1 Categories, Do they know things? Let's find out!

This note follows Category Theory in Context by Riehl.

Definition: A category C consists of

-Collection of objects $X, Y, Z....$

-Collection of morphisms $f, g, h....$

Such that

-Morphism has specified domain and codamin objects: We write $f : X \to Y$

-Each object has designated identity morphisms $I_X : X \to X$

-For $f : X \to Y$, $g : Y \to Z$, there exists a composite morphisms gf such that $gf : X \to Z$.

And subject to the two axioms that

-$I_X f = f I_X$ for $f : X \to Y$

-For $f : X \to Y$,$g : Y \to Z$, $h : Z \to W$, $h(gf) = (hg)f$

Right off the bat, one important thing to recognize is we say *collection* of objects and morphisms. *Not set.* More on this later.

For now, let's consider the two most intuitive examples:

- Sets with functions as morphism forms a category, **Set**
- Groups with homomorphisms as morphism forms a category, **Group**

So in **Set**, $\mathbb{R}$ is a object in the category. The identity morphism $I_{\mathbb{R}}$ is given by the function $f(x) = x$

This is intuitive, because it's easy to think about functions between sets, which of course respects composition and identity.

However, it is important not to fall into the trap of thinking of categories this way, as next example will illustrate.

A group G defines a category **BG**, with single object. The group elements are its morphism, with composition given by multiplication, and identity $e \in G$ acting as the identity morphism.

Umm, what?? Okay, to unpack this, let $G = \{0, 1, 2\}$, endowed with addition mod 3, making it a group.

The category **BG** has the following object.

- G

Remember that moprhims have speicifed domain and codomain, but since the category has only one object, the domain and codomain will just be G.

The morphisms are

- $0 : G \to G$
- $1 : G \to G$
- $2 : G \to G$

The identity morphism is 0. And composition is given by

$$0 \circ 1 = 0 + 1 = 1$$

$$0 \circ 2 = 0 + 2 = 2$$

$$1 \circ 2 = 0$$

$$2 \circ 2 = 1$$

Note that the morphisms are not functions! Morphisms are not necessarily functions, so it's harmful to think about them as functions. Rather, think about them as arrows, connecting objects!

Category in Context has a nice picture for the group BS_3, with S_3 being the symmetric group with $n = 3$.

https://imgur.com/a/h3ZeZJE

2 Small and Large Categories

Let's go back to my point of using the word collection. But first, we have to talk about the Russell's Paradox.

2.1 Russell's Paradox

For most undergraduate courses, sets are taken for as granted. We just say they are collection of objects, and hope that nothing goes wrong.

And for the most part, everything is okay, and you don't have to worry about axiom of choice or other nightmarish sets from set theory.

But Russell's Paradox is an example where such a naive approach fails.

So we want to consider

$$X = \{x : \text{x is a set}\}$$

Namely, the set of all sets.

Is X a set? After all, x where x is a set is an object, so X seems like a collection of objects, which should be a set?

Say X is a set. Then $X \in X$ (!) And this leads to all kinds of troubles.

We define another collection $R = \{x : x \notin x\}$

If $R \in R$, $R \notin R$.

If $R \notin R$, $R \in R$.

Hence, this leads to a paradox, named, Russell's Paradox.

So in ZFC, there is no set X such that $X \in X$.

So there is no set of all sets.

2.2 Back to Small and Large Categories

So in **Set**, the collection of all sets, the objects in the category, is not a set, but a proper class. (Proper class is just a fancy way to denote collections that are not sets)

So we define,

Definition: Category C is small if collection of all objects in C and collection of all morphisms in C forms a set, not a proper class. C is large otherwise.

But not all hope is lost!

Even if the category is large, we can find the next best thing.

Definition: A category is locally small if between any pair of objects, the collection of morphisms between the pair forms a set.

For example, **Set** is locally small, as the collection of all functions between two sets A and B can be viewed as subset of $A \times B$.

3 Other Concepts

There are other concepts, that are similar to one found in other fields applied to categories.

Definition: Isomorphism in a category is morphism $f : X \to Y$ for which there exists morphism $g : Y \to X$ such that $gf = I_X$ and $fg = I_Y$.

Definition: Subcategory D of category C is defined by restricting to a subcollection of objects and subcollection of morphisms subject to the requirement that subcategory D contains all the domain and codamin of any morphisms in D, any identity morphisms of any objects in D, and composite of any composable pair of morphism in D.