# Gambler's Ruin

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## 1 Introduction

A persistent gambler with finite wealth, playing a fair game (that is, each bet has expected value of zero to both sides) will eventually and inevitably go broke against an opponent with infinite wealth. Such as situation is closely related to the concept known as **gambler's ruin**, i.e., a gambler playing a game with negative expected value will eventually go broke, regardless of their betting system. This concept has multiple applications in probability, statistics, and finance. This paper presents some of the results associated to this concept and some visualizations.

## 2 Classic Gambler's Ruin Problem

Consider a coin-flipping game with two players with $p$ and $q = 1 - p$ chance of winning each round, respectively. After each round, the loser transfers one penny to the winner. The game ends when one player has all the pennies. Let the first and second player start with fortunes $\mathrm{n}_1$ and $\mathrm{n}_2$ pennies respectively, what is the probability of the first player winning?[1]

Let $P_i$ denote the probability of the first winning starting with a fortune of $i$, and $N = n_1 + n_2$, the total fortune in play. Notice that, by definition, $P_0 = 0$ and $P_N = 1$. By conditioning on the result of the first outcome, we have

$$P_i = pP_{i+1} + qP_{i-1}. \tag{1}$$

Since $p + q = 1$, rewrite (1) as $pP_i + qP_i = pP_{i+1} + qP_{i-1}$, yielding

$$\begin{aligned} pP_i + qP_i &= pP_{i+1} + qP_{i-1} \\ pP_{i+1} - pP_i &= qP_i - qP_{i-1} \\ P_{i+1} - P_i &= (\frac{q}{p})(P_i - P_{i-1}). \end{aligned} \tag{2}$$

In particular, (2) is $P_2 - P_1 = (\frac{q}{p})(P_1 - P_0) = (\frac{q}{p})P_1$, thus $P_3 - P_2 = (\frac{q}{p})(P_2 - P_1) = (\frac{q}{p})^2 P_1$, and, generally,

$$P_{i+1} - P_i = (\frac{q}{p})^i P_1.$$

Therefore,

$$\begin{aligned} P_{i+1} - P_1 &= P_{i+1} - P_i + P_i - P_{i-1} + ... + P_2 - P_1 \\ &= \sum_{k=1}^{i} (P_{i+1} - P_i) \\ &= \sum_{k=1}^{i} (\frac{q}{p})^k P_1, \end{aligned}$$

yielding

$$P_{i+1} = P_1 + \sum_{k=1}^{i} (\frac{q}{p})^k P_1 = \sum_{k=0}^{i} (\frac{q}{p})^k P_1.$$

[1] This is one of the way to present the Gambler's ruin problem, there are other similar variations, but the derivations presented in this document can all be adapted to fit these variations.

Using the equation for the sum the first $n$ terms of a geometric series (i.e., $\sum_{k=0}^{i} r^k = \frac{1-r^{k+1}}{1-r}$ where $r$ is the common ratio), we have

$$P_{i+1} = \begin{cases} \dfrac{1-(\frac{q}{p})^{i+1}}{1-(\frac{q}{p})} P_1 & \text{if } p \neq q \\ P_1(i+1) & \text{if } p = q. \end{cases} \tag{3}$$

In particular, for $i = N-1$, we have

$$P_N = 1 = \begin{cases} \dfrac{1-(\frac{q}{p})^{N}}{1-(\frac{q}{p})} P_1 & \text{if } p \neq 0.5 \\ NP_1 & \text{if } p = 0.5. \end{cases}$$

Thus,

$$P_1 = \begin{cases} \dfrac{1-(\frac{q}{p})}{1-(\frac{q}{p})^N} & \text{if } p \neq 0.5 \\ \dfrac{1}{N} & \text{if } p = 0.5. \end{cases}$$

Finally, using (3), we get the solution as

$$P_i = \begin{cases} \dfrac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})} \dfrac{1-(\frac{q}{p})}{1-(\frac{q}{p})^N} & \text{if } p \neq q \\ \dfrac{i}{N} & \text{if } p = q. \end{cases} = \begin{cases} \dfrac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} & \text{if } p \neq q \\ \dfrac{i}{N} & \text{if } p = q. \end{cases}$$

## 2.1 Interesting results

When $N \to \infty$ (i.e., in our context, when the second player has an infinite starting fortune), and $p \leq q$ the probability of the first player reaching infinite fortune is

$$\lim_{N\to\infty} P_i = \lim_{N\to\infty} \begin{cases} \dfrac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} & \text{if } p \neq q \\ \dfrac{i}{N} & \text{if } p = q. \end{cases} = 0.$$

When $p > q$, that probability is

$$\lim_{N\to\infty} P_i = \lim_{N\to\infty} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} = 1 - (\frac{q}{p})^i$$

In other words, if the player has a chance of winning less than or equal to 50%, ruin is guaranteed. If $p >$ 50%, the gambler has a positive probability of obtaining an infinite fortune, but may also end up ruined.

With the odds of a **European style roulette**, $p = \frac{18}{37}$, $q = \frac{19}{37}$, and fortunes $n_1 = 10$, $n_2 = 10$, we have

$$P_{10} = \frac{1-(\frac{18}{19})^{10}}{1-(\frac{18}{19})^{20}} = 0.368.$$

That is, a player starting with a fortune of 10 penny that makes roulette bets of 1 penny on black or red has 36.8% chance of reaching 20 before getting ruined. With $n_1 = 100$, $n_2 = 100$, that probability goes down to 0.5%. Using the popular classic strategy of "quitting while we're up", say with $n_1 = 10$, $n_2 = 2$ (i.e., the player will stop playing when he will have made reached 20% of his initial fortune), the probability is 78.5%. The same strategy with $n_1 = 100$, $n_2 = 20$ has 33.8% winning probability. By intuition, it is expected that playing more rounds of a game with a probability of winning under 50% leads to less chance of winning.

## Resources used

[Malaver, 2021] Malaver, J. A. (2021). The gambler's ruin problem.

[Sigman, 2009] Sigman, K. (2009). Gambler's ruin problem.