PicCollage

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1 Example Outlining Algorithm

We want to find $g(s)$ where $s = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10^1 + a_0 10^0$, i.e. a_i is the i^{th} right-most digit in s.

Let's use an example where $s = 8573$.

Suppose we already know that the number of 7's from 0 to 999 is n. Then the number of 7's from 1000 to 1999 should also be n, since it doesn't matter that we are now in the thousands – the thousands digit is 1, so it doesn't affect the number of 7's there are. Similarly the number of 7's from 2000 to 2999, 3000 to 3999, etc. should each be n as well.

The catch here, however, is that when we reach 7000 to 7999, all of these numbers contain a 7. This means that there aren't n numbers with 7, but actually 1000.

This means that from 0 to 7999, we have $7n$ numbers containing a 7 from

$$\{[0, 999], [1000, 1999], [2000, 2999], [3000, 3999], [4000, 4999], [5000, 5999], [6000, 6999]\}$$

and we also have an additional 1000 numbers containing a 7 from $[7000, 7999]$. Hence we have a total of $7n + 1000$ numbers containing 7 from 0 to 7999.

Now, all we need to count are the numbers containing a 7 from 8000 to 8573. The 8 in the thousands digit does not affect the count of numbers containing a 7. Hence, we can just think of this as the equivalent of finding the count of numbers containing a 7 from 0 to 573. Now, assuming that we know the number of 7's from 0 to 99 is m, we can use the same approach. Since the number of 7's in $[0, 99]$ is equivalent to the number of 7's in $[100, 199]$ and so on, there are $5m$ numbers containing 7 from $[0, 499]$. Hence, we have a total of $(7n+1000)+(5m)$ numbers containing 7 from 0 to 8499.

Now, we just need to count the numbers containing a 7 from 8500 to 8573. Again, this means we just need to count the numbers containing a 7 from 0 to 73. Suppose p is the count of the number of 7's from 0 to 9. Then with the same reasoning, we have $7p$ numbers containing a 7 from 0 to 69. We then have the special case for 70 to 73. Since all of the numbers in that range contain a 7, we can just count all the numbers from 0 to 3, which is 4. Hence, in total we have $(7n + 1000) + (5m) + (7p + 4)$ numbers containing 7 from 0 to 8573.

2 Derivation of the calculating the number of 7's from 0 to $10^n - 1$

The main idea driving this process is that we can calculate the number of 7's from 0 to $10^n - 1$.

We can derive this through a little math trick called PIE (Principle of Inclusion and Exclusion):

Suppose $s = a_{n-1} \cdots a_1 a_0$ where a_i are the its digits. Then we can write any number from 0 to $10^n - 1$ in the form of s. For example, if $n = 3$, then 93 can be uniquely written as 093.

Here, we will use PIE, which in our case, says that

(# numbers containing 7) = (# of numbers with at least 1 digit being 7) - (# of numbers with at least 2 digits being 7) + (# of numbers with at least 3 digits being 7) - ...

We can find all of these quantities using our way of representing the number in the form of s.

Let's say we have a generic number $s = a_{n-1} \cdots a_1 a_0$ with at least 1 digit being 7, given that it has to be between 0 to $10^n - 1$. Then one of the digits, a_i, has to be 7, while the rest (there are $n - 1$ of those) can be any number from 0 to 9. Hence there are 10^{n-1} possible assignments to all of the digits such that $a_i = 7$. Now, there are $\binom{n}{1}$ many ways to choose which digit a_i would be the one with a 7. Hence, (# of numbers with at least 1 digit being 7)= $\binom{n}{1}10^{n-1}$.

Similarly, we find that (# of numbers with at least 2 digits being 7)= $\binom{n}{2}10^{n-2}$. We see that the pattern is that (# of numbers with at least m digits being 7)= $\binom{n}{m}10^{n-m}$. Then substituting these values into our equation, we have that

(# numbers containing 7) = $\binom{n}{1}10^{n-1} - \binom{n}{2}10^{n-2} + \binom{n}{3}10^{n-3} - ...$

We can then use another math trick – this equation looks really like the binomial expansion of $(10 - 1)^n = \binom{n}{0}10^{n-0} - \binom{n}{1}10^{n-1} + \binom{n}{2}10^{n-2} - \cdots$. If we do some number manipulation, we see that $10^n - 9^n = 10^n - (10 - 1)^n = \binom{n}{1}10^{n-1} - \binom{n}{2}10^{n-2} + \binom{n}{3}10^{n-3} - \cdots =$(# numbers containing 7), which is exactly what we wanted!

Hence, the number of 7's from 0 to $10^n - 1$ is $10^n - 9^n$.

3 Takeaway

The main takeaway from this process is that we utilize the fact that we can calculate the number of 7's from 0 to $10^n - 1$ in order to break the problem into smaller cases; if we started out with a number s, then we would break this into a problem where we would only need to look at $s[1:]$, and continue to look at smaller suffixes of the problem in order to find our solution.