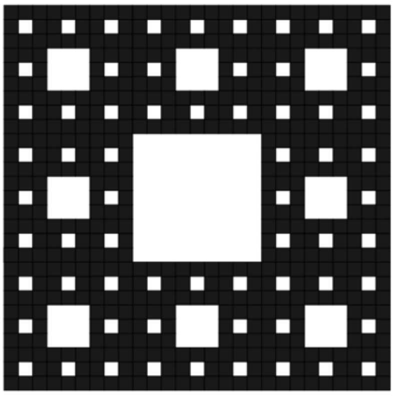
Chapter 16

(16.5) Determine the dimension of the Sierpínski carpet, shown in Fig. 16.6.



Growth factor R	Number of copies N
3	8

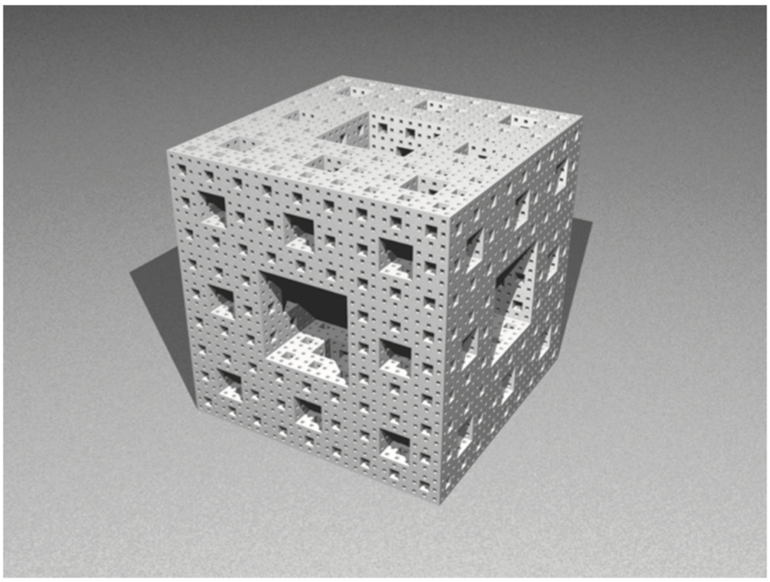
The Hausdorff dimension of the Sierpínski carpet is approximately 1.893.

$$N = R^D$$

$$8 = 3^D$$

$$D = \log_3 8 \approx 1.8928$$

(16.6) Determine the dimension of the Menger sponge, shown in Fig. 16.7.



Growth factor R	Number of copies N
3	20

Suppose we start with a cube and divide it into 27 smaller cubes of equal size (each one, 1/9th the volume of the original cube). We then remove 7 of the smaller cubes from the assemblage: each 1/9th cube at the centre of each face of the original cube, and the 1/9th cube at the centre of the original cube.

At each new iteration of our function, we apply these same steps to all the remaining cubes, removing 7 cubes per cube.

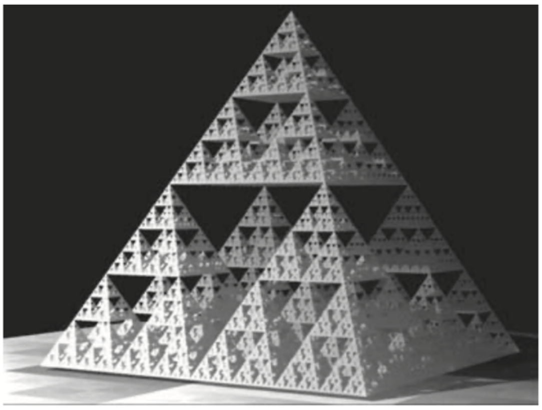
The Hausdorff dimension of the Menger sponge is approximately 2.7268.

$$N = R^D$$

$$20 = 3^D$$

$$D = \log_3 20 \approx 2.7268$$

(16.9) Determine the dimension of the Sierpínski pyramid, shown in Fig. 16.8.



Growth factor R	Number of copies N
2	5

Suppose we start with a pyramid and decide to balance it at it corners on top of 4 other equal sized pyramids (we now have 5 pyramids, forming a new shape with edges twice the length of the original pyramid's). Suppose our new shape has excellent structural integrity, or we have very good glue and lots of patience, and that after balancing the pyramid, we decide that we like the new object we have assembled but want to complicate it further. We make 4 exact copies of the 5-pyramided object and balance the original on them at the corners. We repeat this process until bored or satisfied.

The Hausdorff dimension of the Sierpínski pyramid is approximately 2.3219.

$$N = R^D$$

$$5 = 2^D$$

$$D = \log_2 5 \approx 2.3219$$

(16.3) Consider a Cantor set constructed by removing the middle fifth of each line segment at each step.

(a) Sketch the first several steps in the construction of this Cantor set.

First 6 iterates of the middle-fifths Cantor set (made in Adobe Illustrator):

(b) Determine the dimension of the middle-fifths Cantor set.

Growth factor R	Number of copies N
2.5	2

The Hausdorff dimension of the middle-fifths Cantor set is approximately 0.7565.

$$N = R^D$$

$$2 = 2.5^D$$

$$D = \log_{2.5} 2 \approx 0.7565$$

(16.7) Consider a four-dimensional "cube". (Such an object is sometimes called a hypercube.) If the hypercube were stretched by a factor of three, how many small hypercubes would fit inside the large one?

You could fit 81 small hypercubes inside the large one.

$$N = R^D$$

$$N = 3^4 = 81$$

(16.8) ♯ Suppose one makes a fractal by following the same type of removal process that led to the Menger sponge, Fig. 16.7. However, instead of starting with a cube, start with a four-dimensional hypercube. What is the dimension of the resultant fractal?

I tried to draw this but immediately got into trouble.

This entry on StackExchange was very helpful in figuring this out: https://math.stackexchange.com/questions/2293018/fractal-dimension-for-menger-sponge-in-higher-dimensions

On the StackExchange entry, user wonko suggests this formula for N, (so as to avoid confusion, I've changed some term symbols to differentiate between D, the Hausdorff dimension of the object, and d, the number of dimensions in space: $3^d - 2d - 1$.

Although I am fairly trusting that this is sensible, I don't think I really understand the way this formula works. Is 3 in the first term the scaling factor? Do the subsequent substructions in the second and third terms depend on the scaling factor? Would a different scaling factor require a formula with a different number of terms?

I would love to discuss this more with you after class.

Using wonko's formula, I get:

$d := 4$	4
$3^d - 2 \cdot d - 1$	72
3^4	81
$81 - 72$	9

Where 9 is the number of small hypercubes to be removed from the large hypercubes at every iteration of the Menger-spongifier function (not particularly useful in this case, I was just curious).

We can calculate the Hausdorff dimension of an object using the formula $D = \frac{\ln N}{\ln R}$. Solving the Hausdorff dimension equation for D with $\ln(3^d - 2d - 1)$ as the right-hand numerator results in an approximate Hausdorff dimension for the hypercube Menger sponge of 3.8927.