Fibonacci Numbers Exploration Lab

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§1 Introduction

The Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, $\cdots$, where each term in the sequence (after the second one) is the sum of the preceding two terms. The recursive formula for the Fibonacci numbers is

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_0 = 0 \text{ and } F_1 = 1$$

Many questions have been posed about this sequence, leading to some significant mathematical ideas.

§1.1 Definition of Problem

Fibonacci numbers could be investigated through different topics, such as the closed form of Fibonacci sequence known as Binet's formula, the link between recursive sequence and decimal expansion of fractions, divisibility among indices and periodicity in prime modulus. During this exploration lab, various conjectures have been formed and explored.

§1.2 Real World Applications

Fibonacci numbers are not only related to many mathematical concept such as continued fraction, Euclid's algorithm and Golden ratio, but it is also used in computer science, data analysis, economic and finance. It also describes biological events such as the spiral arrangement of florets in flowers. Fibonacci himself first uses his sequence in the prediction of rabbit population growth, with Fibonacci numbers as the number of rabbits after n generations.

§2 Theorems

§2.1 Closed Form and Binet's Formula

Theorem 2.1

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Proof. We consider the generating function

$$F[x] = \sum_{n=0}^{\infty} x^n \cdot F_n$$

Note that since $F_0 = 0$, we have

$$\sum_{n=0}^{\infty} x^n \cdot F_n = \sum_{n=1}^{\infty} x^n \cdot F_n$$

Now, we consider decomposing the first term from the rest of the summation:

$$\sum_{n=1}^{\infty} x^n \cdot F_n = xF_1 + \sum_{n=2}^{\infty} x^n \cdot F_n$$

We now apply the recursive formula for the Fibonacci numbers:

$$F[x] = xF_1 + \sum_{n=2}^{\infty} x^n \cdot (F_{n-1} + F_{n-2})$$

We can split this into two summations:

$$F[x] = xF_1 + \sum_{n=2}^{\infty} x^n \cdot F_{n-1} + \sum_{n=2}^{\infty} x^n \cdot F_{n-2}$$

First, notice that $F_1 = 1$. Then, we want to write $F[x]$ in terms of itself, so we manipulate our summations a bit:

$$F[x] = x + x\sum_{n=2}^{\infty} x^{n-1} \cdot F_{n-1} + x^2\sum_{n=2}^{\infty} x^{n-2} \cdot F_{n-2}$$

Thus,

$$F[x] = x + xF[x] + x^2F[x]$$
$$F[x]\left(1 - x - x^2\right) = x$$
$$F[x] = \frac{x}{1 - x - x^2} = \frac{-x}{x^2 + x - 1}$$

Solving for the quadratic in the denominator, we find that the solutions are

$$-\alpha = -\frac{1+\sqrt{5}}{2}$$
$$-\beta = -\frac{1-\sqrt{5}}{2}$$

Thus, we write

$$F[x] = \frac{-x}{(x+\alpha)(x+\beta)}$$

We then apply partial fraction decomposition and set

$$F[x] = \frac{y}{x+\alpha} + \frac{x}{x+\beta}$$

Moving these under a common denominator, we find that

$$y(x+\beta) + z(x+\alpha) = -x$$

Substituting $x = -\beta$, we get

$$y(-\beta+\beta) + z(-\beta+\alpha) = -\beta$$
$$z = \frac{\beta}{\beta - \alpha}$$

Similarly, if we plug in $x = -\alpha$

$$y = \frac{\alpha}{\alpha - \beta}$$

Computing, we get

$$z = \frac{\beta}{-\sqrt{5}}$$
$$y = \frac{\alpha}{\sqrt{5}}$$

Back to our equation before we decomposed the fraction:

$$\begin{aligned} F[x] &= \frac{-x}{x^2 + x - 1} \\ &= \frac{-1}{\sqrt{5}}\left(\frac{\alpha}{x+\alpha} - \frac{\beta}{x+\beta}\right) \end{aligned}$$

