Data Structures and Algorithms

Chapter 8

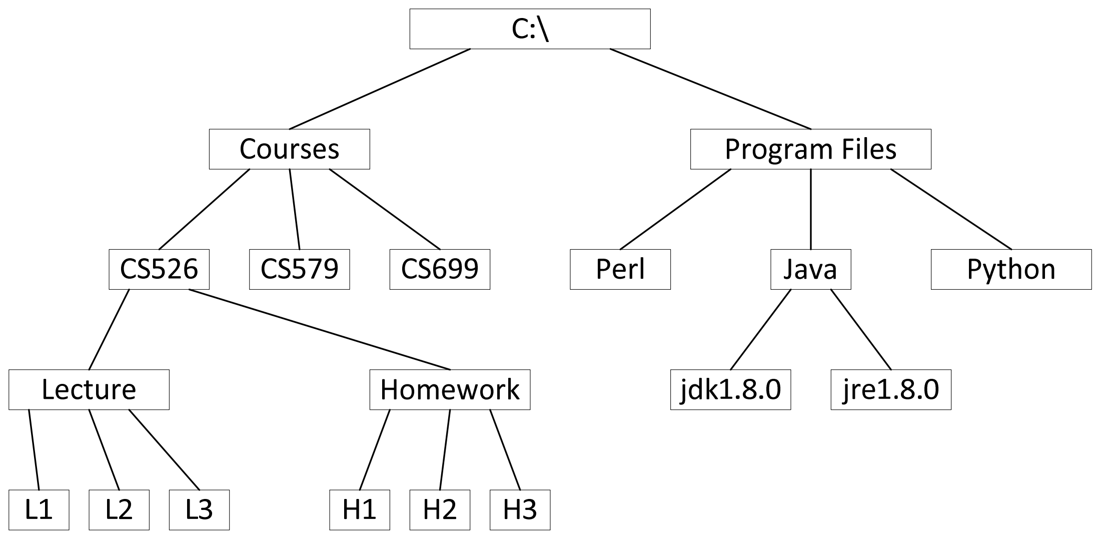
General Trees
Basics

- A *graph* is a set of nodes and a set of edges.
- Formally, a graph $G = (V, E)$, where V is a set of nodes (or vertices) and E is a set of edges.
- Each edge connects two nodes, and is represented as (u, v), where u and v are nodes.

- A ***tree*** is a connected, acyclic, undirected graph with a distinguished node called *root*.
- *Connected*: There is a path from every node to every other node.
- *Acyclic*: There is no cycle
- *Undirected*: Edges have no direction

General Trees
Basics

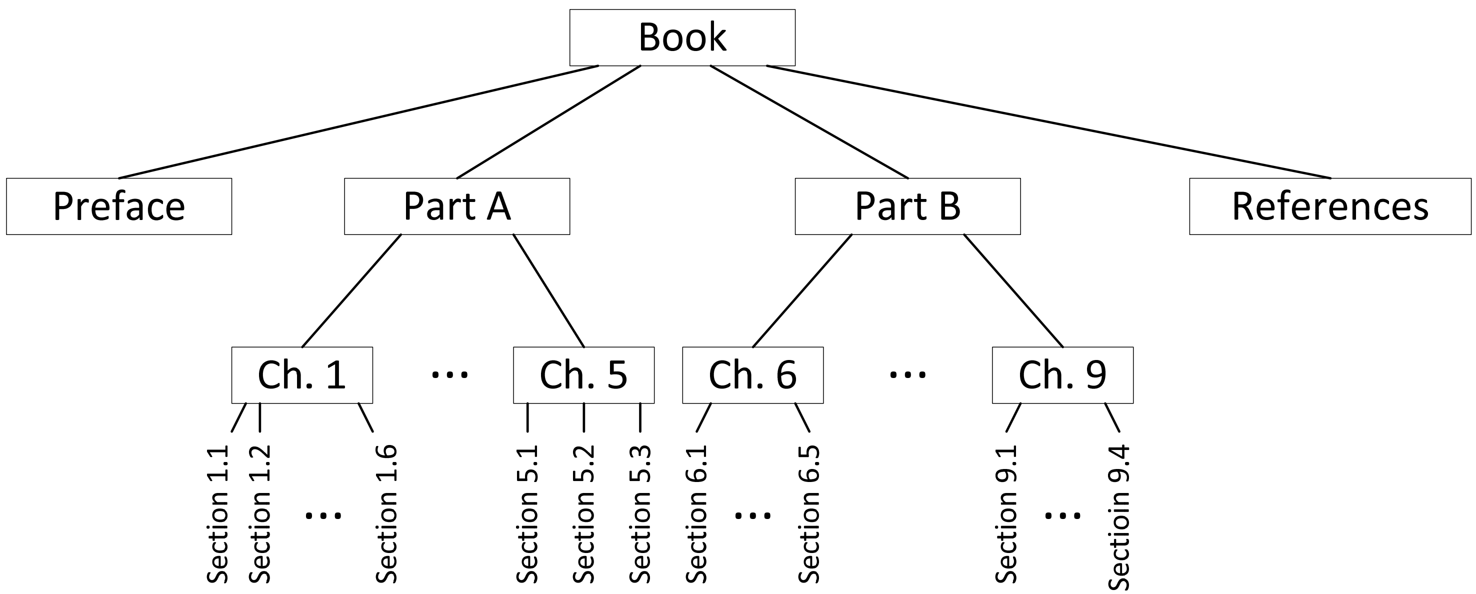
- Example



- Root, parent, child, siblings
- Internal node, external node (or leaf node)
- Ancestor, descendant
- Path

General Trees
Basics

- Ordered tree: There is meaningful ordering among siblings:



General Trees
Basics

- In the subsequent slides, “position” is used in many examples. You can consider it as a “node” or an “element” in the data structure.
- The source programs that come with our textbook uses “position.”
- If you are required to write a program that uses our textbooks source code that uses “position,” I will give you a substitute code that does not use “position.”

General Trees
Tree ADT

- Accessor methods
  - root( ): Returns the position of the root of the tree, or null if the tree is empty.
  - parent(*p*): Returns the position of the parent of position *p*, or null if *p* is the root.
  - children(*p*): Returns the children of position *p*, if any. If the tree is an ordered tree, children are ordered in the result.
  - numChildren(*p*): Returns the number of children of position *p*.

General Trees
Tree ADT

- Query methods
  - isInternal(*p*): Returns true if position *p* is an internal node.
  - isExternal(*p*): Returns true if position *p* is an external node (or a leaf node).
  - isRoot(*p*): Returns true if position *p* is the root of the tree.

General Trees
Tree ADT

- Other general methods

  - size( ): Returns the number of positions (or the elements) in the tree.
  - isEmpty( ): Returns true if the tree does not have any position (or element).
  - iterator( ): Returns an iterator for all elements in the tree. So, the tree is *Iterable*.
  - positions( ): Returns an iterable collection of all positions of the tree.

