COMP 256: Discrete Structures

Logic, Part 2

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Predicates

Propositions by themselves are a bit limited. We need a way to talk about larger *sets* of objects.
A *predicate* is a *function* from some *domain of discourse* to *true* or *false* (its *range*).

$$P(x) = \text{“}x \text{ is an odd number.”}$$

x	$P(x)$
1	T
2	F
$\vdots$	$\vdots$

Universal Quantifier

To make a general statement about a domain, we may want a predicate to hold *for all* elements of the domain:

$$P(x) = \text{“}x \text{ is divisible by 5.”}$$
$$\forall x P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

Note that this statement $\forall x P(x)$ is true or false, and does not depend on any value of x. Instead, its value depends entirely on the domain of discourse D:

$$D = \{5, 10, 15, 20\}$$
$$\forall x P(x) = \text{True}$$

$$D = \{1, 2, 3, 4, 5\}$$
$$\forall x P(x) = \text{False}$$

Existential Quantifier

Another statement we might make has to do with *some* value in the domain of discourse:

$$P(x) = \text{“}x \text{ is odd.”}$$
$$\exists x P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Again, this type of statement depends entirely on the domain D:

$$D = \{2, 4, 6, 8, 10\}$$
$$\exists x P(x) = \text{False}$$

$$D = \{2, 4, 5, 6, 8\}$$
$$\exists x P(x) = \text{True}$$

Nested Quantifiers

Quantifiers may be combined for predicates with more than one variable:

$$
\begin{aligned}
P(x, y) &= x > y \\
\exists x \exists y P(x, y) &= \text{True} \\
\forall x \exists y P(x, y) &= \text{True} \\
\exists x \forall y P(x, y) &= \text{True} \\
\forall x \forall y P(x, y) &= \text{False}
\end{aligned}
$$

Quantified Statements

Predicates may be combined using logic operators, just like propositions.

$$\begin{aligned}
P(x) &= \text{“}x\text{ is even.”} \\
Q(x) &= \text{“}x\text{ is prime.”} \\
P(x) \wedge Q(x) &= \text{“}x\text{ is both even and prime.”} \\
\exists x(P(x) \wedge Q(x)) &= \text{“There's an }x\text{ that is even and prime.”}
\end{aligned}$$

Negation

Using De Morgan's laws, it is easy to derive the negation of any quantifier:

$$\begin{aligned}
\neg\forall x P(x) &\equiv \neg(P(x_1) \wedge \ldots \wedge P(x_n)) \\
&\equiv \neg P(x_1) \vee \ldots \vee \neg P(x_n) \\
&\equiv \exists x \neg P(x) \\
\neg\exists x P(x) &\equiv \neg(P(x_1) \vee \ldots \vee P(x_n)) \\
&\equiv \neg P(x_1) \wedge \ldots \wedge \neg P(x_n) \\
&\equiv \forall x \neg P(x)
\end{aligned}$$

This can be applied to nested quantifiers as well:

$$\begin{aligned}
\neg\forall x \exists y \forall z P(x, y, z) &\equiv \exists x \neg\exists y \forall z P(x, y, z) \\
&\equiv \exists x \forall y \neg\forall z P(x, y, z) \\
&\equiv \exists x \forall y \exists z \neg P(x, y, z)
\end{aligned}$$

Free and Bound Variables

All variables within a predicate statement default to *free*. Recall:

$$P(x) = \text{“}x \text{ is odd.”}$$

x	$P(x)$
1	T
2	F
$\vdots$	$\vdots$

A variable is said to be *bound* whenever it is assigned a value (or set of values as in quantification):

$$\begin{aligned} P(5) &= \text{True} \\ \forall x P(x) &= \text{False} \\ \exists x P(x) &= \text{True} \end{aligned}$$

Expressing Uniqueness

It is often useful to make statements such as "x is the only value such that $P(x)$". How do we accomplish this using only the tools we have?
If x is the *only* value such that $P(x)$, we know that $\forall y \neg P(y)$ when $y \neq x$. We can then write:

$$\exists x P(x) \wedge \forall y (x \neq y \rightarrow \neg P(y))$$

Logical Arguments

We will want to construct valid *logical arguments* that present an *hypothesis* about our domain, and draw a *conclusion*.

$$h_1 \wedge h_2 \wedge \ldots h_n \rightarrow c$$

An argument is *valid* whenever its construction is a tautology:

$$p \wedge q \rightarrow p$$

p	q	$p \wedge q$	$p \wedge q \rightarrow p$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	T

This is an argument form known as *simplification*.

