Building Discount Factor Curves

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Abstract

It is of course obvious that the value of a £1 paid today is — £1. The same amount to be paid at some time T in the future however must have a value today less than £1. This follows from the fact that £1 received today can be invested and therefore be 'worth' more tomorrow. The discount factor function $P(t,T)$ describes the factor by which a cash amount paid at T is 'discounted' in order to obtain the value of the future payment at the time t ($t \leq T$). Thus in terms of $P(t,T)$, the opening argument can be summarised by the statement $P(t,t) \equiv 1$.

How though can we obtain values for the discount factor function for other choices of T? The short answer is that discount factors can be derived from market quoted prices for financial products. This process of turning market quoted rates and prices for financial products into values of the discount factor function, commonly referred to as the 'bootstrapping' of a discount factor curve, is what we will be concerning ourselves with in the sections ahead.

1 Cash Instruments

The simplest market quoted product from which a discount factor can be inferred is the cash deposit.

1.1 Overnight deposits

The 'overnight' deposit is a deposit at the market quoted rate $r_{O/N}$ starting today (denoted t) and ending one good business day later – time $T_{O/N}$ say. Since at $T_{O/N}$ the holder of the deposit can expect to receive $\left(1 + r_{O/N}\langle T_{O/N} - t\rangle\right)$ of the principal invested, the discount factor corresponding to the overnight instrument is

$$P(t, T_{O/N}) = 1/\left(1 + r_{O/N}\langle T_{O/N} - t\rangle\right). \qquad (1)$$

Here $\langle T - t\rangle$ means the day-count fraction from t to T using the appropriate basis.

1.2 Tomorrow-next deposits

The 'tomorrow-next' deposit is a deposit starting on the maturity of the 'overnight' deposit and maturiting one good business day later – time $T_{T/N}$ say. If $r_{T/N}$ denotes the market quoted rate for this instrument, then the tomorrow-next discount factor can be inferred to be

$$P(t, T_{T/N}) = P(t, T_{O/N})/\left(1 + r_{T/N}\langle T_{T/N} - T_{O/N}\rangle\right). \qquad (2)$$

1.3 Deposits of longer terms

The remaining commonly traded short term deposits typically begin on the tomorrow-next end date — usually referred to as the 'spot' date denoted here by T_{SPOT}. If r is the market quoted rate of a deposit starting at time T_{SPOT} and paying at some time T ($T_{SPOT} < T$) then:

$$P(t,T) = P(t, T_{SPOT})/\left(1 + r\langle T - T_{SPOT}\rangle\right). \qquad (3)$$

2 Future instruments

Beyond the cash instruments that make up the very short end of the curve, a popular choice of curve building instrument are the three month IMM futures. We treat these as forward deposits and in determining the value of the discount factor function corresponding to the price of such a forward deposit, we take one of two approaches, depending on how much of the discount factor curve is already known.

2.1 Analytic

Let T_N denote the last T (last known epoch) for which $P(t,T)$ is known. Let T_S and T_E denote the start and maturity dates of the forward deposit respectively. If $T_N >= T_S$ we may solve for$P(t,T_E)$ analytically:

$$P(t,T_E) = P(t,T_S)/\left(1 + F\langle T_E - T_S\rangle\right) \tag{4}$$

with F the convexity adjusted forward rate. Note that $P(t,T_S)$ is unlikely to be known exactly except in the coincidence that we have encountered a cash instrument maturing at precisely that date and it is typically therefore estimated by our interpolation function $f(T_S; P(t,T_{N-1}), P(t,T_N))$ (see Appendix A).

