}{g}\,.$$

35. From Problem 23, $d^2r/dt^2 = -gR^2/r^2$. Since R is a constant, if $r = R + s$, then $d^2r/dt^2 = d^2s/dt^2$ and, using a Taylor series, we get

$$\frac{d^2s}{dt^2} = -g\frac{R^2}{(R+s)^2} = -gR^2(R+s)^{-2} \approx -gR^2[R^{-2} - 2sR^{-3} + \cdots] = -g + \frac{2gs}{R} + \cdots\,.$$

Thus, for R much larger than s, the differential equation is approximated by $d^2s/dt^2 = -g$.

36. **(a)** If ρ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho\,\frac{dV}{dt} = \rho\,\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \rho\left(4\pi r^2\frac{dr}{dt}\right) = \rho S\,\frac{dr}{dt}\,.$$

If dr/dt is a constant, then $dm/dt = kS$ where $\rho\,dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

(b) From Newton's second law, $\dfrac{d}{dt}[mv] = mg$, where v is the velocity of the raindrop. Then

$$m\,\frac{dv}{dt} + v\,\frac{dm}{dt} = mg \qquad \text{or} \qquad \rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v(k4\pi r^2) = \rho\left(\frac{4}{3}\pi r^3\right)g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r}v = g \qquad \text{or} \qquad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0}v = g, \quad k < 0.$$

37. We assume that the plow clears snow at a constant rate of k cubic kilometers per hour. Let t be the time in hours after noon, $x(t)$ the depth in miles of the snow at time t, and $y(t)$ the distance the plow has moved in t hours. Then dy/dt is the velocity of the plow and the assumption gives

$$wx\frac{dy}{dt} = k,$$

where w is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour. Now let t_0 be the number of hours before noon when it started snowing and let s be the constant rate in kilometers per hour at which x increases. Then for $t > -t_0$, $x = s(t + t_0)$. The differential equation then becomes

$$\frac{dy}{dt} = \frac{k}{ws}\,\frac{1}{t + t_0}.$$

Integrating, we obtain

$$y = \frac{k}{ws}\,[\ln(t + t_0) + c\,]$$

where c is a constant. Now when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws}\ln\left(1 + \frac{t}{t_0}\right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$, we obtain

$$\left(1 + \frac{2}{t_0}\right)^2 = \left(1 + \frac{1}{t_0}\right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. Since $t_0 > 0$, we find $t_0 \approx 0.618$ hours ≈ 37 minutes. Thus it started snowing at about 11:23 in the morning.

38. At time t, when the population is 2 million cells, the differential equation $P'(t) = 0.15P(t)$ gives the rate of increase at time t. Thus, when $P(t) = 2$ (million cells), the rate of increase is $P'(t) = 0.15(2) = 0.3$ million cells per hour or 300,000 cells per hour.

39. Setting $A'(t) = -0.002$ and solving $A'(t) = -0.0004332A(t)$ for $A(t)$, we obtain

$$A(t) = \frac{A'(t)}{-0.0004332} = \frac{-0.002}{-0.0004332} \approx 4.6 \text{ grams}.$$

40. (1) : $\dfrac{dP}{dt} = kP$ is linear (2) : $\dfrac{dA}{dt} = kA$ is linear

(3) : $\dfrac{dT}{dt} = k\,(T - T_m)$ is linear (5) : $\dfrac{dx}{dt} = kx\,(n + 1 - x)$ is nonlinear

(6) : $\dfrac{dX}{dt} = k\,(\alpha - X)\,(\beta - X)$ is nonlinear (8) : $\dfrac{dA}{dt} = 6 - \dfrac{A}{100}$ is linear

(10) : $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$ is nonlinear (11) : $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$ is linear

(12) : $\dfrac{d^2s}{dt^2} = -g$ is linear (14) : $m\dfrac{dv}{dt} = mg - kv$ is linear

(15) : $m\dfrac{d^2s}{dt^2} + k\dfrac{ds}{dt} = mg$ is linear (16) : $\dfrac{d^2x}{dt^2} - \dfrac{64}{L}x = 0$ is linear

(17) : linearity or nonlinearity is determined by the manner in which W and T_1 involve x.