This looks very similar to a sum of geometric series equation, so we modify our equations a little bit to fit this form:

$$\frac{a}{1-r}$$

$$\frac{\alpha}{x+\alpha} = \frac{1}{\frac{x}{\alpha}+1}$$

We note that dividing by α is the same as multiplying by $-\beta$

$$\frac{\alpha}{x+\alpha} = \frac{1}{1-\beta x} = \sum_{n=0}^{\infty}(x\beta)^n$$

Similarly, for $\frac{1}{x-\beta}$, we find

$$\frac{1}{x-\beta} = \sum_{n=0}^{\infty}(x\alpha)^n$$

Since we are summing over the same bounds, we can thus combine these two summations:

$$\sum_{n=0}^{\infty} x^n \left(\beta^n - \alpha^n\right)$$

Bringing back our $\frac{-1}{\sqrt{5}}$ term from before, we can move it into our summation to get

$$F[x] = \sum_{n=0}^{\infty} \frac{-1}{\sqrt{5}} x^n \left(\beta^n - \alpha^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} x^n \left(\alpha^n - \beta^n\right)$$

$$= \sum_{n=0}^{\infty} x^n \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Comparing this with our initial generating function,

$$F[x] = \sum_{n=0}^{\infty} x^n F_n$$

we find that the closed form is indeed

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$\square$

§2.2 An Interesting Conjecture on Fractions

Theorem 2.2

$$\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{89}$$

Proof. Let $n \in \mathbb{N}$ and $x = \sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}}$.

$$\begin{aligned}
x &= \sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} \\
&= \left(\sum_{n=1}^{\infty} \frac{F_n}{10^{n+1}}\right) + \frac{F_0}{10} \\
&= \sum_{n=1}^{\infty} \frac{F_n}{10^{n+1}} \\
&= \left(\sum_{n=2}^{\infty} \frac{F_n}{10^{n+1}}\right) + \frac{F_1}{10^2} \\
&= \left(\sum_{n=2}^{\infty} \frac{(F_{n-1} + F_{n-2})}{10^{n+1}}\right) + \frac{1}{10^2} \\
&= \left(\sum_{n=2}^{\infty} \frac{F_{n-1}}{10^{n+1}}\right) + \left(\sum_{n=2}^{\infty} \frac{F_{n-2}}{10^{n+1}}\right) + \frac{1}{10^2} \\
&= \left(\frac{1}{10} \sum_{n=2}^{\infty} \frac{F_{n-1}}{10^{n}}\right) + \left(\frac{1}{10^2} \sum_{n=2}^{\infty} \frac{F_{n-2}}{10^{n-1}}\right) + \frac{1}{10^2} \\
&= \left(\frac{1}{10} \sum_{n=1}^{\infty} \frac{F_{n}}{10^{n+1}}\right) + \left(\frac{1}{10^2} \sum_{n=0}^{\infty} \frac{F_{n}}{10^{n+1}}\right) + \frac{1}{10^2}
\end{aligned}$$

$$= \frac{x}{10} + \frac{x}{10^2} + \frac{1}{10^2}$$

Thus,

$$\begin{aligned}
x &= \frac{1}{10^2} + \frac{1}{10}x + \frac{1}{10^2}x \\
100x &= 1 + 10x + x \\
(100 - 11)x &= 1 \\
x &= \frac{1}{89}
\end{aligned}$$

□

§2.3 Generalization of Decimal Expansion of Fraction

Theorem 2.3

If $A_n = A_{n-1} + A_{n-2} + \cdots + A_{n-k}$ is a recursive relation, then $\sum_{n=0}^{\infty} \frac{A_n}{10^{n+1}} = \frac{8 \cdot 10^{k-1}A_0 + 8 \cdot 10^{k-2}A_1 + \cdots + 8 \cdot 10A_{k-2} + 8A_{k-1} + A_k}{8 \cdot 10^k + 1}$

