Data Structures and Algorithms

Chapter 12

Sorting
Merge-Sort

- A divide-and-conquer algorithm

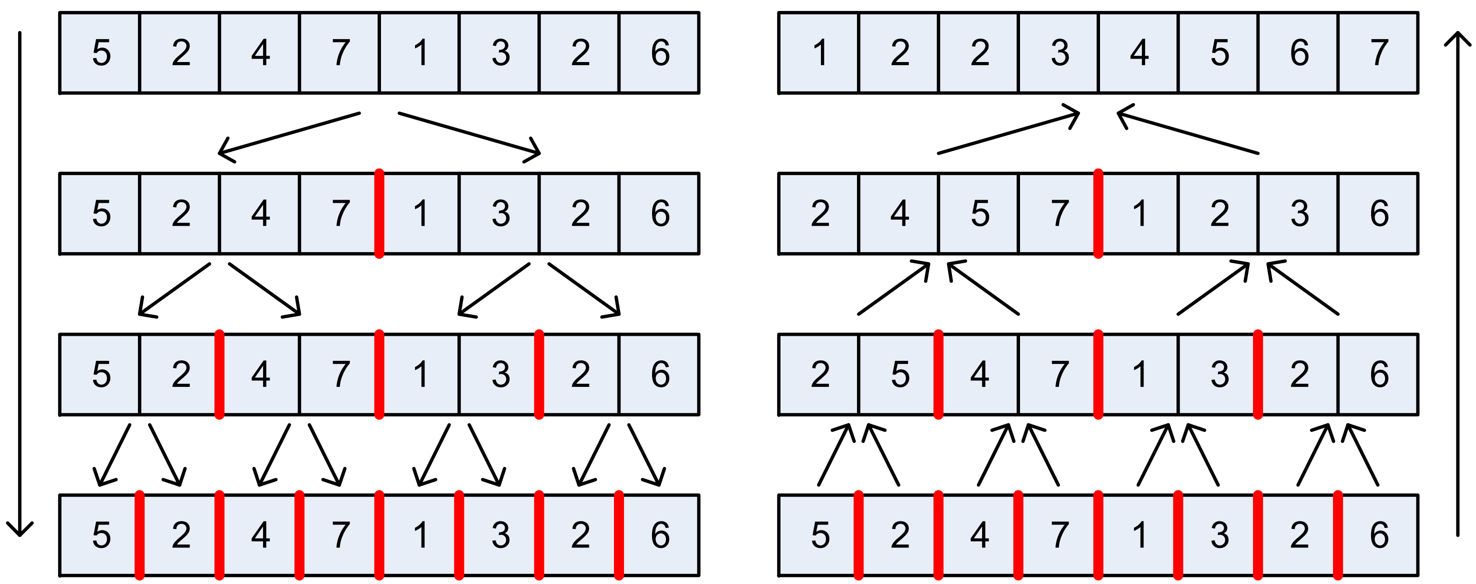
- Divide:
  - If input size is smaller than a certain threshold, solve it using a straightforward method.
  - Otherwise, divide the input into two or more subproblems.
- Conquer: Solve the subproblems recursively.
- Combine: Merge solutions to subproblems to generate a solution to the original problem.

Sorting
Merge-Sort

- Outline of the algorithm:
  1. Divide: If S has zero or one element, return S (because it is already sorted). Otherwise, divide S into two separate arrays, S_1 and S_2, of approximately equal size. S_1 contains the first $\lfloor n/2 \rfloor$ elements of S and S_2 contains the remaining $\lceil n/2 \rceil$ elements.
  2. Conquer: Sort S_1 and S_2 recursively.
  3. Combine: Put the elements back to S by merging the sorted sequences S_1 and S_2 into a sorted sequence.