General Trees
Tree ADT

- Tree interface

```
public interface Tree<E> extends Iterable<E> {
  Position<E> root();
  Position<E> parent(Position<E> p) throws IllegalArgumentException;
  Iterable<Position<E>> children(Position<E> p)
                                throws IllegalArgumentException;
  int numChildren(Position<E> p) throws IllegalArgumentException;
  boolean isInternal(Position<E> p) throws IllegalArgumentException;
  boolean isRoot(Position<E> p) throws IllegalArgumentException;
  int size();
  boolean isEmpty();
  Iterator<E> iterator();
  Iterable<Position<E>> positions();
}
```

General Trees
Tree ADT

- AbstractTree abstract class

```
public abstract class AbstractTree<E> implements Tree<E> {

public boolean isInternal(Position<E> p)
      { return numChildren(p) > 0; }
public boolean isExternal(Position<E> p)
      { return numChildren(p) == 0; }
public boolean isRoot(Position<E> p) { return p == root(); }
public boolean isEmpty() { return size() == 0; }
. . .
}
```

General Trees
Depth and Height

- Depth
  - If *p* is the root, the depth of *p* is 0.
  - Otherwise, the depth of *p* is one plus the depth of its parent.

```
public int depth(Position<E> p) throws IllegalArgumentException
{
  if (isRoot(p))
      return 0;
  else
      return 1 + depth(parent(p));
}
```

Running time = $O(d_p + 1)$
d_p is the depth of p

General Trees
Depth and Height

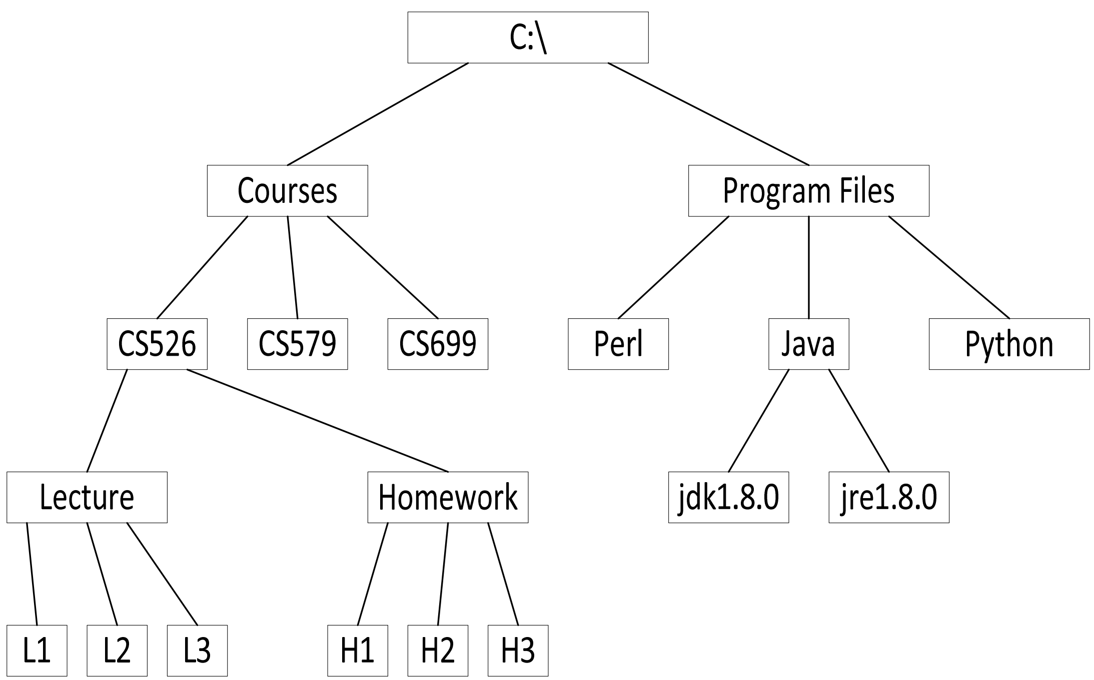
- The *height* of a tree is the length of the longest path from the root downward to an external node.
- Recursive definition:
  - If *p* is a leaf, then the height of *p* is 0.
  - Otherwise, the height of *p* is one more than the maximum of the heights of *p’s* children.

```
public int height(Position<E> p) throws IllegalArgumentException {
  int h = 0;                          // base case if p is external
  for (Position<E> c : children(p))
    h = Math.max(h, 1 + height(c));
  return h;
}
```

Running time = O(n)
n is the number of positions

General Trees
Depth and Height

- Example



- c:\
  - depth 0
  - height 4
- CS526
  - depth 2
  - height 2
- Program Files
  - depth 1
  - height 2

Binary Trees

- A binary tree is an ordered tree with the following properties:
  - Every node has at most two children.
  - Each child node is labeled as being a *left child* or a *right child*.
  - A left child precedes a right child in the order of children of a node

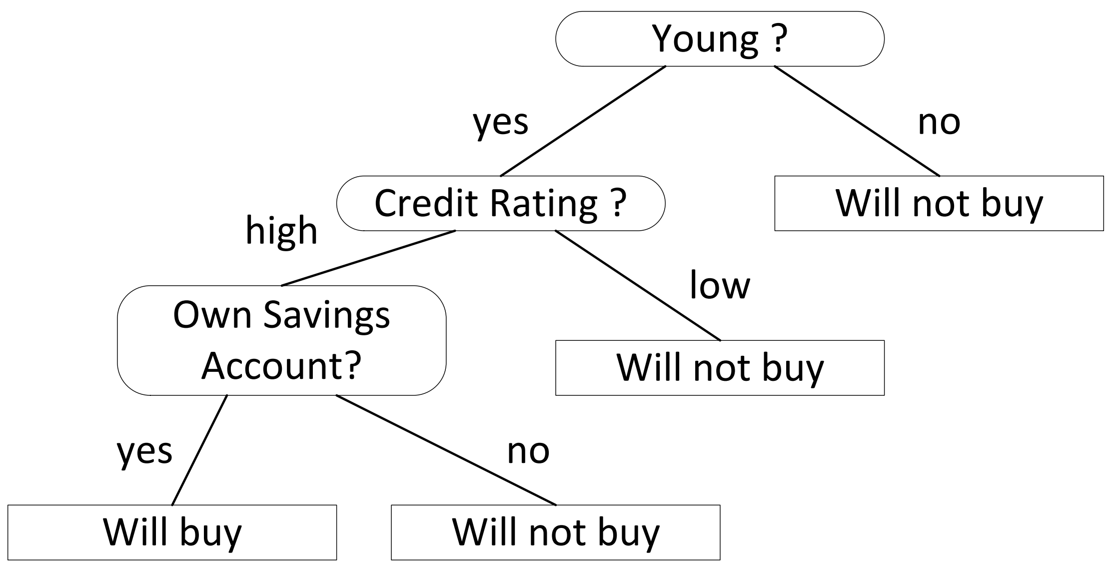
Binary Trees

- The subtree rooted at the left or right child of an internal node *v* is called the *left subtree* or the *right subtree*, respectively, of *v*.