Proof. Let $n \in \mathbb{N}$ and $x = \sum_{n=0}^{\infty} \frac{A_n}{10^{n+1}}$

$$\begin{aligned}
x &= \sum_{n=0}^{\infty} \frac{A_n}{10^{n+1}} \\
&= \sum_{n=k}^{\infty} \frac{(A_{n-1} + A_{n-2} + A_{n-3} + \cdots + A_{n-k})}{10^{n+1}} + \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k}\right) \\
&= \sum_{n=k}^{\infty} \frac{A_{n-1}}{10^{n+1}} + \sum_{n=k}^{\infty} \frac{A_{n-2}}{10^{n+1}} + \cdots + \sum_{n=k}^{\infty} \frac{A_{n-k}}{10^{n+1}} + \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k}\right) \\
&= \frac{1}{10} \sum_{n=k}^{\infty} \frac{A_{n-1}}{10^n} + \frac{1}{10^2} \sum_{n=k}^{\infty} \frac{A_{n-2}}{10^{n-1}} + \cdots \frac{1}{10^k} \sum_{n=k}^{\infty} \frac{A_{n-k}}{10^{n-(k-1)}} + \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k}\right)
\end{aligned}$$

Let $i \in \{1, 2, 3, \cdots, k-1, k-2\}$. Then

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{A_{n-i}}{10^{n-(i-1)}} &= \sum_{n=0}^{\infty} \frac{A_n}{10^{n+1}} - \sum_{n=0}^{k-(i+1)} \frac{A_n}{10^{n+1}} \\
&= x - \sum_{n=0}^{k-(i+1)} \frac{A_n}{10^{n+1}}
\end{aligned}$$

Thus,

$$
\begin{aligned}
x &= \frac{1}{10}(x - \sum_{n=0}^{k-2} \frac{A_n}{10^{n+1}}) + \frac{1}{10^2}(x - \sum_{n=0}^{k-3} \frac{A_n}{10^{n+1}}) + \cdots + \frac{1}{10^{k-1}} \sum_{n=0}^{0} \frac{A_n}{10^{n+1}} + \frac{1}{10^k} x \\
&+ \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k} \right) \\
&= x(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^k}) - \left(\frac{1}{10} \sum_{n=0}^{k-2} \frac{A_n}{10^{n+1}} + \frac{1}{10^2} \sum_{n=0}^{k-3} \frac{A_n}{10^{n+1}} + \cdots + \frac{1}{10^{k-1}} \sum_{n=0}^{0} \frac{A_n}{10^{n+1}} \right) \\
&+ \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k} \right) \\
&= \frac{\frac{x}{10}(1 - (\frac{1}{10})^k)}{1 - \frac{1}{10}} - \left((\frac{A_0}{10}(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{k-1}}) + \frac{A_1}{10^2}(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^{k-2}}) + \cdots + \frac{A_{k-2}}{10^{k-1}}(\frac{1}{10}) \right) \\
&+ \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k} \right) \\
&= \frac{\frac{x}{10}(1 - (\frac{1}{10})^k)}{1 - \frac{1}{10}} - \left(\frac{\frac{A_0}{10^2}(1 - (\frac{1}{10})^{k-1})}{1 - \frac{1}{10}} + \frac{\frac{A_1}{10^3}(1 - (\frac{1}{10})^{k-2})}{1 - \frac{1}{10}} + \cdots + \frac{\frac{A_{k-2}}{10^k}(1 - \frac{1}{10})}{1 - \frac{1}{10}} \right) \\
&+ \left(\frac{A_0}{10} + \frac{A_1}{10^2} + \cdots + \frac{A_{k-1}}{10^k} \right)
\end{aligned}
$$