Sorting
Merge-Sort

- Illustration



Sorting
Merge-Sort

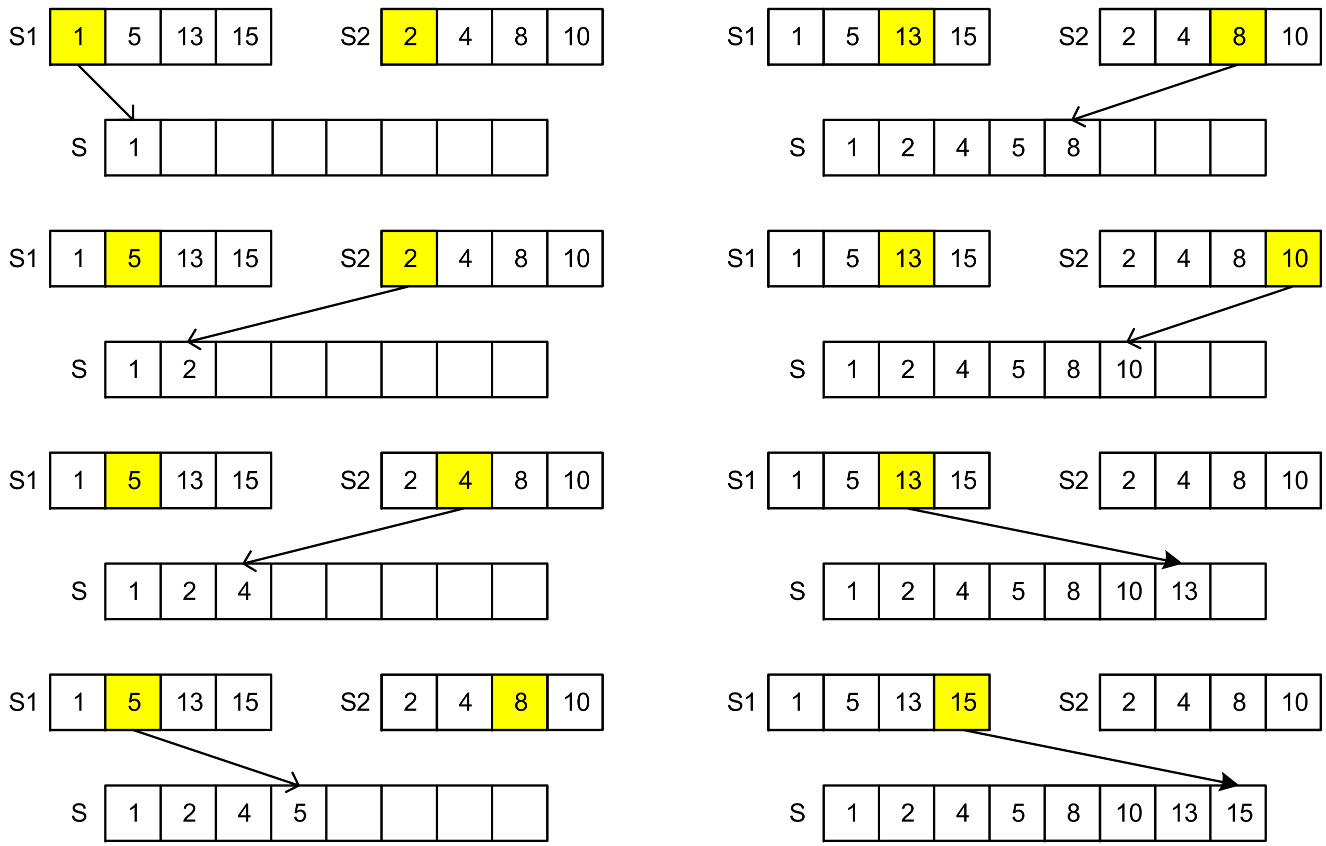
- Array-based implementation

```
public static <K> void merge(K[] S1, K[] S2, K[] S, Comparator<K> comp) {
  int i = 0, j = 0;
  while (i + j < S.length) {
    if (j == S2.length || (i < S1.length && comp.compare(S1[i], S2[j]) < 0))
      S[i+j] = S1[i++];        // copy ith element of S1 and increment i
    else
      S[i+j] = S2[j++];        // copy jth element of S2 and increment j
  }
}
```

