Linear and Quadratic Programming (with CGAL)

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November 1, 2017

Based on slides by Bernd Gärtner

Today

- Quick introduction to Linear Programming (formulation and representation).
- Quick recap on techniques to solve it (no extensive knowledge needed).
- Linear Programming in CGAL.
- Examples.
- Integer Programming (not required for this class, but important).
- Quadratic Convex Programming (CGAL).

Linear Programming (LP)

- Linear programming is a central topic in optimization .
- It provides a powerful tool in modeling many applications .
- LP has attracted most of its attention in optimization during the last six decades for two main reasons :
    - Applicability : There are many real-world applications that can be modeled as linear programming ;
    - Solvability : There are theoretically and practically efficient techniques for solving large-scale problems.

Linear Programming (LP)

We face an optimization problem subject to some constraints.

- **Variables:** describe our choices, the parameters that we are allowed to change;
- **Objective function:** Describes the criterion that we wish to minimize (e.g. cost) or maximize (e.g. profit);
- **Constraints:** Describe the limitations that we have for the choice of the values of the variables.

Linear Programming

Mathematical formulation

- **Problem:** Minimize a linear function in n variables subject to m linear (in)equality constraints!

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- **Example** (n=2, m=5)**:**

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\begin{array}{lrcl}
\text{minimize} & -32y+64 & & \\
\text{subject to} & x+y & \leq & 7 \\
& -x+2y & \leq & 4 \\
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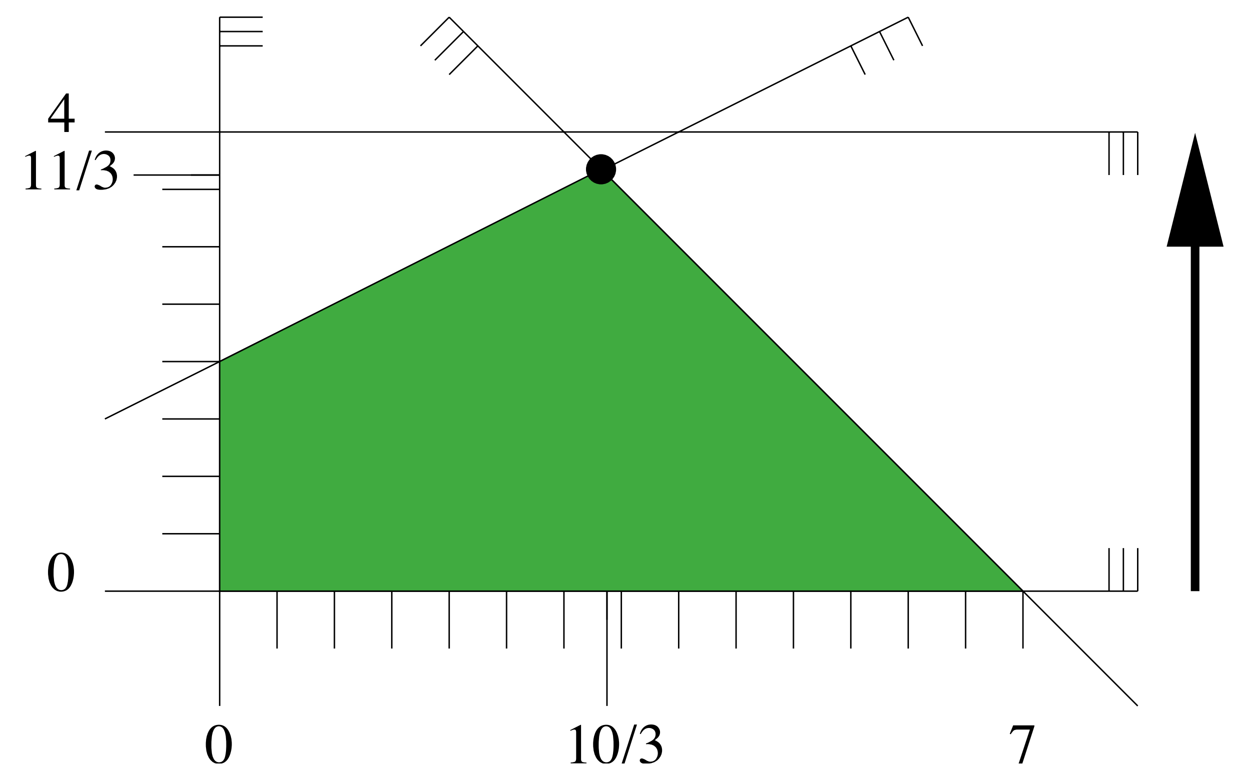

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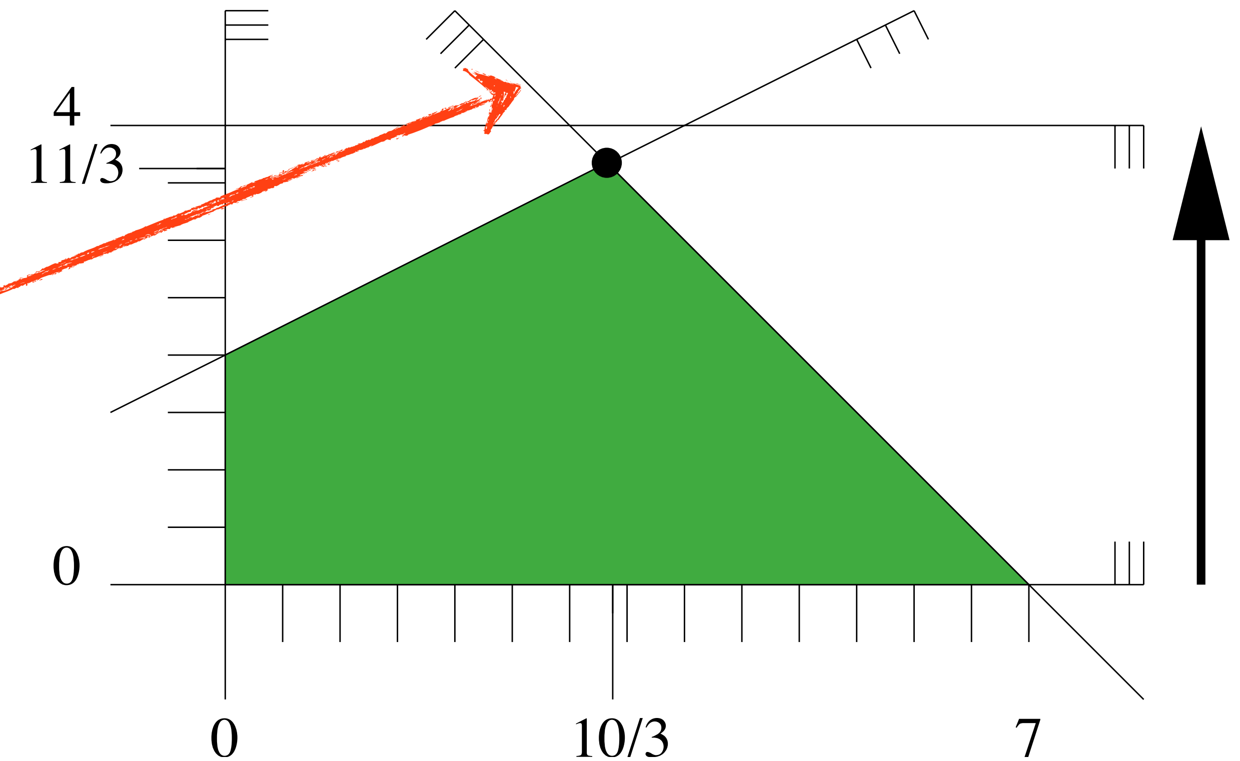

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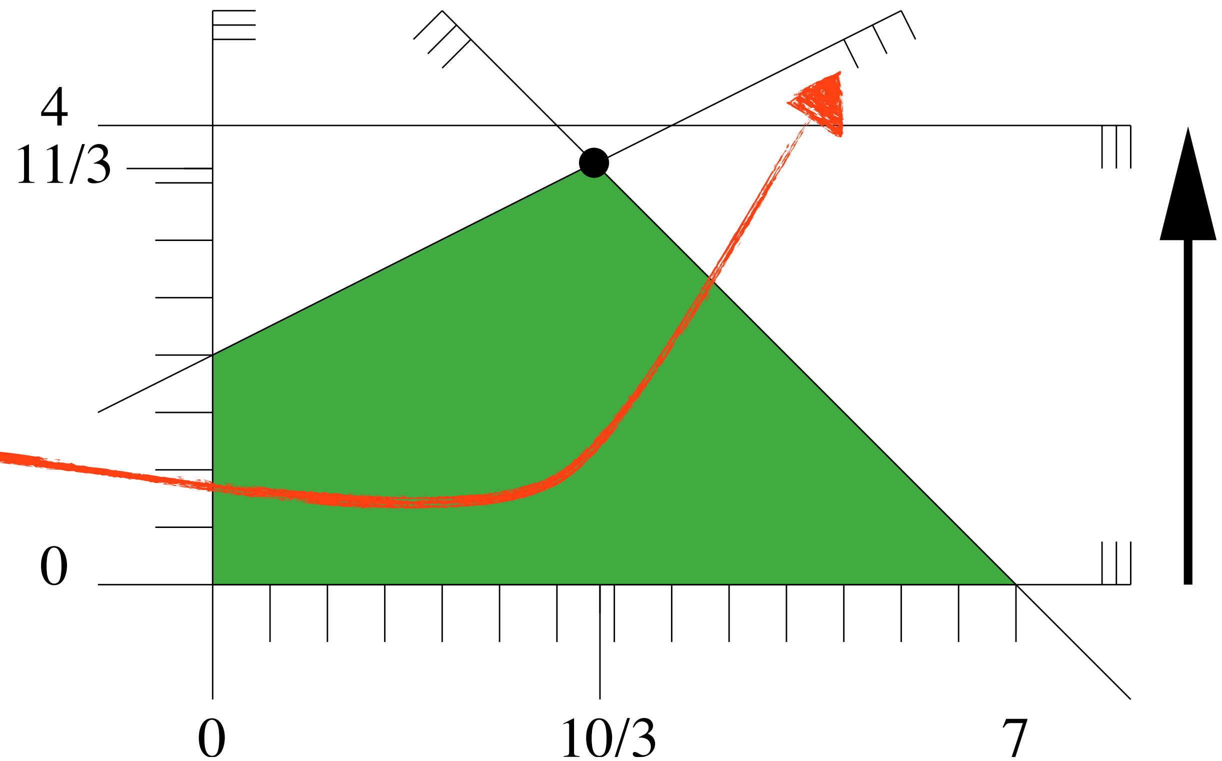

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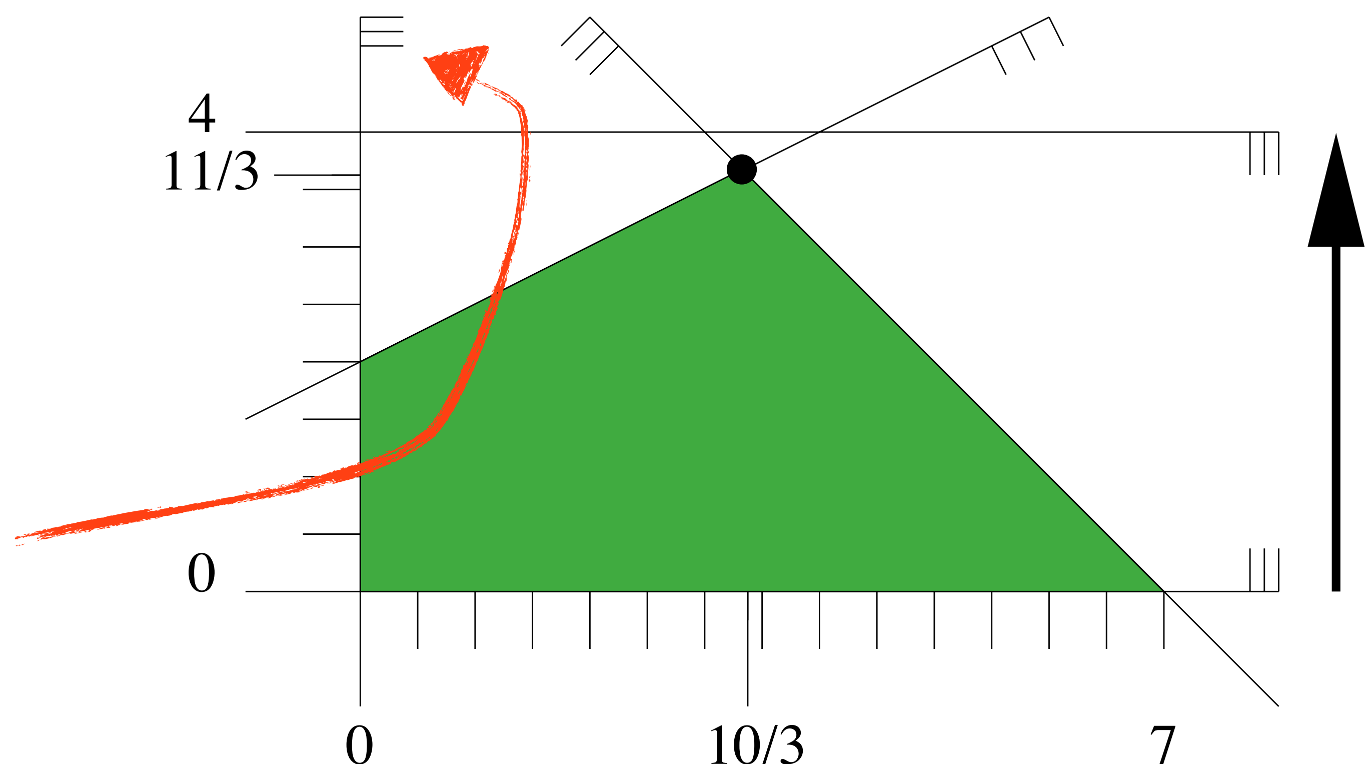

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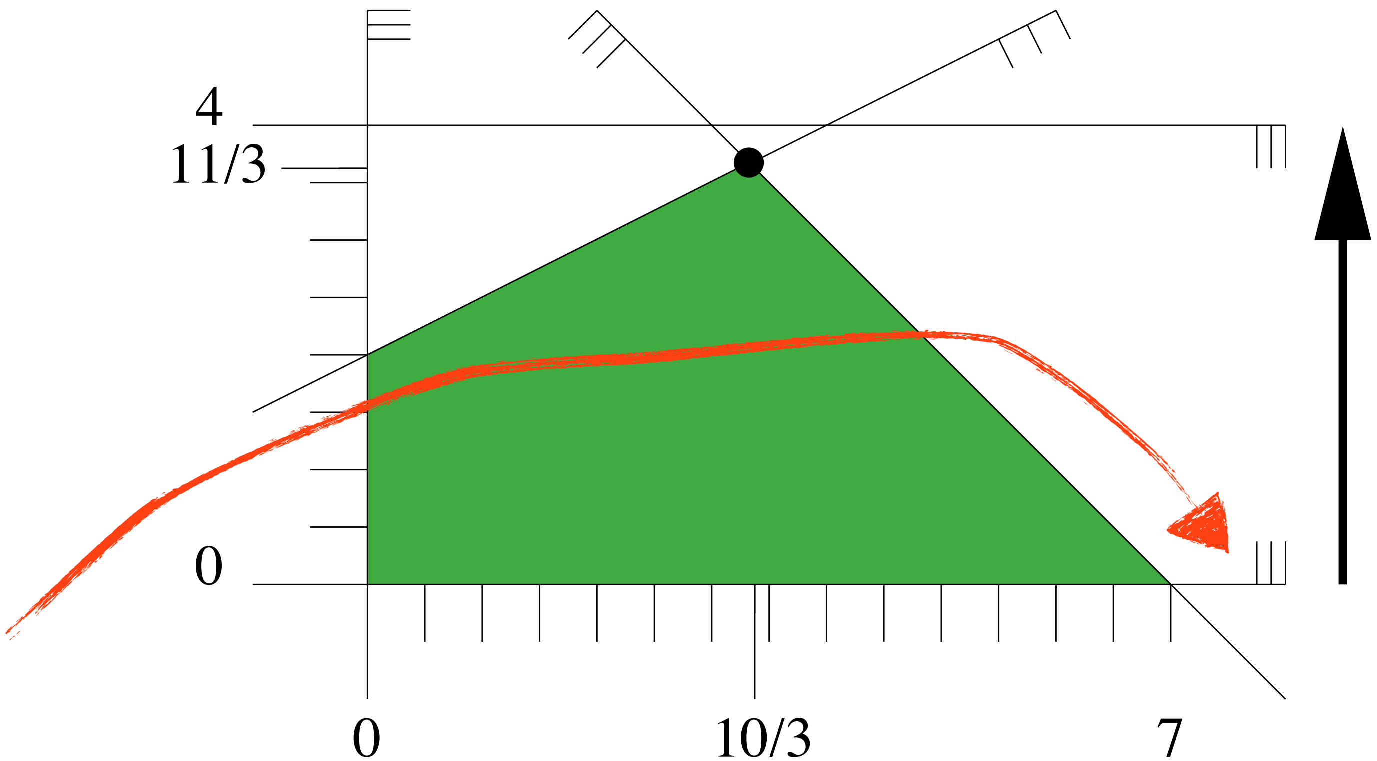

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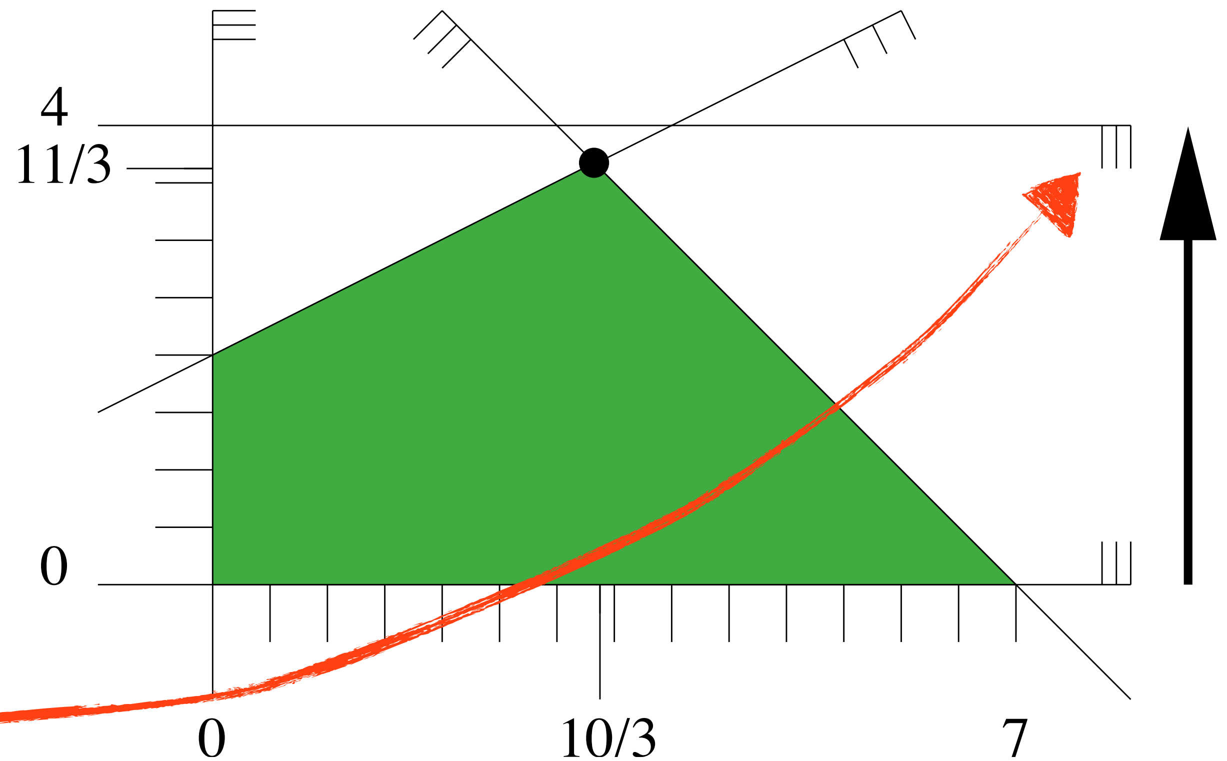

