Chapter 22

Gauss's Law

PowerPoint® Lectures for
University Physics, 14th Edition
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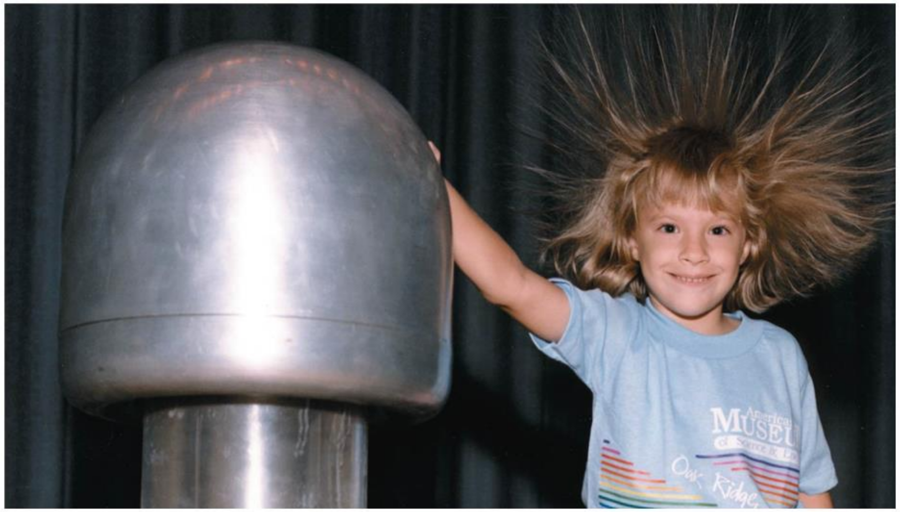
Learning Goals for Chapter 22

Looking forward at …

- how you can determine the amount of charge within a closed surface by examining the electric field on the surface.
- what is meant by electric flux, and how to calculate it.
- how Gauss's law relates the electric flux through a closed surface to the charge enclosed by the surface.
- how to use Gauss's law to calculate the electric field due to a symmetric charge distribution.
- where the charge is located on a charged conductor.

Introduction

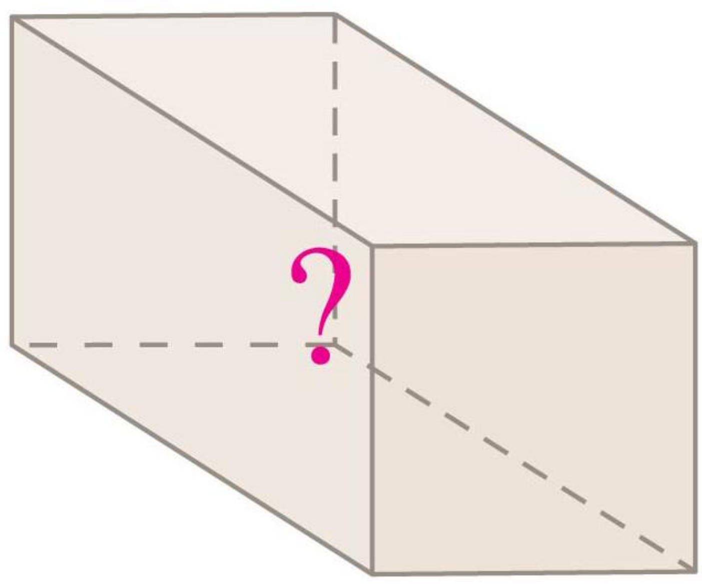
- This child acquires an electric charge by touching the charged metal shell.
- The charged hairs on the child's head repel and stand out.



- What would happen if the child stood *inside* a large, charged metal shell?
- Symmetry properties play an important role in physics.
- Gauss's law will allow us to do electric-field calculations using symmetry principles.

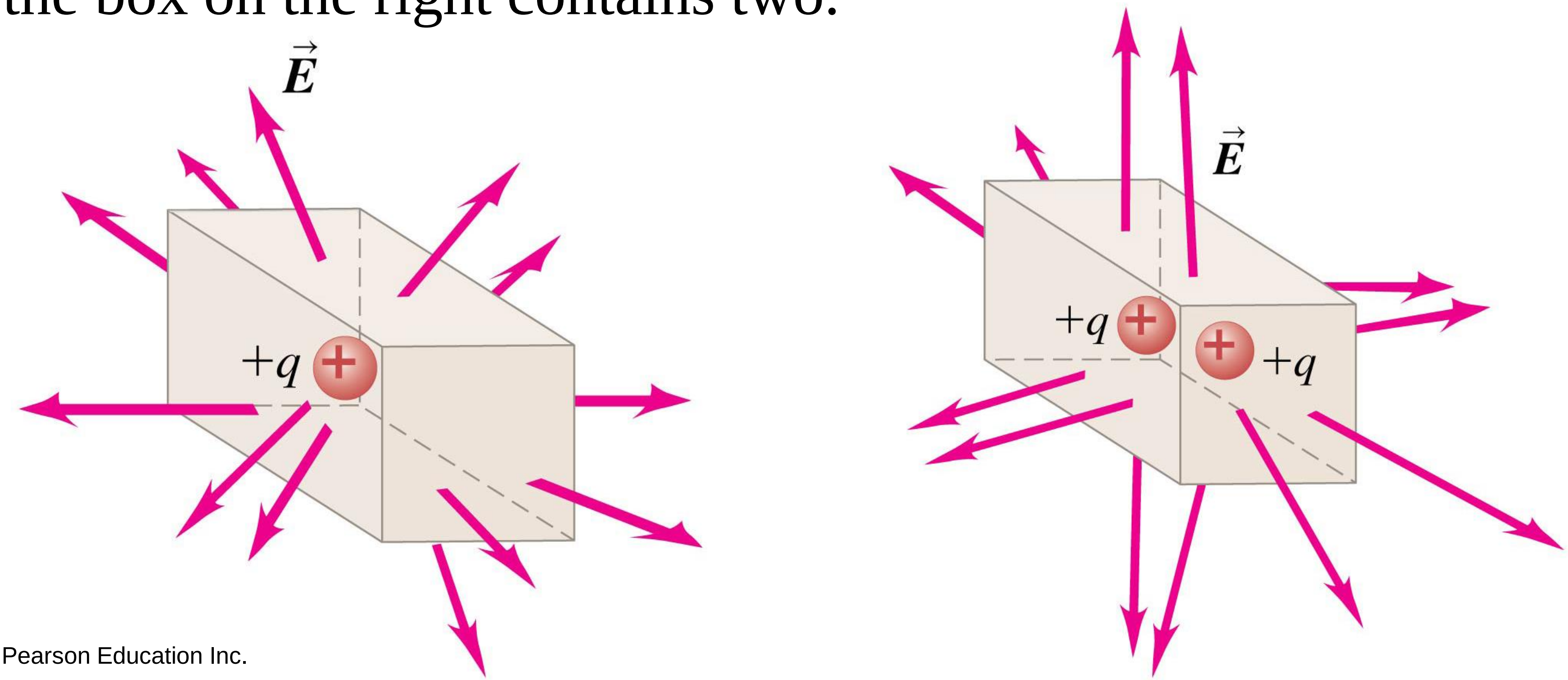
What is Gauss's law all about?

- Given any general distribution of charge, we surround it with an imaginary surface that encloses the charge.
- Then we look at the electric field at various points on this imaginary surface.
- Gauss's law is a relationship between the field at all the points on the surface and the total charge enclosed within the surface.



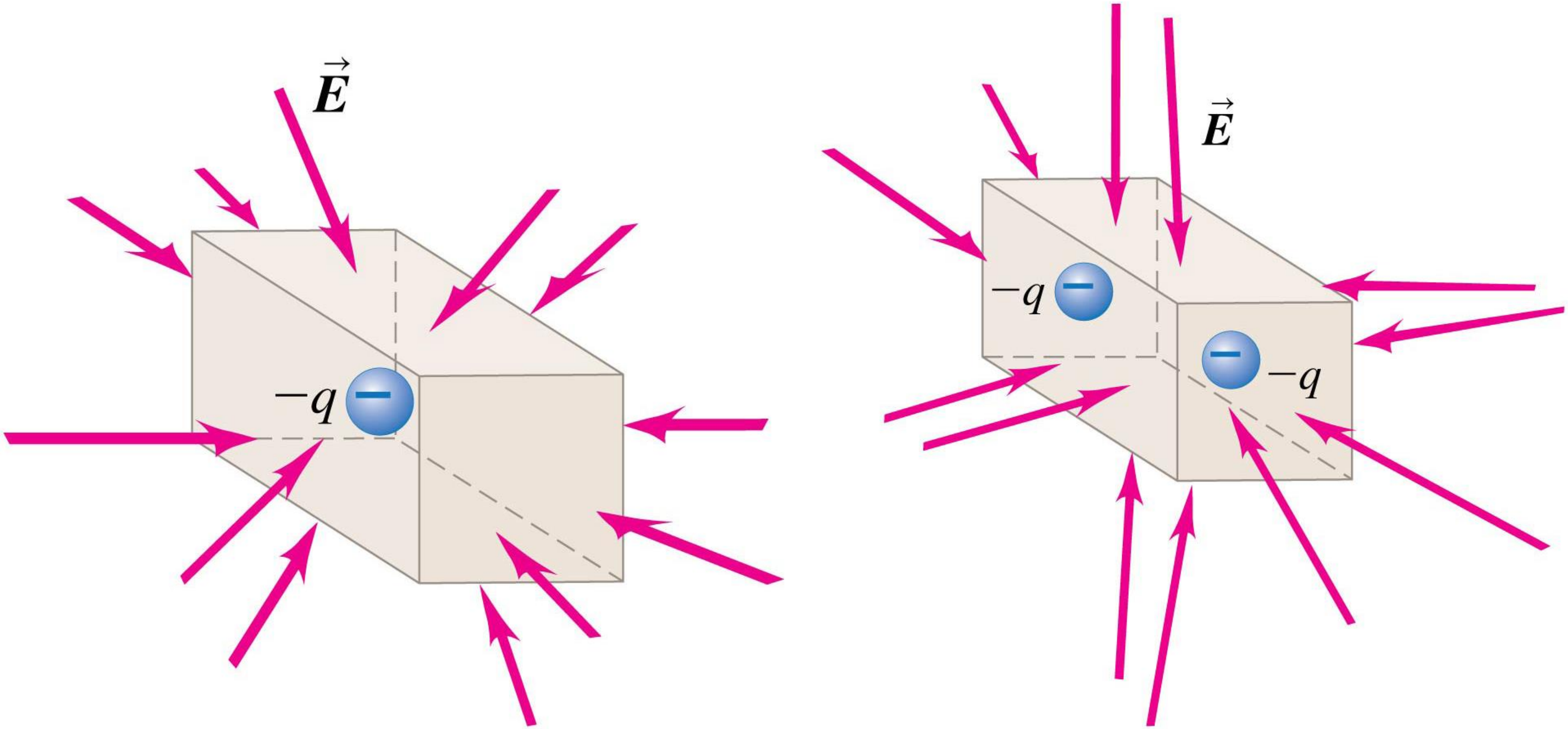
Charge and electric flux

- In both boxes below, there is a *positive* charge within the box, which produces an *outward* pointing **electric flux** through the surface of the box.
- The field patterns on the surfaces of the boxes are different in detail, since the box on the left contains one point charge, and the box on the right contains two.