- Running time: $O(n)$

Sorting
Merge-Sort

- Merge



Sorting
Merge-Sort

- Java implementation

```
public static <K> void mergeSort(K[ ] S, Comparator<K> comp) {
  int n = S.length;
  if (n < 2) return;    // array is trivially sorted
  int mid = n/2;
  K[ ] S1 = Arrays.copyOfRange(S, 0, mid);  // copy of first half
  K[ ] S2 = Arrays.copyOfRange(S, mid, n);  // copy of second half
  mergeSort(S1, comp);                       // sort copy of first half
  mergeSort(S2, comp);                      // sort copy of second half
  merge(S1, S2, S, comp);  // merge sorted halves back into original
}
```

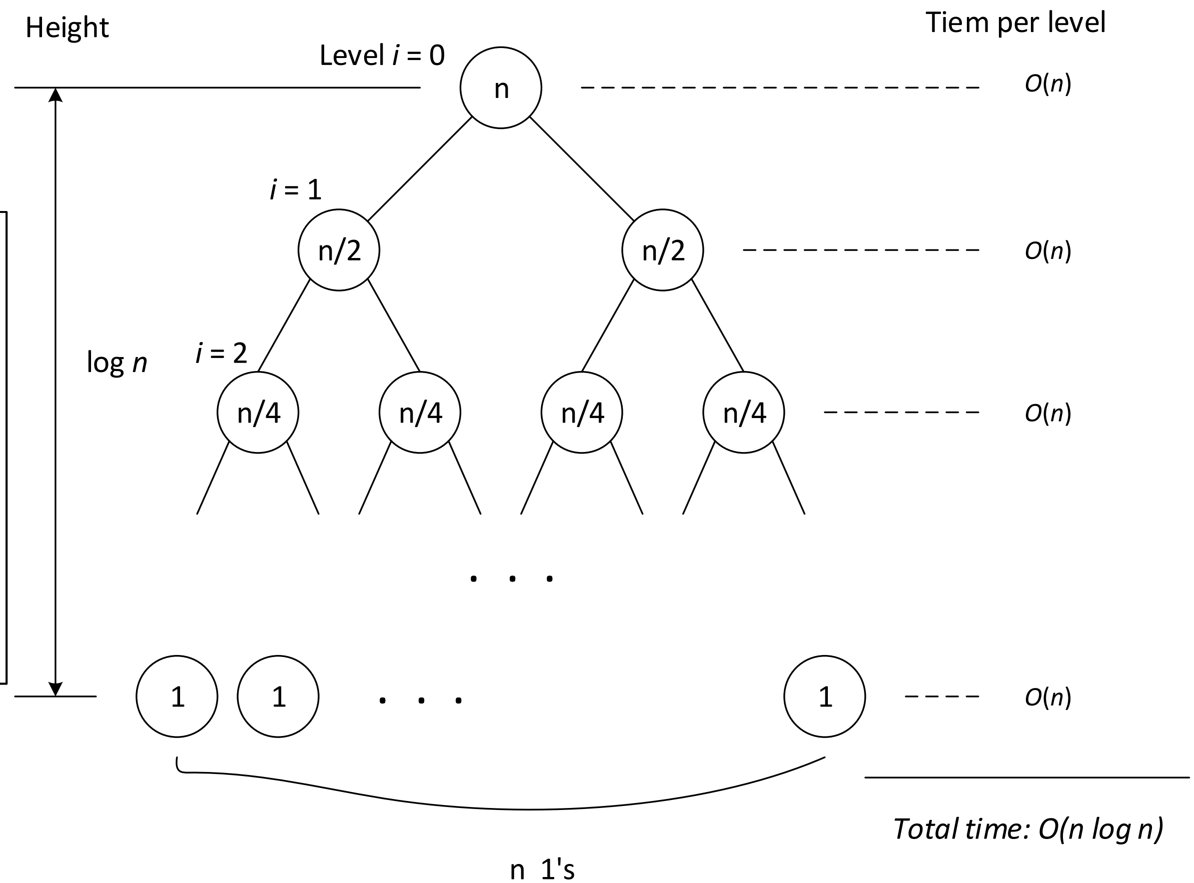
Sorting
Merge-Sort

- Running time analysis
  - Recursive calls are made in lines 7 and 8.
  - Excluding the recursive calls, the program takes O(n).
  - Each recursive call is made on a subarray with $n/2$ elements.
  - The running time of the *mergeSort* on an subarray with n/2 elements is $O(n/2)$.
