Random Processes A1

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1 Question 1

CDF:

$$P(Z <= z) = \begin{cases} 0 & z < 0 \\ 0.5z & 0 \leq z < 1 \\ 0.25 + 0.25z & 1 \leq z < 3 \\ 1 & 3 \geq z \end{cases}$$

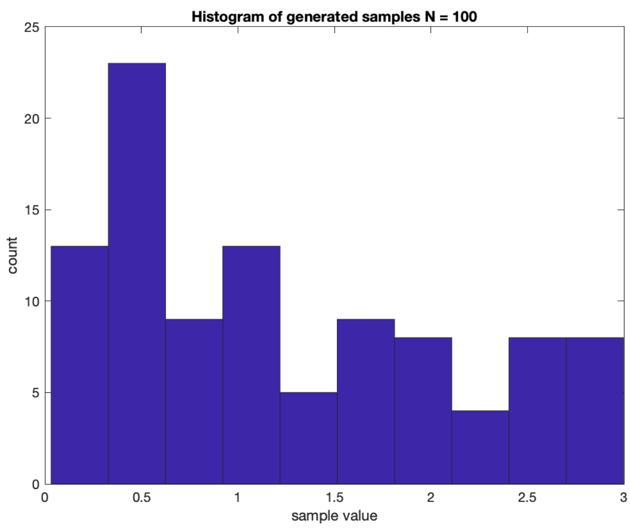
we can then find the PDF by differentiating every interval with respect to z

PDF:

$$P(Z = z) = \begin{cases} 0 & z < 0 \\ 0.5 & 0 \leq z < 1 \\ 0.25 & 1 \leq z < 3 \\ 0 & 3 \geq z \end{cases}$$

1.a

Generate N=100 independent samples of Z and plot the histogram



1.b

Find the average of the samples and compare it to the expected value.

$$\mathbb{E}[z] = \int x\ f(x)\ dx = \int_0^1 0.5x\ dx + \int_1^3 0.25x\ dx = \frac{1}{4}\ x^2\ |_0^1 + \frac{1}{8}x^2\ |_1^3 = \frac{2}{8} + \frac{9}{8} - \frac{1}{8} = 1\ \frac{1}{4}$$

empirical mean is 1.2354

1.c

Find the empirical variance and compare it to the true variance

$$Var(X) = E[X^2] - E[X]^2$$

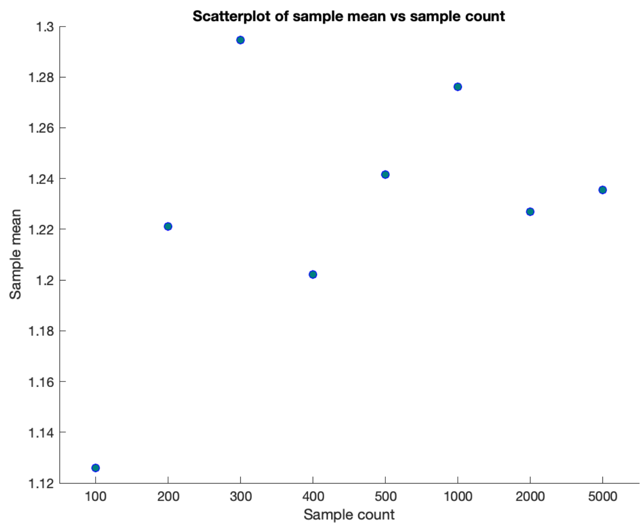
$$E[X^2] = \int x^2\ f(x)dx = \int_0^1 0.5x^2\ dx + \int_1^3 0.25x^2\ dx = \frac{1}{6}x^3|0^1 + \frac{1}{12}x^3|_1^3 = 2 + \frac{1}{3}$$

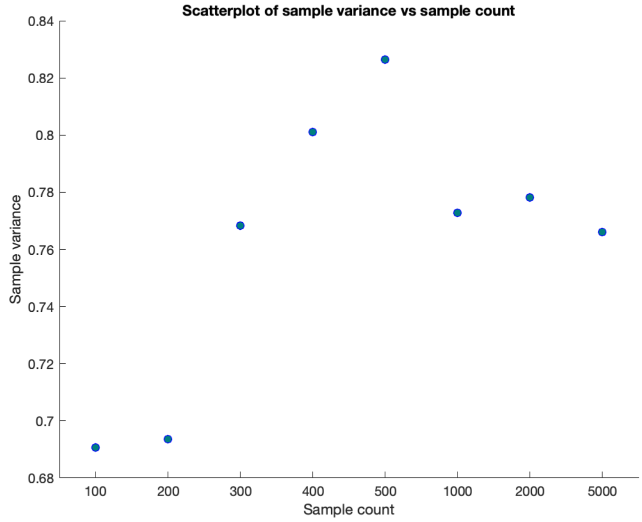
$$2\frac{1}{3} - \mu^2 = 2\frac{1}{3} - 1.56255 \approx 0.771$$

Empirical variance is 0.766

1.d

For N= 100,200,300,400,500,1000,2000,5000 samples of Z find the average and variance and plot them as a function of N. Do they converge as N increases?