- A binary tree is *proper* if each node has either zero or two children. (also referred to as *full binary tree*).
- So, in a proper binary tree, every internal node has exactly two children.
- A binary tree that is not proper is *improper*.

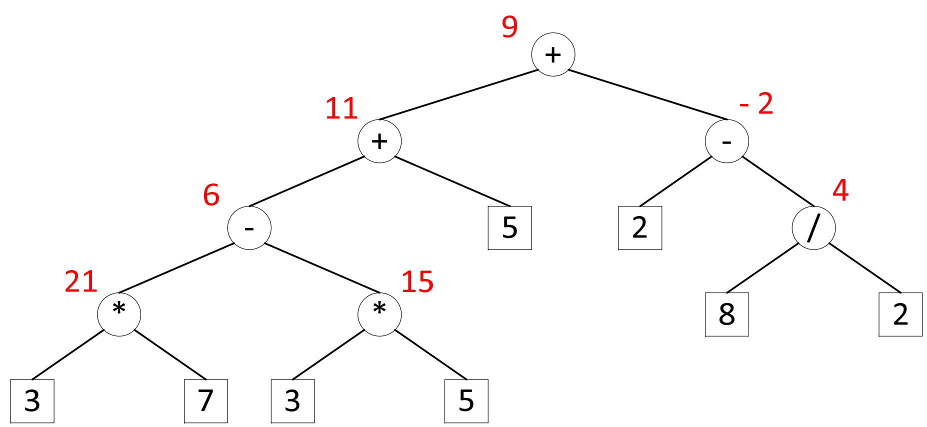
Binary Trees

- Example (a decision tree)



Binary Trees

- Example (arithmetic expression tree)



- ((((3 * 7) – (3 * 5)) + 5) + (2 – (8 / 2)))

Binary Trees

- A binary tree can be recursively defined as follows:

  - A binary tree is either
    - An empty tree, or
    - A nonempty tree with a root node *r* and two binary trees that are the left subtree and the right subtree of *r*. One or both of these subtrees can be empty, by definition.

Binary Trees
ADT

- The binary tree ADT is a specialization of the *Tree ADT.*
- Following additional methods are defined:
  - left(*p*): Returns the position of the left child of *p*. Returns null if *p* has no left child.
  - right(*p*): Returns the position of the right child of *p*. Returns null if *p* has no right child.
  - sibling(*p*): Returns the position of the sibling of *p*. Returns null if *p* has no sibling.

Binary Trees
ADT

- BinaryTree interface

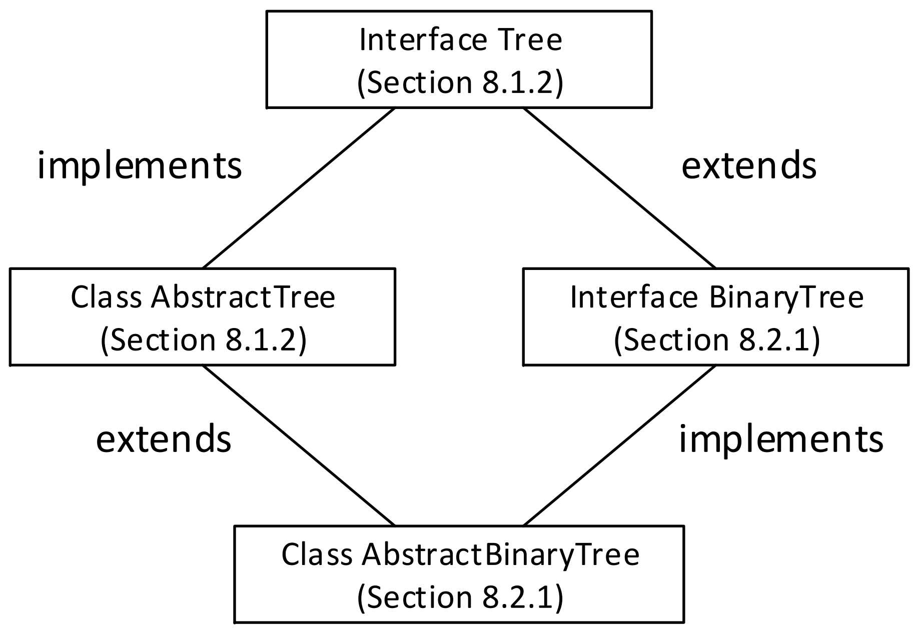
```
public interface BinaryTree<E> extends Tree<E> {

  Position<E> left(Position<E> p) throws
                                  IllegalArgumentException;
  Position<E> right(Position<E> p) throws
                                  IllegalArgumentException;
  Position<E> sibling(Position<E> p) throws
                                  IllegalArgumentException;
}
```

Binary Trees
ADT

- AbstractBinaryTree: extends AtstractTree and implements BinaryTree

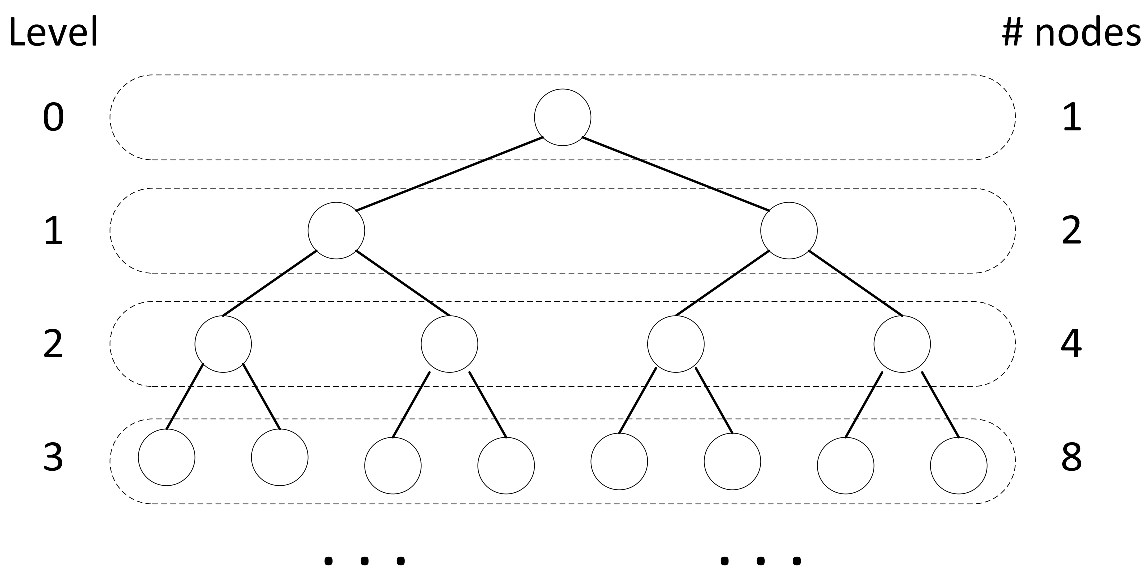
- Additional methods:
  - sibling
  - numChildren
  - children



- AbstractBinaryTree.java

Binary Trees
Binary Tree Properties

- Let *level d* of a binary tree *T* be the set of nodes at depth *d* of *T*.