$$
\begin{aligned}
9 \cdot 10^k x &= x(10^k - 1) - \Big(A_0(10^{k-1} - 1) + A_1(10^{k-2} - 1) + \cdots + A_{k-2}(10 - 1) \Big) \\
&+ \Big(9 \cdot 10^{k-1} A_0 + 9 \cdot 10^{k-2} A_1 + \cdots + 9A_{k-1} \Big) \\
x(9 \cdot 10^k - 10^k + 1) &= - \Big(A_0 10^{k-1} + A_1 10^{k-2} + \cdots + A_{k-2} 10 - (A_0 + A_1 + \cdots + A_{k-2}) \Big) \\
&+ \Big(9 \cdot 10^{k-1} A_0 + 9 \cdot 10^{k-2} A_1 + \cdots + 9A_{k-1} \Big) \\
x(8 \cdot 10^k + 1) &= 8 \cdot 10^{k-1} A_0 + 8 \cdot 10^{k-2} A_1 + \cdots + 8 \cdot 10 A_{k-2} + 9A_{k-1} + (A_0 + A_1 + \cdots + A_{k-2}) \\
&= 8 \cdot 10^{k-1} A_0 + 8 \cdot 10^{k-2} A_1 + \cdots + 8 \cdot 10 A_{k-2} + 9A_{k-1} + (A_k - A_{k-1}) \\
&= 8 \cdot 10^{k-1} A_0 + 8 \cdot 10^{k-2} A_1 + \cdots + 8 \cdot 10 A_{k-2} + 8A_{k-1} + A_k \\
x &= \frac{8 \cdot 10^{k-1} A_0 + 8 \cdot 10^{k-2} A_1 + \cdots + 8 \cdot 10 A_{k-2} + 8A_{k-1} + A_k}{8 \cdot 10^k + 1}
\end{aligned}
$$

□

§2.4 Divisibility Among Indices

Theorem 2.4

$F_n \mid F_{x \cdot n}$ for $n, x \in \mathbb{N}$

Proof. Let F_n be the n^{th} term of Fibonacci sequence.
Using induction, we have, as base case,

$$\begin{aligned} F_{n+2} &= F_n + F_{n+1} \\ F_{n+3} &= F_{n+1} + F_{n+2} \\ &= F_n + 2F_{n+1} \\ F_{n+4} &= F_{n+2} + F_{n+3} \\ &= 2F_n + 3F_{n+1} \end{aligned}$$

We see that,
- Every number in the Fibonacci sequence can be written as a linear equation with F_n and F_{n+1} as constant;
- The coefficients of F_n and F_{n+1} also follow the Fibonacci sequence. We notice that for any number F_k in F_n, the coefficient are $F_{k-(n+1)}$ and F_{k-n}.

Hence, for F_{2n},

$$F_{2n} = F_{n-1}F_n + F_nF_{n+1} = F_n(F_{n-1} + F_{n+1})$$

Thus, our base case is $F_n \mid F_{2 \cdot n}$
For the inductive step, assume that the statement is true for $F_{(x-1) \cdot n}$, in other words, $F_n \mid F_{(x-1) \cdot n}$

We have,

$$F_{xn} = F_{xn-(n+1)} \; F_n + F_{xn-n} \; F_{n+1} = F_{(x-1)n-1} \; F_n + F_{(x-1) \cdot n} \; F_{n+1}$$

Since $F_n \mid F_{(x-1) \cdot n}$, $\exists k \in \mathbb{Z}$ such that $F_{(x-1) \cdot n} = kF_n$.
So,

$$F_{xn} = F_n(F_{(x-1)n-1} + kF_{n+1})$$

Since $(F_{(x-1)n-1} + kF_{n+1}) \in \mathbb{Z}$, $F_n \mid F_{xn}$ □

§2.5 An Interlude on Least Common Multiples

Theorem 2.5

Denote P_x as the period of the Fibonacci numbers mod x. $P_{ab} = LCM[P_a, P_b]$ for relatively prime a and b

	F_0	F_1	$\cdots$	$F_{P_{ab}-1}$	$F_{P_{ab}}$
mod a	a_0	a_1	$\cdots$	1	0
mod b	b_0	b_1	$\cdots$	1	0
mod ab	ab_0	ab_1	$\cdots$	1	0

Proof. Notice that since the Fibonacci sequence is defined recursively for the last two terms, the two terms at the end of the period of any P_x must be 1 and 0.

We know from fundamental modular arithmetic that if $\alpha \equiv 0 \bmod ab$, then $\alpha \equiv 0 \bmod a \equiv 0 \bmod b$. A similar argument can be made for $\beta \equiv 1 \bmod ab$.

As we showed earlier, in P_x, every period ends with 1 followed by 0. Thus, $P_a | P_{ab}$ and $P_b | P_{ab}$.