Chapter 1 in Review

1. $\dfrac{d}{dx}\,c_1e^{10x} = 10\,\overbrace{c_1e^{10x}}^{y};\qquad \dfrac{dy}{dx} = 10y$

2. $\dfrac{d}{dx}(5+c_1e^{-2x}) = -2c_1e^{-2x} = -2(\overbrace{5+c_1e^{-2x}}^{y}-5);\qquad \dfrac{dy}{dx} = -2(y-5)\quad\text{or}\quad \dfrac{dy}{dx} = -2y+10$

3. $\dfrac{d}{dx}(c_1\cos kx + c_2\sin kx) = -kc_1\sin kx + kc_2\cos kx;$

$$\frac{d^2}{dx^2}(c_1\cos kx + c_2\sin kx) = -k^2c_1\cos kx - k^2c_2\sin kx = -k^2(\overbrace{c_1\cos kx + c_2\sin kx}^{y});$$

$$\frac{d^2y}{dx^2} = -k^2y \quad\text{or}\quad \frac{d^2y}{dx^2} + k^2y = 0$$

4. $\dfrac{d}{dx}(c_1\cosh kx + c_2\sinh kx) = kc_1\sinh kx + kc_2\cosh kx;$

$$\frac{d^2}{dx^2}(c_1\cosh kx + c_2\sinh kx) = k^2c_1\cosh kx + k^2c_2\sinh kx = k^2(\overbrace{c_1\cosh kx + c_2\sinh kx}^{y});$$

$$\frac{d^2y}{dx^2} = k^2y \quad\text{or}\quad \frac{d^2y}{dx^2} - k^2y = 0$$

5. $y = c_1e^x + c_2xe^x;\qquad y' = c_1e^x + c_2xe^x + c_2e^x;\qquad y'' = c_1e^x + c_2xe^x + 2c_2e^x;$

$$y'' + y = 2(c_1e^x + c_2xe^x) + 2c_2e^x = 2(c_1e^x + c_2xe^x + c_2e^x) = 2y';\qquad y'' - 2y' + y = 0$$

6. $y' = -c_1e^x\sin x + c_1e^x\cos x + c_2e^x\cos x + c_2e^x\sin x;$

$y'' = -c_1e^x\cos x - c_1e^x\sin x - c_1e^x\sin x + c_1e^x\cos x - c_2e^x\sin x + c_2e^x\cos x + c_2e^x\cos x + c_2e^x\sin x$

$$= -2c_1e^x\sin x + 2c_2e^x\cos x;$$

$$y'' - 2y' = -2c_1e^x\cos x - 2c_2e^x\sin x = -2y;\qquad y'' - 2y' + 2y = 0$$

7. a, d **8.** c **9.** b **10.** a, c **11.** b **12.** a, b, d

13. A few solutions are $y = 0$, $y = c$, and $y = e^x$.

14. Easy solutions to see are $y = 0$ and $y = 3$.

15. The slope of the tangent line at (x, y) is y', so the differential equation is $y' = x^2 + y^2$.

16. The rate at which the slope changes is $dy'/dx = y''$, so the differential equation is $y'' = -y'$ or $y'' + y' = 0$.

17. **(a)** The domain is all real numbers.

(b) Since $y' = 2/3x^{1/3}$, the solution $y = x^{2/3}$ is undefined at $x = 0$. This function is a solution of the differential equation on $(-\infty, 0)$ and also on $(0, \infty)$.

18. **(a)** Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving the equation $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get $y = (2 \pm \sqrt{4 + 4(x^2 - x - 1)}\,)/2 = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}$. Since $x(x - 1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

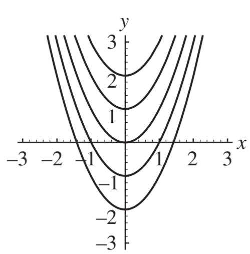
19. Setting $x = x_0$ and $y = 1$ in $y = -2/x + x$, we get

$$1 = -\frac{2}{x_0} + x_0 \qquad \text{or} \qquad x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) = 0.$$

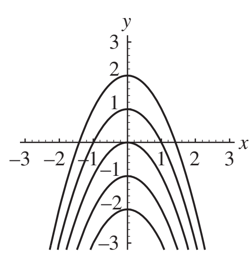
Thus, $x_0 = 2$ or $x_0 = -1$. Since $x \neq 0$ in $y = -2/x + x$, we see that $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(-1) = 1$, on the interval $(-\infty, 0)$ because $-1 < 0$, and $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(2) = 1$, on the interval $(0, \infty)$ because $2 > 0$.

