Kinetic Beam Plasma Instability

Following the discussion in the re-derivation notes, we now compute the kinetic growth rates for arbitrary beam temperatures assuming the boosted Maxwell-Jüttner distribution. That is, we employ

$$f_b(\mathbf{p}) = \frac{e^{-\gamma_b(\epsilon - v_b p_z)/T}}{4\pi\gamma_b m^2 T K_2(m/T)}\,, \tag{1}$$

where $K_2(z)$ is the 2nd order modified Bessel function. Since we will primarily be interested in high temperatures, we will neglect the principle value component, and focus upon only the kinetic term arising from the pole, for which the associated growth rate is

$$\Gamma \simeq -\frac{\pi}{2}\omega_{Pt}\frac{n_b}{n_t}v_b^2 R \quad \text{where} \quad R = m\int d^2p_\perp k\frac{\partial f_b}{\partial p_z}\bigg/\frac{\partial}{\partial p_z}(\omega - kv_z)\bigg|_{p_z \text{ s.t. } \omega - kv_z = 0}\,. \tag{2}$$

Thus the problem is reduced to evaluating the two-dimensional integral over the transverse momenta at the pole. As we will show, this can be done analytically.

Let us begin by noting the following general relations. At the pole we have

$$\omega - kv_z = 0 \quad\Rightarrow\quad v_z = \frac{p_z}{\epsilon} = v_{\rm ph} \equiv \frac{\omega}{k} \quad\Rightarrow\quad p_z = \gamma_{\rm ph}v_{\rm ph}\epsilon_\perp\,, \tag{3}$$

where $\gamma_{\rm ph}$ is the Lorentz factor associated with $v_{\rm ph}$ and $\epsilon_\perp \equiv \sqrt{m^2 + p_\perp^2}$. This immediately implies that at the pole $\epsilon = \gamma_{\rm ph}\epsilon_\perp$. Finally,

$$\gamma_b\left(\epsilon - v_b p_z\right)/T = \gamma_b\gamma_{\rm ph}(1 - v_b v_{\rm ph})\epsilon_\perp/T\,. \tag{4}$$

Then,

$$\frac{\partial}{\partial p_z}(\omega - kv_z)\bigg|_{\rm pole} = -\frac{k}{\gamma_{\rm ph}^3\epsilon_\perp} \tag{5}$$

and

$$k\frac{\partial f_b}{\partial p_z}\bigg|_{\rm pole} = -\frac{k\gamma_b}{T}\left(\frac{p_z}{\epsilon} - v_b\right)f_b\bigg|_{\rm pole} = -\frac{k(v_{\rm ph} - v_b)e^{-\gamma_b\gamma_{\rm ph}(1 - v_b v_{\rm ph})\epsilon_\perp/T}}{4\pi m^2 T^2 K_2(m/T)}\,. \tag{6}$$

Inserting the above into the expression for R yields the integral:

$$\begin{aligned} R &= \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_b)}{4\pi m T^2 K_2(m/T)}\int d^2p_\perp\epsilon_\perp e^{-\gamma_b\gamma_{\rm ph}(1 - v_b v_{\rm ph})\epsilon_\perp/T} \\ &= \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_b)}{4mT^2 K_2(m/T)}\int_0^\infty dp_\perp^2\epsilon_\perp e^{-\mathcal{G}\epsilon_\perp/m}\,, \end{aligned} \tag{7}$$

where $\mathcal{G} \equiv \gamma_b \gamma_{\rm ph}(1 - v_b v_{\rm ph}) m/T$ is independent of $p_\perp$. This is of the form,

$$\int_0^\infty dx \sqrt{1+x} e^{-a\sqrt{1+x}} = \frac{2}{a^3}\left(a^2 + 2a + 2\right) e^{-a} . \tag{8}$$

Thus,

$$R = \frac{\gamma_{\rm ph}^3 m^2 (v_{\rm ph} - v_b)}{2T^2 K_2(m/T)} \left(\mathcal{G}^2 + 2\mathcal{G} + 2\right) \frac{e^{-\mathcal{G}}}{\mathcal{G}^3} , \tag{9}$$

and therefore the kinetic growth rate is

$$\Gamma \simeq -\omega_{Pt} \frac{n_b}{n_t} \frac{\pi \gamma_{\rm ph}^3 m^2 v_b^2 (v_{\rm ph} - v_b)}{4T^2 K_2(m/T)} \left(\mathcal{G}^2 + 2\mathcal{G} + 2\right) \frac{e^{-\mathcal{G}}}{\mathcal{G}^3} . \tag{10}$$

Defining the typical maximal growth rate scale,

$$\Gamma_0 \equiv \omega_{Pt} \frac{n_b}{n_t} \frac{\gamma_b m v_b^2}{T} \tag{11}$$

this simplifies to

$$\Gamma \simeq -\Gamma_0 \frac{\pi \gamma_{\rm ph}^3 m (v_{\rm ph} - v_b)}{4\gamma_b T K_2(m/T)} \left(\mathcal{G}^2 + 2\mathcal{G} + 2\right) \frac{e^{-\mathcal{G}}}{\mathcal{G}^3} . \tag{12}$$

Finding the maximal growth rate in general is quite difficult. However, we can estimate it the limit of high and low temperature. We do this, and then plot the general case below.

