Week 6 class 3 - Chaos and Fractals

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10/02/2022

Today's goals

- Midterm check-in
- Complex numbers
- 2d iteration

Complex Numbers

Among the values 2, √2, √-1, ∞, which would you personally consider to really exist? (Q from Grant Sanderson)

Let's start by thinking about square roots. Take a positive number: $\sqrt{4} \pm 2$.

Okay, what about the square root of a negative number? Well, we can't get out of trouble in the same way, so let's define our way out of trouble: $\sqrt{-1} = i$, where we define i such that $i^2 = -1$. The number i is a unit. We know it as the **imaginary unit**.

For just a moment, let's take for granted that it makes sense to use i. If we allow ourselves to do so, we discover that i has some very interesting properties.

- if we take i to be the imaginary unit, then we discover that we can make new numbers along the imaginary number line, numbers like $2i$, or $3.7i$. We can also make numbers that combine imaginary numbers and real numbers, like $3 + 4i$. These numbers are known as complex numbers.

We need to figure out how to add and multiply complex numbers. (It's traditional to use Z to denote complex numbers in mathematics).

$Z_1 = 3 + 4i$

$Z_2 = 2 - 6i$

Addition with complex numbers:

$(3 + 4i) + (2 - 6i) = (3 + 2) + (4 - 6)i$

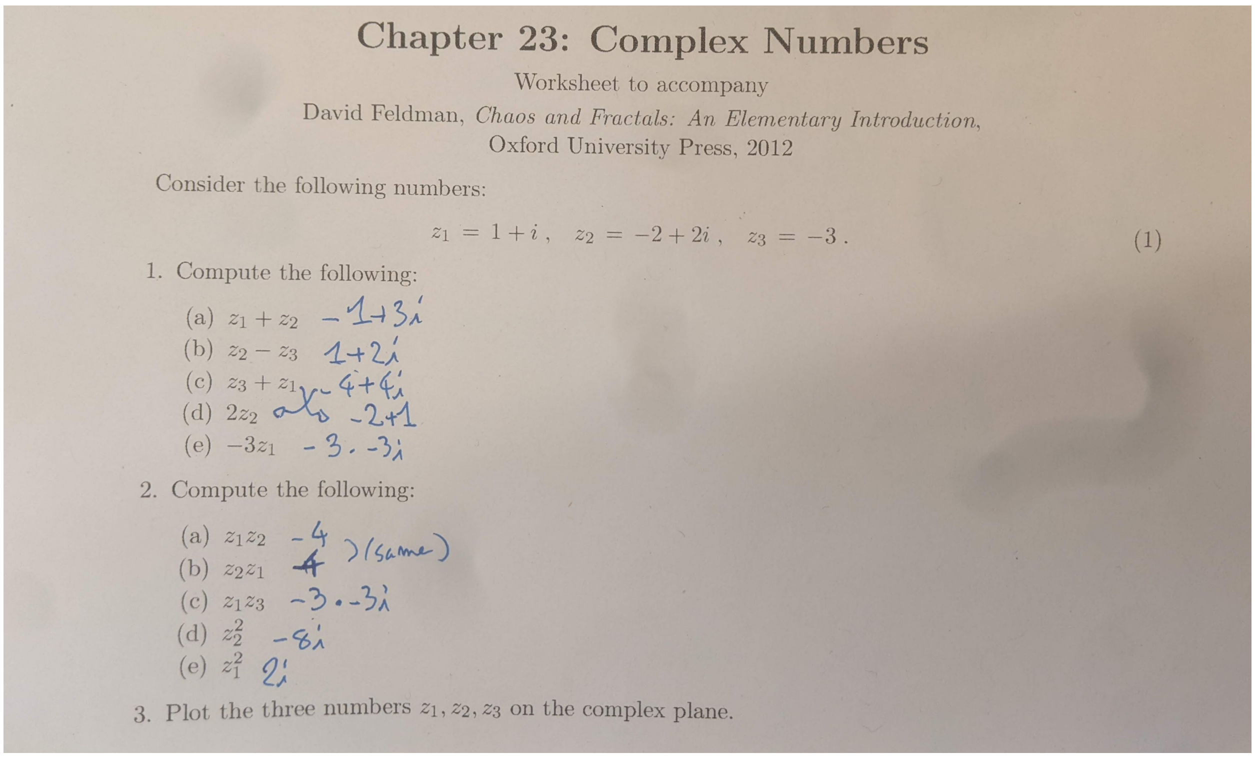
Multiplication with complex numbers:

$(3 + 4i) \cdot (2 - 6i)$

$= (3) \cdot (2) + (3) \cdot (-6i) + (4i) \cdot (2) + (4i) \cdot (-6i)$

$= 6 - 18i + 8i - 24i^2$ Note: The conversion is that if a term is raised to a power without parentheses, the last symbol is raised to that power, not the rest of what it's connected to. For example: $2x^2 = 2 \cdot x^2$ and $(2x)^2 = 2x \cdot 2x$.