2.2 Numeric

In the event $T_N < T_S$ we resort to a 'root-solving' technique to find $P(t,T_E)$. Rearranging equation 4 we see that F is related to $P(t,T_E)$ by

$$F = \left(P(t,T_S)/P(t,T_E) - 1\right)\left(1/\langle T_E - T_S\rangle\right).$$

The idea is to refine guesses of $P(t,T_E)$ until we find the $P(t,T_E)$ such that

$$F^{implied} = \left(P(t,T_S)/P(t,T_E) - 1\right)\left(1/\langle T_E - T_S\rangle\right) = (1 - Q)$$

where in this procedure, we estimate $P(t,T_S)$ via our interpolation function

$$f\left(T_S; P(t,T_N), P(t,T_E)\right).$$

Putting this another way, we seek the $P(t,T_E)$ such that

$$w\left(P(t,T_S), P(t,T_E)\right) = F^{implied} - (1 - Q) = 0. \tag{5}$$

Our preferred algorithm for the root-solving procedure is the 'Newton-Raphson' method for its efficiency. In order to apply this method, we will require the value of w at the point $(P(t,T_S), P(t,T_E))$ and its derivative there. The derivative follows easily from the chain-rule (see Appendix B):

$$\begin{aligned}\frac{dw}{dP(t,T_E)} &= \frac{\partial w}{\partial P(t,T_S)}\frac{dP(t,T_S)}{dP(t,T_E)} + \frac{\partial w}{\partial P(t,T_E)} \\ &= \left(1/P(t,T_E)\langle T_E - T_S\rangle\right)\frac{\partial P(t,T_S)}{\partial P(t,T_E)} \\ &\quad - P(t,T_S)/\left(P(t,T_E)^2\langle T_E - T_S\rangle\right).\end{aligned} \tag{6}$$

Note that the partial derivative

$$\frac{\partial P(t,T_S)}{\partial P(t,T_E)} = \frac{\partial f(T_S; P(t,T_N), P(t,T_E))}{\partial P(t,T_E)}$$

that is, is determined by the interpolation function f (see Appendix A).

3 Swap Instruments

The futures strip described previously typically makes up the 'mid-section' of the discount factor curve. For the values of $P(t,T)$ for values of T a year or greater, swap instruments are typically chosen over futures. Suppose a swap starting at $T_0^S = T_{SPOT}$. If the fixed side of the swap has N_{fixed} payments at times T_i^S, $1 < i < N_{fixed}$ with final payment date $T_{end}^S = T_{N_{fixed}}^S$ and the market quoted swap rate is r, then at t, the value of the fixed side is

$$\sum_{i=1}^{N_{fixed}} r\langle T_i^S - T_{i-1}^S\rangle P(t,T_i^S). \tag{7}$$

Suppose the float side also starting at T_0^S and with final payment at T_{end}^S has N_{float} payments. Let $T_{i-1}^{S'}$ and $T_i^{S'}$ be the index reset dates of the ith payment period ($i = 1,\ldots,N_{float}$) and T_{i-1}^S, T_i^S denote in this case, the float payment dates. Then the value of the floating side of the swap is

$$\sum_{i=1}^{N_{float}} \left(\frac{P(t,T_{i-1}^{S'})}{P(t,T_i^{S'})} - 1\right)\left(\frac{1}{\langle T_i^{S'} - T_{i-1}^{S'}\rangle}\right)\langle T_i^S - T_{i-1}^S\rangle P(t,T_i^S). \tag{8}$$

A common simplification made when building discount factor curves is that the index reset dates and float payment dates are aligned. That is, $T_i^{S'} = T_i^S$ for all i. This is the so-called 'swap' convention. In this case the value of the float side becomes

$$\sum_{i=1}^{N_{float}} P(t,T_{i-1}^S) - P(t,T_i^S)$$

and after expansion of the sum, further reduces to the very simple notional exchange:

$$P(t,T_0^S) - P(t,T_{end}^S). \tag{9}$$

To find $P(t,T) = P(t,T_{end}^S)$ we utilise the fact that the net value of the swap at the beginning of it's lifetime must be 0. That is from equations 7 and 9

$$\left(\sum_{i=1}^{N_{fixed}} r\langle T_i^S - T_{i-1}^S\rangle P(t,T_i^S)\right) - \left(P(t,T_0^S) - P(t,T_{end}^S)\right) = 0$$

