# Linear and Quadratic Programming (with CGAL)

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*October 31, 2018*

*Based on slides by Bernd Gärtner*

# Today

- Quick introduction to Linear Programming (formulation and representation).
- Quick recap on techniques to solve it (no extensive knowledge needed).
- Linear Programming in CGAL.
- Examples
- Quadratic Convex Programming.

# Linear Programming (LP)

- Linear programming is a central topic in optimization .
- It provides a powerful tool in modeling many applications .
- LP has attracted most of its attention in optimization during the last six decades for two main reasons :
  - Applicability : There are many real-world applications that can be modeled as linear programming ;
  - Solvability : There are theoretically and practically efficient techniques for solving large-scale problems.

# Linear Programming (LP)

**We face an optimization problem subject to some constraints.**

- **Variables:** describe our choices, the parameters that we are allowed to change;

- **Objective function:** Describes the criterion that we wish to minimize (e.g. cost) or maximize (e.g. profit);

- **Constraints:** Describe the limitations that we have for the choice of the values of the variables.

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## Mathematical formulation

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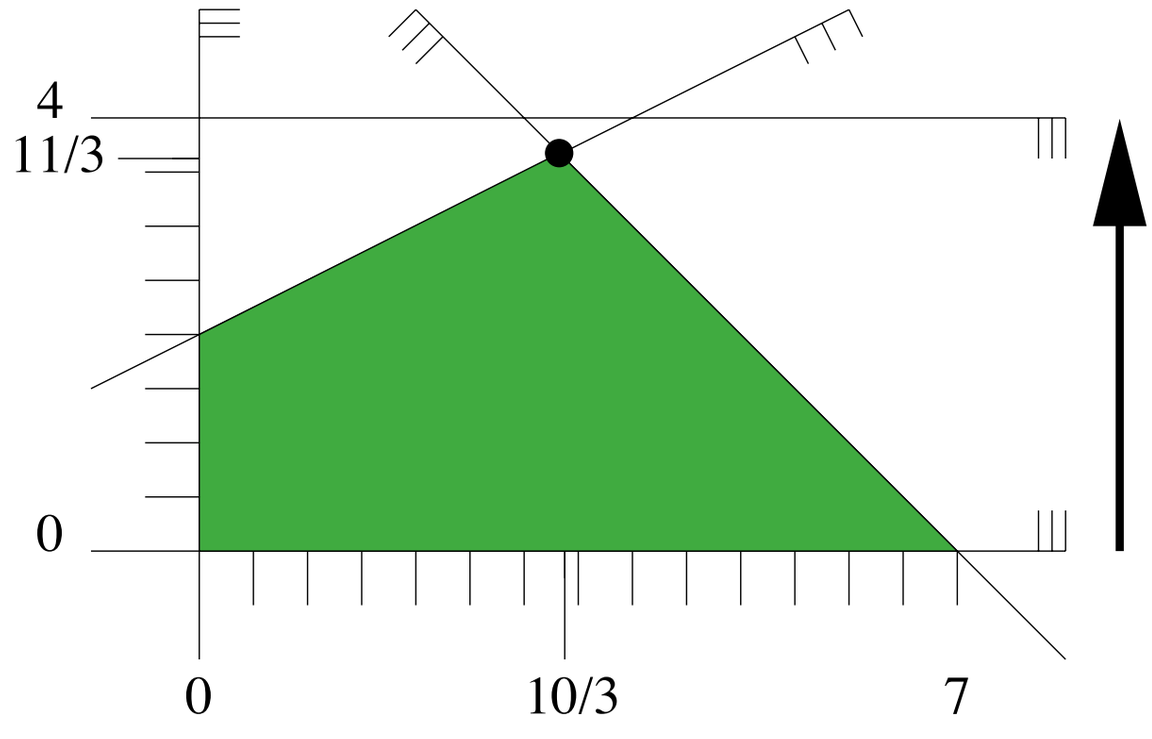

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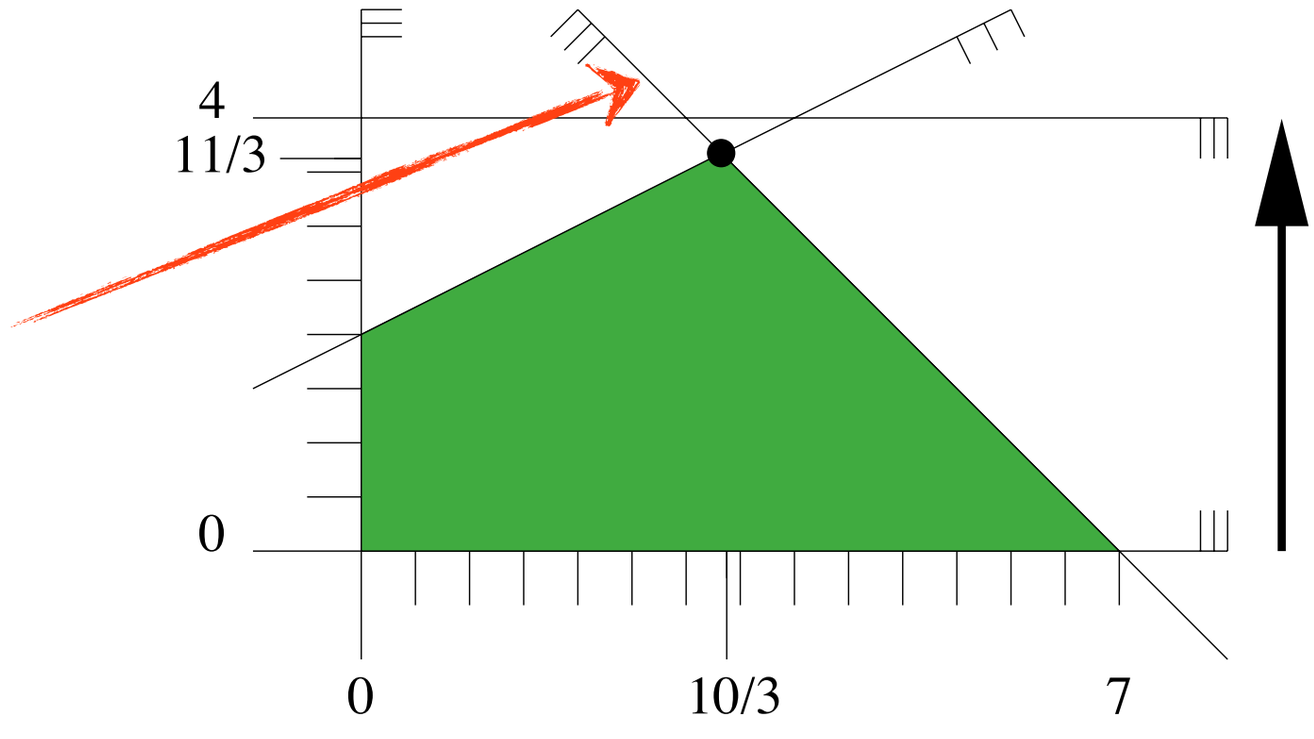

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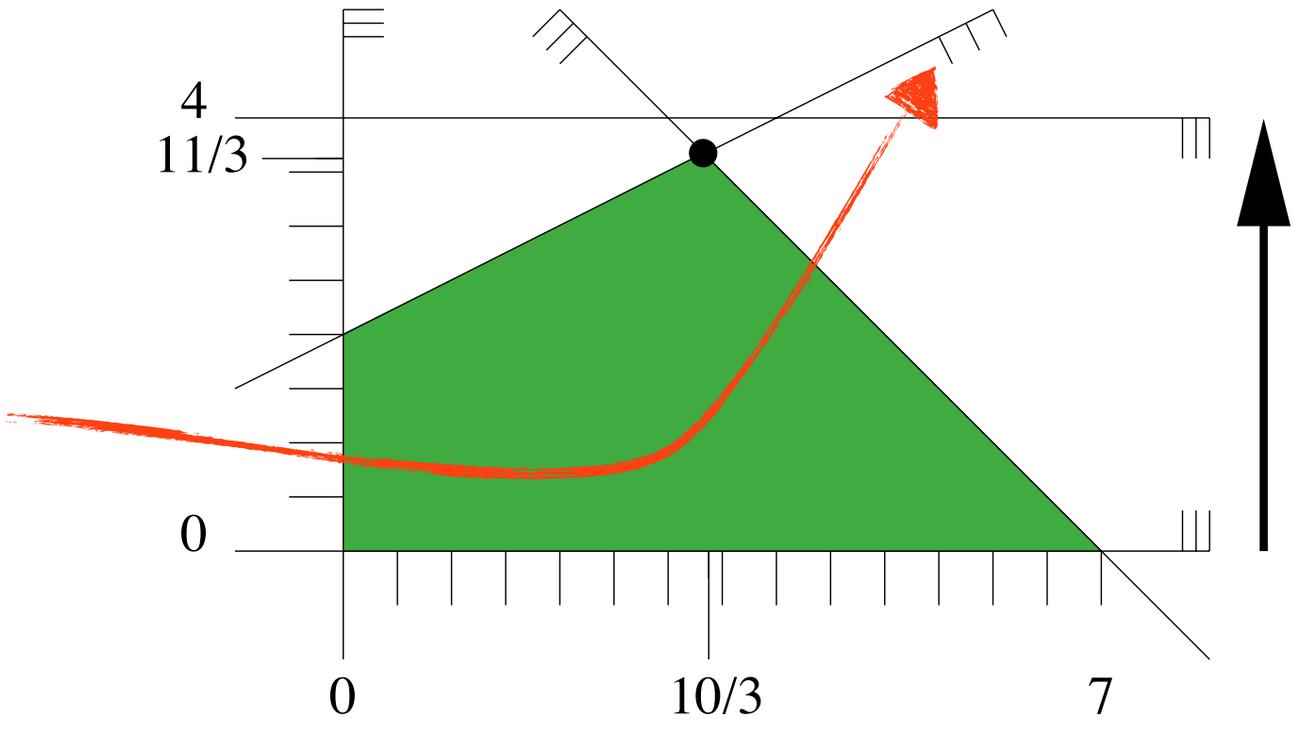

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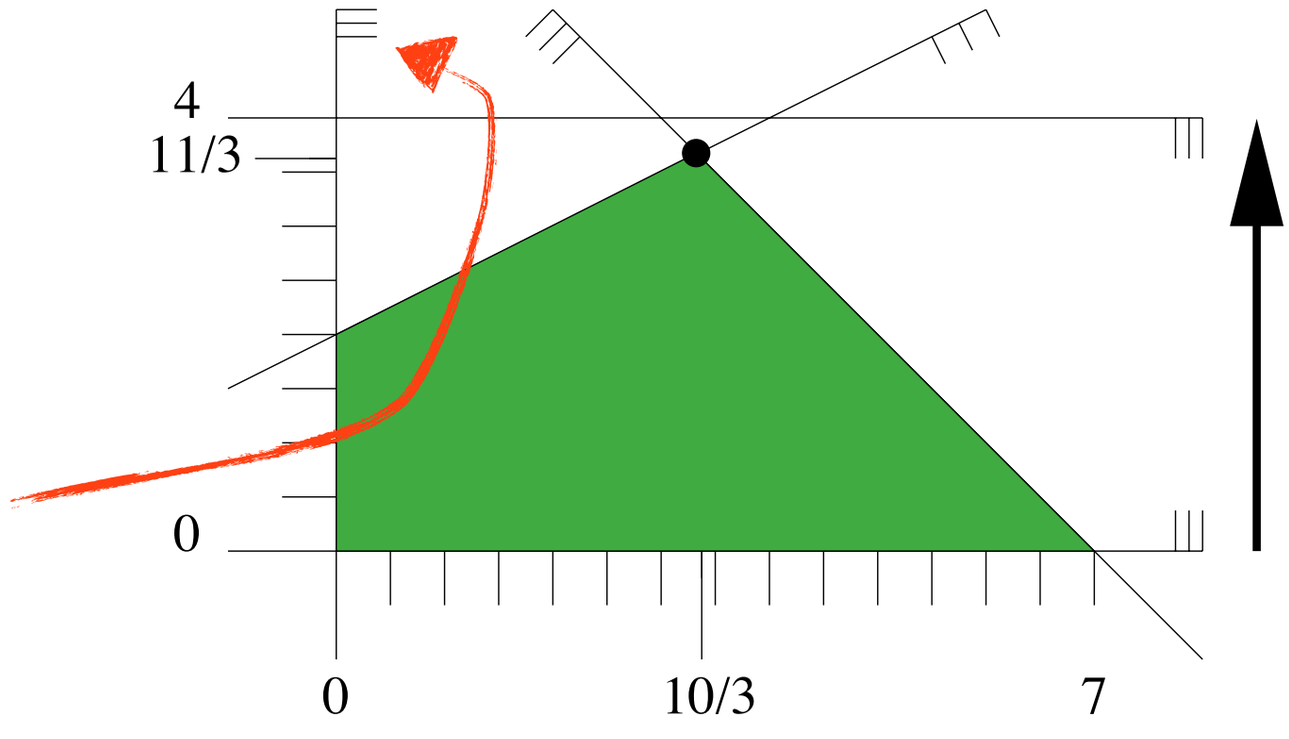

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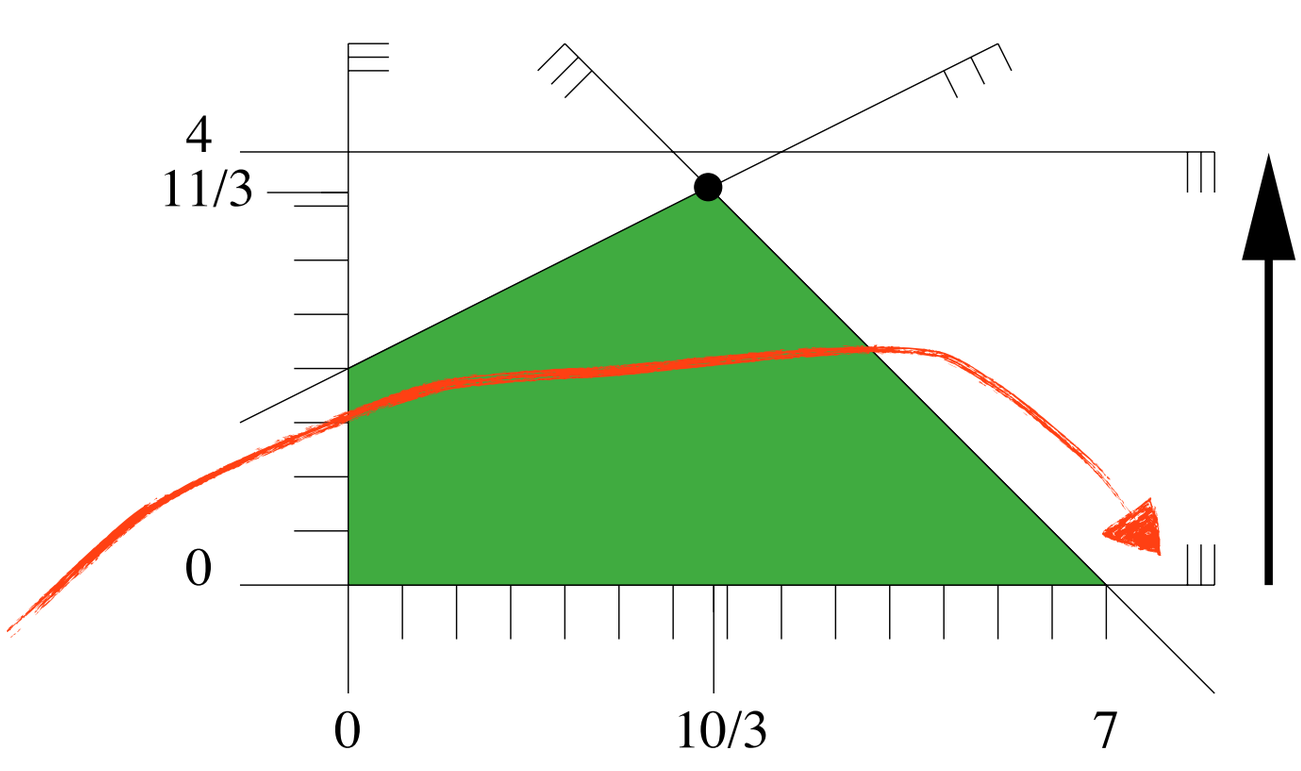

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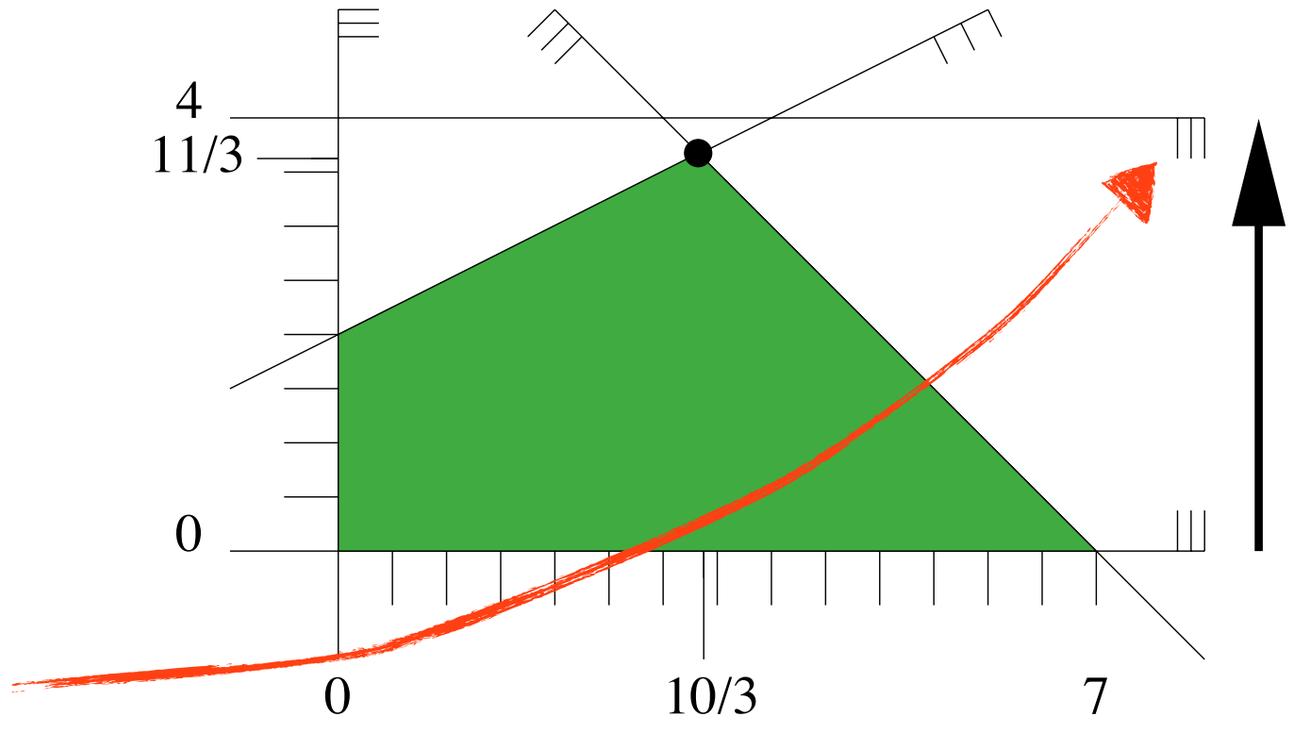