As shown in the plots the sample mean and sample variance converge to the true variance and true mean.

2 Question 2

Consider two biased dice each with the following PMF:

$$P(X = 1) = P(X = 2) = 0.25$$

$$P(X = 3) = P(X = 4) = P(X = 5) = P(X = 6) = 0.125$$

2.a

Generate $N = 100$ independent samples of the two dice tosses denoted by (X, Y), and show the joint empirical PMF in a table.

Below, we show the joint empirical PMF table of (X, Y) for $N = 100$ independent tosses of both dice. Likewise, we also include the marginal probabilities of X and Y, denoted as Px and Py, respectively.

MATLAB OUTPUT 2.a:

```
Pxy =

0.0500    0.0900    0.0100    0.0500    0.0300    0.0300
0.0100    0.1000    0.0200    0.0100    0.0100    0.0300
0.0300    0.0600    0.0300    0.0100    0.0100    0.0400
0.0500    0.0300         0    0.0400    0.0200    0.0100
0.0300    0.0100    0.0200    0.0100         0    0.0100
0.0100    0.0600    0.0200    0.0400         0    0.0200


Px =

0.1800    0.3500    0.1000    0.1600    0.0700    0.1400
```

```
Py =

0.2600
0.1800
0.1800
0.1500
0.0800
0.1500
```

2.b

In order for (X, Y) to be independent, the following must hold:

$$P(X, Y) = P(X)P(Y).$$

We show empirically that this is true by multiplying the empirical marginal PMF's, Px and Py, element wise, i.e. $Px_1 \times Py_1 = Pxy_{1,1}, Px_1 \times Py_2 = Pxy_{1,2}, \ldots Px_6 \times Py_6 = Pxy_{6,6}$. Below we show the output and the accompanying MATLAB code.

MATLAB OUTPUT 2.b:

```
ind_check_XY =

0.0468    0.0910    0.0260    0.0416    0.0182    0.0364
0.0324    0.0630    0.0180    0.0288    0.0126    0.0252
0.0324    0.0630    0.0180    0.0288    0.0126    0.0252
0.0270    0.0525    0.0150    0.0240    0.0105    0.0210
0.0144    0.0280    0.0080    0.0128    0.0056    0.0112
0.0270    0.0525    0.0150    0.0240    0.0105    0.0210
```

Comparing the product of the marginals to the empirical joint PMF table from part 2.a, there is well enough of a match between every element to confirm independence. However, it is important to note that due to the small sample size, i.e. $N = 100$ independent dice tosses, the product of the marginals and the empirical joint PMF do not fully match. If the sample size was larger, for instance when $N = 100,000$, we get the following output:

```
Pxy =

0.0625    0.0627    0.0320    0.0314    0.0303    0.0309
0.0624    0.0619    0.0317    0.0306    0.0311    0.0326
0.0311    0.0314    0.0154    0.0151    0.0152    0.0161
0.0315    0.0310    0.0155    0.0159    0.0164    0.0154
0.0311    0.0306    0.0158    0.0155    0.0158    0.0158
0.0313    0.0311    0.0154    0.0156    0.0163    0.0156

ind_check_XY =

0.0624    0.0621    0.0314    0.0310    0.0312    0.0316
```

```
    0.0625    0.0622    0.0315    0.0311    0.0313    0.0316
    0.0310    0.0309    0.0156    0.0154    0.0155    0.0157
    0.0314    0.0313    0.0158    0.0156    0.0157    0.0159
    0.0311    0.0310    0.0157    0.0155    0.0156    0.0158
    0.0313    0.0312    0.0158    0.0156    0.0157    0.0159
```

which has a significantly higher match.

2.c

In a similar manner to part 2.a, we generate the empirical joint PMF table for $(Z1, Z2)$. The MATLAB output, denoted as Pzz, and accompanying code is listed below.

Following the same process as in part 2.b, we check whether $(Z1, Z2)$ are independent by multiplying the marginals $Pz1$ and $Pz2$ element wise. In other words, for $(Z1, Z2)$ to be independent, the following condition mus be met:

$$P(Z1, Z2) = P(Z1)P(Z2).$$

Below we show the MATLAB output of the marginals and their element wise products. From the mismatch between the product of the marginals and the empirical joint PMF of $(Z1, Z2)$, it is clear that $(Z1, Z2)$ are not independent.