where in this equation, the T_i^S are the payment dates of the fixed side of the swap. To do this, we find by a root-solving technique, the $P(t,T_{end}^S)$ such that

$$r^{implied} = \frac{P(t,T_0^S) - P(t,T_{end}^S)}{\sum_{i=1}^{N_{fixed}} P(t,T_i^S)\langle T_i^S - T_{i-1}^S\rangle} = r \tag{10}$$

or putting this another way, we seek the $P(t,T_{end}^S)$ such that

$$w(P(t,T_0^S), P(t,T_1^S),\ldots,P(t,T_{end}^S)) = r^{implied} - r = 0. \tag{11}$$

As always, we prefer the Newton-Raphson method for it's efficiency. As well as the value of w at the point $(P(t,T_0^S),\ldots P(t,T_{end}^S))$ we will be required to compute it's derivative there with respect to the final curve epoch $P(t,T_N^C)$

($T_N^C = T_{end}^S$). Note that in equation 11, the $P(t, T_i^S)$ are *dependant* variables because they are related to the *indepedent* variables $P(t, T_0^C), \ldots, P(t, T_N^C)$ through the interpolation function f:

$$P(t, T_i^S) = f(T_i^S; P(t, T_0^C), \ldots, P(t, T_N^C))). \tag{12}$$

Now, by the chain rule (see Appendix B) we have

$$\frac{\partial w}{\partial P(t, T_N^C)} = \frac{\partial w}{\partial P(t, T_0^S)} \left(\frac{\partial P(t, T_0^S)}{\partial P(t, T_N^C)} \right) + \sum_{k=1}^{N_{fixed}} \frac{\partial w}{\partial P(t, T_k^S)} \left(\frac{\partial P(t, T_k^S)}{\partial P(t, T_N^C)} \right) \tag{13}$$

where the factors in brackets can be obtained from the interpolation function f (see Appendix A) and the remaining factors are given by

$$\frac{\partial w}{\partial P(t, T_0^S)} = \frac{1}{\sum_{i=1}^{N_{fixed}} P(t, T_i^S) \langle T_i^S - T_{i-1}^S \rangle} \tag{14}$$

$$\frac{\partial w}{\partial P(t, T_k^S)} = -\frac{1}{\sum_{i=1}^{N_{fixed}} P(t, T_i^S) \langle T_i^S - T_{i-1}^S \rangle} \left(\delta_{k, N_{fixed}} + r^{implied} \langle T_k^S - T_{k-1}^S \rangle \right). \tag{15}$$

4 Cutovers

4.1 Futures over cash - Preservation of the 3M cash rate

Let the next IMM future deposit starting beyond the curve build date t, be denoted IMM_1. Suppose T_S to be the start date of this forward deposit and T_E to be the deposit payment date. If F is the convexity adjusted forward rate and $P(t, T_S)$ and $P(t, T_E)$ the values of the discount factor function at T_S and T_E respectively then,

$$P(t, T_E) = \frac{P(t, T_S)}{1 + F\langle T_E - T_S \rangle}. \tag{16}$$

Consider now the $3M$ cash deposit. This is a deposit at the market quoted rate r_{3M} (say), starting at T_{SPOT} and ending at T_{3M}. Note that $T_{SPOT} \leq T_S$ and if $T_{SPOT} \neq T_S$ then, $T_S < T_{3M} < T_E$. If follows that if $P(t, T_{SPOT})$ and $P(t, T_S)$ were known, $P(t, T_E)$ would be fixed by the formula above and we could predict a value for $P(t, T_{3M})$ from our interpolation function $f(t; T_{3M}, P(T_S), P(T_E))$. Of course,

$$P(t, T_{3M}) = \frac{P(t, T_{SPOT})}{1 + r_{3M} \langle T_{3M} - T_{SPOT} \rangle}$$

