Math 156, Midterm Section 4

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Problem 1 Suppose that $A = \{\alpha, \odot, \Delta\}$, $B = \{\delta, \oplus\}$. Find that cardinality of $|P(P(A)) \times B|$.

$$\begin{aligned} |A| &= 3 \text{ and } |B| = 2 \\ |P(A)| &= 8 \text{ and so } |P(A) \times B| = |P(A)| \cdot |B| \\ |P(A)| \cdot |B| &= 24 \\ |(P(|P(A) \times B|)) &= 2^{24} \end{aligned}$$

Problem 2 Use a truth table to show that the following statements are logically equivalent. $P \implies (Q \wedge R) = (P \implies Q) \wedge (P \implies R)$ For $P \implies (Q \wedge R)$

P	Q	R	$Q \wedge R$	$P \implies (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

and for $(P \implies Q)$ and for $(P \implies R)$

P	Q	R	$P \implies Q$	$P \implies R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

So we see that $(P \implies Q) \wedge (P \implies R)$ and $P \implies (Q \wedge R)$ share the same truth table.

P	Q	R	$(P \implies Q) \wedge (P \implies R)$	$P \implies (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Problem 3 Prove that if a is an odd integer, then $a^2 \equiv 1(\mod 8)$.

If a is an odd integer, then there exists some $k \in \mathbb{Z}$ such that $a = 2k+1$. Then $a^2 = 4k^2+4k+1$ and we have $(4k^2+4k+1) \equiv 1(\mod 8)$. Then $8|(4k^2+4k+1)-1$ and so $8|4k^2+4k$. Since $4k^2+4k = 4k(k+1)$, then $8|4k(k+1)$. The proof proceeds by cases.

If k is even, then $k = 2\alpha, \alpha \in \mathbb{Z}$. Then $4k(k+1) = 8\alpha(2\alpha+1)$ which is clearly divisible by 8.

If k is odd, then $k = 2\alpha+1, \alpha \in \mathbb{Z}$. Then $4k(k+1) = (8\alpha+4)(2\alpha+2) = 16(\alpha^2)+24\alpha+8 = 8(2(\alpha^2)+12\alpha+1)$, which is clearly divisible by 8.

Therefore $8|a^2-1$, and so $a^2 \equiv 1(\mod 8)$.

Problem 4 Prove that all integer numbers of the form $7^n - 2^n$ are divisible by 5.

Proof by Induction

Base Case: Suppose $n = 1$. Then $7^n - 2^n = 7 - 2 = 5$, which is clearly divisible by 5.

Inductive Case: Assume $7^n - 2^n$ is divisible by 5. Then letting $k = n+1$, we will prove that $7^k - 2^k$ is also divisible by 5.

Since $k = n + 1$ then $7^k - 2^k = 7 \cdot 7^n - 2 \cdot 2^n$. Since $2 = 7 - 5$ we have $7 \cdot 7^n - (7-5) \cdot 2^n$ which expands to $7 \cdot 7^n - 7 \cdot 2^n - 5 \cdot 2^n = 7(7^n - 2^n) - 5 \cdot 2^n$. Since $5 | 7^n - 2^n$ then it divides $7(7^n - 2^n)$, and it clearly divides $5 \cdot 2^n$, so $5 | 7^k - 2^k$.

Therefore all integer numbers of the form $7^n - 2^n$ are divisible by 5.

Problem 5 Let $x, y \in \mathbb{R}^+$. Prove that if $x \leq y$ then $x^2 \leq y^2$.

Direct Proof If $x \leq y$ then clearly $x \cdot x \leq y \cdot x$. Likewise, $x \cdot y \leq y \cdot y$. But then we have $x^2 \leq x \cdot y \leq y^2$, and so $x^2 \leq y^2$.

Proof by Contrapositive Assume instead $y^2 < x^2$. Then we see that $0 < x^2 - y^2$ which is a difference of squares so $0 < (x + y)(x - y)$ and so dividing both sides by $(x + y)$, we see that $0 < x - y$ and so $y < x$.

Proof by Contradiction Assume that $x \leq y$ and $x^2 > y^2$. Then by the logic of the direct proof we may multiply both sides of the inequality $x \leq y$ by x to see that $x \cdot x \leq x \cdot y$ and by y to see that $y \cdot x \leq y \cdot y$, which yields $x^2 \leq y^2$, a contradiction.

Problem 6 There exists an $n \in Naturals$ such that $11 | 2^n - 1$

As an example, take $n = 10$, since $2^{10} = 1024$, and then $1024 - 1 = 1023$, which is $93 \cdot 11$.

Problem 7 Suppose $a, b \in \mathbb{Z}$. Prove that $a \equiv b(\mod 10)$ if and only if $a \equiv b(\mod 2)$ and $a \equiv b(\mod 5)$.

Since this is a bi-conditional, first we will show that if $a \equiv b(\mod 10)$ then $a \equiv b(\mod 2)$ and $a \equiv b(\mod 5)$.

If $a \equiv b(\mod 10)$, then $10 | a - b$, and so $a - b = 10k$ for some $k \in \mathbb{Z}$. Since $2 | 10$, then it is clear that $2 | 10k$ and so $2 | a - b$. Therefore $a \equiv b(\mod 2)$. Similarly, since $5 | 10$, then $5 | 10k$ and $5 | a - b$. Therefore $a \equiv b(\mod 5)$.

Conversely, say $a \equiv b(\mod 2)$ and $a \equiv b(\mod 5)$. Then $2|a-b$ and so $a-b=2\alpha,\ \alpha \in \mathbb{Z}$ and $5|a-b$, so $a-b=5\beta,\ \beta \in \mathbb{Z}$.

Finally, notice that $2\alpha = 5\beta$, and since $a-b|a-b$, then we have $2\alpha|5\beta$. Since $2 \nmid 5$, then $2|\beta$, and $\beta = 2\gamma,\ \gamma \in \mathbb{Z}$. Therefore, $a-b = 5 \cdot 2 \cdot \gamma$, which is $a-b = 10\gamma$, which is clearly divisible by 10. Therefore $a \equiv b(\mod 10)$.