CONSTRAINED OPTIMIZATION

You — YOUR OPPONENT

1st way.

rock — rock → 0
rock — paper → −1
rock — scissor → 1.

paper — rock → 1
paper — paper → 0
paper — scissor → −1

scissor — rock → −1
scissor — paper → 1.
scissor — scissor → 0

2nd way. (Pure Strategy)

(P2) Player2

(P1) Player 1.

	Rock	Paper	Scissor
Rock	0	−1	1
Paper	1	0	−1
Scissor	−1	1	0

Payoff Table. outcome for Player 1.
(Matrix)

A.

skew symmetric

$A^T = -A$

payoff for the 2nd player.

meaning if Player 1 wins, Player 2 loses & vice versa.

Q: What should you do? (How to win this game?)

- played once: no clear winning strategy.
- repeated many times: $\mathbb{P}(\text{winning}) = 1/3$.

John Nash: Mixed Strategy.

($\mathbb{P}$) should play each pure strategy (if play infinitely many times).

Probability

eg: $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ ← Player 1.

rock — every 2 times, pick Rock

paper — every 6 times pick paper

scissor — every 3 times, pick scissor.

John Nash: There's an EQUILIBRIUM of MIXED STRATEGY.

eg. $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ ← Player 2.

Given what the other player does, You CANNOT DO BETTER.
(suppose that the other player is also maximizing gain/minimizing loss).

Payoff matrix.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{matrix} \to R \\ \to P \\ \to S \end{matrix}$$
: probabilities Player 1's gonna play.

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{matrix} \to R \\ \to P \\ \to S \end{matrix}$$
: probabilities Player 2's gonna play.

A way to construct the score for each player.

$$\vec{a} \cdot (A\vec{b}) = (A\vec{b}) \cdot \vec{a}$$
$$= (A\vec{b})^T \vec{a}$$
$$= \vec{b}^T A^T \vec{a}.$$
$$= -\vec{b}^T A \vec{a}$$
$$= -\vec{b} \cdot (A\vec{a}).$$

$$\rightarrow \boxed{\vec{a} \cdot (A\vec{b}) \neq \vec{b} \cdot (A\vec{a})}$$

$$\underset{-\vec{b} \cdot (A\vec{a}).}{\shortparallel}$$

Goal. Define the score that makes sense.

e.g. $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow$ just rock for player 2.

$$A\vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

@ $\vec{a} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 1$ always win

$\vec{a}$ should be $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for player 1.

@ if $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \vec{a} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = -1$ lose.

just play scissor

@ if $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \vec{a} \cdot \begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix} = 0$ draw.

just play rock.

$\rightarrow \vec{a} \cdot (A\vec{b})$ = score for PLAYER 1 if player 1 plays $\vec{a}$ & player 2 plays $\vec{b}$

e.g. $\vec{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \leftarrow$ Player 2.

$A\vec{b} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \rightarrow$ Q. what's the strategy for Player 1?

best strategy for Player 1: (guess work).

$\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \vec{a} \cdot (A\vec{b}) = \frac{1}{2}.$

just play paper.

how to know this is the best score & the best strategy?

Formally

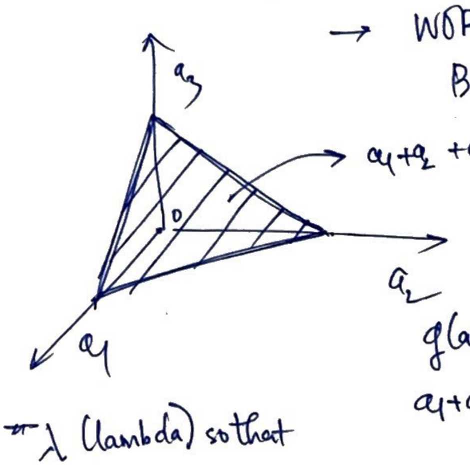
$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \quad \begin{pmatrix} a_1 + a_2 + a_3 = 1 \\ a_i \geq 0 \end{pmatrix}$

$$\vec{a} \cdot (A\vec{b}) = -\frac{1}{2} a_1 + \frac{1}{2} a_2 = f(a_1, a_2, a_3)$$

$\rightarrow \nabla f = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$. $\rightarrow$ cannot set $\nabla f = 0$

$\rightarrow$ NO CRITICAL POINTS.

$\rightarrow$ WORK ON THE BOUNDARY.



