Math 156, Sec 4, H.W. 5 *Franchi-Pereira, Philip*

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Problem 1 Prove that if x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x}+\sqrt{y}$.

Suppose for sake of contradiction that if x and y are positive real numbers, that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. Squaring both sides of the equation, $(\sqrt{x+y})^2 = (\sqrt{x} + \sqrt{y})^2$, we see that $(\sqrt{x+y})^2 = x + y$, but $(\sqrt{x} + \sqrt{y})^2 = (\sqrt{x} + \sqrt{y})(\sqrt{x} + \sqrt{y}) = x + 2\sqrt{xy} + y$, which is a contradiction.

Problem 2 Let x be a positive real number. Prove that if $x - \frac{2}{x} > 1$, then $x > 2$.

Suppose for sake of contradiction that if $x - \frac{2}{x} > 1$, then $x \leq 2$. Since x is a positive real number and $x \leq 2$, it is either 1 or 2.

a) In the case where $x = 1$, we have $1 - \frac{2}{1} = 1 - 2 = -1$, but $-1 > 1$ is a contradiction.

b) In the case where $x = 2$, we have $2 - \frac{2}{2} = 2 - 1 = 1$, but $1 > 1$ is also contradiction.

Therefore, x must be greater than 2.

Problem 3 Suppose $x \in \mathbb{Z}$. Then x is odd if and only if $3x + 6$ is odd.

First we will show that if x is odd, then $2x + 6$ is odd. By definition if x is odd, then there exists $k \in \mathbb{Z}$ such that $x = 2k + 1$. Then $3x + 6 = 3(2k + 1) + 6 = 6k + 9 = 2(3k + 4) + 1$. Since $3k + 4$ is an integer, then $2(3k + 4) + 1$ is odd.

Conversely, if $3x+6$ is odd, then x is odd. To prove the contrapositive, suppose that x is even. Then there exists some $k \in \mathbb{Z}$ such that $x = 2k$, and so $3x+6 = 6k+6 = 2(3k+3)$. Since $3k+3$ is an integer, $2(3k+3)$ is even.

Problem 4 Suppose $x, y \in \mathbb{Z}$. Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.

First we will show that if $x^3 + x^2y = y^2 + xy$ then $y = x^2$ or $y = -x$. We may factor this equation to be $x^2(x + y) = y(x + y)$. Then there are two cases:

- **a)** If $y + x = 0$, then we have $y = -x$, and $x^2(0) = y(0) = 0$.
- **b)** If $y + x \neq 0$ we may cancel $(x + y)$ from both sides, and we have $y = x^2$.

Conversely, if $y = x^2$ or $y = -x$, then $x^3 + x^2y = y^2 + xy$. This can be shown directly.

- **a)** If $y = x^2$, then we have $x^3 + (x^2)(x^2) = (x^2)^2 + x(x^2)$ which becomes $x^3 + x^4 = x^4 + x^3$.
- **b)** If $y = -x$ we have $x^3 + x^2(-x) = (-x)^2 + x(-x)$ which becomes $x^3 - x^3 = x^2 - x^2 = 0$.