# Integration

## The Poincare inequality

$$\int_\Omega |\nabla f(x)|^2 \, dx \geq C \int_\Omega f(x) \, dx$$
$$\frac{\int_\Omega |\nabla f(x)|^2 \, dx}{\int_\Omega f(x) \, dx} \geq C \text{ (the Poincare constant)}$$

💡 Example (The isoperimetric inequality): The following inequality is called an isoperimetric inequality, a geometric inequality

$$\frac{l^2}{S} \geq 4\pi \text{ (the Poincare constant)}$$

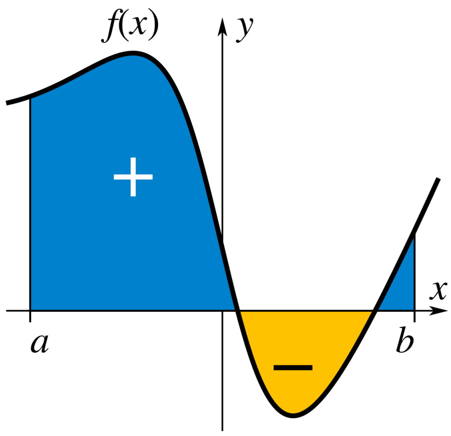
Isos means “equal” in Greek (e.g., isosceles triangle — having 2 sides of equal length). Isoperimetric means having the same perimeter. The isoperimetic problem tries to answer the question what planar region whose boundary is a curve of a certain length has the largest area? The inequality above says that the ratio between the squared length of a closed curve, $l^2$, and the area of the planar region bounded by that curve, $S$, is at least $4\pi$. The equality holds (dấu bằng xảy ra) when the closed curve we're talking about is a circle. [Source]

This inequality is crucial in Hamiltonian/Lagrangian Mechanics, which is crucial in Quantum Physics. To know these, we must learn Calculus of Variations (and a lot of other stuff).

# Review of Single Variable Calculus

Integration in $\mathbb{R}$

> Definite integral $\int_a^b f(x) \, dx$ (returns a number) → compute it numerically → can be approximated by the Riemann sum




Source: Wikipedia

Indefinite integral $\int f(x)\,dx$ (returns a family of functions) → require knowledge about the antiderivative

e.g., $\int x\,dx = \frac{x^2}{2} + C$

Fundamental Theorem of Calculus (FTC)

$F(x) = \int_a^x f(t)\,dt + C$ (do not know the formula, but still can do it with the antiderivative)

$F'(x) = f(x)$

# Transition to Integration in $\mathbb{R}^n$

Arc length of the curve $\vec{\gamma}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ (can be understood as distance, taking into account all the movements):

$$\int_a^b ||\vec{\gamma}'(t)||\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

Displacement (where you are):

$$\int_a^b \vec{\gamma}'(t)\,dt + \vec{\gamma}(a) = \vec{\gamma}(b)$$

Same technical challenge as 1D problems'.

# Integration in $\mathbb{R}^n$

$$F : \mathbb{R}^2 \to \mathbb{R}^n$$

$$D_{(x,y)}F = \begin{bmatrix} \partial_x F_1(x,y) & \partial_y F_1(x,y) \\ \partial_x F_2(x,y) & \partial_y F_2(x,y) \\ \vdots & \vdots \end{bmatrix}$$

## Double Riemann Sum

### Activity 1 & 2: Partition rectangles into sub-rectangles

Assume a rectangular domain $R = [0, 6] \times [2, 4]$. This is just the set $R = \{(x, y) : 0 \le x \le 6, 2 \le y \le 5\}$. Until now it's just a rectangle; we haven't partitioned it yet. Let's break it into parts. How? For example, divide the interval on the x-axis, $[0, 6]$, into 3 subintervals of equal size, and divide the interval on the y-axis, $[2, 4]$, into 2 subintervals of equal size.

1. 🔗Let $0 = x_0 < x_1 < x_2 < x_3 = 6$ be the endpoints of the subintervals of $[0, 6]$ after partitioning. What is the length $\Delta x$ of each subinterval $[x_{i-1}, x_i]$ for $i$ from 1 to 3?