Cold Beams ($T \ll m$)

In the limits of a cold beam, $G \gg 1$, and we need only keep the $\mathcal{G}^2$ term in the parentheses in Equation (12). Furthermore, we can expand $\mathcal{G}$ in $\delta p \equiv (\gamma_{\rm ph} v_{\rm ph} - \gamma_b v_b) m$:

$$\begin{aligned}
\mathcal{G} &= \frac{\epsilon_b \epsilon_{\rm ph} - p_b p_{\rm ph}}{mT} \\
&= \frac{1}{mT}\left(\epsilon_b \sqrt{\epsilon_b^2 + 2p_b \delta p + \delta p^2} - p_b^2 - p_b \delta p\right) \\
&= \frac{1}{mT}\left[\epsilon_b^2 \left(1 + \frac{p_b \delta p}{\epsilon_b^2} + \frac{\delta p^2}{2\epsilon_b^2} - \frac{p_b^2 \delta p^2}{2\epsilon_b^4} + \ldots\right) - p_b^2 - p_b \delta p\right] \\
&= \frac{m}{T}\left(1 + \frac{\delta p^2}{2\epsilon_b^2}\right) + \ldots ,
\end{aligned} \tag{13}$$

where $p_{\rm ph} \equiv \gamma_{\rm ph} v_{\rm ph} m$ and $\epsilon_{\rm ph} \equiv \sqrt{m^2 + p_{\rm ph}^2}$. Similarly,

$$\begin{aligned}
\frac{\delta p}{m} &= \gamma_{\rm ph} v_{\rm ph} - \gamma_b v_b \\
&= \frac{v_b + \delta v}{\sqrt{1 - v_b^2 - 2 v_b \delta v + \dots}} - \gamma_b v_b \\
&= \gamma_b \frac{v_b + \delta v}{\sqrt{1 - 2\gamma_b^2 v_b \delta v + \dots}} - \gamma_b v_b \\
&\simeq \gamma_b \left(v_b + \delta v\right) \left(1 + \gamma_b^2 v_b \delta v\right) - \gamma_b v_b \\
&= \left(\gamma_b + \gamma_b^3 v_b^2\right) \delta v \\
&= \gamma_b^3 \delta v \,,
\end{aligned} \tag{14}$$

and thus $\delta v \equiv v_{\rm ph} - v_b \simeq \delta p / \gamma_b^3 m$.

At low T, m/T is large, and we may use the asymptotic expansion of $K_2(z)$ around large z,

$$K_2(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{15}{8z} + \dots\right) \quad \Rightarrow \quad K_2\left(\frac{m}{T}\right) \simeq \sqrt{\frac{\pi}{2}} \left(\frac{T}{m}\right)^{1/2} e^{-m/T} \,, \tag{15}$$

(Abramowitz & Stegun, p. 378).

Since the coefficient is already first order in $\gamma_{\rm ph}(v_{\rm ph} - v_b) m \simeq \delta p$, we now have

$$\Gamma \simeq -\Gamma_0 \sqrt{\frac{\pi}{8}} \left(\frac{m}{T}\right)^{1/2} \frac{\delta p}{m \gamma_b} e^{-\delta p^2 m / 2 \epsilon_b^2 T} \,. \tag{16}$$

This is trivially maximized at $\delta p = -\epsilon_b \sqrt{T/m} = -\gamma_b m \sqrt{T/m}$, and thus

$$\Gamma_M \simeq \Gamma_0 \sqrt{\frac{\pi}{8e}} \simeq 0.4 \Gamma_0 \,. \tag{17}$$

The temperature that defines "cold" beams here is set by the validity of the expansions made. In all cases, this is satisfied if $T \ll m$. The above result was obtained repeatedly in the oblique re-derivation notes.

Hot Beams ($T \gg m$)

In the opposite limit, $\mathcal{G} \ll 1$, and thus we need only keep the ~ 1 term in the parentheses in Equation (12). In this case, the maximum growth rate is determined by

$$\frac{\partial}{\partial v_{\rm ph}}\gamma_{\rm ph}^3(v_{\rm ph}-v_b)\frac{e^{-\mathcal{G}}}{\mathcal{G}^3} = \left[3\gamma_{\rm ph}^2 v_{\rm ph}(v_{\rm ph}-v_b)+1-(v_{\rm ph}-v_b)\left(\frac{3}{\mathcal{G}}+1\right)\frac{\partial\mathcal{G}}{\partial v_{\rm ph}}\right]\gamma_{\rm ph}^3\frac{e^{-\mathcal{G}}}{\mathcal{G}^3}$$
$$\simeq \left[3\gamma_{\rm ph}^2 v_{\rm ph}(v_{\rm ph}-v_b)+1-3\frac{(v_{\rm ph}-v_b)}{\mathcal{G}}\frac{\partial\mathcal{G}}{\partial v_{\rm ph}}\right]\gamma_{\rm ph}^3\frac{e^{-\mathcal{G}}}{\mathcal{G}^3} = 0\,, \tag{18}$$

where we have kept only the largest terms in the T expansion in the last line. Inserting $\partial\mathcal{G}/\partial v_{\rm ph} = \gamma_b\gamma_{\rm ph}^3(v_{\rm ph}-v_b)m/T$ this reduces to

$$\mathcal{G}\left[1+3\gamma_{\rm ph}^2 v_{\rm ph}(v_{\rm ph}-v_b)\right]-3\gamma_{\rm ph}^3(v_{\rm ph}-v_b)^2\frac{m}{T} = 0\,. \tag{19}$$