  - As the successive recursive calls are made, the size of subarray becomes $n/2$, $n/4$, $n/8$, …, and so on, and eventually it becomes 1.
  - This can be represented as a recursion tree.

Sorting
Merge-Sort

- Running time analysis

- Each level takes O(n)
- There are (log n + 1) levels
- Total running time = $O(n)$ (log n + 1) = $O(n)$(log n) + $O(n)$ = $O(n \log n)$

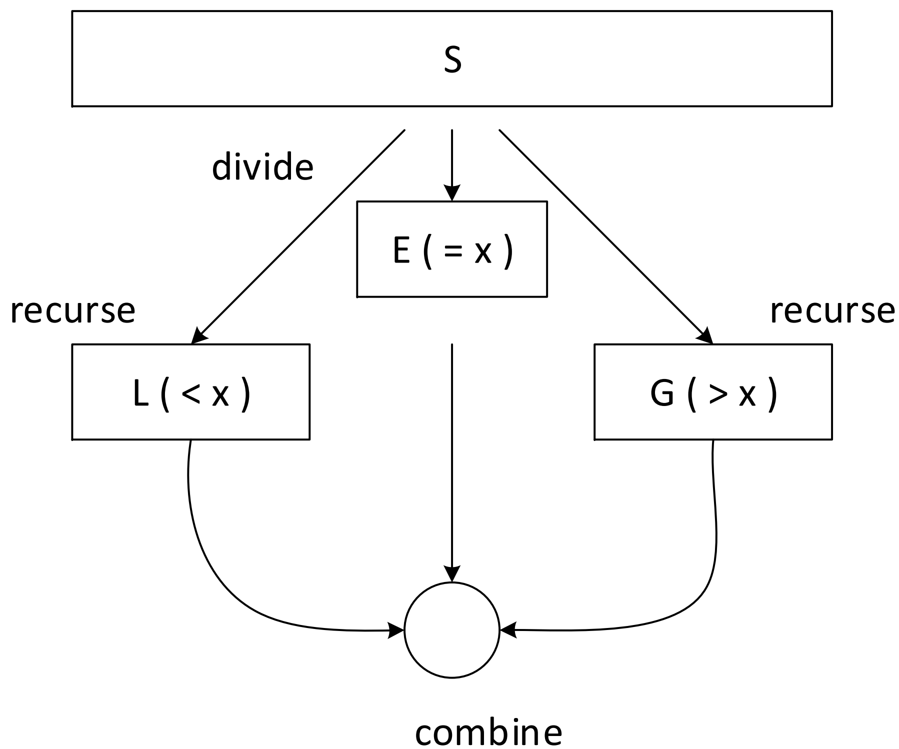


Sorting
Quick-Sort

- Outline
  - Divide: If *S* has only one element, return. Otherwise, remove all elements from *S* and put them into three sequences:
    - *L*: This sequence contains the elements that are less than *x*.
    - *E*: This sequence contains the elements that are equal to *x*.
    - *G*: This sequence contains the elements that are greater than *x*.
  - If the elements in *S* are distinct, then *E* has only one element, which is *x*.
  - Conquer: Recursively sort *L* and *G*.
  - Combine: Put back the elements from the three parts into *S* in order.
- The element *x* is called *pivot*.

