COMP 256: Discrete Structures

Logic, Part 1

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Propositions

Propositions are statements that are either *true* or *false*.

$$
\begin{aligned}
p &= \text{“The weather is lovely.”} \\
q &= \text{“I will go running.”} \\
r &= \text{“Where's the beef?”} \\
s &= \text{“Order me a hamburger.”}
\end{aligned}
$$

p and q are propositions, r and s are not.

Compound Statements

Propostions may be joined by logical connectors to form *compound statements*.

$$\begin{aligned} p &= \text{“The weather is lovely.”} \\ q &= \text{“I will go running.”} \\ r &= \text{“I will go swimming.”} \end{aligned}$$

$$\begin{aligned} \neg p &= \text{“The weather is not lovely.”} \\ p \wedge q &= \text{“The weather is lovely and I will go running.”} \\ p \vee r &= \text{“The weather is lovely or I will go swimming.”} \\ q \oplus r &= \text{“I will go running or I will go swimming.”} \end{aligned}$$

Truth Tables

If things get confusing, use a *truth table* to figure out the value of a statement.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$
F	F	T	F	F	F
F	T	T	F	T	T
T	F	F	F	T	T
T	T	F	T	T	F

Example: What are the possible values for $\neg(p \oplus q) \vee (p \wedge q)$?

p	q	$p \oplus q$	$\neg(p \oplus q)$	$p \wedge q$	$\neg(p \oplus q) \vee (p \wedge q)$
F	F	F	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
T	T	F	T	T	T

Conditional Statements

$$\begin{aligned} p &= \text{“The weather is lovely.”} \\ q &= \text{“I will go running.”} \\ p \to q &= \text{“If the weather is lovely, then I will go running.”} \end{aligned}$$

Here, p is known as the *hypothesis*, and q the *conclusion*.

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

If the weather is not lovely (that is, $\neg p$), we say that $p \to q$ is *vacuously* true.

Converse, Inverse, Contrapositive

There are some common statements relating to the conditional $p \to q$:

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
F	F	T	T	T	T
F	T	T	F	F	T
T	F	F	T	T	F
T	T	T	T	T	T

Biconditional

The *biconditional* is a conjunction of a conditional, and its converse:

$$p \leftrightarrow q \equiv p \rightarrow q \wedge q \rightarrow p$$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

"I will go running *iff* the weather is lovely."

Tautology/Contradiction

A *tautology* is a logical statement that is always true:

$$p \vee \neg p$$

A *contradition* is a logical statement that is always false:

$$p \wedge \neg p$$

Logical Equivalence

Two expressions may be said to be *logically equivalent* if their truth values are the same for every combination of propositional values:

$$\neg(p \oplus q) \equiv \neg(p \oplus q) \vee (p \wedge q)$$

p	q	$p \oplus q$	$\neg(p \oplus q)$	$p \wedge q$	$\neg(p \oplus q) \vee (p \wedge q)$
F	F	F	T	F	T
F	T	T	F	F	F
T	F	T	F	F	F
T	T	F	T	T	T

De Morgan's Laws

Two of the most famous logical equivalences are *De Morgan's Laws*:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

p	q	$\neg(p \wedge q)$	$\neg(p \vee q)$	$\neg p \vee \neg q$	$\neg p \wedge \neg q$
F	F	T	T	T	T
F	T	T	F	T	F
T	F	T	F	T	F
T	T	F	F	F	F

Idempotency

Another basic law if logical equivalence, *idempotency* states that a proposition does not change its value when involved in a conjunction:

$$p \wedge p \equiv p$$
$$p \vee p \equiv p$$

Associativity

Much like $+$ and $\times$, $\vee$ and $\wedge$ obey associativity laws:

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$
$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

Commutativity

Again, like $+$ and $\times$, $\vee$ and $\wedge$ obey commutativity laws:

$$p \vee q \equiv q \vee p$$
$$p \wedge q \equiv q \wedge p$$

Distributivity

Both $\vee$ and $\wedge$ distribute over parenthesis:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Identity

Much like 0 for $+$, and 1 for $\times$, *false* is an identity value for $\vee$, and *true* for $\wedge$:

$$p \vee \mathsf{F} \equiv p$$

$$p \wedge \mathsf{T} \equiv p$$

Domination

We can draw conclusions about certain logical statements without ever knowing the value of some propositions:

$$p \vee \mathsf{T} \equiv \mathsf{T}$$
$$p \wedge \mathsf{F} \equiv \mathsf{F}$$

Conditional Identities

Reconsider the idea of vacuous truth for a conditional. From this, we can conclude that if the hypothesis is *false*, a conditional statement is *true*. Otherwise, that conditional is dependent on the conclusion (takes its value):

$$p \rightarrow q \equiv \neg p \vee q$$

Similarly, we may derive (from many of the rules previously stated):

$$\begin{aligned}
p \leftrightarrow q &\equiv p \rightarrow q \wedge q \rightarrow p \\
&\equiv (\neg p \vee q) \wedge (\neg q \vee p) \\
&\equiv (\neg q \wedge (\neg p \vee q)) \vee (p \wedge (\neg p \vee q)) \\
&\equiv ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \vee ((p \wedge \neg p) \vee (p \wedge q)) \\
&\equiv ((\neg q \wedge \neg p) \vee \mathrm{F}) \vee (\mathrm{F} \vee (p \wedge q)) \\
&\equiv (p \wedge q) \vee (\neg p \wedge \neg q)
\end{aligned}$$

TEX

Do you want to learn TEX this semester?

- Time during labs will be devoted to introducing typesetting.
- *All* homeworks to be turned in as PDF, formatted using TEX.