Here we will only verify that

$$\gamma_{\rm ph}^2 \simeq \frac{\gamma_b^2}{2} \quad \Rightarrow \quad v_{\rm ph}-v_b = -\frac{1}{(v_{\rm ph}+v_b)\gamma_b^2} \simeq -\frac{1}{2\gamma_b^2}\,, \tag{20}$$

and

$$\mathcal{G} \simeq \frac{\gamma_b^2}{\sqrt{2}}\left[1-v_b\left(v_b-\frac{1}{2\gamma_b^2}\right)\right]\frac{m}{T} \simeq \frac{\gamma_b^2}{\sqrt{2}}\left(\frac{1}{\gamma_b^2}+\frac{v_b}{2\gamma_b^2}\right)\frac{m}{T} \simeq \frac{3}{2\sqrt{2}}\frac{m}{T}\,, \tag{21}$$

provides a solution to this:

$$\frac{3}{2\sqrt{2}}\frac{m}{T}\left[1+\frac{3}{2}\gamma_b^2\left(v_b-\frac{1}{2\gamma_b^2}\right)\left(-\frac{1}{2\gamma_b^2}\right)\right]-\frac{3}{2\sqrt{2}}\gamma_b^4\left(-\frac{1}{2\gamma_b^2}\right)^2\frac{m}{T}$$
$$= \frac{3}{2\sqrt{2}}\frac{m}{T}\left(1-\frac{3v_b}{4}-\frac{1}{4}+\frac{3}{8\gamma_b^2}\right) \simeq 0\,. \tag{22}$$

Finally, note that in the $T \gg m$ limit, we may use the asymptotic expansion of $K_2(z)$ around small z:

$$K_2(z) \simeq \frac{2}{z^2}-\frac{1}{2}+\ldots \quad \Rightarrow \quad K_2\left(\frac{m}{T}\right) \simeq 2\left(\frac{T}{m}\right)^2\,. \tag{23}$$

Inserting this into the growth rate gives

$$\Gamma \simeq -\Gamma_0\frac{\pi\gamma_{\rm ph}^3(v_{\rm ph}-v_b)}{4\gamma_b}\left(\frac{m}{T}\right)^3\frac{e^{-\mathcal{G}}}{\mathcal{G}^3} \simeq \Gamma_0\frac{\pi}{27}e^{-3m/2\sqrt{2}T} \simeq 0.1\Gamma_0\,. \tag{24}$$

This is a little less than found in the cold beam case, though only marginally so. Most importantly, the general form of growth rate is identical.

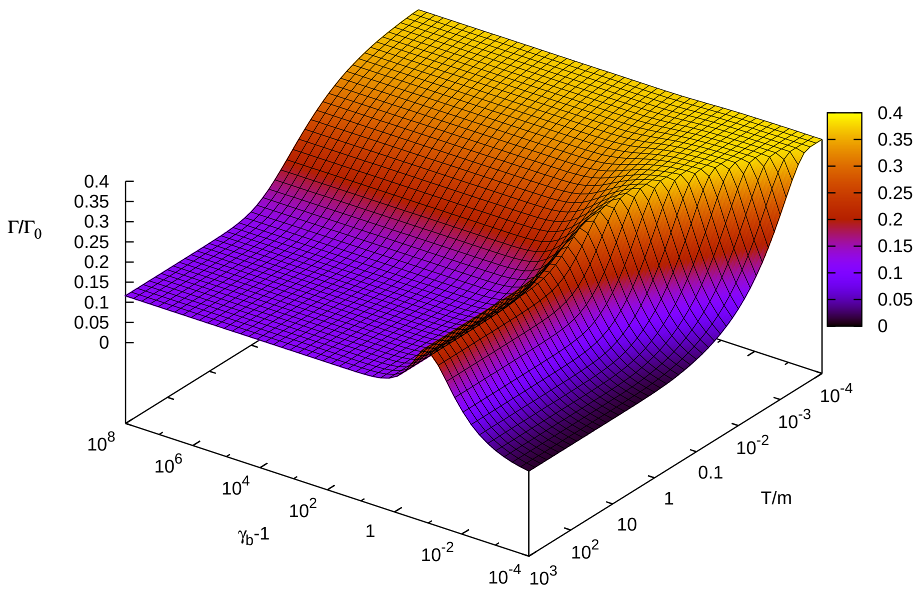

Figure 1: Kinetic growth rate, normalized by $\Gamma_0 \equiv \omega_{Pt}(n_b/n_t)\gamma_b m v_b^2/T$ as a function of T (in units of m) and γ_b. Note the evolution from 0.4 at low T to 0.1 at high T.

Arbitrary Beam Temperature

Finding the general maximum of the growth rate in Equation (12) is sufficiently difficult that I have not been able to do it analytically. However, we may still do this numerically. Figure 1 shows this as a function of T and γ_b over ranges large in both.