- The maximum number of nodes at level *d* is 2^d.

Binary Trees

Binary Tree Properties

- n: the number of nodes in T
- n_E: the number of external nodes in T
- n_I: the number of internal nodes in T
- h: the height of T

- $h + 1 \le n \le 2^{h+1} - 1$
- $1 \le n_E \le 2^h$
- $h \le n_I \le 2^h - 1$
- $\log(n + 1) - 1 \le h \le n - 1$

Binary Trees

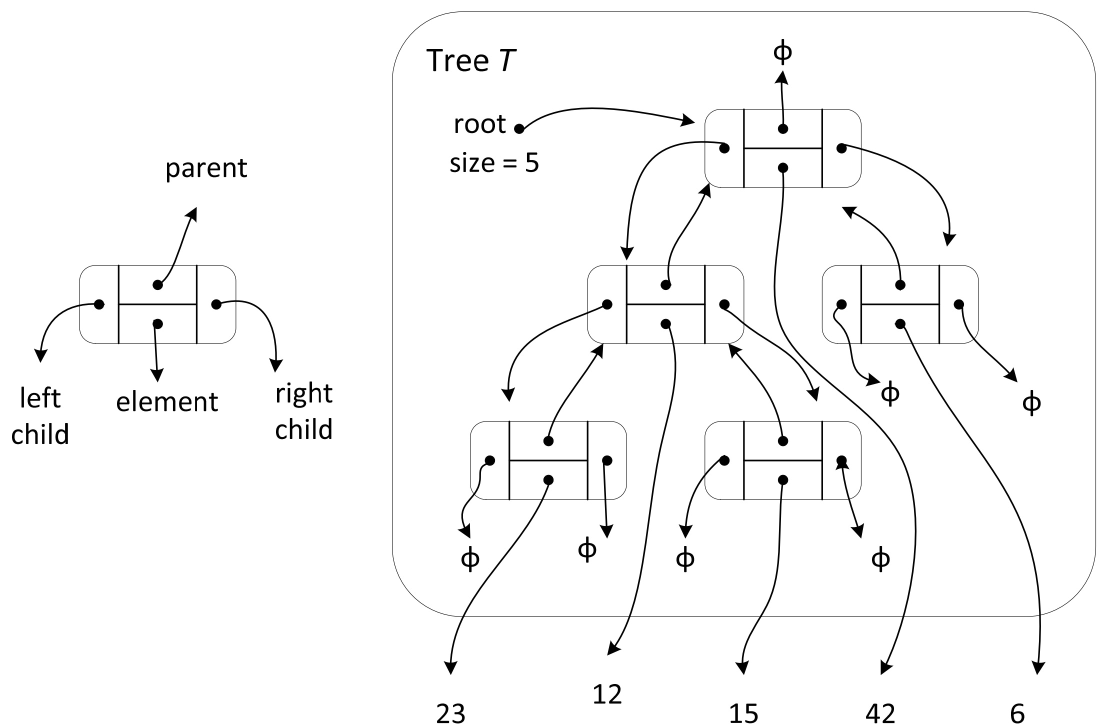
Binary Tree Properties

- If T is a proper binary tree:

- $2h + 1 \le n \le 2^{h+1} - 1$
- $h + 1 \le n_E \le 2^h$
- $h \le n_I \le 2^h - 1$
- $\log(n + 1) - 1 \le h \le (n - 1)/2$
- $n_E = n_I + 1$

Binary Trees
Implementation Using Linked Structure

- A node has the following linked structure.



Binary Trees
Implementation Using Linked Structure

- LinkedBinaryTree extends AbstractBinaryTree abstract class with the following update methods:
  - addRoot(*e*): Creates a new node with element *e* and make it the root of an empty tree. Returns the position of the root. An error occurs if the tree is not empty.
  - addLeft(*p*, *e*): Creates a new node with element *e* and make it a left child of position *p*. Returns the position of the new node (left child). An error occurs if *p* already has a left child.
  - addRight(*p*, *e*): Creates a new node with element *e* and make it a right child of position *p*. Returns the position of the new node (right child). An error occurs if *p* already has a right child.

Binary Trees
Implementation Using Linked Structure

- Update methods (continued):
  - set(*p*, *e*): Replaces the element of *p* with element *e.* Returns the previously stored element.
  - attach(*p*, T_1, T_2): Attaches internal structure of T_1 and T_2 as the left subtree and the right subtree, respectively, of a leaf node position *p* and resets T_1 and T_2 to empty trees. If *p* is not a leaf node, an error occurs.
  - remove(*p*): Removes the node at position *p*, replacing it with its child (if any). Returns the element that had been stored at *p*. An error occurs if *p* has two children.

Binary Trees
Implementation Using Linked Structure

- A node is a *position* (instance variables shown below)

```
protected static class Node<E> implements Position<E> {
  private E element;          // an element stored at this node
  private Node<E> parent;     // a reference to the parent node (if any)
  private Node<E> left;       // a reference to the left child (if any)
  private Node<E> right;      // a reference to the right child (if any)
```

- LinkedBinaryTree has two instance variables

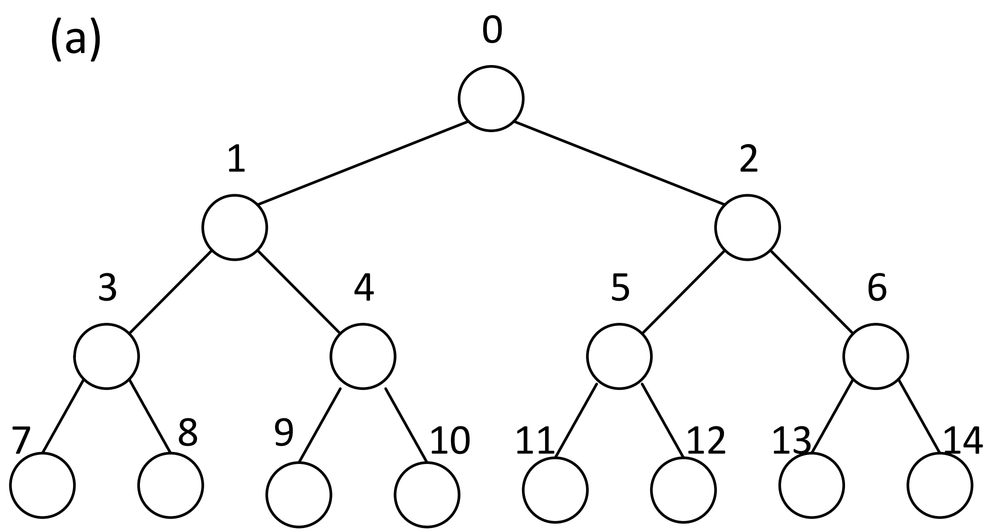
```
protected Node<E> root = null;
private int size = 0;
```

- LinkedBinaryTree.java

Binary Trees
Implementation Using Array

- Nodes are stored in an array.
- *Level numbering* scheme is used.