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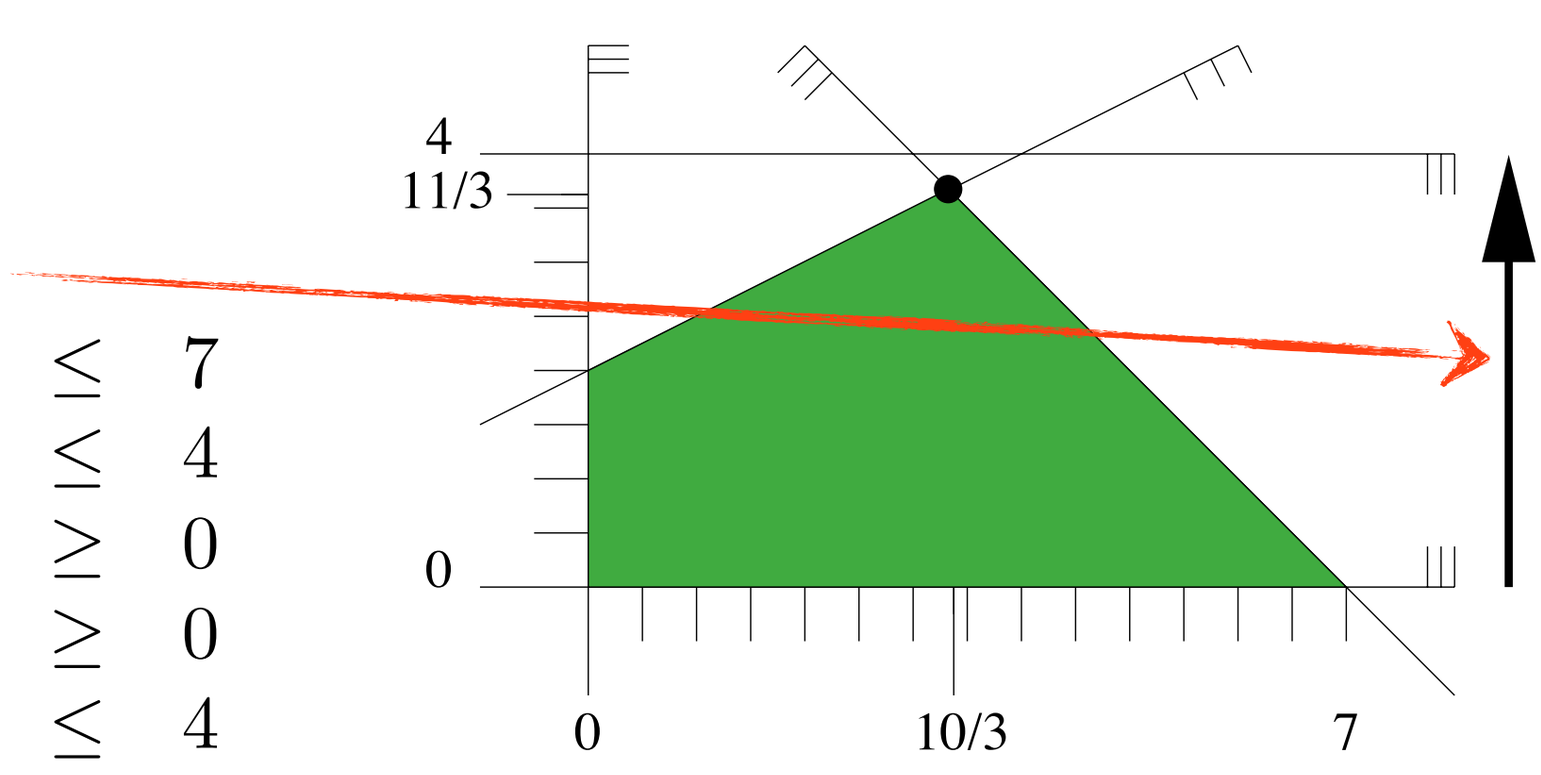

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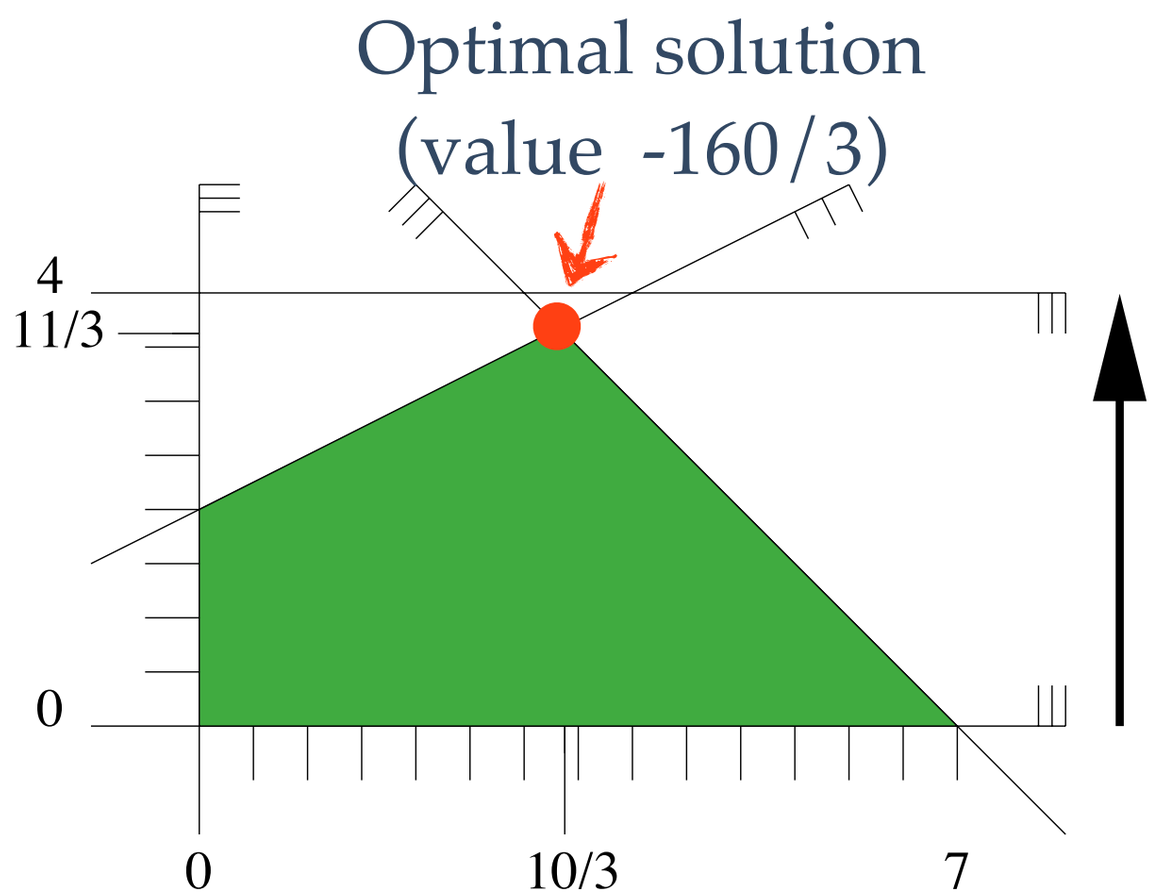

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- **Unbounded linear programs:**

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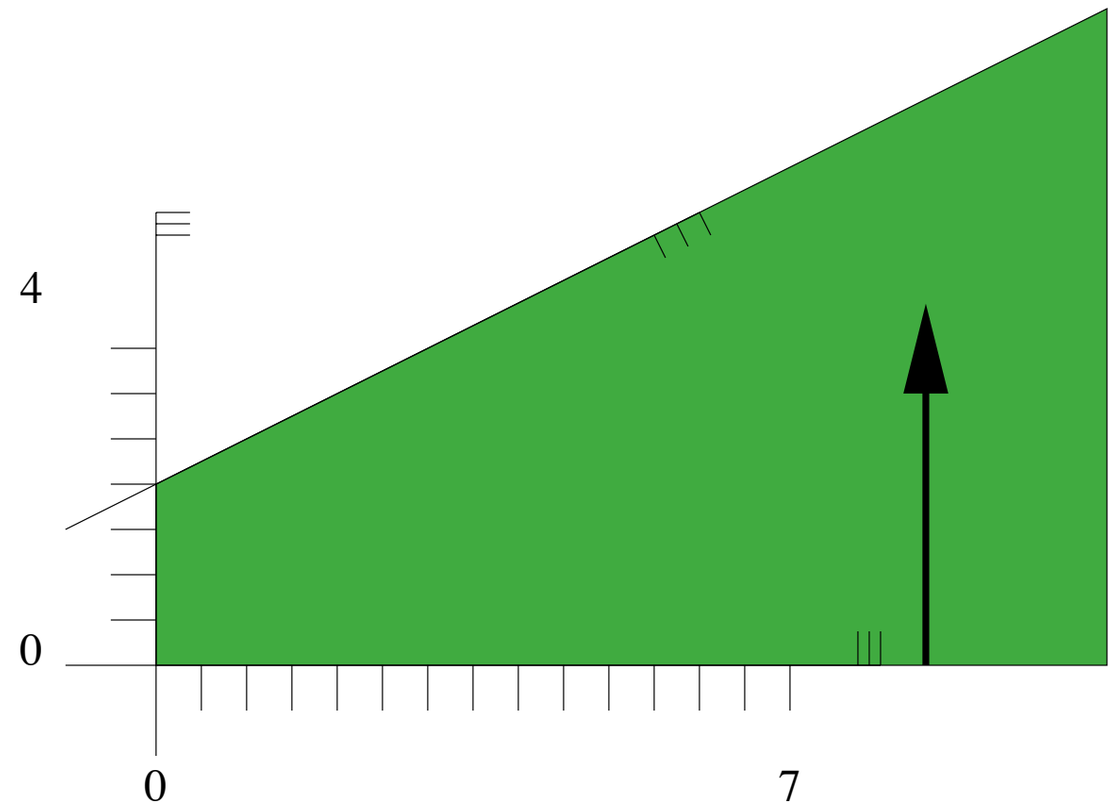

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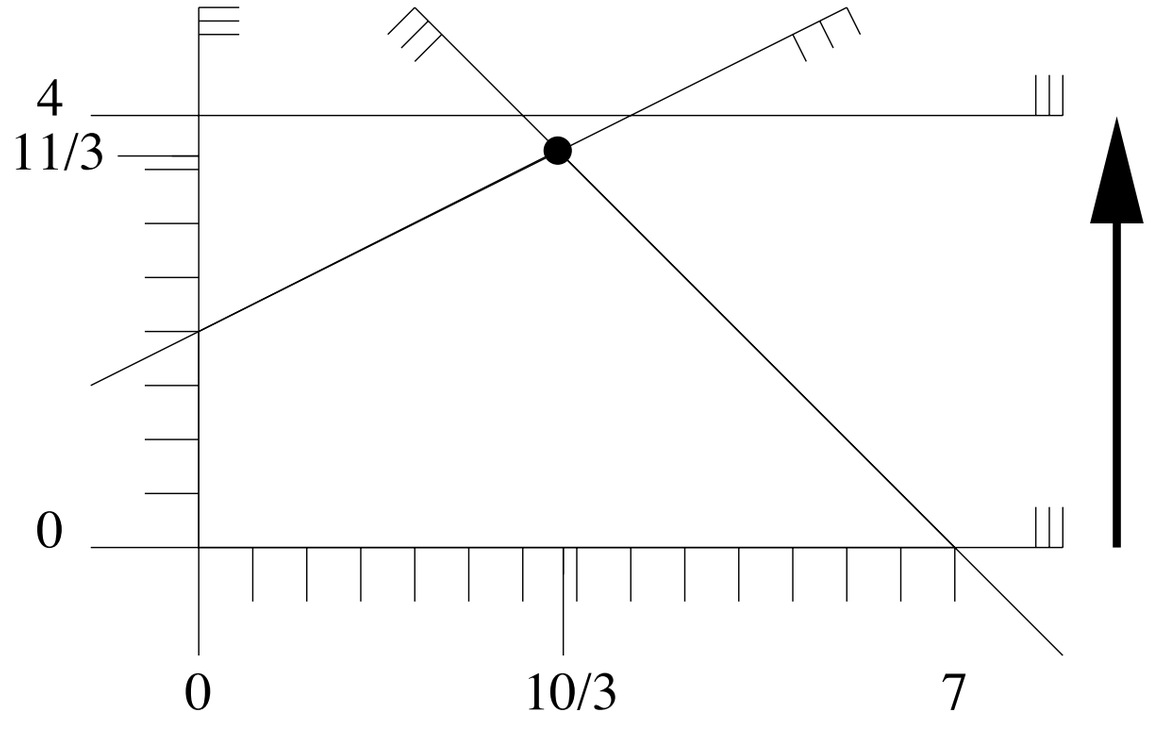

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\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}}_{\text{A}}
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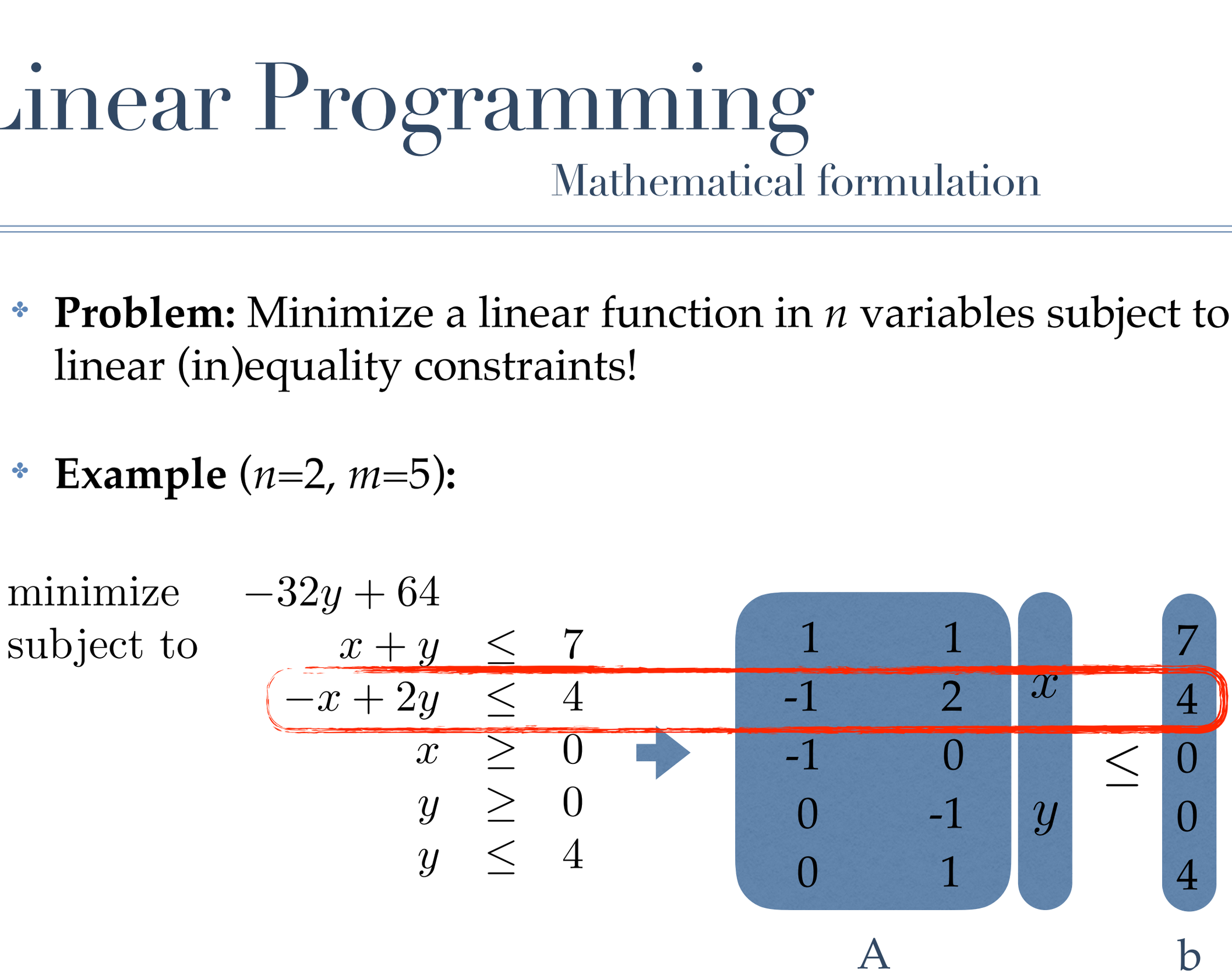

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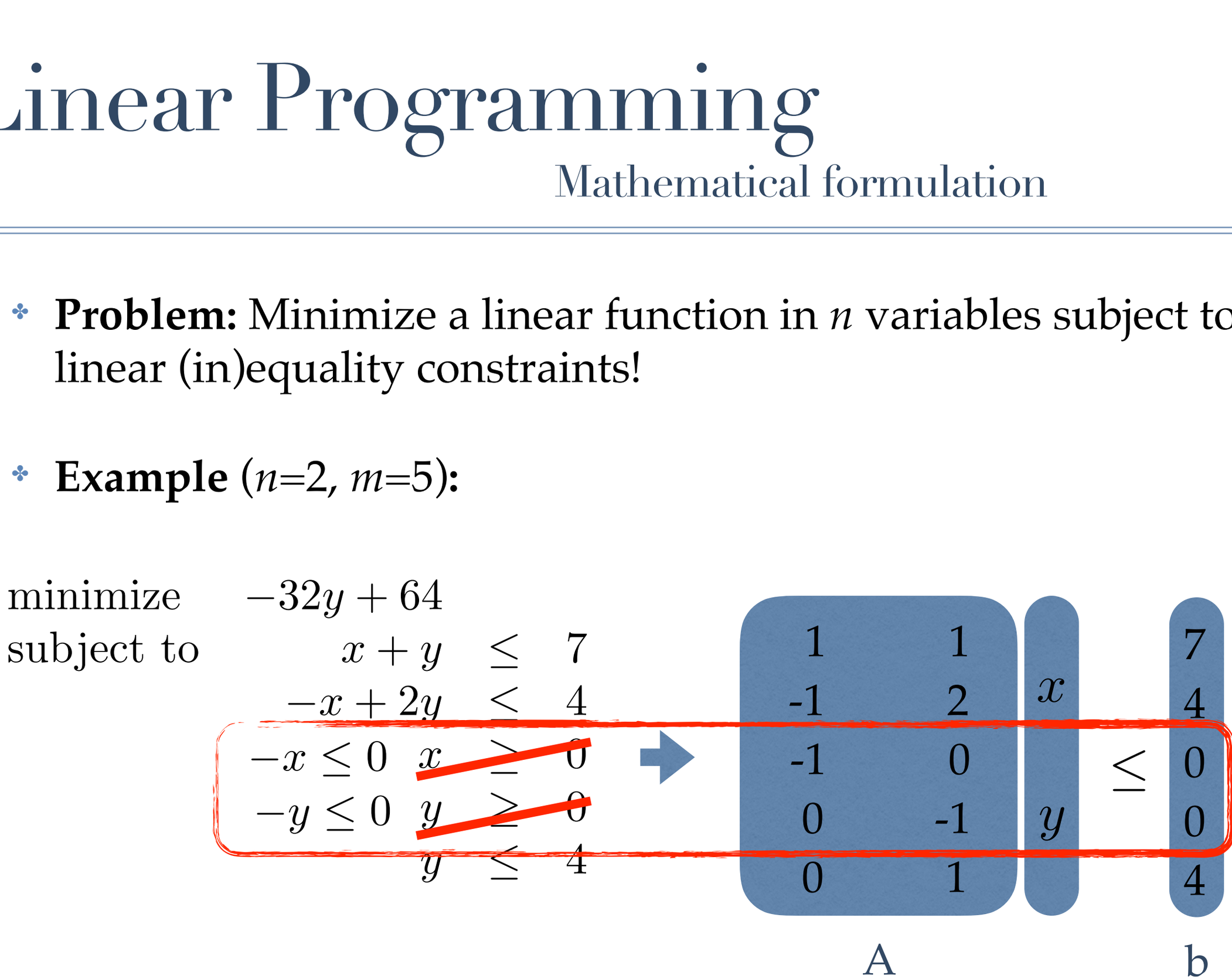

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$$\mathbf{c}^T \begin{bmatrix} 0 \\ -32 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} + \mathbf{c}_0 \; 64$$

