M311S24 Problem Set 3 *Franchi-Pereira, Philip*

Problem 1 Extend the Chinese Remainder Theorem to find the solutions in $\mathbb{Z}_{210}$ of the following equations.

$$2x \equiv_7 3 \qquad x \equiv_5 4 \qquad 3x \equiv_6 3$$

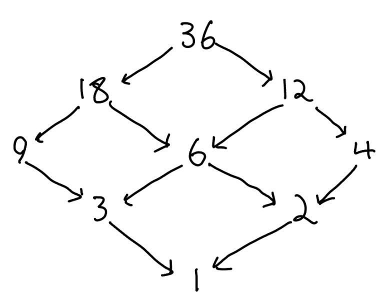
Solution: To start, we simplify the first equation to isolate x by multiplying both sides by $2^{-}1$ to get $x \equiv_7 5$. Then we compute the solutions to $x \equiv_7 5$ and $x \equiv_5 4$. Using the extended Euclidean algorithm, we get that $(7, 5) = 1$, with values for $\lambda = -2$ and $\omega = 3$ $-2 \cdot 7 + 3 \cdot 5 = 1$. So, the solutions to the first twi congruences some multiple of $5 \cdot 15 + 4 \cdot -14 = 19$ in $\mathbb{Z}_{35}$.

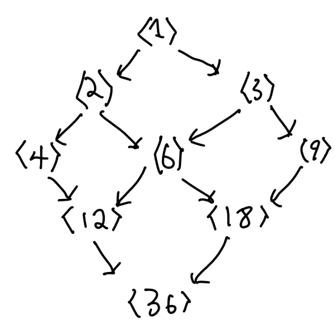
Next, notice that the equation $3x \equiv_6 3$ has three solutions, which by hand we compute to be $1, 3,$ and 5. So we will actually be solving three systems of equations, using the same method: applying the Extended Euclidean Algorithm on (35,6) to find values for λ and ω, which yield $\lambda = -1$ and $\omega = 6$, then solving $x = r_{210}((19 \cdot 6 \cdot 6) + (b \cdot 35 \cdot -1)) = r_{210}(684 - 35b)$

$$\begin{array}{lll} x \equiv_{35} 19 & x \equiv_6 1 & r_{210}(684 - 35) = 19 \\ x \equiv_{35} 19 & x \equiv_6 3 & r_{210}(684 - 35 \cdot 3) = 159 \\ x \equiv_{35} 19 & x \equiv_6 5 & r_{210}(684 - 35 \cdot 5) = 89 \end{array}$$

So, we have three solutions in $\mathbb{Z}_{210}$, $19, 89, 159$.

Problem 2 Give the divisors and subgroup diagrams for 36.





Problem 2c Give the table of subgroups and generators for 36.

group	generators
$\mathbb{Z}_{36}$	{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35}
$2\mathbb{Z}_{36}$	{2, 10, 14, 22, 26, 34}
$3\mathbb{Z}_{36}$	{3,15, 21, 33 }
$4\mathbb{Z}_{36}$	{8, 16, 20, 28, 32}
$6\mathbb{Z}_{36}$	{6, 30}
$9\mathbb{Z}_{36}$	{9, 27}
$12\mathbb{Z}_{36}$	{12, 24}
$18\mathbb{Z}_{36}$	{18}
$\{0\} = 36\mathbb{Z}_{36}$	{0}

Problem 3 Let $k \in \mathbb{Z}_n$. Show that the additive order of k is $\frac{n}{(k,n)}$.

The additive order of k is defined as an $m \in \mathbb{Z}_n$ such that $m \cdot k \equiv_n 0$. Therefore $n|mk$, and so $mk = \alpha n$. Letting $d = (k, n)$ we have $k = h_k d$ and $n = h_n d$. So, $mh_k d = \alpha h_n d$ and cancelling $mh_k = \alpha h_n$. Since $h_k|mh_k$ and $mh_k|\alpha h_n$ we have $h_k|\alpha h_n$. But, $(h_n, h_k) = 1$ so we have $h_k|\alpha$ and $\alpha = \beta h_k$ for some $\beta \in \mathbb{Z}$. Therefore $mh_k = \beta h_k h_n$ and cancelling we have $m = \beta h_n$. So m is a multiple of h_n, but since it is the smallest positive multiple,W we take $\beta = 1$ and have $m = h_n$, which is $m = \frac{n}{(k,n)}$

Problem 4a Show that $(n-1)! \equiv_n -1$ if n is prime and $(n-1!) \equiv_n 0$ if n is composite and $n > 4$.