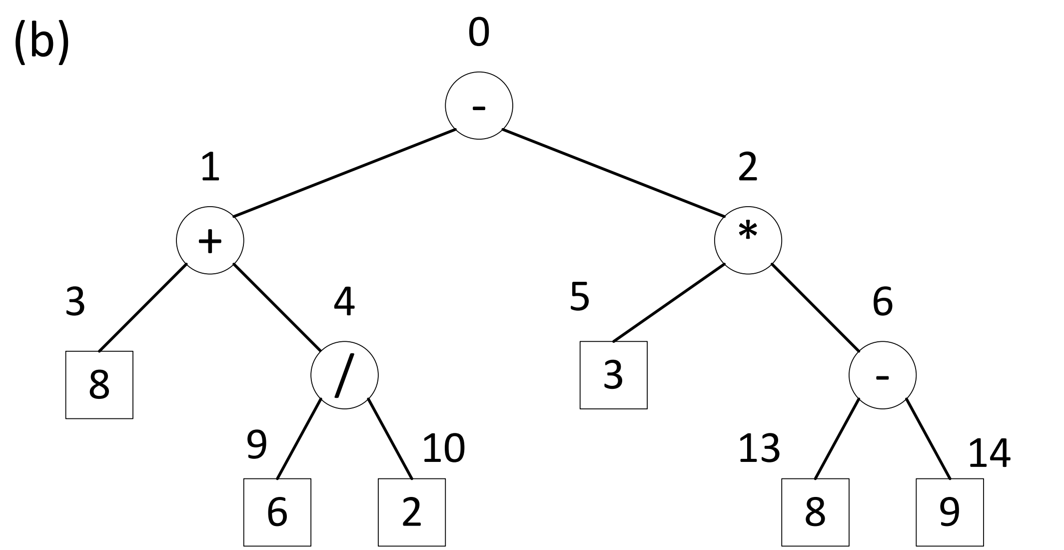
(a)



- A number above a node is the index in the array.

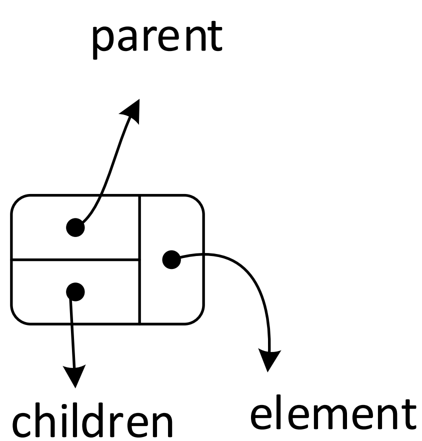
Binary Trees
Implementation Using Array

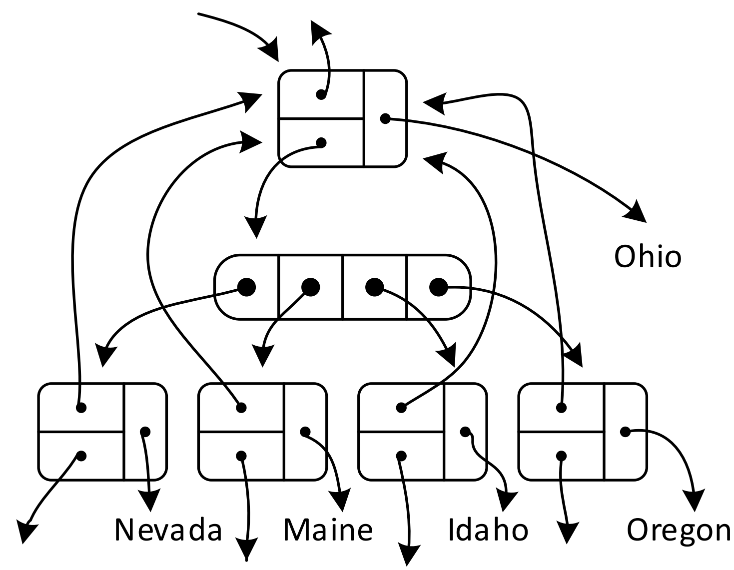
- Example



-	+	*	8	/	3	-			6	2			8	9
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Binary Trees
Linked Structure for General Trees





