The role of controllability and observability in partial pole placement by the method of receptances

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Abstract

In classical control the concept of controllability is used to ensure that the control has the flexibility to change the entire dynamics of the system without excessive control effort. In vibration control the designer is usually confronted with the scenario that only a small number of poles are required to be relocated away from some resonance frequencies. With full controllability the system may suffer from "spill-over" of poles which are not intended to be relocated. Partial pole placement exploits the concept of partial controllability, where the system is controllable only with respect to the desired relocated poles. This principle allows partial pole placement by state feedback control without specifying of the immovable eigenpairs. However, an analytical model of the system has to be available.

The method of receptances was developed to avoid using analytical modeling. Measured modal data are used instead of finite element stiffness and mass matrices of large dimensions. The original formulation which used the Sherman-Morrison formula for matrix rank-one modification was applicable only to the single input control. Moreover, the method could not handle partial pole placement since the receptances do not exist at the immovable poles of the open loop system. The concept of observability was used in the recent direct reformulation of the method without the use of the Sherman-Morrison formula. Invariance of poles was achieved by virtue of designed unobservability. Additionally, straightforward extension to the multi-input control was achieved. Also, the same theory of receptance-based partial pole placement by virtue of partial observability was derived in the perspective of continuous systems, which provides further support that the method of receptances is based on the response model of continuous structures obtained from measured experimental data, requiring no discrete or continuous analytical models, model reduction or state estimation.

1 Introduction

The motion of the n degrees of freedom viscously damped system may be represented by the set of second-order differential equations

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are $.n \times n$.real matrices, $\mathbf{M} = \mathbf{M}^T$, $\mathbf{C} = \mathbf{C}^T \geq 0$ and $\mathbf{K} = \mathbf{K}^T \geq 0$, and dots denote derivatives with respect to time. By the application of state feedback control, the dynamic behaviour may be altered as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{b}u(t) \tag{2}$$

where

$$u(t) = \mathbf{f}^T\dot{\mathbf{x}} + \mathbf{g}^T\mathbf{x} \tag{3}$$

and where $\mathbf{b}$, $\mathbf{f}$ and $\mathbf{g}$ are real vectors denoting force-distribution and control-gain terms. It is assumed that the open and closed loop systems are simple.

It is well known that (1) has exponential solutions of the form

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t} \tag{4}$$

for certain values of λ and constant vectors $\mathbf{v}$. Substituting (4) in (1) gives

$$\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right)\mathbf{v} = \mathbf{0}. \tag{5}$$

The quadratic eigenvalue problem (5) of the open loop system has a self-conjugate set of $2n$ poles, $\{\lambda_k\}_{k=1}^{2n}$, with corresponding non-vanishing eigenvectors $\{\mathbf{v}_k\}_{k=1}^{2n}$ that satisfy (5).

Similarly exponential solutions of the form

$$\mathbf{x}(t) = \mathbf{w}e^{\mu t} \tag{6}$$

applied to the closed loop system (2) lead to the right eigenvalue problem

$$\left(\mu^2\mathbf{M} + \mu\left(\mathbf{C} - \mathbf{b}\mathbf{f}^T\right) + \mathbf{K} - \mathbf{b}\mathbf{g}^T\right)\mathbf{w} = \mathbf{0} \tag{7}$$

The self-conjugate set of $2n$ poles, $\{\mu_k\}_{k=1}^{2n}$, with corresponding non-vanishing eigenvectors $\{\mathbf{w}_k\}_{k=1}^{2n}$ that satisfy (5) are the eigenpairs of the closed loop system.

The feedback control matrices $\mathbf{b}\mathbf{f}^T$ and $\mathbf{b}\mathbf{g}^T$ are usually asymmetric, so that there exists a left eigenvalue problem

$$\mathbf{z}^T\left(\mu^2\mathbf{M} + \mu\left(\mathbf{C} - \mathbf{b}\mathbf{f}^T\right) + \mathbf{K} - \mathbf{b}\mathbf{g}^T\right) = \mathbf{0}. \tag{8}$$

The system (2) is said to be completely controllable if

$$\operatorname{rank}\left\{\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}, \quad \mathbf{b}\right\} = n \tag{9}$$

for every eigenvalue λ_k, $k = 1, 2, \cdots, 2n$ [1, 2].