Approximate Oblique Instability

Here we take a stab at the first corrections due to the Oblique instability to the growth rate. The important new aspect is allowing $\mathbf{k}$ and $\mathbf{v}_b$ to only be approximately parallel. We can do this in a couple different ways, but here we will assume that it is $\mathbf{v}_b$, and correspondingly $\mathbf{p}_b$, that take on the non-z component. In this case, we may set $\mathbf{v}_b = v_{bz}\hat{\mathbf{z}} + v_{bx}\hat{\mathbf{x}}$. We will approximate the more general oblique instability by assuming $|v_{bx}| \ll |v_{bz}|$.

In this frame we retain the location of the pole: $p_z = \gamma_{\rm ph} v_{\rm ph} \epsilon_\perp$ and $\epsilon = \gamma_{\rm ph}\epsilon_\perp$, where $\epsilon_\perp = \sqrt{m^2 + p_\perp^2}$. We also have

$$\frac{\partial}{\partial p_z}(\omega - k v_z)\bigg|_{\rm pole} = -k\left(\frac{1}{\epsilon} - \frac{p_z^2}{\epsilon^3}\right)\bigg|_{\rm pole} = -\frac{k\epsilon_\perp^2}{\epsilon^3}\bigg|_{\rm pole} = -\frac{k}{\gamma_{\rm ph}^3 \epsilon_\perp}\,. \tag{25}$$

Similar, but slightly different, we also have

$$\gamma_b(\epsilon - v_{bz}p_z - v_{bx}p_x)/T|_{\rm pole} = \gamma_b\gamma_{\rm ph}(1 - v_{bz}v_{\rm ph})\epsilon_\perp/T - \gamma_b v_{bx} p_x/T\,, \tag{26}$$

and therefore,

$$\begin{aligned}\mathbf{k}\cdot\frac{\partial f_b}{\partial \mathbf{p}}\bigg|_{\rm pole} &= k\frac{\partial f_b}{\partial p_z}\bigg|_{\rm pole} = -\frac{k\gamma_b}{T}\left(\frac{p_z}{\epsilon} - v_{bz}\right) f_b\bigg|_{\rm pole} \\ &= -\frac{k(v_{\rm ph} - v_{bz})e^{-\gamma_b\gamma_{\rm ph}(1-v_{bz}v_{\rm ph})\epsilon_\perp/T} e^{\gamma_b v_{bx} p_x/T}}{4\pi m^2 T^2 K_2(m/T)}\,,\end{aligned} \tag{27}$$

where $v_b \to v_{bz}$ and the only new term is the final exponential factor which depends upon p_x.

In terms of the above we have for the residue term:

$$R = \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_{bz})}{4\pi m T^2 K_2(m/T)} \int d^2p_\perp \epsilon_\perp e^{-\gamma_b\gamma_{\rm ph}(1-v_{bz}v_{\rm ph})\epsilon_\perp/T} e^{\gamma_b v_{bx} p_x/T}\,. \tag{28}$$

Unlike the $\mathbf{k} \parallel \mathbf{v}_b$ case, we now have an asymmetric term within the integral. To perform this explicitly, let us write this in polar coordinates, $p_\perp$ and $\phi_\perp$:

$$\begin{aligned} R &= \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_{bz})}{4\pi m T^2 K_2(m/T)} \int_0^{2\pi} d\phi_\perp \int_0^\infty dp_\perp p_\perp \epsilon_\perp e^{-\gamma_b\gamma_{\rm ph}(1-v_{bz}v_{\rm ph})\epsilon_\perp/T} e^{\gamma_b v_{bx} p_\perp \cos\phi_\perp/T} \\ &= \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_{bz})}{4\pi m T^2 K_2(m/T)} \int_0^\infty dp_\perp p_\perp \epsilon_\perp e^{-\gamma_b\gamma_{\rm ph}(1-v_{bz}v_{\rm ph})\epsilon_\perp/T} \int_0^{2\pi} d\phi_\perp e^{\gamma_b v_{bx} p_\perp \cos\phi_\perp/T} \\ &= \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_{bz})}{4 m T^2 K_2(m/T)} \int_0^\infty dp_\perp^2 \epsilon_\perp e^{-\gamma_b\gamma_{\rm ph}(1-v_{bz}v_{\rm ph})\epsilon_\perp/T} I_0\left(\frac{\gamma_b v_{bx} p_\perp}{T}\right)\,,\end{aligned} \tag{29}$$

where $I_0(z)$ is the 0th order modified Bessel function.

Performing the remaining integral is difficult. However, since we are only interested in $v_{bx} \ll 1/\gamma_b \ll v_{bz} \simeq 1$, we can use the small-$z$ expansion of $I_0(z)$:

$$I_0\left(\frac{\gamma_b v_{bx} p_\perp}{T}\right) \simeq 1 + \frac{1}{4}\left(\frac{\gamma_b v_{bx} p_\perp}{T}\right)^2 + \frac{1}{64}\left(\frac{\gamma_b v_{bx} p_\perp}{T}\right)^4 + \dots, \tag{30}$$

which could have easily been derived by expanding the p_x term in the $\phi_\perp$ integral in R above. Thus, we have

$$R \simeq \frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_{bz})}{4mT^2K_2(m/T)} \int_0^\infty dp_\perp^2 \epsilon_\perp e^{-\gamma_b\gamma_{\rm ph}(1-v_{bz}v_{\rm ph})\epsilon_\perp/T}\left[1 + \frac{\gamma_b^2 v_{bx}^2}{4T^2}p_\perp^2\right]. \tag{31}$$