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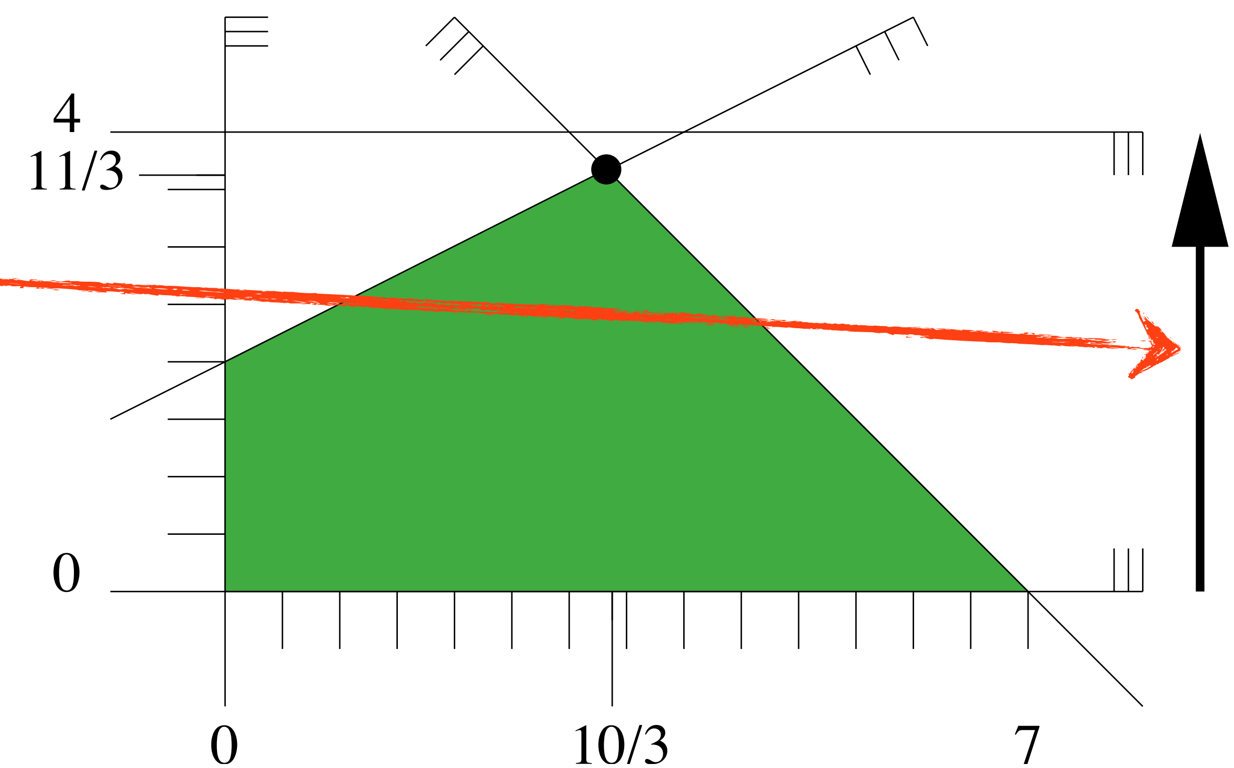

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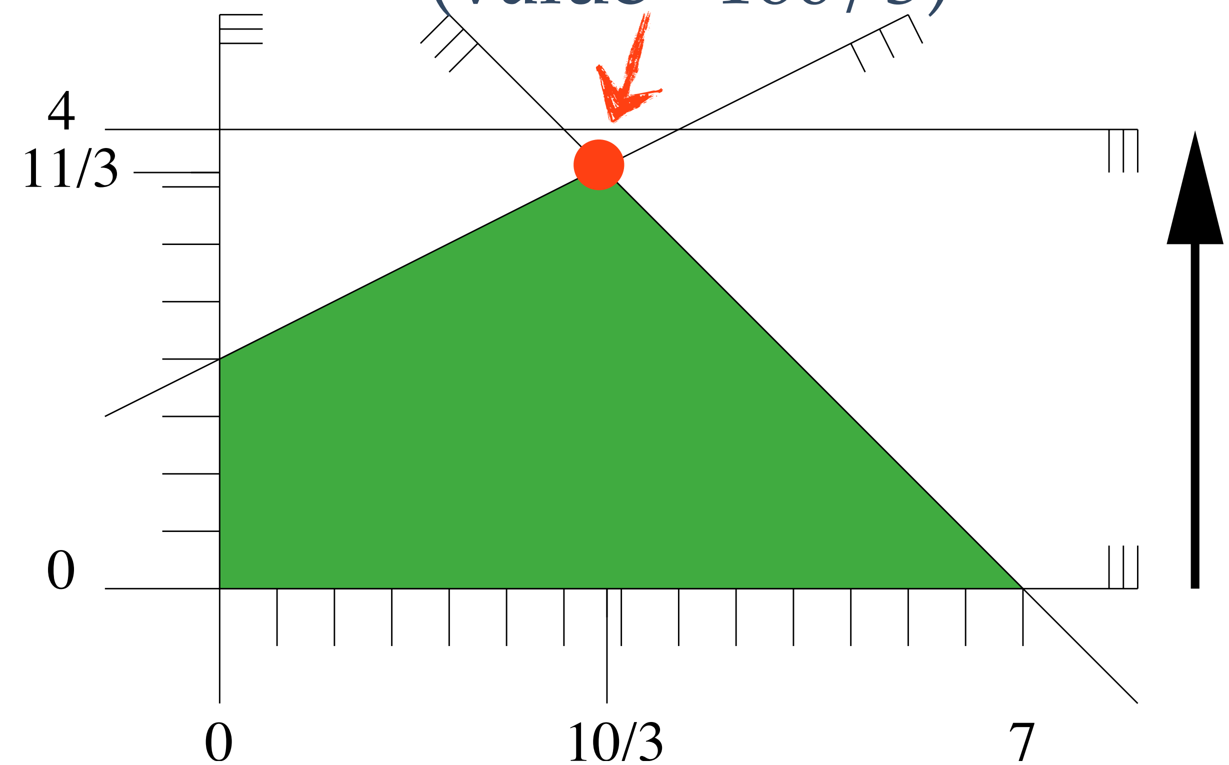
Optimal solution
(value -160/3)



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- **Unbounded linear programs:**

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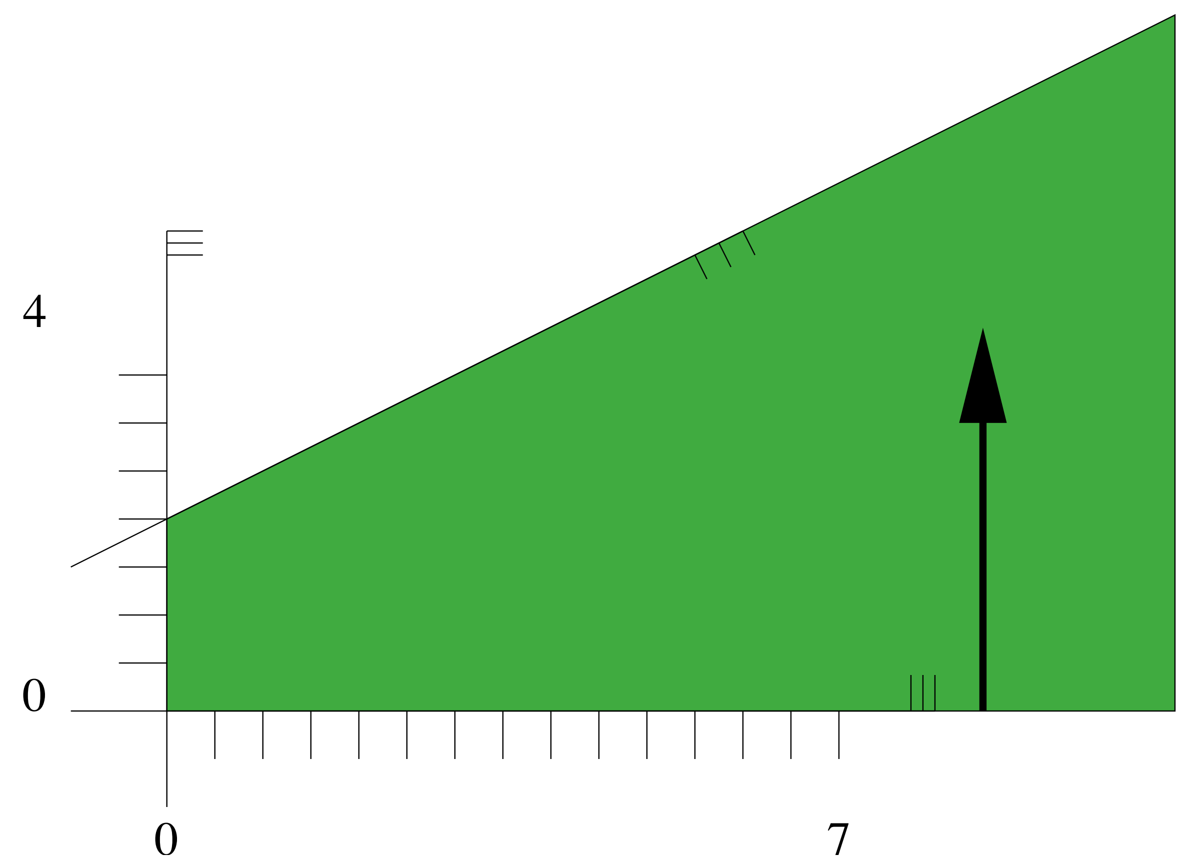

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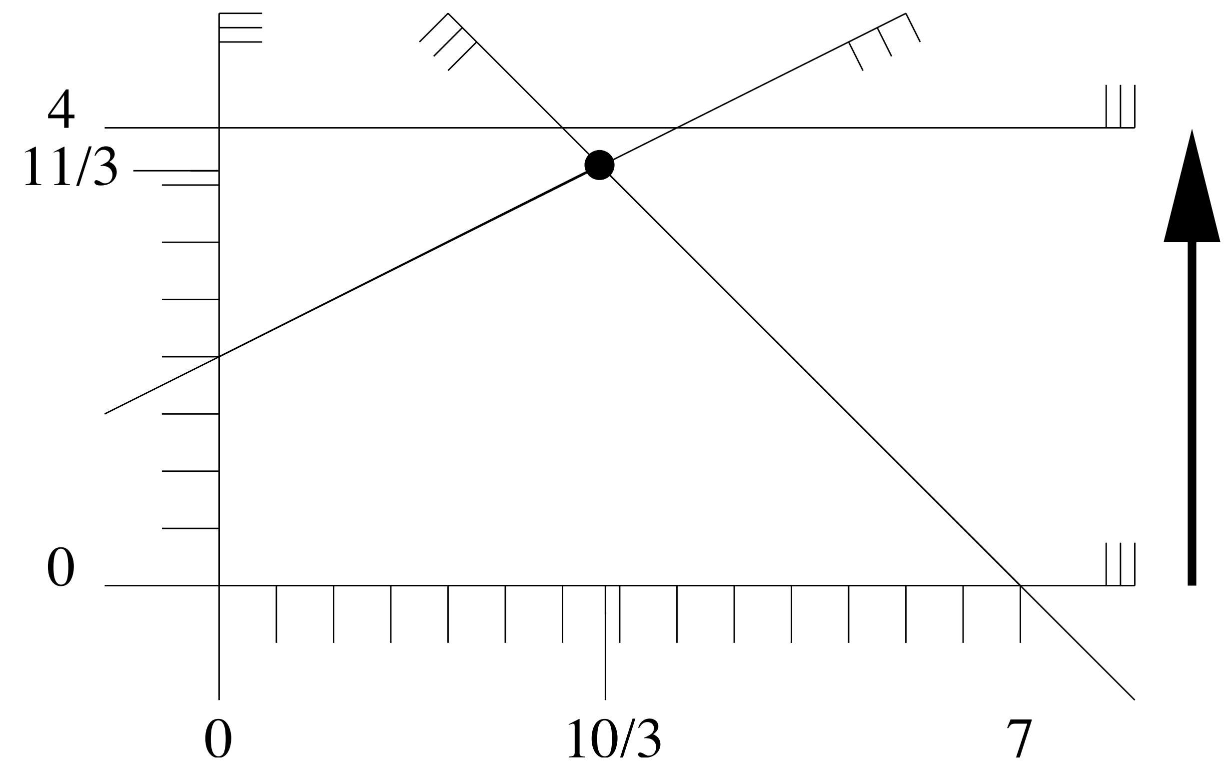

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$$\overset{c^T}{\begin{pmatrix}0\\-32\end{pmatrix}}\begin{pmatrix}x & y\end{pmatrix} + \overset{c_0}{\begin{pmatrix}64\end{pmatrix}}$$

$$\underset{A}{\begin{pmatrix}1 & 1\\-1 & 2\\-1 & 0\\0 & -1\\0 & 1\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix} \le \underset{b}{\begin{pmatrix}7\\4\\0\\0\\4\end{pmatrix}}$$

Linear Programming

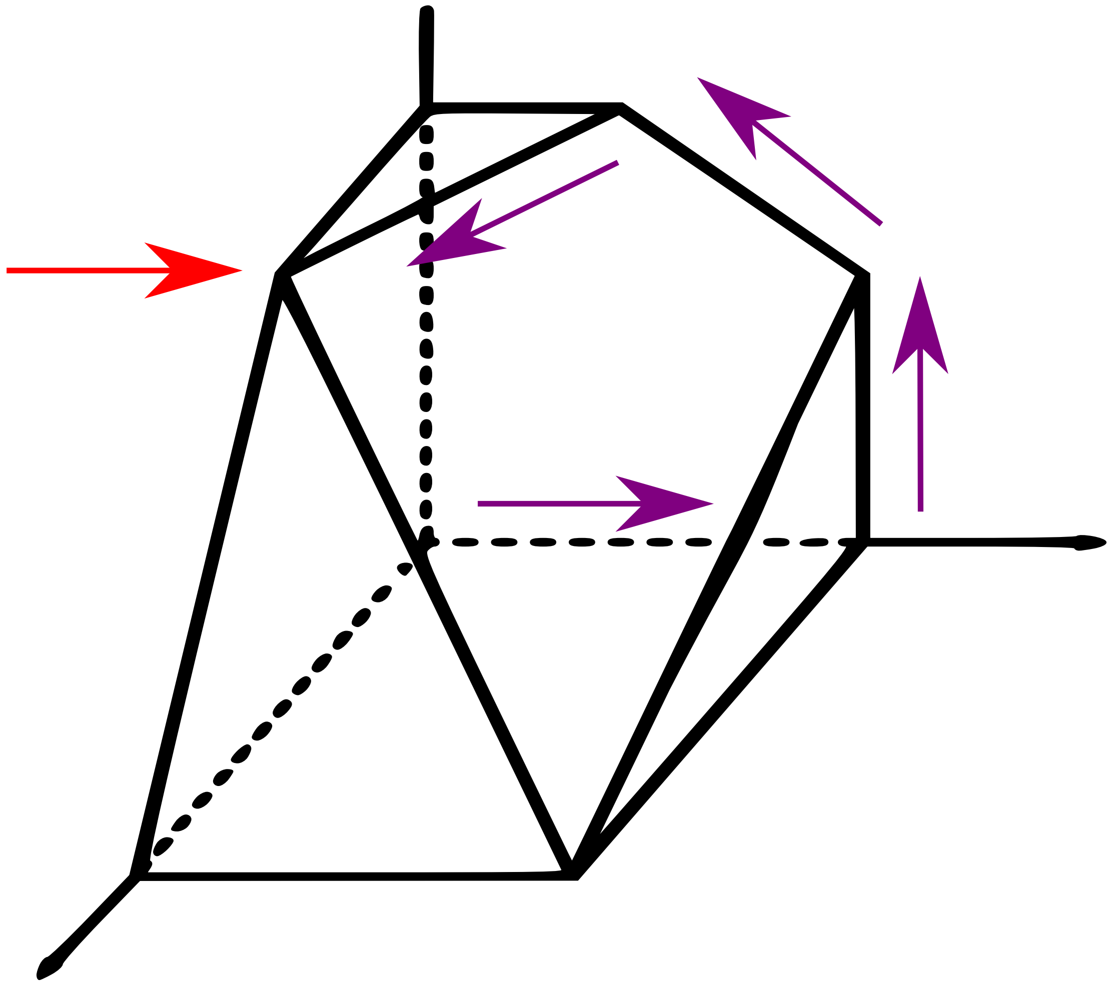
Mathematical formulation

- **General form of LP:**

$$\begin{array}{ll} \text{minimize} & c^T x + c_0 \\ \text{subject to} & Ax \gtreqless b \\ & l \leq x \leq u \end{array}$$

$$(x, c, l, u \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c_0 \in \mathbb{R})$$

Linear Programming



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- The constraints define a convex polyhedron

$$P = \{x | Ax \leq b\}$$

Linear Programming

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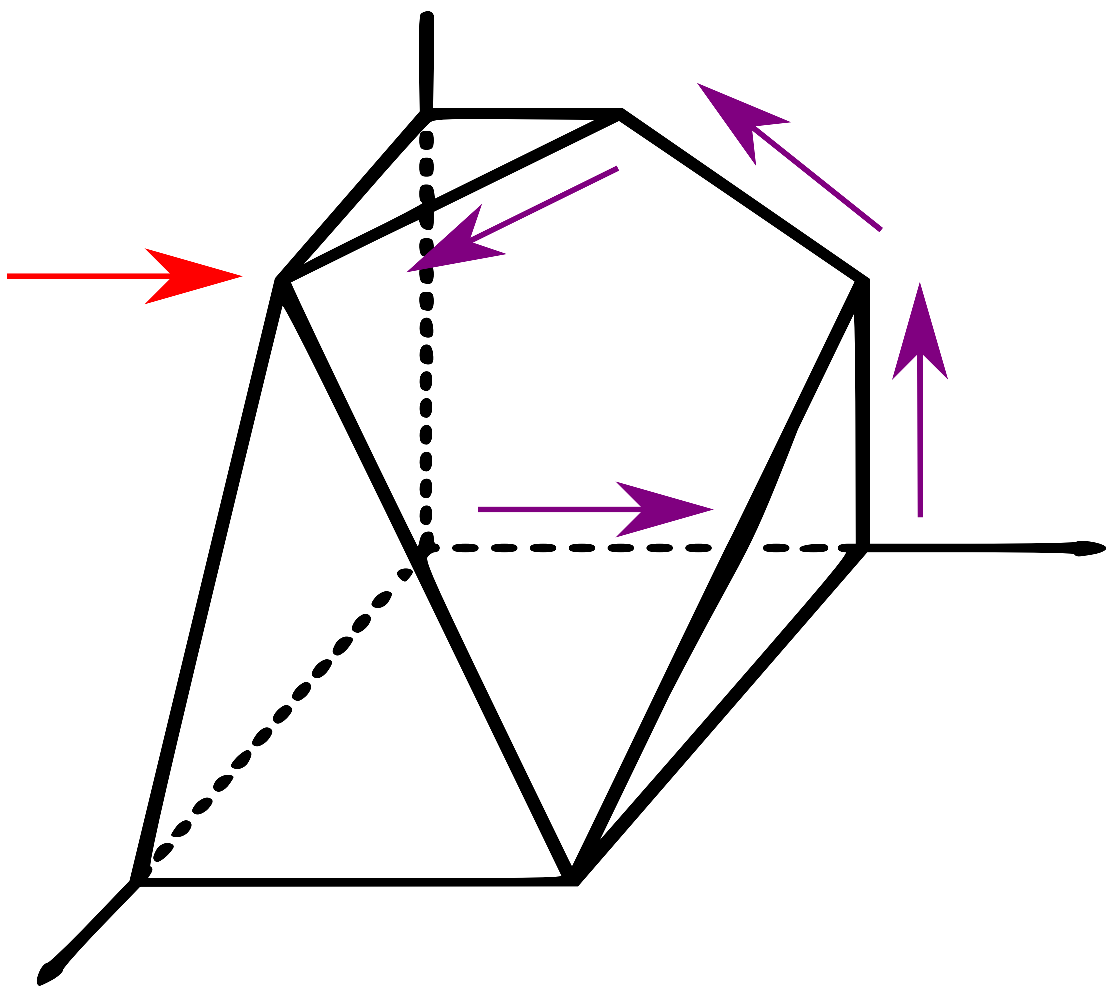

- The constraints define a convex polyhedron
$$P = \{x | Ax \le b\}$$
- Simplex-type algorithms walk along the edges of P to vertices with better objective value.
- To decide on the next edge we use a *pivot rule.*

Linear Programming

Complexity of Simplex-type algorithms with *n* variables and *m* constraints.