and so the idea is to 'vary' $P(t, T_S)$ until

$$r_{3M}^{implied} = \left(\frac{P(t, T_{SPOT})}{f(T_{3M}; P(t, T_S), P(t, T_E))} - 1 \right) \left(\frac{1}{\langle T_{3M} - T_{SPOT} \rangle} \right) = r_{3M} \tag{17}$$

or putting this another way, we seek the $P(t, T_S)$ such that

$$w\left(P(t, T_{SPOT}), P(t, T_S)\right) = r_{3M}^{implied} - r_{3M} = 0. \tag{18}$$

An abundance of root solving techniques exist that could be applied to this problem; we prefer the Newton-Raphson method for its efficiency. In order to apply this method, aside from the value of w at the point $(P(t,T_{SPOT}), P(t,T_S))$ which can be found by equation 17, we will be required to compute the partial derivative $\frac{\partial w}{\partial P(t,T_S)}$.

From the definition of w and application of the chain-rule (see Appendix B) we find:

$$\frac{\partial w}{\partial P(t,T_S)} = \frac{\partial w}{\partial P(t,T_{SPOT})}\frac{\partial P(t,T_{SPOT})}{\partial P(t,T_S)} + \frac{\partial w}{\partial P(t,T_{3M})}\frac{\partial P(t,T_{3M})}{\partial P(t,T_S)}.$$

Carrying out differentiations we get:

$$\frac{\partial w}{\partial P(t,T_S)} = \left(\frac{-P(t,T_{SPOT})}{P(t,T_{3M})^2\langle T_{3M}-T_{SPOT}\rangle}\right)\frac{\partial P(t,T_{3M})}{\partial P(t,T_S)} \tag{19}$$

and so it remains to compute the value of $\frac{\partial P(t,T_{3M})}{\partial P(t,T_S)}$.

To compute $\frac{\partial P(t,T_{3M})}{\partial P(t,T_S)}$, recall $P(t,T_E) = \frac{P(t,T_S)}{1+F\langle T_E-T_S\rangle}$ by equation 16. That is, $P(t,T_E)$ is a single variabled function: $P(t,T_E) = y\left(P(t,T_S)\right)$. If we define $u = P(t,T_S)$, $x(u) = u$ then for the interpolation scheme f, we may write

$$P(t,T_{3M}) = f\left(T_{3M}; P(t,T_S), P(t,T_E))\right) = f(x(u), y(u))$$

so that,

$$\begin{aligned}
\frac{\partial P(t,T_{3M})}{\partial P(t,T_S)} &= \frac{dP(t,T_{3M})}{dP(t,T_S)} \\
&= \frac{df(x(u),y(u))}{du} \\
&= \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} \\
&= \frac{\partial f}{\partial x}\times 1 + \frac{\partial f}{\partial y}\left(\frac{1}{1+F\langle T_E-T_S\rangle}\right) \\
&= \frac{\partial f}{\partial P(t,T_S)} + \frac{\partial f}{\partial P(t,T_E)}\left(\frac{1}{1+F\langle T_E-T_S\rangle}\right)
\end{aligned} \tag{20}$$

and finally by combining equations 19 and 20 we get:

$$\begin{aligned}
\frac{\partial w}{\partial P(t,T_S)} = & \\
& \left(\frac{-P(t,T_{SPOT})}{P(t,T_{3M})^2\langle T_{3M}-T_{SPOT}\rangle}\right) \\
& \times\left(\frac{\partial f}{\partial P(t,T_S)} + \frac{\partial f}{\partial P(t,T_E)}\left(\frac{1}{1+F\langle T_E-T_S\rangle}\right)\right)
\end{aligned} \tag{21}$$

(see Appendix A for the partial derivatives of f).

A Interpolation

Interpolation is the process of estimating the values of a function $y(x)$ for arguments between $x_0,\ldots,x_n$ at which the values $y_0,\ldots,y_n$ are known.