For a certain eigenvalue λ, $\operatorname{rank}\left\{\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}, \quad \mathbf{b}\right\} < n$ if and only if there exists a nonzero vector $\mathbf{v}$ such that

$$\mathbf{v}^T\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}, \quad \mathbf{b}\right) = \mathbf{0}. \tag{10}$$

That is,

$$\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right)\mathbf{v} = \mathbf{0} \tag{11}$$

and

$$\mathbf{v}^T\mathbf{b} = 0 \tag{12}$$

which indicates that $\mathbf{v}$ is an eigenvector of equation (5) corresponding to the eigenvalue λ and it is orthogonal to $\mathbf{b}$ vector. That means that the eigenpair $(\lambda, \mathbf{v})$ of (5) satisfying (12) is uncontrollable. The system (2), with some eigenpairs satisfying (9) and the remaining (12), is designated as partially controllable.

The concept of observability is dual to the concept of controllability. The system (2) is completely observable if

$$\operatorname{rank}\begin{Bmatrix} \lambda\mathbf{f}^T + \mathbf{g}^T \\ \lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K} \end{Bmatrix} = n \tag{13}$$

for every eigenvalue $\lambda_k,\ k = 1, 2, \cdots, 2n$ [1].

Similarly, the eigenpair $(\lambda, \mathbf{v})$ of (5) is unobservable when

$$\left(\lambda\mathbf{f}^T + \mathbf{g}^T\right)\mathbf{v} = 0 \tag{14}$$

and the system (2) is partially observable if some of its eigenpairs satisfy (13) and the others satisfy (14).

To regulate the dynamic of the open loop system (1) it is frequently required to alter a subset of eigenvalues. Since the eigenvalues may be ordered arbitrarily, without loss of generality we may assume that the $2m \le 2n$ poles of the self-conjugate set $\{\lambda_k\}_{k=1}^{2m}$ associated with (5) are required to be changed to a predetermined self-conjugate set $\{\mu_k\}_{k=1}^{2m}$ by the applied control force. To avoid spillover it is further required that $\{\mu_k\}_{k=2m+1}^{2n} = \{\lambda_k\}_{k=2m+1}^{2n}$. These conditions may be thus written in the form

$$\mu_k = \begin{cases} \mu_k & k = 1, 2, ..., 2m \\ \lambda_k & k = 2m+1, 2m+2, ..., 2n. \end{cases} \tag{15}$$

which consists of the well-known partial pole placement problem and is of practical value in suppressing vibration and stabilising dynamic systems.

Partial pole placement exploits the concept of partial controllability or partial observability, where the system is controllable or observable only with respect to the desired relocated poles. This principle allows partial pole placement by state feedback control without specifying of the immovable eigenpairs.

2 Partial pole placement in matrix-based control

By virtue of an orthogonaility relation for symmetric definite quadratic pencil, Datta et al. [2] obtained a closed form solution to partial pole placement by choosing

$$\mathbf{f} = \mathbf{MV\Lambda\beta}, \tag{16}$$

$$\mathbf{g} = -\mathbf{KV\beta} \tag{17}$$

where

$$\mathbf{\Lambda} = diag\left(\lambda_k, \quad k = 1, 2, ..., 2m\right), \tag{18}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{2m} \end{bmatrix} \tag{19}$$

and

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{2m} \end{pmatrix}^T. \tag{20}$$

and where $\beta_i,\ i = 1, 2, \cdots, 2m$ are arbitrary. It is ensured that that $\mathbf{f}$ and $\mathbf{g}$ given in (16) and (17) respectively with any choice of $\boldsymbol{\beta}$ will satisfy (14) and hence the last $2n - 2m$ modes are unobservable and consequently unaltered. To specify the first $2m$ eigenvalues to $\{\mu_k\}_{k=1}^{2m}$, β_i are chosen as

$$\beta_i = \frac{1}{\mathbf{b}^T\mathbf{v}_i}\frac{\mu_i - \lambda_i}{\lambda_i}\prod_{\substack{k=1 \\ k\ne i}}^{2m}\frac{\mu_k - \lambda_i}{\lambda_k - \lambda_i}, \quad i = 1, 2, ..., 2m \tag{21}$$

Essentially, the partial pole placement approach described by Datta et al. [2] exploits the partial observability condition. Following the partial pole placement approach described by Datta et al. [2], the partial pole placement by the use of multi-input control was developed in [3, 4], the robust partial pole placement problem was investigated in [5, 6], and the problem of partial pole placement with time delay was considered in [7, 8].