In addition to the integral in the parallel case, the first order oblique term involves an integral of the form

$$\int_0^\infty dx\, x\sqrt{1+x}e^{-a\sqrt{1+x}} = \frac{4}{a^5}\left(a^3 + 5a^2 + 12a + 12\right). \tag{32}$$

Defining $\mathcal{G} \equiv \gamma_b\gamma_{\rm ph}(1 - v_{bz}v_{\rm ph})m/T$, analogous to the parallel case, we have then

$$R \simeq \frac{\gamma_{\rm ph}^3 m^2(v_{\rm ph} - v_{bz})}{2T^2K_2(m/T)}\left(\frac{\mathcal{G}^2 + 2\mathcal{G} + 2}{\mathcal{G}^3} + \frac{\gamma_b^2 v_{bx}^2}{2}\frac{m^2}{T^2}\frac{\mathcal{G}^3 + 5\mathcal{G}^2 + 12\mathcal{G} + 12}{\mathcal{G}^5}\right)e^{-\mathcal{G}}, \tag{33}$$

with the corresponding growth rate

$$\Gamma = -\Gamma_0\frac{\pi\gamma_{\rm ph}^3 m(v_{\rm ph} - v_{bz})}{4\gamma_b T K_2(m/T)}\left(\frac{\mathcal{G}^2 + 2\mathcal{G} + 2}{\mathcal{G}^3} + \frac{\gamma_b^2 v_{bx}^2}{2}\frac{m^2}{T^2}\frac{\mathcal{G}^3 + 5\mathcal{G}^2 + 12\mathcal{G} + 12}{\mathcal{G}^5}\right)e^{-\mathcal{G}}, \tag{34}$$

where here $\Gamma_0 \equiv \omega_{Pt}(n_b/n_t)\gamma_b m v_{bz}^2/T$, i.e., again v_{bz} replaces v_z.

As with the parallel case, computing the maximum of this is difficult. Thus again we will restrict ourselves to various limiting cases.

Cold Beams ($T \ll m$)

At low T, and thus large $\mathcal{G}$, this reduces to

$$\Gamma \simeq -\Gamma_0\sqrt{\frac{\pi}{8}}\left(\frac{m}{T}\right)^{3/2}\frac{\gamma_{\rm ph}^3(v_{\rm ph} - v_{bz})}{\gamma_b}\left(\frac{1}{\mathcal{G}} + \frac{\gamma_b^2 v_{bx}^2}{2}\frac{m^2}{T^2}\frac{1}{\mathcal{G}^2}\right)e^{m/T-\mathcal{G}}. \tag{35}$$

The updated approximate form of δp being

$$\begin{aligned}
\frac{\delta p}{m} &= \gamma_{\rm ph} v_{\rm ph} - \gamma_b v_{bz} \\
&= \frac{v_{bz} + \delta v}{\sqrt{1 - v_b^2 - 2v_{bz}\delta v + \ldots}} - \gamma_b v_{bz} \\
&\simeq \gamma_b (v_{bz} + \delta v)\left(1 + \gamma_b^2 v_{bz}\delta v\right) - \gamma_b v_{bz} \\
&= \left(\gamma_b + \gamma_b^3 v_{bz}^2\right)\delta v \\
&= \gamma_b^3 (1 - v_{bx}^2)\delta v \\
&\simeq \gamma_b^3 \delta v\,,
\end{aligned} \tag{36}$$

and $\mathcal{G}$ unchanged with the above definition of δp, we have

$$\Gamma \simeq \Gamma_0 \sqrt{\frac{\pi}{8}} \left(\frac{m}{T}\right)^{1/2} \frac{\delta p}{\gamma_b m} \left(1 + \frac{\gamma_b^2 v_{bx}^2}{2}\right) e^{-\delta p^2/2\epsilon_b^2}\,. \tag{37}$$

which is also maximized when $\delta p = -\epsilon_b$, yielding,

$$\Gamma_M \simeq \Gamma_0 \sqrt{\frac{\pi}{8e}} \left(1 + \frac{\gamma_b^2 v_{bx}^2}{2}\right) \simeq 0.4 \left(1 + \frac{\gamma_b^2 v_{bx}^2}{2}\right) \Gamma_0\,. \tag{38}$$

Thus, when $v_{bx} \simeq 1/\gamma_b$, the growth rate is roughly 150% that of the beam plasma instability. Note that we have only assumed that $v_{bx} \ll v_{bz}$, and thus this has not been forbidden by our approximate expansions thus far. Unlike in the zero-temperature limit (i.e., when the principle value term dominates the beam contribution to the dispersion relation) the functional form does not change, with $\Gamma_M \propto \gamma_b/T$ in both cases.