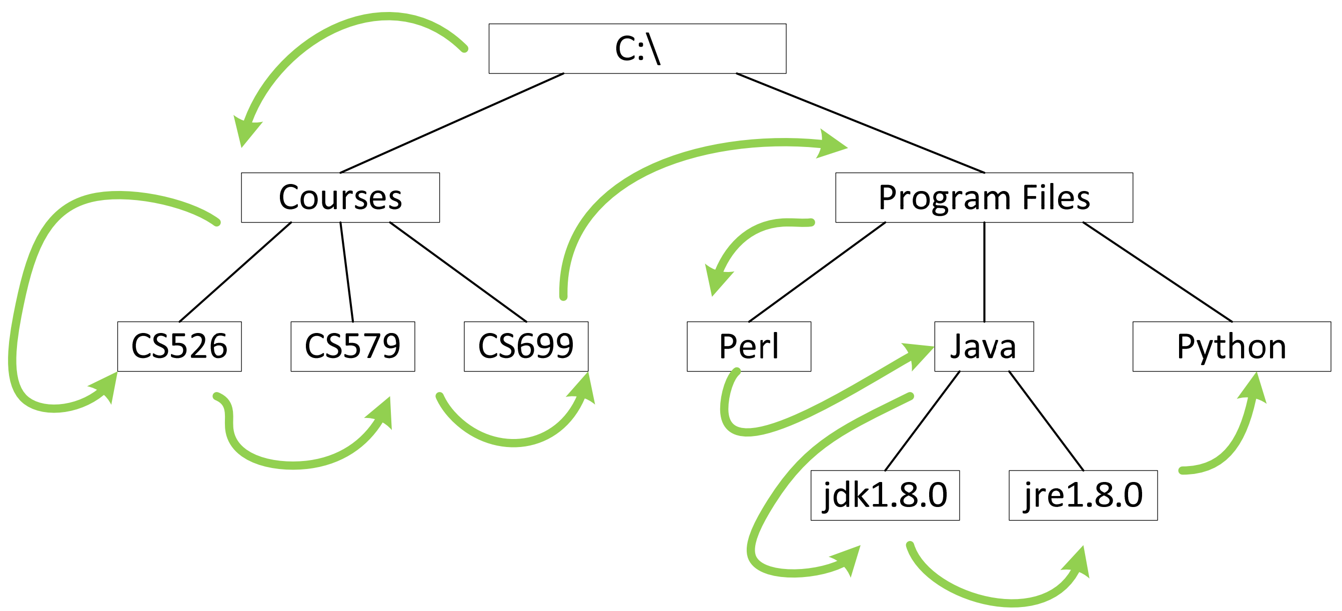
Binary Trees
Tree Traversal

- A *traversal* of a tree *T* is a systematic way of visiting all positions in *T*.
- Preorder tree traversal:
  - visit the root
  - visit all children

```
Algorithm preorder(p)
    visit p
    for each child c in children(p)
        preorder(c)
```

Binary Trees
Tree Traversal

- Preorder tree traversal illustration:



Binary Trees
Tree Traversal

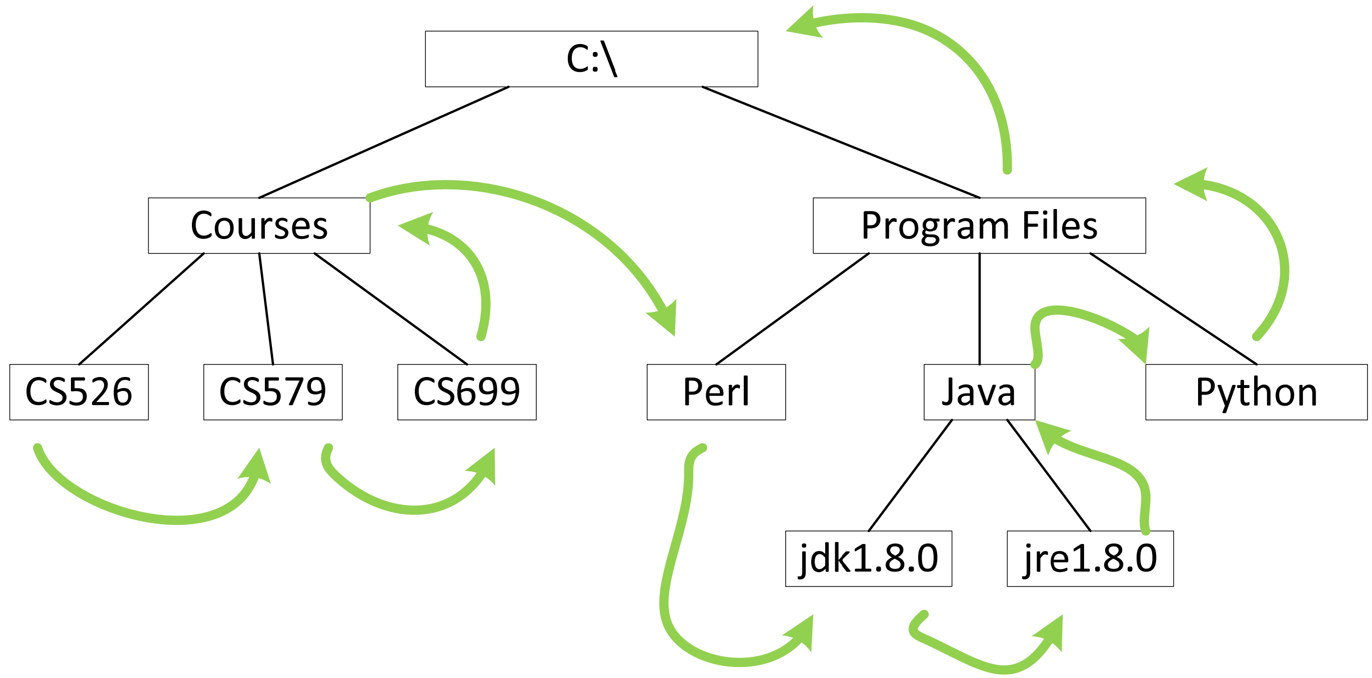
- Postorder tree traversal:
  - Visit all children (recursively)
  - Visit the root

Algorithm postorder(*p*)
    for each child *c* in *children*(*p*)
        postorder(*c*)
    visit *p*

Binary Trees

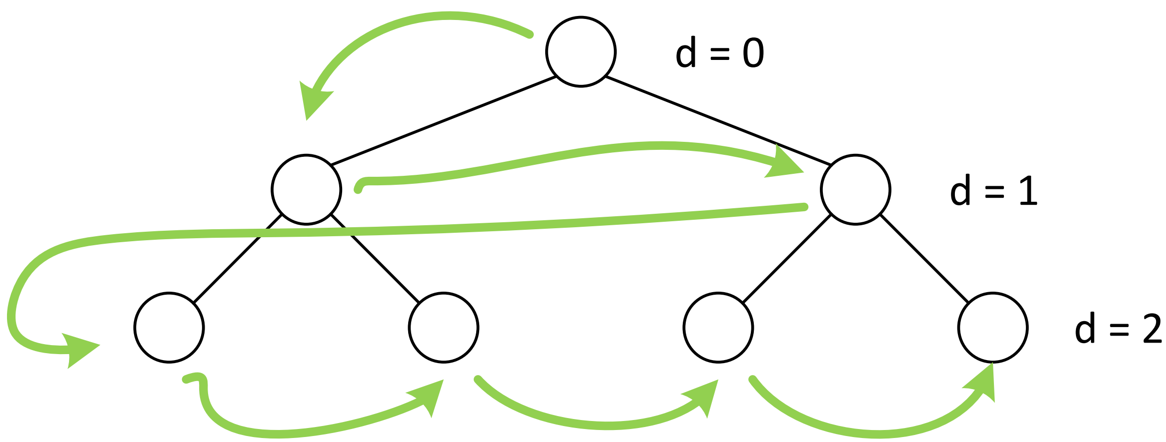
Tree Traversal

- Postorder tree traversal illustration



Binary Trees
Tree Traversal

- Breadth-first tree traversal
  - Also called *breadth-first search* or *BFS*
  - Visits all positions at depth *d* before visiting positions at depth *d* + 1.