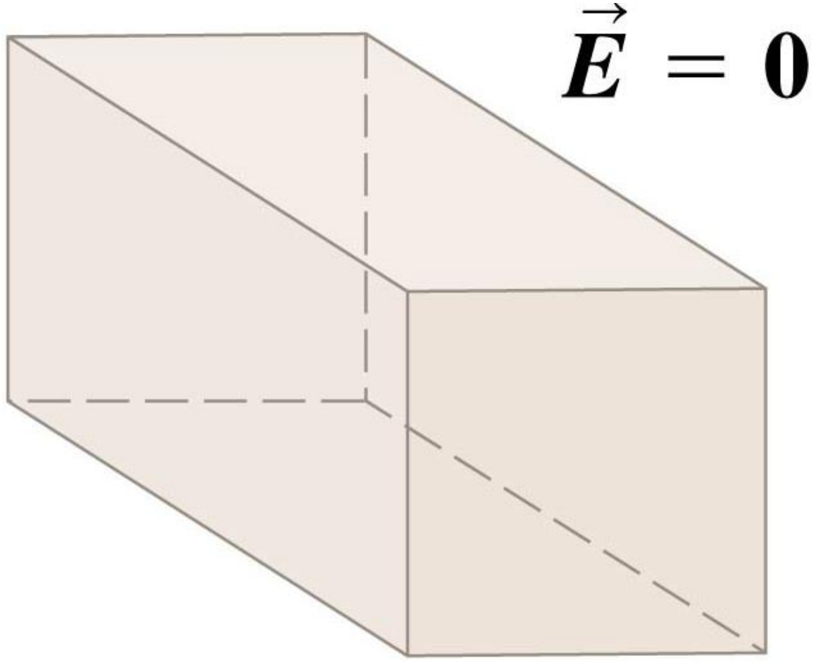
Charge and electric flux

- When there are negative charges inside the box, there is an *inward* pointing electric flux on the surface.



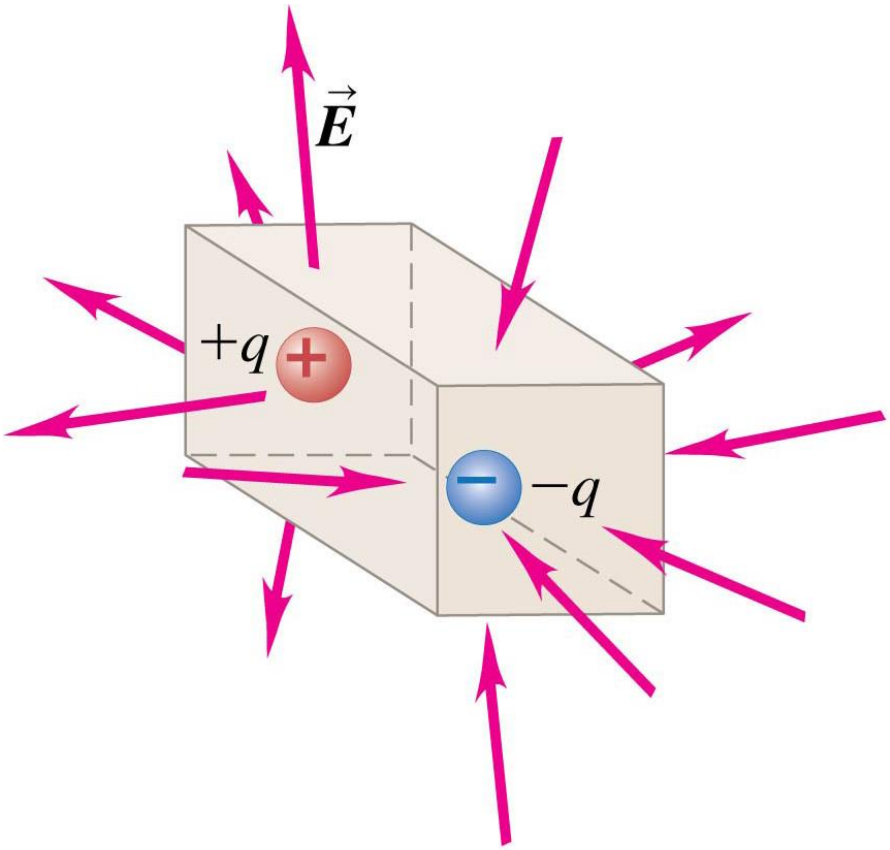
Zero net charge inside a box: Case 1 of 3

- What happens if there is zero charge inside the box?
- If the box is empty and the electric field is zero everywhere, then there is no electric flux into or out of the box.



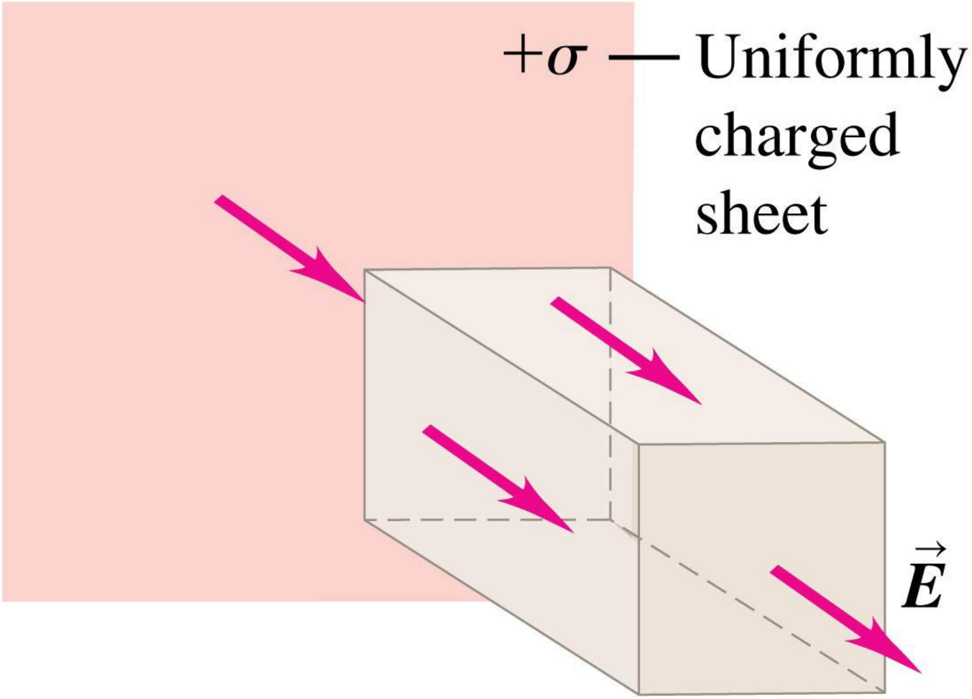
Zero net charge inside a box: Case 2 of 3

- What happens if there is zero *net* charge inside the box?
- There is an electric field, but it "flows into" the box on half of its surface and "flows out of" the box on the other half.
- Hence there is no *net* electric flux into or out of the box.



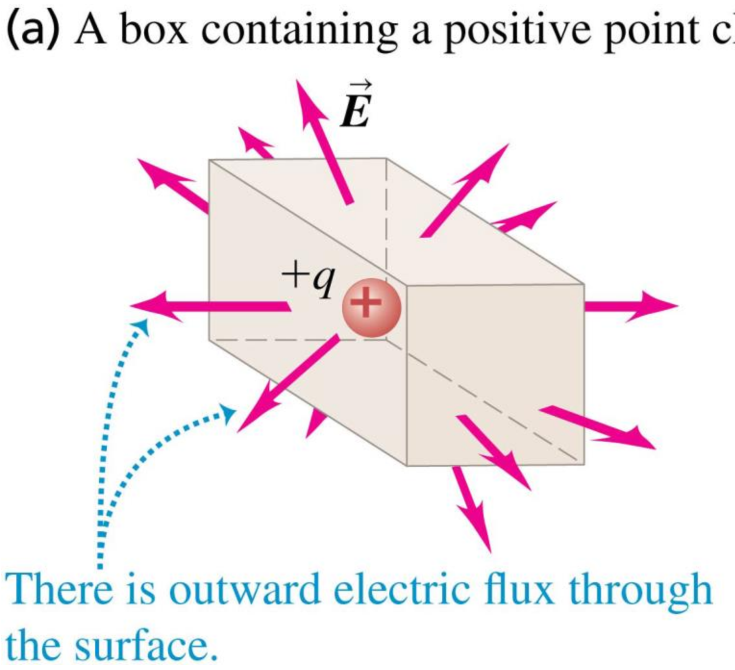
Zero net charge inside a box: Case 3 of 3

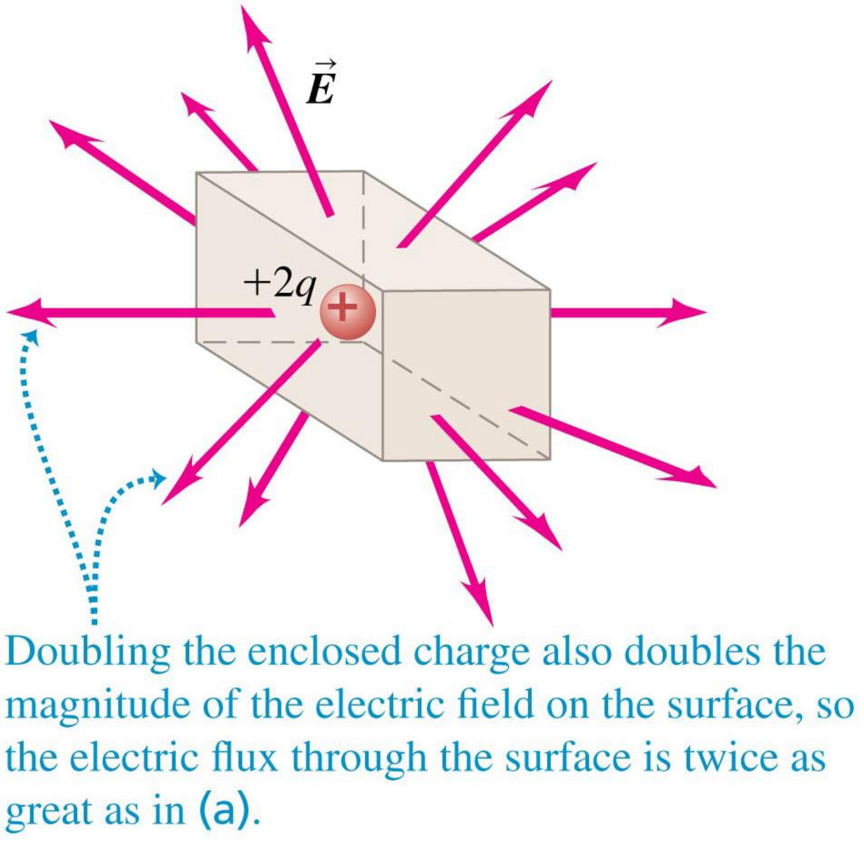
- What happens if there is charge *near* the box, but not inside it?
- On one end of the box, the flux points into the box; on the opposite end, the flux points out of the box; and on the sides, the field is parallel to the surface and so the flux is zero.
- The *net* electric flux through the box is zero.



What affects the flux through a box?