A.1 Linear interpolation

In this scheme, if $x_{i-1} \leq x < x_i$ we estimate $y(x)$ by

$$y = \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)(y_i - y_{i-1}) + y_{i-1}.$$

If we define the quantity R by $R = \frac{x-x_{i-1}}{x_i-x_{i-1}}$, then in terms of R we find

$$y = R\,(y_i - y_{i-1}) + y_{i-1}. \tag{22}$$

Note that it is easily shown that $1-\left(\frac{x_i - x}{x_i - x_{i-1}}\right) \equiv R$

In addition to estimating $y(x)$ by equation 22 we also have that if $x = x_i$ for some i then $y = y(x_i)$ and $\frac{\partial y}{\partial y_i} = 1$. Otherwise, if $x_{i-1} < x < x_i$ then

$$\frac{\partial y}{\partial y_{i-1}} = 1 - R \tag{23}$$

$$\frac{\partial y}{\partial y_i} = R. \tag{24}$$

A.2 Log-linear interpolation

In this scheme, we estimate $y(x)$ by

$$y = e^{ln(y_{i-1})+(ln(y_i)-ln(y_{i-1}))R}. \tag{25}$$

In addition to estimating $y(x)$ by equation 25 we also have that if $x = x_i$ for some i then $y = y(x_i)$ and $\frac{\partial y}{\partial y_i} = 1$. Otherwise, if $x_{i-1} < x < x_i$ then

$$\frac{\partial y}{\partial y_{i-1}} = \frac{y}{y_{i-1}}(1 - R) \tag{26}$$

$$\frac{\partial y}{\partial y_i} = \frac{y}{y_i}R. \tag{27}$$

A.3 Linear on zero interpolation

In this scheme, we estimate $y(x)$ in the following way. First, if $i - 1 = 0$ then

$$y = y_i^{\left(\frac{x-x_0}{x_i-x_{i-1}}\right)}. \tag{28}$$

In this case it follows that

$$\frac{\partial y}{y_{i-1}} = 0 \tag{29}$$

$$\frac{\partial y}{y_i} = \frac{x - x_0}{x_i - x_{i-1}} y_i^{\left(\frac{x-x_0}{x_i-x_{i-1}}\right)-1}. \tag{30}$$

Else define

$$z_i = \frac{-ln(y_i)}{x_i - x_0}$$

$$z_{i-1} = \frac{-ln(y_{i-1})}{x_{i-1} - x_0}.$$

Then,

$$y = e^{-(z_{i-1}+R(z_i-z_{i-1}))(x-x_0)}. \tag{31}$$

In addition to estimating $y(x)$ by equation 31 we also have that if $x = x_i$ for some i then $y = y(x_i)$ and $\frac{\partial y}{\partial y_i} = 1$. Otherwise, if $x_{i-1} < x < x_i$ then

$$\frac{\partial y}{\partial y_{i-1}} = \frac{y}{y_{i-1}} \left(\frac{x-x_0}{x_{i-1}-x_0} \right) (1-R) \tag{32}$$

$$\frac{\partial y}{\partial y_i} = \frac{y}{y_i} \left(\frac{x-x_0}{x_i-x_0} \right) R. \tag{33}$$

A.4 Cubic spline interpolation

Another popular interpolation method, popular because the curves it produces are particularly smooth, is to let the fitting function be a piecewise union of cubic polynomials. That is we define a polynomial P_i on each interval $[a_{i-1}, a_i]$ such that the endpoints of the polynomial pass through the ordinates $\{y_i\}$and that the first and second derivatives of the cubic match with the next cubic along - i.e.:

$$P_i(x_i) = y_i \tag{34}$$

$$P_{i-1}(x_i) = y_i \tag{35}$$

$$\frac{d}{dx}P_i(x_{i-1}) = \frac{d}{dx}P_{i-1}(x_i) \tag{36}$$

$$\frac{d^2}{dx^2}P_i(x_{i-1}) = \frac{d^2}{dx^2}P_{i-1}(x_i) \tag{37}$$

for all i. By imposing conditions on the values of the derivative at the very endpoints of the function x_0 and x_{N-1} there are sufficiently many conditions for the coefficients of all the cubics to be determined uniquely by solving a linear system of equations. The exact form of this linear system varies from one source to another. In this note, we use the form found in [**?**].

