2.1 Approximate dynamics

The complete dynamics of the field equation (1.3) can be handled only with a numeric approach, but we can adopt some approximations in order to study it analytically.

First of all, it is convenient to rewrite the field ϕ by separating the mean value from the fluctuations,

$$\phi(\boldsymbol{x}, t) = \phi_0(t) + \eta(\boldsymbol{x}, t) \qquad \text{with } \eta(\boldsymbol{x}, t) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k} \neq 0} \tilde{\phi}_{\boldsymbol{k}} e^{i \boldsymbol{k} \cdot \boldsymbol{x}},$$

where $\phi_0(t) \equiv \langle \phi(\boldsymbol{x}, t) \rangle$ and clearly $\langle \eta(\boldsymbol{x}, t) \rangle = 0$.

The equation of motion after the split reads

$$\ddot{\phi}_0 + \xi^2 \phi_0 + \phi_0^3 + \left(\Box + \xi^2 + 3\phi_0^2 \right) \eta + 3\phi_0 \eta^2 + \eta^3 = 0, \tag{2.1}$$

with $\phi_0 = \phi_0(t)$, $\eta = \eta(\boldsymbol{x}, t)$ and $\Box = \partial^2/\partial t^2 - \sum_i \partial^2/\partial x_i^2$ is the Dalambertian operator.

Starting from the above definitions we can adopt various approximations, from the simpler to the more accurate ones.

2.1.1 Zeroth order dynamics

In the zero-order approximation there are no fluctuations at all, $\eta(\boldsymbol{x}, t) = \dot{\eta}(\boldsymbol{x}, t) = 0$, and equation (2.1) reduces to

$$\ddot{\phi}_0 + \xi^2 \phi_0 + \phi_0^3 = 0 \tag{2.2}$$

which has an exact solution in terms of the *cnoidal* function

$$\phi_0(t) = \phi_c(t) \equiv \phi_0(0) \ \mathrm{cn} \left(t\sqrt{\xi^2 + \phi_0^2(0)}, k \right), \quad k \equiv \frac{\phi_0(0)}{\sqrt{2\left(\xi^2 + \phi_0^2(0)\right)}}, \tag{2.3}$$

where $\mathrm{cn}(x, k)$ is the Jacobi cosinus of *modulus* k and we have assumed the initial condition $\dot{\phi}_0(0) = 0$, which can always be chosen without loss of generality by taking an appropriate origin for the time axis.

The solution satisfies the periodicity condition

$$\phi_c(t) = -\phi_c(t + P), \ \ \text{with} \ \ P = \frac{2K(k)}{\sqrt{\xi^2 + \phi_0^2}} \ \ \text{and} \ \ K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

where P is the half-period and $K(k)$ is the complete elliptic integral of the first kind.

Since we are adopting the rescaling described in section 1.1.1, the initial condition is fixed to $\phi_0(0) = 1$ and the generic solution is thus

$$\phi_c(t; \xi) = \ \mathrm{cn} \left(t\sqrt{\xi^2 + 1}, \frac{1}{\sqrt{2\left(\xi^2 + 1\right)}} \right). \tag{2.4}$$

The trend of equation (2.4) is plotted in Figure 2.1 for different values of ξ.