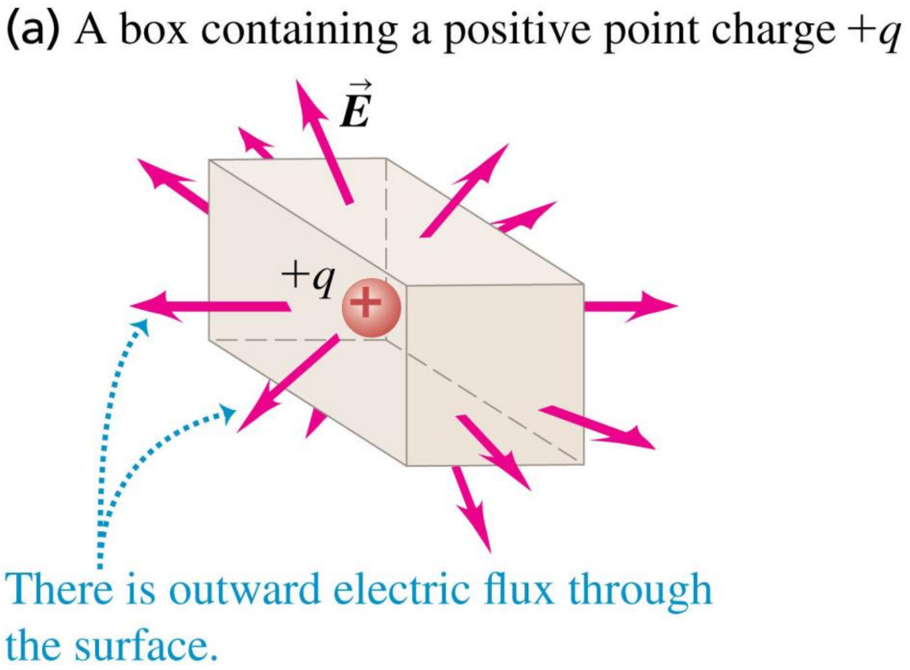
- The net electric flux is *directly proportional* to the net amount of charge enclosed within the surface.

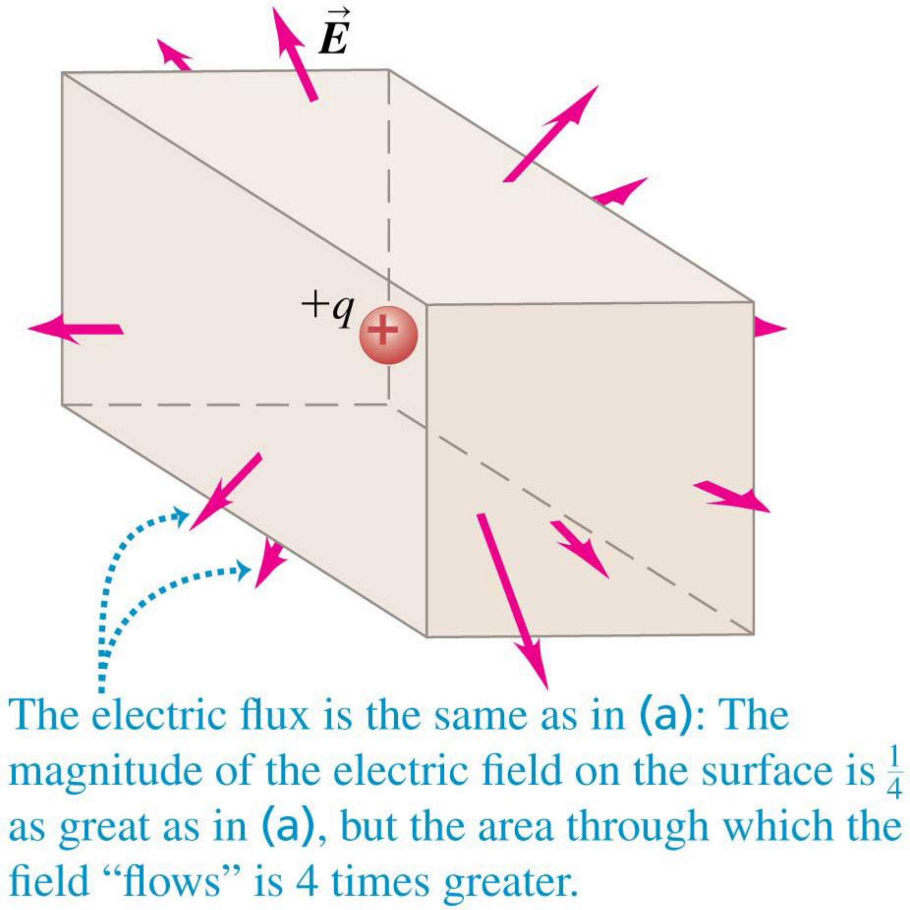




What affects the flux through a box?

- The net electric flux is *independent* of the size of the closed surface.

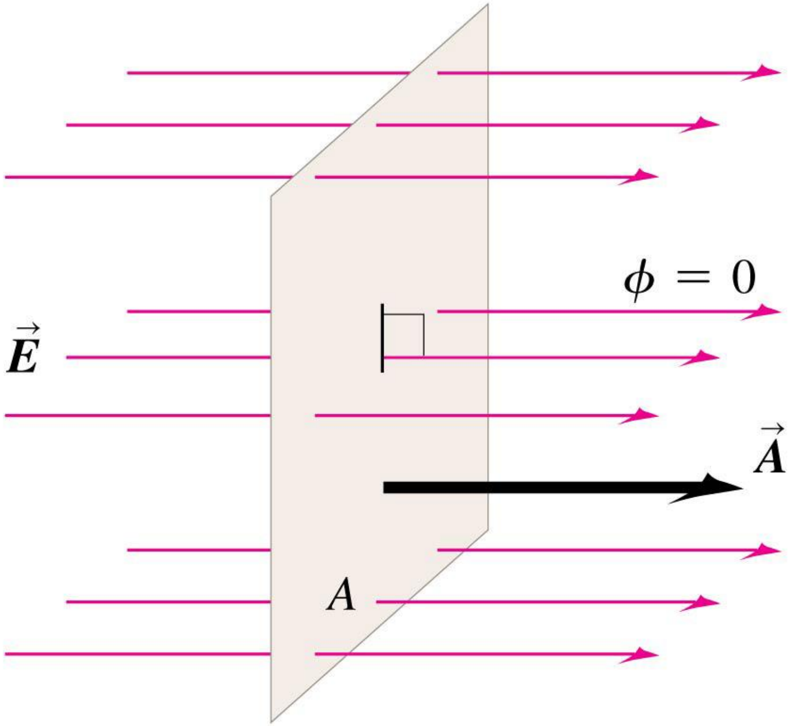




Calculating electric flux

- Consider a flat area perpendicular to a uniform electric field.
- Increasing the area means that more electric field lines pass through the area, increasing the flux.
- A stronger field means more closely spaced lines, and therefore more flux.

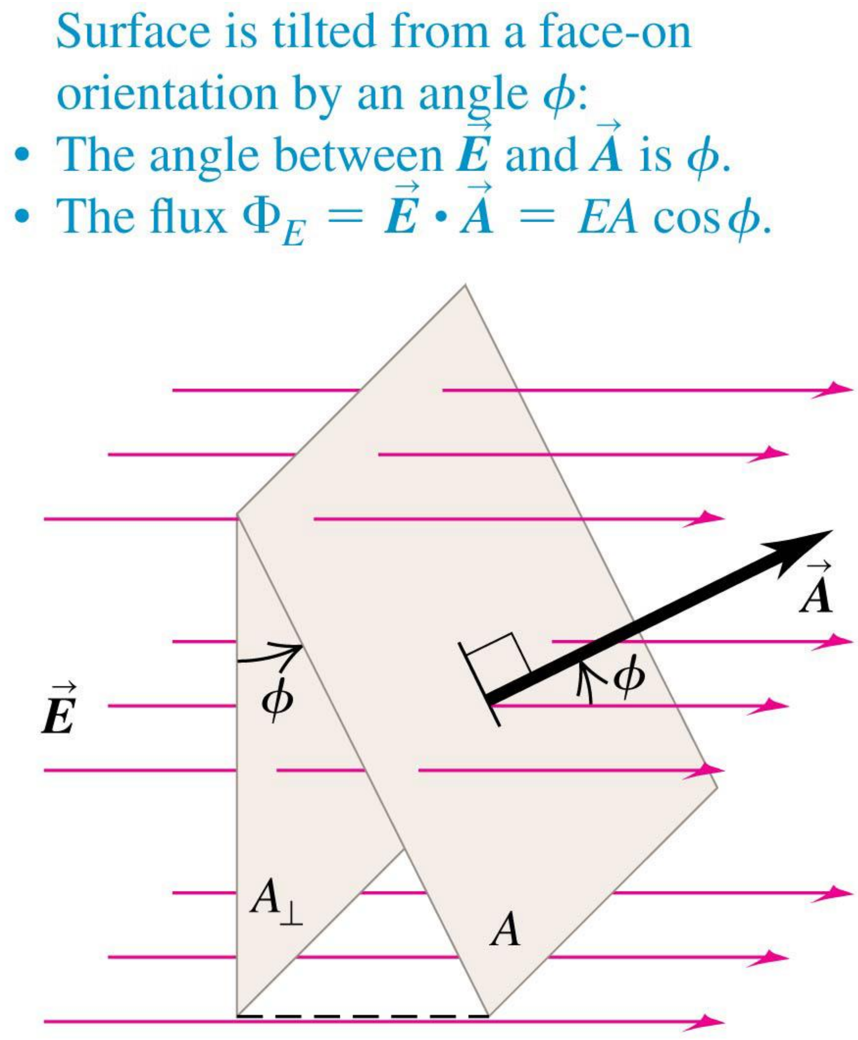
Surface is face-on to electric field:
- $\vec{E}$ and $\vec{A}$ are parallel (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 0$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$.



