Exploitation of Symmetry in Polynomial System Solving

Philip Hossu[1] | Advisers: Dr. Hualong Feng[1], Jeff Sommars[2], Dr. Jan Verschelde[2]

[1]Illinois Institute of Technology, [2]University of Illinois at Chicago

College of Science
ILLINOIS INSTITUTE OF TECHNOLOGY

Introduction/Background

I. Basics of Homotopy Continuation

Example:

$f(x_1, x_2)$ represents the system which we wish to solve, called the target system. It is evident that the maximum number of solutions to this system would be 9 from the max degrees.

$$f(x_1,x_2) = \begin{cases} x_1^2x_2 + x_2^2 + x_1 + 1 \\ x_1x_2^2 + x_1^2 + x_2 + 1 \end{cases}$$

The target system is highly symmetric, shown below. *Our code automatically generates the symmetry present in an input system so the user doesn't have to determine this on their own.*

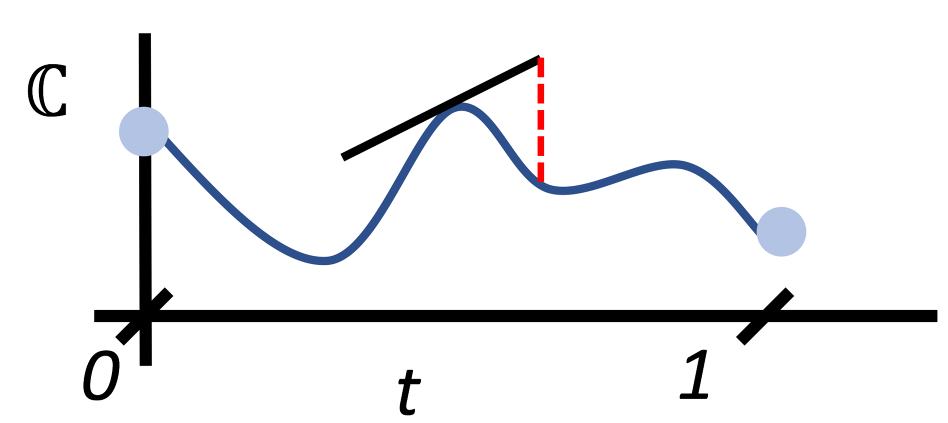
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{bmatrix} = f\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$g(x_1, x_2)$ represents the start system, created by assigning random complex coefficients to each term in the target system.

$$g(x_1,x_2) = \begin{cases} (c_1x_1 + c_2)(c_3x_1 + c_4x_2 + c_5)(c_6x_2 + c_7) \\ (c_1x_2 + c_2)(c_3x_2 + c_4x_1 + c_5)(c_6x_1 + c_7) \end{cases}$$

Using the homotopy $h(x_1, x_2)$ and a series of predictor and corrector methods, solutions to the start system can be tracked to solutions of the target system.