Logical Arguments (Continued)

Arguments may also be written in a more straight forward notation:

$$\begin{array}{rl} & p \\ & q \\ & r \\ \hline \therefore & p \wedge q \wedge r \end{array}$$

This is equivalent to writing:

$$p \wedge q \wedge r \rightarrow p \wedge q \wedge r$$

(Which is clearly a tautology.)

Modus Ponens

$$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

If we know the hypothesis is true, and we know that an implication is true, clearly the implication's conclusion must be true.

p	q	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	T

Modus Tollens

$$\begin{array}{rl} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array}$$

If we know a conclusion is false, and we know that an implication is true, clearly the implication's hypothesis must be false.

p	q	$\neg q \wedge (p \rightarrow q)$	$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$
F	F	T	T
F	T	F	T
T	F	F	T
T	T	F	T

Addition

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

If a proposition is true, then either it, or some other proposition is true.

p	q	$p \to p \vee q$
F	F	T
F	T	T
T	F	T
T	T	T

Simplification

$$\frac{p \wedge q}{\therefore p}$$

If two propositions are true, then one of those propositions is true.

p	q	$p \wedge q \rightarrow p$
F	F	T
F	T	T
T	F	T
T	T	T

Conjunction

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

If multiple propositions are true, than all of those propositions are true.

p	q	$p \wedge q \rightarrow p \wedge q$
F	F	T
F	T	T
T	F	T
T	T	T

Hypothetical Syllogism

$$\begin{array}{r} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Implication is transitive.

p	q	r	$p \to q \land q \to r$	$(p \to q \land q \to r) \to (p \to r)$
F	F	F	T	T
F	F	T	T	T
F	T	F	F	T
F	T	T	T	T
T	F	F	F	T
T	F	T	F	T
T	T	F	F	T
T	T	T	T	T

Disjunctive Syllogism

$$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

If at least one of two propositions is true, and the first proposition is not true, the second proposition must be true.

p	q	$(p \vee q) \wedge \neg p$	$(p \vee q) \wedge \neg p \to q$
F	F	F	T
F	T	T	T
T	F	F	T
T	T	F	T

Resolution

$$\begin{array}{r l} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

If p is false, q must be true for $p \vee q$ to be true. Likewise if p is true, r must be true in order for $\neg p \vee r$ to be true. Therefore, either q or r must be true.

p	q	r	$(p \vee q) \wedge (\neg p \vee r)$	$(p \vee q) \wedge (\neg p \vee r) \to q \vee r$
F	F	F	F	T
F	F	T	F	T
F	T	F	T	T
F	T	T	T	T
T	F	F	F	T
T	F	T	T	T
T	T	F	F	T
T	T	T	T	T

Universal Instantiation

$$\begin{array}{rl} & a \text{ is an arbitrary element in the domain} \\ & \forall x P(x) \\ \hline \therefore & P(a) \end{array}$$

If a predicate applies to all elements, it applies to any element.

Universal Generalization

$$\begin{array}{rl} & a \text{ is an arbitrary element in the domain} \\ & P(a) \\ \hline \therefore & \forall x P(x) \end{array}$$

If a predicate applies to any arbitrary element, it applies to all elements.

Existential Instantiation

$$\frac{\exists x P(x)}{\therefore a \text{ is an element } \wedge P(a)}$$

If there is an element for which a predicate holds, it should be nameable.

Existential generalization

$$\begin{array}{rl} & a \text{ is an element (possibly arbitrary)} \\ & P(a) \\ \hline \therefore & \exists x P(x) \end{array}$$

If there is an element for which the predicate holds, there is at least one element for which the predicate holds.

Affirming the Conclusion

$$\begin{array}{rl} & p \to q \\ & \underline{q \qquad\quad} \\ \therefore & p \end{array}$$

Just because a conclusion is true, does not make the hypothesis true:
If you study really hard, then you will pass.
You passed.
Therefore, you studied really hard.

Denying the Hypothesis

$$\begin{array}{rl} & p \to q \\ & \underline{\neg p \qquad} \\ \therefore & \neg q \end{array}$$

If the hypothesis is untrue, we can conclude nothing about the conclusion:
If you a mathematician, then you can understand logic.
You are not a mathematician.
Therefore, you cannot understand logic.

Begging the Question

I will teach you how to use this phrase correctly!

$$\begin{aligned} p &= \text{“Logic is confusing.”} \\ q &= \text{“You find logic confusing.”} \\ p \to q &= \text{“You find logic confusing because it is.”} \end{aligned}$$

Nobody ever declared that “henceforth, it shall be that logic is confusing!” Your confusion over logic is completely your own fault!