Calculating electric flux

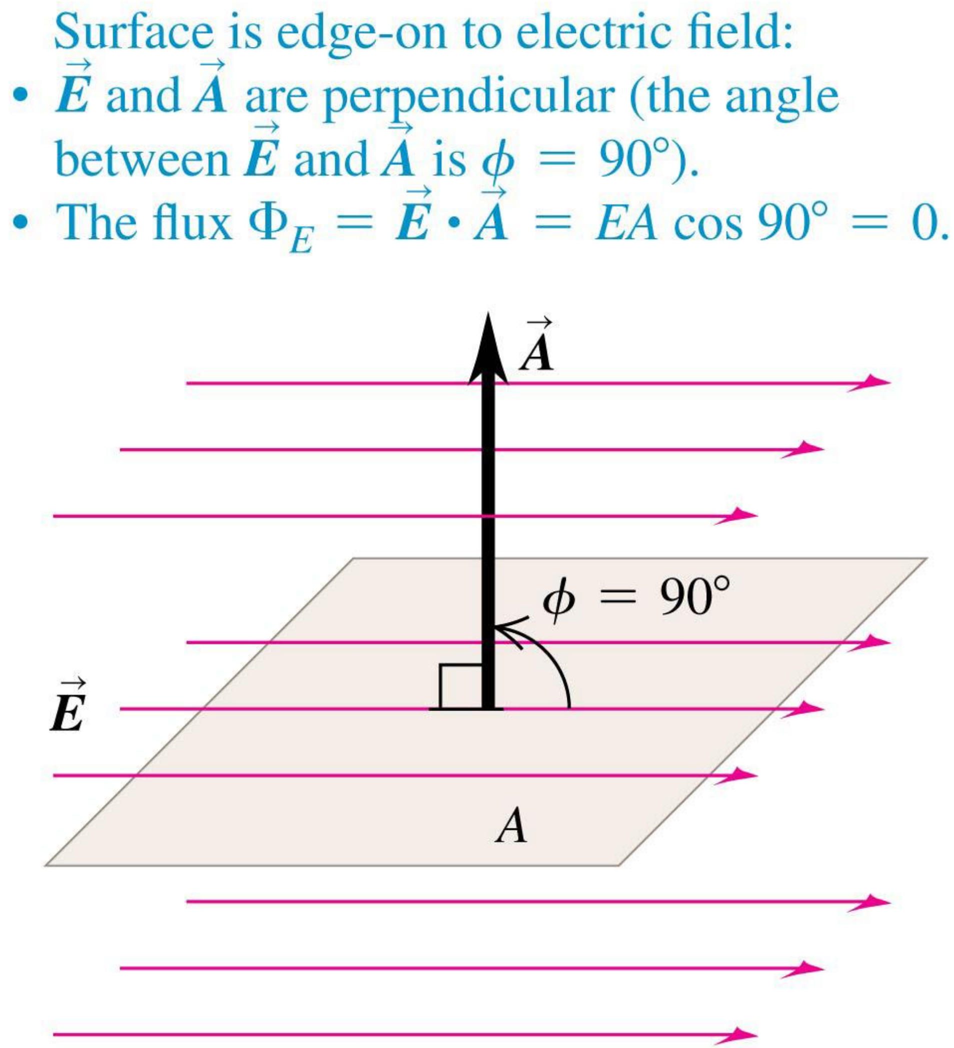
- If the area is not perpendicular to the field, then fewer field lines pass through it.

- In this case the area that counts is the silhouette area that we see when looking in the direction of the field.



Calculating electric flux

- If the area is edge-on to the field, then the area is perpendicular to the field and the flux is zero.



Flux of a nonuniform electric field

- In general, the flux through a surface must be computed using a **surface integral** over the area:

Magnitude of electric field $\vec{E}$

Component of $\vec{E}$ perpendicular to surface

Electric flux through a surface

$$\underbrace{\Phi_E}_{\text{Electric flux through a surface}} = \int \overbrace{E}^{\text{Magnitude of electric field }\vec{E}} \cos \underbrace{\phi}_{\text{Angle between }\vec{E}\text{ and normal to surface}} \, \underbrace{dA}_{\text{Element of surface area}} = \int \overbrace{E_\perp}^{\text{Component of }\vec{E}\text{ perpendicular to surface}} \, \underbrace{dA}_{\text{Element of surface area}} = \int \vec{E} \cdot \underbrace{d\vec{A}}_{\text{Vector element of surface area}}$$

Angle between $\vec{E}$ and normal to surface

Element of surface area

Vector element of surface area

- The SI unit for electric flux is $1\ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}$.

Example 1

A square that has 10-cm-long edges is centered on the *x* axis in a region where there exists a uniform electric field given by $\vec{E} = (2.00\,\frac{kN}{C})\hat{\imath}$.

(a) What is the electric flux of this electric field through the surface of a square if the normal to the surface is in the +*x* direction?

(b) What is the electric flux through the same square surface if the normal to the surface makes a 60° angle with the *y* axis and an angle of 90° with the *z* axis?

Example 1 (Solution)

Picture the Problem The definition of electric flux is $\phi = \oint_S \vec{E} \cdot \hat{n}\,dA$. We can apply this definition to find the electric flux through the square in its two orientations.

(a) What is the electric flux of this electric field through the surface of a square if the normal to the surface is in the $+x$ direction?

Apply the definition of ϕ to find the flux of the field when the square is parallel to the yz plane:

$$\phi = \oint_S (2.00\,\mathrm{kN/C})\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}}\,dA$$

$$= (2.00\,\mathrm{kN/C})\oint_S dA$$

$$= (2.00\,\mathrm{kN/C})(0.100\,\mathrm{m})^2$$

$$= \boxed{20.0\,\mathrm{N \cdot m^2/C}}$$

Example 1 (Solution)

(b) What is the electric flux through the same square surface if the normal to the surface makes a 60° angle with the y axis and an angle of 90° with the z axis?

Proceed as in (a) with
$\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{n}} = \cos 30°$:

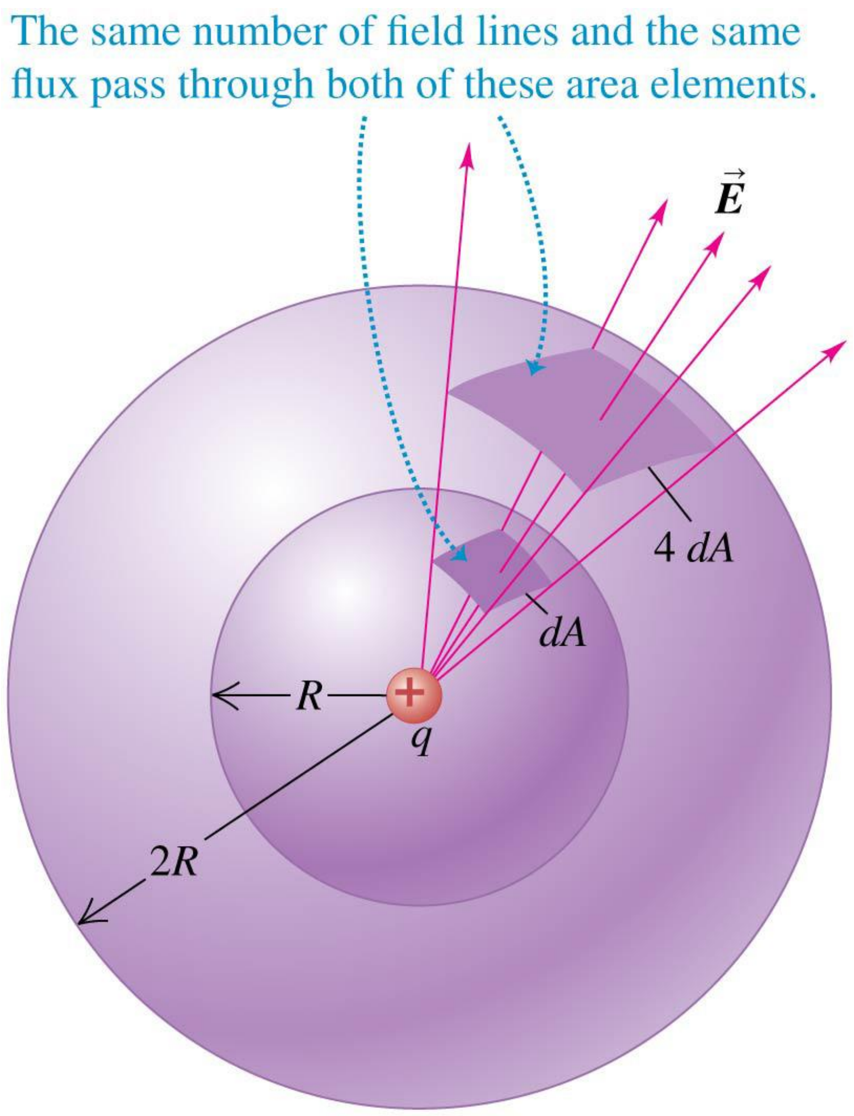
$$\begin{aligned}\phi &= \oint_S (2.00\,\text{kN/C}) \cos 30° \, dA \\ &= (2.00\,\text{kN/C}) \cos 30° \oint_S dA \\ &= (2.00\,\text{kN/C})(0.100\,\text{m})^2 \cos 30° \\ &= \boxed{17\,\text{N} \cdot \text{m}^2/\text{C}}\end{aligned}$$

Gauss's law

- Carl Friedrich Gauss helped develop several branches of mathematics, including differential geometry, real analysis, and number theory.
- The "bell curve" of statistics is one of his inventions.
- Gauss also made state-of-the-art investigations of the earth's magnetism and calculated the orbit of the first asteroid to be discovered.
- While completely equivalent to Coulomb's law, **Gauss's law** provides a different way to express the relationship between electric charge and electric field.

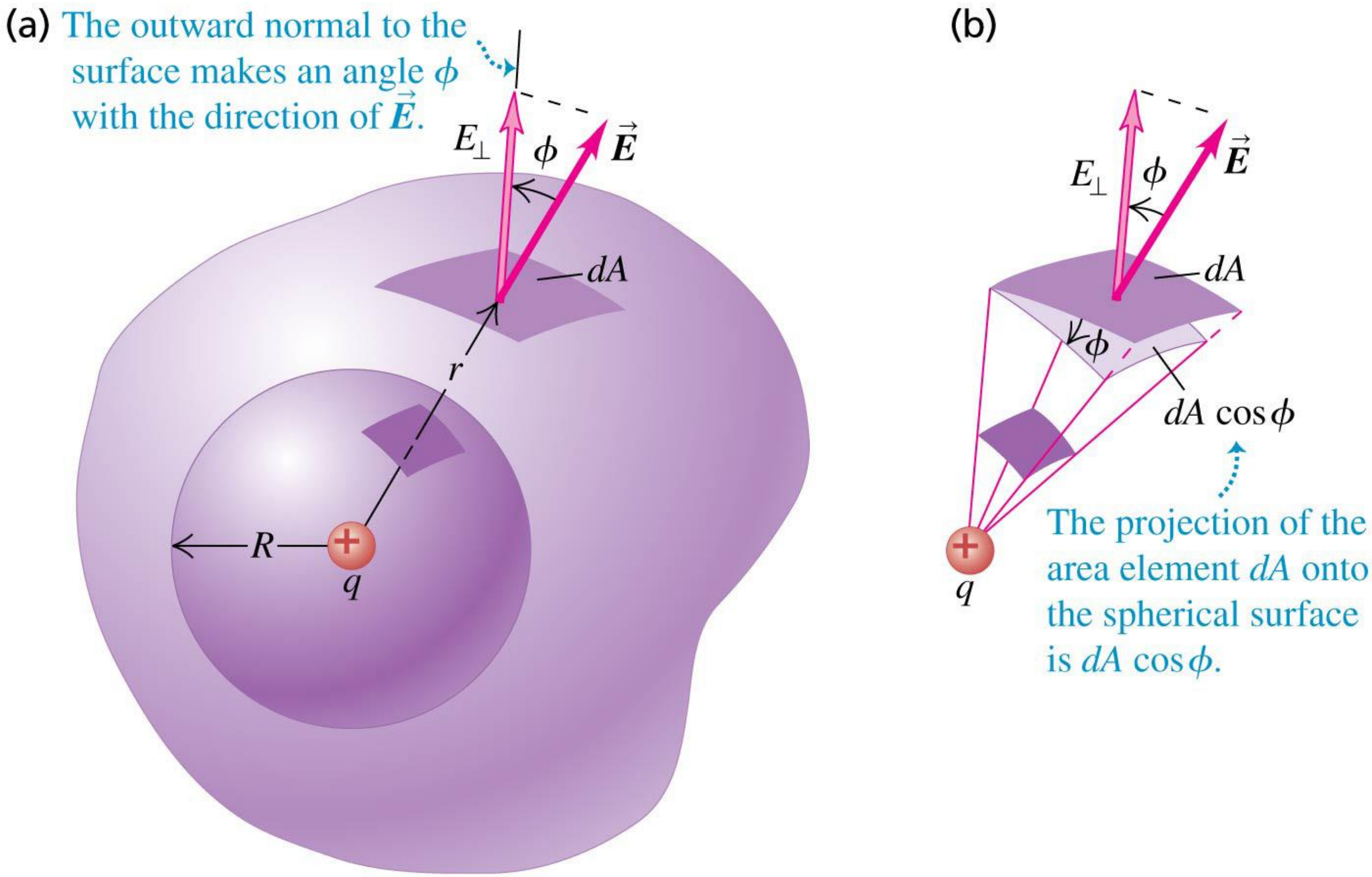
Point charge centered in a spherical surface

- Shown is the projection of an element of area *dA* of a sphere of radius *R* onto a concentric sphere of radius *2R*.
- The area element on the larger sphere is 4 *dA*, but the electric field magnitude is ¼ as great on the sphere of radius 2*R* as on the sphere of radius *R*.
  - **Hence the electric flux is the *same* for both areas and is *independent* of the radius of the sphere.**



Point charge inside a nonspherical surface

- As before, the flux is independent of the surface and depends only on the charge inside.