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- Exponential running time in the worst case! (exp. on $min\{m,n\}$)

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- O(*max*{*m*, *n*}) if *min*{*m*,*n*} is *O*(1) (hidden constant exponential in *min*{*m*,*n*})

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- O(*max{m, n}*) if *min{m,n}* is *O*(1) (hidden constant exponential in *min{m,n}*)

- CGAL: Gives exact solutions! Works well as long as *min{m,n}* ≤ 50 (or up to 100)

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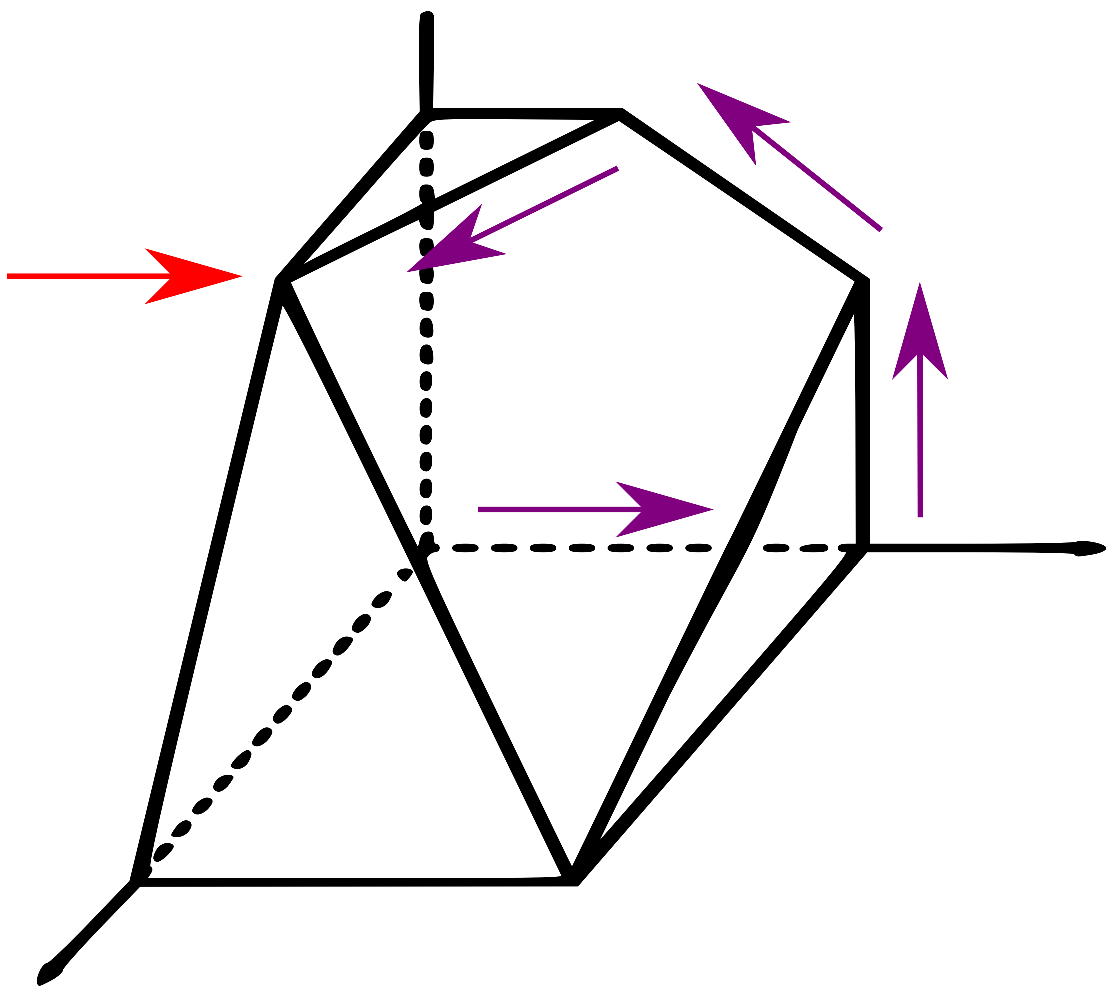
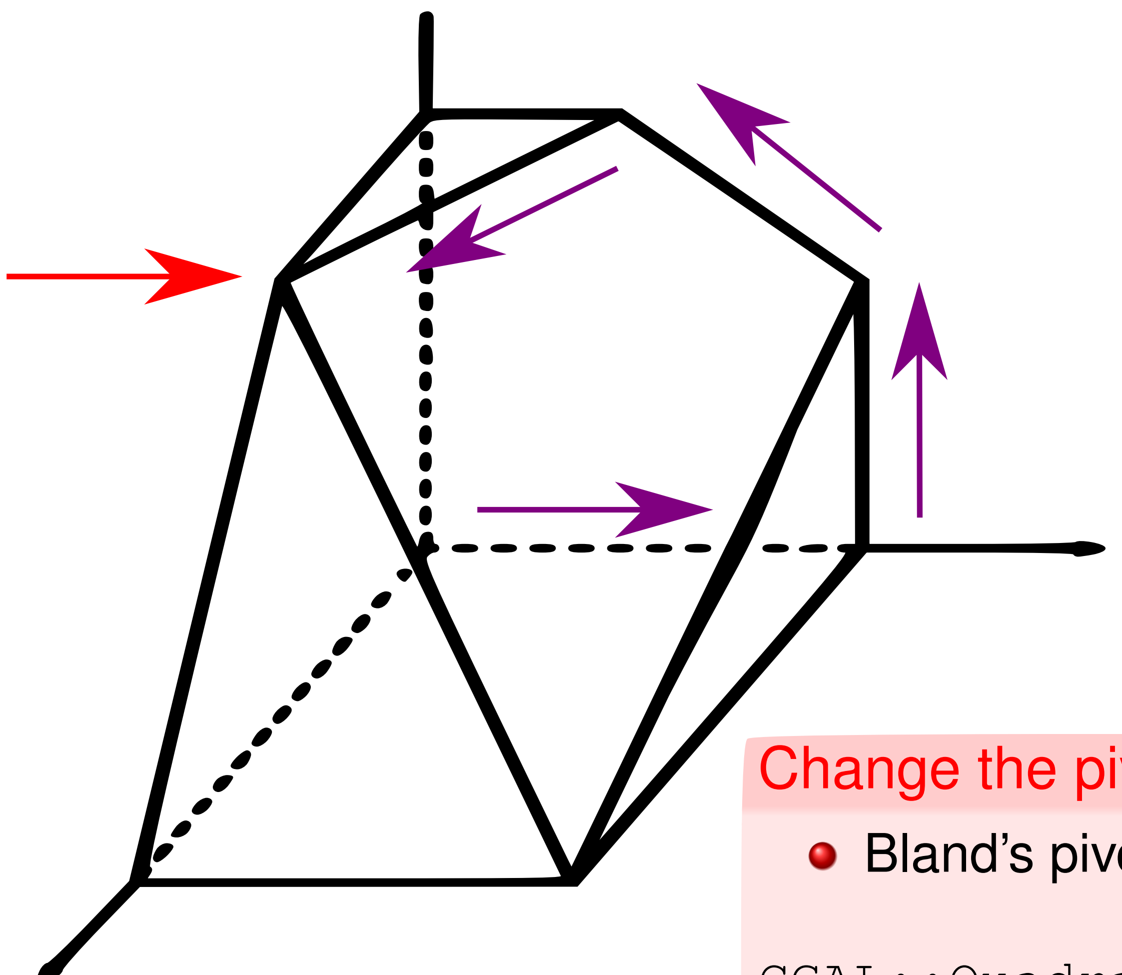


The default pivot rule strategy might fail:

- it is deterministic, so it picks the same improving edge every time.
- If the algorithm reaches a vertex for the second time, then cycles forever.

Linear Programming



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Change the pivot rule

- Bland's pivot rule avoids cycling (but it is slower. . .)

```
CGAL::Quadratic_program_options options;
options.set_pricing_strategy(CGAL::QP_BLAND);
Solution s = CGAL::SOLVER(program, ET(), options);
```

Linear Programming

- **General form of LP in CGAL:**

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≤, =, or ≥ (individually for each constraint)

variables

$$(x, c, l, u \in \mathbb{R}^n, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \quad c_0 \in \mathbb{R})$$

objective function

shift

lower and upper bounds

right-hand side

constraint matrix

Linear Programming ... in CGAL

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \\ Ax \le b & \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right. \end{array}$$

- **Preamble:** Choice of input type and exact internal number type

Gnu Multi-precision Library (GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpz.h>

// choose exact integral type
typedef CGAL::Gmpz ET;




// program and solution types
typedef CGAL::Quadratic_program<int> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
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input type

exact internal type

for linear *and* quadratic programs

GMP used internally

Linear Programming ... in CGAL

$$\text{minimize} \quad -32y + 64 = c^T x + c_0$$
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input type

Inside, the LP solver uses quotients of the data type specified here.

exact internal type

for linear *and* quadratic programs

GMP used internally

Linear Programming ... in CGAL

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- **Setup:** Enter the program data

```
int main() {
  // by default, we have a nonnegative LP with Ax <= b
  Program lp (CGAL::SMALLER, true, 0, false, 0);

  // now set the non-default entries
  const int X = 0;
  const int Y = 1;
  lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
  lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
  lp.set_u(Y, true, 4);                                   //      y <= 4
  lp.set_c(Y, -32);                                       // -32y
  lp.set_c0(64);                                          // +64
```

$l =$ (0,0,..., 0)

$u = (\infty,\infty,\ldots, \infty)$

last argument: value

variable index (0,1,...)

constraint index (0,1,...)

Linear Programming ... in CGAL

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  lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
  lp.set_u(Y, true, 4);                                   //       y <= 4
  lp.set_c(Y, -32);                                       // -32y
  lp.set_c0(64);                                          // +64
```

$l =$ $(0,0,\ldots,0)$ $u = (\infty,\infty,\ldots,\infty)$

variable index (0,1,...)

constraint index (0,1,...)

last argument: value

Linear Programming ... in CGAL

$$
\begin{array}{ll}
\text{minimize} & -32y + 64 = c^T x + c_0 \\
\text{subject to} & \\
\end{array}
$$

$$
Ax \le b \left\{ \begin{array}{rcl}
x + y & \le & 7 \\
\boxed{-x + 2y} & \boxed{\le} & \boxed{4} \\
x & \ge & 0 \\
y & \ge & 0 \\
y & \le & 4
\end{array} \right.
$$

- **Setup:** Enter the program data

```
int main() {
  // by default, we have a nonnegative LP with Ax <= b
  Program lp (CGAL::SMALLER, true, 0, false, 0);

  // now set the non-default entries
  const int X = 0;
  const int Y = 1;
  lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
  lp.set_a(X, 1, -1); lp.set_a(Y, 1, 2); lp.set_b(1, 4);  // -x + 2y <= 4
  lp.set_u(Y, true, 4);                                   //       y <= 4
  lp.set_c(Y, -32);                                       // -32y
  lp.set_c0(64);                                          // +64
```

$l =$ $(0,0,\ldots, 0)$ $u = (\infty,\infty,\ldots, \infty)$

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Linear Programming ... in CGAL

$$\begin{array}{ll} \text{minimize} & -32y + 64 = c^T x + c_0 \\ \text{subject to} & \\ Ax \le b & \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right. \end{array}$$

- **Setup:** Enter the program data

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int main() {
  // by default, we have a nonnegative LP with Ax <= b
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  lp.set_a(X, 0,  1); lp.set_a(Y, 0, 1); lp.set_b(0, 7);  //  x + y  <= 7
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  lp.set_u(Y, true, 4);                                   //       y <= 4
  lp.set_c(Y, -32);                                       // -32y
  lp.set_c0(64);                                          // +64
```

$l = (0,0,\ldots,0)$

$u = (\infty,\infty,\ldots,\infty)$

variable index (0,1,...)

constraint index (0,1,...)

last argument: value

Linear Programming
... in CGAL

minimize $\boxed{-32y + 64 = c^T x + c_0}$

subject to

$$Ax \le b \left\{ \begin{array}{rcl} x + y & \le & 7 \\ -x + 2y & \le & 4 \\ x & \ge & 0 \\ y & \ge & 0 \\ y & \le & 4 \end{array} \right.$$

- **Setup:** Enter the program data

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int main() {
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  lp.set_u(Y, true, 4);                                    //       y <= 4
  lp.set_c(Y, -32);                                        // -32y
  lp.set_c0(64);                                           // +64
```

$l =$ $(0,0,\ldots, 0)$

$u = (\infty,\infty,\ldots, \infty)$

variable index (0,1,...)

constraint index (0,1,...)

last argument: value

Linear Programming ... in CGAL

- **Solve:** Call the linear programming solver and output solution

```
 // solve the program, using ET as the exact type
  Solution s = CGAL::solve_linear_program(lp, ET());
  assert (s.solves_linear_program(lp));

  // output solution
  std::cout << s;
  return 0;
}
```

independent verification

Linear Programming
… in CGAL

- **Solve:** Call the linear programming solver and output solution

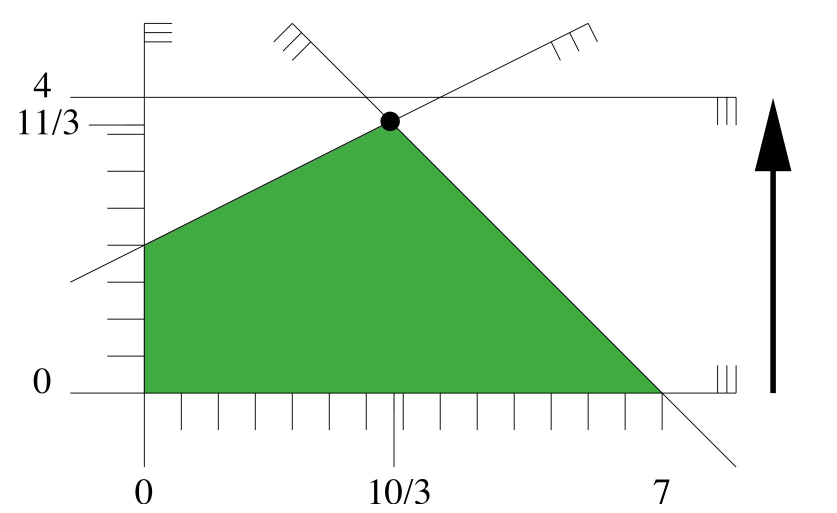
```
  // solve the program, using ET as the exact type
  Solution s = CGAL::solve_linear_program(lp, ET());
  assert (s.solves_linear_program(lp));

  // output solution
  std::cout << s;
  return 0;
}
```

independent verification

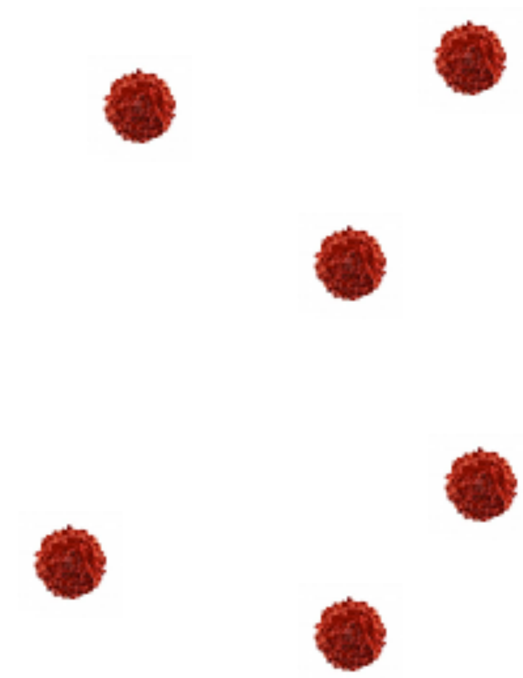
- **Output:**