The period is also defined as the least index of the Fibonacci numbers such that the sequence repeats indefinitely, and thus we take the least common multiple of P_a and P_b to construct the period of ab. □

§2.6 Sum of Fibonacci numbers

Theorem 2.6

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

Proof.

$$\begin{aligned}
x &= \sum_{i=0}^{n} F_i \\
&= F_0 + 2\sum_{i=0}^{n-2} F_i + F_{n-1} \\
&= F_0 + 2\sum_{i=0}^{n-2} F_i + F_{n-1} + (F_{n-1} - F_{n-1} + 2F_n - 2F_n) \\
&= F_0 + 2\sum_{i=0}^{n} F_i - F_{n-1} - 2F_n \\
&= F_0 + 2x - (F_{n-1} + F_n) - F_n \\
&= -F_0 + F_{n+1} + F_n \\
&= F_{n+2} - 1
\end{aligned}$$

□

§2.7 Fibonacci Sequence and the Golden Ratio

Theorem 2.7

The continued fraction of $\phi = \frac{1+\sqrt{5}}{2}$ is $[\bar{1}]$.

The convergents of this continued fraction are $\frac{F_{n+1}}{F_n}$.

Proof. The continued fraction of $\frac{1+\sqrt{5}}{2}$ is

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

We notice that

$$\frac{1+\sqrt{5}}{2} = [\bar{1}]$$

$$\begin{aligned}
\text{So let } \phi &= \frac{1+\sqrt{5}}{2} \\
\text{Then } \phi &= 1+\frac{1}{\phi} \\
\phi^2 &= \phi+1 \\
\phi^2-\phi-1 &= 0 \\
\phi &= \frac{1\pm\sqrt{5}}{2}
\end{aligned}$$

Using the magic box to find the convergents $\frac{P_n}{Q_n}$,

a_n	-	-	1	1	1	1	1
P_n	0	1	1	2	3	5	8
Q_n	1	0	1	1	2	3	5

Knowing that $P_k = a_kP_{k-1} + P_{k-2}$ and $Q_k = a_kQ_{k-1} + Q_{k-2}$, and since $a_k = 1$, Then $P_k = P_{k-1} + P_{k-2}$ and $Q_k = Q_{k-1} + Q_{k-2}$, which corresponds exactly to the Fibonacci sequence.

□

§3 Open Conjectures

§3.1 Period of powers of prime

Theorem 3.1

Denote K_p as the period of F_n (modp), $\quad\forall$ p where p is a prime, $K_{p^a} = K_p \cdot p^{a-1}$

Numerical examples:

mod n	period	mod n	period	mod n	period	mod n	period
2	3	3	8	5	20	7	16
4	6	9	24	25	100	49	112
8	12	27	72	125	500	343	784
16	24	81	216	625	2500	2401	5488

Proof. Take a = 2

Consider the first K_p terms:

$$1, 1, 2, \cdots, F_{K_p-1}, F_{K_p}$$
$$\therefore F_{K_p-1} = 1(\text{mod}p) \text{ and } F_{K_p} = 0(\text{mod}p)$$

Then $F_{K_p-1} = pq + 1$ and $F_{K_p} = pm$ for some $q, m \in \mathbb{N}$

$$\begin{aligned}
&\implies F_{K_p+1} = p(q+m) + 1 \\
&\implies F_{K_p+2} = p(q+2m) + 1 \\
&\implies F_{K_p+3} = p(2q+3m) + 2 \\
&\implies F_{K_p+4} = p(3q+5m) + 3
\end{aligned}$$

We see a pattern in the coefficients of q and m:

$$\begin{aligned}
\therefore F_{2K_p} &= p(F_{K_p}q + (F_{K_p}+1)m) + F_{K_p} \\
&= p\left((pm)q + (p(q+m)+1)m\right) + pm \\
&= p\left((pm)q + p(p(q+m)+1)m\right) + pm \\
&\equiv (pm + pm) \mod p^2 \\
&\equiv (2pm) \mod p^2
\end{aligned}$$

Now, we will use induction.