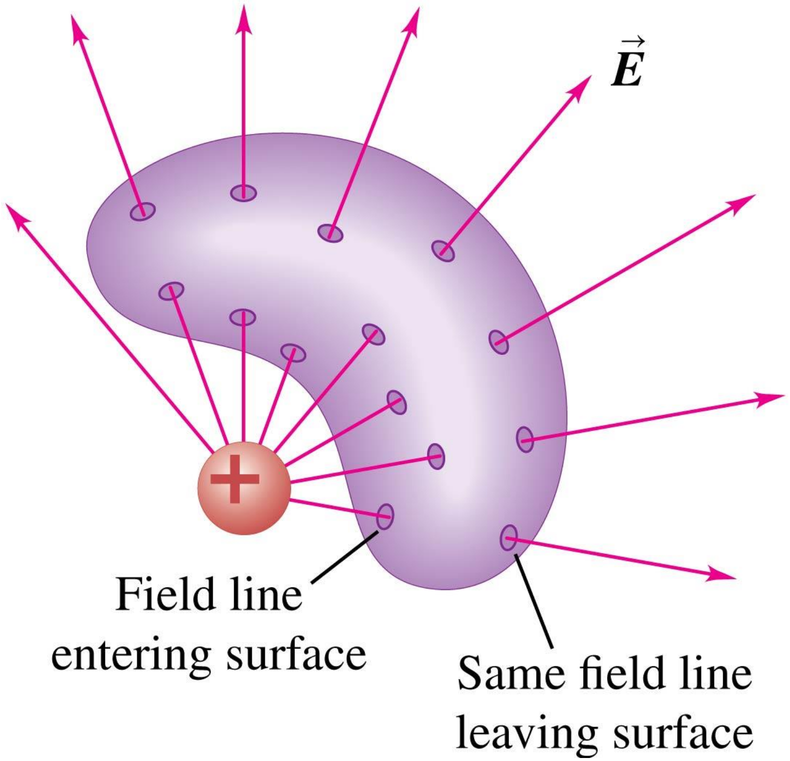


Gauss's law in a vacuum

- For a closed surface enclosing no charge:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 0$$

- The figure shows a point charge *outside* a closed surface that encloses no charge.
- If an electric field line from the external charge enters the surface at one point, it must leave at another.



General form of Gauss’s law

- Let Q_{encl} be the total charge enclosed by a surface.
- Gauss’s law states that **the total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ε_0:**

Gauss’s law:

$$\Phi_E = \underbrace{\oint \vec{E} \cdot d\vec{A}}_{} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

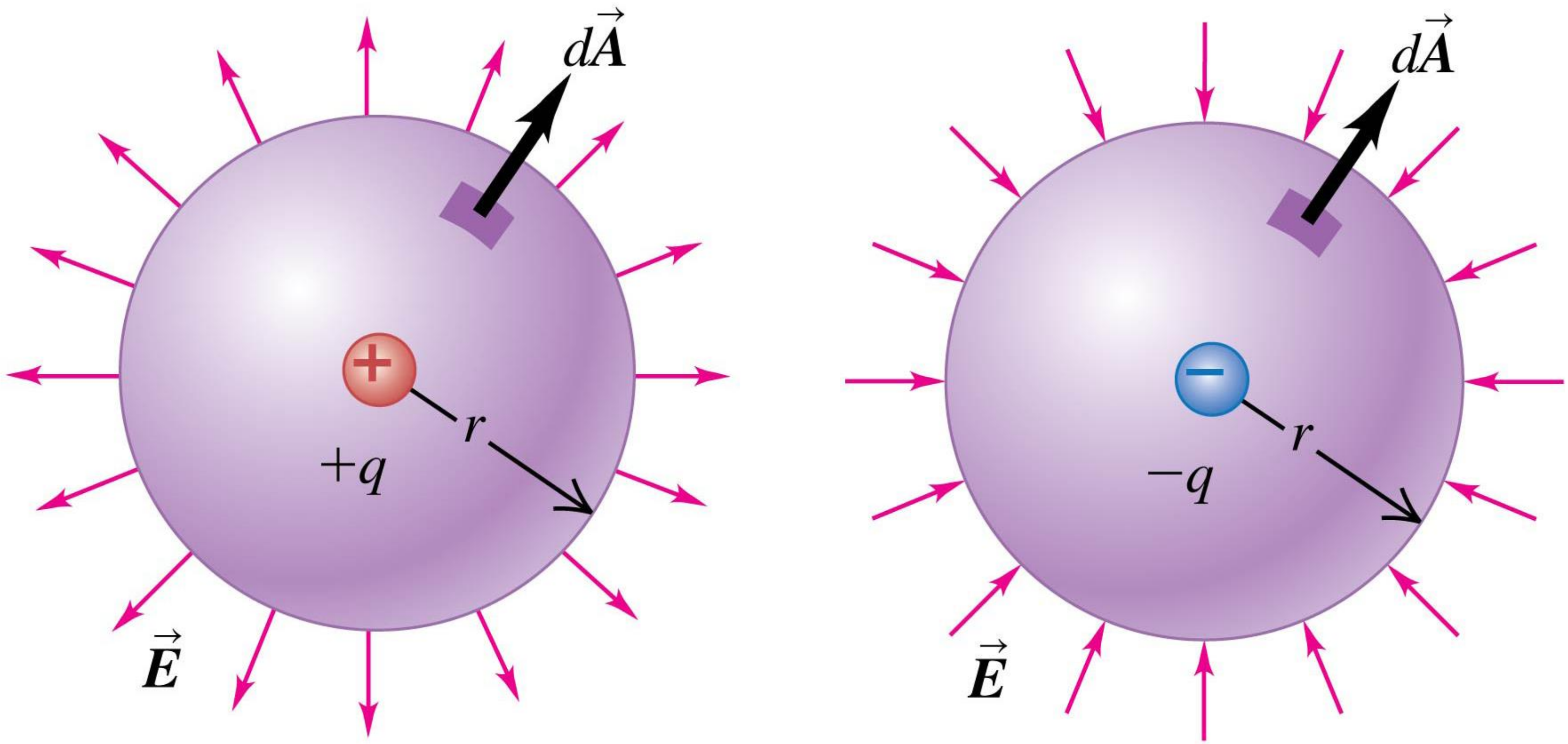
Total charge enclosed by surface

Electric flux through a closed surface of area A = surface integral of $\vec{E}$

Electric constant

Positive and negative flux

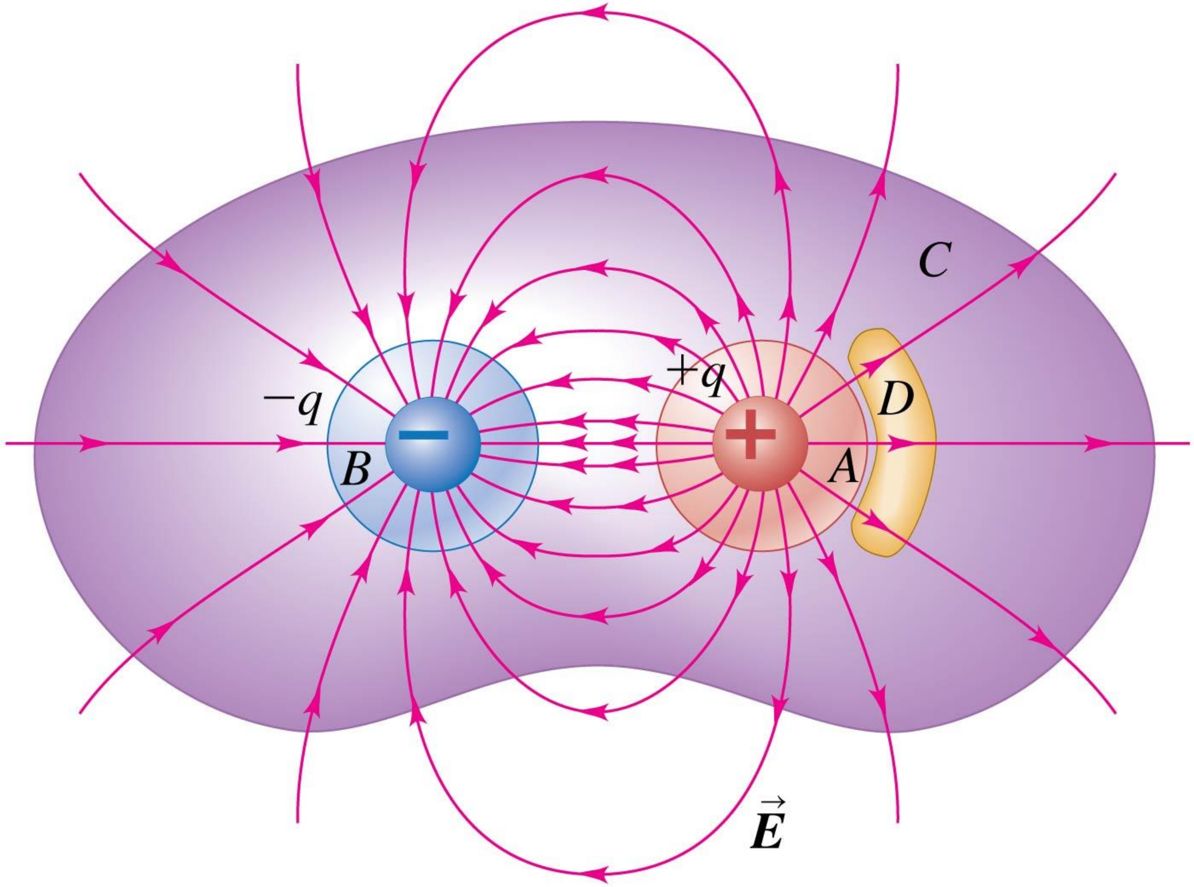
- In the case of a single point charge at the center of a spherical surface, Gauss's law follows simply from Coulomb's law.
- A surface around a positive charge has a ***positive (outward) flux***, and a surface around a negative charge has ***a negative (inward) flux***.