A couple exercises with complex numbers:



Chapter 23: Complex Numbers

Worksheet to accompany

David Feldman, *Chaos and Fractals: An Elementary Introduction*,
Oxford University Press, 2012

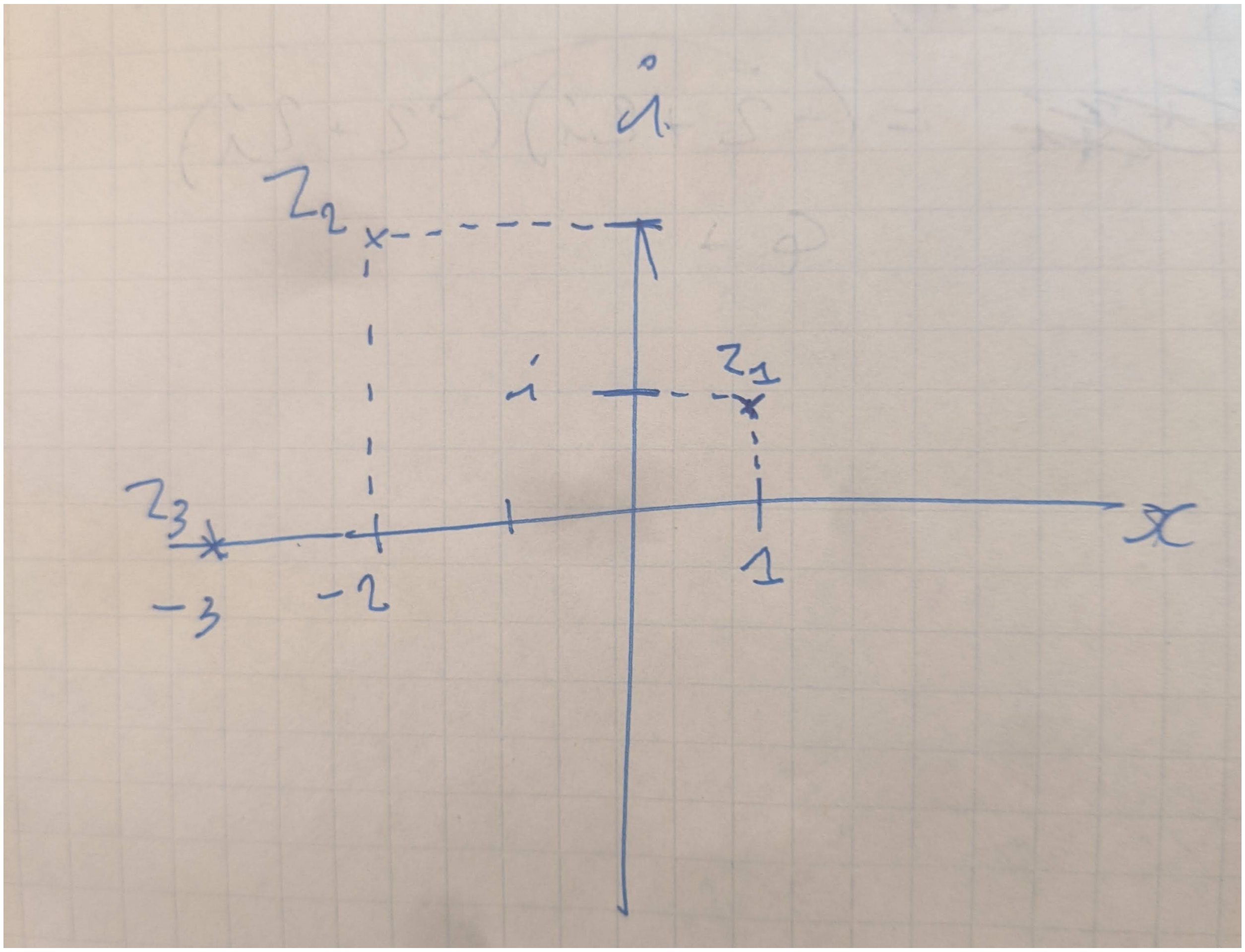
Consider the following numbers:

$$z_1 = 1 + i\,, \quad z_2 = -2 + 2i\,, \quad z_3 = -3\,. \tag{1}$$

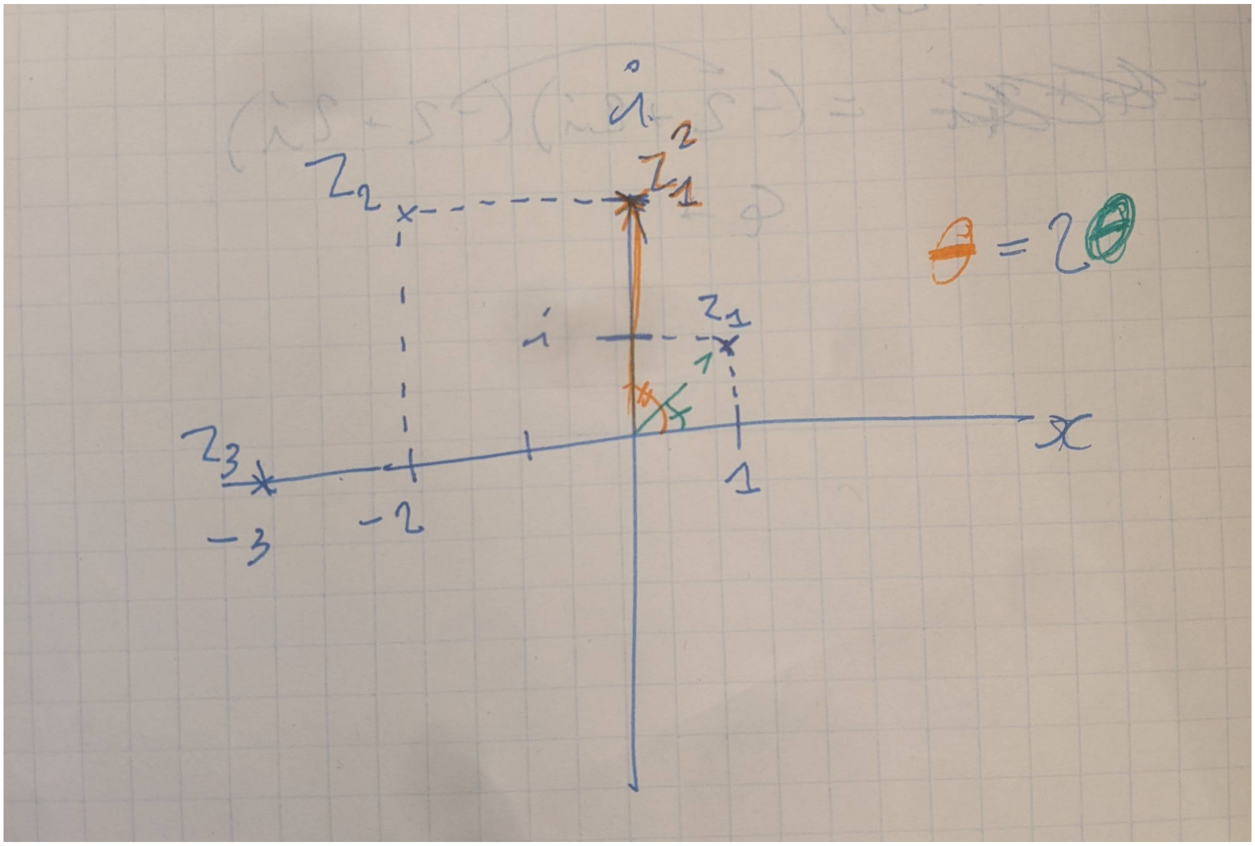
1. Compute the following:
   (a) $z_1 + z_2$ $-1+3i$
   (b) $z_2 - z_3$ $1+2i$
   (c) $z_3 + z_1$ $-4+4i$
   (d) $2z_2$ $-2+1$
   (e) $-3z_1$ $-3 \cdot -3i$
2. Compute the following:
   (a) $z_1 z_2$ -4
   (b) $z_2 z_1$ 4 (same)
   (c) $z_1 z_3$ $-3 \cdot -3i$
   (d) z_2^2 $-8i$
   (e) z_1^2 $2i$
3. Plot the three numbers z_1, z_2, z_3 on the complex plane.

The complex plane

Since complex numbers have two dimensions, the imaginary and real parts, we can plot them in two dimensions in a space called the complex plane. In complex plane plots, the x axis is typically the real number line, and the y axis is the imaginary number line.