Given x, let i be such that $a_{i-1} < x < a_i$. Then our formulation says that our cubic for this i-th segment is

$$\begin{aligned} p(x) &= \frac{c_{i-1} * (a_i - x)^3}{6h_i} \\ &+ \frac{c_i(x-a_{i-1})^3}{6h_i(a_i-a_{i-1})} \\ &+ (y_{i-1} - \frac{c_{i-1}h_i^2}{6})(\frac{a_i - x}{h_i}) \\ &+ (y_i - \frac{c_i h^2}{6})(\frac{x-a_{i-1}}{h_i}) \end{aligned} \tag{38}$$

where c is a set of vectors linearly dependent on the ordinates $\{y_i\}$that we will determine and h_i is the width of the segment $(= a_i - a_{i-1})$

Because c is a linear function of the $\{y_i\}$and p is a linear function of the cs, it follows that the p is a linear function of the $\{y_i\}$also. Thus once we have found all the partial derivatives $\frac{\partial c_i}{\partial y_j}$, we may easily find $\frac{\partial p}{\partial y_j}$.

c is determined by the linear system of equations $Ac = b$ where A is square tridiagonal matrix whose values are dependent only on the segment widths h_i and each b is a linear combination of the y_i. More specifically

$$\begin{aligned} b_0 &= d_{left} \\ b_i &= \frac{6}{h_i + h_{i+1}}\left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) \\ b_n &= d_{right} \end{aligned} \tag{39}$$

where d_{left} and d_{right} are constants dependent only on the choice of the value of the derivatives at the endpoints of the curve [1]. A is invertible, let $F = A^{-1}$. Then we have

$$\begin{aligned} c_i &= F_{i,0}d_{left} + F_{i,N-1}d_{right} + 6\frac{F_{i,1}}{h_1(H_1 + h_1)}F_{i,1}y_0 \\ &+ 6\left(-\frac{F_{i,1}}{H_1(H_1 + h_1)} - \frac{F_{i,1}}{h_1(H_1 + h_1)} + \frac{F_{i,2}}{h_2(H_2 + H_2)}\right)y_1 \\ &+ 6\left(\Sigma_{j=2}^{N-3}\left(\frac{F_{i,j-1}}{H_{j-1}(H_{j-1} + h_{j-1})} - \frac{F_{i,j}}{H_j(H_j + h_j)} - \frac{F_{i,j}}{h_j(H_j + h_j)} + \frac{F_{i,j+1}}{h_{j+1}(H_{j+1} + h_{j+1})}\right)\right)y_j \\ &+ 6\left(-\frac{F_{i,N-2}}{H_{N-2}(H_{N-2} + h_{N-2})} - \frac{F_{i,N-1}}{h_{N-2}(H_{N-2} + h_{N-2})} + \frac{F_{i,N-3}}{H_{N-3}(H_{N-3} + h_{N-3})}\right)y_{N-2} \\ &+ 6\frac{F_{i,N-2}}{H_{N-2}(H_{N-2} + h_{N-2})}y_{N-1} \end{aligned} \tag{40}$$

where $h_i = a_i - a_{i-1}$ as before and $H_i = h_{i+1}$. Thus the c_i are of the form:

$$c_i = l_i + \Sigma_j k_{i,j} y_j \tag{41}$$

for some known constants $k_{i,j}$ and l_i. This means we have

$$\frac{\partial c_i}{\partial y_j} = k_{i,j} \tag{42}$$

Plugging this into our original formula for p we have

$$\begin{aligned} \frac{\partial p}{\partial y_j}(x) &= \frac{(a_i - x)^3}{6h_i}k_{i-1,j} \\ &+ (\frac{(x - a_{i-1})^3}{6h_i}k_{i,j} \\ &+ (\delta_{i-1,j} - \frac{k_{i-1,j}h_i^2}{6})\frac{x_i - x}{a_i - a_{i-1}} \\ &+ (\delta_{i,j} - \frac{k_{i,j}h_i^2}{6})\frac{x_i - x}{a_i - a_{i-1}} \end{aligned} \tag{43}$$