Applications of Gauss's law

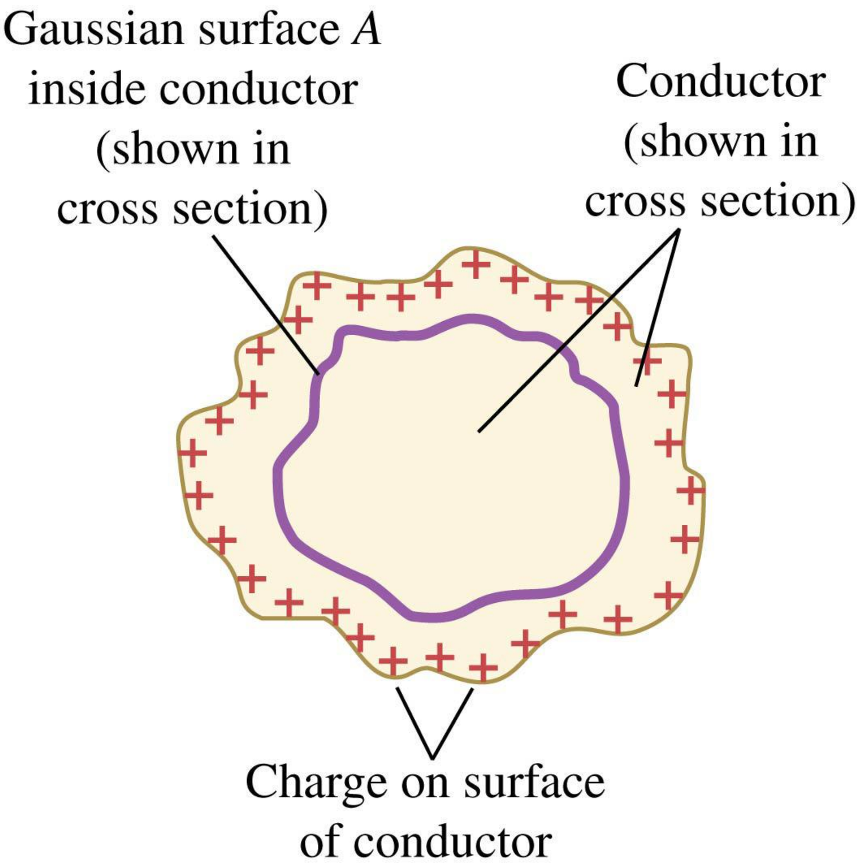
- Without having to do any integration, we can use Gauss's law to determine the electric flux through the closed surfaces in the diagram.

- $\Phi_{EA} = +q/\varepsilon_0$
- $\Phi_{EB} = -q/\varepsilon_0$
- $\Phi_{EC} = 0$
- $\Phi_{ED} = 0$



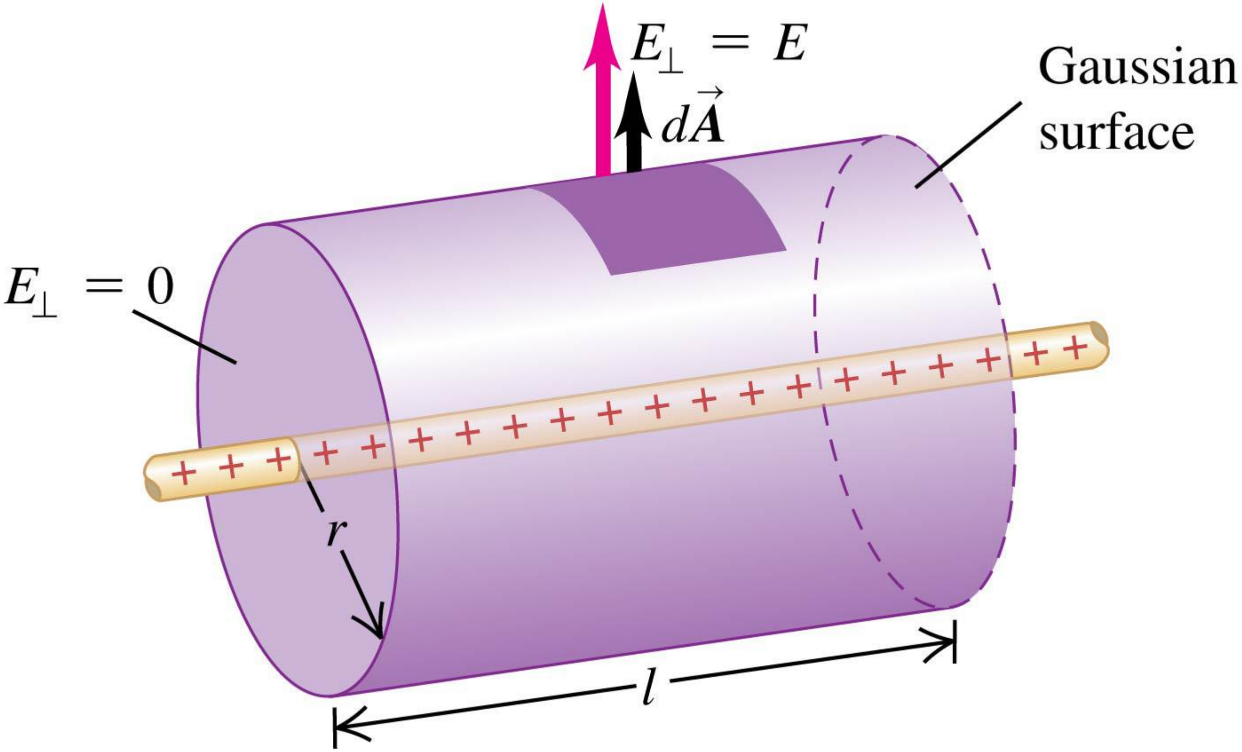
Applications of Gauss's law

- Suppose we construct a Gaussian surface inside a conductor.
- Because $\vec{E} = 0$ everywhere on this surface, Gauss's law requires that the net charge inside the surface is zero.
- Under electrostatic conditions (charges not in motion), any excess charge on a solid conductor resides entirely on the conductor's surface.



Field of a uniform line charge

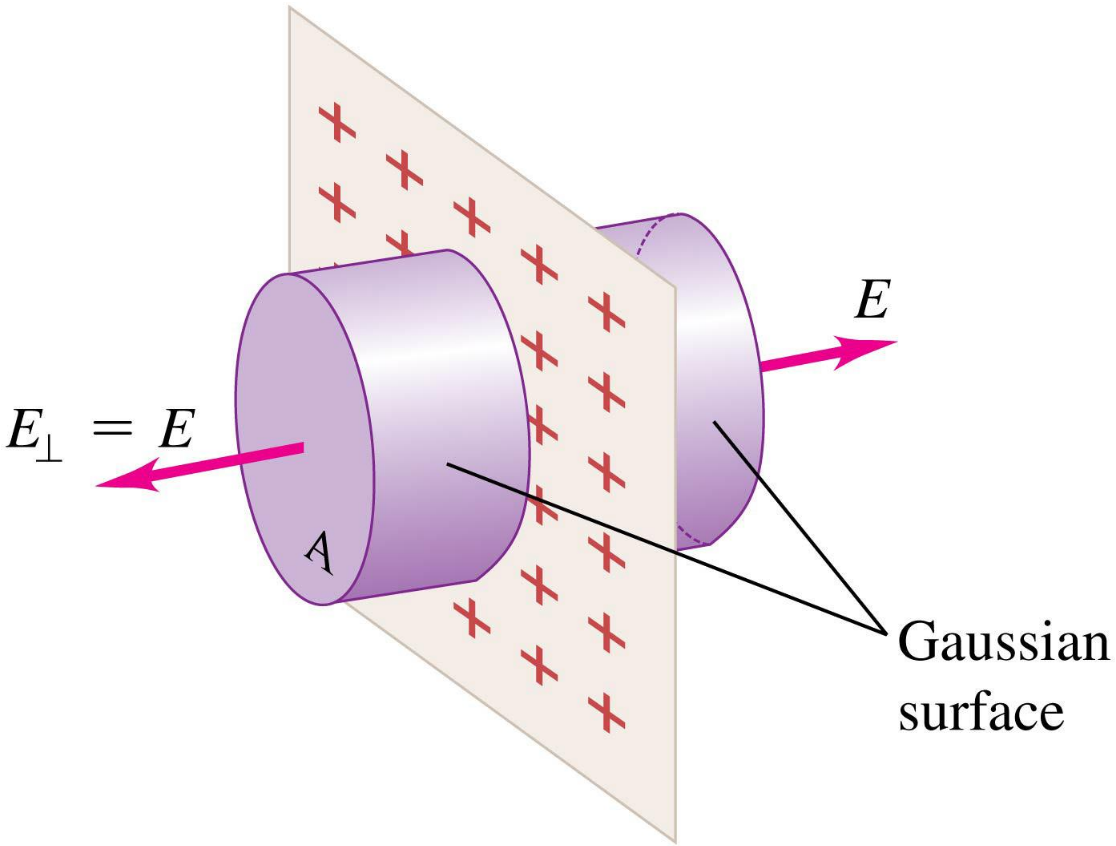
- Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is λ (assumed positive).
- Using Gauss's law, we can find the electric field: $E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$



Field of an infinite plane sheet of charge

- Gauss's law can be used to find the electric field caused by a thin, flat, infinite sheet with a uniform positive surface charge density σ.

$$E = \frac{\sigma}{2\epsilon_0}$$



Example 2

Careful measurement of the electric field at the surface of a black box indicates that the net outward electric flux through the surface of the box is 6.0 kN·m^2/C.

(a) What is the net charge inside the box?

(b) If the net outward electric flux through the surface of the box were zero, could you conclude that there were no charges inside the box? Explain your answer.

Picture the Problem We can use Gauss's law in terms of ϵ_0 to find the net charge inside the box.

Example 2 (Solution)

(a) Apply Gauss's law in terms of ϵ_0 to find the net charge inside the box:

$$\phi_{\text{net}} = \frac{Q_{\text{inside}}}{\epsilon_0} \Rightarrow Q_{\text{inside}} = \epsilon_0\, \phi_{\text{net}}$$

Substitute numerical values and evaluate Q_{inside} :

$$Q_{\text{inside}} = \left(8.854 \times 10^{-12}\, \frac{\text{C}^2}{\text{N} \cdot \text{m}^2}\right)\left(6.0\, \frac{\text{kN} \cdot \text{m}^2}{\text{C}}\right) = \boxed{5.3 \times 10^{-8}\ \text{C}}$$

(b) You can only conclude that the net charge is zero. There may be an equal number of positive and negative charges present inside the box.

Example 3

A thin non-conducting spherical shell of radius R_1 has a total charge q_1 that is uniformly distributed on its surface. A second, larger thin non-conducting spherical shell of radius R_2 that is coaxial with the first has a charge q_2 that is uniformly distributed on its surface.

(a) Use Gauss's law to obtain expressions for the electric field in each of the three regions: $r < R_1$, $R_1 < r < R_2$, and $r > R_2$.

(b) What should the ratio of the charges q_1/q_2 and the relative signs for q_1 and q_2 be for the electric field to be zero throughout the region $r > R_2$?

(c) Sketch the electric field lines for the situation in Part (*b*) when q_1 is positive.

Picture the Problem To find E_n in these three regions we can choose Gaussian surfaces of appropriate radii and apply Gauss's law. On each of these surfaces, E_r is constant and Gauss's law relates E_r to the total charge inside the surface.

Example 3 (Solution)

(a) Use Gauss's law to find the electric field in the region $r < R_1$:

$$\oint_S E_n dA = \frac{1}{\epsilon_0} Q_{\text{inside}}$$

and

$\vec{E}_{r<R_1} = \dfrac{Q_{\text{inside}}}{\epsilon_0 A}\hat{r}$ where $\hat{r}$ is a unit radial vector.