20. From the differential equation, $y'(1) = 1^2 + [y(1)]^2 = 1 + (-1)^2 = 2 > 0$, so $y(x)$ is increasing in some neighborhood of $x = 1$. From $y'' = 2x + 2yy'$ we have $y''(1) = 2(1) + 2(-1)(2) = -2 < 0$, so $y(x)$ is concave down in some neighborhood of $x = 1$.

21. **(a)**



$y = x^2 + c_1$



$y = -x^2 + c_2$

(b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

(c) Pasting together x^2, $x \geq 0$, and $-x^2$, $x \leq 0$, we get $y = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0\,. \end{cases}$

22. The slope of the tangent line is $y'\big|_{(-1,4)} = 6\sqrt{4} + 5(-1)^3 = 7$.

23. Differentiating $y = x\sin x + x\cos x$ we get

$$y' = x\cos x + \sin x - x\sin x + \cos x$$

and

$$\begin{aligned} y'' &= -x\sin x + \cos x + \cos x - x\cos x - \sin x - \sin x \\ &= -x\sin x - x\cos x + 2\cos x - 2\sin x. \end{aligned}$$

Thus

$$y'' + y = -x\sin x - x\cos x + 2\cos x - 2\sin x + x\sin x + x\cos x = 2\cos x - 2\sin x.$$

An interval of definition for the solution is $(-\infty, \infty)$.

24. Differentiating $y = x\sin x + (\cos x)\ln(\cos x)$ we get

$$\begin{aligned} y' &= x\cos x + \sin x + \cos x\left(\frac{-\sin x}{\cos x}\right) - (\sin x)\ln(\cos x) \\ &= x\cos x + \sin x - \sin x - (\sin x)\ln(\cos x) \\ &= x\cos x - (\sin x)\ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y'' &= -x\sin x + \cos x - \sin x\left(\frac{-\sin x}{\cos x}\right) - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{1 - \cos^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \sec x - \cos x - (\cos x)\ln(\cos x) \\ &= -x\sin x + \sec x - (\cos x)\ln(\cos x). \end{aligned}$$

Thus

$$y'' + y = -x\sin x + \sec x - (\cos x)\ln(\cos x) + x\sin x + (\cos x)\ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

25. Differentiating $y = \sin(\ln x)$ we obtain $y' = \cos(\ln x)/x$ and $y'' = -[\sin(\ln x) + \cos(\ln x)]/x^2$. Then

$$x^2y'' + xy' + y = x^2\left(-\frac{\sin(\ln x) + \cos(\ln x)}{x^2}\right) + x\frac{\cos(\ln x)}{x} + \sin(\ln x) = 0.$$

An interval of definition for the solution is $(0, \infty)$.

26. Differentiating $y = \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$ we obtain

$$y' = \cos(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right) + \ln(\cos(\ln x))\left(-\frac{\sin(\ln x)}{x}\right) + \ln x\frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x}$$

$$= -\frac{\ln(\cos(\ln x))\sin(\ln x)}{x} + \frac{(\ln x)\cos(\ln x)}{x}$$

and

$$y'' = -x\left[\ln(\cos(\ln x))\frac{\cos(\ln x)}{x} + \sin(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right)\right]\frac{1}{x^2}$$

$$+ \ln(\cos(\ln x))\sin(\ln x)\frac{1}{x^2} + x\left[(\ln x)\left(-\frac{\sin(\ln x)}{x}\right) + \frac{\cos(\ln x)}{x}\right]\frac{1}{x^2} - (\ln x)\cos(\ln x)\frac{1}{x^2}$$

$$= \frac{1}{x^2}\Bigg[-\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x)$$

$$- (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x)\Bigg].$$

Then

$$x^2y'' + xy' + y = -\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x) - (\ln x)\sin(\ln x)$$

$$+ \cos(\ln x) - (\ln x)\cos(\ln x) - \ln(\cos(\ln x))\sin(\ln x)$$

$$+ (\ln x)\cos(\ln x) + \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$$

$$= \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) = \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x).$$

