$p(x)$: densità di probabilità , $x \in R$

(in breve pdf = probability density function)

$$P(x) = \int_{-\infty}^{x} dx' \, p(x') \quad : \text{ comulative probability}$$

$P'(x) = p(x) \geq 0 \quad \Rightarrow \quad P(x)$ monotonica non decrescente

$\Rightarrow \quad \exists \, P^{-1}$ uniforme random

$\Rightarrow \quad x = P^{-1}(\rho) \quad , \quad 0 \leq \rho \leq 1$

Esempio:

$$p(x) = \begin{cases} e^{-x} , & x \geq 0 \\ 0 , & x < 0 \end{cases} \qquad P(x) = 1 - e^{-x} \quad , \quad x \geq 0$$

$$= 1 - \rho \quad , \quad 0 \leq \quad \leq 1$$

$(1 - \rho$ è distribuito come $\rho)$ $\qquad e^{-x} = \rho \quad , \quad x = -\log \rho$

Ma se $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (gaussiana)

$P(x) =$ error function, $P^{-1} = ?$

Tuttavia se $p(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$ (gaussiana isotropica in 2D)

allora $p(x,y)\,dx\,dy = \frac{r}{2\pi} e^{-r^2/2}\,dr\,d\theta$, $\quad x = r\cos\theta$, $y = r\sin\theta$

e $r e^{-r^2/2} = \frac{d}{dr} e^{-r^2/2}$! $\Rightarrow 1 - e^{-r^2/2} = 1 - \rho_1$, $r = \sqrt{-2\log\rho_1}$, $0 \le \rho_1 \le 1$

$\theta = 2\pi\rho_2$ $\quad 0 \le \rho_2 \le 1$

Ancora meglio:

$$\frac{1}{2\pi} e^{-r^2/2}\, r\,dr = \frac{1}{\pi}\tilde{r}\,d\tilde{r}, \quad 0 \le \tilde{r} \le 1 \quad \left(\begin{array}{l} r \to 0,\ \tilde{r} \to 1 \\ r \to \infty,\ \tilde{r} \to 0 \end{array} \right)$$

quindi

$$\tilde{r}\frac{d\tilde{r}}{dr} = \frac{1}{2}\frac{d\tilde{r}^2}{dr} = \frac{1}{2} r e^{-r^2/2} = \frac{1}{2}\frac{d}{dr} e^{-r^2/2}$$

$$\tilde{r}^2 = e^{-r^2/2} \quad , \quad r = \sqrt{-2\log \tilde{r}^2}$$

e

$$-1 \le \tilde{x} \le 1 \atop -1 \le \hat{y} \le 1 \quad , \quad d\tilde{x}\, d\tilde{y} = \tilde{r}\, d\tilde{r}\, d\vartheta \quad , \quad \begin{matrix} \tilde{x} = \tilde{r}\cos\vartheta \\ \hat{y} = \tilde{r}\sin\vartheta \end{matrix}$$

$$\boxed{\begin{matrix} x = r\cos\vartheta = \dfrac{r}{\tilde{r}}\hat{x} \\ y = r\sin\vartheta = \dfrac{r}{\tilde{r}}\hat{y} \end{matrix}}$$