Hot Beams ($T \gg m$)

At high T, and thus small $\mathcal{G}$, the growth rate is roughly

$$\Gamma = -\Gamma_0 \frac{\pi \gamma_{\rm ph}^3 (v_{\rm ph} - v_{bz})}{4\gamma_b} \left(\frac{m}{T}\right)^3 \frac{e^{-\mathcal{G}}}{\mathcal{G}^3} \left(1 + 3\frac{\gamma_b^2 v_{bx}^2}{\mathcal{G}^2}\frac{m^2}{T^2}\right)\,. \tag{39}$$

Assuming that v_{bx} is small, and thus doesn't appreciably affect the maximization, we may insert the $v_{\rm ph}$ found for the parallel case, finding

$$\Gamma_M \simeq \Gamma_0 \frac{\pi}{27} \left(1 + \frac{8}{3}\gamma_b^2 v_{bx}^2\right) e^{-3m/2\sqrt{2}T}\,. \tag{40}$$

Again, this is enhanced by a factor of order unity in comparison to the parallel case when $v_{bx} \simeq 1/\gamma_b$. In this case the enhancement is marginally larger, nearly a factor of 4.

General Oblique Instability

The above treatment was unsatisfactory because we were limited to $v_{bx} \ll 1/\gamma_b$ but this was precisely where we saw the largest effect. That is, it appears the largest enhancements are likely to be outside of the regime of validity to which we had restricted ourselves. However, it is possible to compute the integral in Equation (28) directly, using a little Lorentzian slight of hand. Here we do this.

The integral we wish to perform has the form:

$$I_\perp \equiv \int d^2p_\perp \epsilon_\perp e^{-\mathcal{G}(\epsilon_\perp - wp_x)/m} \tag{41}$$

where $w \equiv v_{bx}/\gamma_{\rm ph}(1-v_{bz}v_{\rm ph}) = \gamma_b v_{bx} m/\mathcal{G}T \leq 1$. That the inequality is guaranteed, regardless of v_{bx}, may be seen trivially form the form of Equation (26), which, being simply the energy in the beam frame, is always positive definite. However, this appears just as boosted distribution itself. Thus, we may imagine boosting by w along the x-axis, removing the anisotropic term from the exponential.

That is, set $p'_x = \gamma_w(p_x - w\epsilon_\perp)$ and $p'_y = p_y$, constituting a change of variables in p_x alone. Then, it is trivial to demonstrate:

$$\epsilon'_\perp = \gamma_w(\epsilon_\perp - wp_x) \quad \epsilon_\perp = \gamma_w(\epsilon'_\perp + wp'_x) \quad \text{and} \quad dp_x dp_y = \frac{\epsilon_\perp}{\epsilon'_\perp} dp'_x dp'_y\,. \tag{42}$$

Therefore, we have

$$\begin{aligned} I_\perp &= \int d^2p'_\perp \frac{\epsilon_\perp^2}{\epsilon'_\perp} e^{-\mathcal{G}'\epsilon'_\perp/m} = \gamma_w^2 \int d^2p'_\perp \left(\epsilon'_\perp + 2w\frac{p'_x}{\epsilon'_\perp} + w^2\frac{p'^2_x}{\epsilon'_\perp}\right) e^{-\mathcal{G}'\epsilon'_\perp/m} \\ &= \gamma_w^2 \int d^2p'_\perp \left(\epsilon'_\perp + \frac{w^2}{2}\frac{p'^2_\perp}{\epsilon'_\perp}\right) e^{-\mathcal{G}'\epsilon'_\perp/m} \\ &= \pi\gamma_w^2 \int_0^\infty dp'^2_\perp \left(\epsilon'_\perp + \frac{w^2}{2}\frac{p'^2_\perp}{\epsilon'_\perp}\right) e^{-\mathcal{G}'\epsilon'_\perp/m}\,, \end{aligned} \tag{43}$$

where $\mathcal{G}' \equiv \mathcal{G}/\gamma_w$ and in the second line we used the fact that $\epsilon'_\perp e^{-\mathcal{G}'\epsilon'_\perp/m}$ is isotropic in $\mathbf{p}'_\perp$. Here we make use of both the earlier integral and

$$\int_0^\infty dx \frac{x}{\sqrt{1+x}} e^{-a\sqrt{1+x}} = \frac{4(a+1)}{a^3} e^{-a}\,, \tag{44}$$

to obtain

$$I_\perp = \frac{2\pi\gamma_w^2 m^3}{\mathcal{G}^3}\left[\left(a^2 + 2a + 2\right) + \frac{w^2}{2}(2a+2)\right]. \tag{45}$$

Inserting this into the definition for R then yields

$$R = \frac{\gamma_{\rm ph}^3 m^2 (v_{\rm ph} - v_{bz})}{2T^2 K_2(m/T)} \gamma_w^2 \left[\left(\mathcal{G}'^2 + 2\mathcal{G}' + 2 \right) + \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}^2} \frac{m^2}{T^2} (2\mathcal{G}' + 2) \right] \frac{e^{-\mathcal{G}'}}{\mathcal{G}'^3} \tag{46}$$

and therefore,

$$\Gamma \simeq -\Gamma_0 \frac{\pi \gamma_w^2 \gamma_{\rm ph}^3 m (v_{\rm ph} - v_{bz})}{4\gamma_b T K_2(m/T)} \left[\left(\mathcal{G}'^2 + 2\mathcal{G}' + 2 \right) + \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}^2} \frac{m^2}{T^2} (2\mathcal{G}' + 2) \right] \frac{e^{-\mathcal{G}'}}{\mathcal{G}'^3} \, . \tag{47}$$