By inspection of the solutions (16) and (17), analytical model of the system has to be available to construct the control gains. Also, we realise in general we may choose for example $\mathbf{b} = \mathbf{e}_1$, where $\mathbf{e}_k$ is the k-th unit vector of appropriate dimension. However, the control gains $\mathbf{f}$ and are generally fully populated vectors. The physical meaning is that the state feedback control may be implemented in general by one actuator and n sensors measuring the complete state of the system in real time.

3 Partial pole placement by the method of receptances

The method of receptances in active vibration control [9] was developed to avoid to need for analytical models. Measured modal data are used instead of finite element stiffness and mass matrices of large dimensions. Also, the damping is embedded in the measured modal data rather than ignoring it or using an assumed damping matrix. Moreover, there is no need to estimate the unmeasured state using an observer or a Kalman filter and no need for model reduction. The original formulation [9] which used the Sherman-Morrison formula for matrix rank-one modification was applicable only to the single input control.

Based on partial controllability, a partial pole placement approach using measured receptances for single-input and multi-input state feedback control was proposed by Tehrani et al. [10]. For multi-input control case, Sherman-Morrison-Woodbury formula was used, and partial pole assignment was implemented by the sequential assignment of poles, ensuring at each step that previously assigned eigenvalues were unaffected by spillover of the most recently assigned pair of poles.

Using the concept of observability, a new partial pole placement approach based on the measured receptances was formulated [11] without the use of the Sherman-Morrison formula. Invariance of poles was achieved by virtue of designed unobservability. The vectors of control gains that assign the poles $\{\mu_k\}_{k=1}^{2m}$ while keeping the other poles $\{\mu_k\}_{k=2m+1}^{2n}$ unchanged is given by

$$\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{0} \end{pmatrix} \tag{22}$$

where

$$\mathbf{P} = \begin{bmatrix} \mu_1 \mathbf{r}_1^T & \mathbf{r}_1^T \\ \mu_2 \mathbf{r}_2^T & \mathbf{r}_2^T \\ \vdots & \vdots \\ \mu_{2m} \mathbf{r}_{2m}^T & \mathbf{r}_{2m}^T \end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix} \mu_{2m+1} \mathbf{v}_1^T & \mathbf{v}_{2m+1}^T \\ \mu_{2m+2} \mathbf{v}_2^T & \mathbf{v}_{2m+2}^T \\ \vdots & \vdots \\ \mu_{2n} \mathbf{v}_{2n}^T & \mathbf{v}_{2n}^T \end{bmatrix}, \ \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{2m} \ \text{and} \ \mathbf{r}_k = \mathbf{H}(\mu_k)\mathbf{b} \tag{23}$$

and $\mathbf{H}(\mu_k)$ is the receptance at the prescribed frequency μ_k. $\mathbf{b}$ is the force distribution vector. Straightforward extension to the multi-input control was achieved.

4 Partial pole placement in Continuous systems

The matrix-based partial pole placement approach requires analytical models, which are in general obtained by discretising continuous systems using finite element methodologies. The partial pole placement approach by the method of receptances mentioned above is completely based on the response model of continuous structures obtained from measured experimental data. Also, the theory was verified by several experiments on continuous structures. However, the theoretical derivation is based on discretised analytical models.

Alternatively, the receptance-based partial pole placement approach can be derived by the use of continuous modelling of linear elastic structures, which further supports that the receptance-based method does not rely on continuous or discrete analytical models.

In analogy to pole assignment by the method of receptances for finite dimensional systems it is generally possible to assign $2m$ poles of a continuous linear elastic system and retain $2(n-m)$ poles by sensing the state of the system at n distinct points and by applying an appropriate control force $\mathbf{b}u(t)$ with $u(t)$ given by (3). It has been shown in [11] that the control gains are determined by (22) where the elements of $h_{ij}(s)$ are the receptances between exponential excitation at x_j and the displacement response at x_i. Such receptances may be either determined analytically if the theoretical model of the system is given or determined experimentally by modal test if a prototype of the structure is available. If the receptances are exact the assignment of poles is exact as well and suffers from no discretisation or model reduction errors.