```
status:            OPTIMAL
objective value:   -160/3
variable values:
  0:  10/3
  1:  11/3
```



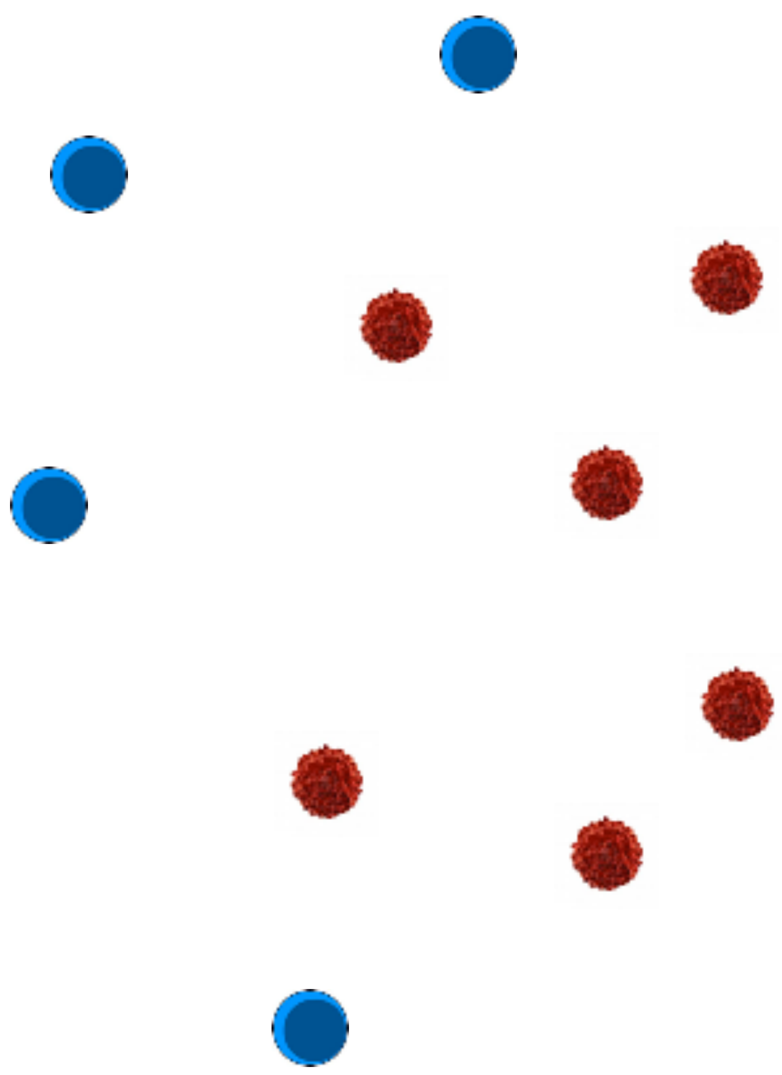
Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red)



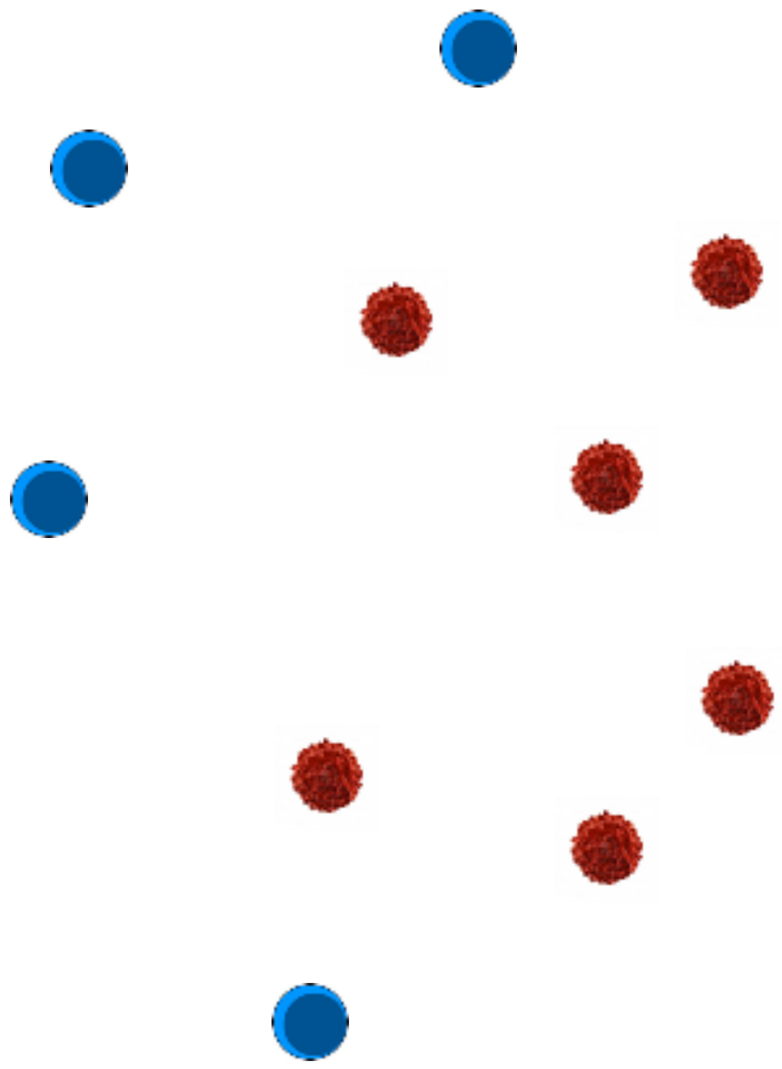
Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)



Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)
- **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.

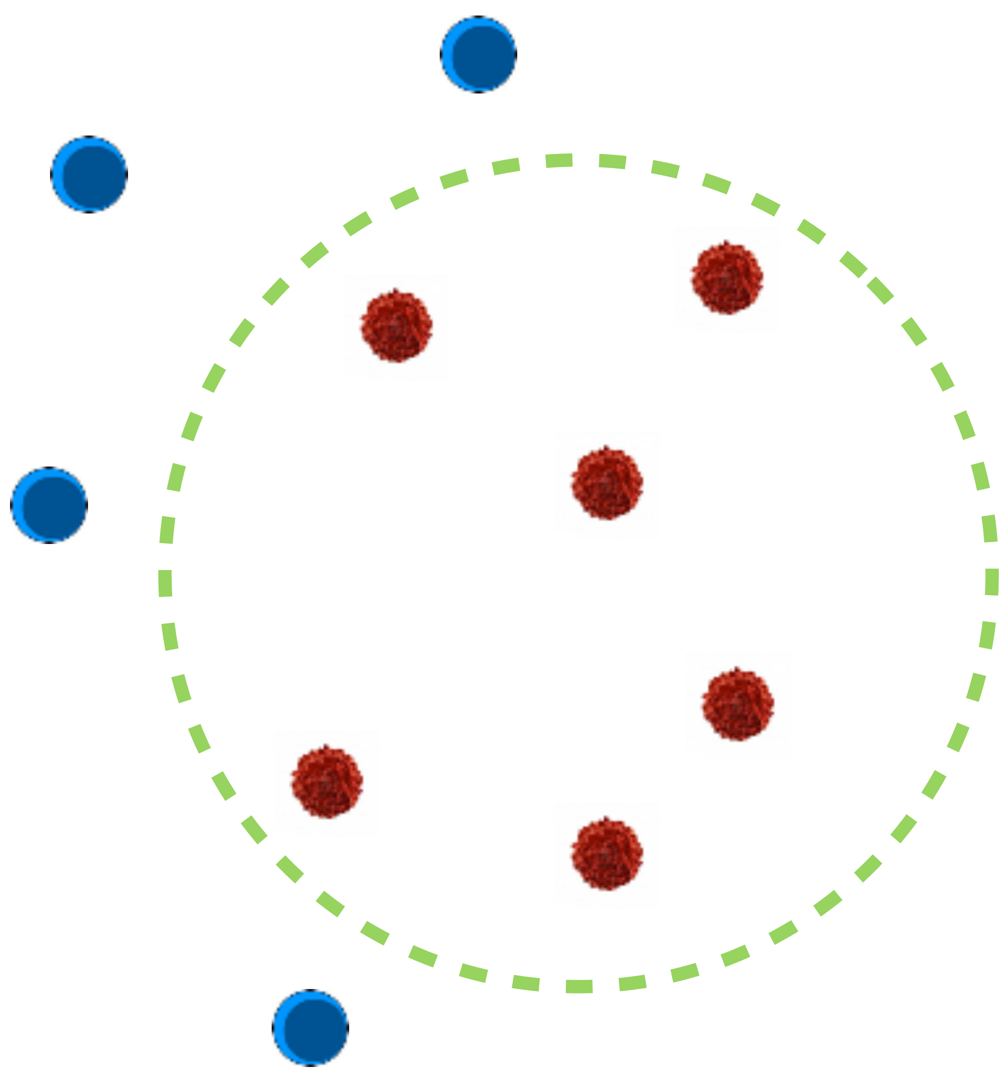


Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)

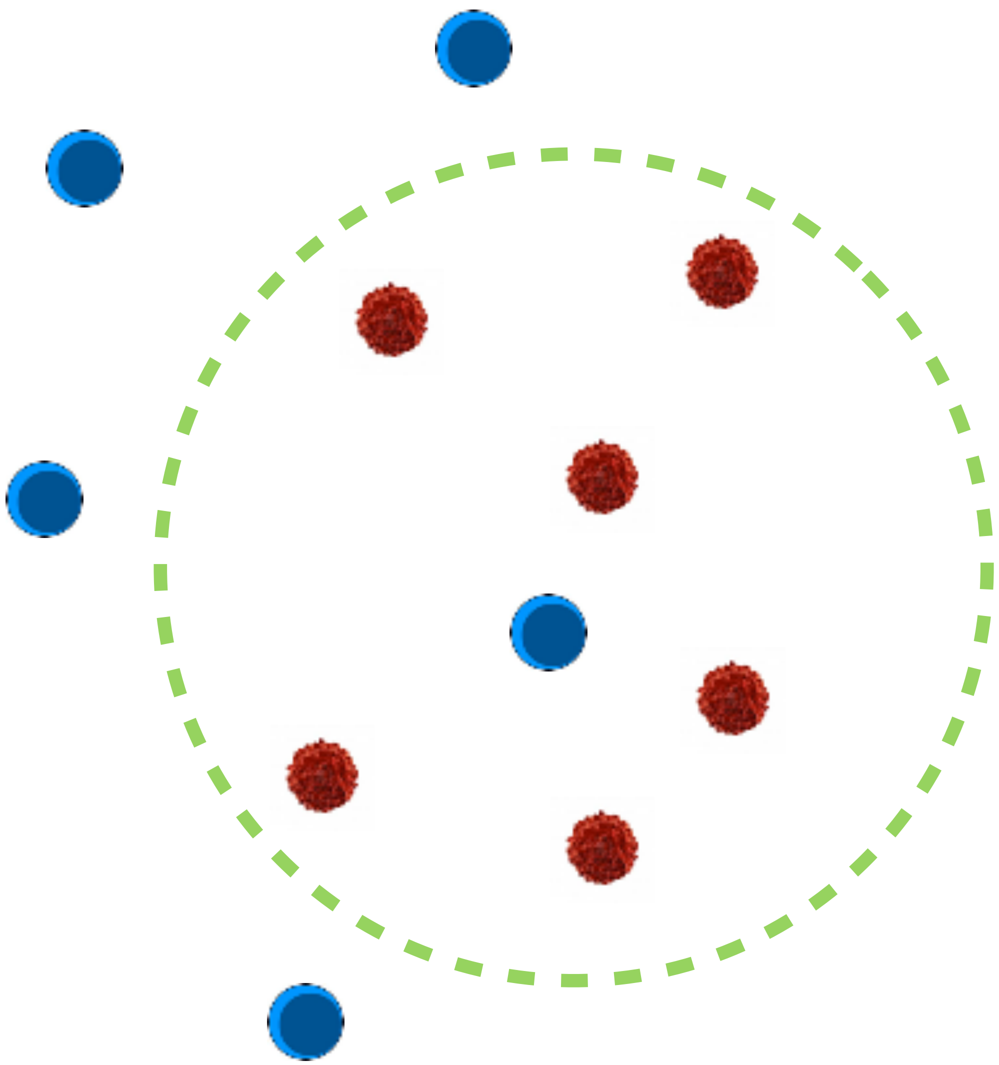
- **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.

- This may be possible…



Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)
- **Wanted:** center and radius of exposure so that all cancer cells are killed and all healthy cells are unaffected.
- This may be possible... or not.



Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)

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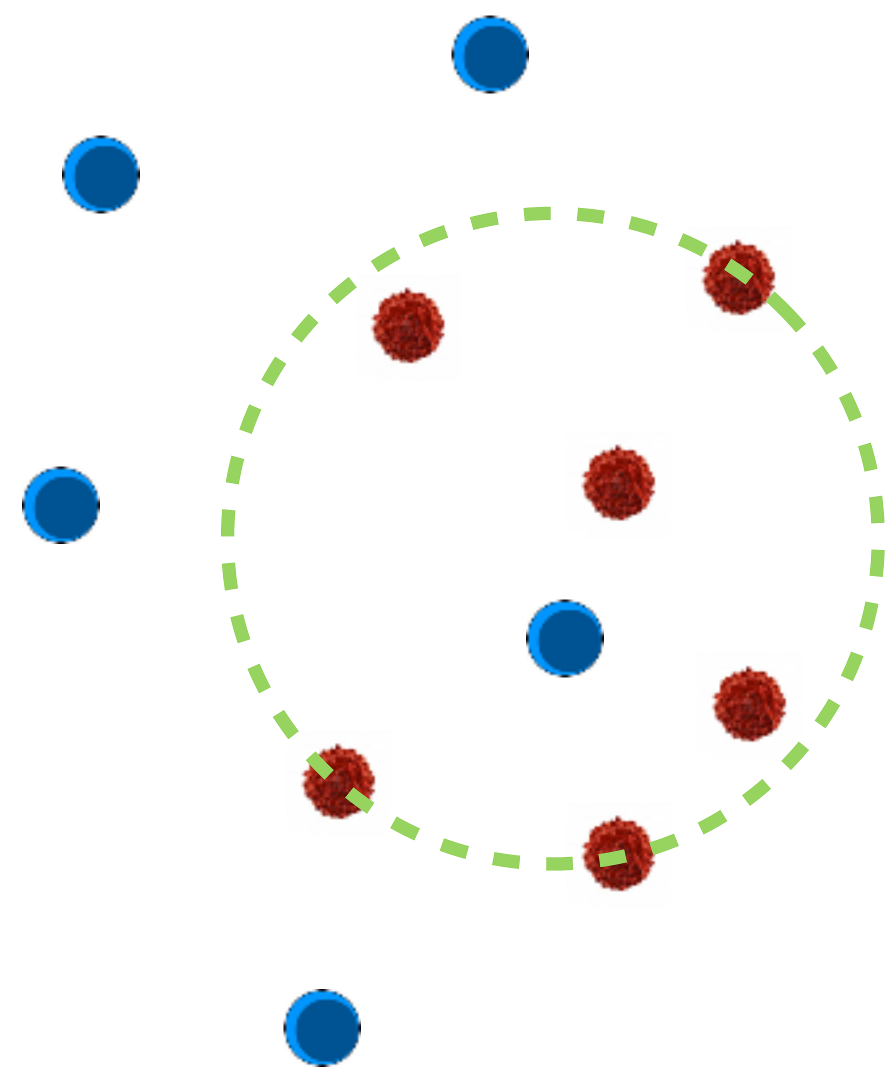

- Minimum Enclosing circle is not enough to solve the problem.

Linear Programming Application I: Cancer Therapy

- **Given:** locations of cancer cells (red) and healthy cells (blue)
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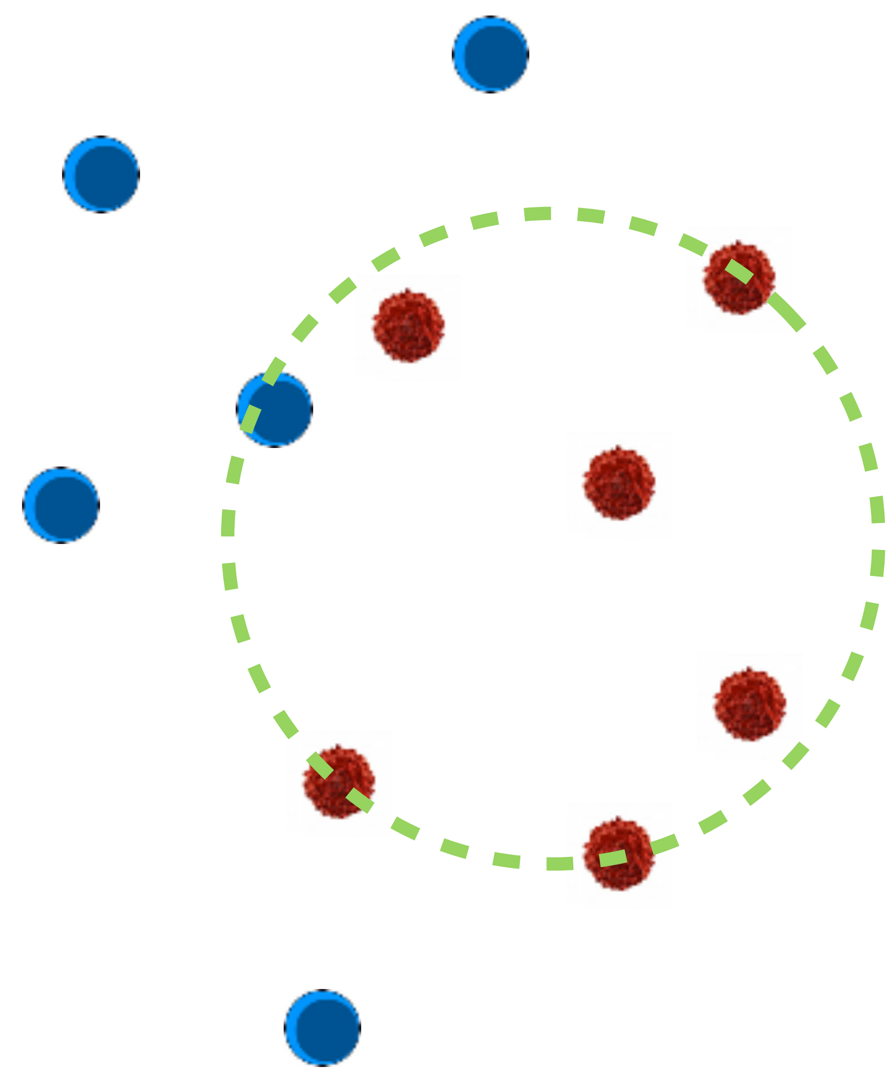

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Linear Programming Application I: Cancer Therapy

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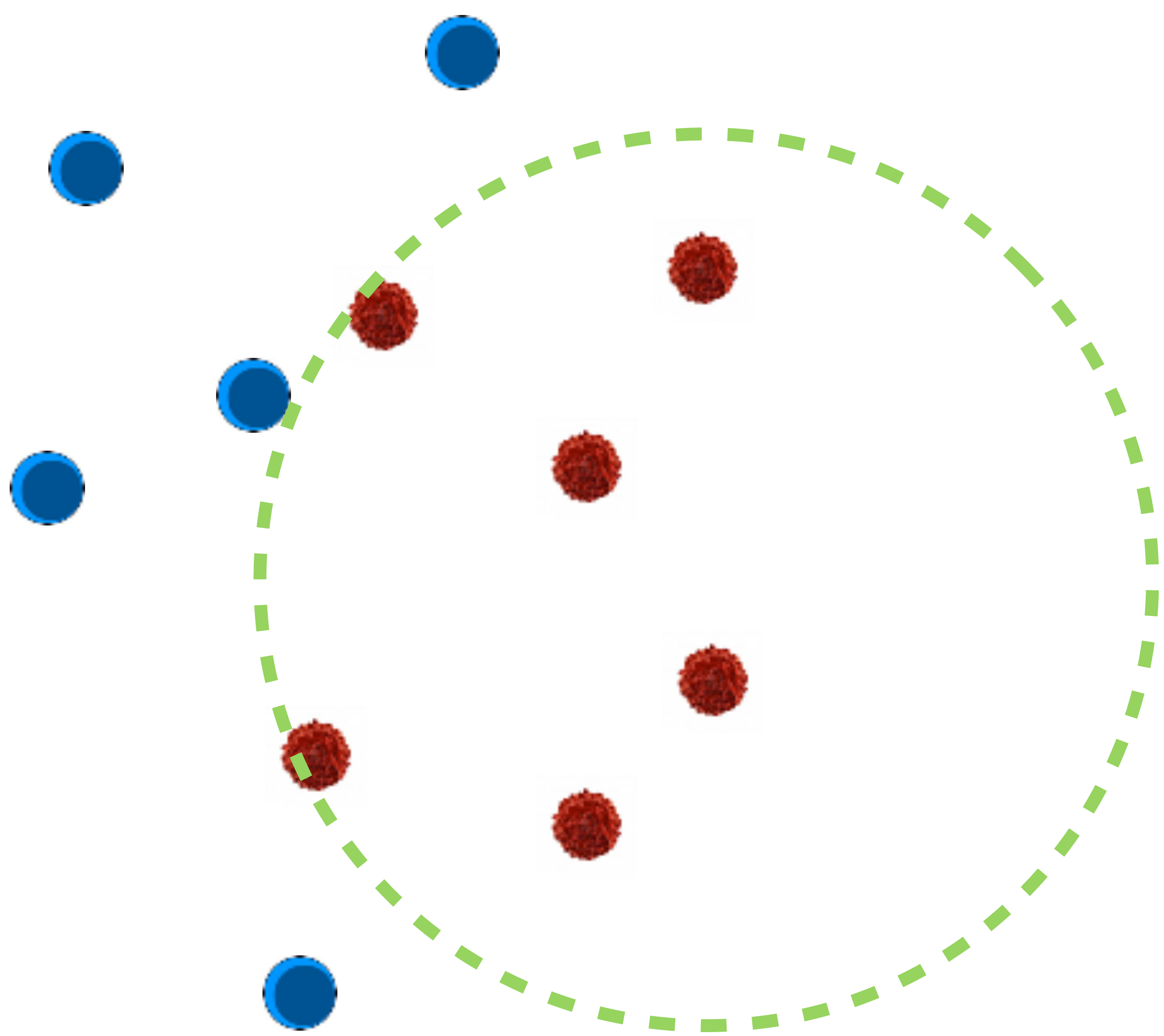
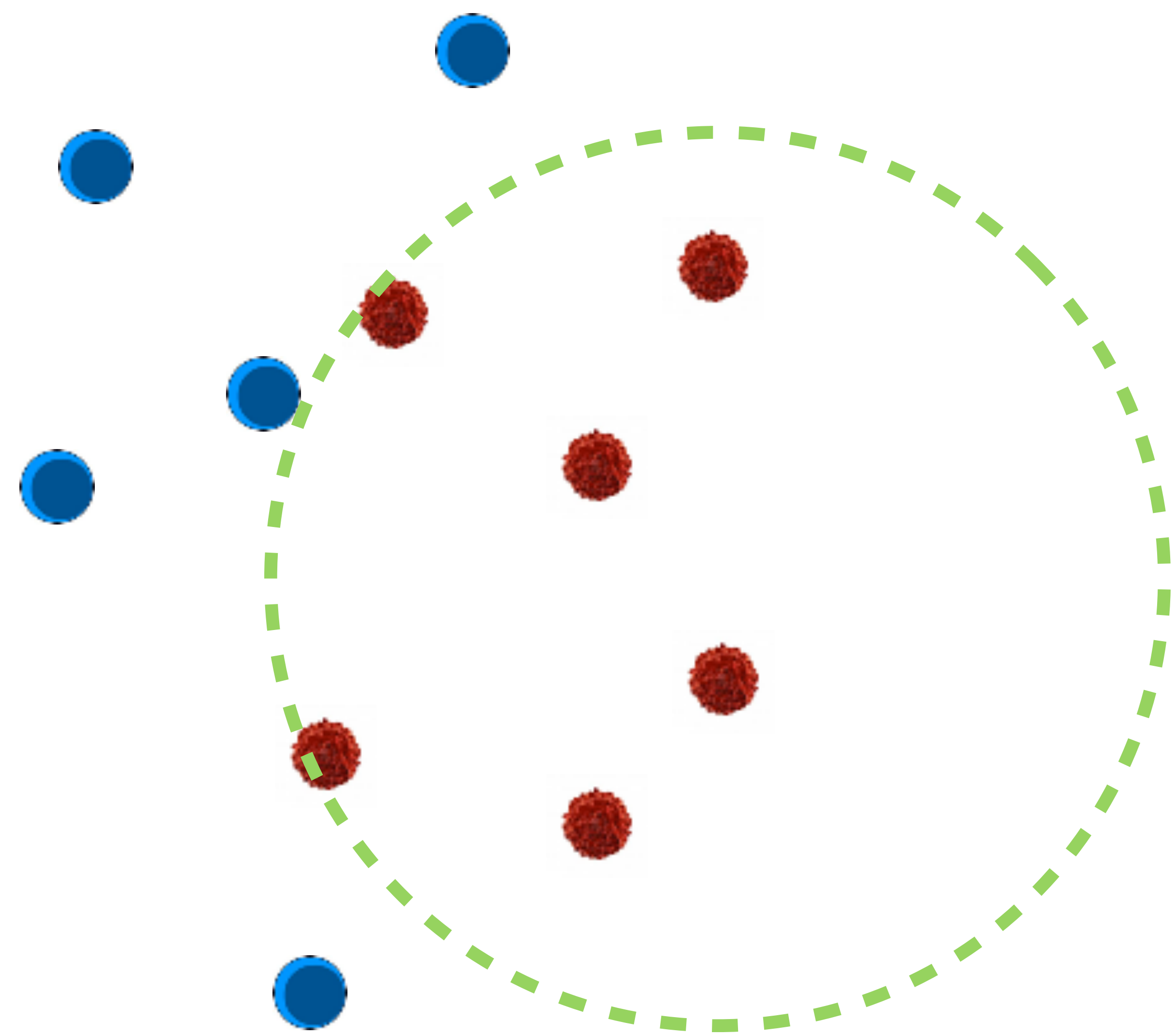


- Minimum Enclosing circle is not enough to solve the problem.

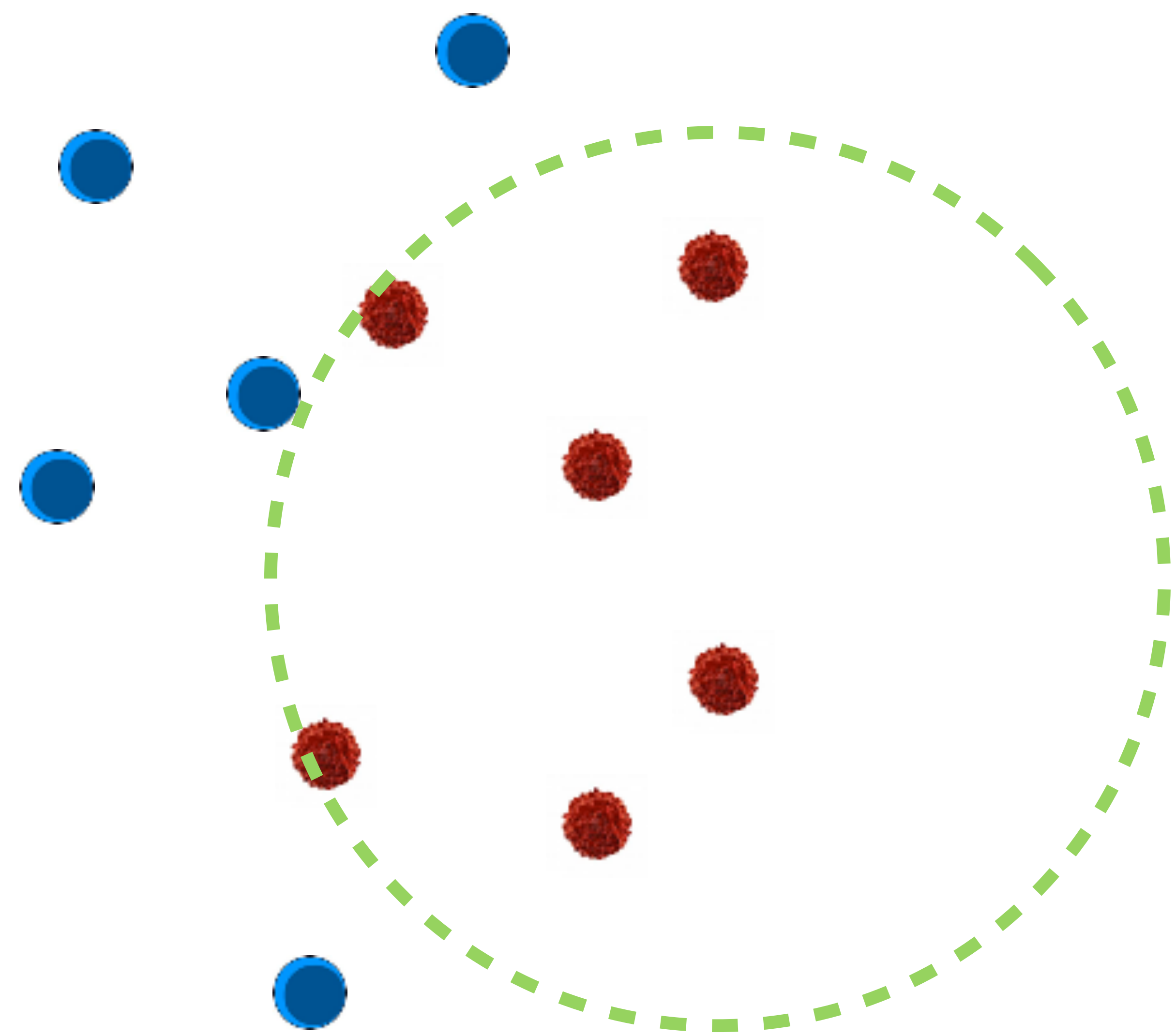
Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets R and B in the plane, does there exist a disk that contains R and is disjoint from B?



Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets *R* and *B* in the plane, does there exist a disk that contains *R* and is disjoint from *B*?
- **Variables:** Parameters used to define a circle.

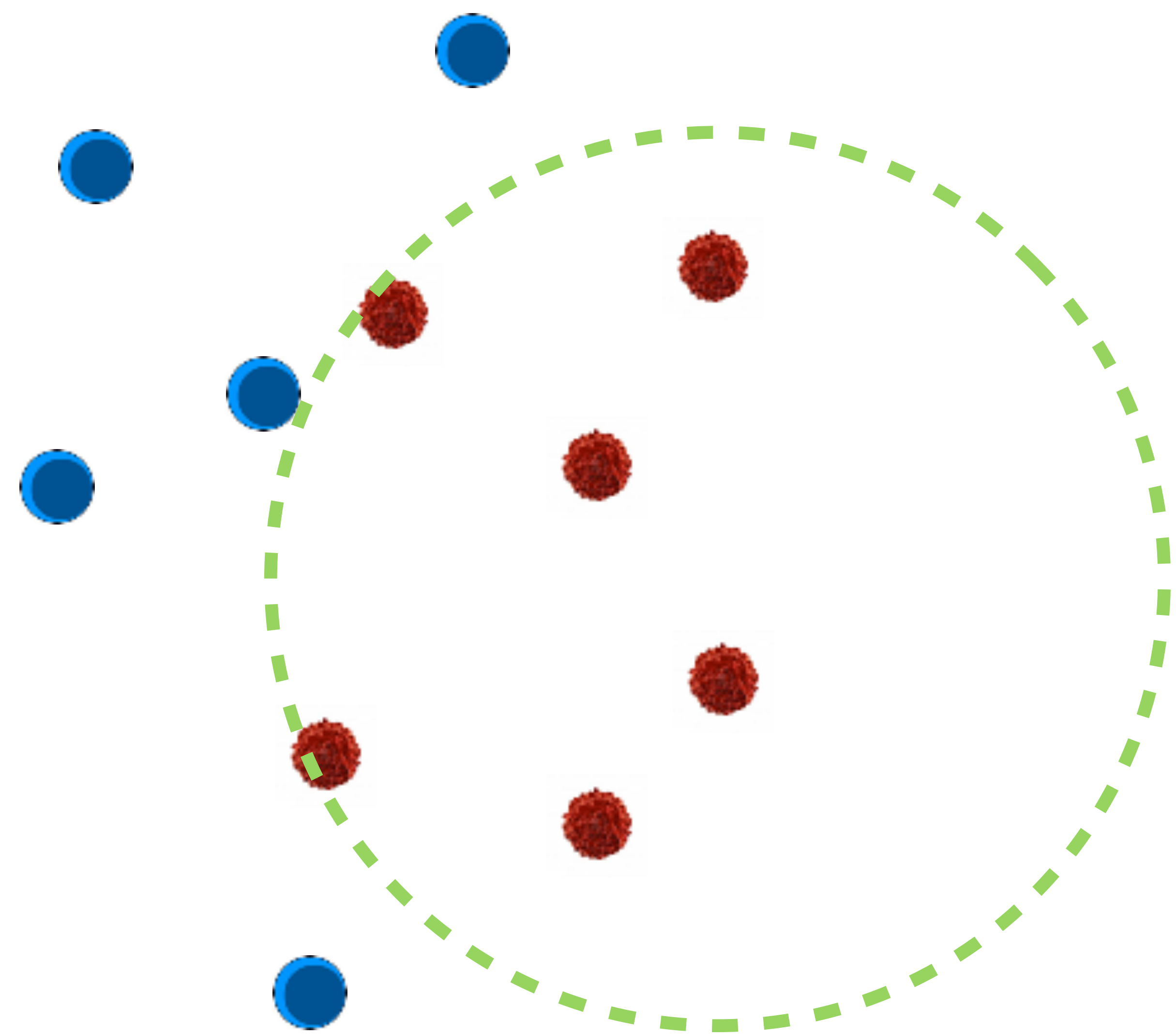


Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets R and B in the plane, does there exist a disk that contains R and is disjoint from B?

- **Variables:** Parameters used to define a circle.

- **Constraints:** Blue points should be outside, red points inside.



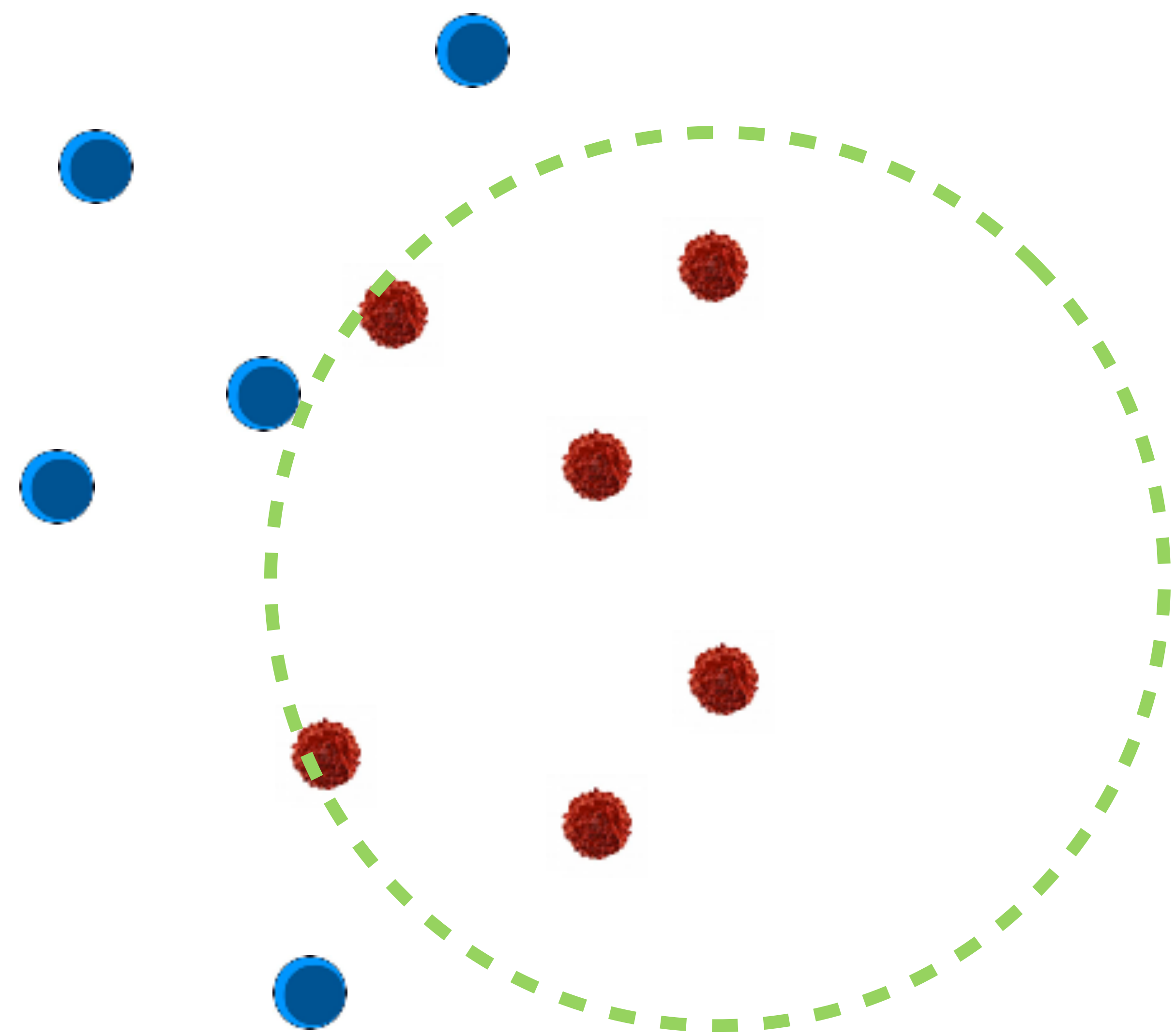
Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets R and B in the plane, does there exist a disk that contains R and is disjoint from B?

- **Variables:** Parameters used to define a circle.

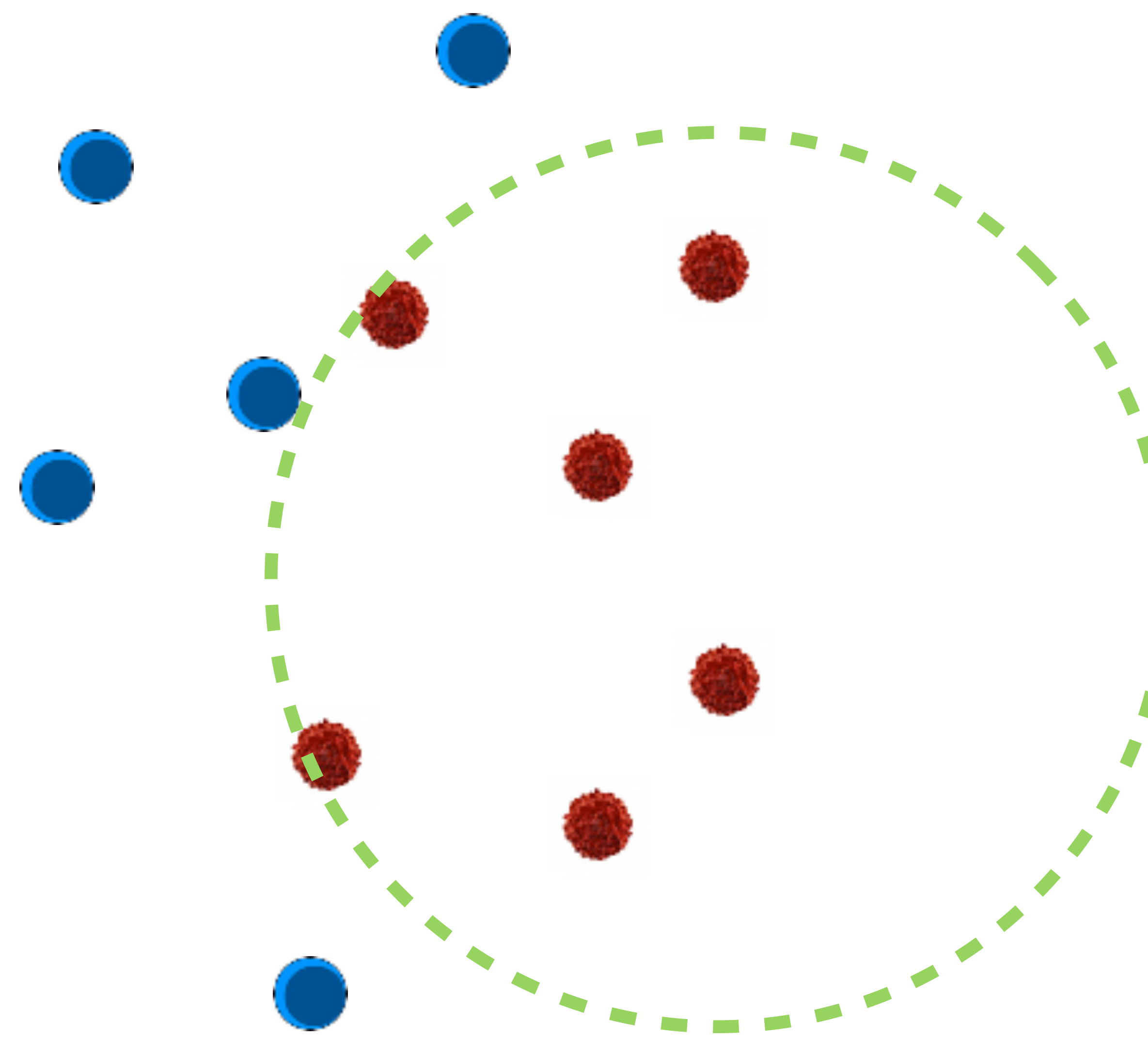
- **Constraints:** Blue points should be outside, red points inside.

- **Problem:** Constraints are not linear…

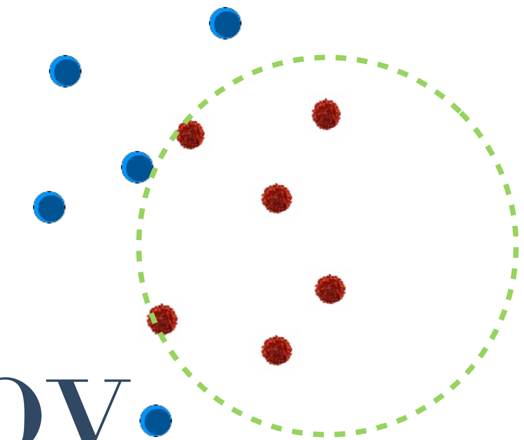


Linear Programming Application I: Cancer Therapy

- **The geometric problem:** Given two finite sets R and B in the plane, does there exist a disk that contains R and is disjoint from B?
- **Variables:** Parameters used to define a circle.
- **Constraints:** Blue points should be outside, red points inside.
- **Problem:** Constraints are not linear… or are they?

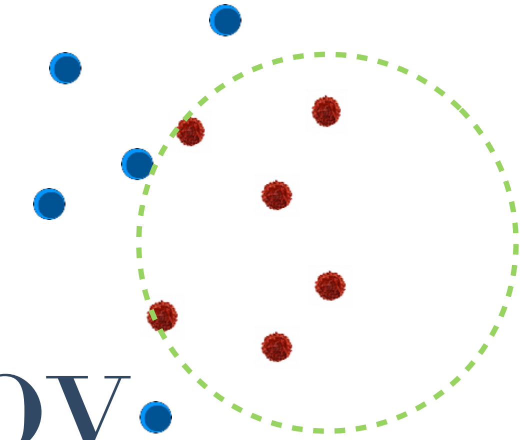


Linear Programming Application I: Cancer Therapy



- **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

Linear Programming Application I: Cancer Therapy



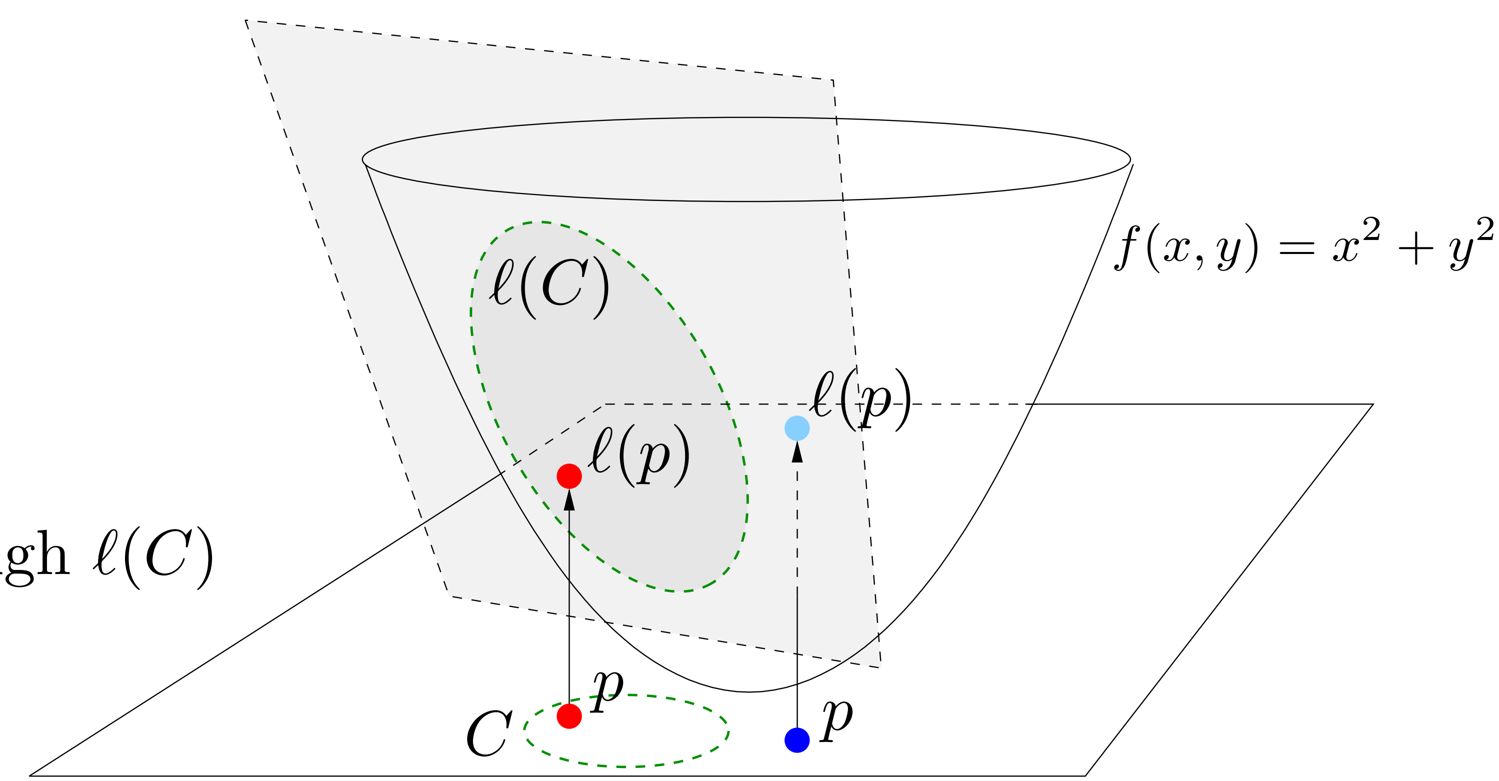
- **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

- Apply *lifting map* $\ell : (x, y) \mapsto (x, y, x^2 + y^2)$