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- **General form of LP:**

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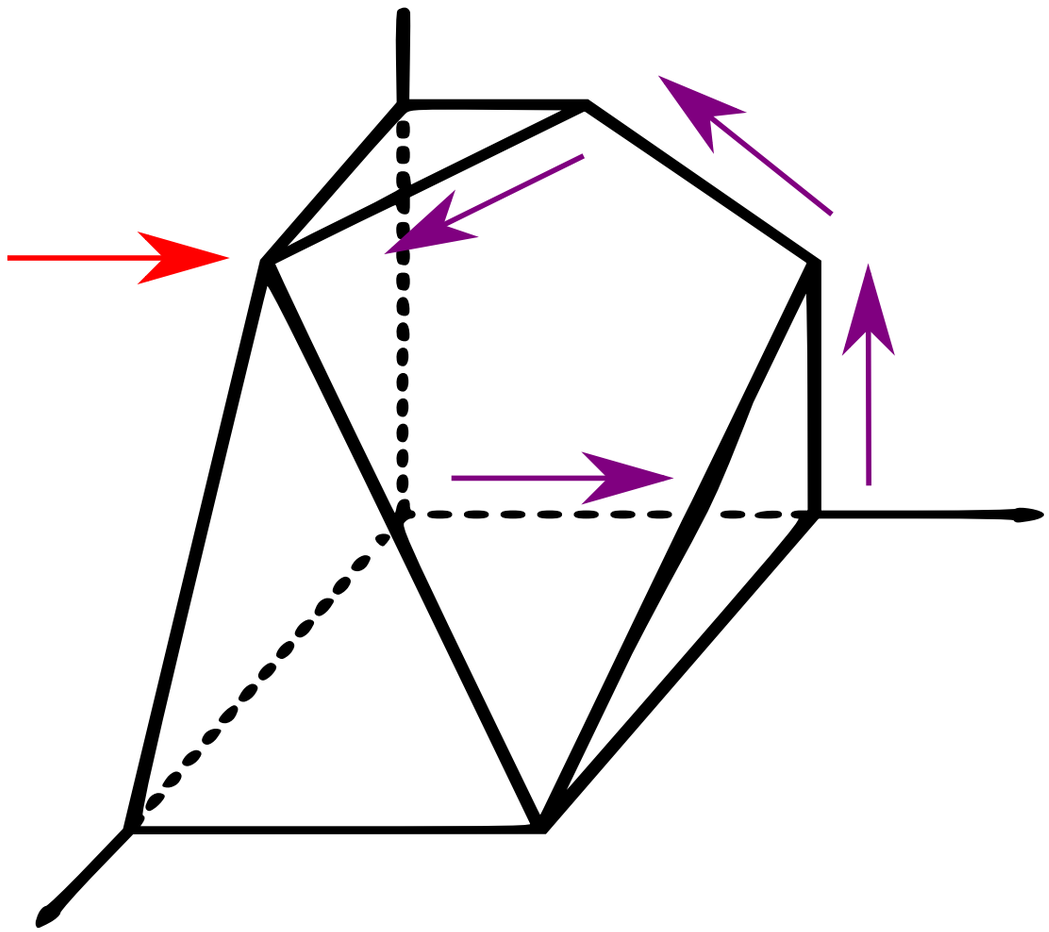

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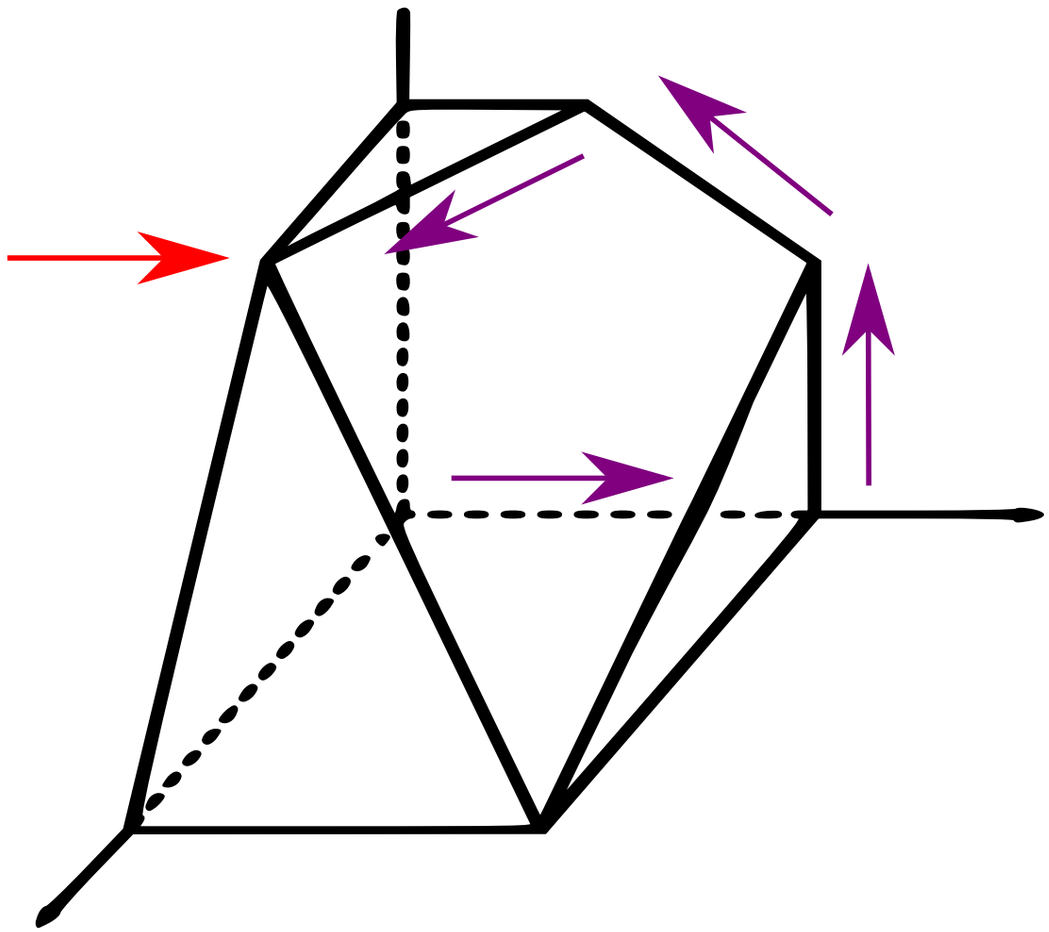

- The constraints define a convex polyhedron
$$P = \{x | Ax \leq b\}$$
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**Complexity of Simplex-type algorithms with *n* variables and *m* constraints.**

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*min*{*m*,*n*}: 0 20 200

Easy! No Way!

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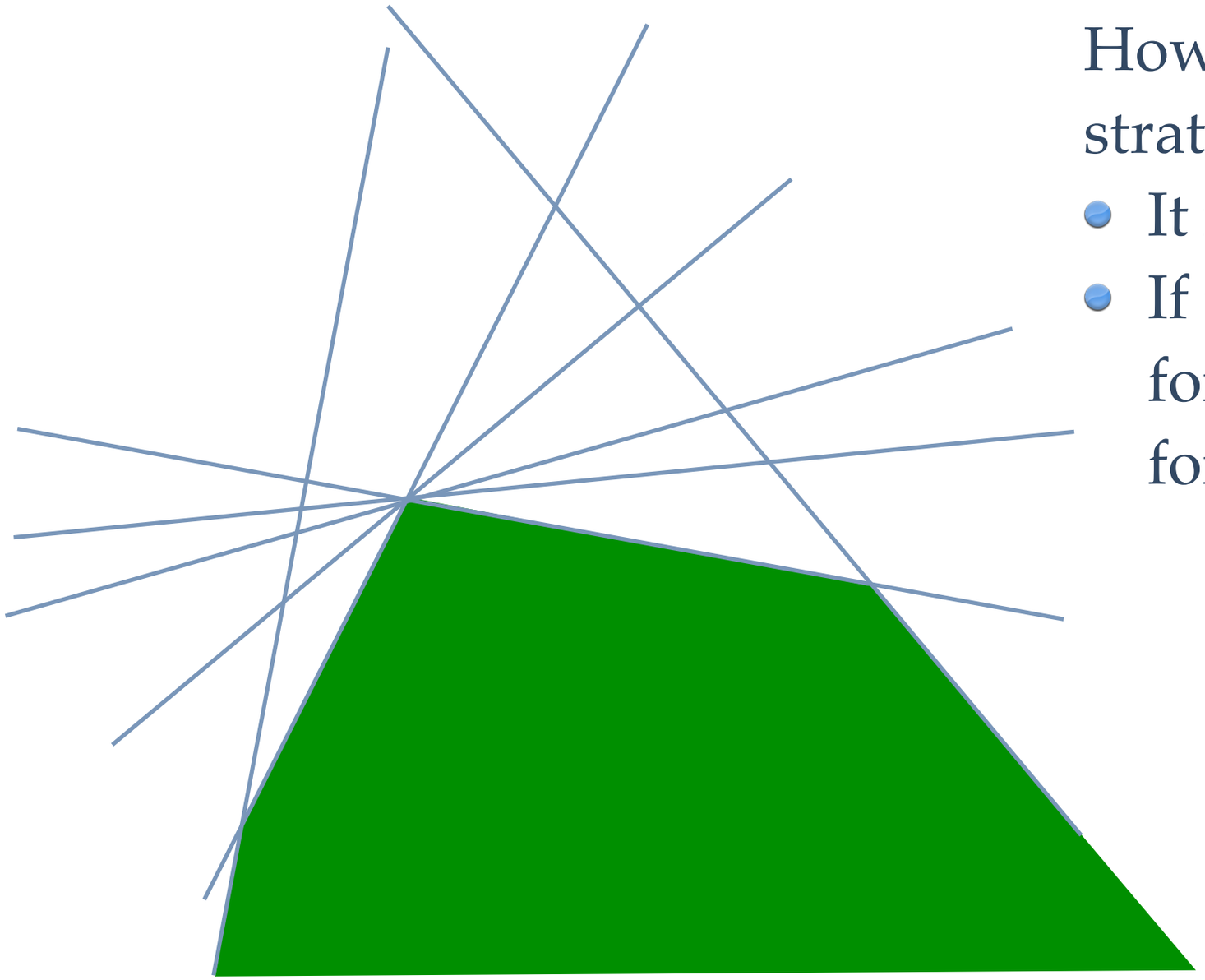


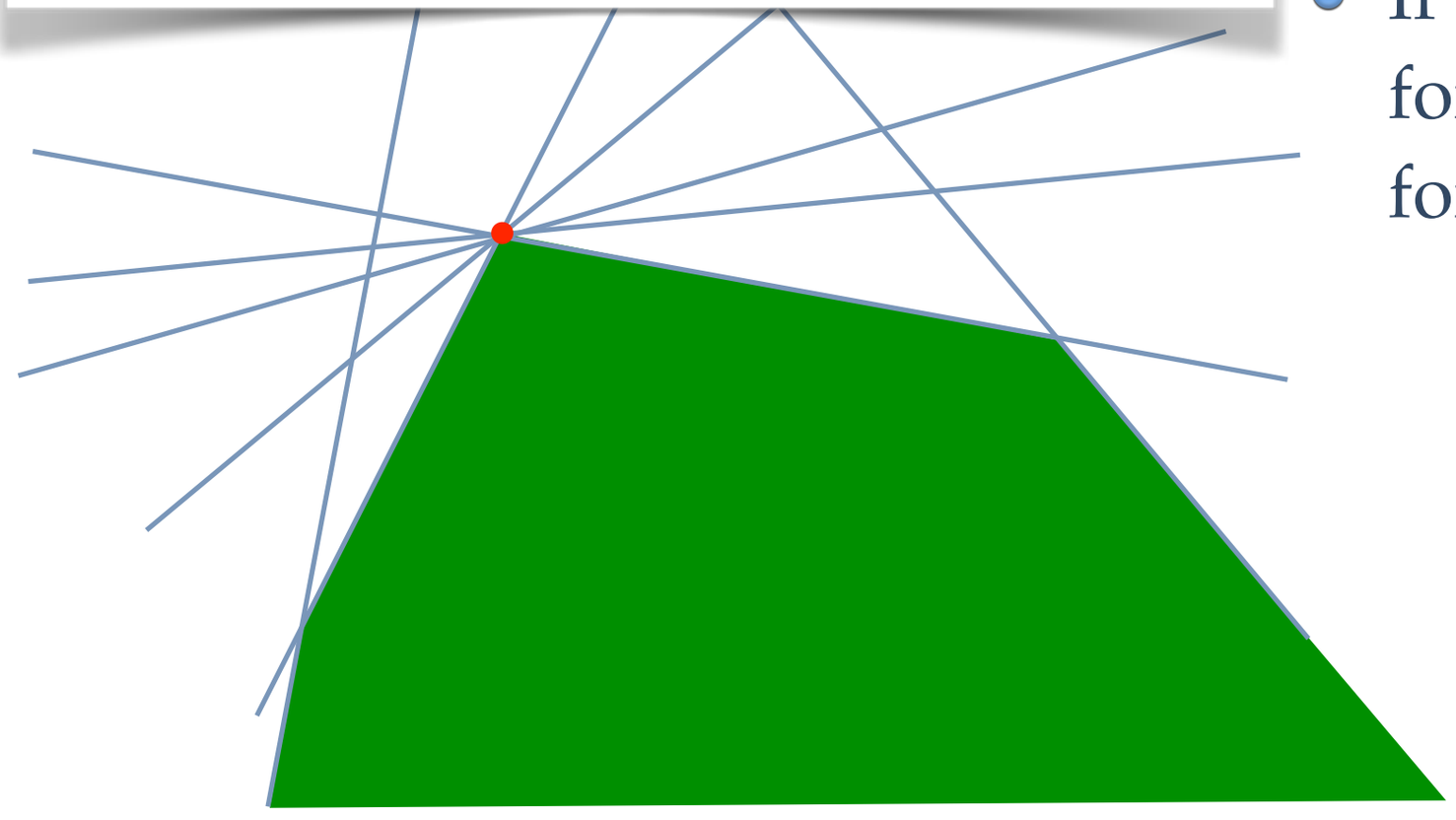
However, the default pivot rule strategy might fail:

- It is deterministic.
- If the algorithm reaches a vertex for the second time, then cycles forever.

# Pivot Rules (how to avoid cycles)

We recommend always to use the default pivot rule.

- Combinatorially a vertex is the intersection of $d$ planes.
- The same point can be represented by many $d$-tuples



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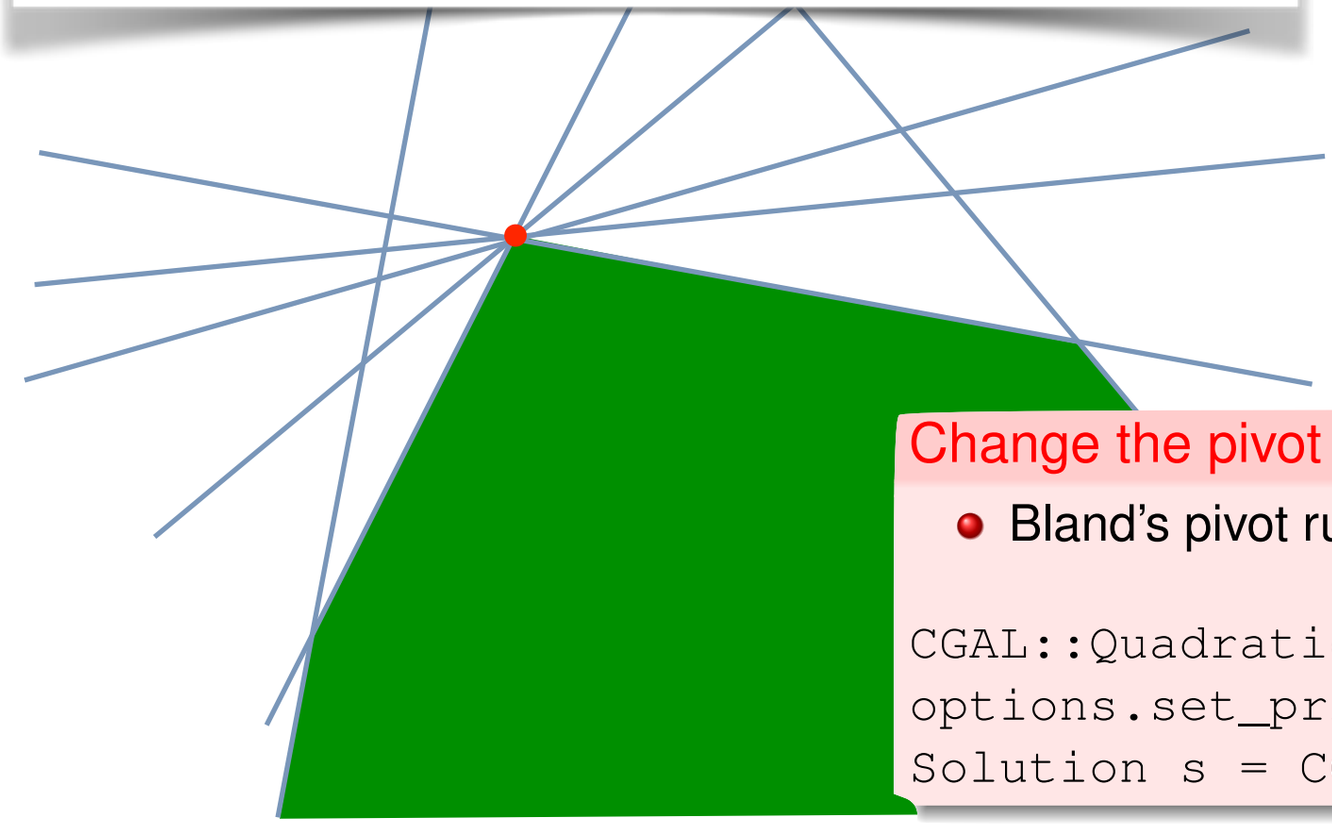

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**Change the pivot rule**

- Bland's pivot rule avoids cycling (but it is slower...)

```
CGAL::Quadratic_program_options options;
options.set_pricing_strategy(CGAL::QP_BLAND);
Solution s = CGAL::SOLVER(program, ET(), options);
```

# Linear Programming

- **General form of LP in CGAL:**

$$\begin{array}{ll} \text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \begin{matrix} \geq \\ \leq \end{matrix} b \\ & l < x < u \end{array}$$

$\leq$, $=$, or $\geq$ (individually for each constraint)

variables

$(x, c, l, u \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c_0 \in \mathbb{R})$

objective function

lower and upper bounds

constraint matrix

right-hand side

shift

# Linear Programming … in CGAL

$$\begin{aligned} \text{minimize} \quad & -32y + 64 = c^T x + c_0 \\ \text{subject to} \quad & Ax \leq b \begin{cases} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{cases} \end{aligned}$$

- **Preamble:** Choice of input type and exact internal number type

Gnu Multi-precision Library (GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose exact integral type
typedef CGAL::Gmpz ET;



// program and solution types
typedef CGAL::Quadratic_program<int> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

input type

exact internal type

for linear *and* quadratic programs

GMP used internally

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typedef CGAL::Quadratic_program_solution<ET> Solution;
```

input type

Inside, the LP solver uses quotients of the data type specified here.

exact internal type

for linear *and* quadratic programs

GMP used internally

# Linear Programming … in CGAL

$$\begin{aligned} \text{minimize} \quad & -32y + 64 = c^T x + c_0 \\ \text{subject to} \quad & Ax \leq b \begin{cases} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{cases} \end{aligned}$$

- **Preamble:** Choice of input type and exact internal number type

Gnu Multi-precision Library (GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose exact integral type
typedef CGAL::Gmpz ET;



// program and solution types
typedef CGAL::Quadratic_program<int> Program;
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```

input type

exact internal type

for linear *and* quadratic programs

GMP used internally

# Linear Programming … in CGAL

$$\begin{array}{lll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \left\{ \begin{array}{rcl} x + y & \leq & 7 \\ -x + 2y & \leq & 4 \\ x & \geq & 0 \\ y & \geq & 0 \\ y & \leq & 4 \end{array} \right. \end{array}$$

$$Ax \leq b$$

- **Setup:** Enter the program data

```
int main() {
  // by default, we have a nonnegative LP with Ax <= b
  Program lp (CGAL::SMALLER, true,0,false,0);

  // now set the non-default entries
  const int X = 0;
  const int Y = 1;
  lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
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  lp.set_u(Y, true, 4);                                   //       y <= 4
  lp.set_c(Y, -32);                                       // -32y
  lp.set_c0(64);                                          // +64
```

$l =$ (0,0,..., 0)

$u = (\infty,\infty,\ldots,\infty)$

variable index (0,1,...)

constraint index (0,1,...)

last argument: value

# Linear Programming … in CGAL

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# Linear Programming … in CGAL

- **Solve:** Call the linear programming solver and output solution

```
// solve the program, using ET as the exact type
 Solution s = CGAL::solve_linear_program(lp, ET());
 assert (s.solves_linear_program(lp));