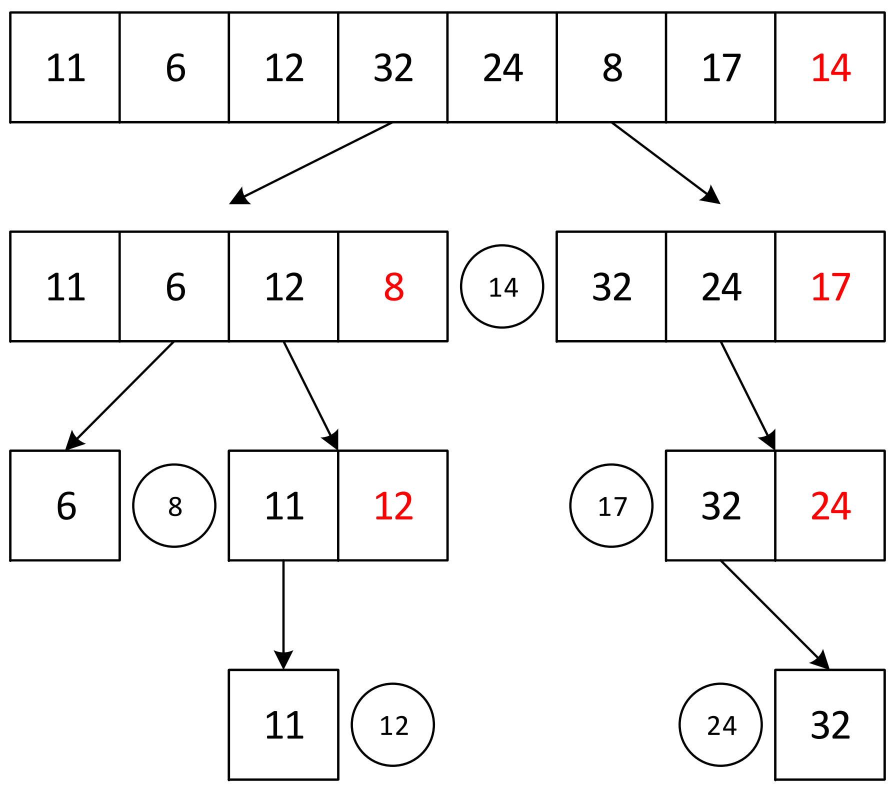
Sorting
Quick-Sort

- Outline



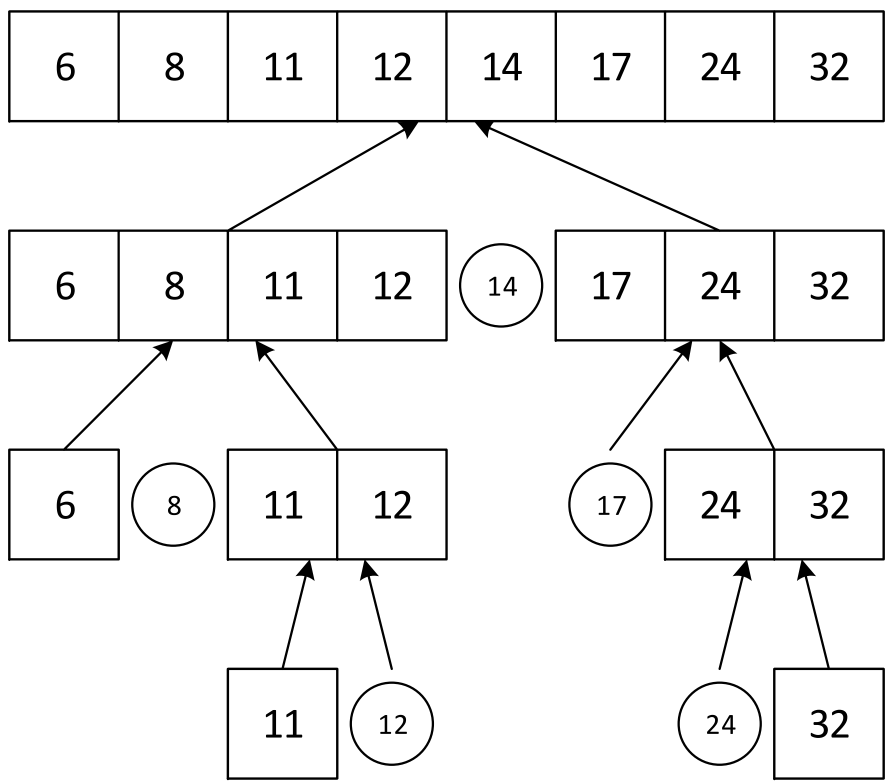
Sorting
Quick-Sort

- Illustration



Sorting
Quick-Sort

- Illustration (continued)