The above differs from the beam plasma instability only due to the minor difference in the definition of $\mathcal{G}'$ and the second term. When $v_{bx} = 0$, it is precisely the beam plasma instability growth rate. When $v_{bx} \ll 1/\gamma_b \simeq 1/\gamma_{\rm ph}$,

$$\gamma_w \simeq 1 + \frac{v_{bx}^2}{2\gamma_{\rm ph}^2 (1 - v_{bz} v_{\rm ph})^2} = 1 + \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}^2} \frac{m^2}{T^2} \quad \Rightarrow \quad \mathcal{G}' \simeq \mathcal{G} - \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}} \frac{m^2}{T^2} \tag{48}$$

and thus, to lowest order in $\gamma_b v_{bx}$,

$$\begin{aligned}
\Gamma &\simeq -\Gamma_0 \frac{\pi \gamma_{\rm ph}^3 m (v_{\rm ph} - v_{bz})}{4\gamma_b T K_2(m/T)} \frac{e^{-\mathcal{G}}}{\mathcal{G}^3} \left(1 + \frac{\gamma_b^2 v_{bx}^2}{\mathcal{G}^2} \frac{m^2}{T^2} \right) \\
&\qquad \times \left[\left(\mathcal{G}^2 + 2\mathcal{G} + 2 - \gamma_b^2 v_{bx}^2 \frac{m^2}{T^2} - \frac{\gamma_b^2 v_{bx}}{\mathcal{G}} \frac{m^2}{T^2} \right) + \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}^2} \frac{m^2}{T^2} (2\mathcal{G} + 2) \right] \\
&\qquad\qquad \times \left(1 + 3 \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}^2} \frac{m^2}{T^2} \right) \left(1 + \frac{\gamma_b^2 v_{bx}^2}{2\mathcal{G}} \frac{m^2}{T^2} \right) \\
&\simeq -\Gamma_0 \frac{\pi \gamma_{\rm ph}^3 m (v_{\rm ph} - v_{bz})}{4\gamma_b T K_2(m/T)} \frac{e^{-\mathcal{G}}}{\mathcal{G}^3} \\
&\qquad \times \left(\mathcal{G}^2 + 2\mathcal{G} + 2 + \frac{\gamma_b^2 v_{bx}^2}{2} \frac{m^2}{T^2} \frac{2 - 2\mathcal{G}^2}{\mathcal{G}^2} \right) \left(1 + \frac{\gamma_b^2 v_{bx}^2}{2} \frac{m^2}{T^2} \frac{\mathcal{G} + 5}{\mathcal{G}^2} \right) \\
&\simeq -\Gamma_0 \frac{\pi \gamma_{\rm ph}^3 m (v_{\rm ph} - v_{bz})}{4\gamma_b T K_2(m/T)} e^{-\mathcal{G}} \\
&\qquad \times \left(\frac{\mathcal{G}^2 + 2\mathcal{G} + 2}{\mathcal{G}^3} + \frac{\gamma_b^2 v_{bx}^2}{2} \frac{m^2}{T^2} \frac{\mathcal{G}^3 + 5\mathcal{G}^2 + 12\mathcal{G} + 12}{\mathcal{G}^5} \right) ,
\end{aligned} \tag{49}$$

as anticipated by the preceding section.

As before, we wish to maximize this. However now we would do so simultaneously in both v_{bx}^2 (corresponding to $k_{bx}^2 v_b^2 / k^2$) and $v_{\rm ph}$. While this may be

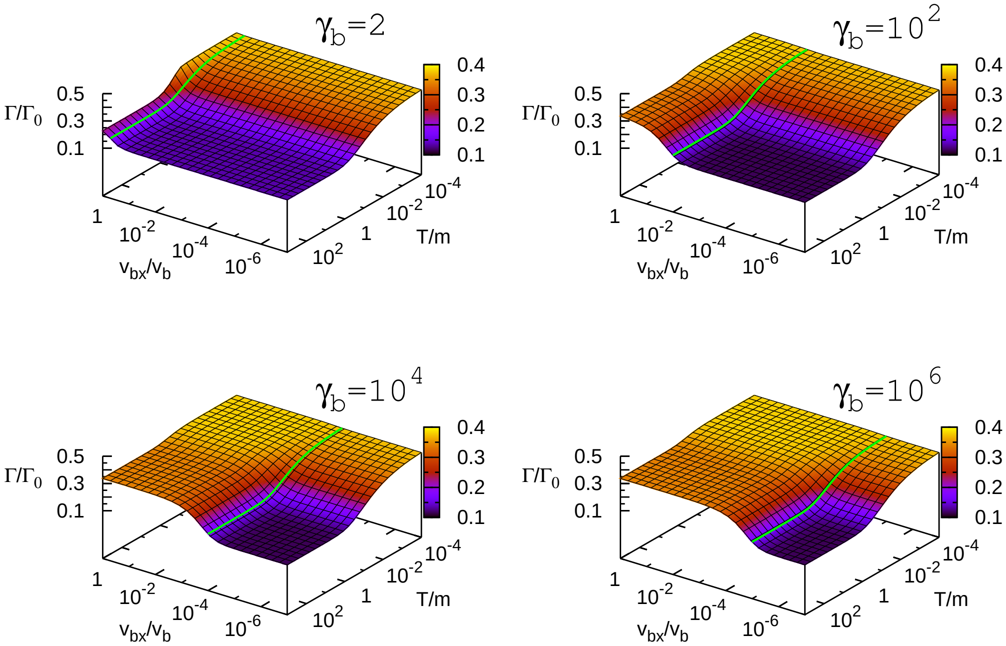

Figure 2: Oblique growth rate maximized over $v_{\rm ph}$ as a function of v_{bx} and T for various γ_b. In each the green line shows the chord along which $v_{bx}/v_b = 1/\gamma_b$. In all cases, the maximum growth rate occurs when $v_{bx}/v_b \gg 1/\gamma_b$.