  // output solution
  std::cout << s;
  return 0;
}
```

independent verification

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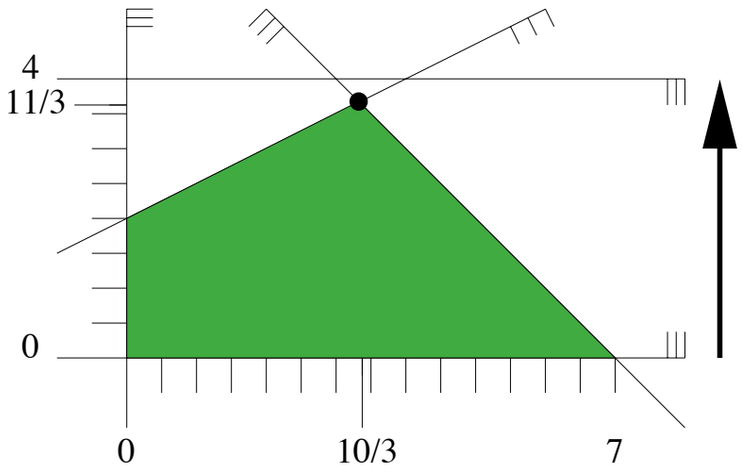
}
```

independent verification

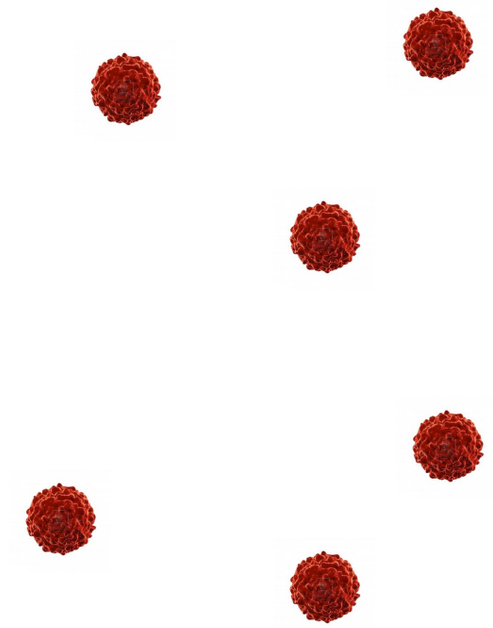
- **Output:**

```
status:                 OPTIMAL
objective value: -160/3
variable values:
   0: 10/3
   1: 11/3
```



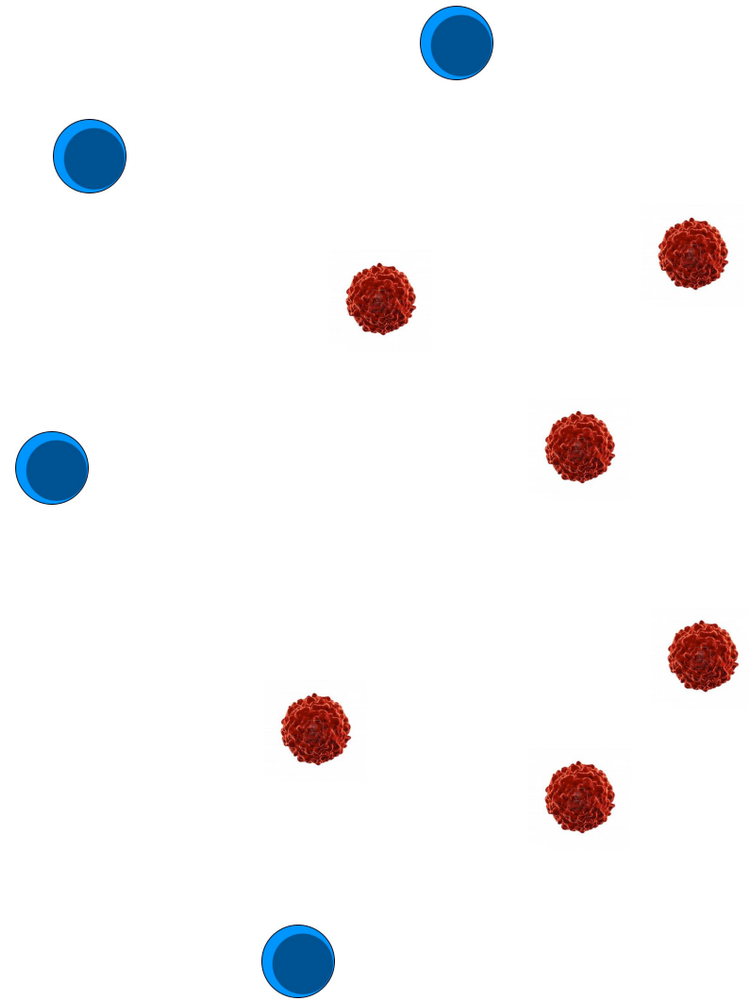
# Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red)



# Linear Programming Application I: Cancer Therapy

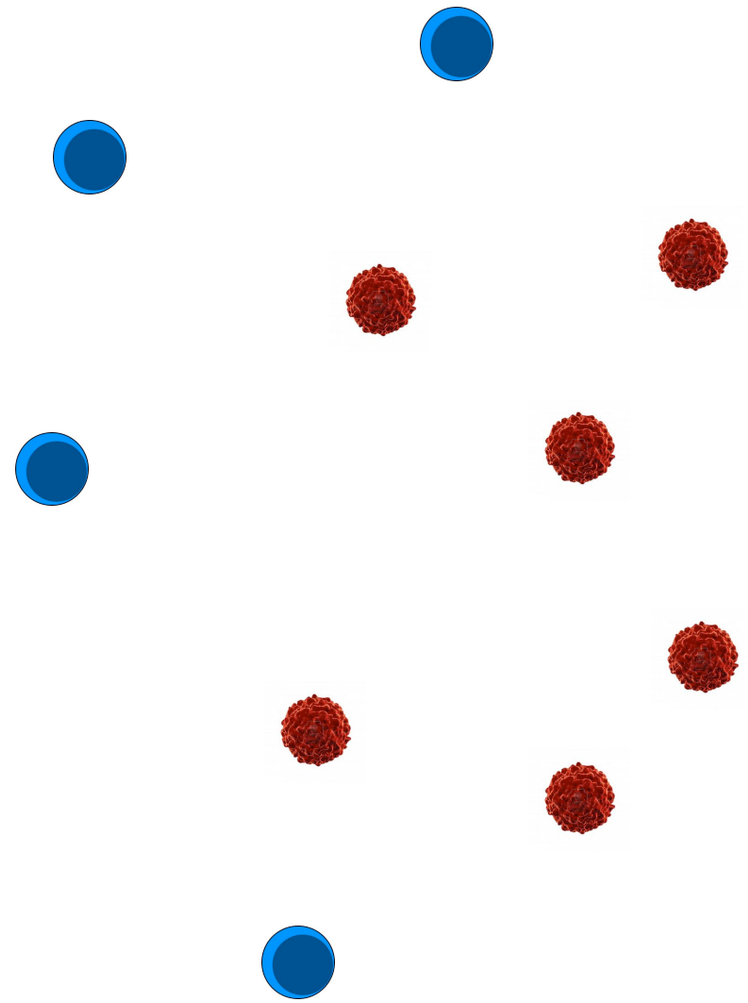
- **Given:** locations of cancer cells (red) and healthy cells (blue)



# Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)

- **Wanted:** center and radius of exposure  so that all cancer cells are killed and all healthy cells are unaffected.



# Linear Programming Application I: Cancer Therapy

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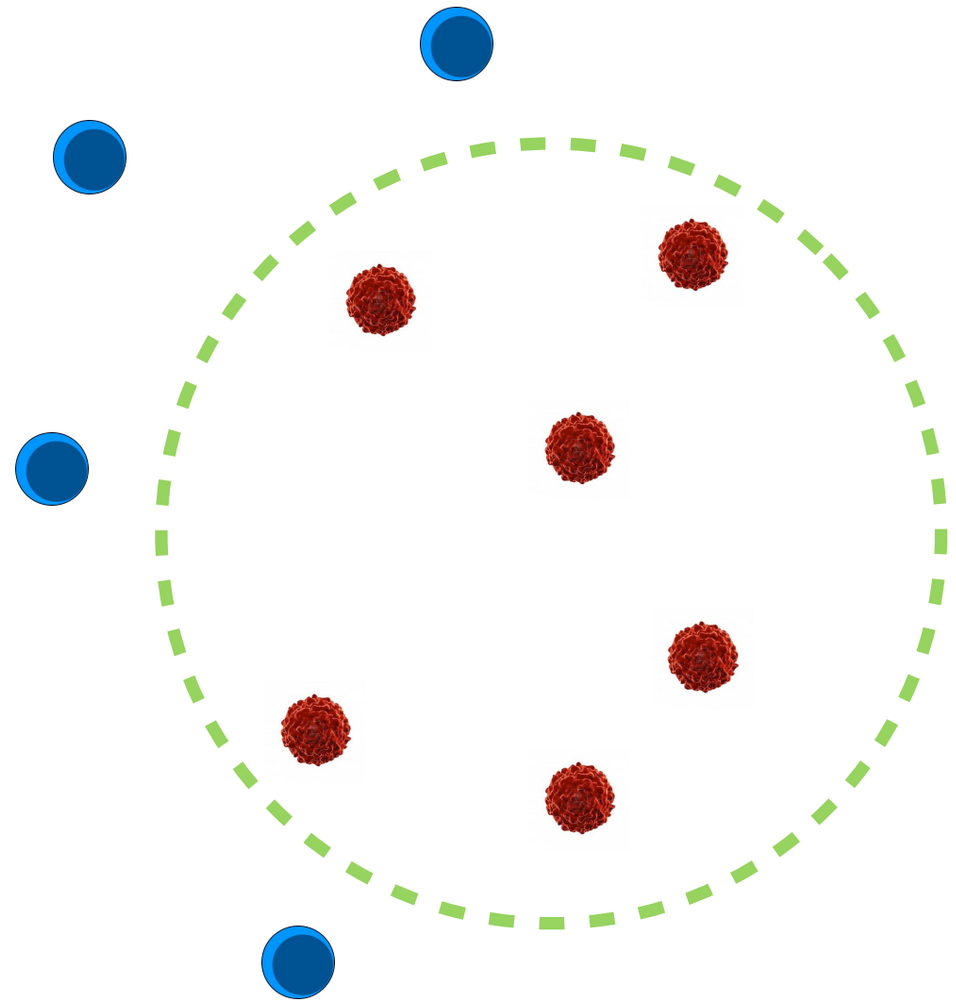
- This may be possible...



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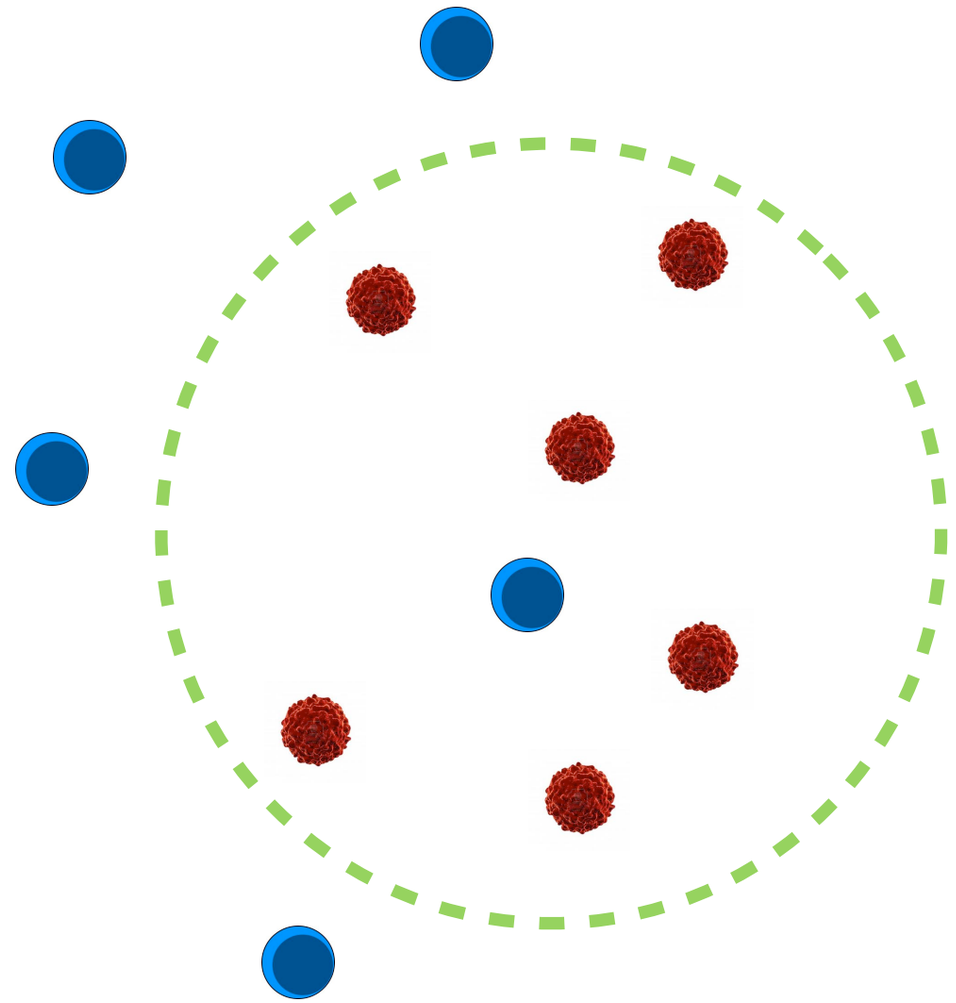

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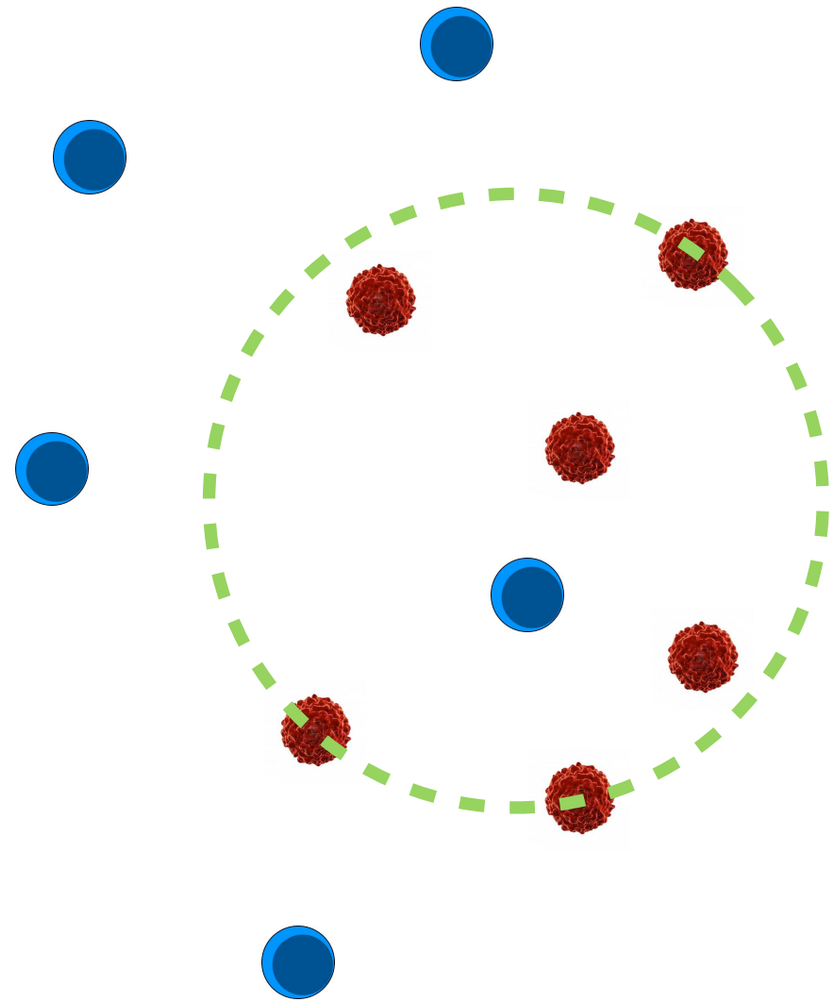

- Minimum Enclosing circle is not enough to solve the problem.

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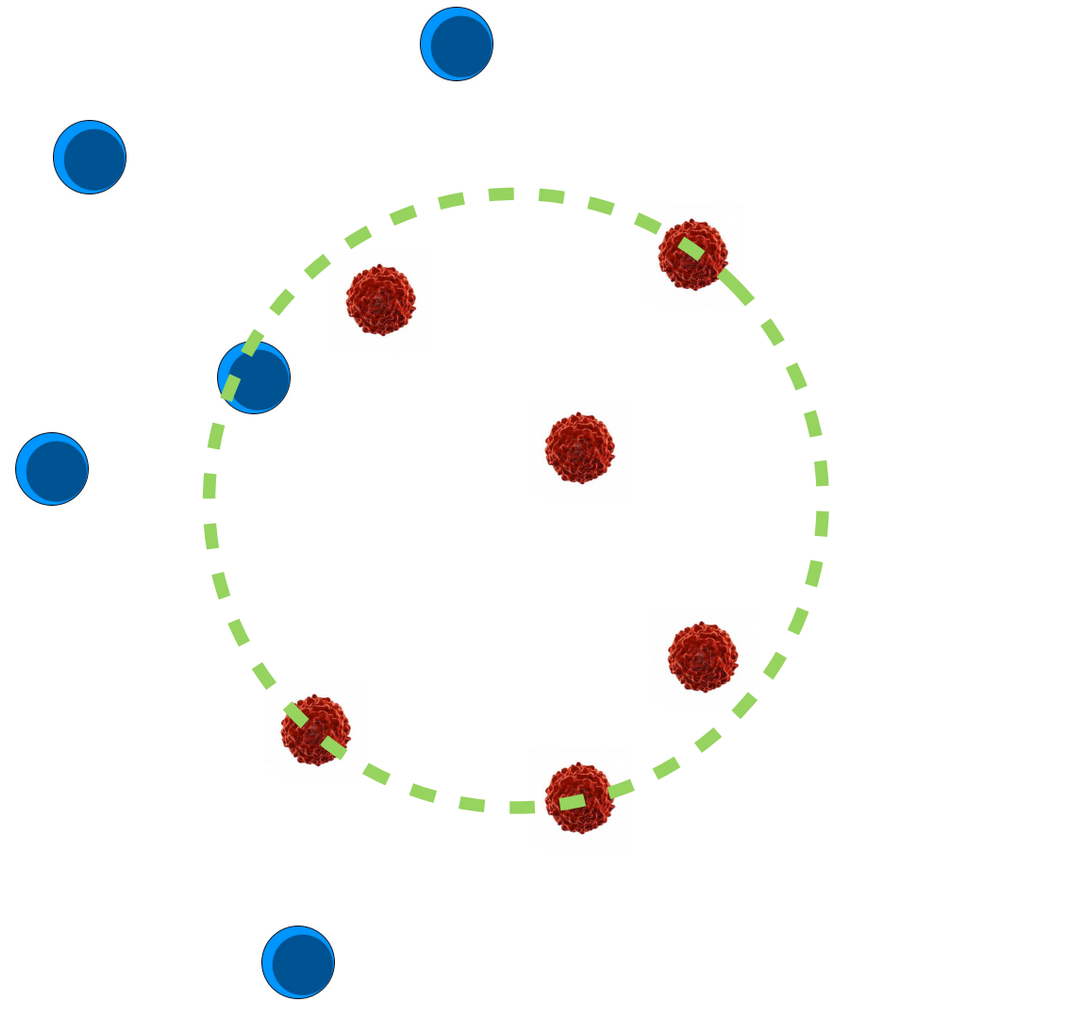