Sorting
Quick-Sort

- The quick-sort algorithm as discussed in the previous slides is not an in-place sorting algorithm.
- An array-based, in-place quicksort algorithm is described in Section 12.2.2 (page 553).
- The “partition” example in the next slides illustrates in-place sorting.

Sorting
Quick-Sort

- The “divide” step is usually called *partition*.
- Partitioning array *S* with *n* elements.
  – S[n – 1] is used as the pivot
  – Keeps two pointers, *left* and *right*
  – *Left* begins at S[0] and moves right until it meets the first element that is equal to or larger than the pivot, *right marker*.
  – *Right* begins at S[n – 2] and moves right until it meets the first element that is equal to or smaller than the pivot, *left marker*.
  – Left marker and right marker are swapped.
  – Repeat this until left and right cross each other
  – Left marker is swapped with pivot.

Sorting

Quick-Sort

- Partitioning illustration

85	24	63	45	17	31	96	50

l *r*

85	24	63	45	17	31	96	50

l ⟷ *r*
swap

31	24	63	45	17	85	96	50

l ⟷ *r*
swap

31	24	17	45	63	85	96	50

r < *l* ⟷

l and r crossed*; stop; swap S*[*l*] with pivot

31	24	17	45	50	85	96	63

Sorting
Quick-Sort

- Running time analysis
  - Can use the same method we used for merge-sort (i.e., use a recursion tree).
  - In merge-sort, we always have a balanced divide.
  - In quick-sort, depending on the pivot value, there may be a very unbalanced partitioning
  - In the best case:
    - Always balanced partitioning is created.
    - Running time is $O(n \log n)$
    - Even when partitions are not completely balanced (for example 1 : 9), the running time is still $O(n \log n)$

Sorting
Quick-Sort

- Running time analysis (continued)
  - In the worst case:
    - We always have an extremely unbalanced partitioning, i.e., no element on one side and $n - 1$ elements on the other side.
    - This occurs if an array is already sorted and the last element is chosen as a pivot.
    - Running time is $O(n^2)$.

Sorting
Quick-Sort

- Improvement
  - Randomized quick-sort: pivot is chosen randomly
  - *median-of-three* method: the median of the first element, the middle element, and the last element is used as a pivot.
  - When the input size becomes smaller than a certain threshold, we stop the recursion and sort that subarray using insertion-sort. There is no known one threshold value that is considered best. Our textbook suggests 50 and some experiments showed that a value around 15 is a reasonably good choice.

Sorting
Lower Bound for sorting

- The running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$ in the worst case.