To demonstrate the result we consider the problem of natural frequency modification of a fixed-free axially vibrating uniform rod of unit length where the control force is applied at the end the rod, $x=1$. The eigenvalue problem of the open loop system is

$$\begin{cases} v'' + \omega^2 v = 0 \quad 0 < x < 1 \\ v(0) = 0 \quad v'(1) = 0 \end{cases} \tag{24}$$

Suppose that we wish to assign the natural frequencies $\{\omega_k\}_{k=1}^{m}$ to the frequencies $\{\sigma_k\}_{k=1}^{m}$ by using the control force

$$u(t) = \sin \sigma t \int_0^1 gw dx \tag{25}$$

where

$$g(x) = \sum_{k=1}^{m} g_k \delta(x - x_k) \tag{26}$$

w. is the mode shape of the closed loop system associated with σ and $\delta(x - x_k)$ is the Dirac delta function. Note that for the natural frequency modification problem, the velocity feedback control gain $f(x) = 0$ since the closed loop system is undamped.

It thus follows from (25) and (26) that

$$\int_0^1 gw dx = \mathbf{g}^T \mathbf{w} \tag{27}$$

where

$$\mathbf{g} = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \end{pmatrix}^T \text{ and } \mathbf{w} = \begin{pmatrix} w(x_1) & w(x_2) & \cdots & w(x_n) \end{pmatrix}^T \quad . \tag{28}$$

Then, the eigenvalue problem associated with the closed loop system is

$$\begin{cases} w'' + \sigma^2 w = 0 \quad 0 < x < 1 \\ w(0) = 0 \quad w'(1) = \int_0^1 gw dx = \mathbf{g}^T \mathbf{w} \end{cases} \tag{29}$$

According to [12], if the control gain vector $\mathbf{g}$ is chosen such that

$$\mathbf{g}^T \mathbf{v}_k = 0, \quad k = m+1, m+2, \cdots, n \tag{30}$$

where

$$\mathbf{v}_k = \begin{pmatrix} v_k(x_1) & v_k(x_2) & \cdots & v_k(x_n) \end{pmatrix}^T \quad , \tag{31}$$

then

$$\{\sigma_k \quad w_k\} = \{\omega_k \quad v_k\}, \quad k = m+1, m+2, \cdots, n \tag{32}$$

For the first m assigned natural frequencies, we have

$$\mathbf{g}^T \mathbf{w}_k = 1, \quad k = 1, 2, \cdots, m \tag{33}$$

The equations in (30) and (33) are exactly the same as (22) in the case of partial natural frequency assignment.

As shown in [12], $h(x_i, 1, \sigma) = w_k(x_i)$, where $w_k(x_i)$ is defined by (29) and (33). To maintain real arithmetic we consider here the frequency response function to sinusoidal excitation at the end of the rod,

$$h(x, 1, \sigma) = \frac{\sin \sigma x}{\sigma \cos \sigma}. \tag{34}$$

Equations (22) and (34) with $\mathbf{b} = \mathbf{e}_n$ determine the control gain vector $\mathbf{g}$ as demonstrated in the following example.

Example

Consider the case where $n = m = 1$, $\mathbf{b} = b_1 = 1$ and $\boldsymbol{\xi} = \xi_1 = 1$. Suppose that we want to assign the frequency σ_1 to the axially vibrating rod. Then from (34)

$$h(1, 1, \sigma_1) = \frac{\sin \sigma_1}{\sigma_1 \cos \sigma_1} \tag{35}$$

and solving (33) with $\mathbf{b} = 1$ gives

$$g_1 = h^{-1}(1, 1, \sigma_1) = \frac{\sigma_1 \cos \sigma_1}{\sin \sigma_1} \tag{36}$$

It thus follows that the eigenvalue problem of the closed loop system is

$$\begin{cases} w'' + \sigma^2 w = 0 \quad 0 < x < 1 \\ w(0) = 0 \quad w'(1) = g_1 w(1) \end{cases} \tag{37}$$

where g_1 is given by (36).