Because $Q_{\text{inside}} = 0$:

$$\vec{E}_{r<R_1} = \boxed{0}$$

Apply Gauss's law in the region $R_1 < r < R_2$:

$$\vec{E}_{R_1<r<R_1} = \frac{q_1}{\epsilon_0 \left(4\pi r^2\right)}\hat{r} = \boxed{\frac{kq_1}{r^2}\hat{r}}$$

Using Gauss's law, find the electric field in the region $r > R_2$:

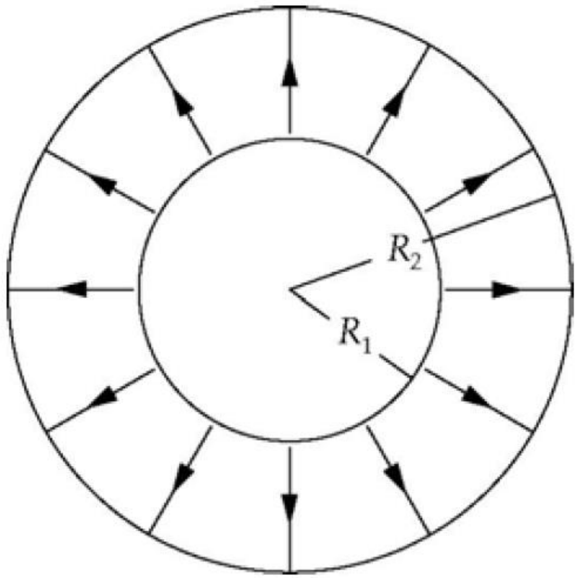
$$\vec{E}_{r>R_2} = \frac{q_1 + q_2}{\epsilon_0 \left(4\pi r^2\right)}\hat{r} = \boxed{\frac{k\left(q_1 + q_2\right)}{r^2}\hat{r}}$$

(b) Set $E_{r>R_2} = 0$ to obtain:

$$q_1 + q_2 = 0 \Rightarrow \frac{q_1}{q_2} = \boxed{-1}$$

Example 3 (solution)

(c) The electric field lines for the situation in (b) with q_1 positive is shown to the right.



Example 4

An infinitely long cylindrical conductor has radius *R* and uniform surface charge density σ.

(a) In terms of σ and *R*, what is the charge per unit length λ for the cylinder?

(b) In terms of σ, what is the magnitude of the electric field produced by the charged cylinder at a distance $r > R$ from its axis?

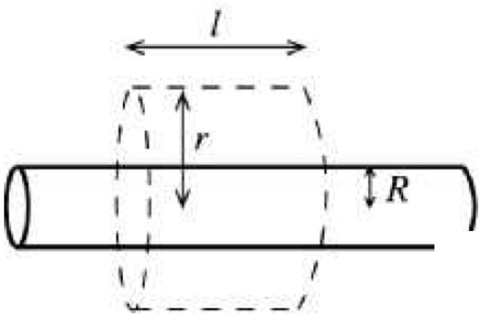
(c) Express the result of part (b) in terms of λ and show that the electric field outside the cylinder is the same as if all the charge were on the axis.

IDENTIFY: Apply Gauss's law to a Gaussian surface and calculate *E*.

Example 4 (Solution)

(a) SET UP: Consider the charge on a length l of the cylinder. This can be expressed as $q = \lambda l$. But since the surface area is $2\pi Rl$ it can also be expressed as $q = \sigma 2\pi Rl$. These two expressions must be equal, so $\lambda l = \sigma 2\pi Rl$ and $\lambda = 2\pi R\sigma$.

(b) Apply Gauss's law to a Gaussian surface that is a cylinder of length l, radius r, and whose axis coincides with the axis of the charge distributior



EXECUTE:

$Q_{\text{encl}} = \sigma(2\pi Rl)$

$\Phi_E = 2\pi rlE$

$\Phi_E = \dfrac{Q_{\text{encl}}}{\epsilon_0}$ gives $2\pi rlE = \dfrac{\sigma(2\pi Rl)}{\epsilon_0}$

$$E = \frac{\sigma R}{\epsilon_0 r}$$

(c) EVALUATE:

$\sigma = \dfrac{\lambda}{2\pi R}$, so $E = \dfrac{\sigma R}{\epsilon_0 r} = \dfrac{R}{\epsilon_0 r}\left(\dfrac{\lambda}{2\pi R}\right) = \dfrac{\lambda}{2\pi\epsilon_0 r}$, the same as for an infinite line of charge that is along the axis of the cylinder.

Example 5

Two very long uniform lines of charge are parallel and are separated by 0.300 m. Each line of charge has charge per unit length +5.20 uC/m. What magnitude of force does one line of charge exert on a 0.0500-m section of the other line of charge?

- Each line lies in the electric field of the other line, and therefore each line experiences a force due to the other line.

- The field of one line is $E = \frac{\lambda}{2\pi\epsilon_0 r}$. For charge $dq = \lambda\, dx$ on one line, the force on it due to the other line is $dF = Edq$. The total force is

$$F = \int Edq = E\int dq = Eq.$$

Example 5 (Solution)

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{5.20 \times 10^{-6}\ \text{C/m}}{2\pi(8.854 \times 10^{-12}\ \text{C}^2/(\text{N}\cdot\text{m}^2))(0.300\ \text{m})} = 3.116 \times 10^5\ \text{N/C}.$$

The force on one line due to the other is $F = Eq$, where

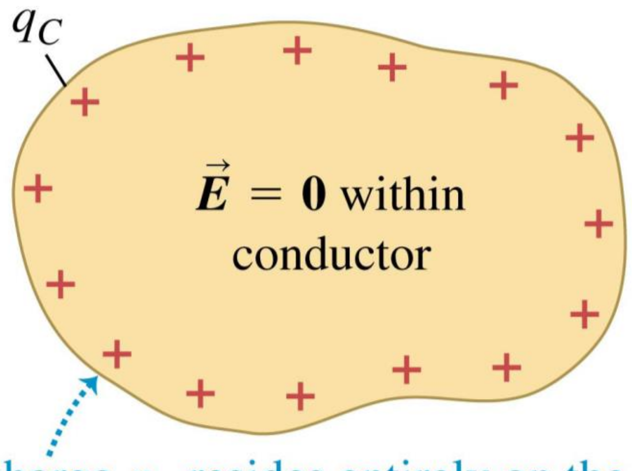
$$\boldsymbol{q = \lambda\,(0.0500\ \text{m}) = 2.60.10^{-7}\ \text{C.}}$$

The net force is

- $\boldsymbol{F = Eq = (3.116 \text{ x } 10^5\,\text{N/C})(2.60.10^{-7}\,\text{C}) = 0.0810\ \text{N.}}$

Charges on conductors

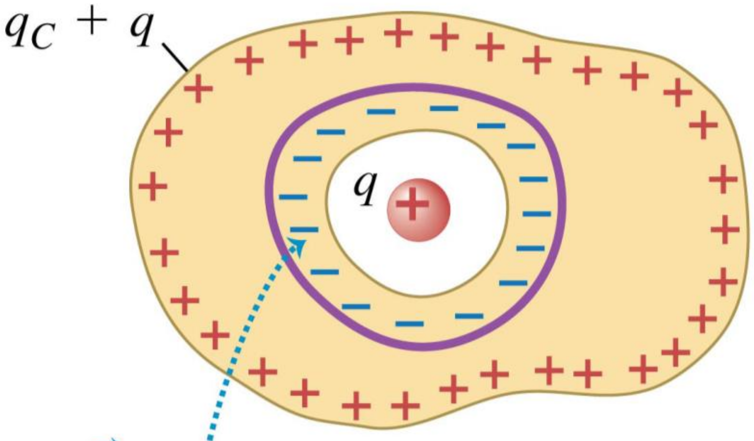
- Consider a solid conductor with a hollow cavity inside.
- If there is no charge within the cavity, we can use a Gaussian surface such as *A* to show that the net charge on the surface of the cavity must be zero, because $\vec{E} = 0$ everywhere on the Gaussian surface.



The charge q_C resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = \mathbf{0}$ within the conductor.

Charges on conductors

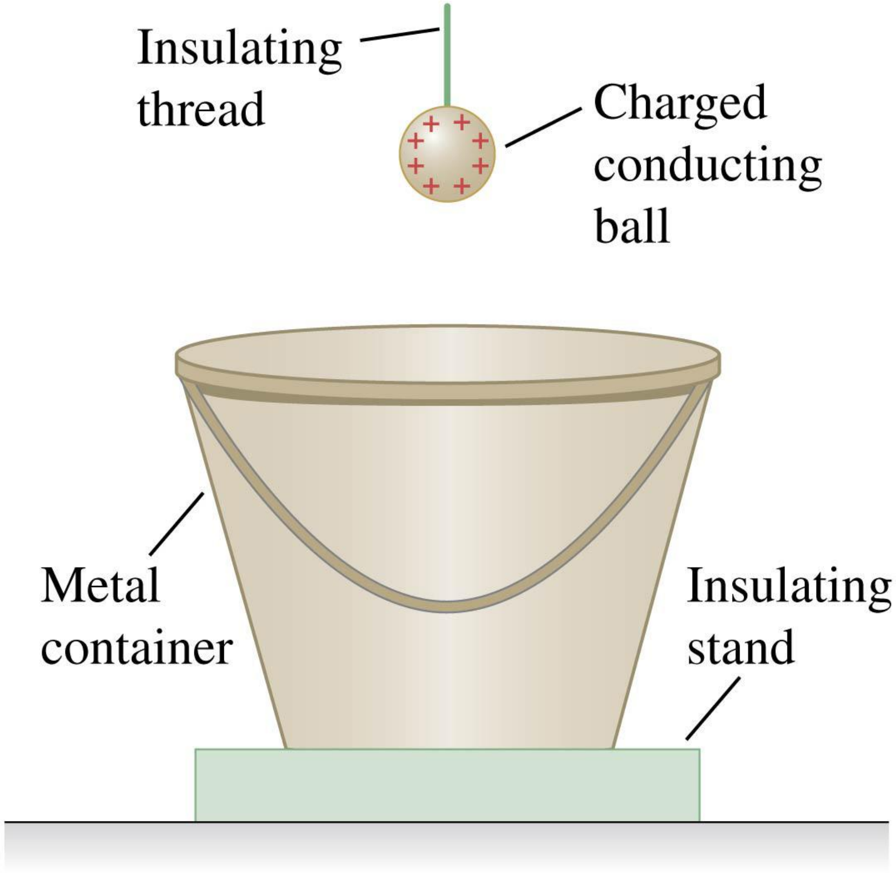
- Suppose we place a small body with a charge q inside a cavity within a conductor. The conductor is uncharged and is insulated from the charge q.
- According to Gauss's law the total there must be a charge $-q$ distributed on the surface of the cavity, drawn there by the charge q inside the cavity.
- The total charge on the conductor must remain zero, so a charge $+q$ must appear on its outer surface.



For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.