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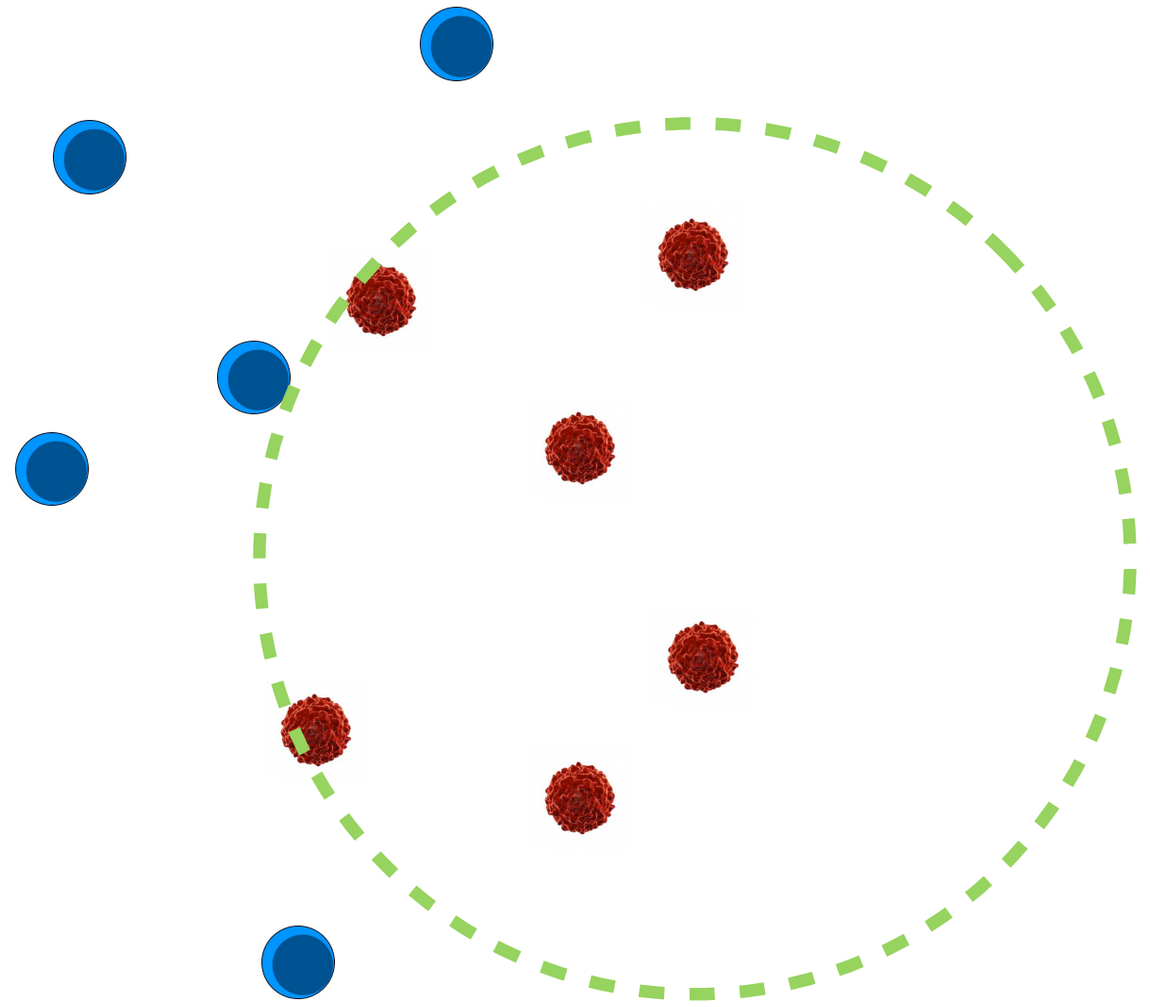

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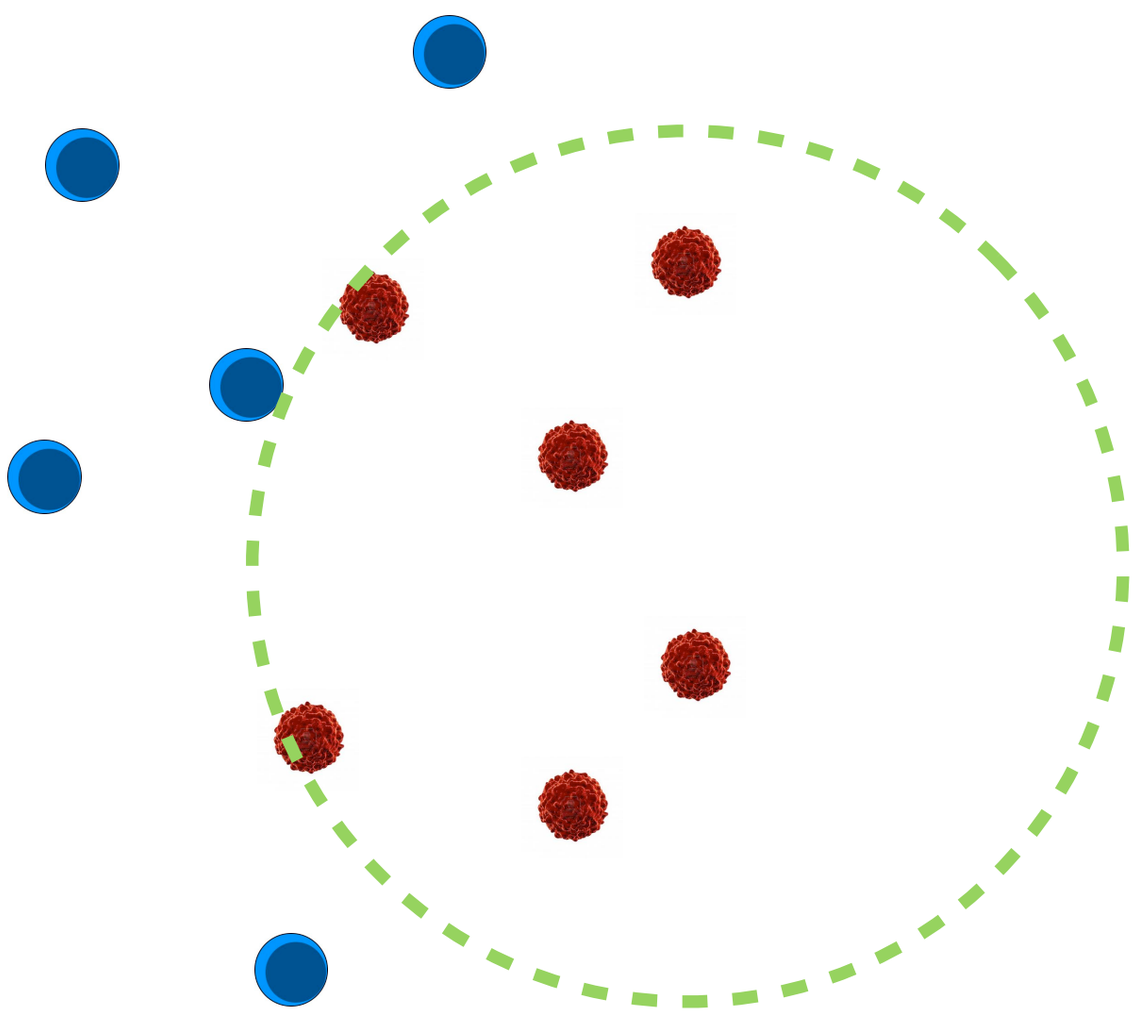
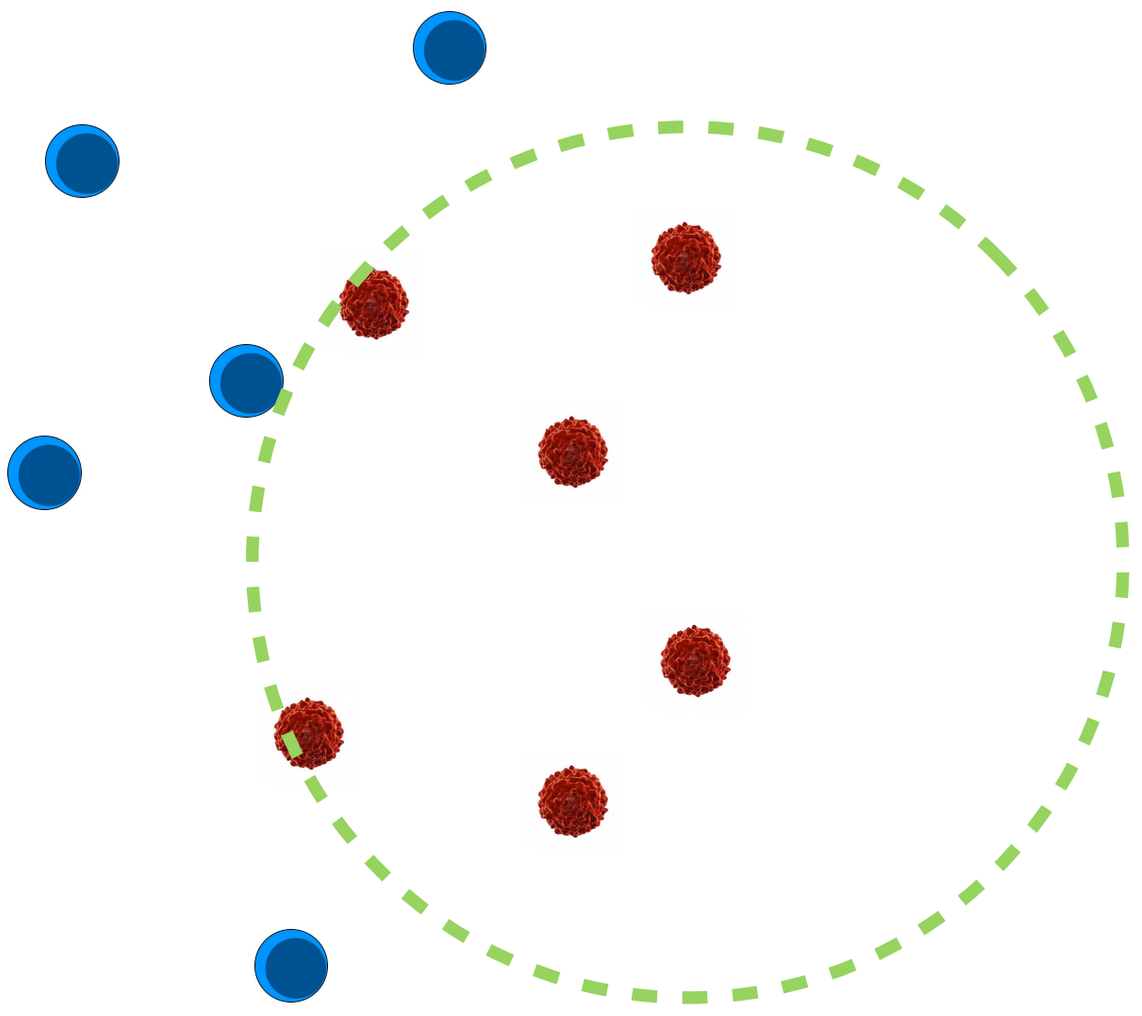
# Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets $R$ and $B$ in the plane, does there exist a disk that contains $R$ and is disjoint from $B$?



# Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets $R$ and $B$ in the plane, does there exist a disk that contains $R$ and is disjoint from $B$?
- **Variables:** Parameters used to define a circle.



# Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets $R$ and $B$ in the plane, does there exist a disk that contains $R$ and is disjoint from $B$?
- **Variables:** Parameters used to define a circle.
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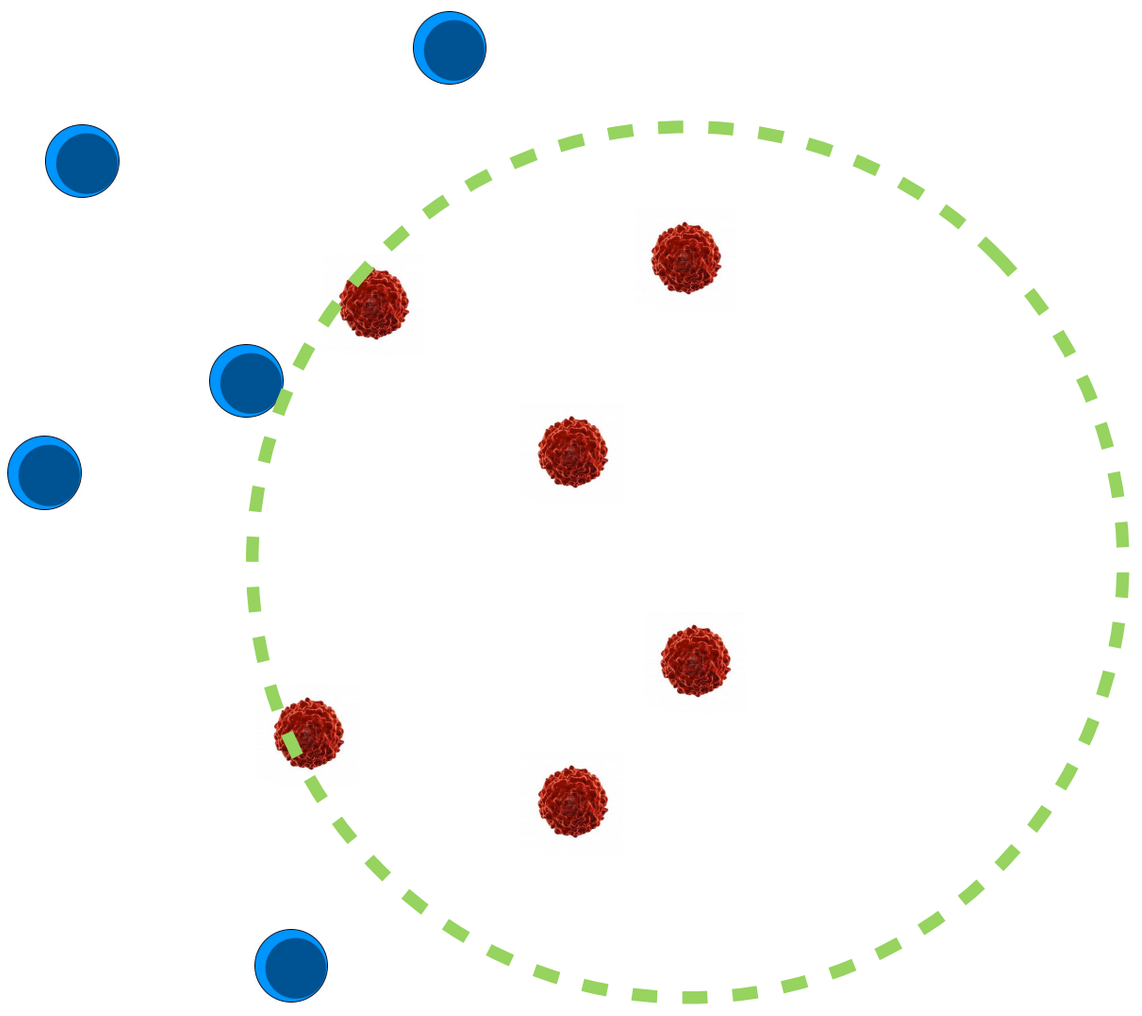

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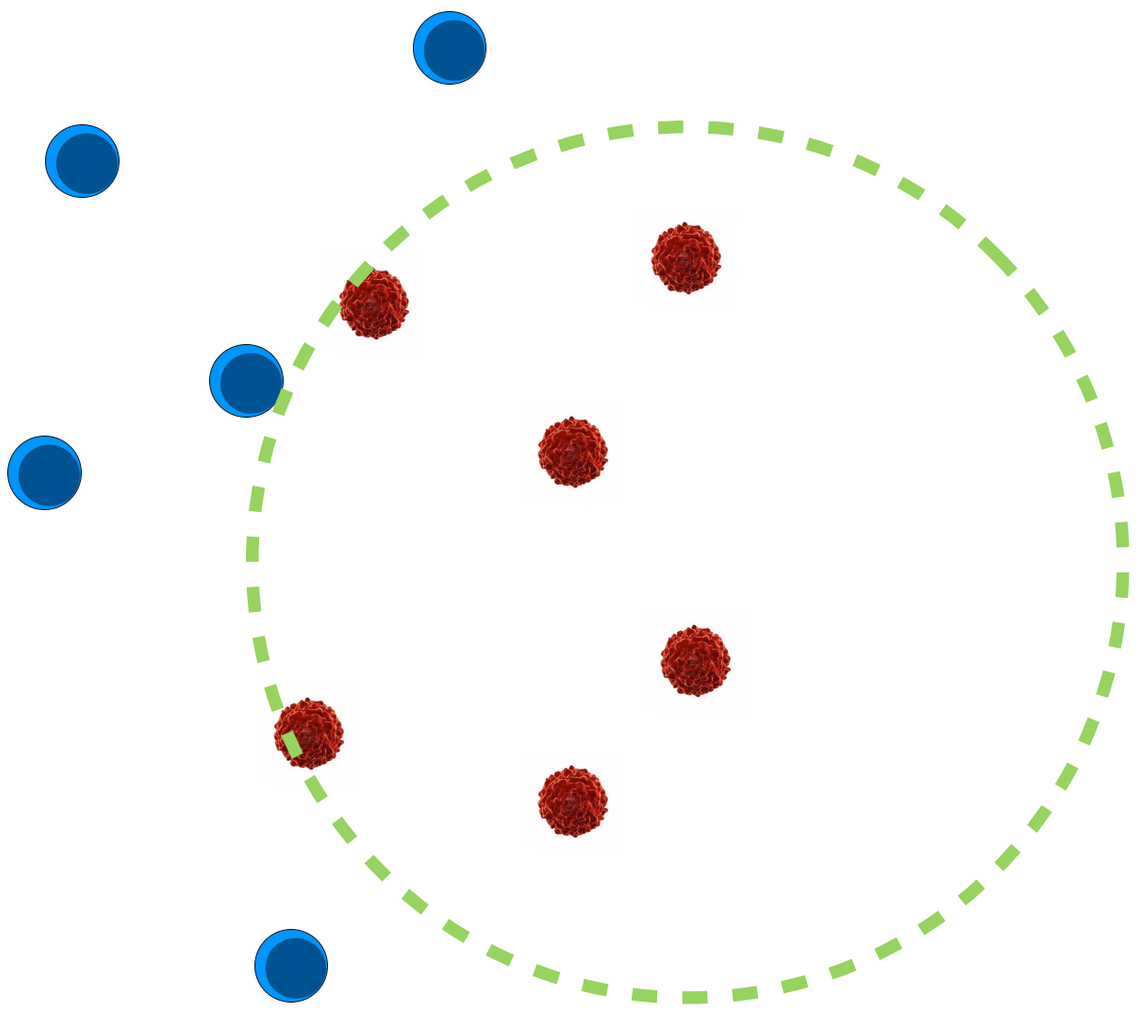
- **Problem:** Constraints are not linear…