   For $i$ from 1 to 3, the subintervals are $[x_0, x_1], [x_1, x_2], [x_2, x_3]$. The length of each subinterval is

   $$\Delta x = \frac{6 - 0}{3} = 2.$$

2. Explicitly identify $x_0, x_1, x_2, x_3$.

   $$\begin{aligned} &x_0 = 0 \\ &x_1 = x_0 + \Delta x = 0 + 2 = 2 \\ &x_2 = x_1 + \Delta x = x_0 + 2\Delta x = 0 + 2 \cdot 2 = 4 \\ &x_3 = 6\,(= x_2 + \Delta x = x_0 + 3\Delta x). \end{aligned}$$

   General formula (same for $y_j$, replace $i$ by $j$)

   $$x_i = x_0 + i \cdot \frac{b - a}{n}$$

$a$ and $b$ are endpoints; $n$ is the number of intervals

3. Let $2 = y_0 < y_1 < y_2 = 4$ be the endpoints of the subintervals of $[2, 4]$ after partitioning. What is the length $\Delta y$ of each subinterval $[y_{j-1}, y_j]$ for $j$ from 1 to 2? Explicitly identify $y_0, y_1, y_2$.

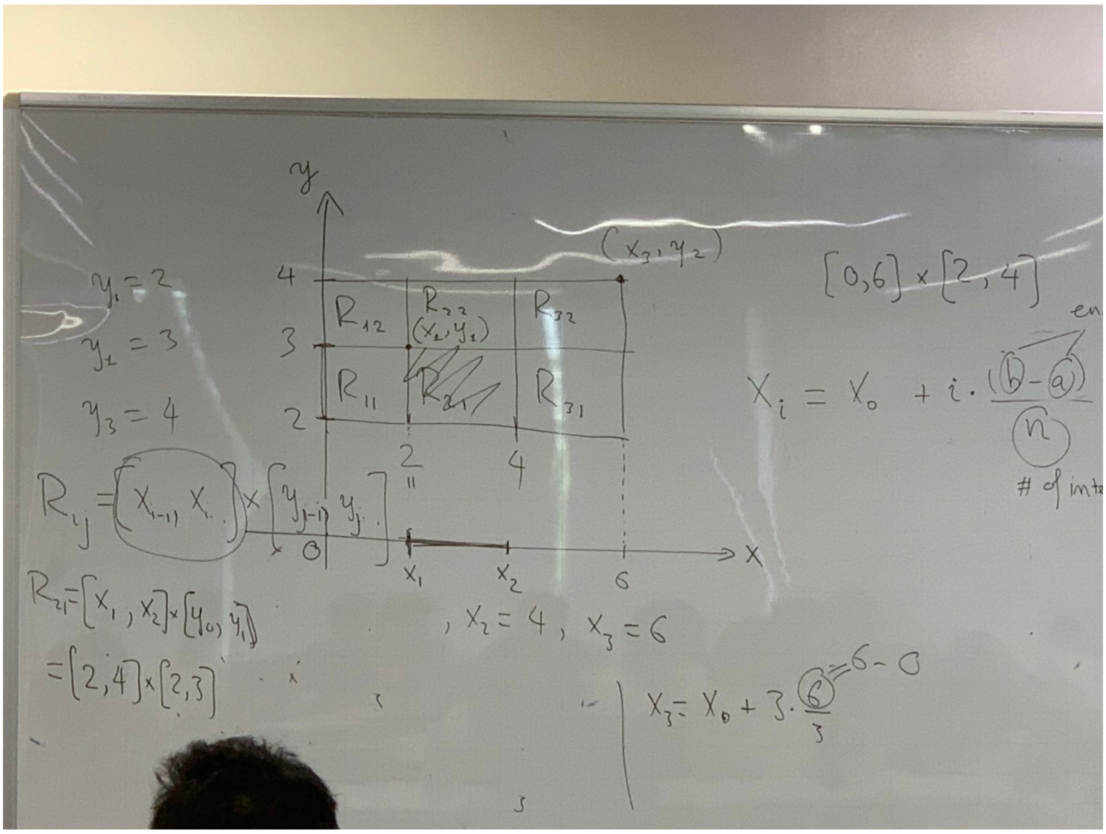
   For $j$ from 0 to 2, the subintervals are $[y_0, y_1]$, $[y_1, y_2]$. The length of each subinterval is

