Seminar on moduli of K3 surfaces

Organizational information. The organizational meeting will tentatively be on March 28, 2024, 2:00PM in N0.003, and on Zoom, at https://zoom.us/j/93737503454. After the meeting, the talk sign-up form will be posted online, and updated as necessary (e.g. if talks need to be moved, etc, during the semester).

The regular seminar meeting will be on Wednesday, 14:00-16:00 in N0.003. *The seminar will begin on the second week of the term, and so the first meeting will be April 18, 2024.*

Guideline for preparing talks. Email me at

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to arrange a meeting the week before your talk. During this meeting we can discuss questions you have about the material, and how best to present it.

Remember, the goal is for everyone to learn the material! In particular, most of your participation in the seminar will be as an audience member, so it is in everyone's interest for the talks to be understandable. Therefore, plan your talk to take 75 minutes without questions, so that it can take 90 minutes with questions.

Some of the talks are best planned with the speaker of the next talk. In these cases, it is strongly encouraged for you to reach out to that speaker, and discuss how the material will be split up.

It is also good idea to rehearse your talk in front of a friend or colleague. This can be also combined with a meeting to discuss with the speaker of the next talk.

Lecture notes. Please email lecture notes (that is, what you plan to write on the board during your talk) to me, preferably before but certainly no later than, the day of your talk. Scans of neatly handwritten notes are fine.

General references. The following references are broadly useful, and cover many of the specific topics listed below. Additional references are also listed for each individual talk.

Reference 1: *Géométrie des surfaces: modules et périodes*, Séminaire Palaiseau, Astérisque, Volume 126, 1985

Reference 2: Lecture notes on K3 surfaces, https://www.math.uni-bonn.de/people/huybrech/K3Global.pdf, by Daniel Huybrechts

0.1. **List of lectures.** A list of lectures for the seminar is as follows:

(1) *Introduction to K3 surfaces.* This lecture will describe the geometry of K3 surfaces, polarizations and Picard group, cohomology, examples of some moduli spaces in low degree, and where K3 surfaces sit in the Enriques-Kodaira classification.
Reference 1: *Complex Algebraic Surfaces*, by Beauville
Reference 2: *Compact Complex Surfaces*, by Barth, Peters, and Van de Ven

(2) *Introduction to K3 surfaces II.* This lecture will describe linear systems of curves on K3 surfaces, the nef cone, vanishing results for cohomology, existence of K3 surfaces of all degree, etc.
Reference: Above, and Huybrecht's notes

(3) *Deformations of complex manifolds.* This and the next lecture will discuss deformations of complex manifolds, the Kodaira-Spencer map, conditions for the existence of a Kuranishi family, and give some proofs.
Reference 1: Notes at https://www.math.stonybrook.edu/~cschnell/pdf/notes/kodaira.pdf, by Schnell
Reference 2: Notes at https://www1.mat.uniroma1.it/people/manetti/rend344B.pdf, by Manetti

(4) *Period mapping and period domain.* This lecture will continue the previous one on deformations, and apply the result to analytic K3 surfaces. Then, the holomorphic period map and period domain for polarized and unpolarized K3 surfaces will be described, along with the notion of a marking, and the local Torelli theorem will be proven.
References: Astérique, Huybrecht's notes, and the above

(5) *Variations of Hodge structure.* This lecture will define variations of Hodge structure generally, describe how they arise from families of smooth projective varieties, and explain the general origin of the Griffiths transversality condition.
Reference: *Period Mappings and Period Domains*, Chapter 4, Carlson, Müller-Stach, Peters

(6) *The Baily-Borel theorem.* This lecture will outline the theorem of Baily and Borel—that the target of the period map is a quasiprojective variety. The Borel algebraicity theorem will also be mentioned. [This talk is pretty hard. It should only be taken by someone familiar with Lie algebras/groups.]

Reference 1: *Compactification of Arithmetic Quotients of Bounded Symmetric Domains*, by Baily and Borel

Reference 2: Lecture notes at https://virtualmath1.stanford.edu/~conrad/shimsem/2013Notes/BailyBorelcompactification.pdf, by Lipnowski

(7) *Degenerations of K3 surfaces.* The Kulikov-Persson-Pinkham theorem describing nice degenerations of K3 surfaces will be described (and the proof outlined, if time permits). Examples of Kulikov models will be given.

Reference 1: *Degeneration of Surfaces with Trivial Canonical Bundle*, Persson and Pinkham

Reference 2: *Birational Geometry of Degenerations*, introductory material of book by Friedman

(8) *Some mixed Hodge theory.* This lecture will discuss Deligne's theory of mixed Hodge structures, with an eye towards the examples appearing for degenerations of K3 surfaces.

Reference 1: *Théorie de Hodge: II*, by Deligne

Reference 2: Survey at https://arxiv.org/pdf/1412.8499.pdf, by Filippini, Ruddat, and Thompson

(9) *Toric varieties.* This lecture will describe the fundamentals on toric varieties, including fans and polytopes.

Reference 1: *Introduction to toric varieties*, by Fulton

Reference 1: Lecture notes at https://www2.math.ethz.ch/education/bachelor/lectures/fs2015/math/alg_geom/brasselet, by Jean-Paul Brasselet

(10) *Anticanonical pairs.* The theory of "log Calabi-Yau" surfaces with an anticanonical cycle (the building blocks of Kulikov models) will be outlined, and their Torelli theorem discussed.

Reference: Survey article at https://arxiv.org/pdf/1502.02560.pdf, by Friedman

(11) *Toroidal compactifications of the period domain.* This lecture will describe the construction of toroidal compactifications of Type IV arithmetic quotients, with examples relevant to moduli of polarized K3 surfaces.

Reference 1: https://webspace.science.uu.nl/~looij101/ivarrdmj2a.pdf, by Looijenga

Reference 2: Talk to me. The general reference (Ash-Mumford-Rapaport-Tai) and the written references are very difficult, but for Type IV domains, everything is much easier than the general case.

(12) *Global Torelli theorem.* This lecture will prove the global Torelli theorem for K3 surfaces, following Friedman or Beauville.

Reference 1: *A New Proof of the Global Torelli Theorem for K3 Surfaces*, by Friedman

Reference 2: Astérisque survey, Chapters XIII, IX, X by Beauville

(13) *Lattice-polarized K3 surfaces, and Enriques surfaces.* Moduli spaces and period domains of lattice-polarized and Enriques surfaces will be described.

Reference 1: *Mirror symmetry for lattice-polarized K3 surfaces*, by Dolgachev, with some minor corrections, discussed in Section 1 of `https://arxiv.org/pdf/2101.12186v1.pdf`

Reference 2: `https://arxiv.org/pdf/1502.02723.pdf`, by Gritsenko and Hulek