$$p \left\{ \begin{array}{l} \text{inside} \\ \text{on} \\ \text{outside} \end{array} \right\} C$$

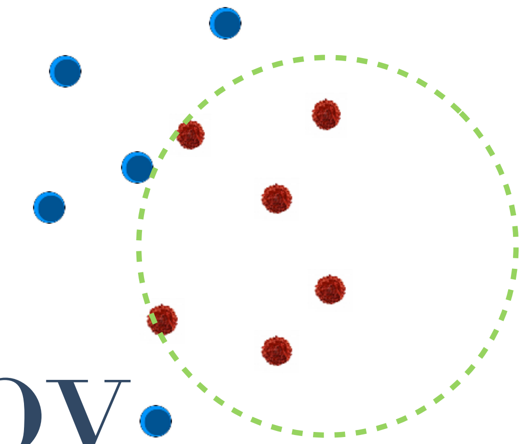
$$\Updownarrow$$

$$\ell(p) \left\{ \begin{array}{l} \text{below} \\ \text{on} \\ \text{above} \end{array} \right\} \text{the plane through } \ell(C)$$



[Ref: Section 2.3]

Linear Programming
Application I: Cancer Therapy



- **Problem:** We want to represent the property of being inside of a circle as a linear constraint.

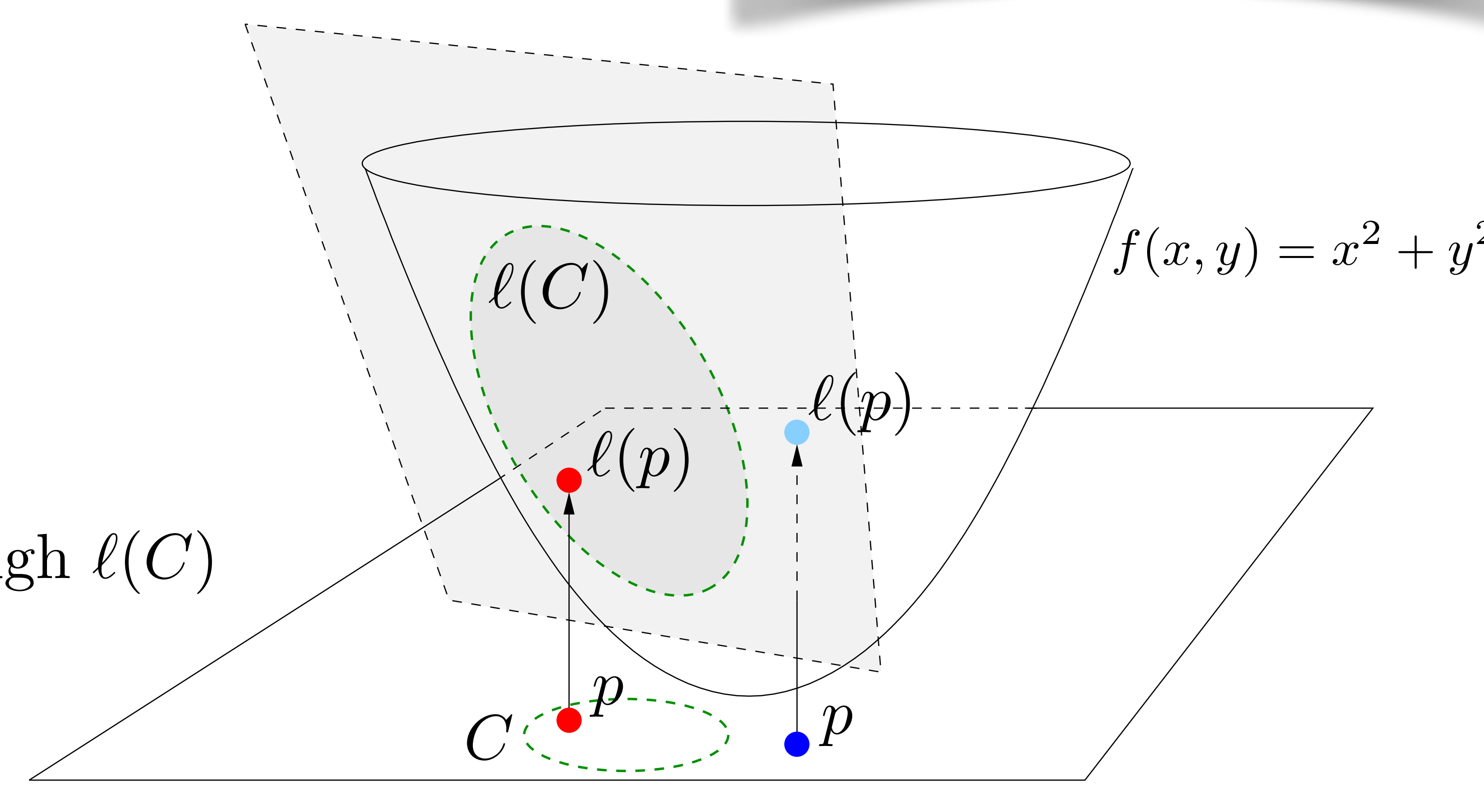
- Apply *lifting map* $\ell : (x, y) \mapsto (x, y, x^2 + y^2)$

Every circle has a corresponding plane

$$p \begin{Bmatrix} \text{inside} \\ \text{on} \\ \text{outside} \end{Bmatrix} C$$

$$\Updownarrow$$

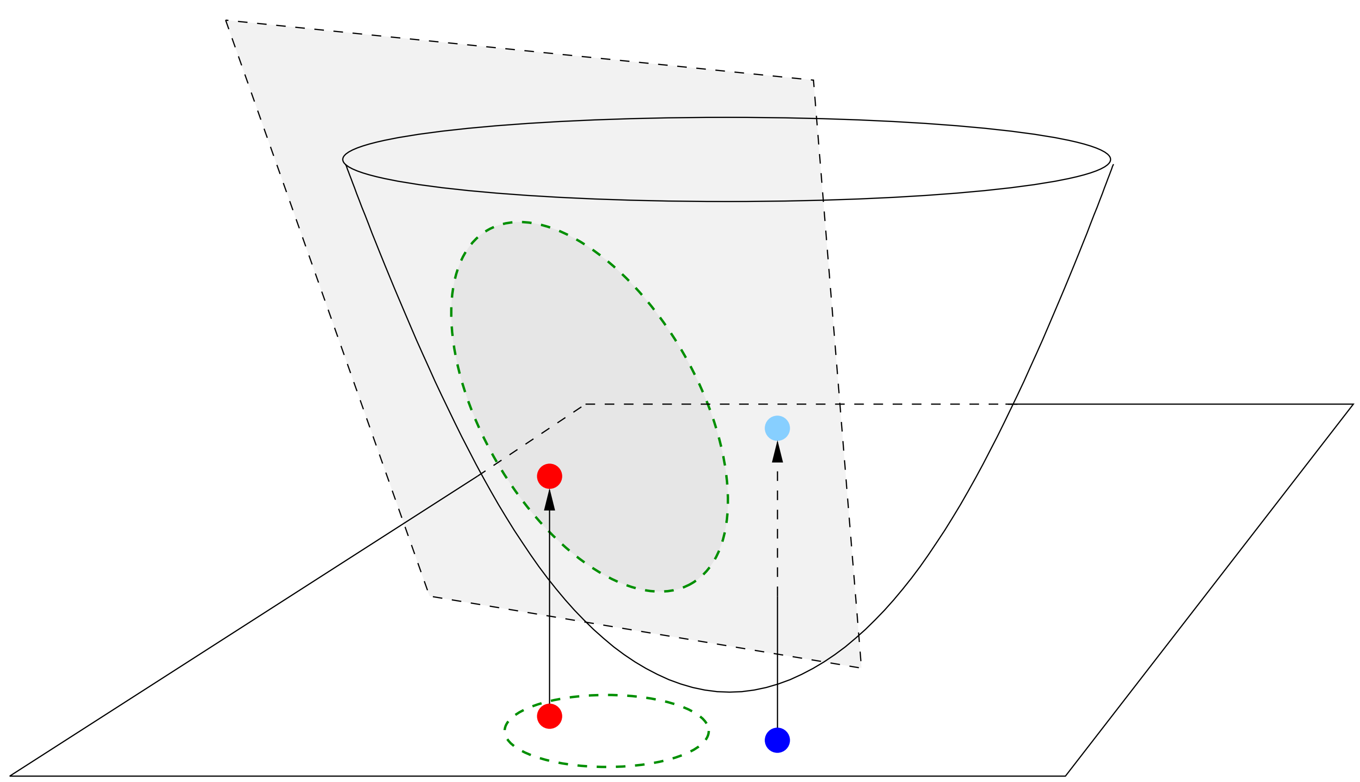
$$\ell(p) \begin{Bmatrix} \text{below} \\ \text{on} \\ \text{above} \end{Bmatrix} \text{the plane through } \ell(C)$$



[Ref: Section 2.3]

Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?



Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?
- This can be solved with linear programming!

Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

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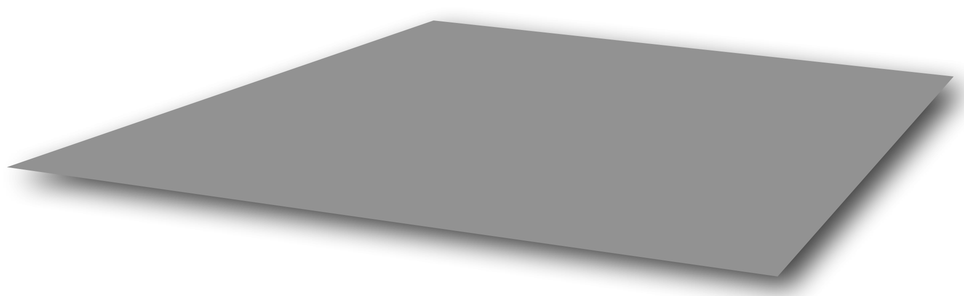
plane: $z = \alpha x + \beta y + \gamma$



Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

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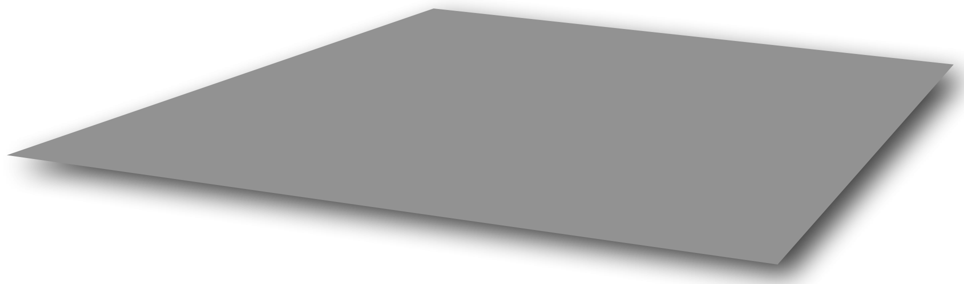
- Find α, β, γ, δ ($\delta > 0$) such that...

plane: $z = \alpha x + \beta y + \gamma$



Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?
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- Find α, β, γ, $\delta > 0$ such that...

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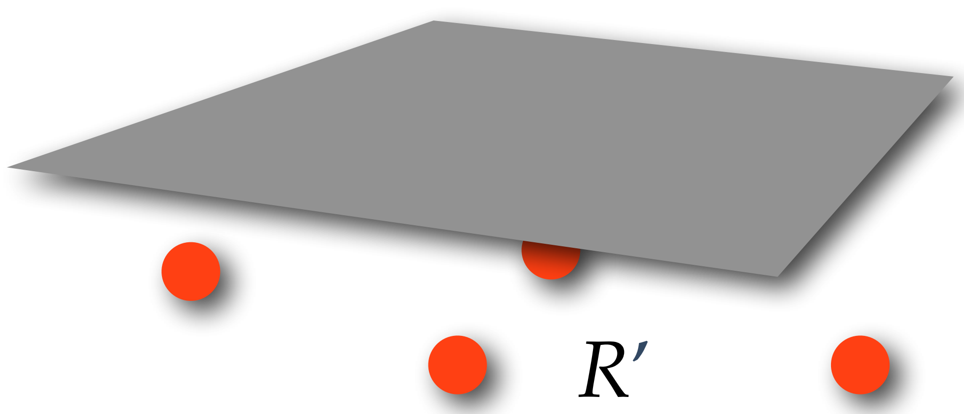

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

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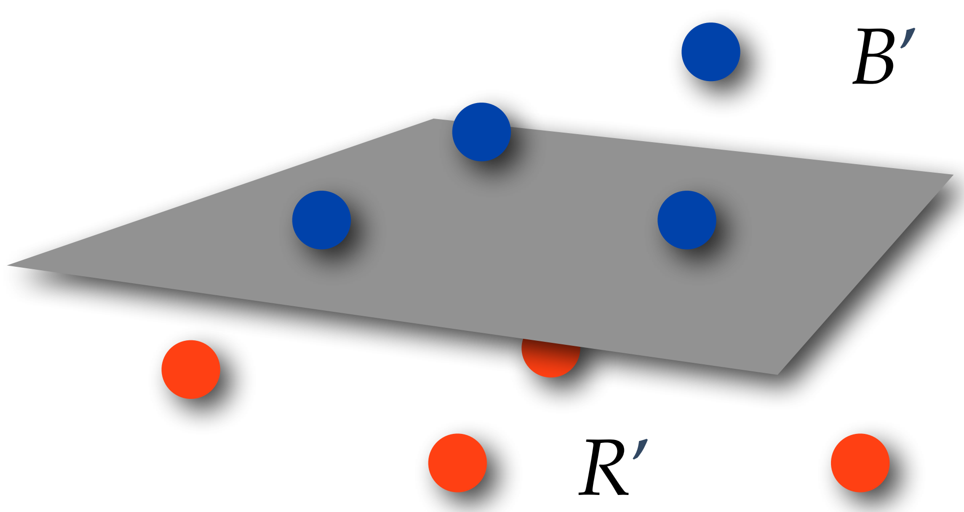

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Linear Programming Application I: Cancer Therapy

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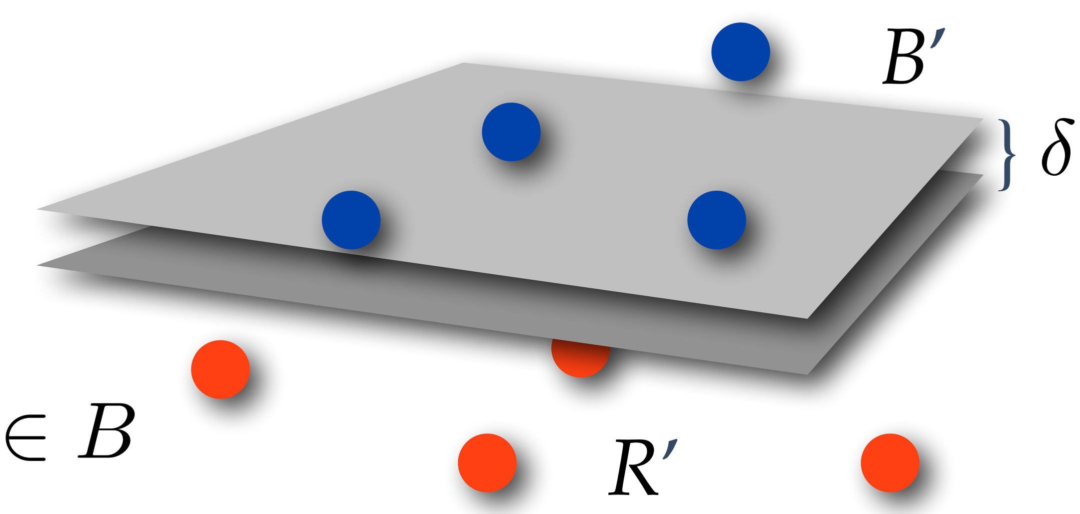

$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

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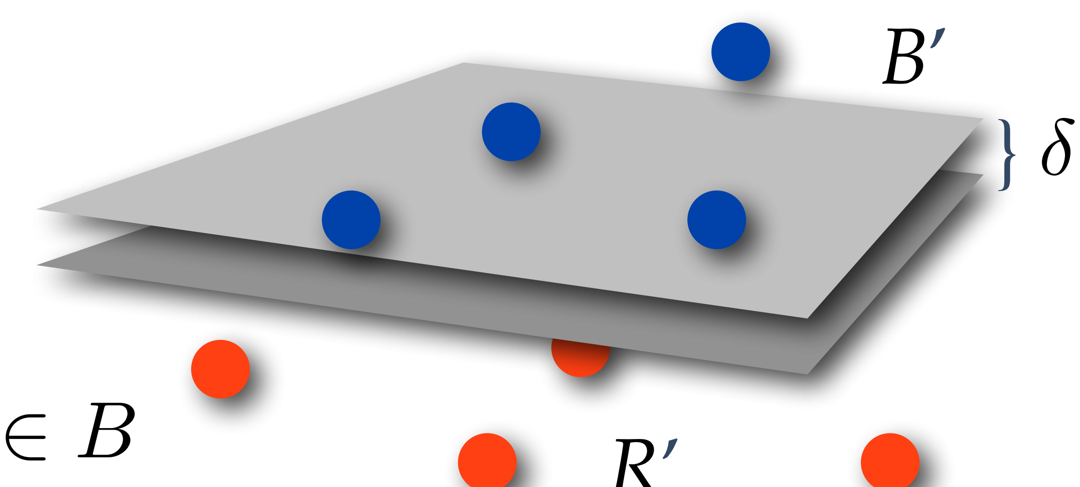
plane: $z = \alpha x + \beta y + \gamma$



$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

These are linear constraints!

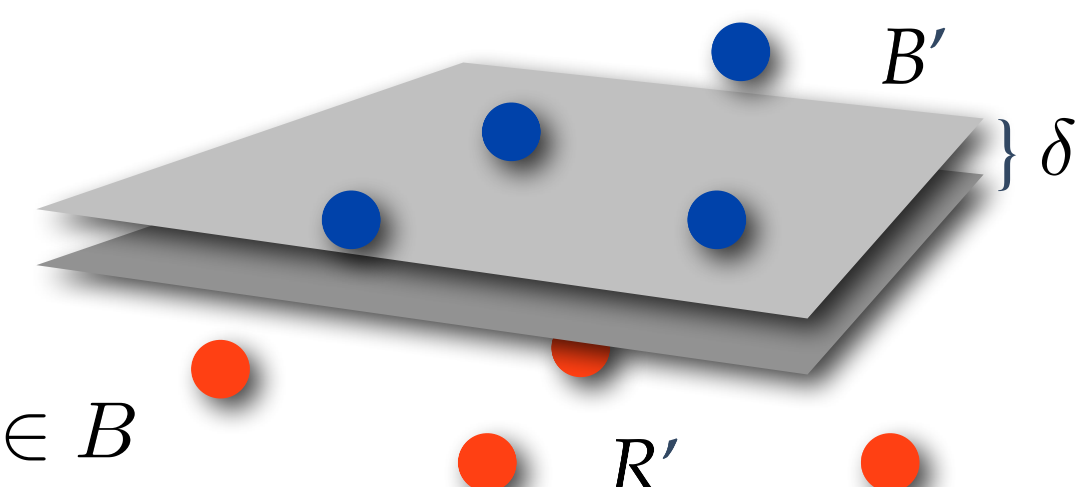
Linear Programming
Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

- This can be solved with linear programming!

- Find α, β, γ, $\delta > 0$ such that...

plane: $z = \alpha x + \beta y + \gamma$



$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

$(3, 6) \in R$

These are linear constraints!

Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?
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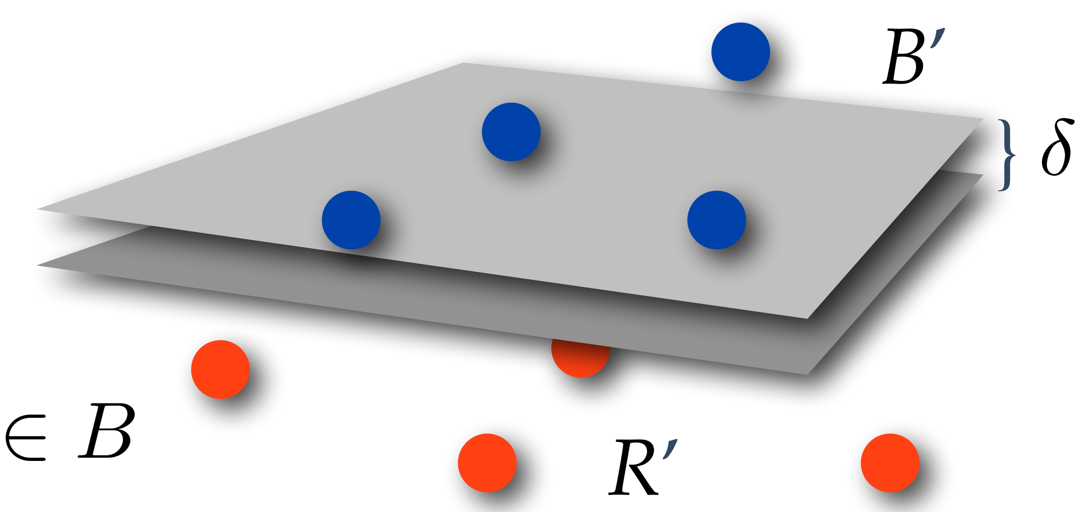
- Find α, β, γ, $\delta > 0$ such that...

plane: $z = \alpha x + \beta y + \gamma$



$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

$$(3, 6) \in R \longrightarrow 3^2 + 6^2 \leq 3\alpha + 6\beta + \gamma$$

These are linear constraints!

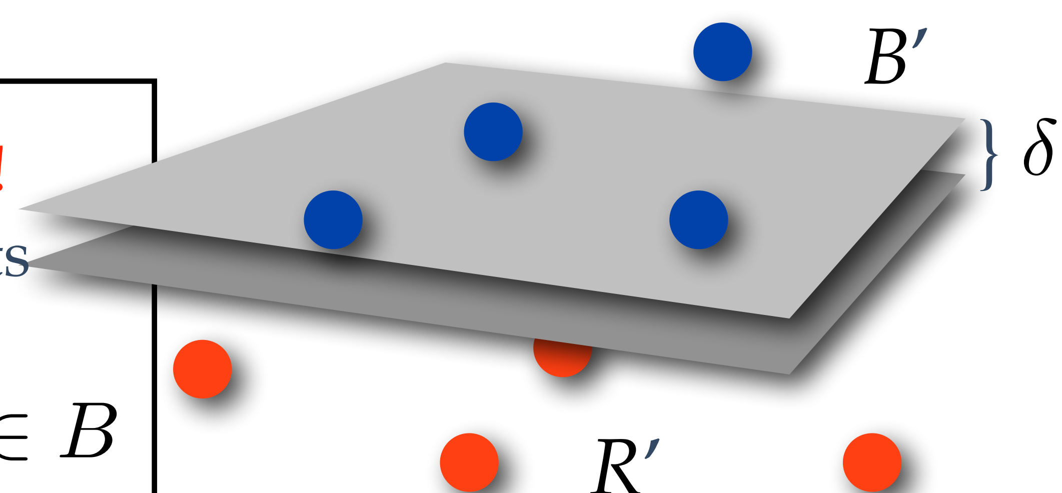
Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?

- This can be solved with linear programming!

- Find α, β, γ, $\delta > 0$ such that...

plane: $z = \alpha x + \beta y + \gamma$



maximize δ

4 variables!!

subject to

$|B| + |R|$ constraints