$a_1 + a_2 + a_3 = 1.$

$g(a_1, a_2, a_3) = a_1 + a_2 + a_3 - 1.$

Find λ (lambda) so that

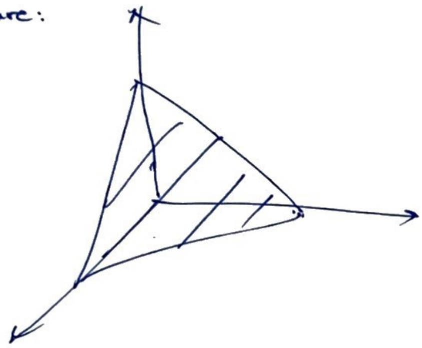
$$\nabla f(a_1, a_2, a_3) = \lambda \nabla g(a_1, a_2, a_3)$$

$$\nabla f(a_1,a_2,a_3) = \lambda \nabla g(a_1,a_2,a_3)$$

$$\rightarrow \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\downarrow$ no such λ exists

The picture:



<u>Check on the boundary:</u>

① $a_1 + a_3 = 1$, $a_2 = 0$.

$\rightarrow f(a_1,a_2,a_3) = -\frac{a_1}{2} + \frac{a_2}{2} = -\frac{a_1}{2}$.

max at $a_1 = 0,\ a_2 = 0,\ a_3 = 1$.

$f(0,0,1) = 0$.

② $a_1 + a_2 = 1$, $a_3 = 0$.

$\rightarrow f(a_1,a_2,a_3) = -\frac{a_1}{2} + \frac{a_2}{2} = \frac{1}{2} - a_1$.

$\rightarrow$ max at $(0,1,0)$

$f(0,1,0) = \frac{1}{2}$.

③ $a_2 + a_3 = 1$, $a_1 = 0$

$\rightarrow f(a_1,a_2,a_3) = -\frac{a_1}{2} + \frac{a_2}{2} = \frac{a_2}{2}$.

$\rightarrow$ max at $(0,1,0)$.

$f(0,1,0) = \frac{1}{2}$.

$\rightarrow$ Player ① should choose $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ if

Player ② chooses $\vec{b} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$

General setup:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

$$f_1(\vec{a},\vec{b}) = \vec{a} \cdot (A\vec{b}).$$

$$f_2(\vec{a},\vec{b}) = \vec{b} \cdot (A\cancel{\vec{b}}\,\vec{a}).$$

Player 1 tries to maximize f_1

Player 2 tries to maximize $f_2 = -f_1$

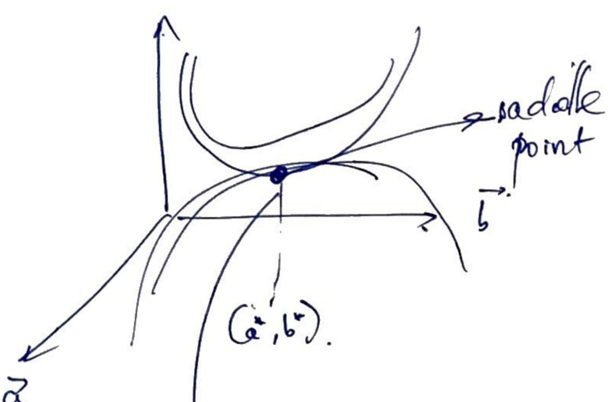
$\Leftrightarrow$ ~~minimize~~ maximize $-f_1$

$\Leftrightarrow$ minimize f_1.

$\rightarrow$ The game ends iff

$$\underbrace{\max_{\vec{a}}}_{\text{max in } \vec{a} \text{ direction}} \ \underbrace{\min_{\vec{b}} f_1(\vec{a},\vec{b})}_{\text{min in } \vec{b} \text{ direction}}.$$

$$\Leftrightarrow \frac{\partial f_1}{\partial \vec{a}} = 0 \ \ \& \ \ \frac{\partial f_1}{\partial \vec{b}} = 0.$$



$[H_f](a^*,b^*) < 0$

$$[H_f] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow \text{has both positive \& negative eigenvalues.}$$

$\vec{a} \cdot \cancel{Hf}$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \vec{b} = f(\vec{a}, \vec{b})$$

$$\cancel{\frac{\partial f}{\partial a}} =$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_3 - b_2 \\ b_1 - b_3 \\ b_2 - b_1 \end{pmatrix}.$$

$$= a_1(b_3 - b_2) + a_2(b_1 - b_3) + a_3(b_2 - b_1).$$

$$\nabla f = \begin{pmatrix} b_3 - b_2 \\ b_1 - b_3 \\ b_2 - b_1 \\ a_2 - a_3 \\ a_3 - a_1 \\ a_1 - a_2. \end{pmatrix} = 0$$

$$\rightarrow \quad a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = \frac{1}{2}.$$