Algolab 2018
Winter Games

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Objective

Learn how to solve a problem given by a textual description.
This includes:

- appropriate problem modeling
- choice of suitable (combinatorial) algorithms, and
- implementation.

Your Task:

- read the 3 problem descriptions
- sketch your approach (modeling, algorithms)

Use them all!

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- no overlap — only “close neighbors” matter
- maximize radius — optimization problem?

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- only the closest cannon pair matters

Use them all! – Solution

Try them all!

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Check only the distance to the nearest neighbor for each cannon

- Compute the Delaunay triangulation $\rightarrow \Theta(n \log n)$
- Iterate over the edges of the triangulation $\rightarrow \Theta(n)$

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Implementation detail

- Squared distances fit into `double`:

$$d^2 = d_x^2 + d_y^2 \leq (2^{25})^2 + (2^{25})^2 = 2^{51}$$

Downhill course

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- graph problem: cannons are vertices, put an edge whenever two ranges overlap
- find maximum independent set
  - in general — NP-complete
  - bipartite graphs — König's Theorem, Matching
  - special cases — trivial

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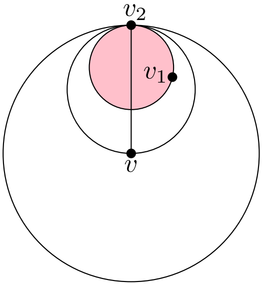

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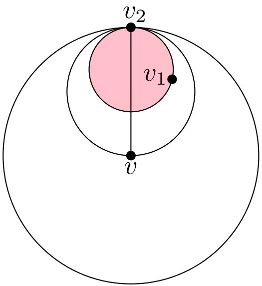

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Lemma

Let v_1 be a nearest neighbor and v_2 be a second nearest neighbor of v. Then at least one of vv_2 or v_1v_2 is an edge of the Delaunay triangulation.

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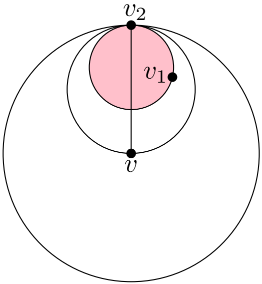

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- maximal degree 2

$\Rightarrow$ the graph G must be a disjoint union of paths and cycles

$\Rightarrow$ greedy/ad-hoc solution for every component $\rightarrow \Theta(n)$

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Key ideas

- optimization problem find a larger radius for each cannon
  - the improvement is linear in the radius
  - lower bounds for the radii
  - implicit upper bounds: no overlap

Software update – Solution

Linear program with n variables and $n + \binom{n}{2}$ constraints:

- *Variables*: Operation range (radius) of every snow cannon

$$r_i \geq \left\lfloor \frac{\texttt{closest\_pair\_dist}}{2} \right\rfloor, \quad i = 1, \ldots, n$$

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Implementation details

- **Input type**: Exact type with sqrt