$$x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B$$

$$x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R$$

Linear Programming Application I: Cancer Therapy

- **The geometric problem (lifted space):** Given the lifted sets R' and B' in space, is there a plane that has R' below/on it and B' above?
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- Find α, β, γ, δ > 0 such that...

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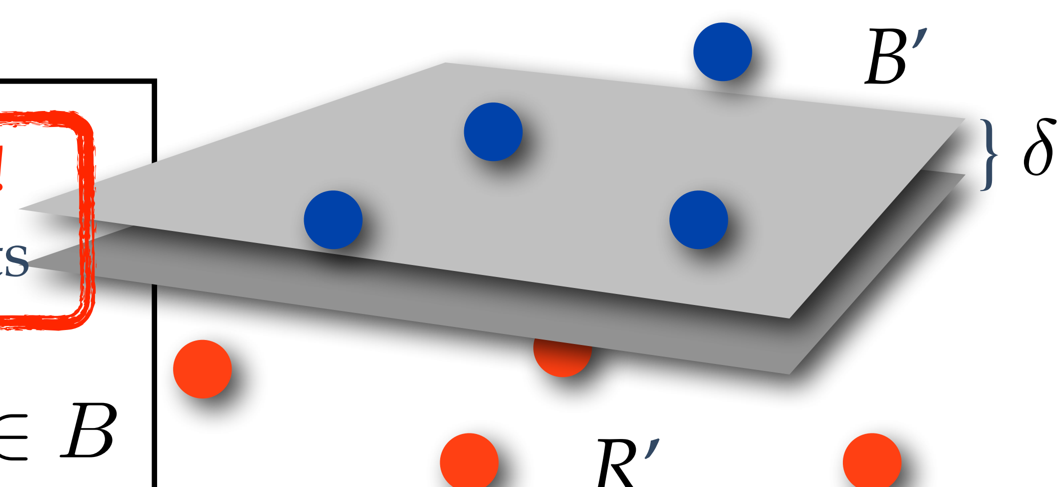


$$
\begin{aligned}
&\text{maximize } \delta \\
&\text{subject to} \\
&\quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\
&\quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R
\end{aligned}
$$

4 variables!!

|B|+|R| constraints

Linear Programming Application I: Cancer Therapy

- **Fact:** Exposure is possible if and only if the following linear program has positive value.

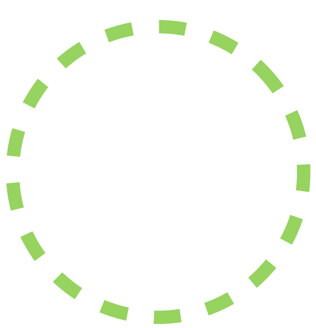
$$\boxed{\begin{array}{l} \text{maximize } \delta \\ \text{subject to} \\ \quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\ \quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \end{array}}$$

Linear Programming Application I: Cancer Therapy

- **Fact:** Exposure is possible if and only if the following linear program has positive value.

$$
\boxed{
\begin{aligned}
&\text{maximize } \delta \\
&\text{subject to} \\
&\quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\
&\quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R
\end{aligned}
}
$$

- **Reconstructing the exposure from an optimal solution** (α, β, γ, δ):



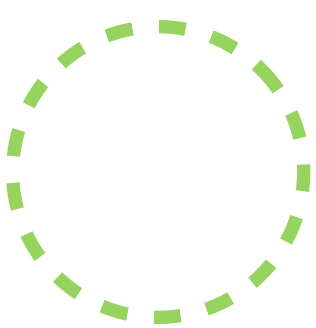
$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

Linear Programming Application I: Cancer Therapy

- **Fact:** Exposure is possible if and only if the following linear program has positive value.

$$\boxed{\begin{aligned} &\text{maximize } \delta \\ &\text{subject to} \\ &\quad x^2 + y^2 \geq \alpha x + \beta y + \gamma + \delta, \quad (x, y) \in B \\ &\quad x^2 + y^2 \leq \alpha x + \beta y + \gamma, \quad (x, y) \in R \end{aligned}}$$

- **Reconstructing the exposure from an optimal solution** (α, β, γ, δ):



$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

$$= \{(x, y) : (x - \frac{\alpha}{2})^2 + (y - \frac{\beta}{2})^2 = \gamma + \frac{\alpha^2}{4} + \frac{\beta^2}{4}\}$$

Linear Programming Application I: Cancer Therapy

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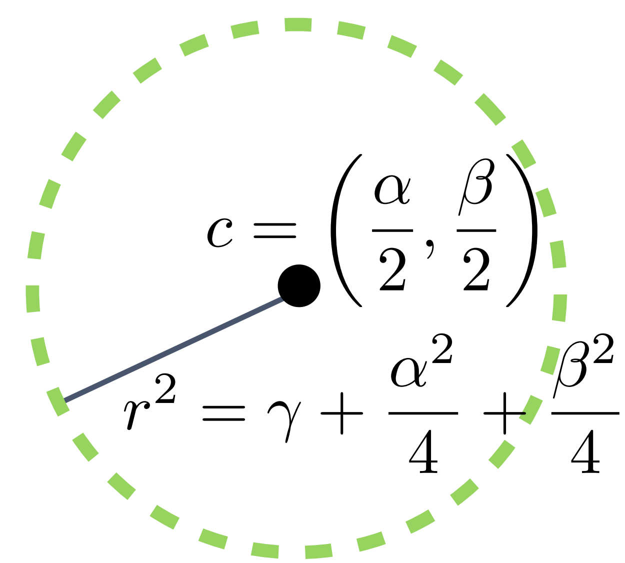
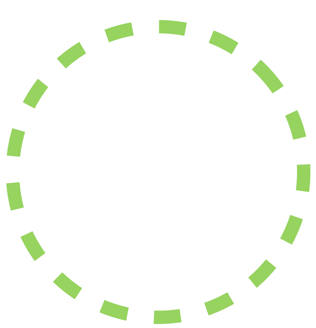
$$



- **Reconstructing the exposure from an optimal solution** (α, β, γ, δ):



$$= \{(x, y) : x^2 + y^2 = \alpha x + \beta y + \gamma\}$$

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Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL:**

$$\begin{array}{llcll} \text{minimize} & -\delta & & & \\ \text{subject to} & x^2+y^2 & \geq & \alpha x+\beta y+\gamma+\delta, & (x,y)\in B \\ & x^2+y^2 & \leq & \alpha x+\beta y+\gamma, & (x,y)\in R \\ & \delta & \leq & 1 & \end{array}$$

Avoids unbounded program

maximize $c^T x \rightarrow$ minimize $-c^T x$ and negate resulting value

$$\begin{array}{llll} \text{minimize} & -\delta & & \\ \text{subject to} & x^2+y^2 & \geq & \alpha x + \beta y + \gamma + \delta, \quad (x,y) \in B \\ & x^2+y^2 & \leq & \alpha x + \beta y + \gamma, \quad (x,y) \in R \\ & \delta & \leq & 1 \end{array}$$

Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL: Setup and Solve** (Preamble as before)

