20/13
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Geometric and Binomial Series

1 Geometric Series

One of the most important series in all of calculus is the geometric series, defined by

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

which converges for all $|x| < 1$ (please note that this is a strict inequality).

Notice that the sum also converges for any value of x such that $|f(x)| < 1$:

$$\sum_{n=0}^{\infty} (f(x))^n = \frac{1}{1-f(x)}$$

1. What function does the following sum converge to?

$$\sum_{n=0}^{\infty} (-1)^n x^n$$

2. Express the following function as an infinite series.

$$\frac{1}{1+x^2}$$

3. Show that the infinite series you found in the previous part "works" by doing the following:

   (a) Start with the equation

   $$\frac{1}{1+x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

   using the coeficients a_i that you found in the previous problem. Now multiply both sides by $1+x^2$. What does the left-hand side become? What about the right (leave as the product of two polynomials)?

   (b) Begin to expand the right-hand side by collecting like powers of x. Show that when you do this all of the powers of x greater than zero dissapear. Explain how this "confirms" that the infinite series really is equal to $\frac{1}{1+x^2}$.

2 Generalized Binomial Series

We can define the Binomial Coefficients in terms of factorials using

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

These numbers are extremely useful and they show up in all sorts of unexpected places. One of the places that they show up is in the **binomial expansion**:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Notice that the above formula only makes sense when n is an integer greater than or equal to zero. We would like a way to extend this relationship between binomial expansions and series even for negative or fractional values of n. One way to do this is to redefine the binomial coefficients $\binom{n}{k}$ so that they do not depend on factorials, which are only defined for non-negative integers.

1. Show that for n and k integers such that $n \geq k \geq 0$ an alternate way to define the binomial coefficients is given by

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(3)(2)(1)}$$

2. Using this new definition, show that if n is a non-negative integer then $\binom{n}{k} = 0$ whenever $k > n$. Explain how this allows use to express the binomial expansion as the infinite series

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k$$

3. Does $\binom{n}{k}$ ever equal zero for fractional or negative values of n? Please justify your answer.

4. Naively we would like to use this more general binomial coefficient to define a **Binomial Series** valid for all values of n, negative, fractional, positive. We would like something like the following expression:

$$(a+b)^{m/n} = \sum_{k=0}^{\infty} \binom{m/n}{k} a^{n-k} b^k$$

Unfortunately, this won't work because the powers of a don't depend on m, only on n. If we tried to just replace n by m/n to give

$$(a+b)^{m/n} = \sum_{k=0}^{\infty} \binom{m/n}{k} a^{(m/n)-k} b^k$$

then we would have the issue that now the powers of a would be fractional. This seems to be a big problem. This problem would go away if we could find some value of a that was always the same regardless of the power. Find such a value. It might be useful to recall that $0^0 = 1$.

5. After all this hard work, we arrive at the following unproven definition of the Binomial Series: if $|x| < 1$

$$(1+x)^{m/n} = \sum_{k=0}^{\infty} \binom{m/n}{k} x^k$$

We are going to use this series to find an expansion for $\sqrt{1-x}$ and then argue informally that the series "works" by showing that it squares to $1-x$.

If the lack of rigor in any of what we're doing seems in some sense illegal, rest assured that this is almost exactly how Newton reasoned and worked with series. We are not going to worry about subtle issues like convergence right now, but we are going to blindly trust that anything that holds true for finite calculations also holds true for infinite calculations. While this is not true in general, it is in this particular case. Hey, if it's good enough for Newton it should be good enough for us.

(a) Show that you can write $\sqrt{1-x}$ as $(1+(-x))^a$ for some value of a. What is this value?

(b) Use the definition of the general binomial coefficients to calculate $\binom{1/2}{0}, \binom{1/2}{1}, \binom{1/2}{2}, \binom{1/2}{3}$.

(c) Write out the first four terms in the expansion for $\sqrt{1-x}$. Your answer should look like

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

You need to determine the coefficients, a_i.

(d) Confirm that this series looks like it should work by squaring both sides of the following equation:

$$(\sqrt{1-x})^2 = (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)^2$$

To square the right hand side, collect like powers of x:

$$\begin{aligned}(a_0 + a_1x + a_2x^2 + a_3x^3 \cdots)^2 &= (a_0 + a_1x + a_2x^2 + a_3x^3 \cdots) \times (a_0 + a_1x + a_2x^2 + a_3x^3 \cdots) \\ &= a_0a_0 + a_0a_1x + a_0a_1x + a_0a_2x^2 + a_0a_2x^2 + (a_1x)(a_1x) \\ &\quad + a_0a_3x^3 + a_0a_3x^3 + (a_1x)(a_2x^2) + (a_1x)(a_2x^2) + \cdots \\ &= (a_0a_0) + (a_0a_1x + a_0a_1x) + (a_0a_2x^2 + a_0a_2x^2 + a_1^2x^2) \\ &\quad + (a_0a_3x^3 + a_0a_3x^3 + a_1a_2x^3 + a_1a_2x^3) + \cdots \\ &= a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + (2a_0a_3 + 2a_1a_2)x^3 + \cdots\end{aligned}$$

Show that $a_0^2 = 1, 2a_0a_1 = -1$ and that the coëfficients of the higher powers of x are equal to zero.

6. We now want to use this series to approximate $\sqrt{15}$.

   (a) Begin by rewriting $\sqrt{15} = \sqrt{16-1} = \sqrt{16(1-\frac{1}{16})}$. Explain why this is equal to $4\sqrt{1-\frac{1}{16}}$.

   (b) Use the first four terms of the binomial series that you found above to approximate $\sqrt{1-\frac{1}{16}}$.

   (c) Multiply the value that you found in the previous part by 4. Explain why this should be an approximation to $\sqrt{15}$.

   (d) Use a calculator or computer to calculate $\sqrt{15}$. How close is the answer that you got above? Bear in mind you only used the first four terms of an infinite polynomial. Explain what you would do if you wanted to increase the accuracy of your approximation.

3 Connection Between Geometric and Binomial Series

We saw in the first part that for $|x| < 1$ the geometric series is given by

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

We saw in the second part that for $|x| < 1$ and any value of n the binomial series is given by

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

1. Rewrite $\frac{1}{1-x}$ so that it looks like a binomial series $(1+X)^a$. How is X related to x and what is the value of a?

2. Use the previous part to show that

$$\sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \binom{-1}{k} (-x)^k$$

3. Use the previous part and compare coefficients to show that for all $k = 0, 1, 2, 3\ldots$ the following holds

$$\binom{-1}{k} (-1)^k = 1$$