[1] In the case of the so-called *natural spline*, we set the derivatives at the end-points to be zero, and have $d_{left} = d_{right} = 0.0$

B The general chain rule

B.1 The most general case

Suppose that w is a function of the variables $x_1, x_2, x_3, \ldots, x_m$, and that each of these is a function of the variables $t_1, t_2, \ldots, t_n$. If all of these functions have continuous first order partial derivatives, then

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_i}. \tag{44}$$

In the situation of the above theorem, we refer to w as the *dependent* variable, the $x_1, \ldots, x_m$ as *intermediate* variables and the $t_1, \ldots, t_n$ as the *independent* variables.

B.2 The simplest multivariable case

The simplest multivariable chain rule situtation involves a function $w = f(x, y)$ where x and y are functions of the same single variable $t : x = g(t)$ and $y = h(t)$. The composite function $f(g(t), h(t))$ is then a single-variable function of t, and the following theorem expresses its derivative in terms of the partial derivatives of f and the ordinary derivatives of g and h

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}. \tag{45}$$

C Solving on price vs.solving on rates

What is a 'fair' cash deposit rate? A fair cash rate is the interest rate at which the value today of a deposit at that rate has no value. In the notation of section 1.3 that is:

$$V(t,T) = (1 + r\,\langle T - T_{SPOT}\rangle)\,P(t,T) - P(t,T_s) = 0. \tag{46}$$

The partial derivatives of V with respect to the discount factors at the curve epochs are:

$$\begin{aligned}\frac{\partial V(t,T)}{\partial P(t,T_i^C)} &= \frac{\partial V(t,T)}{\partial P(t,T_{SPOT})}\frac{\partial P(t,T_{SPOT})}{\partial P(t,T_i^C)} + \frac{\partial V(t,T)}{\partial P(t,T)}\frac{\partial P(t,T)}{\partial P(t,T_i^C)} \\ &= -\frac{\partial P(t,T_{SPOT})}{\partial P(t,T_i^C)} + (1 + r\,\langle T - T_{SPOT}\rangle)\,\frac{\partial P(t,T)}{\partial P(t,T_i^C)}. \end{aligned} \tag{47}$$

What is a 'fair' swap rate? A fair swap rate is the rate at which the value today of a swap at that rate has no value. In the notation of section 3 that is:

$$V(t,T) = P(t,T_0^S) - P(t,T_N^S) - r\sum_{i=1}^{N} P(t,T_i^S)\,\langle T_i - T_{i-1}\rangle = 0. \tag{48}$$

The partial derivatives of V with respect to the discount factors at the curve epochs are:

$$\frac{\partial V(t,T)}{\partial P(t,T_i^C)} = \frac{\partial V(t,T)}{\partial P(t,T_0^S)}\frac{\partial P(t,T_0^S)}{\partial P(t,T_i^C)} + \sum_{k=1}^{N}\frac{\partial V(t,T)}{\partial P(t,T_k^S)}\frac{\partial P(t,T_k^S)}{\partial P(t,T_i^C)}$$

$$= \frac{\partial P(t, T_0^S)}{\partial P(t, T_i^C)} - \sum_{k=1}^{N} \left(\delta_{k,N} + r \left\langle T_k - T_{k-1} \right\rangle\right) \frac{\partial P(t, T_k^S)}{\partial P(t, T_i^C)}. \quad (49)$$