```
int main() {
  // by default, we have an LP with Ax <= b and no bounds for
  // the four variables alpha, beta, gamma, delta
  Program lp (CGAL::SMALLER, false, 0, false, 0);
  const int alpha = 0;
  const int beta  = 1;
  const int gamma = 2;
  const int delta = 3;

  // number of red and blue points
  int m; std::cin >> m;
  int n; std::cin >> n;

  // read the red points (cancer cells)
  for (int i=0; i<m; ++i) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up <= constraint for point inside/on circle:
    //   -alpha x - beta y - gamma <= -x^2 - y^2
    lp.set_a (alpha, i, -x);
    lp.set_a (beta,  i, -y);
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  }
```

```
  // read the blue points (healthy cells)
  for (int j=0; j<n; ++j) {
    int x; std::cin >> x;
    int y; std::cin >> y;
    // set up <= constraint for point outside circle:
    //   alpha x + beta y + gamma + delta <= x^2 + y^2
    lp.set_a (alpha, m+j, x);
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  }

  // objective function: -delta (the solver minimizes)
  lp.set_c(delta, -1);

  // enforce a bounded problem:
  lp.set_u (delta, true, 1);

  // solve the program, using ET as the exact type
  Solution s = CGAL::solve_linear_program(lp, ET());
  assert (s.solves_linear_program(lp));
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\begin{array}{lrcll}
\text{minimize} & -\delta & & & \\
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Linear Programming Application I: Cancer Therapy

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Linear Programming Application I: Cancer Therapy

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\end{array}
$$

Linear Programming Application I: Cancer Therapy

- **Implementation in CGAL: Output**

negate resulting value!

```
  // output exposure center and radius, if they exist
  if (s.is_optimal() && (s.objective_value() < 0)) {
    // *opt := alpha, *(opt+1) := beta, *(opt+2) := gamma
    CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
      opt = s.variable_values_begin();
    CGAL::Quotient<ET> alpha_opt = *opt +alpha;// +0
    CGAL::Quotient<ET> beta_opt = *(opt+beta);// +1
    CGAL::Quotient<ET> gamma_opt = *(opt+gamma);// +2
    std::cout << "There is a valid exposure:\n";
    std::cout << " Center = ("        // (alpha/2, beta/2)
         << alpha_opt/2 << ", " <<  beta_opt/2
         << ")\n";
    std::cout << " Squared Radius = " // gamma + alpha^2/4 + beta^2/4
         << gamma_opt + alpha_opt*alpha_opt/4 + beta_opt*beta_opt/4 << "\n";
  } else
    std::cout << "There is no valid exposure.";
  std::cout << "\n";
  return 0;
}
```

$$\begin{array}{llcll} \text{minimize} & -\delta & & & \\ \text{subject to} & x^2+y^2 & \geq & \alpha x+\beta y+\gamma+\delta, & (x,y)\in B \\ & x^2+y^2 & \leq & \alpha x+\beta y+\gamma, & (x,y)\in R \\ & \delta & \leq & 1 & \end{array}$$

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    CGAL::Quadratic_program_solution<ET>::Variable_value_iterator
      opt = s.variable_values_begin();
    CGAL::Quotient<ET> alpha_opt = *opt +alpha;// +0
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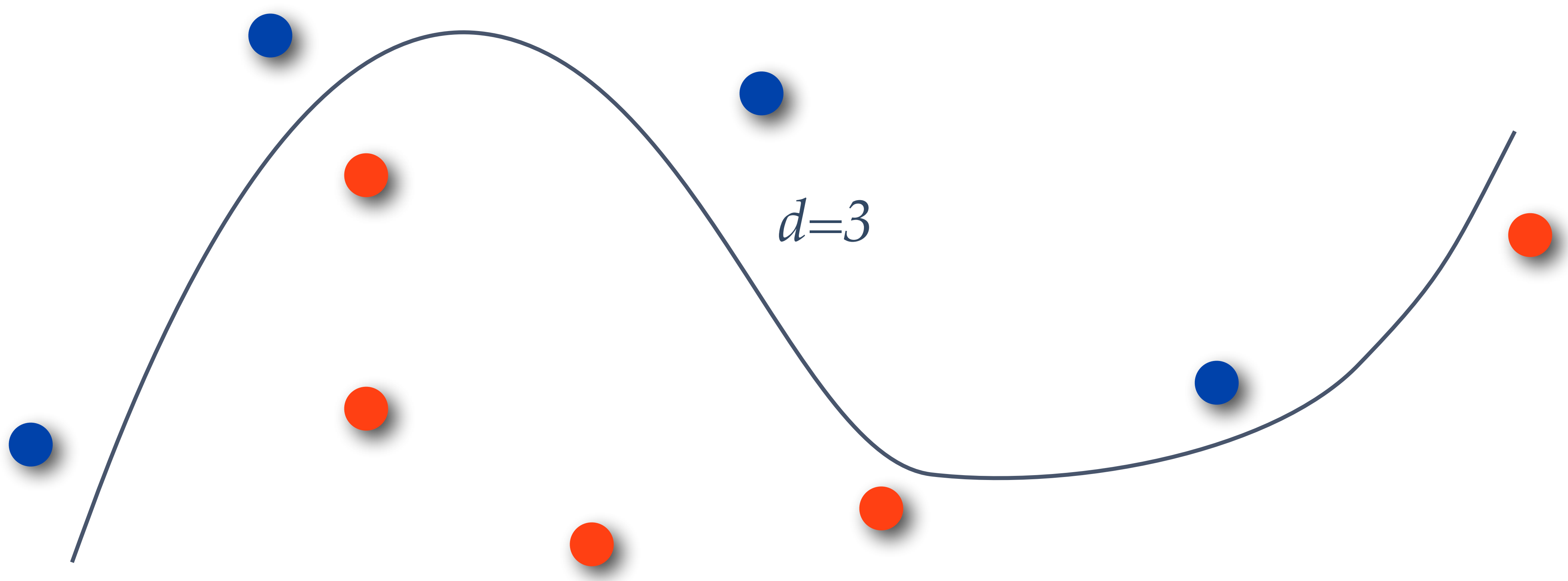
"Pointer" to first variable of optimal solution

The quotient **`*(opt+i)`** is the value of the variable x_i in the optimal solution

Linear Programming Beyond Cancer Therapy

- Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree *d*?



Linear Programming Beyond Cancer Therapy

- Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree d?

- Polynomial of degree 3:

$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j$$

Linear Programming Beyond Cancer Therapy

- Given a set of R of red and a set B of blue points, can they be separated by the zero set of a polynomial of degree *d*?

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- Linear programming formulation: find *a,b,c,d,e,f,g,h,i,j* such that

$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \leq 0, \quad (x, y) \in B$$
$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j \geq 0, \quad (x, y) \in R$$

Linear Programming Beyond Cancer Therapy

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- Polynomial of degree 3:

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- Linear programming formulation: find *a*,*b*,*c*,*d*,*e*,*f*,*g*,*h*,*i*,*j* such that

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- This is linear separability in 9-dimensional space, under the generalized lifting map $(x,y) \rightarrow (x^3, x^2y, xy^2, y^3, x^2, xy, y^2, x, y)$

Linear Programming Further Applications

Not necessary for the class, but good to know

- Linear programming relaxations for hard combinatorial problems

Linear Programming Further Applications

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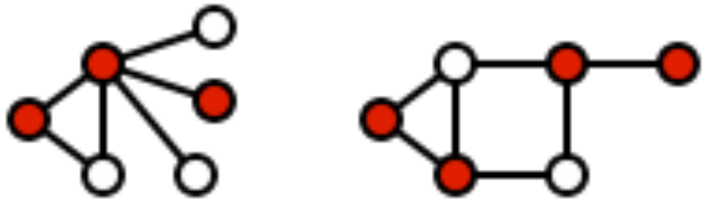
- Linear programming relaxations for hard combinatorial problems
- **Vertex Cover:** Given a graph G=(V,E), find a smallest subset of vertices (a vertex cover) such that every edge is incident to one vertex of the cover.



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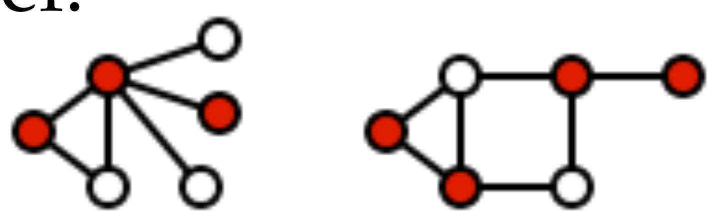

$x_i \in \{0,1\}$

- Formulation as "LP": x_i indicates whether vertex i is in the cover (0: not in the cover, 1: in the cover):

$$
\begin{array}{lrcl}
\text{minimize} & \sum_{i=1}^{n} x_i & & \\
\text{subject to} & x_i + x_j & \geq & 1 \quad \forall \{i,j\} \in E \\
& 0 \leq x_i \leq 1 & & \forall i \in V
\end{array}
$$

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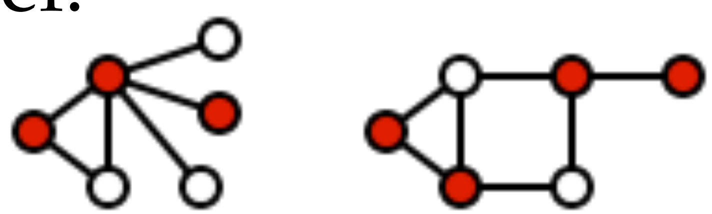

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not an LP!

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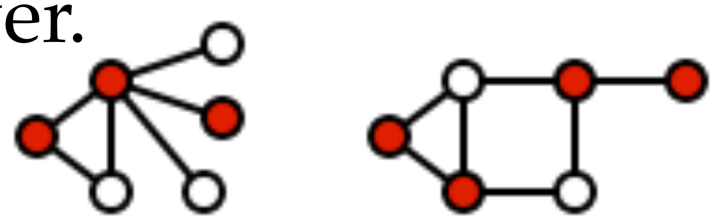

- Let $x^*_1, x^*_2, \ldots, x^*_n$ be an optimal solution of the *LP relaxation*

$$
\begin{array}{lrcl}
\text{minimize} & \sum_{i=1}^{n} x_i & & \\
\text{subject to} & x_i + x_j & \geq & 1 \quad \forall \{i,j\} \in E \\
& 0 \leq x_i \leq 1 & & \quad \forall i \in V
\end{array}
$$

Linear Programming Further Applications

Not necessary for the class, but good to know

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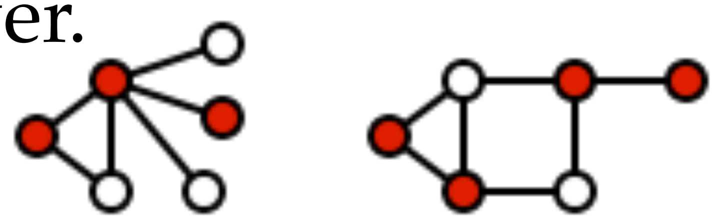

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$$

- **Theorem:** $C = \{i : x^*_i \geq 1/2\}$ is a vertex cover of size at most 2 opt.

[Ref]

Linear vs. Integer Programming

- Often, applications lead to linear programs with the additional requirement of *integral solutions* (e.g. vertex cover)

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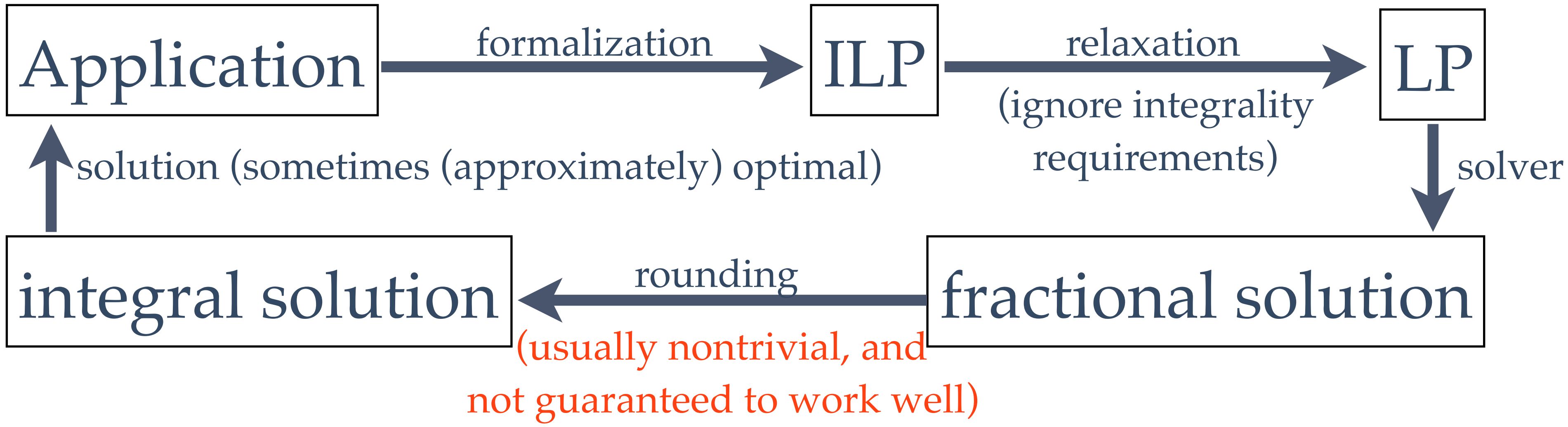
Linear vs. Integer Programming

- Often, applications lead to linear programs with the additional requirement of *integral solutions* (e.g. vertex cover)
- Such programs are called *integer linear programs* (ILP) and are in general much harder to solve than linear programs (NP-hard)
- Typical approach (e.g. vertex cover):



Quadratic Programming (QP)

- **Problem:** Minimize a convex quadratic function in n variables subject to m linear (in)equality constraints!