$$\Delta y = \frac{4-2}{2} = 1.$$

   The endpoints are

$$\begin{aligned} y_0 &= 2 \\ y_1 &= y_0 + \Delta y = 2 + 1 = 3 \\ y_2 &= 4\,(= y_1 + \Delta y = y_0 + 2\Delta y = 2 + 2 \cdot 1 = 4). \end{aligned}$$

4. Label the subrectangles.



💡 Example: Let $f(x, y) = 100 - x^2 - y^2$ be defined on the rectangular domain $R = [a, b] \times [c, d]$. Partition the interval $[a, b]$ into four uniformly sized subintervals and the interval $[c, d]$ into three evenly sized subintervals.

1. Let $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$. What is the length $\Delta x$ of each subinterval $[x_{i-1}, x_i]$? Your answer should be in terms of $a$ and $b$.

$$\Delta x = \frac{x_4 - x_0}{4} = \frac{b - a}{4}.$$

2. Let $c = y_0 < y_1 < y_2 < y_3 = d$. What is the length $\Delta y$ of each subinterval $[y_{j-1}, y_j]$? Your answer should be in terms of $c$ and $d$.

$$\Delta y = \frac{y_3 - y_0}{3} = \frac{d - c}{3}.$$

3. The partitions of the intervals $[a, b]$ and $[c, d]$ partition the rectangle $R$ into subrectangles. How many subrectangles are there?

$$4 \cdot 3 = 12 \text{ subrectangles}.$$

4. Let $R_{ij}$ denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. What is area $\Delta A$ of each subrectangle?

$$\Delta A = \Delta x \Delta y = \frac{b - a}{4} \cdot \frac{d - c}{3} = \frac{(b - a)(c - d)}{12}.$$

5. Now let $[a, b] = [0, 8]$ and $[c, d] = [2, 6]$. Let $(x_{11}^*, y_{11}^*)$ be the point in the upper right corner of the subrectangle $R_{11}$. Calculate the product $f(x_{11}^*, y_{11}^*)\Delta A$. Explain, geometrically, what this product represents.

   We have $\Delta x = \frac{8-0}{4} = 2$ and $\Delta y = \frac{6-2}{3} = \frac{4}{3}$, so

$$\begin{aligned} x_1 &= x_0 + 1 \cdot \Delta x = 0 + 1 \cdot 2 = 2 \\ y_1 &= y_0 + 1 \cdot \Delta y = 2 + 1 \cdot \frac{4}{3} = \frac{10}{3} \end{aligned}$$

   The subrectangle $R_{11}$ is

$$[x_0, x_1] \times [y_0, y_1] = [0, 2] \times [2, \frac{10}{3}],$$

and

$$f(x_{11}^*, y_{11}^*) = f(2, \frac{10}{3}) = 100 - 2^2 - (\frac{10}{3})^2 \approx 84.9.$$

The area of each subrectangle is $\Delta x \Delta y = 2 \cdot \frac{4}{3} = \frac{8}{3}$.

The product

$$f(x_{11}^*, y_{11}^*)\Delta A \approx 84.9 \cdot \frac{8}{3} \approx 226.4.$$

Geometrically, this is the volume of a rectangular box with the base as the subrectangle $R_{11}$ and the height $f(x_{11}^*, y_{11}^*)$.

6. If we were to add all the values $f(x_{ij}^*, y_{ij}^*)\Delta A$ for each $i$ and $j$, what does the resulting number approximate about the surface defined by $f$ on the domain $R$? (You don't actually need to add these values.)

   If we were to add all such values, we get the following sum

   $$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

   This sum approximates the volume of the solid under the surface defined by $f$ on the domain $R$.