6

Exercises on Galton Watson processes

Exercice 6.1

Prove that for a subcritical GW process($m < 1$) the mean total progeny is

$$\mathbb{E}\left[\bar{X}\right] = \frac{1}{1-m}$$

Exercice 6.2

Assume that $\sigma^2 := \mathrm{Var}(\xi) < +\infty$. Show that

$$\mathrm{Var}(X_{n+1}) = m^n \sigma^2 + m^2 \,\mathrm{Var}(X_n)\,, \tag{6.1}$$

and then that

$$\mathrm{Var}(X_n) = \begin{cases} \frac{\sigma^2 m^n (m^n - 1)}{m^2 - m} & \text{if } m \neq 1\,. \\ n\sigma^2 & \text{if } m = 1. \end{cases} \tag{6.2}$$

Show that if $m > 1$ then the martingale $W_n = \frac{X_n}{m^n}$ is UI.

Solution de l'Exercice 6.2

By the conditional variance formula

$$\begin{aligned} \mathrm{Var}(X_{n+1}) &= \mathrm{Var}(\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right]) + \mathbb{E}\left[(X_{n+1} - \mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right])^2\right] \\ &= m^2 \,\mathrm{Var}(X_n) + \mathbb{E}\left[\mathbb{E}\left[(X_{n+1} - mX_n)^2 \mid X_n\right]\right] \\ &= m^2 \,\mathrm{Var}(X_n) + \mathbb{E}\left[X_n \sigma^2\right]. \end{aligned}$$

Exercice 6.3

The Galton Watson process with immigration is defined by the recurrence

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n+1)} + Y_{n+1}$$

where the $(\xi_i^{(k)}, k \geq 1, i \geq 1)$ are IID distributed as xi and are independent from $(Y_k, k \geq 1)$ IID distributed as Y. In this model $\xi_i^{(n+1)}$ is the number of children of the i-th individual of the n-th generation, and Y_n is the number of immigrants in the n-th generation. We assume that $0 < m = \mathbb{E}[\xi] < +\infty$, $\forall j \mathbb{P}(\xi = j) < 1$ and $0 < \lambda = \mathbb{E}[Y] < +\infty$.

1. Prove that one has

$$X_n = Z_n + U_n^{(1)} + \cdots + U_n^{(n)}, \tag{6.3}$$

with Z_n the number of descendants at generation n of the initial individual, $U_n^{(i)}$ is the number of descendants at generation n of immigrants that arrived at generation i, and all these processes are independent.

2. Let $V_n = m^{-n} X_n$. Show that

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = m X_n + \lambda, \tag{6.4}$$

after defining precisely $\mathcal{F}_n$.

3. Show that V_n is a positive submartingale.

4. Show that

$$\mathbb{E}[X_n] = \frac{m^n(m + \lambda - 1) - \lambda}{m - 1}, \tag{6.5}$$

and infer that $C := \sup \mathbb{E}[V_n] < +\infty$.

5. Show that there exists a rv V, $0 \leq V < +\infty$ a.e. and $V_n \to V$ a.e.

6. Recall that $W_n = m^{-n} Z_n \to W$ a.e. for a positive finite rv W. Let $\beta = \frac{m+\lambda-1}{m-1}$ and let

$$T = \sum_{k=1}^{Y} W_k, \tag{6.6}$$

with $(W_k, k \geq 1)$ IID distributed as W. Show that

$$m^{-n} U_n^{(i)} \to m^{-i} T^{(i)} a.e. \quad \text{with} \quad T^{(i)} \stackrel{d}{=} T. \tag{6.7}$$

Deduce that

$$V \geq U := W + \sum_{i=1}^{+\infty} m^{-i} T^{(i)} \quad a.e. \tag{6.8}$$

7. Using independence in (6.3), compute for $\lambda > 0$, $\mathbb{E}\left[e^{-\lambda V_n}\right]$ Combining the inequality for a posivitve random variable

$$-\log \mathbb{E}\left[e^{-X}\right] \leq \mathbb{E}\left[X\right]$$

with the fact that $\mathbb{E}\left[m^{-n} U_n^{(i)}\right] = m^{-i}\mathbb{E}\left[T\right]$, show that

$$\mathbb{E}\left[e^{-\lambda V}\right] = \lim_{n \to +\infty} \mathbb{E}\left[e^{-\lambda V_n}\right] = \mathbb{E}\left[e^{-\lambda U}\right], \tag{6.9}$$

and deduce from it that $V = U$ a.e.

8. Show that if $\mathbb{E}\left[\xi \log^+ \xi\right] < +\infty$ the $V > 0$ a.e. and that if $\mathbb{E}\left[\xi \log^+ \xi\right] = +\infty$ then $V = 0$ a.e.