Assume true for $(p-1)^{th}$ period

$$\begin{aligned} F_{(p-1)\cdot K_p} &= (p-1)pm \equiv 0 \mod p \\ \implies F_{(p-1)\cdot K_p+1} &= (p-1)(pm+pq)+1 \\ \implies F_{(p-1)\cdot K_p+2} &= (p-1)(2pm+pq)+1 \\ \implies F_{(p-1)\cdot K_p+3} &= (p-1)(3pm+2pq)+2 \\ \therefore F_{(p-1)\cdot K_p-1} &= (p-1)pq+1 \equiv 1 \mod p \end{aligned}$$

We observe same patterns as above and we can write,

$$F_{(p)\cdot K_p-1} = (p-1)(F_{K_p}\cdot pm + F_{K_p}\cdot pq) + F_{K_p}$$

$$\begin{aligned} \therefore F_{(p)\cdot K_p} &= (p-1)(F_{K_p}\cdot pm + (F_{K_p+1})\cdot pq) + F_{K_p} \\ &= (p-1)\big((p(q+m)+1)pm + (p)^2 mq\big) + pm \\ &\equiv ((p-1)pm + pm) \bmod p^2 \\ &\equiv 0 \bmod p^2 \\ \therefore F_{p\cdot K_p} &\equiv 0 \bmod p^2 \end{aligned}$$

$$\therefore K_{p^2} \leq pK_p$$

Consider,

$$F_{K_{p+x}} = p(F_x\cdot p + F_{x+1}\cdot m) + F_x \quad with \quad 0 < x < K_p$$

Now,

$$\begin{aligned} p(F_x\cdot p + F_{x+1}\cdot m) \equiv 0 \quad &\mod p \\ And \quad (F_x) \equiv 0 \quad &\mod p, \\ only \quad if \quad F_{x-1} \not\equiv 1 \quad &\mod p \end{aligned}$$

But if $F_{K_{p+x}}$ is the period then

$$F_{x-1} \equiv 1 \mod p^2$$

$$\implies F_{x-1} \equiv 1 \mod p$$
$$\rightarrow\leftarrow$$
$$\therefore F_{K_{p+x}} \not\equiv 0 \mod p^2$$

$\implies$ There are no numbers between F_{K_p} and $F_{K_{p+x}}$ that are congruent to 0 $(mod \quad p^2)$
So period of p^2 is either K_p or pK_p.

So now the question is how to prove that $F_{K_p} \neq F_{K_{p^2}}$

For that we could either show that $(m, p) = 1$

$\implies p^2 \nmid (pm)$
$\implies F_{K_p} \neq F_{K_{p^2}}$

Or the lemma $F_n \mid F_{x \cdot n}$ could be useful.

If we can prove this, then we can further extend it to get the required result. □

§3.2 Periodicity in Odd Modulus

Theorem 3.2
If the Fibonacci sequence F_n is rewritten in modulo m where m $\in \mathbb{N}$ is odd and greater than 1, then the period of the F_n mod m sequence must be a positive even number.

Proof. We will prove the converse, that if the period of F_n mod m is odd then m must be even. Let $m \in \mathbb{N}$ s.t. $m > 1$. Let k be the period of F_n mod m s.t $k \in \mathbb{N}$ is odd.

Let k be the period of F_n mod m s.t. $k \in \mathbb{N}$ is odd.

Then $k \in \mathbb{N}$. Note that if $x = 0$, then $k = 1 \implies m = 1$ which contradicts the assumption that $m > 1$. Also note that if $x < 0$, then $k < 0 \implies k \notin \mathbb{N}$ which contradicts the assumption that $k \in \mathbb{N}$.

Since k is the period of F_n mod m and $F_0 \equiv 0 \mod m$ and $F_1 \equiv 1 \mod m$, $F_k \equiv 0 \mod m$ and $F_{k+1} \equiv 1 \mod m$.

Then $F_{k-1} \equiv F_{k+1} - F_k \equiv -1 \mod m$, $F_{k-2} \equiv F_k - F_{k-1} \equiv 1 \mod m$, $F_{k-3} \equiv F_{k-1} - F_{k-2} \equiv -2 \mod m$, $\cdots$, $\mathrm{F}_{k-w} \equiv (-1)^{w+1} F_w \mod m$.

