Domino Magic
Threefold Problem Set

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ETH Zürich

Threefold Problem Set

Goal: simulate the thinking steps of a six hour exam in one hour.

Exam Preparation Intro (17:15-17:35)

First Hour: (17:35-18:20)

- 3 problems on paper
- think about them
- sketch your solution on paper
- no coding required
- provide us with some feedback

Break: (18:20-18:30)

- fill out the questions on eduapp

Second Hour: (18:30-19:00)

- solution discussion
- feedback collection

At the end: (19:00-...)

- Q &A session

Fetch two sheets of paper at the entrance!

Go to https://eduapp-app1.ethz.ch/ or use ETH Edu App for the second hour.

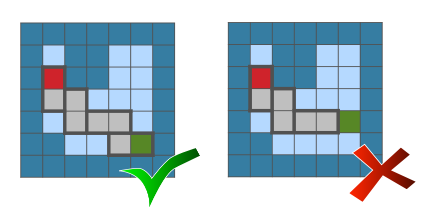
Domino Snake – The problem

Given:

- $h \times w$ grid with obstacles
- p queries of point pairs $((q, r), (s, t))$

Wanted:

- **y** or **n** per query: Does a domino snake between the two points exist?



Rephrased in graph theory language:
What corresponds to a *domino snake*?

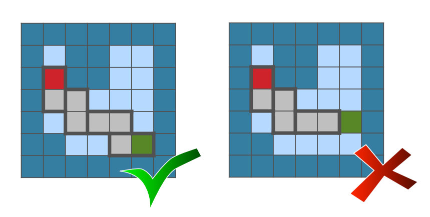
- a (q, r)-(s, t)-path on the 4-neighborhood grid graph with holes
- this path needs to have even length (i.e. an even number of vertices)

Is even length necessary and sufficient?

- *necessary*: odd length paths can not be tiled into dominos of area 2.
- *sufficient*: a path $P = (p_1, p_2, \ldots, p_l)$ of even length $l = 2k$ can always be tiled into dominoes of the form (p_1, p_2), (p_3, p_4), $\ldots$, (p_{l-1}, p_l).

Domino Snake – Handling a single query

BFS can answer if any path from (q, r) to (s, t) exists and if possible gives us the shortest such path.

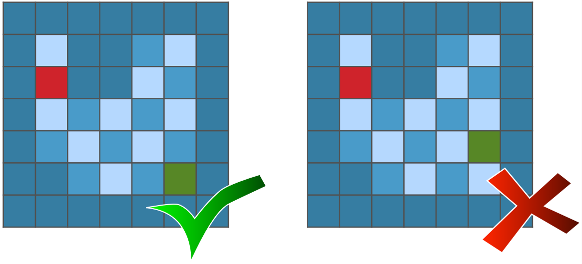


What if this path is of odd length?

Can we add a detour to find a path of the right parity?

No! Chessboard coloring argument.

We require: $(q + r) \not\equiv_2 (s + t)$



Domino Snake – Handling multiple queries

$p = 1$: check that BFS from (q, r) visits (s, t) and $(q + r) \not\equiv_2 (s + t)$.

$p > 1$:

- Running BFS over and over is expensive $\to \mathcal{O}(hwp)$
- We do not really need the shortest path.
- We just ask: Are (q, r) and (s, t) in the same connected component?
- Precompute the connected components in $\mathcal{O}(hw)$.
- Each of the p queries can then be answered in $\mathcal{O}(1)$ by checking
  - $component((q, r)) \stackrel{?}{=} component((s, t))$
  - $(q + r) \not\equiv_2 (s + t)$.

Overall runtime: $\mathcal{O}(hw + p)$

New Tiles – The problem

Problem

Given a $h \times w$ matrix of 0's and 1's.

Find the maximum number of non-overlapping 2×2 matrices of the form: $\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$

Is there a greedy solution, ideas?

Greedily take every 2x2 free space.

Counterexample:

```
0110
1111
1111
```

Greedy gives answer 1. Maximum is 2.

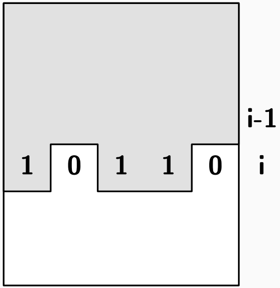
(We ignore the zeros at the boundary from now on.)