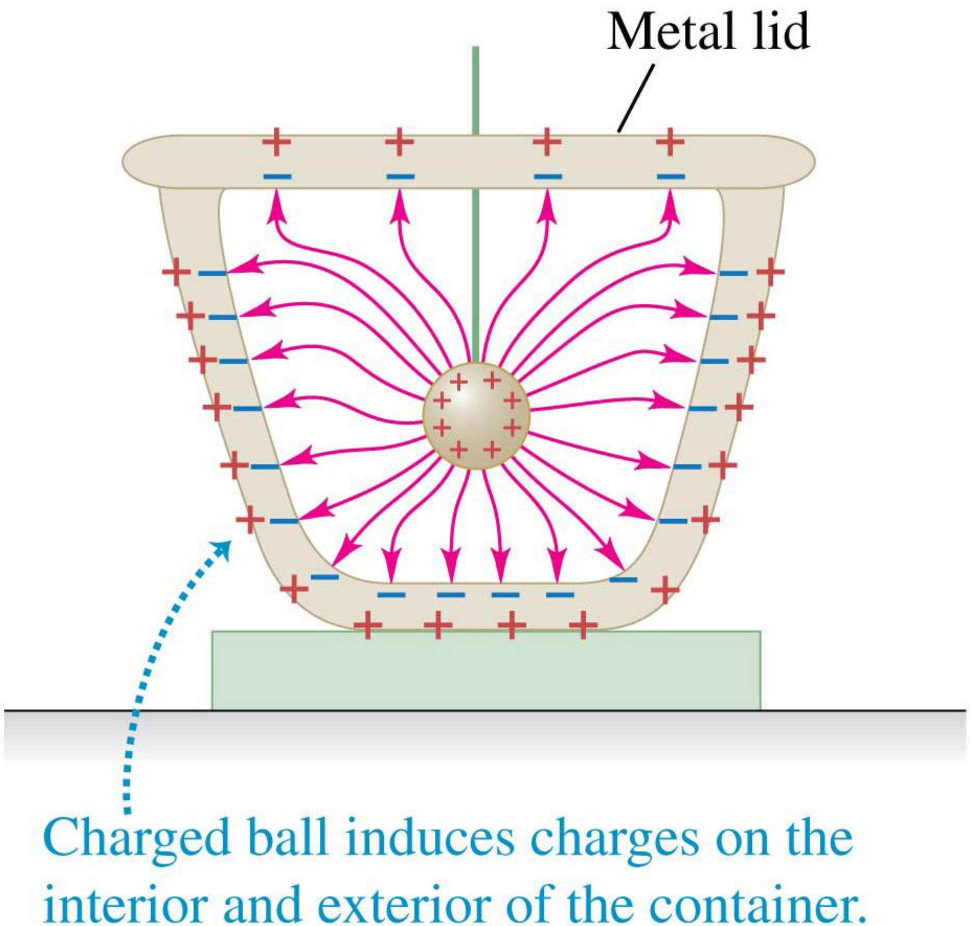
Faraday's icepail experiment: Slide 1 of 3

- We now consider Faraday's historic **icepail experiment**.
- We mount a conducting container on an insulating stand.
- The container is initially uncharged.
- Then we hang a charged metal ball from an insulating thread, and lower it into the container.



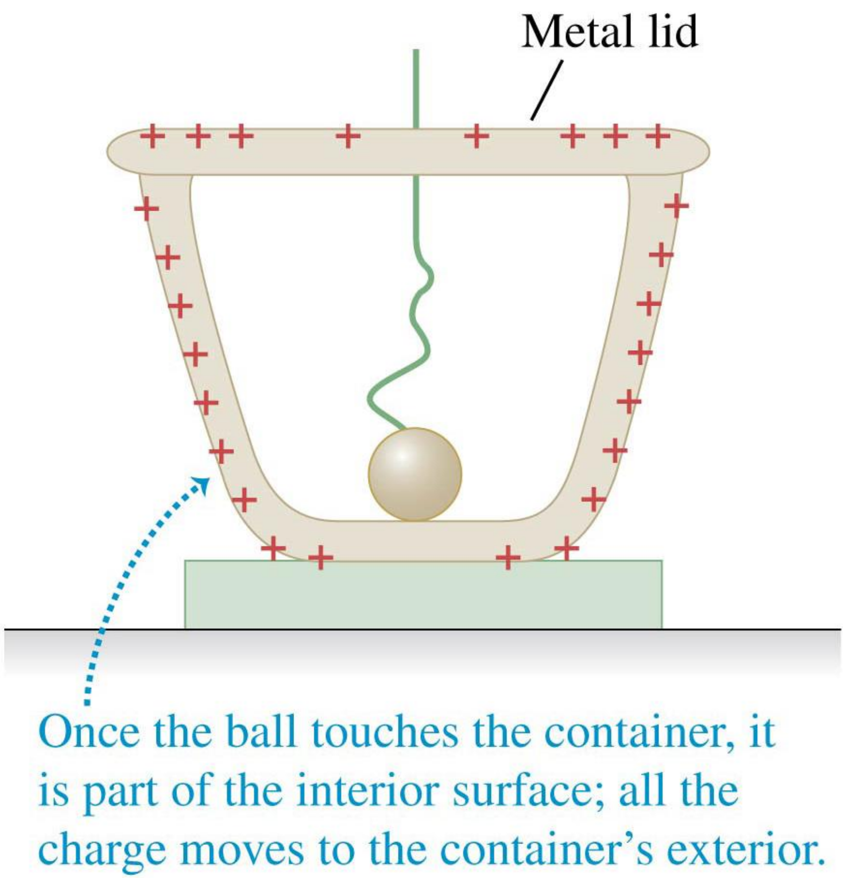
Faraday's icepail experiment: Slide 2 of 3

- We lower the ball into the container, and put the lid on.
- Charges are induced on the walls of the container, as shown.



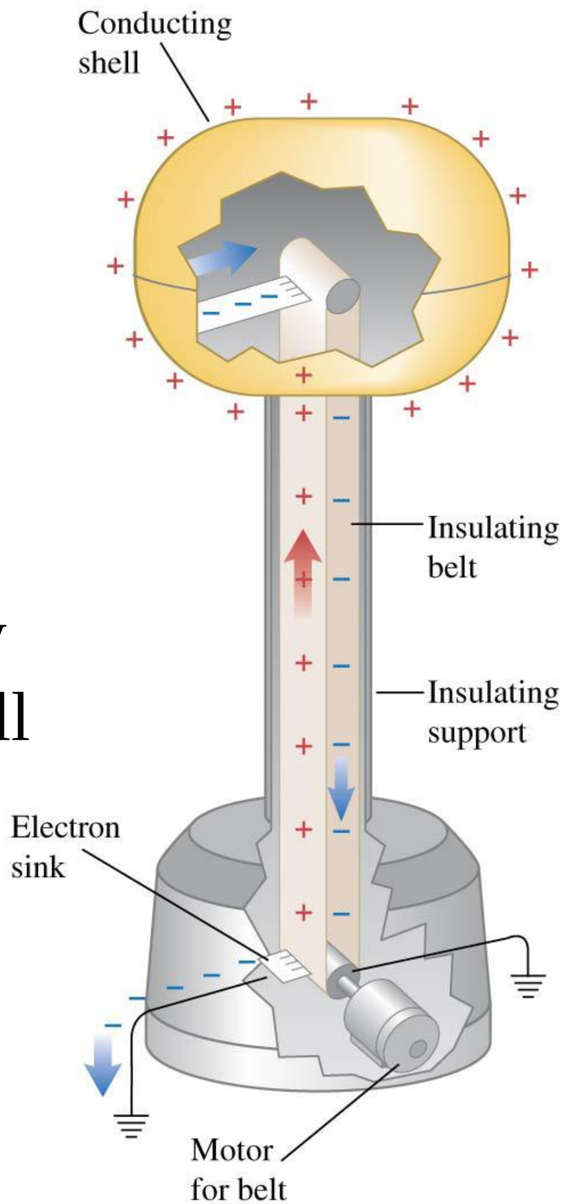
Faraday's icepail experiment: Slide 3 of 3

- We now let the ball touch the inner wall.
- The surface of the ball becomes part of the cavity surface, thus, according to Gauss's law, the ball must lose all its charge.
- Finally, we pull the ball out; we find that it has indeed lost all its charge.



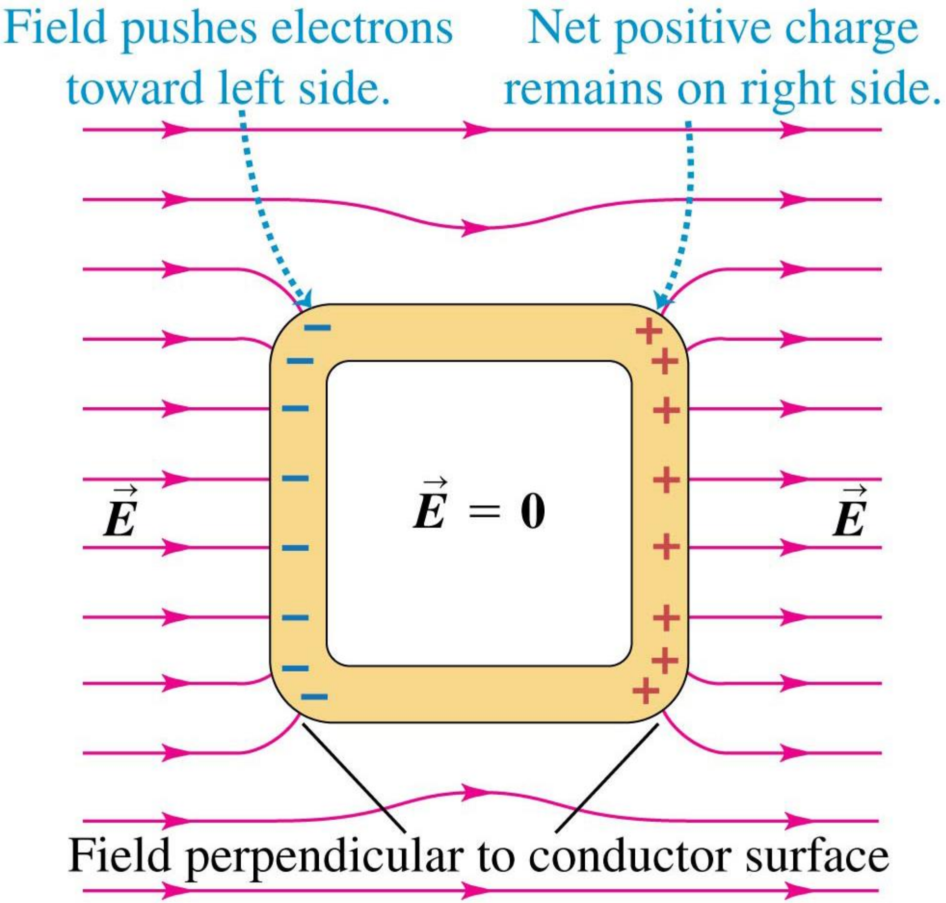
The Van de Graaff generator

- The **Van de Graaff generator** operates on the same principle as in Faraday's icepail experiment.
- The electron sink at the bottom draws electrons from the belt, giving it a positive charge.
- At the top the belt attracts electrons away from the conducting shell, giving the shell a positive charge.



Electrostatic shielding

- A conducting box is immersed in a uniform electric field.
- The field of the induced charges on the box combines with the uniform field to give *zero* total field inside the box.



Electrostatic shielding

- Suppose we have an object that we want to protect from electric fields.
- We surround the object with a conducting box, called a Faraday cage.
- Little to no electric field can penetrate inside the box.
- The person in the photograph is protected from the powerful electric discharge.