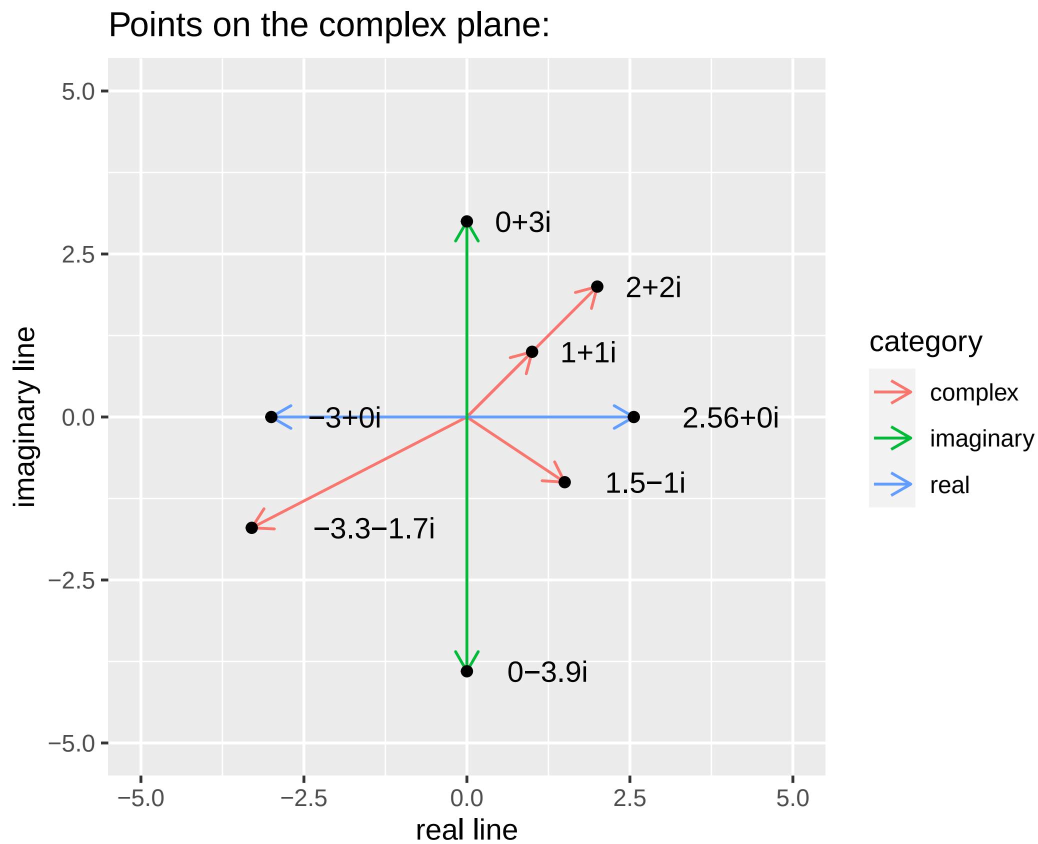
i
z_2
z_1
i
z_3
x
-3
-2
1



Let's plot some complex numbers on the complex plane!

```
numbers <- c(1+1i, 2+2i, -3.3-1.7i, 1.5-1i, -3+0i, 2.56+0i, 3i, -3.9i)

# let's split the complex numbers into real and imaginary parts to feed to ggplot
real <- Re(numbers)
imaginary <- Im(numbers)
category <- as.factor(c("complex", "complex", "complex", "complex", "real", "real", "imaginary"

name <- numbers

tibble(real, imaginary, name, category) %>% ggplot(aes(real, imaginary)) +
  geom_segment(lineend = "round", linejoin = "mitre",
               arrow = arrow(length = unit(0.3, "cm")), x=0, y=0,
               xend=real, yend=imaginary, aes(colour=category)) +
  geom_point() +
  geom_text(aes(label=name), hjust=-.5, vjust = .5) +
  labs(title= "Points on the complex plane:", x="real line", y = "imaginary line") +
  lims(x=c(-5, 5), y=c(-5, 5)) +
  coord_equal()
```

```
## Don't know how to automatically pick scale for object of type complex. Defaulting to contin
```