MATLAB OUTPUT 2.c:

```
Pzz =

 0         0         0         0         0    0.0100         0         0         0         0         0
 0         0         0         0    0.0300         0    0.0600         0         0         0         0
 0         0         0    0.0500         0    0.0100         0    0.0200         0         0         0
 0         0    0.0300         0    0.0300         0    0.0200         0    0.0400         0         0
 0    0.0100         0    0.0600         0         0         0    0.0100         0         0         0
0.0500         0    0.1000         0    0.0300         0    0.0400         0         0         0    0.0200
 0    0.0900         0    0.0200         0    0.0100         0    0.0200         0    0.0100         0
 0         0    0.0100         0    0.0100         0    0.0100         0    0.0100         0         0
 0         0         0    0.0500         0    0.0100         0    0.0400         0         0         0
 0         0         0         0    0.0300         0    0.0300         0         0         0         0
 0         0         0         0         0    0.0300         0         0         0         0         0


Pz1 =

0.0500    0.1000    0.1400    0.1800    0.1300    0.0700    0.1600    0.0900    0.0500    0.0100    0.0200


Pz2 =

0.0100
0.0900
0.0800
0.1200
0.0800
0.2400
0.1500
0.0400
0.1000
0.0600
0.0300

ind_check_ZZ =

0.0005    0.0010    0.0014    0.0018    0.0013    0.0007    0.0016    0.0009    0.0005    0.0001    0.0002
0.0045    0.0090    0.0126    0.0162    0.0117    0.0063    0.0144    0.0081    0.0045    0.0009    0.0018
0.0040    0.0080    0.0112    0.0144    0.0104    0.0056    0.0128    0.0072    0.0040    0.0008    0.0016
0.0060    0.0120    0.0168    0.0216    0.0156    0.0084    0.0192    0.0108    0.0060    0.0012    0.0024
0.0040    0.0080    0.0112    0.0144    0.0104    0.0056    0.0128    0.0072    0.0040    0.0008    0.0016
0.0120    0.0240    0.0336    0.0432    0.0312    0.0168    0.0384    0.0216    0.0120    0.0024    0.0048
0.0075    0.0150    0.0210    0.0270    0.0195    0.0105    0.0240    0.0135    0.0075    0.0015    0.0030
0.0020    0.0040    0.0056    0.0072    0.0052    0.0028    0.0064    0.0036    0.0020    0.0004    0.0008
0.0050    0.0100    0.0140    0.0180    0.0130    0.0070    0.0160    0.0090    0.0050    0.0010    0.0020
0.0030    0.0060    0.0084    0.0108    0.0078    0.0042    0.0096    0.0054    0.0030    0.0006    0.0012
0.0015    0.0030    0.0042    0.0054    0.0039    0.0021    0.0048    0.0027    0.0015    0.0003    0.0006
```

Again, if we increase the sample size to $N = 100,000$, it becomes more clear that the above condition on independence does not hold for $(Z1, Z2)$.

Pzz =

0	0	0	0	0	0.0313	0	0	0	0	0
0	0	0	0	0.0311	0	0.0311	0	0	0	0
0	0	0	0.0315	0	0.0306	0	0.0154	0	0	0
0	0	0.0311	0	0.0310	0	0.0158	0	0.0156	0	0
0	0.0624	0	0.0314	0	0.0155	0	0.0155	0	0.0163	0
0.0625	0	0.0619	0	0.0154	0	0.0159	0	0.0158	0	0.0156
0	0.0627	0	0.0317	0	0.0151	0	0.0164	0	0.0158	0
0	0	0.0320	0	0.0306	0	0.0152	0	0.0154	0	0
0	0	0	0.0314	0	0.0311	0	0.0161	0	0	0
0	0	0	0	0.0303	0	0.0326	0	0	0	0
0	0	0	0	0	0.0309	0	0	0	0	0

ind_check_ZZ =

0.0020	0.0039	0.0039	0.0039	0.0043	0.0048	0.0035	0.0020	0.0015	0.0010	0.0005
0.0039	0.0078	0.0078	0.0078	0.0086	0.0096	0.0069	0.0039	0.0029	0.0020	0.0010
0.0048	0.0097	0.0097	0.0098	0.0107	0.0120	0.0086	0.0049	0.0036	0.0025	0.0012
0.0058	0.0117	0.0117	0.0118	0.0129	0.0145	0.0103	0.0059	0.0044	0.0030	0.0015
0.0088	0.0176	0.0176	0.0178	0.0195	0.0218	0.0156	0.0089	0.0066	0.0045	0.0022
0.0117	0.0234	0.0234	0.0236	0.0259	0.0289	0.0207	0.0119	0.0088	0.0060	0.0029
0.0089	0.0177	0.0177	0.0179	0.0196	0.0219	0.0157	0.0090	0.0066	0.0045	0.0022
0.0058	0.0116	0.0116	0.0117	0.0129	0.0144	0.0103	0.0059	0.0044	0.0030	0.0015
0.0049	0.0098	0.0098	0.0099	0.0109	0.0122	0.0087	0.0050	0.0037	0.0025	0.0012
0.0039	0.0079	0.0079	0.0079	0.0087	0.0097	0.0070	0.0040	0.0029	0.0020	0.0010
0.0019	0.0039	0.0039	0.0039	0.0043	0.0048	0.0034	0.0020	0.0014	0.0010	0.0005