To obtain an interval of definition, we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos(\ln x) > 0$. Since $\cos x > 0$ when $-\pi/2 < x < \pi/2$, we require $-\pi/2 < \ln x < \pi/2$. Since e^x is an increasing function, this is equivalent to $e^{-\pi/2} < x < e^{\pi/2}$. Thus, an interval of definition is $(e^{-\pi/2}, e^{\pi/2})$. (Much of this problem is more easily done using a computer algebra system such as *Mathematica* or *Maple*.)

In Problems 27 - 30 we use (12) of Section 1.1 and the Product Rule.

27.

$$y = e^{\cos x}\int_0^x te^{-\cos t}\,dt$$

$$\frac{dy}{dx} = e^{\cos x}\left(xe^{-\cos x}\right) - \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt$$

$$\frac{dy}{dx} + (\sin x)\,y = e^{\cos x}xe^{-\cos x} - \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt + \sin x\left(e^{\cos x}\int_0^x te^{-\cos t}\,dt\right)$$

$$= x - \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt + \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt = x$$

28.

$$y = e^{x^2}\int_0^x e^{t-t^2}\,dt$$

$$\frac{dy}{dx} = e^{x^2}e^{x-x^2} + 2xe^{x^2}\int_0^x e^{t-t^2}\,dt$$

$$\frac{dy}{dx} - 2xy = e^{x^2}e^{x-x^2} + 2xe^{x^2}\int_0^x e^{t-t^2}\,dt - 2x\left(e^{x^2}\int_0^x e^{t-t^2}\,dt\right) = e^x$$

29.

$$y = x\int_1^x \frac{e^{-t}}{t}\,dt$$

$$y' = x\frac{e^{-x}}{x} + \int_1^x \frac{e^{-t}}{t}\,dt = e^{-x} + \int_1^x \frac{e^{-t}}{t}\,dt$$

$$y'' = -e^{-x} + \frac{e^{-x}}{x}$$

$$\begin{aligned}x^2y'' + \left(x^2 - x\right)y' + (1-x)\,y &= \left(-x^2e^{-x} + xe^{-x}\right)\\ &\quad + \left(x^2e^{-x} + x^2\int_1^x \frac{e^{-t}}{t}\,dt - xe^{-x} - x\int_1^x \frac{e^{-t}}{t}\,dt\right)\\ &\quad + \left(x\int_1^x \frac{e^{-t}}{t}\,dt - x^2\int_1^x \frac{e^{-t}}{t}\,dt\right) = 0\end{aligned}$$

30.

$$\begin{aligned}y &= \sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt\\ y' &= \sin x\left(e^{x^2}\cos x\right) + \cos x\int_0^x e^{t^2}\cos t\,dt - \cos x\left(e^{x^2}\sin x\right) + \sin x\int_0^x e^{t^2}\sin t\,dt\\ &= \cos x\int_0^x e^{t^2}\cos t\,dt + \sin x\int_0^x e^{t^2}\sin t\,dt\\ y'' &= \cos x\left(e^{x^2}\cos x\right) - \sin x\int_0^x e^{t^2}\cos t\,dt + \sin x\left(e^{x^2}\sin x\right) + \cos x\int_0^x e^{t^2}\sin t\,dt\\ &= e^{x^2}\left(\cos^2 x + \sin^2 x\right) - \left(\overbrace{\sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt}^{y}\right)\\ &= e^{x^2} - y\end{aligned}$$

$$y'' + y = e^{x^2} - y + y = e^{x^2}$$

31. Using implicit differentiation we get

$$\begin{aligned} x^3y^3 &= x^3+1 \\ 3x^2\cdot y^3 + x^3\cdot 3y^2\frac{dy}{dx} &= 3x^2 \\ \frac{3x^2y^3}{3x^2y^2} + \frac{x^3 3y^2}{3x^2y^2}\frac{dy}{dx} &= \frac{3x^2}{3x^2y^2} \\ y + x\frac{dy}{dx} &= \frac{1}{y^2} \end{aligned}$$