Since $\mathbb{Z}_n$ is a commutative ring, every element a in $\mathbb{Z}_n$ except for $1, n-1$, and 0 have a distinct inverse a^{-1} such that $a \cdot a^{-1} = 1$. Since $(n-1)!$ is the product of every integer from 1 to $n-1$, we have the product $(n-1)! \equiv_n 1 \cdot (a_1 a_1^{-1}) \cdot (a_2 a_2^{-1}) \cdots (n-1) \equiv_n (1) \cdot (1)^{(\frac{n-2}{2})} \cdot (n-1) \equiv_n n-1$.

If instead n is composite, then there exist some integers a, b such that $n = ab$. We have two cases: either $a = b$ or $a \neq b$. If $a \neq b$ then ab is a product in $(n-1)!$, so $n|(n-1)!$ and $(n-1)! \equiv_n 0$.

If instead we have $a = b$, then $n = a^2$. If a is composite, such that $a = pq$, then we have that $n = ppqq$, but since p is a term in $(n-1)!$ and pqq is a term in $(n-1)!$ then $ppqq|(n-1)!$.

Finally, if a is a prime and $n = aa$, then a is clearly a term in $(n-1)!$, but so is $2a$, when $n > 4$. Then $2aa|(n-1)!$, so $(n-1!) \equiv_n 0$.

Problem 4b Find $k \in \mathbb{Z}_{101}$ such that $96! \equiv_{101} k$.

For a prime p, we have shown that $(p-1)! \equiv_p p-1$. Observe that for the prime $p = 101$ the solution to $96! \equiv_{101} k$ can be found by applying the inverses of $100, 99, 98, 97$ to the equivalence $100! \equiv_{101} 1$. Since 101 is prime, then the greatest common divisor between it and each of those values is 1. To find the inverses, we note that since $\lambda n + \omega(101) = 1$, then in $\mathbb{Z}_{101}$, $\omega(101) = 0$ and $\lambda n = 1$, so λ is our inverse. We know the inverse to 100 from Wilson's Theorem, and 3 applications of the Extended Euclidean Algorithm show that:

$$100^{-1} = 100 \qquad 99^{-1} = 50 \qquad 98^{-1} = -34 \qquad 97^{-1} = 25$$

And so

$$\begin{aligned}(100 \cdot 50 \cdot -34 \cdot 25) \cdot (100!) \equiv_{101} 96! &\equiv_{101} 100 \cdot (100 \cdot 50 \cdot -34 \cdot 25)\\ &\equiv_{101} 50 \cdot -34 \cdot 25 \equiv_{101} (100) \cdot -17 \cdot 25\\ &\equiv_{101} (100) \cdot -17 \cdot 25 \equiv_{101} (100) \cdot 84 \cdot 25\\ &\equiv_{101} (100) \cdot 21 \cdot 25 \cdot 4 \equiv_{101} (100) \cdot (100) \cdot 21\\ &\equiv_{101} 21\end{aligned}$$

So $96! \equiv_{101} 21$

Problem 5 Compute $13^{(243^{(65^{35})})} \pmod{200}$

The factors of 200 are $2^3 \cdot 5^2$. Since 13 is prime, then $(200, 13) = 1$. By Euler's Theorem, $13^{\varphi(200)} \equiv_{200} 1$ and we can compute $\varphi(200) = 200 \cdot (1 - \frac{1}{2})(1 - \frac{1}{5}) = 200 \cdot \frac{4}{10} = 80$. So $13^{80} \equiv_{200} 1$. Using the division algorithm, we can re-express $13^{(243^{(65^{35})})}$ as $13^{k(80)+r} \equiv_{200} (13^{80})^k(13^r) \equiv_{200} (1^k)(13^r) \equiv_{200} 13^r$ for some $k, r \in \mathbb{Z}$.