Quadratic Programming

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- **Example** (n=2, m=5):

$$
\begin{array}{lrcr}
\text{minimize} & x^2 + 4y^2 - 32y + 64 & & \\
\text{subject to} & x + y & \leq & 7 \\
& -x + 2y & \leq & 4 \\
& x & \geq & 0 \\
& y & \geq & 0 \\
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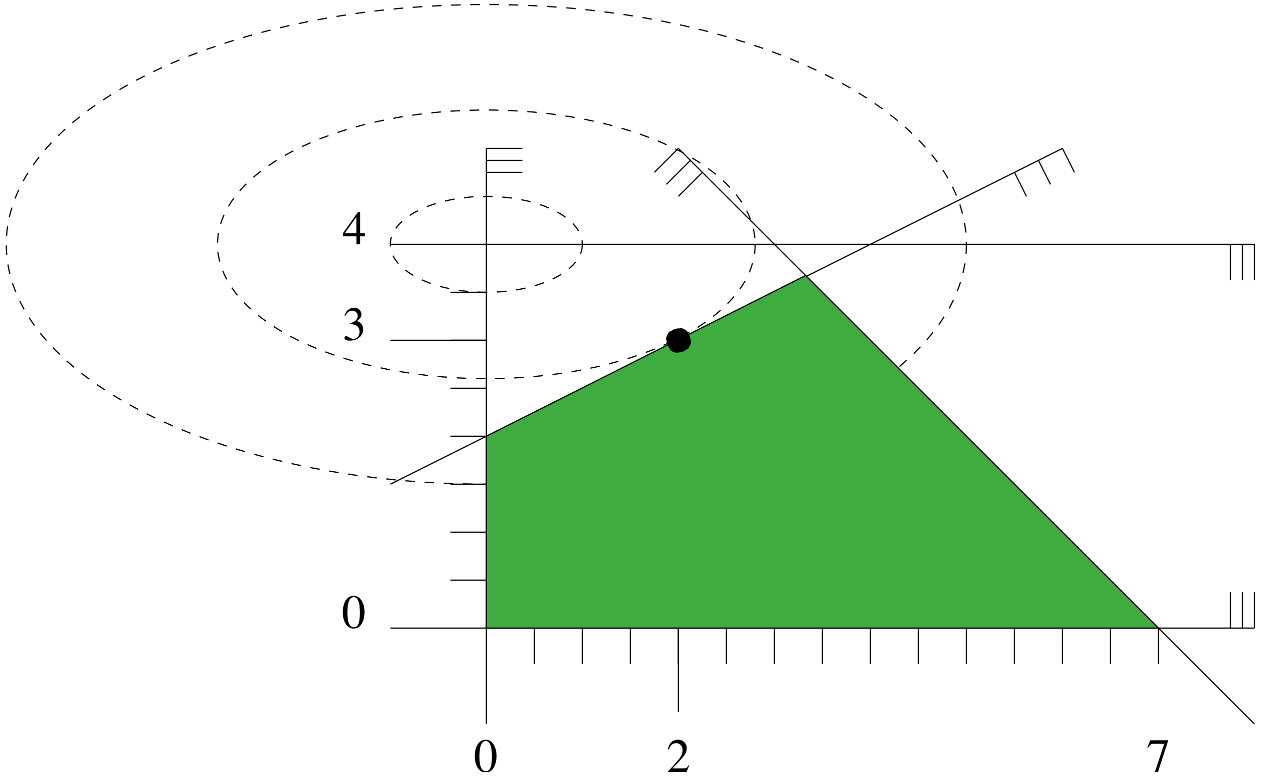
$$



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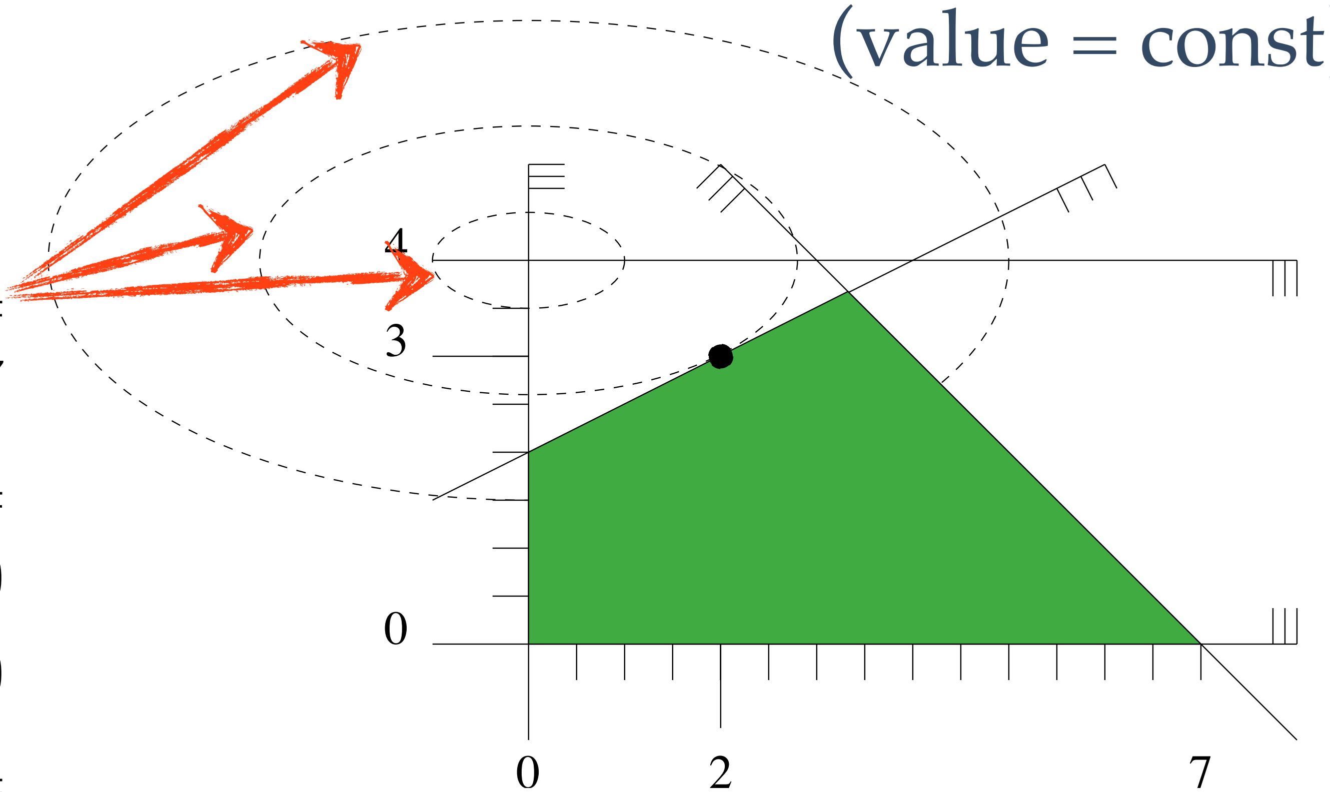
isolines
(value = const)



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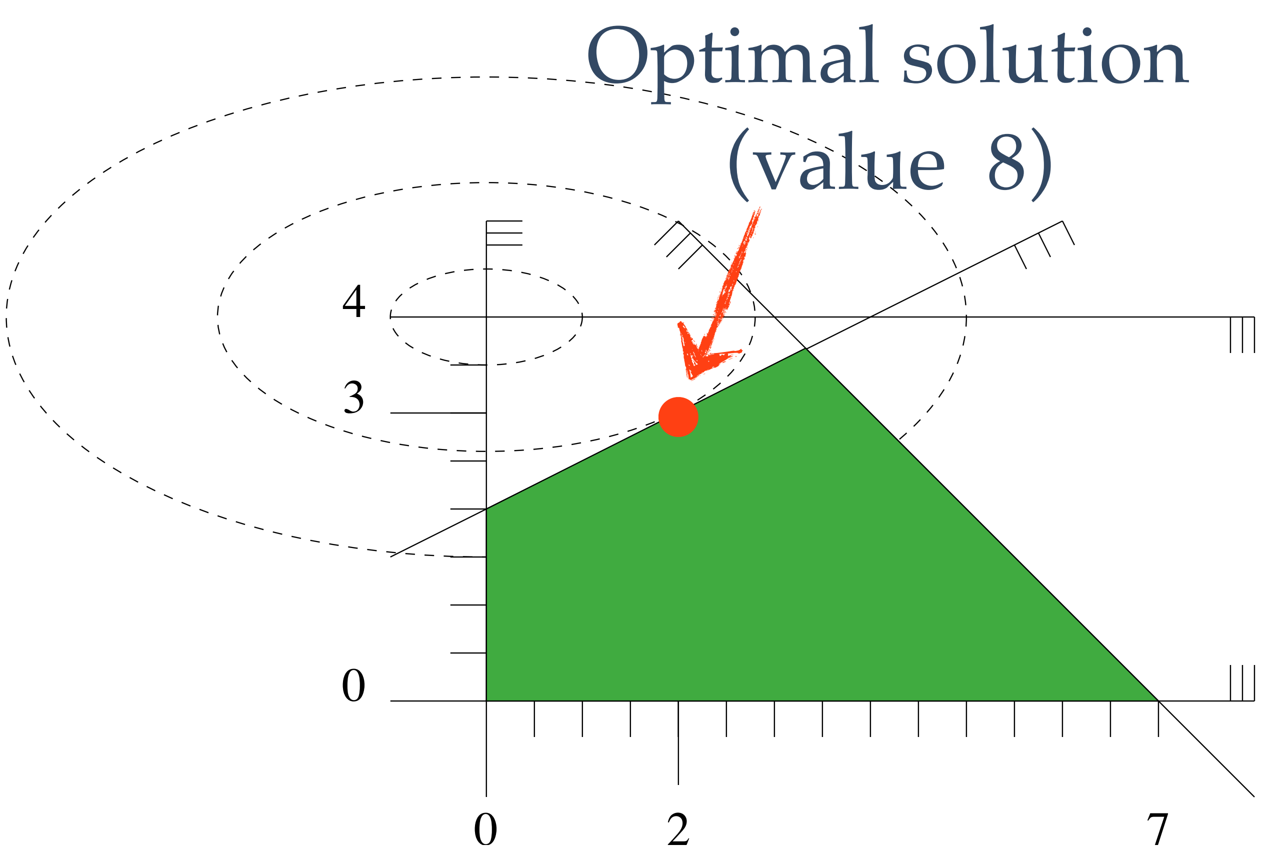

Quadratic Programming ... in CGAL

- **General form of QP in CGAL:**

$$
\begin{array}{ll}
\text{minimize} & x^T D x + c^T x + c_0 \\
\text{subject to} & A x \gtreqless b \\
 & l \leq x \leq u
\end{array}
$$

$(D \in \mathbb{R}^{n \times n}$ symmetric positive semidefinite$)$

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- **Warning:** if D is not positive semidefinite, the quadratic objective function is not convex, and we know not how to solve it.
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- **Relax:** In the applications, we know from theory that *D* is "good"

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$$\begin{array}{lrcr} \text{minimize} & \multicolumn{3}{l}{x^2 + 4y^2 - 32y + 64} \\ \text{subject to} & x + y & \leq & 7 \\ & -x + 2y & \leq & 4 \\ & x & \geq & 0 \\ & y & \geq & 0 \\ & y & \leq & 4 \end{array} \quad \Rightarrow \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \checkmark$$

Quadratic Programming Application: Low-Risk Investment

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- **Example:** n=2

	r_i
Swatch shares	10% (0.1)
Credit Suisse shares	51% (0.51)

v_{ij}	Swatch shares	Credit Suisse shares
Swatch shares	0.09	-0.05
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Negative correlation: if CS does worse than expected, Swatch will probably do better, and vice versa

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Read as: standard deviation of return rate is $\sqrt{0.25} = 0.5$
(actual return rate could easily be off by 0.5)

Quadratic Programming Application: Low-Risk Investment

- **Investment strategy:**

$$(x_1, x_2, \ldots, x_n), \quad \sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0 \forall i$$

Meaning: An x_i fraction of your money goes into investment *i*

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- **Expected return rate of this strategy:**

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- **Example:** half the money in Swatch shares, half in Credit Suisse shares; expected return rate is $\frac{1}{2} \cdot 0.1 + \frac{1}{2} \cdot 0.51 = 0.305 = 30.5\%$

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- **Risk of this strategy:**

Straightforward calculations

$$\mathrm{Var}[\sum_{i=1}^{n} x_i R_i] = \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} x_i x_j = x^T D x$$

$$D = (v_{ij})_{1 \leq i,j \leq n} \text{is the } \textit{covariance matrix}$$

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- **Example:** half-half Swatch/CS has risk $\dfrac{0.09 - 2 \cdot 0.05 + 0.25}{4} = 0.06$

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less than each individual risk!

- **Example:** half-half Swatch/CS has risk $\dfrac{0.09 - 2 \cdot 0.05 + 0.25}{4} = 0.06$

Quadratic Programming Application: Low-Risk Investment

- **The risk-tolerant case:** Find the investment strategy with lowest risk that guarantees expected return rate ρ at least!

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} x_i x_j \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \ge \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \ge 0, \quad i = 1, \ldots, n \end{aligned}$$

risk

expected return rate

strategy

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\begin{aligned}
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\text{subject to} \quad & \underbrace{\sum_{i=1}^{n} r_i x_i}_{\text{expected return rate}} \geq \rho \\
& \sum_{i=1}^{n} x_i = 1 \\
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strategy

Fact: A covariance matrix is positive semidefinite, so this is indeed a convex QP.

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\text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \geq \rho \qquad \boxed{\text{expected return rate}} \\
& \sum_{i=1}^{n} x_i = 1 \\
\boxed{\text{strategy}} \quad & x_i \geq 0, \quad i = 1, \ldots, n
\end{aligned}
$$

Fact: A covariance matrix is positive semidefinite, so this is indeed a convex QP.

- **Example:** $\rho = 0.4$: 26.8% Swatch, 73.2% Credit Suisse; risk $= 0.121$

Low-Risk Investment Example... in CGAL

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_ix_j = x^TDx \\ \text{subject to} \quad & \sum_{i=1}^{n} r_ix_i \ge \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \ge 0, \quad i = 1,\dots,n \end{aligned}$$

- **Preamble:** This time, the input is rational…

Gnu Multi-precision Library (GMP)

```
#include <iostream>
#include <cassert>
#include <CGAL/basic.h>
#include <CGAL/QP_models.h>
#include <CGAL/QP_functions.h>
#include <CGAL/Gmpq.h>

// choose exact Rational type
typedef CGAL::Gmpq ET;


// program and solution types
typedef CGAL::Quadratic_program<ET> Program;
typedef CGAL::Quadratic_program_solution<ET> Solution;
```

Low-Risk Investment Example... in CGAL

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\text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_i x_j = x^T D x \\
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& \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0, \quad i = 1, \ldots, n
\end{aligned}
$$

- **Input:** Desired expected return

```
int main() {
  // read minimum expected return rate
  std::cout << "What is your desired expected return rate? ";
  double rho; std::cin >> rho;
```

for example, $0.4 \approx 40\%$

Low-Risk Investment Example... in CGAL

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- **Setup:** Make sure to enter matrix *2D* (customary in QP solvers)!

```
// by default, we have a nonnegative QP with Ax >= b
Program qp (CGAL::LARGER, true, 0, false, 0);

// now set the non-default entries:
const int sw = 0;
const int cs = 1;

// constraint on expected return: 0.1 sw + 0.51 cs >= rho
qp.set_a(sw, 0, ET(1)/10);
qp.set_a(cs, 0, ET(51)/100);
qp.set_b(    0, rho);

// strategy constraint: sw + cs = 1
qp.set_a(sw, 1, 1);
qp.set_a(cs, 1, 1);
qp.set_b(    1, 1);
qp.set_r(    1, CGAL::EQUAL); // override default >=

// objective function: 0.09 sw^2 - 0.05 sw cs - 0.05 cs sw + 0.25 cs^2
// we need to specify the entries of the symmetric matrix 2D, on and below the diagonal
qp.set_d(sw, sw,  ET(18)/100); //  0.09 sw^2
qp.set_d(cs, sw, -ET(1)/10);   // -0.05 cs sw
qp.set_d(cs, cs,  ET(1)/2);    //  0.25 cs^2
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Low-Risk Investment Example… in CGAL

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\text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_ix_j = x^TDx \\
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// now set the non-default entries:
const int sw = 0;
const int cs = 1;

// constraint on expected return: 0.1 sw + 0.51 cs >= rho
qp.set_a(sw, 0, ET(1)/10);
qp.set_a(cs, 0, ET(51)/100);
qp.set_b(    0, rho);

// strategy constraint: sw + cs = 1
qp.set_a(sw, 1, 1);
qp.set_a(cs, 1, 1);
qp.set_b(    1, 1);
qp.set_r(    1, CGAL::EQUAL); // override default >=

// objective function: 0.09 sw^2 - 0.05 sw cs - 0.05 cs sw + 0.25 cs^2
// we need to specify the entries of the symmetric matrix 2D, on and below the diagonal
qp.set_d(sw, sw,  ET(18)/100); //  0.09 sw^2
qp.set_d(cs, sw, -ET(1)/10);   // -0.05 cs sw
qp.set_d(cs, cs,  ET(1)/2);    //  0.25 cs^2
```

j ≤ i in **`set_d(i, j)`**

Low-Risk Investment Example... in CGAL

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_ix_j = x^TDx \\ \text{subject to} \quad & \sum_{i=1}^{n} r_ix_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n \end{aligned}$$

- **Solve:** ...as nonnegative quadratic program (a little faster)

```
 // solve the program, using ET as the exact type
assert(qp.is_nonnegative());
Solution s = CGAL::solve_nonnegative_quadratic_program(qp, ET());
assert (s.solves_quadratic_program(qp));
```

Independent verifications.
Only do something in Debug mode.

Low-Risk Investment Example… in CGAL

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\text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_ix_j = x^TDx \\
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Low-Risk Investment Example... in CGAL

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}x_i x_j = x^T D x \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n \end{aligned}$$

- **Output:** query solution status; if feasible, output strategy / risk

```
// output
  if (s.status() == CGAL::QP_INFEASIBLE) {
    std::cout << "Expected return rate " << rho << " cannot be achieved.\n";
  } else {
    assert (s.status() == CGAL::QP_OPTIMAL);
    Solution::Variable_value_iterator opt = s.variable_values_begin();
    ET sw_ratio = opt->numerator() / opt->denominator();
    ET cs_ratio = (opt+1)->numerator() / (opt+1)->denominator();
    ET risk = s.objective_value().numerator() / s.objective_value().denominator();
    double sw_percent = ceil_to_double(100 * *opt);
    std::cout << "Minimum risk investment strategy:\n"
          << sw_ratio << " ~ " << sw_percent << "%" << " Swatch\n"
          << cs_ratio << " ~ " << 100-sw_percent << "%" << " CS\n"
          << "Risk = " << risk
          << " ~ ." << ceil_to_double(100 * s.objective_value())
          << "\n";
  }
```

Low-Risk Investment Example... in CGAL

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij} x_i x_j = x^T D x \\ \text{subject to} \quad & \sum_{i=1}^{n} r_i x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \quad i = 1, \ldots, n \end{aligned}$$

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Low-Risk Investment Example... in CGAL

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Low-Risk Investment Example... in CGAL

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          << cs_ratio << " ~ " << 100-sw_percent << "%" << " CS\n"
          << "Risk = " << risk
          << " ~ ." << ceil_to_double(100 * s.objective_value())
          << "\n";
  }
```

Known Bug :=(

- You can't reliably copy or assign instances of the class **CGAL::Quadratic_program_solution<ET>**
- **Workaround 1:** If you want to pass or return such instances to/from a function, pass a pointer to the instance instead!
- **Workaround 2:** If you want to assign a new solution to an existing instance... don't do it!

Sources and Further Reading

- **LP/QP Solver:** Online manual at **`www.cgal.org`**: Online Manual → Combinatorial Algorithms → Linear and Quadratic Programming Solver

- **Cancer Therapy:** J. O'Rourke, S. Kosaraju, and N. Megiddo: Computing Circular Separabiliy, *Discrete & Computational Geometry* 1:105-113 (1986)

- **Low-Risk Investment:** H. Markowitz: Portfolio Selection, *Journal of Finance* 7(1): 77-91 (1952)