Binary Trees
Tree Traversal

- Breadth-first tree traversal (continued)

```
Algorithm breadthfirst( )
    initialize Q to contain the root of the tree
    while Q is not empty
       p = Q.dequeue( )  // remove the oldest entry in Q
       visit p
       for each child c in children(p)
          Q.enqueue(c)    // add all children of p to the rear of Q
```

- Running time
  - Each node is enqued and dequeued once each.
  - O(n)

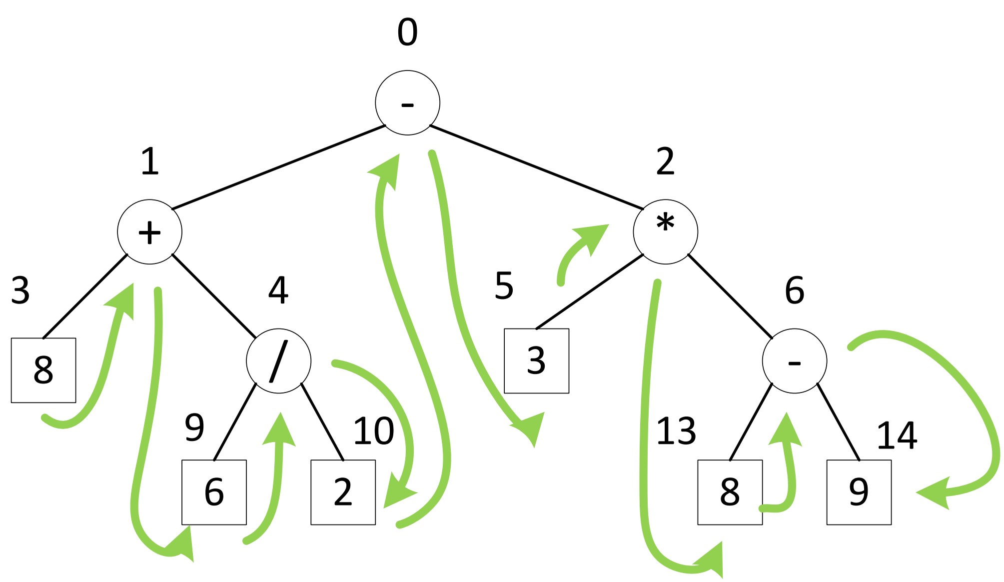
Binary Trees
Tree Traversal

- Inorder tree traversal of binary tree
  – Visit the left subtree
  – Visit the root
  – Visit the right subtree

```
Algorithm inorder(p)
        if p has a left child lc    // visit left subtree
            inorder(lc)
        visit p
        if p has a right child rc   // visit right subtree
            inorder(rc)
```

Binary Trees
Tree Traversal

- Inorder tree traversal of binary tree illustration:



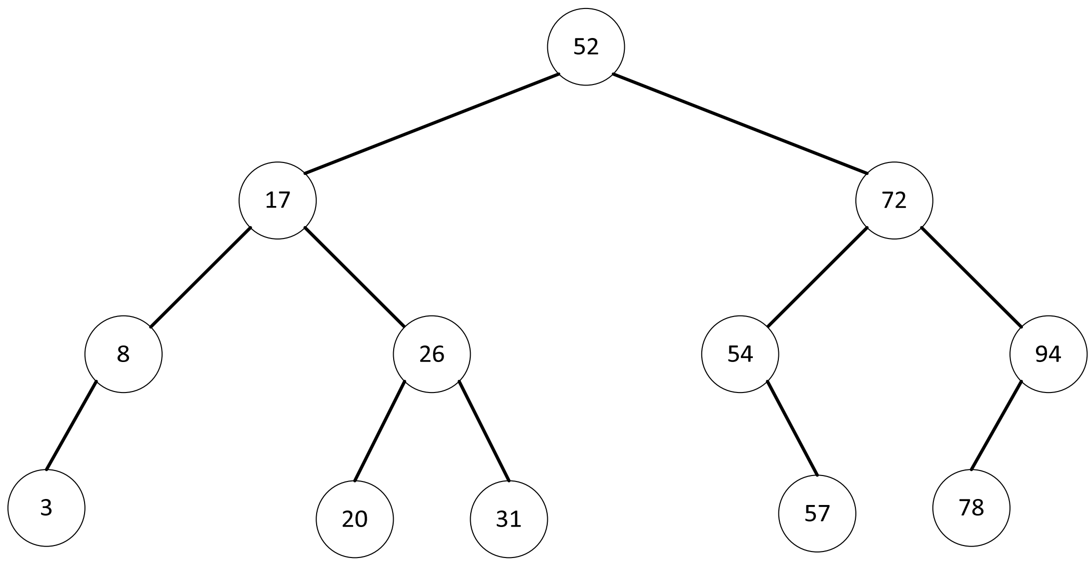
- Inorder tree traversal generates: 8 + 6 / 2 – 3 * 8 – 9
- Correct expression without parentheses

Binary Trees
Binary Search Tree

- A binary search tree is a binary tree with additional properties:
  - Each position *p* stores an element, denoted as *e*(*p*).
  - All elements in the left subtree of a position *p* (if any) are less than *e*(*p*).
  - All elements in the right subtree of a position *p* (if any) are greater than *e*(*p*).

Binary Trees
Binary Search Tree

- A binary search tree example:



- Inorder tree traversal generates:

  3, 8, 17, 20, 26, 31, 52, 54, 57, 72, 78, 94

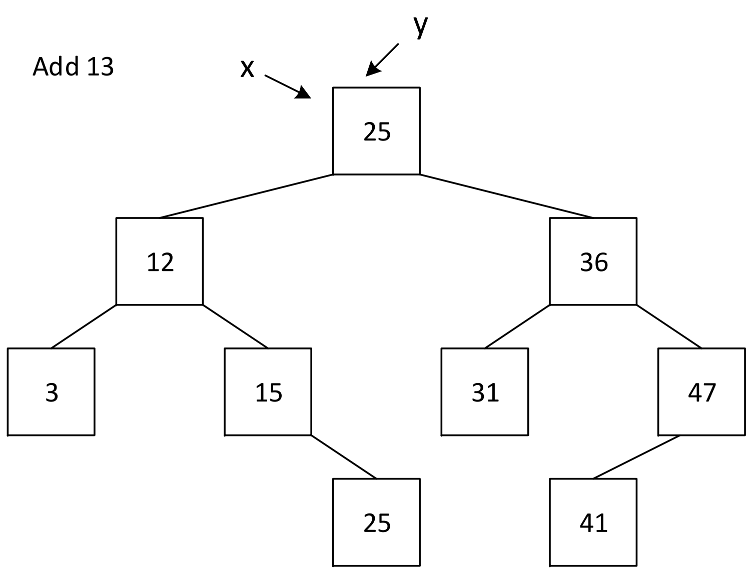
Binary Trees
Binary Search Tree

```
Algorithm add(p, e) // an incomplete code
    if p == null // this is an empty tree
        create a new node with e and make it the root of the tree
    x = p; y = x; // y follows x
    while (x is not null) {
      if (the element of x) is the same as e, return null
      else if (the element of x) > e{
            y = x;        x = left child of x;
        }
        else {
            y = x;        x = right child of x;
        }
    } // end of while
```