So to find r, we solve $243^{(65^{35})} \equiv_{80} r$, and since the prime factors of $80 = 2^4 \cdot 5$ and $243 = 3^5$, then applying Euler's theorem again we see that $\varphi(80) = 32$ and $243^{32} \equiv_{80} 1$. So applying the division algorithm again we will find an r_2 such that $243^{65^{35}} \equiv_{80} 243^{k(32)+r_2}$.

We must next solve $65^{35} \equiv_{32} r_2$, and again the factors of $65 = 13 \cdot 5$ and $32 = 2^5$, so $\varphi(32) = 16$ and $65^{16} \equiv_{32} 1$. Since $35 = 2 \cdot 16 + 3$, we have $65^{35} \equiv_{32} 65^3$, which a calculator shows equals 274625 in $\mathbb{Z}$.

Notice next that $65^{355} \equiv_{32} 274625 \equiv_{32} 1$, so working backwards we have $r_2 = 1$, so $243^{65^{35}} \equiv_{80} 243^1$. Then $r \equiv_{80} 243 \equiv_{80} 3$, so $13^{(243^{(65^{35})})} \equiv_{200} 35^{243} \equiv_{200} 35^3$, and since $13^3 = 2197$, we have $2197 \equiv_{200} 197$.

Problem 6a Show that for all primes p, $n^{k(p-1)+1} - n \equiv_p 0$.

Since p is prime, either p is a factor of n or $(n, p) = 1$. If it is a factor, then $p|n$ and so clearly $p|n^{k(p-1)+1}$, so $n^{k(p-1)+1} - n \equiv_p 0$

If instead $(n, p) = 1$ then we can rewrite $n^{k(p-1)+1} - n$ as $n \cdot n^{k(p-1)} - n$, and since Euler's theorem states that $\phi(p) = p - 1$, we have $n^{k(\phi(p))} \equiv_p n^{(\phi(p))^k} \equiv_p 1$ so the equation becomes $n \cdot (1^k) - n$ which is $n - n = 0$ and $n^{k(p-1)+1} - n \equiv_p 0$.

Problem 6b Show that for all n, $1919190|n^{37} - n$.

By the previous problem, we have $n^{k(p-1)+1} - n \equiv_p 0$ for any prime p. We can rewrite $n^{37} - n = n \cdot n^{36+1} - n$, and find factors of 36 such that $k(p-1) = 36$. The values for a prime p such that $(p-1)|36$ are $P = \{2, 3, 5, 7, 13, 19, 37\}$. So, for all $p \in P$ and all $n \in \mathbb{Z}$, the congruence $n^{37} - n \equiv_p 0$ has a solution, and we have $p|n^{37} - n$. Since every $p \in P$ is prime, then the product of all $p \in P$, $p_1 \cdot p_2 \cdots p_n | n^{37} - 1$, which is 1919190. Therefore, $1919190|n^{37} - n$.

Problem 7 Show that for all $n \geq 2$, $2^n \not\equiv_n 0$.

Suppose instead that $2^n \equiv_n 1$. Let p be the smallest prime factor of n. It is clear that n must be odd, and so p must be odd as well. Then by Euler's theorem we have $2^{(p-1)} \equiv_p 1$ as well as $2^n \equiv_p = 1$. Then let $d = (n, p-1)$. If $d \neq 1$ then we have a contradiction, so $d = 1$. But then by Bézout's identity we have $\lambda(p-1) + \omega n = 1$. Observe then that

$$\begin{aligned} 2^n \equiv_p 2^{(p-1)} &\equiv_p (2^n)^\omega \equiv_p \left(2^{(p-1)}\right)^\lambda \\ 1 \cdot (2^n)^\omega &\equiv_p (2^n)^\omega \cdot \left(2^{(p-1)}\right)^\lambda \\ 2^{\lambda(p-1)+\omega n} &\equiv_p 2^1 \equiv_p 1 \end{aligned}$$

which is a contradiction, since there is no odd prime where $2 \equiv_p 1$