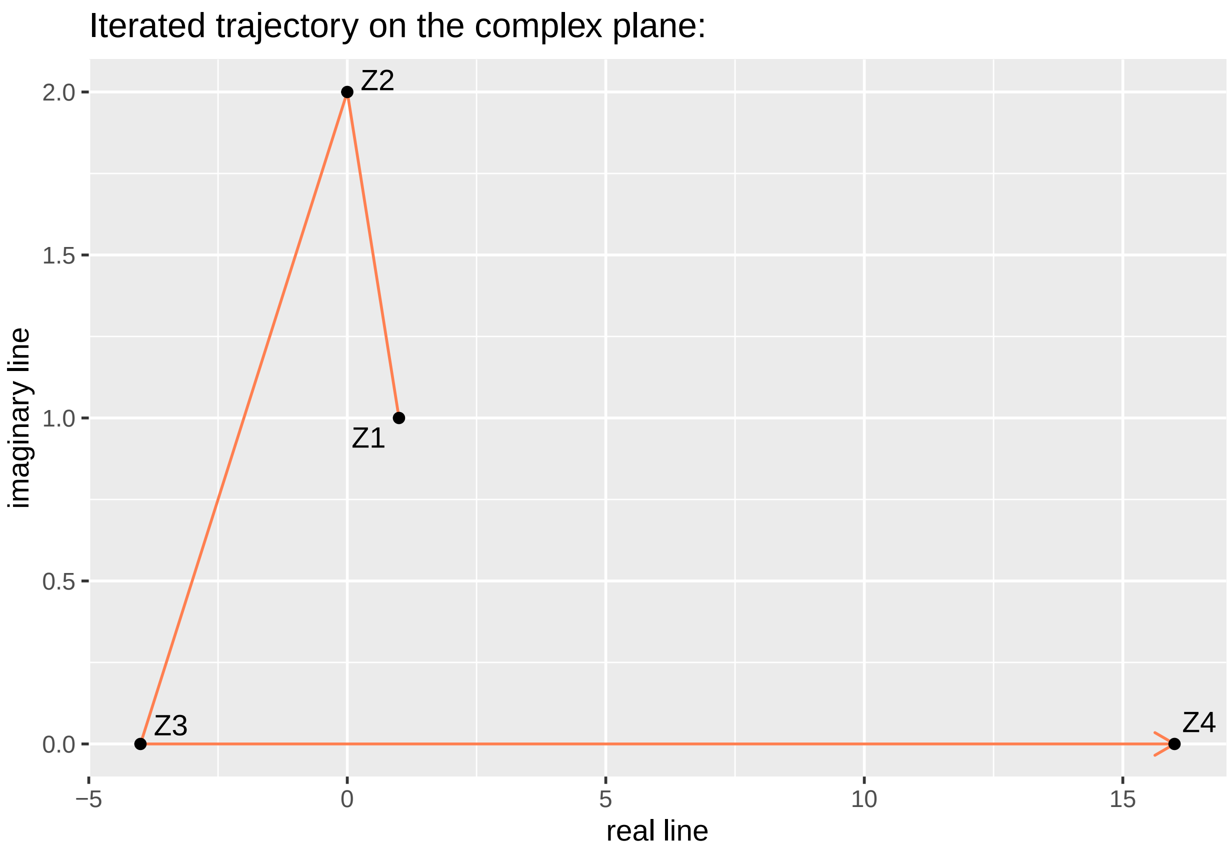
Okay, pretty neat, now let's experiment. What if we take a complex number and square it, and then iterate, squaring its square, then that square, then the next, etc?

Squaring a complex number has the effect of doubling its angle from an axis on the complex plane.

```
n <- 3
x_0 <- 1+1i

numbers <- x_0

for(i in 1:n){
  numbers <- c(numbers, tail(numbers, 1)^2)
}

# let's split the complex numbers into real and imaginary parts to feed to ggplot
real <- Re(numbers)
imaginary <- Im(numbers)

name <- rep("Z", length(numbers))
name <- paste0(name, as.character(1:length(numbers)))

complex_sequence <- tibble(real, imaginary, name)
complex_sequence
```

```
## # A tibble: 4 x 3
##    real imaginary name
##   <dbl>     <dbl> <chr>
## 1     1         1 Z1
## 2     0         2 Z2
## 3    -4         0 Z3
## 4    16         0 Z4
```

```
complex_sequence %>% ggplot(aes(real, imaginary)) +
  geom_path(lineend = "round", linejoin = "mitre",
               arrow = arrow(length = unit(0.3, "cm")),
            colour="coral") +
  geom_point() +
  geom_text_repel(aes(label=name)) +
  labs(title= "Iterated trajectory on the complex plane:", x="real line", y = "imaginary line")
```




```
complex_sequence %>% ggplot(aes(real, imaginary)) +
  geom_segment(lineend = "round", linejoin = "mitre",
               arrow = arrow(length = unit(0.3, "cm")), x=0, y=0,
               xend=real, yend=imaginary,
               colour = "springgreen3") +
  geom_point() +
```

```
geom_text_repel(aes(label=name)) +
labs(title= "Points on the complex plane:", x="real line", y = "imaginary line")
```