possible in principle, in practice this has proved prohibitively difficult analytically (i.e., I can't seem to do it easily). However, from numerical calculations, shown in Figure 2, where the growth rate is maximized over $v_{\rm ph}$, it appears the maximum growth rate for a given γ_b, and T, occurs at $v_{bx}/v_b \lesssim 1/\gamma_b$, the limit of the region of validity of the approximations made in the preceding section. This suggests that we might estimate the maximal oblique growth rates by taking the limit as $v_{bx} \gg v_b/\gamma_b$.

However, obtaining asymptotic expansions for the growth rate in the high and low temperature limits that can be easily solved for the maximum growth rate is difficult. Even with the tremendous numerical insight provided by Figure 2, it is unclear how to make significant progress. However, some general remarks can be

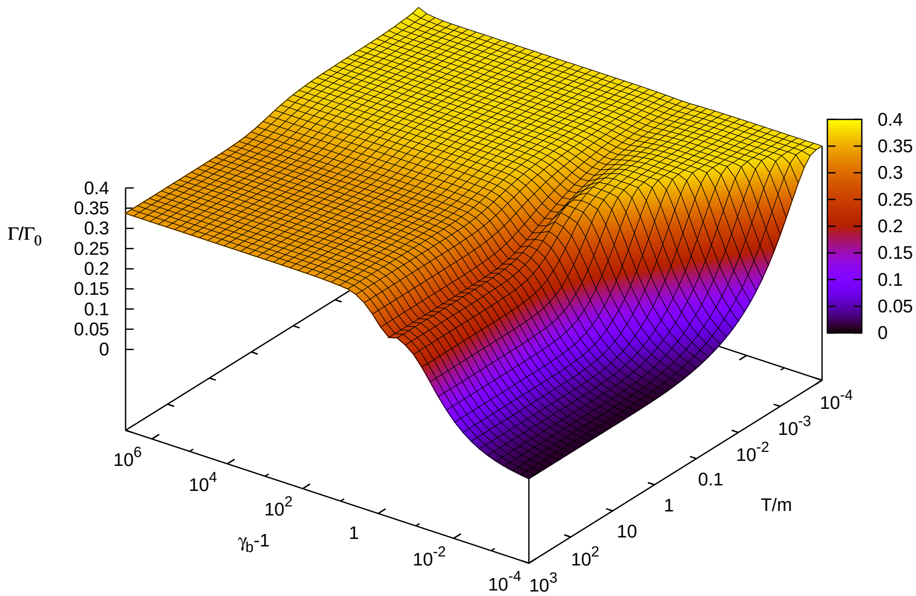

Figure 3: Oblique kinetic growth rate, maximized over v_{bx}/v_b and ω/k, normalized by $\Gamma_0 \equiv \omega_{Pt}(n_b/n_t)\gamma_b m v_b^2/T$ as a function of T (in units of m) and γ_b. Note that unlike the beam plasma case shown in Figure 1, at high γ_b the transition between high and low temperature is only marginal, constituting a roughly 10% reduction.

made. It appears at the maximum growth rate, we have to leading order:

$$v_{\rm ph} \simeq \sqrt{2\delta Z_x}\,,\quad v_{bx} = (1-\delta Z_x)v_b\,,\quad v_{bz} \simeq \sqrt{2\delta Z_x}\,v_b\,,$$
$$v_{bz} - v_{\rm ph} \simeq \begin{cases} \dfrac{\sqrt{T/m}}{\gamma_b} & T \ll m \\ \dfrac{1}{2\gamma_b} & T \gg m \end{cases}\,,\quad \gamma_{\rm ph} \simeq 1 + \delta Z_x\,,\quad \mathcal{G}' \sim \frac{m}{T}\,, \tag{50}$$
$$\text{and}\quad w \sim 1\,,$$

where $\delta Z_x \equiv 1 - v_{bx}/v_b$. Inserting these into Equation (47) suffices to show that there exists growth rates that are $\propto \Gamma_0$, with

$$\Gamma_M \simeq \begin{cases} 0.38\Gamma_0 & T \ll m \\ 0.34\Gamma_0 & T \gg m \end{cases}\,, \tag{51}$$

for $\gamma_b \gtrsim 10$, as seen in Figure (2).

Maximizing over v_{bx}/v_b as well results in the growth rates show in Figure 3. Unlike the beam plasma case, the difference between hot and cold relativistic beams is much more marginal, differing by roughly 10%. At low γ_b, a difference still persists, but is dominated by the effective destruction of the beam itself as $T/m \gg \gamma_b$.