This can be visualized through a table:

F_1	F_2	F_3	F_4	$\cdots$	$F_{\frac{k-1}{2}}$	$F_{\frac{k+1}{2}}$	$\cdots$	F_{k-4}	F_{k-3}	F_{k-2}	F_{k-1}
1	1	2	3	$\cdots$	$F_{\frac{k-1}{2}}$	$F_{\frac{k+1}{2}}$	$\cdots$	-3	2	-1	1

Since k is odd, $k \equiv\ 1 \bmod 4$ or $k \equiv\ 3 \bmod 4$.

1. Suppose $k \equiv\ 1 \bmod 4 \implies$ k = 4q + 1 for some $q \in \mathbb{N}$.

Then $\frac{k-1}{2} = 2q$
$\implies F_{\frac{k-1}{2}} \equiv\ (-1)^{2q+1} F_{k-\frac{k-1}{2}} \bmod m$
$\implies F_{\frac{k-1}{2}} \equiv\ -F_{\frac{k+1}{2}} \bmod m$

We also get that $\frac{k+1}{2} = 2q + 1$
$\implies F_{\frac{k+1}{2}} \equiv\ (-1)^{2q+2} F_{k-\frac{k+1}{2}} \bmod m$
$\implies F_{\frac{k+1}{2}} \equiv\ F_{\frac{k-1}{2}} \bmod m$

Thus, $F_{\frac{k-1}{2}} \equiv\ -F_{\frac{k+1}{2}}\ \equiv\ F_{\frac{k+1}{2}} \bmod m \implies 2\mathrm{F}_{\frac{k+1}{2}} \equiv\ 0 \bmod m$

If $(m, 2) = 1$, then $F_{\frac{k+1}{2}} \equiv\ 0 \bmod m \implies F_{\frac{k-1}{2}} \equiv\ 0 \bmod m$

Then $F_{\frac{k-1}{2}} + F_{\frac{k+1}{2}} \equiv\ F_{\frac{k+3}{2}} \equiv\ 0 \bmod m$. We see that all terms following $F_{\frac{k-1}{2}}$ are congruent to 0 mod m because $F_{n+1} = F_n + F_{n-1} \forall n \in \mathbb{N}$ and there exists two consecutive terms that are congruent to 0 mod m.

This is not possible because $F_{k-1} \equiv\ 1 \bmod m$ and $k - 1 > \frac{k-1}{2}$.
Thus, $(m, 2) \neq 1 \implies$ m is even.

2. Suppose $k \equiv\ 3 \bmod 4 \implies k = 4q + 3$ for some q $\in \mathbb{N}$.

Then $\frac{k-1}{2} = 2q + 1$
$\implies F_{\frac{k-1}{2}} \equiv\ (-1)^{2q+2} F_{k-\frac{k-1}{2}} \bmod m$
$\implies F_{\frac{k-1}{2}} \equiv\ F_{\frac{k+1}{2}} \bmod m$

We also get that $\frac{k+1}{2} = 2q$
$\implies F_{\frac{k+1}{2}} \equiv\ (-1)^{2q+1} F_{k-\frac{k+1}{2}} \bmod m$
$\implies F_{\frac{k+1}{2}} \equiv\ -F_{\frac{k-1}{2}} \bmod m$

Thus, $F_{\frac{k+1}{2}} \equiv\ -F_{\frac{k-1}{2}}\ \equiv_{\frac{k-1}{2}} \bmod m \implies 2\mathrm{F}_{\frac{k-1}{2}} \equiv \bmod m$

If $(m, 2) = 1$, then $F_{\frac{k-1}{2}} \equiv\ 0 \bmod m \implies F_{\frac{k+1}{2}} \equiv\ 0 \bmod m$

Then $F_{\frac{k-1}{2}} + F_{\frac{k+1}{2}} \equiv\ F_{\frac{k+3}{2}} \equiv\ 0 \bmod m$. We see that all terms following $F_{\frac{k-1}{2}}$ are congruent to 0 mod m because $F_{n+1} = F_n + F_{n-1} \forall n \in \mathbb{N}$ and there exists two consecutive terms that are congruent to 0 mod m.

This is not possible because $F_{k-1} \equiv\ 1 \bmod m$ and $k - 1 > \frac{k-1}{2}$.
Thus, $(m, 2) \neq 1 \implies$ m is even.

$\square$