32. Using implicit differentiation we get

$$\begin{aligned} (x-5)^2+y^2 &= 1 \\ 2(x-5)+2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2(x-5) \\ y\frac{dy}{dx} &= -(x-5) \\ \left(y\frac{dy}{dx}\right)^2 &= (x-5)^2 \\ \left(\frac{dy}{dx}\right)^2 &= \frac{(x-5)^2}{y^2} \end{aligned}$$

Now from the original equation, isolating the first term leads to $(x-5)^2 = 1-y^2$. Continuing from the last line of our proof we now have

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x-5)^2}{y^2} = \frac{1-y^2}{y^2} = \frac{1}{y^2} - 1$$

Adding 1 to both sides leads to the desired result.

33. Using implicit differentiation we get

$$\begin{aligned} y^3+3y &= 1-3x \\ 3y^2y' + 3y' &= -3 \\ y^2y' + y' &= -1 \\ (y^2+1)y' &= -1 \\ y' &= \frac{-1}{y^2+1} \end{aligned}$$

Differentiating the last line and remembering to use the quotient rule on the right side leads to

$$y'' = \frac{2yy'}{(y^2+1)^2}$$

Now since $y' = -1\big/\big(y^2+1\big)$ we can write the last equation as

$$y'' = \frac{2y}{(y^2+1)^2}y' = \frac{2y}{(y^2+1)^2}\frac{-1}{(y^2+1)} = 2y\left(\frac{-1}{y^2+1}\right)^3 = 2y(y')^3$$

which is what we wanted to show.

34. Using implicit differentiation we get

$$\begin{aligned} y &= e^{xy} \\ y' &= e^{xy}(y+xy') \\ y' &= ye^{xy} + xe^{xy}y' \\ (1-xe^{xy})y' &= ye^{xy} \end{aligned}$$

Now since $y = e^{xy}$, substitute this into the last line to get

$$(1-xy)y' = yy$$

or $(1-xy)y' = y^2$ which is what we wanted to show.

In Problem 35–38, $y = c_1e^{3x} + c_2e^{-x} - 2x$ is given as a two-parameter family of solutions of the second-order differential equation $y'' - 2y' - 3y = 6x + 4$.

35. If $y(0) = 0$ and $y'(0) = 0$, then

$$\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 - c_2 - 2 &= 0 \end{aligned}$$

so $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus $y = \frac{1}{2}\,e^{3x} - half\,e^{-x} - 2x$.

36. If $y(0) = 1$ and $y'(0) = -3$, then

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - c_2 - 2 &= -3 \end{aligned}$$

so $c_1 = 0$ and $c_2 = 1$. Thus $y = e^{-x} - 2x$.

37. If $y(1) = 4$ and $y'(1) = -2$, then

$$\begin{aligned} c_1e^3 + c_2e^{-1} - 2 &= 4 \\ 3c_1e^3 - c_2e^{-1} - 2 &= -2 \end{aligned}$$

so $c_1 = \frac{3}{2}\,e^{-3}$ and $c_2 = \frac{9}{2}\,e$. Thus $y = \frac{3}{2}\,e^{3x-3} + \frac{9}{2}\,e^{-x+1} - 2x$.

38. If $y(-1) = 0$ and $y'(-1) = 1$, then

$$c_1 e^{-3} + c_2 e + 2 = 0$$
$$3c_1 e^{-3} - c_2 e - 2 = 1$$

so $c_1 = \frac{1}{4}\, e^3$ and $c_2 = -\frac{9}{4}, e^{-1}$. Thus $y = \frac{1}{4}\, e^{3x+3} - \frac{9}{4}\, e^{-x-1} - 2x$.

39. From the graph we see that estimates for y_0 and y_1 are $y_0 = -3$ and $y_1 = 0$.

40. The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w}\sqrt{2gh}\,.$$

Using $A_0 = \pi(0.6/0.0125)^2 = 4.906^{-4}$, $A_w = \pi(0.6)^2 = 1.134$, and $g = 9.8$, this becomes

$$\frac{dh}{dt} = -\frac{c4.906^{-4}}{1.134}\sqrt{19.6h} = -\frac{c}{522}\sqrt{h}\,.$$