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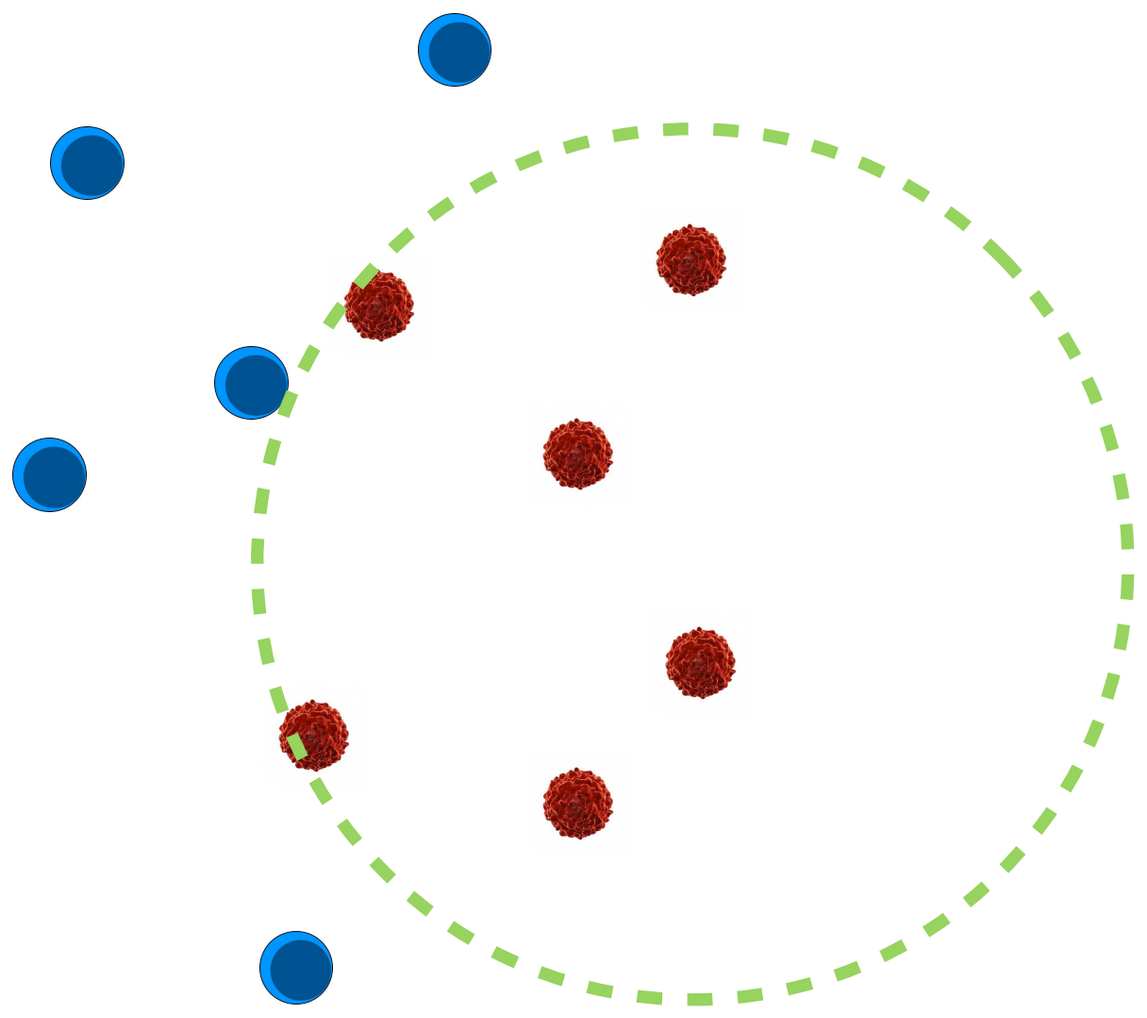
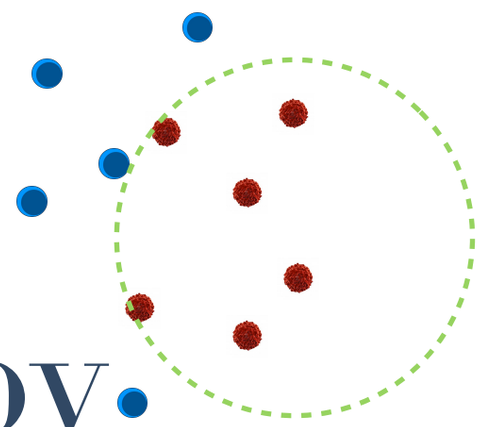
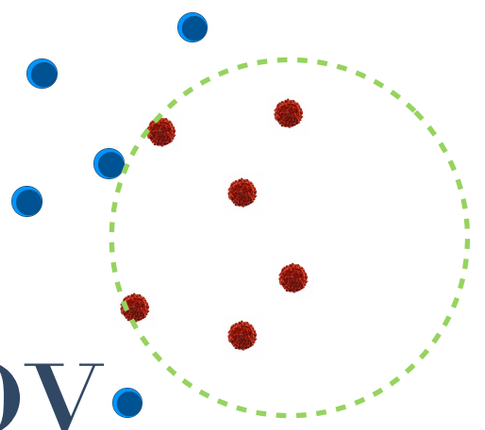


# Linear Programming Application I: Cancer Therapy



- **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

# Linear Programming
# Application I: Cancer Therapy



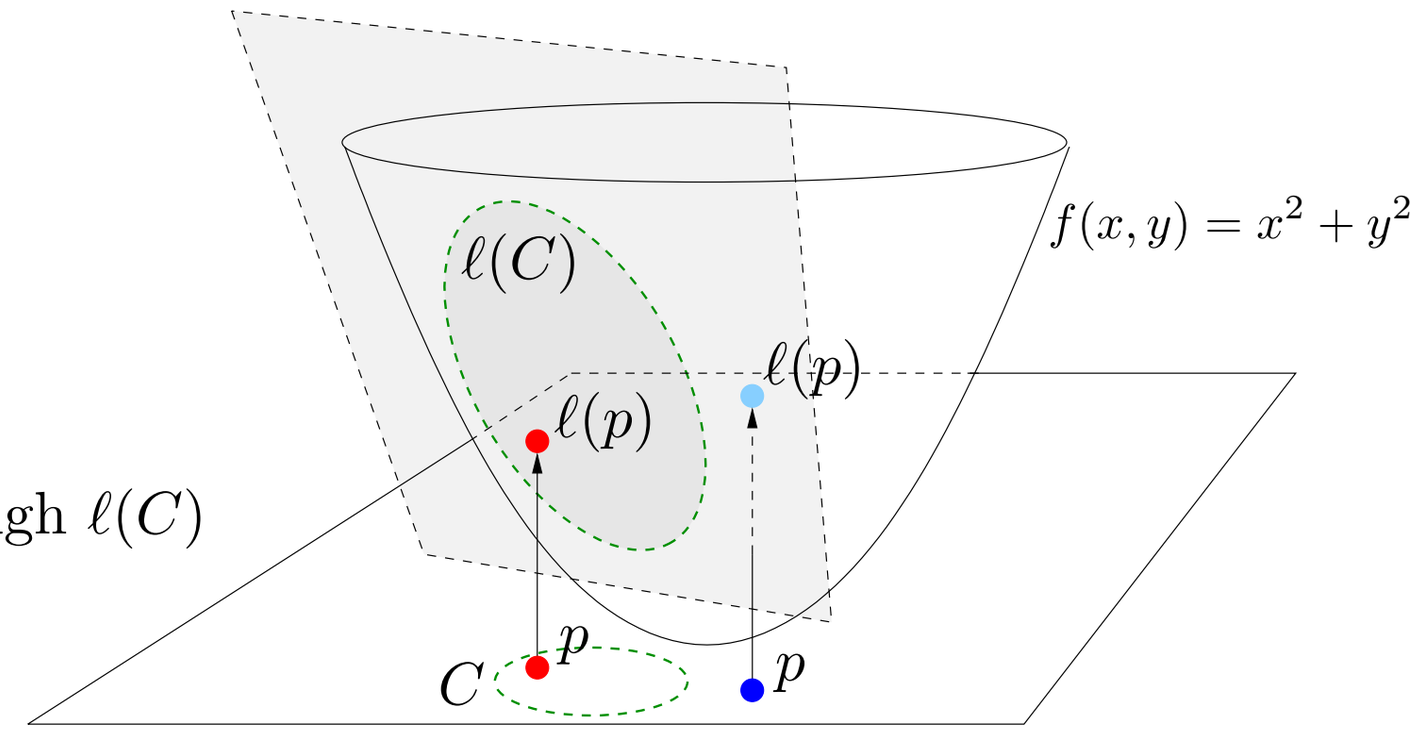
- **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

- Apply *lifting map* $\ell : (x, y) \mapsto (x, y, x^2 + y^2)$

$$p \left\{ \begin{array}{l} \text{inside} \\ \text{on} \\ \text{outside} \end{array} \right\} C$$

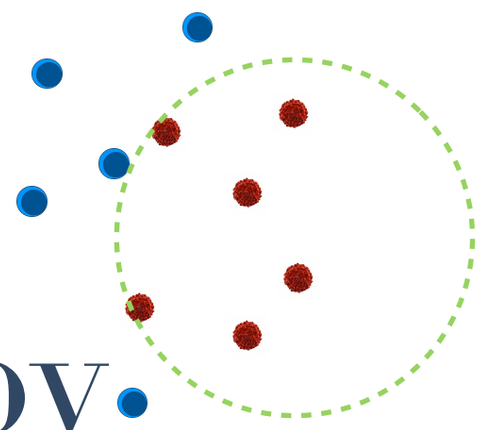
$$\Updownarrow$$

$$\ell(p) \left\{ \begin{array}{l} \text{below} \\ \text{on} \\ \text{above} \end{array} \right\} \text{the plane through } \ell(C)$$




[Ref: Section 2.3]

# Linear Programming
# Application I: Cancer Therapy



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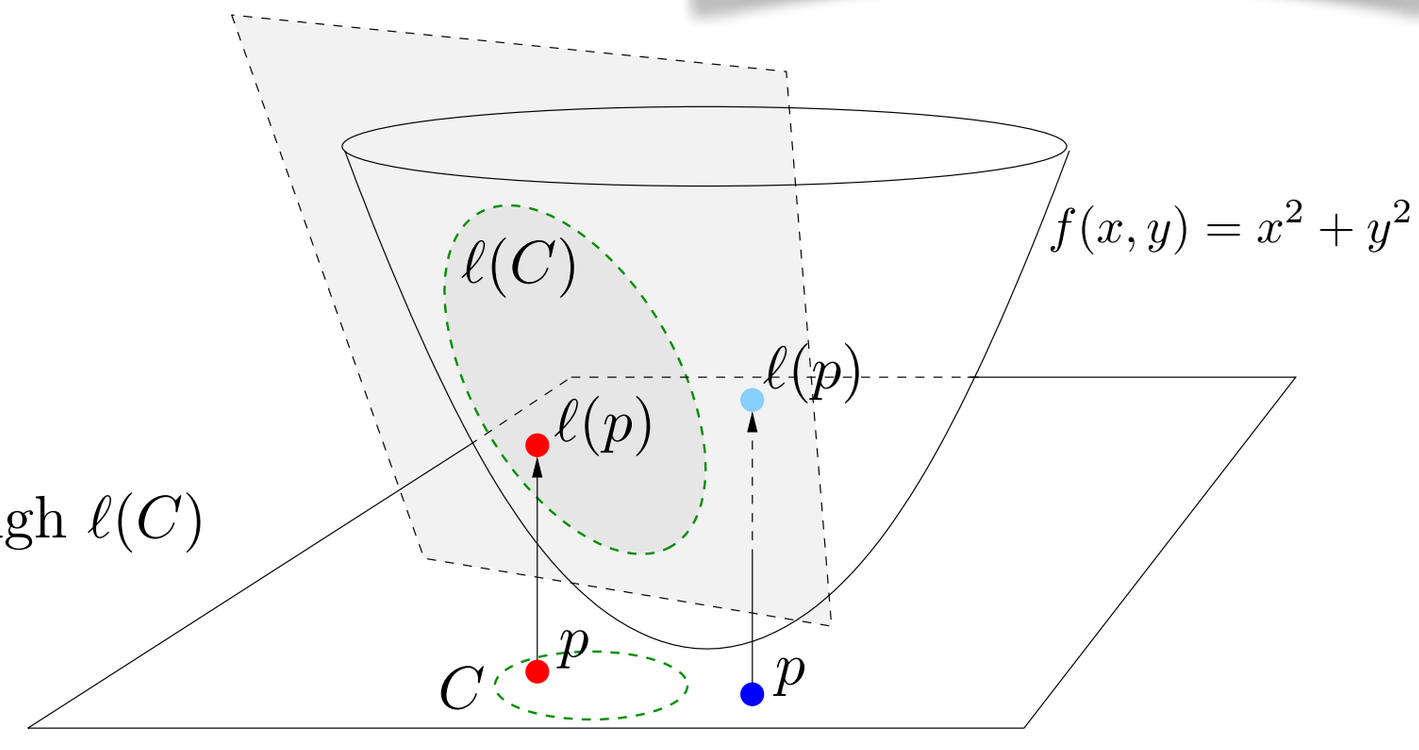
Every circle has a corresponding plane

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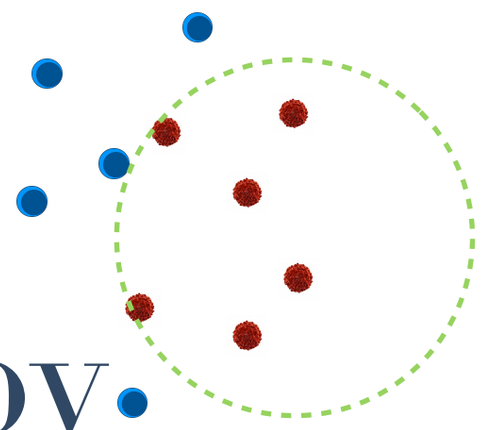
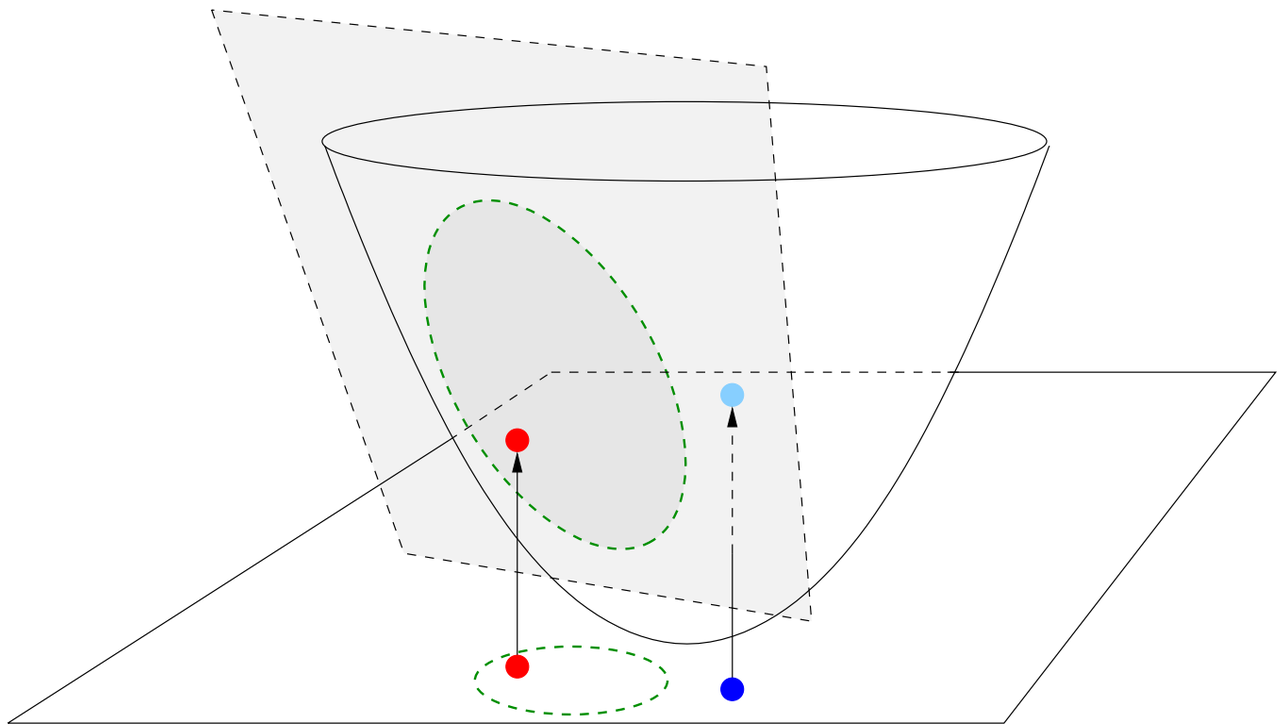
# Linear Programming Application I: Cancer Therapy



- **The geometric problem (lifted space):** Given the lifted sets $R'$ and $B'$ in space, is there a plane that has $R'$ below/on it and $B'$ above?



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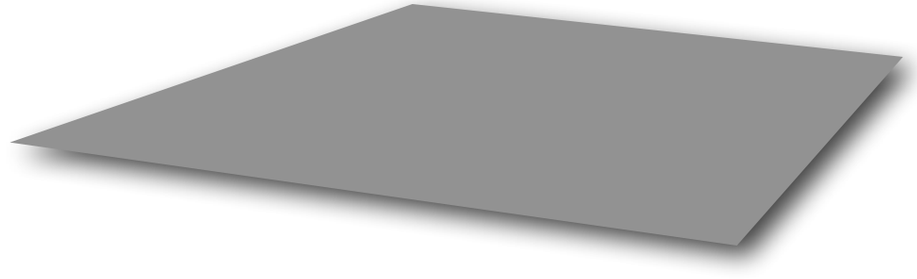
plane: $z = \alpha x + \beta y + \gamma$



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- **The geometric problem (lifted space):** Given the lifted sets $R'$ and $B'$ in space, is there a plane that has $R'$ below/on it and $B'$ above?

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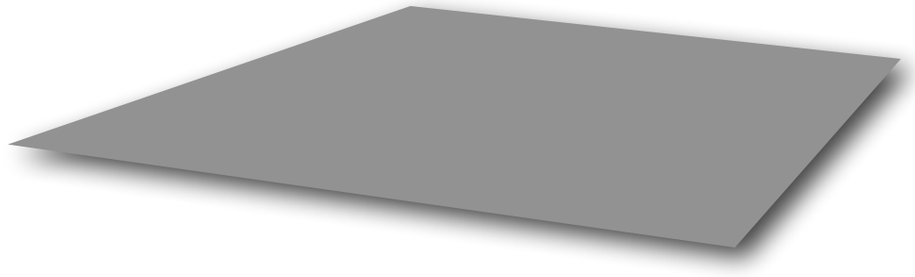
- Find $\alpha$, $\beta$, $\gamma$, $\delta$ ($\delta > 0$) such that...

$$\text{plane: } z = \alpha x + \beta y + \gamma$$



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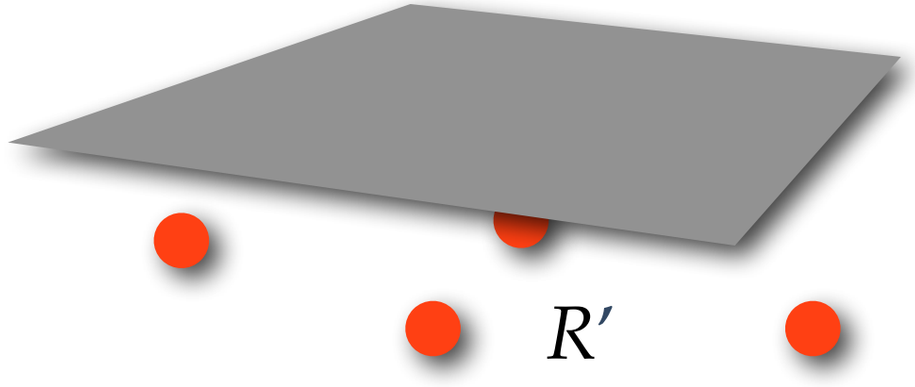


$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$

$(x, y, x^2 + y^2) \in R'$

# Linear Programming Application I: Cancer Therapy

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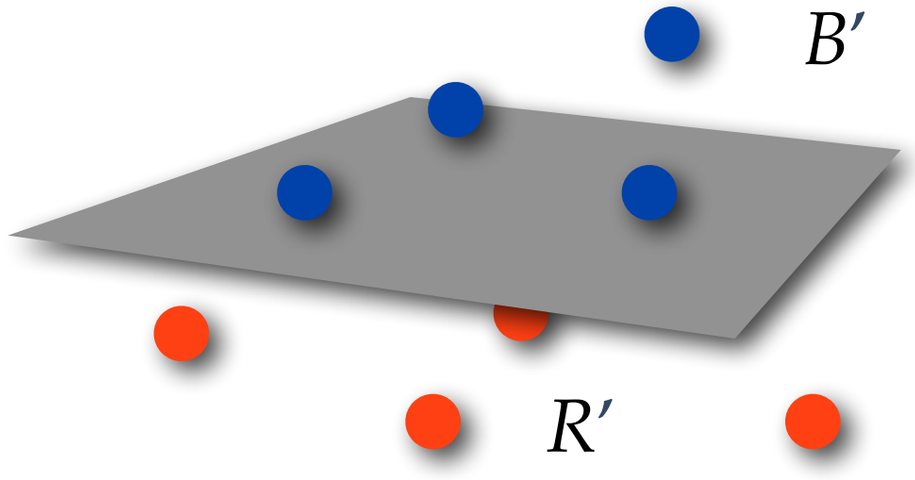


$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

# Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets $R'$ and $B'$ in space, is there a plane that has $R'$ below/on it and $B'$ above?