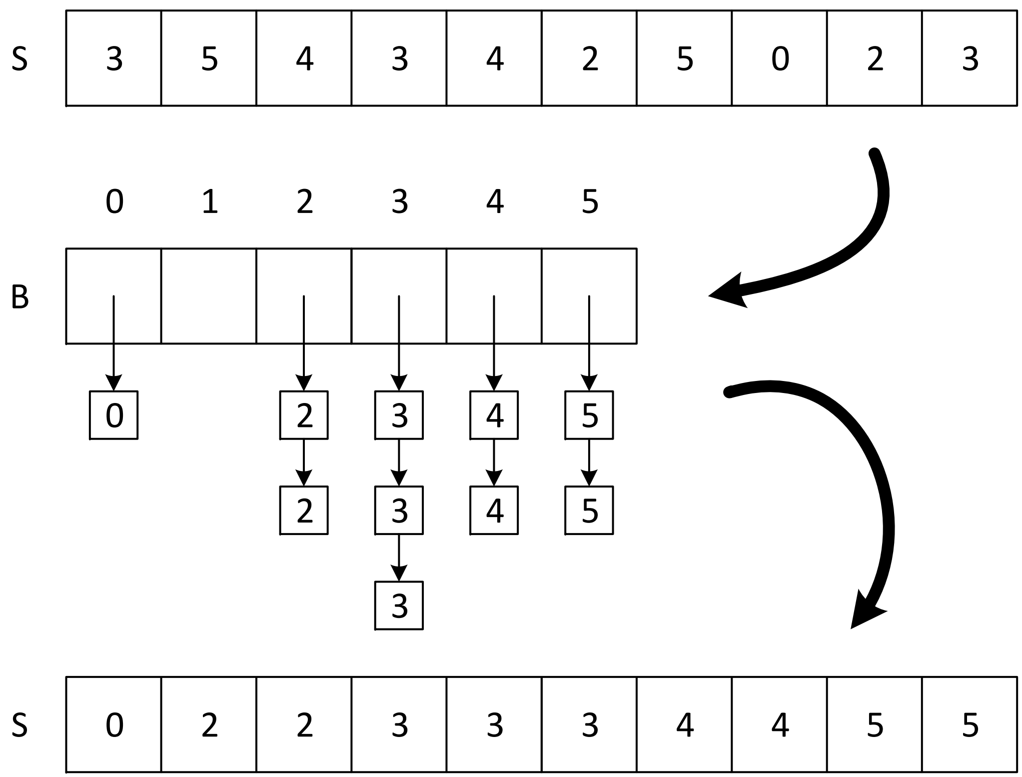
- Linear-time sorting: counting-sort, bucket-sort, radix-sort.
- Will discuss bucket-sort and radix-sort.

Sorting
Bucket-Sort

- Sorts a sequence of elements in a linear time with a constraint.
- Constraint:
  - The elements are integers in the range [0, N – 1], for some integer $N \geq 2$.
  - If the elements to be sorted are objects, then the objects must have integer keys with total ordering.

Sorting
Bucket-Sort

- Illustration (N = 6)



Sorting
Bucket-Sort

- Pseudocode

Algoritm bucketSort(*S*)
Input: Sequence *S* of entries with integer keys in range [0, *N* – 1]
Output: Sequence *S* sorted in nondecreasing order of keys

create an empty array *B* of size *N*
for each entry *e* in *S* do
  let *k* be the key of *e*
  remove *e* from *S* and add it to the end of bucket *B*[*k*], which is a sequence
for *i* = 0 to *N – 1* do
  for each entry in sequence *B*[*i*] do
    remove *e* from *B*[*i*] and insert it at the end of *S*

Sorting

Stable Sorting

- Let S = ((k_0, v_0), (k_1, v_1), …, (k_{n-1}, v_{n-1})).
- Assume there are two entries (k_i, v_i) and (k_j, v_j) with an identical key, i.e, $k_i = k_j$, $i \neq j$
- We say a sorting algorithm is *stable* if (k_i, v_i) precedes (k_j, v_j) in S before sorting, then (k_i, v_i) also precedes (k_j, v_j) in S after sorting.

- Example:
  - S = ((9, W), (4, F), (7, H), (4, A), (2, P)) before sorting
  - S = ((2, P), (4, F), (4, A), (7, H), (9, W)) after sorting

- The bucket-sort described earlier is stable if S and B behave as queues.

Sorting
Radix-Sort

- Illustration:
  - Sorting three digit numbers
  - Each column is sorted using a stable sorting algorithm

456		932		912		148
723		912		723		239
148		723		932		456
239	→	745	→	239	→	648
932		456		745		723
912		148		148		745
648		648		648		912
745		239		456		932

Sorting
Comparison

- Running times

Running Time (average)	Sorting Algorithms
O(n)	bucket-sort, radix-sort
O(n log n)	heap-sort, quick-sort, merge-sort
$O(n^2)$	insertion-sort

Sorting
Comparison

- Insertion-Sort
  - When the number of elements is small (typically less than 50), insertion-sort is very efficient.
  - Insertion-sort is very efficient for an “almost” sorted sequence.
  - In general, due to its quadratic running time, insertion-sort is not a good choice except for the situations listed above.