New Tiles – Dynamic programming

We look at bitmasks: j-th bit is 1 iff j-th column may be used (regardless of the input)

State of the subproblem: $[i, bitmask]$, $1 \leq i \leq h$, $0 \leq bitmask \leq 2^w - 1$

$DP[i][bitmask]$:= maximum number of 2x2 matrices we can place on top of the first i rows where the i-th row is constrained to the $bitmask$.



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Initialization:

- In a single row, no tiles fit: $DP[1][bitmask] = 0$, for any $bitmask$
- If we do not use the new row at all: $DP[i][\underbrace{00\ldots0}_{w}] = \max\limits_{bitmask}(DP[i-1][bitmask])$

New Tiles – Dynamic programming

Recurrent formula for placing new tiles:

We have to take some 2×2 matrices in the $i-1, i$ strip

1	1	1	1	0	0	1	0	0	0	0	1	1	0	0	lookup in row i-1
0	0	0	0	1	1	0	1	1	1	1	0	0	1	1	state in row i

0	1	1	0	1	1	1	1	1	1	1	1	0	1	1	row i-1 of the input
0	1	1	1	1	1	0	1	1	1	1	0	1	1	1	row i of the input

Try only those bitmasks that have even number of consecutive 1s.

All others are covered by removing single bits and not adding anything new.

Check whether this maximal amount of 2×2 matrices is compatible with the input matrix. If yes, check whether this would lead to a larger tiling.

New Tiles – Dynamic programming

Recurrent formula for placing new tiles:

We have to take some 2×2 matrices in the $i-1, i$ strip

1	1	1	1	0	0	1	0	0	0	0	1	1	0	0	lookup in row i-1
0	0	0	0	1	1	0	1	1	1	1	0	0	1	1	state in row i

0	1	1	0	1	1	1	1	1	1	1	1	0	1	1	row i-1 of the input
0	1	1	1	1	1	0	1	1	1	1	0	1	1	1	row i of the input

For the example:

$$DP[i][000011011110011] \leftarrow DP[i-1][111100100001100] + 4$$

New Tiles – Dynamic programming

Recurrent formula for placing new tiles:

We have to take some 2×2 matrices in the $i-1, i$ strip

- We might always leave a square of the new row empty.
- In general: if the bitmask fits in row i and $i-1$ of the input, try to fix all squares.
- Take the lexicographical order to compute in topological order.

$$
\begin{aligned}
DP[i][bitmask] = \max(&\max_{j \in \{j \mid j\text{-th bit is set in } bitmask\}} DP[i][bitmask \text{ with } j\text{-th bit unset}], \\
&DP[i-1][\text{negated } bitmask] + bitcount(bitmask)/2)
\end{aligned}
$$

New Tiles – Dynamic programming

Runtime:

Size of the table: $2^w \cdot h$.

Update step per entry: two loops over w (one in the initialization, one for the check).

Runtime: $\mathcal{O}(2^w \cdot h \cdot w)$, which is fast enough.

Slower solution for first subtask:

For smaller w, we could also check for a given bitmask all of its subsets, resulting in running time $\mathcal{O}(3^w \cdot w \cdot h)$.

Implement it yourself:

Try it out! This task is available on the judge.

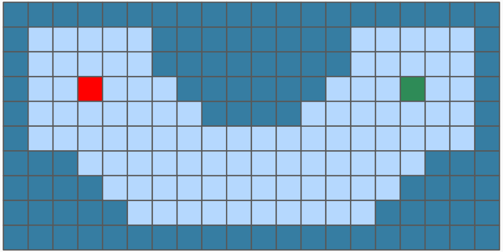
Snakes strike backe – The problem

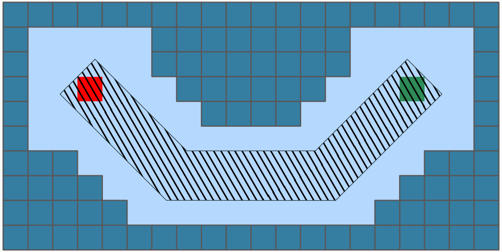
Given:

- $h \times w$ grid with obstacles (snake cages) and entrance/exit pair.
- Minimum path width p, safety distance $p/2$. (Cases: $p = 1,\ p = 2,\ p \leq 30$)

Wanted:

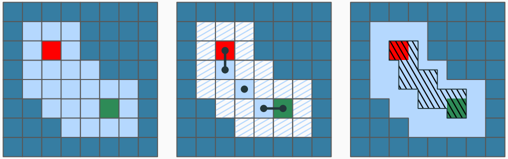
- yes or no: Is there a path between the entrance and the exit with minimum width p which keeps clear from obstacles by $p/2$?





Snakes strike back – Case $p = 1$

Rough Idea:
Use graph from Domino Snake, but delete squares which are adjacent to snake cages.

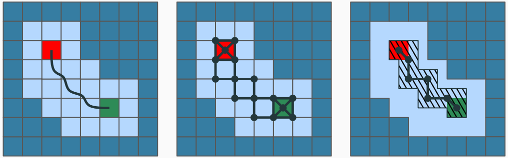


Problem: Fails, graph is disconnected!

Need better grasp on path width p and safety distance $p/2$.

Equivalent: **Curve** (has 0 width!), Safety distance: $\boldsymbol{p}$.

Solution Approach:
If there is a curve: Shift and bend, such that it runs along the boundary of cells which are not adjacent to snake cages.

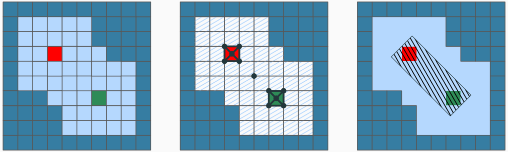


Solution ($\boldsymbol{p} = 1$):
BFS on grid graph given by vertices and by boundary segments with distance ≥ 1 to snake cage boundarys.

Can we adapt for larger p?

Snakes strike back – Case $p = 2$

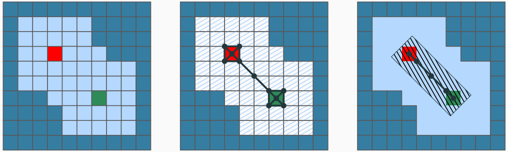
Rough Idea (first case solution adapted):
BFS on grid graph given by vertices and by boundary segments with distance ≥ 2 to snake cage boundarys.



Problem: When can we pass between two obstacles / their vertices?

- Δx or Δy at least $2p$, or
- $\sqrt{(\Delta x)^2 + (\Delta y)^2} \geq 2p$.

Solution Approach:
Because $p = 2$, second case only appears for $\Delta x = \Delta y = 3$. Hence add diagonals between vertices u, v of the same cell.



Solution ($p = 2$):
BFS on grid graph given by

- vertices and boundary segments with safety distance ≥ 2,
- diagonals between vertices if they belong to the same cell.

Snakes strike back – Arbitrary p

Question: Can we extend the $p = 2$ solution to arbitrary p?

Recall: Main problem for $p = 2$ were obstacles which lie diagonally apart, i.e. pairs $(\Delta x, \Delta y)$ such that $\sqrt{(\Delta x)^2 + (\Delta y)^2} \geq 2p$, but $\Delta x, \Delta y < 2p$.

Answer: For $p > 2$ there will be many different kinds of pairs $(\Delta x, \Delta y)$, not only one! Things start to get messy!

Look at the problem statement again:

Originally: Is there a path of width p which keeps clear from obstacles by $p/2$?

Redefined: Is there a path of width 0 which keeps clear from obstacles by p?

Redefinition II: Is there a path of width $2p$ which keeps clear from obstacles by 0?

Snakes strike back – Arbitrary p

Draw path with a brush of Diameter $2p$.

This sounds familiar:

H1N1 – How to move a disk D without colliding with a given point set P?

Move disk along Voronoi Diagram edges / in the Delaunay triangulation.

Problem: Here our obstacles are cells, not points.

Solution: Replace every snake cage square by its 4 vertices.

Careful:

- We need vertices of each square! (Only considering the convex & concave vertices of the boundary of the full obstacle is not enough.)