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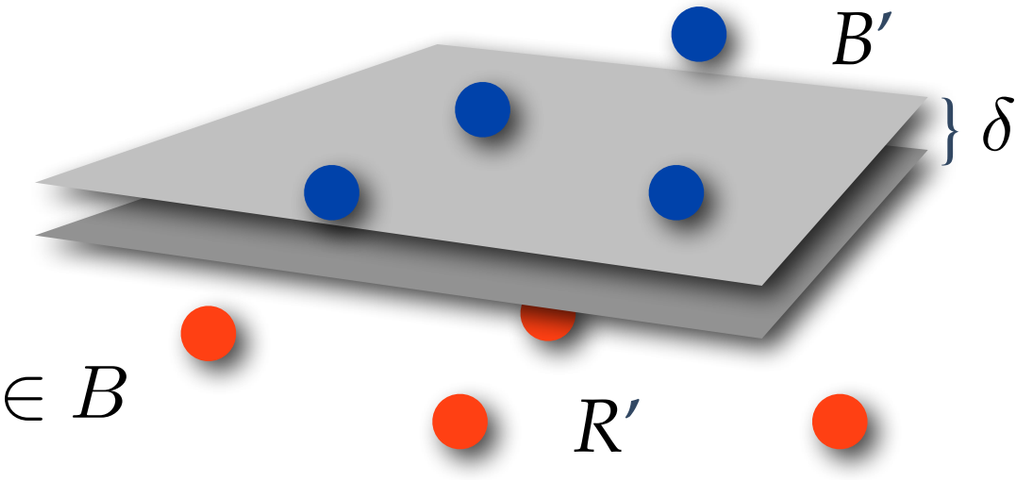


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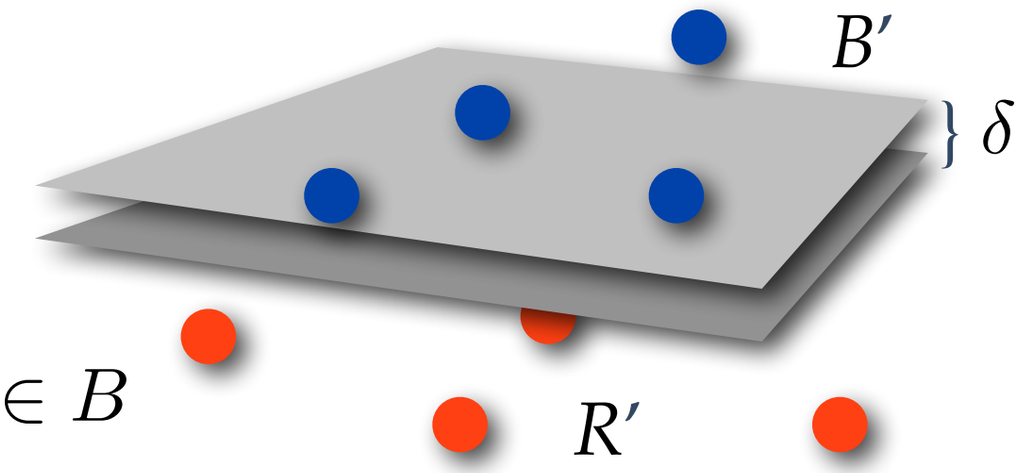


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These are linear constraints!

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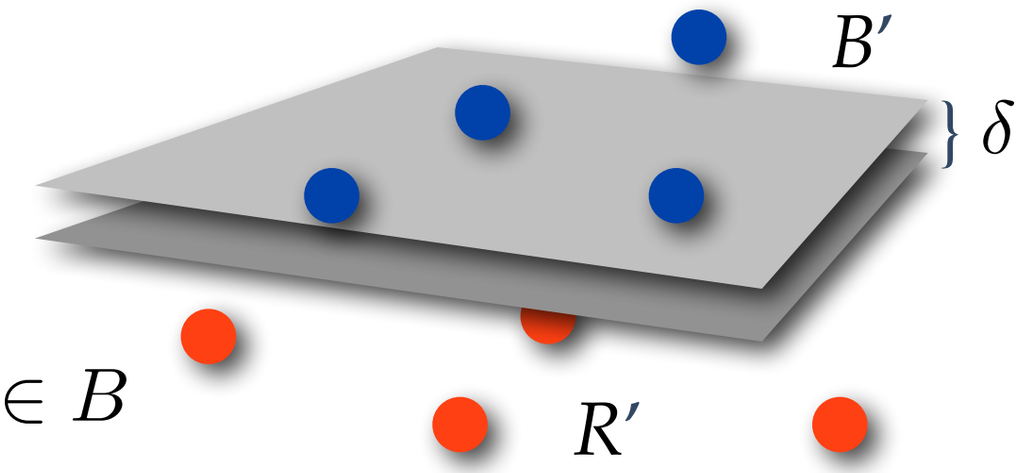


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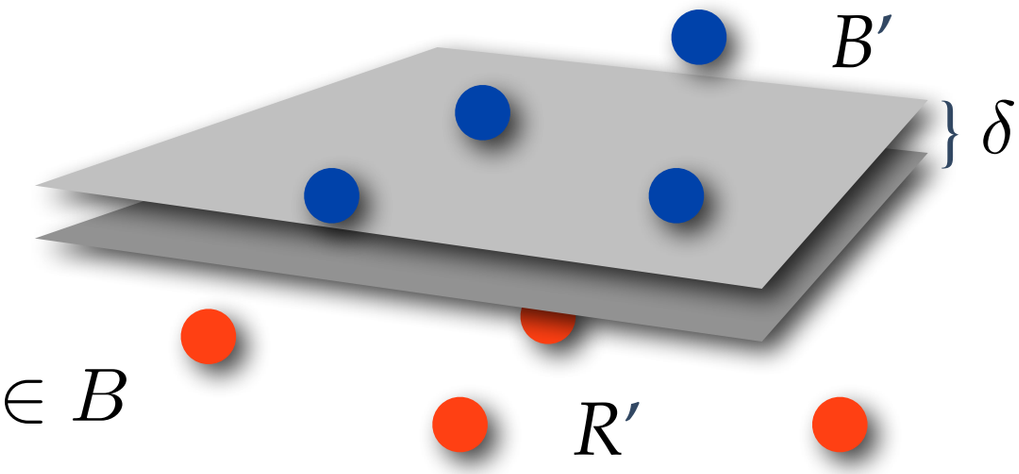


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$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

$$(3, 6) \in R \longrightarrow 3^2 + 6^2 \leq 3\alpha + 6\beta + \gamma$$

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maximize $\delta$

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4 variables!!

$|B| + |R|$ constraints

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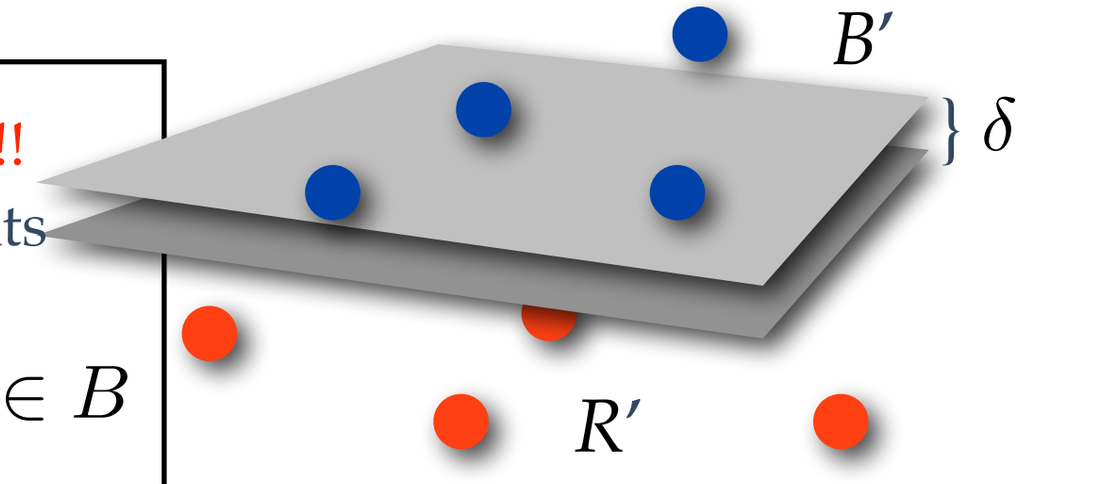

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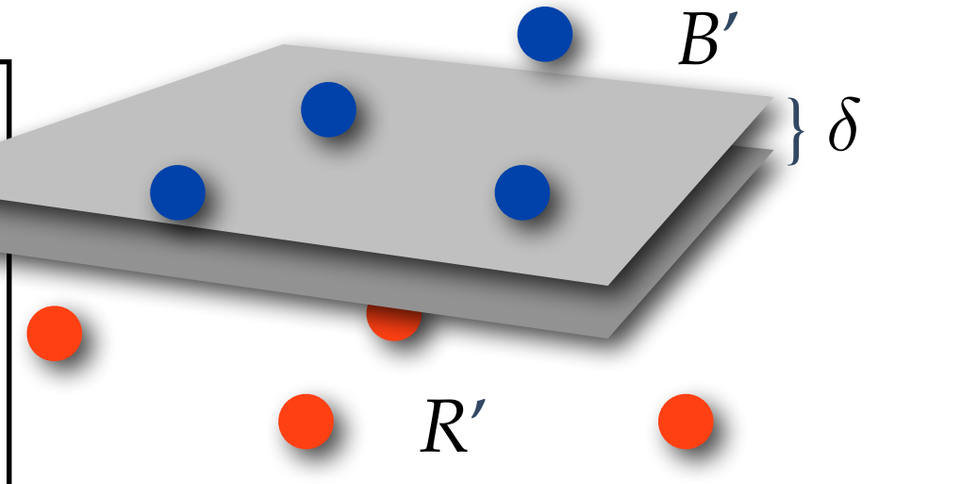

# Linear Programming Application I: Cancer Therapy

- **Fact:** Exposure is possible if and only if the following linear program has positive value.

$$\begin{aligned}
&\text{maximize } \delta \\
&\text{subject to} \\
&\quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\
&\quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R
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- **Reconstructing the exposure from an optimal solution** ($\alpha$, $\beta$, $\gamma$, $\delta$):

$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

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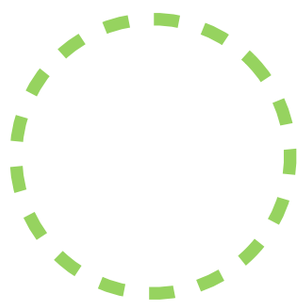

$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

$$= \{(x, y) : (x - \frac{\alpha}{2})^2 + (y - \frac{\beta}{2})^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4}\}$$

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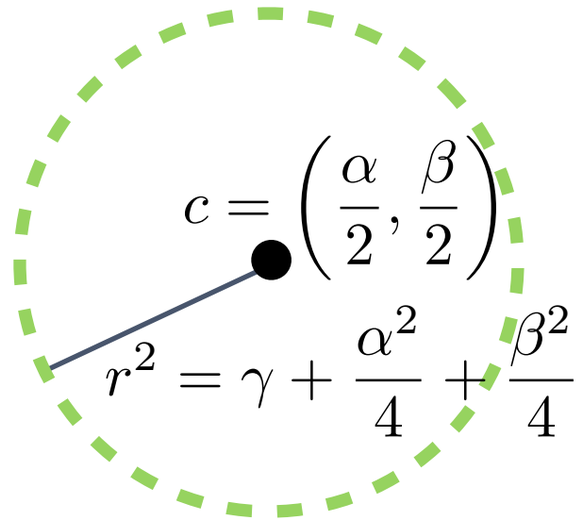


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&= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\} \\
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# Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL:**

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\begin{array}{lrcll}
\text{minimize} & -\delta & & & \\
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& x^2+y^2 & \leq & \alpha x+\beta y+\gamma, & (x,y)\in R \\
& \delta & \leq & 1 &
\end{array}
$$

Avoids unbounded program

maximize $c^T x \rightarrow$ minimize $-c^T x$ and negate resulting value

$$
\begin{array}{lrcll}
\text{minimize} & -\delta & & & \\
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# Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL: Setup and Solve** (Preamble as before)

```
int main() {
  // by default, we have an LP with Ax <= b and no bounds for
  // the four variables alpha, beta, gamma, delta
  Program lp (CGAL::SMALLER, false, 0, false, 0);
  const int alpha = 0;
  const int beta  = 1;
  const int gamma = 2;
  const int delta = 3;

  // number of red and blue points
  int m; std::cin >> m;
  int n; std::cin >> n;

  // read the red points (cancer cells)
  for (int i=0; i<m; ++i) {
    int x; std::cin >> x;
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    // set up <= constraint for point inside/on circle:
    //   -alpha x - beta y - gamma <= -x^2 - y^2
    lp.set_a (alpha, i, -x);
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  // read the blue points (healthy cells)
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    lp.set_a (alpha, m+j, x);
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  // objective function: -delta (the solver minimizes)
  lp.set_c(delta, -1);

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```

Important: Be careful with the indices!
If you set constraint *i*, it enlarges your matrix to have *i* rows. This can blow up the running time if *i* is unnecessarily large. The same goes for variables. Limit yourself to use consecutive indices starting from 0.

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\end{array}
$$

# Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL: Setup and Solve** (Preamble as before)

```
int main() {
  // by default, we have an LP with Ax <= b and no bounds for
  // the four variables alpha, beta, gamma, delta
  Program lp (CGAL::SMALLER, false, 0, false, 0);
  const int alpha = 0;
  const int beta  = 1;
  const int gamma = 2;
  const int delta = 3;

  // number of red and blue points
  int m; std::cin >> m;
  int n; std::cin >> n;

  // read the red points (cancer cells)
  for (int i=0; i<m; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up <= constraint for point inside/on circle:
    //   -alpha x - beta y - gamma <= -x^2 - y^2
    lp.set_a (alpha, i, -x);
    lp.set_a (beta,  i, -y);
    lp.set_a (gamma, i, -1);
    lp.set_b (       i, -x*x - y*y);
  }
```

```
  // read the blue points (healthy cells)
  for (int j=0; j<n; ++j) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up <= constraint for point outside circle:
    //   alpha x + beta y + gamma + delta <= x^2 + y^2
    lp.set_a (alpha, m+j, x);
    lp.set_a (beta,  m+j, y);
    lp.set_a (gamma, m+j, 1);
    lp.set_a (delta, m+j, 1);
    lp.set_b (       m+j, x*x + y*y);
  }

  // objective function: -delta (the solver minimizes)
  lp.set_c(delta, -1);

  // enforce a bounded problem:
  lp.set_u (delta, true, 1);

  // solve the program, using ET as the exact type
  Solution s = CGAL::solve_linear_program(lp, ET());
  assert (s.solves_linear_program(lp));
```

$$
\begin{array}{lrcll}
\text{minimize} & -\delta & & & \\
\text{subject to} & x^2+y^2 & \geq & \alpha x+\beta y+\gamma+\delta, & (x,y)\in B \\
& x^2+y^2 & \leq & \alpha x+\beta y+\gamma, & (x,y)\in R \\
& \delta & \leq & 1 &
\end{array}
$$

# Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL: Output**

negate resulting value!

```
// output exposure center and radius, if they exist
if (s.is_optimal() && (s.objective_value() < 0)) {
  // *opt := alpha, *(opt+1) := beta, *(opt+2) := gamma
  CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
    opt = s.variable_values_begin();
  CGAL::Quotient<ET> alpha_opt = *(opt+alpha);// +0
  CGAL::Quotient<ET> beta_opt = *(opt+beta);// +1
  CGAL::Quotient<ET> gamma_opt = *(opt+gamma);// +2
  std::cout << "There is a valid exposure:\n";
  std::cout << " Center = ("        // (alpha/2, beta/2)
       << alpha_opt/2 << ", " <<  beta_opt/2
       << ")\n";
  std::cout << " Squared Radius = " // gamma + alpha^2/4 + beta^2/4
       << gamma_opt + alpha_opt*alpha_opt/4 + beta_opt*beta_opt/4 << "\n";
} else
  std::cout << "There is no valid exposure.";
std::cout << "\n";
return 0;
}
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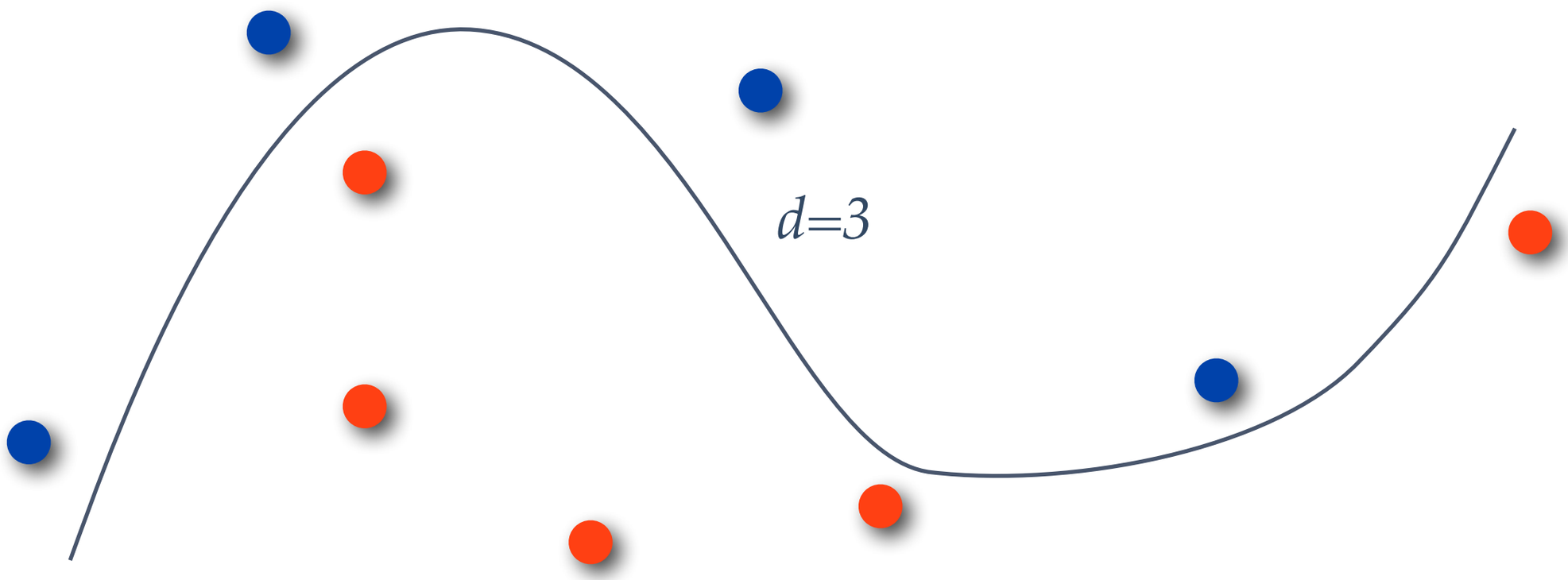
negate resulting value!

Points to first variable of optimal solution

The quotient **`*(opt+i)`** is the value of the variable $x_i$ in the optimal solution

# Linear Programming Beyond Cancer Therapy

- Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree *d*?



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- Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree *d*?

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$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

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$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \leq 0, \quad (x, y) \in B$$
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- This is linear separability in 9-dimensional space, under the generalized lifting map $(x, y) \rightarrow (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y)$

# Linear Programming Further Applications

Not necessary for the class, but good to know

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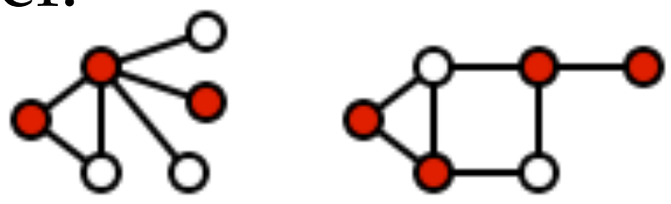

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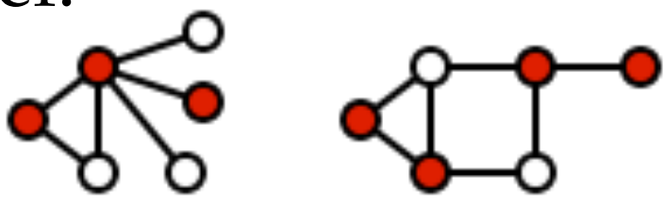

$x_i \in \{0, 1\}$

- Formulation as "LP": $x_i$ indicates whether vertex i is in the cover (0: not in the cover, 1: in the cover):

$$\begin{array}{lrcll} \text{minimize} & \sum_{i=1}^{n} x_i & & & \\ \text{subject to} & x_i + x_j & \geq & 1 & \forall \{i,j\} \in E \\ & 0 \leq x_i \leq 1 & & & \forall i \in V \end{array}$$

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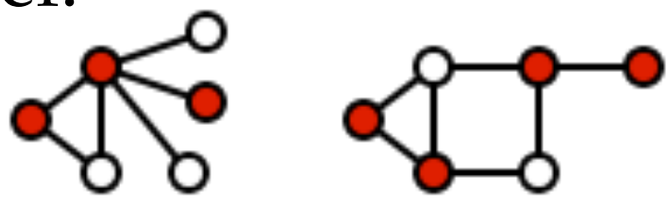

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← not an LP!