The general solution of (37) is

$$w(x) = A \sin \sigma x + B \cos \sigma x \tag{38}$$

where A and B are arbitrary constants. The first boundary condition of (37) gives

$$w(0) = B = 0 \tag{39}$$

so that

$$w(x) = A \sin \sigma x \tag{40}$$

and

$$w'(x) = A\sigma \cos \sigma x \tag{41}$$

The second boundary condition of (37) gives

$$A\sigma \cos \sigma = A\frac{\sigma_1 \cos \sigma_1 \sin \sigma}{\sin \sigma_1} \tag{42}$$

by virtue (36), (40) and (41). Clearly, equation (42) holds for $\sigma = \sigma_1$ and hence the natural frequency σ_1 has been assigned by the control (22). The control force in general affects the remaining natural frequencies of the rod.

Examples of larger dimensions where partial poles placement are implemented are given in [12]. Although the analysis is performed for an axially vibrating rod, the theory can be extended to continuous beams, plates, etc.

5 Conclusion

In the paper, the application of controllability and observability in partial pole placement is reviewed. The approach of partial pole placement in matrix-based control requires analytical models and exploits partial observability. By contrast, the partial pole placement approach by the method of receptances renders analytical models unnecessary, and uncontrollability or unobservability conditions may be used to prevent unwanted spill-over effects. The receptance-based partial pole placement approach by virtue of partial observability can not only be derived by the use of discretised analytical models but also continuous modelling of linear elastic structures. In both cases the receptance-based method is shown not to rely on continuous or discrete analytical models.

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References

[1] A.J. Laub, W.F. Arnold, *Controllability and observability criteria for multivariable linear second-order models*, IEEE Transactions on Automatic Control, Vol. AC-29, No. 2, (1984), pp. 163-165.

[2] B.N. Datta, S. Elhay, Y.M. Ram, *Orthogonality and partial pole assignment for the symmetric definite quadratic pencil*, Linear Algebra and its Applications, Vol. 257, No. 0, (1997), pp. 29-48.

[3] Y.M. Ram, S. Elhay, *Pole assignment in vibratory systems by multi-input control*, Journal of Sound and Vibration, Vol. 230, No. 2, (2000), pp. 309-321.

[4] B.N. Datta, D.R. Sarkissian, *Multi-input partial eigenvalue assignment for the symmetric quadratic pencil*, Proceedings of the 1999 American Control Conference (99ACC), IEEE, San Diego, CA, USA, (1999), pp. 2244-2247.

[5] S. Xu, J. Qian, *Orthogonal basis selection method for robust partial eigenvalue assignment problem in second-order control systems*, Journal of Sound and Vibration, Vol. 317, No. 1-2, (2008), pp. 1.

[6] S. Brahma, B. Datta, *An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures*, Journal of Sound and Vibration, Vol. 324, No. 3-5, (2009), pp. 471-489.

[7] Y.M. Ram, J.E. Mottershead, M.G. Tehrani, *Partial pole placement with time delay in structures using the receptance and the system matrices*, Linear Algebra and its Applications, Vol. 434, No. 7, (2011), pp. 1689-1696.

[8] K.V. Singh, R. Dey, B.N. Datta, *Partial eigenvalue assignment and its stability in a time delayed system*, Mechanical Systems and Signal Processing, Vol. 42, No. 1–2, (2014), pp. 247-257.

[9] Y.M. Ram, J.E. Mottershead, *Receptance method in active vibration control,* AIAA Journal, Vol. 45, No. 3, (2007), pp. 562-567.

[10] M. Ghandchi Tehrani, R.N.R. Elliott, J.E. Mottershead, *Partial pole placement in structures by the method of receptances: Theory and experiments*, Journal of Sound and Vibration, Vol. 329, No. 24, (2010), pp. 5017-5035.

[11] Y.M. Ram, J.E. Mottershead, *Multiple-input active vibration control by partial pole placement using the method of receptances*, Mechanical Systems and Signal Processing, Vol. 40, No. 2, (2013), pp. 727-735.

[12] A. Maha, Y.M. Ram, *The Method of Receptances for Continuous Rods*, Journal of Applied Mechanics, Vol. 81, No. 7, (2014), pp. 071009-071009.