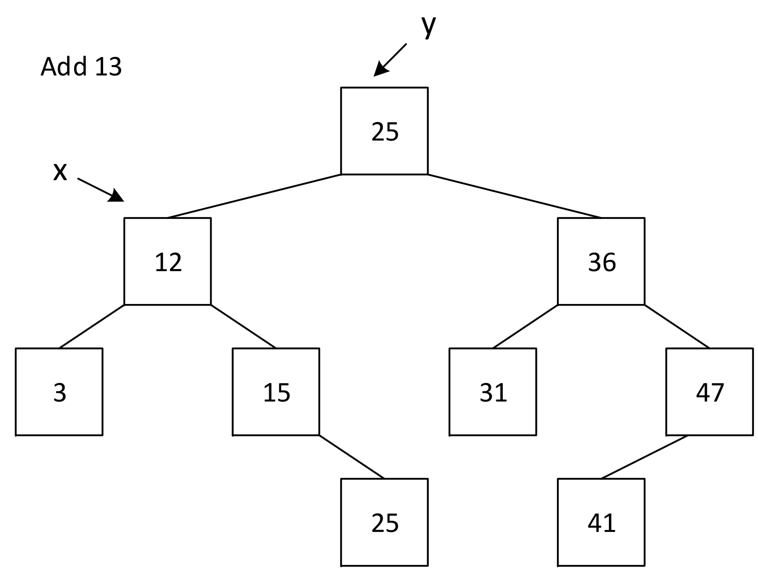
Binary Trees
Binary Search Tree

add(root, 13)

Add 13



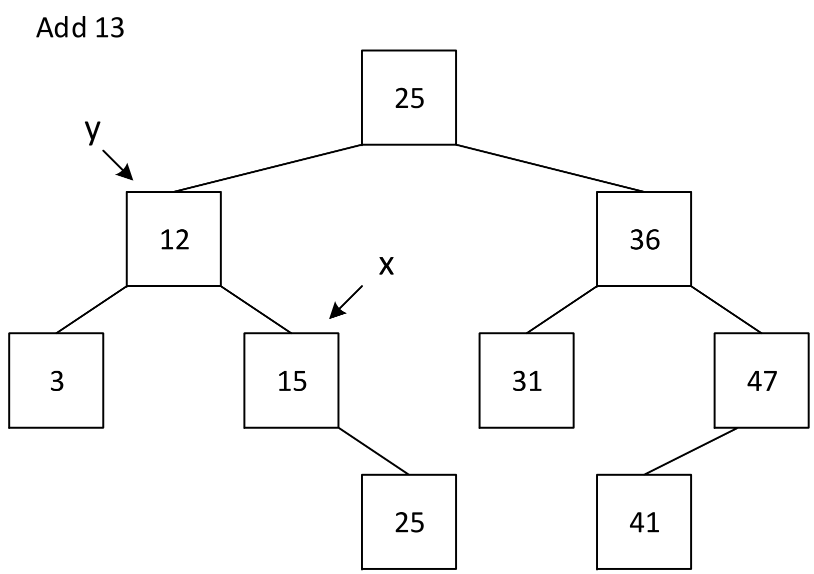
Add 13

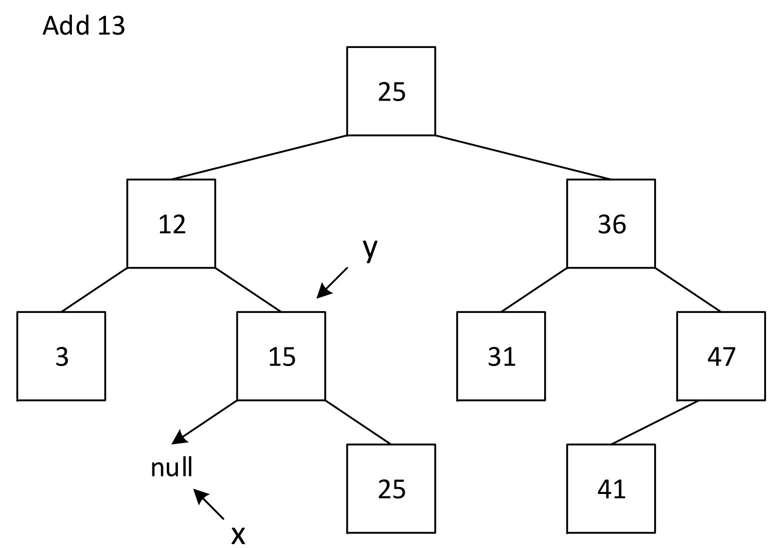


Binary Trees

Binary Search Tree

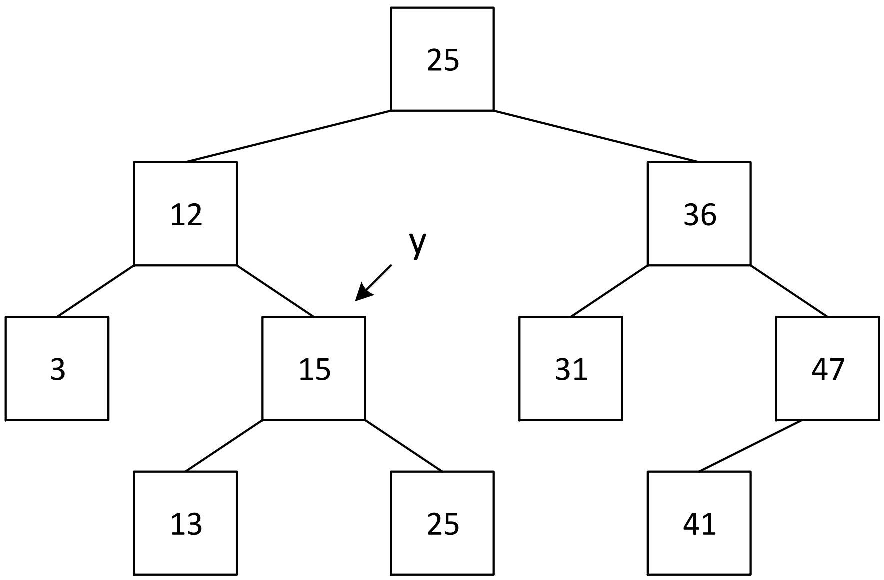
add(root, 13)





Binary Trees
Binary Search Tree

add(root, 13)



References

- M.T. Goodrich, R. Tamassia, and M.H. Goldwasser, “Data Structures and Algorithms in Java,” Sixth Edition, Wiley, 2014.