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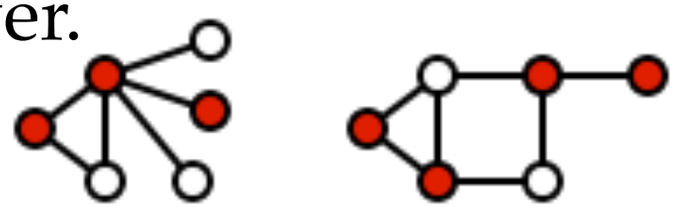

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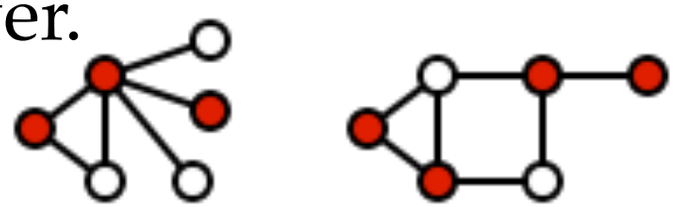

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- **Theorem:** $C = \{i: x^*_i \geq 1/2\}$ is a vertex cover of size at most 2 opt.

[Ref]

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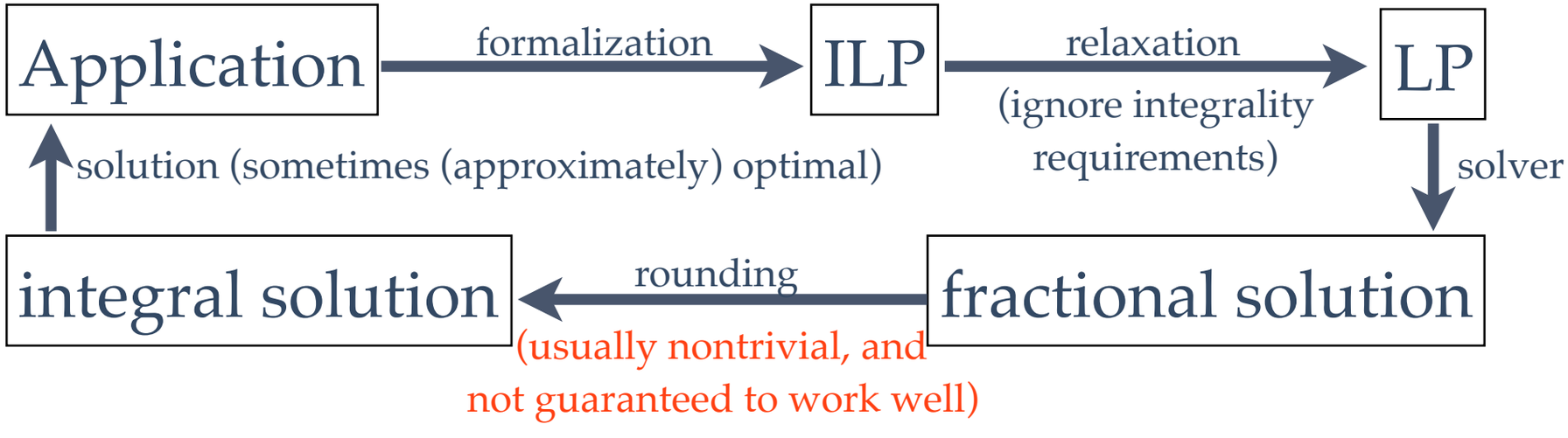

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- **Example** (n=2, m=5):

$$\begin{array}{lrcr}
\text{minimize} & x^2 + 4y^2 - 32y + 64 & & \\
\text{subject to} & x + y & \leq & 7 \\
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& x & \geq & 0 \\
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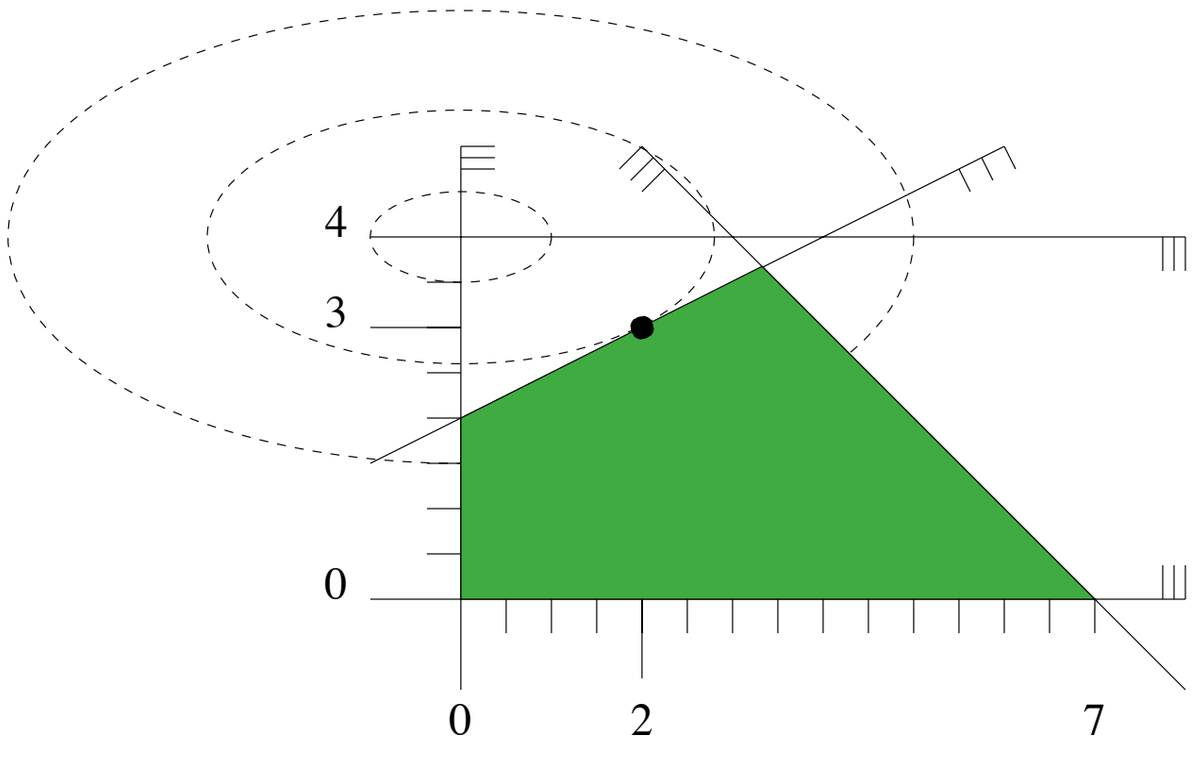

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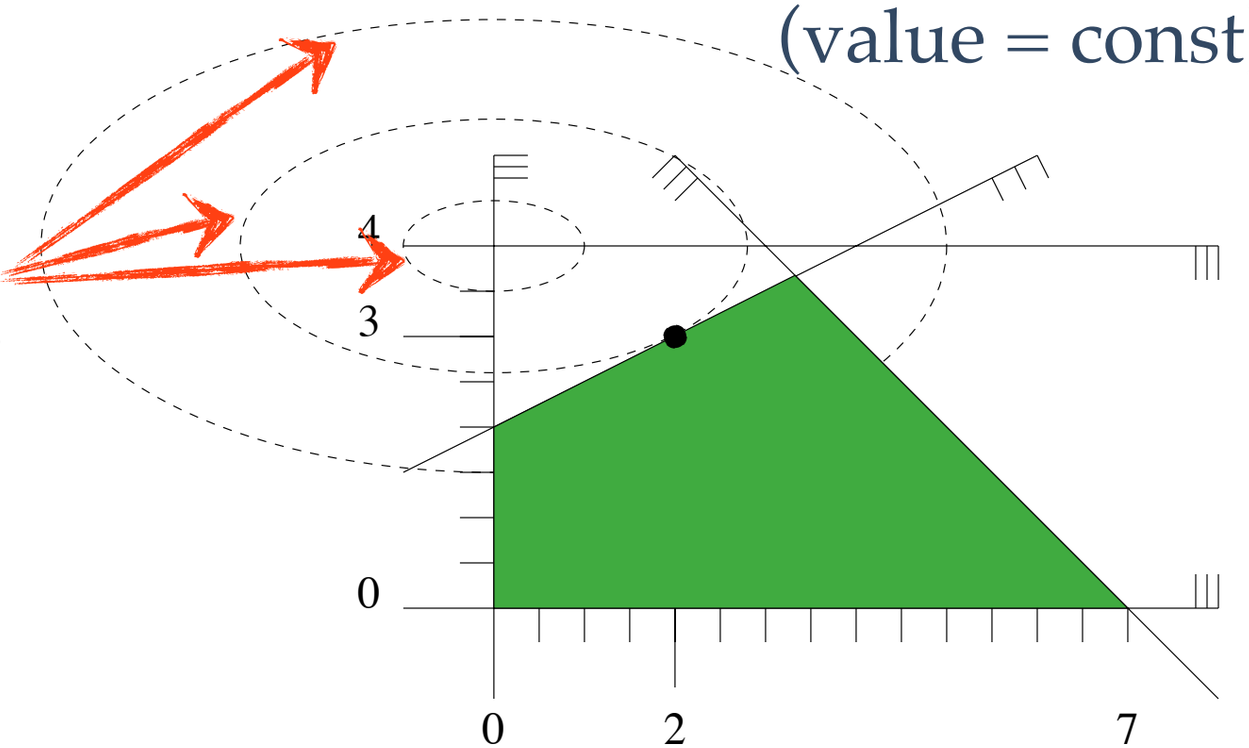

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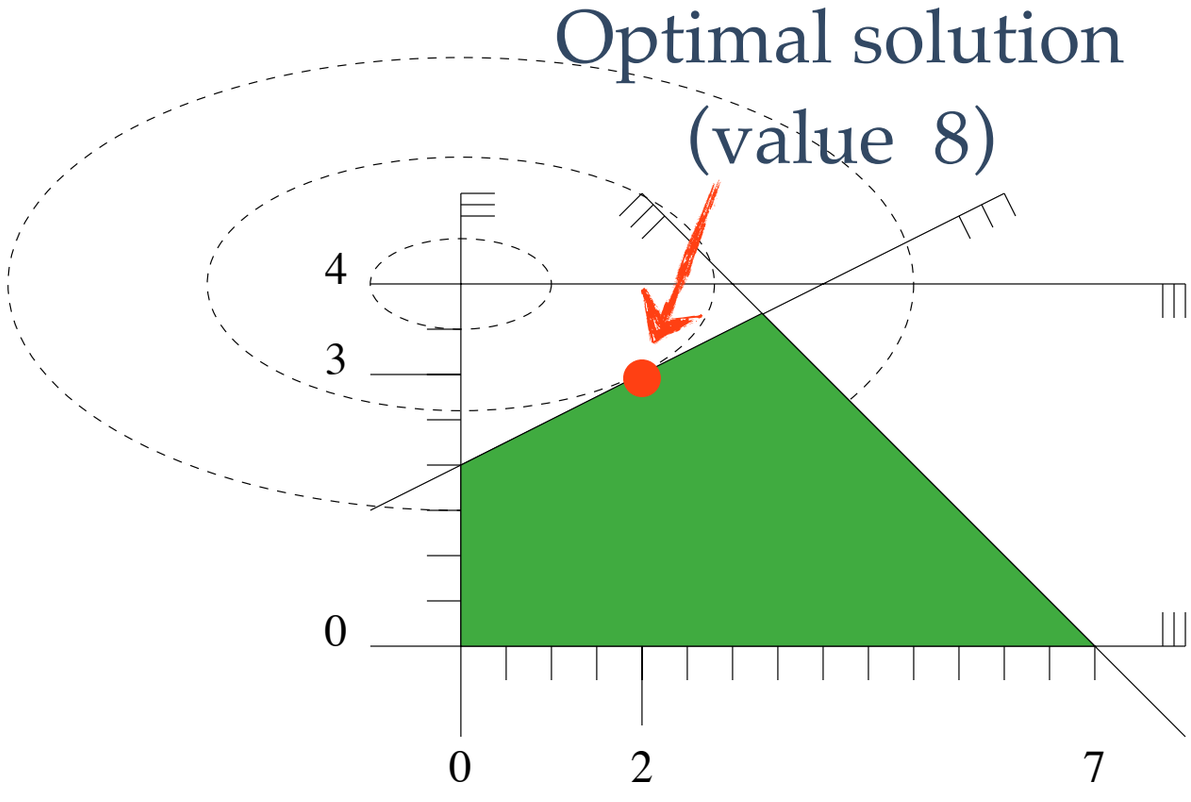

# Quadratic Programming … in CGAL

- **General form of QP in CGAL:**

$$\begin{aligned} \text{minimize} \quad & x^T D x + c^T x + c_0 \\ \text{subject to} \quad & Ax \begin{matrix} \geq \\ \leq \end{matrix} b \\ & l < x < u \end{aligned}$$

$(D \in \mathbb{R}^{n \times n}$ symmetric positive semidefinite$)$

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- **Relax:** In the applications, we know from theory that *D* is "good"

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- **Risk-tolerant strategy:** Minimize the risk under a given lower bound for the expected return!

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- **Example:** $n=2$

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|---|---|---|
| Swatch shares | 0.09 | -0.05 |
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Negative correlation: if CS does worse than expected, Swatch will probably do better, and vice versa

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Read as: standard deviation of return rate is $\sqrt{0.25} = 0.5$
(actual return rate could easily be off by 0.5)

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- **Investment strategy:**

$$(x_1, x_2, \ldots, x_n), \quad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0 \forall i$$

Meaning: An $x_i$ fraction of your money goes into investment *i*

# Quadratic Programming Application: Low-Risk Investment

- **Investment strategy:**

$$(x_1, x_2, \ldots, x_n), \quad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0 \forall i$$

Meaning: An $x_i$ fraction of your money goes into investment $i$

- **Expected return rate of this strategy:**

$$E[\sum_{i=1}^{n} x_i R_i] = \sum_{i=1}^{n} x_i E[R_i] = \sum_{i=1}^{n} r_i x_i$$

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- **Example:** half the money in Swatch shares, half in Credit Suisse shares; expected return rate is

$$\frac{1}{2} \cdot 0.1 + \frac{1}{2} \cdot 0.51 = 0.305 = 30.5\%$$

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- **Risk of this strategy:**

Straightforward calculations

$$\mathrm{Var}[\sum_{i=1}^{n} x_i R_i] = \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij} x_i x_j = x^T D x$$

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less than each individual risk!

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# Quadratic Programming Application: Low-Risk Investment

- **The risk-tolerant case:** Find the investment strategy with lowest risk that guarantees expected return rate $\rho$ at least!

$$\text{minimize} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_i x_j \longleftarrow \boxed{\text{risk}}$$

$$\text{subject to} \quad \sum_{i=1}^{n} r_i x_i \geq \rho$$

expected return rate

$$\sum_{i=1}^{n} x_i = 1$$

strategy

$$x_i \geq 0, \quad i = 1, \ldots, n$$

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& \sum_{i=1}^{n} x_i = 1 \quad \leftarrow \text{strategy} \\
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- **Example:** $\rho = 0.4$: 26.8% Swatch, 73.2% Credit Suisse; risk $= 0.121$

# Low-Risk Investment Example… in CGAL

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} x_i x_j \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n \end{aligned}$$

- **Preamble:** This time, the input is rational…

Gnu Multi-precision Library (GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpq.h>

// choose exact Rational type
typedef CGAL::Gmpq ET;


// program and solution types
typedef CGAL::Quadratic_program<ET> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

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- **Input:** Desired expected return

```
int main() {
  // read minimum expected return rate
  std::cout << "What is your desired expected return rate? ";
  double rho; std::cin >> rho;
```

for example, 0.4 ≈ 40%

# Low-Risk Investment Example… in CGAL

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- **Setup:** Make sure to enter matrix $2 \cdot D$ (customary in QP solvers)!

```
// by default, we have a nonnegative QP with Ax >= b
Program qp (CGAL::LARGER, true, 0, false, 0);

// now set the non-default entries:
const int sw = 0;
const int cs = 1;

// constraint on expected return: 0.1 sw + 0.51 cs >= rho
qp.set_a(sw, 0, ET(1)/10);
qp.set_a(cs, 0, ET(51)/100);
qp.set_b(    0, rho);

// strategy constraint: sw + cs = 1
qp.set_a(sw, 1, 1);
qp.set_a(cs, 1, 1);
qp.set_b(    1, 1);
qp.set_r(    1, CGAL::EQUAL); // override default >=

// objective function: 0.09 sw^2 - 0.05 sw cs - 0.05 cs sw + 0.25 cs^2
// we need to specify the entries of the symmetric matrix 2D, on and below the diagonal
qp.set_d(sw, sw,  ET(18)/100); //  0.09 sw^2
qp.set_d(cs, sw, -ET(1)/10);   // -0.05 cs sw
qp.set_d(cs, cs,  ET(1)/2);    //  0.25 cs^2
```

$j \leq i$ in **`set_d(i, j)`**

# Low-Risk Investment Example… in CGAL

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} x_i x_j \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n \end{aligned}$$

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- **Solve:** ...as nonnegative quadratic program (a little faster)

```
// solve the program, using ET as the exact type
Solution s = CGAL::solve_nonnegative_quadratic_program(qp, ET());
assert (s.solves_quadratic_program(qp));
```

independent verification

The nonnegative solvers ignore global lower and upper bounds (as specified by the constructor) and replace them by nonnegativity constraints only. On the other hand, adding these bounds to the matrix explicitly blows up your LP.

In such a case, you are better off using global bounds and the "normal" solver (not nonnegative). Anyway, we recommend to use the "normal" solver as a default and switch to the nonnegative one only to check out whether it makes a difference (in many cases, it won't).

# Low-Risk Investment Example… in CGAL

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\text{minimize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} x_i x_j \\
\text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \geq \rho \\
& \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0, \quad i = 1, \ldots, n
\end{aligned}$$

- **Output:** query solution status; if feasible, output strategy / risk

```
// output
  if (s.status() == CGAL::QP_INFEASIBLE) {
    std::cout << "Expected return rate " << rho << " cannot be achieved.\n";
  } else {
    assert (s.status() == CGAL::QP_OPTIMAL);
    Solution::Variable_value_iterator opt = s.variable_values_begin();
    double sw_percent = ceil_to_double(100 * *opt);
    double cs_percent = ceil_to_double(100 * *(opt+1));
    double risk = ceil_to_double(1000 * s.objective_value());
    std::cout << "Minimum risk investment strategy:\n"
          << "~" << sw_percent << "%" << " Swatch\n"
          << "~" << 100-sw_percent << "%" << " Credit Suisse\n"
          << "Risk <= 0." << risk << "\n";
  }
```

# Known Bug :=(

- You can't reliably copy or assign instances of the class **CGAL::Quadratic_program_solution<ET>**

- **Workaround 1:** If you want to pass or return such instances to/from a function, pass a pointer to the instance instead!

- **Workaround 2:** If you want to assign a new solution to an existing instance... don't do it!

# Sources and Further Reading

- **LP/QP Solver:** Online manual at **`www.cgal.org`**: Online Manual → Combinatorial Algorithms → Linear and Quadratic Programming Solver

- **Cancer Therapy:** J. O'Rourke, S. Kosaraju, and N. Megiddo: Computing Circular Separabiliy, *Discrete & Computational Geometry* 1:105-113 (1986)

- **Low-Risk Investment:** H. Markowitz: Portfolio Selection, *Journal of Finance* 7(1): 77-91 (1952)