Sorting
Comparison

- Heap-Sort
  - Heap-sort runs in $O(n \log n)$ in the worst case.
  - It works well on small- and medium-sized sequences.
  - It can be made an in-place sorting algorithm.
  - Its performance is poorer than that of quick-sort and merge-sort on large sequences.
  - Heap-sort is not a stable sorting algorithm.

Sorting
Comparison

- Quick-Sort
  - Worst-case running time is $O(n^2)$.
  - Experimental studies showed quick-sort outperformed heap-sort and merge-sort.
  - Quick-sort has been a default algorithm as a general-purpose, in-memory sorting algorithm.
  - It was used in *C* libraries.
  - Java uses it as the standard sorting algorithm for sorting arrays of primitive types.

Sorting
Comparison

- Merge-Sort
  - Worst-case running time is $O(n \log n)$.
  - It is difficult to make merge-sort an in-place sorting algorithm. So, it is less attractive than heap-sort or quick-sort.
  - Merge-sort is an excellent algorithm for sorting data that resides on the disk (or storage outside the main memory).

Sorting
Comparison

- Tim-Sort
  - Tim-sort is a hybrid algorithm which uses a bottom-up merge-sort and insertion-sort.
  - Tim-sort has been the standard sorting algorithm in Python since 2003.
  - Java uses Tim-sort for sorting arrays of objects.

Sorting
Comparison

- Bucket-Sort and Radix-Sort
  - Excellent for sorting entries with small integer keys, character strings, or *d*-tuple keys from a small range.

Selection Problem

- Selection problem: Given a set S of n comparable elements and an integer k, $1 \le k \le n$, find the element $e \in S$ that is larger than exactly $k - 1$ elements of S.
- The k^{th} smallest element is also referred to as the k^{th} *order statistic*.
- We assume S is a sequence.

- Will discuss *randomized quick-select*, which runs in $O(n)$ expected time.
- Similar to the randomized quick-sort algorithm.

Selection Problem

- Pseudocode

```
Algorithm quickSelect (S, k) // find the kth order statistic
if n == 1  // n is the size of S
  return the (first) element
pick a random pivot element x of S and divide S into three subsequences:
L, storing the elements in S less than x
E, storing the elements in S equal to x
G, storing the elements in S greater than x
if k ≤ |L| then                    // case 1
  return quickSelect(L, k)
else if k ≤ |L| + |E|              // case 2
  return x
else                               // case 3
  return quickSelect(G, k - |L| - |E|)
```

Selection Problem

- Illustration (Case 1: if $k \leq |L|$)

  Find 5th order statistic.

  pivot = 9

- After partition:

k = 5 < |L|, recurse on L with k = 5

7	3	5	1	6	2	9	9	13	15	17	10

L

E

G

|L| = 6

|G| = 4

|E| = 2

Selection Problem

- Illustration (Case 2: else if $k \le |L| + |E|$)

  Find 7th order statistic.

  pivot = 9

- After partition:

k = 7 ≤ |L| + |E|, return 9

7	3	5	1	6	2	9	9	13	15	17	10

L

|L| = 6

E

|E| = 2

G

|G| = 4

Selection Problem

- Illustration (Case 3: else if $k > |L| + |E|$)

  Find 10th order statistic.

  pivot = 9

- After partition:

k = 10 > |L| + |E|, recurse on G with k = 2

7	3	5	1	6	2	9	9	13	15	17	10

L

$|L| = 6$

E

$|E| = 2$

G

$|G| = 4$

References

- M.T. Goodrich, R. Tamassia, and M.H. Goldwasser, "Data Structures and Algorithms in Java," Sixth Edition, Wiley, 2014.