LINEAR ALGEBRA REVIEW

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1. Vectors, Matrices, and Linear Systems

1.1. Vectors in Euclidean Space.

- **Euclidean n-space**, denoted $\mathbb{R}^n$, consists of all ordered n-tuples of real numbers. Each n-tuple x can be regarded as a point $(x_1, x_2, \ldots, x_n)$ and represented graphically as a dot, or regarded as a vector. In vector form, x would be written as $\vec{x} = [x_1, x_2, \ldots, x_n]$, illustrated by an arrow. The n-tuple $\vec{0} = [0, 0, \ldots, 0]$ is called the **zero vector**.
- Vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ can be added and subtracted. They can also be multiplied by scalars $r \in \mathbb{R}$. The addition or scalar multiplication is performed on the components of the vectors.
- We say that two vectors are **parallel** if one vector is a scalar multiple of another.
- A **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector of the form $r_1\vec{v}_1, r_2\vec{v}_2, \ldots, r_n\vec{v}_n$ for $r_i \in \mathbb{R}$. The set of all linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is denoted $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$.
- Every vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of the **standard basis vectors** $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ where $\vec{e}_i$ has 1 as its i^{th} component and zeros for all other components.

1.2. The Norm and the Dot Product.

- Let $\vec{v} = [v_1, v_2, \ldots, v_n]$ be a vector in $\mathbb{R}^n$. The **norm** or **magnitude** of $\vec{v}$ is denoted $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
- Some properties of the norm are:
  (1) $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.
  (2) $\|r\vec{v}\| = |r|\|\vec{v}\|$.
  (3) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

- A vector in $\mathbb{R}^n$ is a **unit vector** if and only if it has magnitude 1. In order to make a vector $\vec{v} = [v_1, v_2, \ldots, v_n]$ a unit vector, divide it by its magnitude:
$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{[v_1, v_2, \ldots, v_n]}{\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$$
- The **dot product** of two vectors is equal to the product of their magnitudes with the cosine and the angle between them. For two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ given by $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$, we define:
$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \|\vec{v}\| \|\vec{w}\| \cos(\theta)$$
- We can find the angle between two vectors using this definition:
$$\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}\right) = \cos^{-1}\left(\frac{v_1 w_1 + v_2 w_2 + \cdots + v_n w_n}{\|\vec{v}\| \|\vec{w}\|}\right)$$
- Two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are **orthogonal** or **perpendicular** if and only if $\vec{v} \cdot \vec{w} = 0$.

1.3. **Matrices and Their Algebra.**

- An $m \times n$ **matrix** is an ordered rectangular array of numbers containing m rows and n columns. We say that an $m \times 1$ matrix is a **column vector** with m components, and a $1 \times n$ matrix is a **row vector** with n components.
- The product $A\vec{b}$ of an $m \times n$ matrix A and a column vector $\vec{b}$ with n rows is a column vector. This column vector is equal to a linear combination of the column vectors of A where the scalar coefficient of the j^{th} column vector of A is b_j.
- The product AB of an $m \times n$ matrix A and an $n \times s$ matrix B is the $m \times s$ matrix C whose j^{th} column is A times the j^{th} column of B. The entry c_{ij} is the dot product of the i^{th} row vector of A and the j^{th} column vector of B. In general, $AB \neq BA$.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, then $A + B$ is the matrix of that size with entry $a_{ij} + b_{ij}$ in the i^{th} row and j^{th} column.
- For any matrix A and scalar r, the matrix rA is found by multiplying each entry in A by r.
- The **transpose** of an $m \times n$ matrix A is the $m \times n$ matrix A^T, which has its k^{th} row vector as the k^{th} column vector of A.

1.4. **Solving Systems of Linear Equations.**

- A linear system has an associated **augmented matrix**, having the coefficient matrix of the system on the left of the partition and the column vector of constants on the right of the partition.
- The elementary row operations on a matrix are as follows:

(1) Interchanging two rows.

(2) Multiplication of a row by a nonzero scalar.

(3) Addition of a multiple of a row to a different row.

- Matrices A and B are **row equivalent** if and only if A can be transformed into B by a sequence of elementary row operations. We denote this $A \sim B$.
- If $A\vec{x} = \vec{b}$ and $H\vec{x} = \vec{c}$ are systems such that the augmented matrices $[\,A \,|\, \vec{b}\,]$ and $[\,H \,|\, \vec{c}\,]$ are row equivalent, then the systems $A\vec{x} = \vec{b}$ and $H\vec{x} = \vec{c}$ have the same solution set.
- A matrix is in **row-echelon form** if:

  (1) All rows containing only zeros are grouped together at the bottom of the matrix.

  (2) The first nonzero element (called a **pivot**) in any row appears in a column to the right of the first nonzero element in any preceding row.
- A matrix is in **reduced row-echelon form** if it is in row-echelon form and each pivot is 1 and the only nonzero element in its column. Every matrix is row-equivalent to a unique matrix in reduced row-echelon form.
- We can solve linear systems by augmenting a system's matrix with the **identity matrix** and reducing. If the matrix can be put into reduced echelon form, the solutions are on the augmented portion. This technique is called the **Gauss-Jordan** method.
- A linear system $A\vec{x} = \vec{b}$ has no solutions if and only if after reducing the matrix $[\,A \,|\, \vec{b}\,]$ so that A is in row-echelon form, there exists a row with only zero entries to the left of the partition but with a nonzero entry to the right of the partition. We call this type of system **inconsistent**.
- Let A be an $m \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ is **consistent** if and only if the vector $\vec{b} \in \mathbb{R}^n$ is in the span of the column vectors of A.
- The following theorem describes consistent systems. Let $A\vec{x} = \vec{b}$ be consistent and suppose $[\,A \,|\, \vec{b}\,] \sim [\,H \,|\, \vec{c}\,]$. Then:

  Theorem 1.1. *If every column of H contains a pivot, then the system has a unique solution. If some column of H has no pivot, then the system has infinitely many solutions, with as many free variables as there are pivot-free columns in H.*
- An **elementary matrix** E is obtained by applying a single elementary row operation to an identity matrix I. Multiplication of a matrix A by E on the left performs the same elementary row operation on A.