$$h(x_1,x_2) = g(x_1,x_2)(1-t) + f(x_1,x_2)(t), t \in [0,1]$$



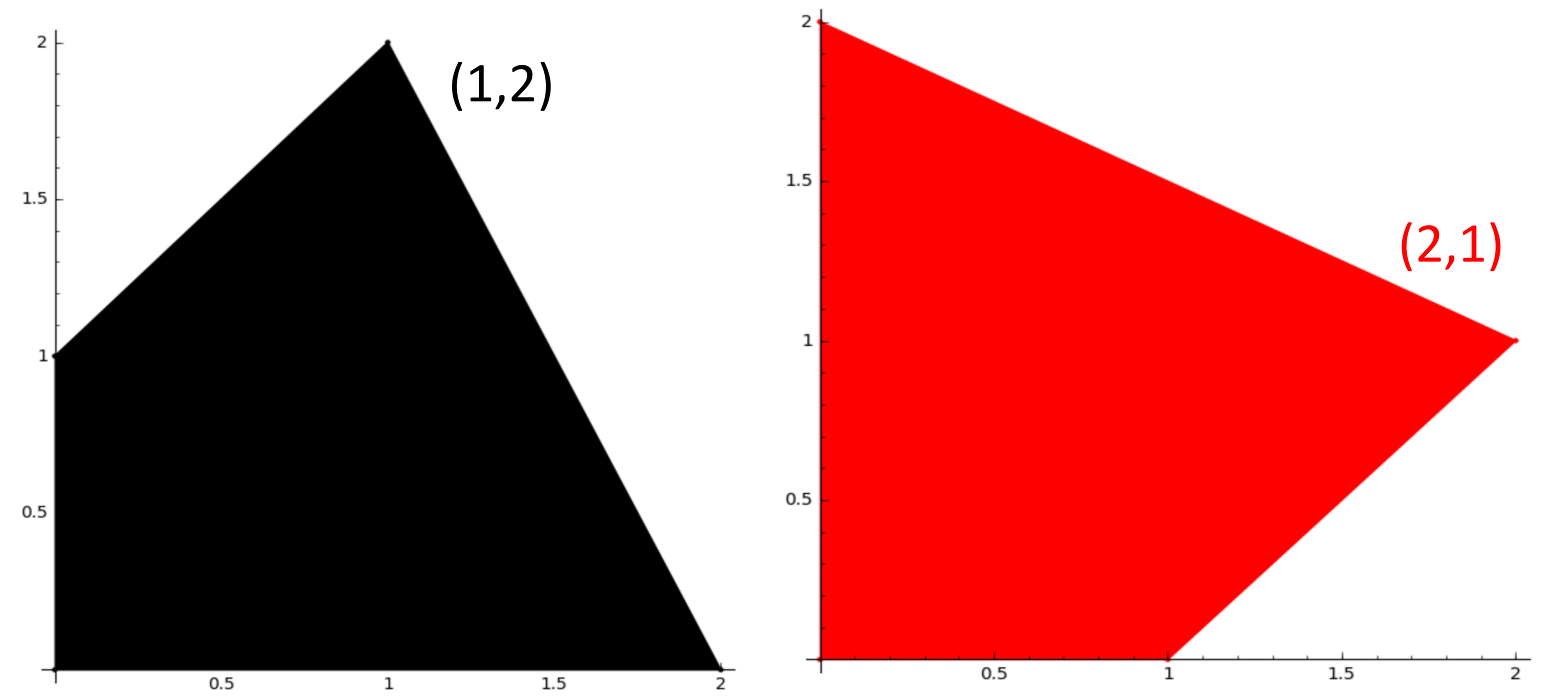
However, if symmetry is exploited, the number of path tracks is far fewer than 9. *Our code automatically sends the generated symmetry to PHCpack which can exploit its presence.*

For a symmetric polynomial system, we aim to construct a symmetric homotopy so only the generating solution paths need to be tracked.

II. Polyhedral Homotopy Continuation, Symmetric Lifting

The support sets for $f(x_1, x_2)$ denote the presence and degree of each variable in each equation and can be plotted.

$$\textbf{\textit{Black}}\ \textit{Polytope}\ (f_1)\ \textit{Support} \rightarrow \{(2,1),(0,2),(1,0),(0,0)\}$$
$$\textbf{\textit{Red}}\ \textit{Polytope}\ (f_2)\ \textit{Support} \rightarrow \{(1,2),(2,0),(0,1),(0,0)\}$$



A symmetric lifting function must be applied to every vertex of these polytopes before the mixed volume can be calculated.

$$\textbf{\textit{Black}}\ \textit{Polytope}\ (f_1) \rightarrow \{(2,1,\mathbf{0}),(0,2,\mathbf{1}),(1,0,\mathbf{7}),(0,0,\mathbf{0})\}$$
$$\textbf{\textit{Red}}\ \textit{Polytope}\ (f_2) \rightarrow \{(1,2,\mathbf{0}),(2,0,\mathbf{1}),(0,1,\mathbf{7}),(0,0,\mathbf{0})\}$$

Polyhedral homotopy induced by the lifting:

$$x_1^2x_2t^0 + x_2^2t^1 + x_1t^7 + t^0 = 0$$
$$x_1x_2^2t^0 + x_1^2t^1 + x_2t^7 + t^0 = 0$$

Calculation of the start system for (-1, 2, 2):

$$(\mathbf{-1,2,2}) \cdot (\boldsymbol{x_1^2x_2t^0}) = \mathbf{0} \qquad (-1,2,2) \cdot (x_1x_2^2t^0) = 3$$
$$(-1,2,2) \cdot (x_2^2t^1) = 6 \qquad (\mathbf{-1,2,2}) \cdot (\boldsymbol{x_1^2t^1}) = \mathbf{0}$$
$$(-1,2,2) \cdot (x_1t^7) = 13 \qquad (-1,2,2) \cdot (x_2t^7) = 16$$
$$(\mathbf{-1,2,2}) \cdot (\boldsymbol{t^0}) = \mathbf{0} \qquad (\mathbf{-1,2,2}) \cdot (\boldsymbol{t^0}) = \mathbf{0}$$

Vector: (-1, 2, 2)

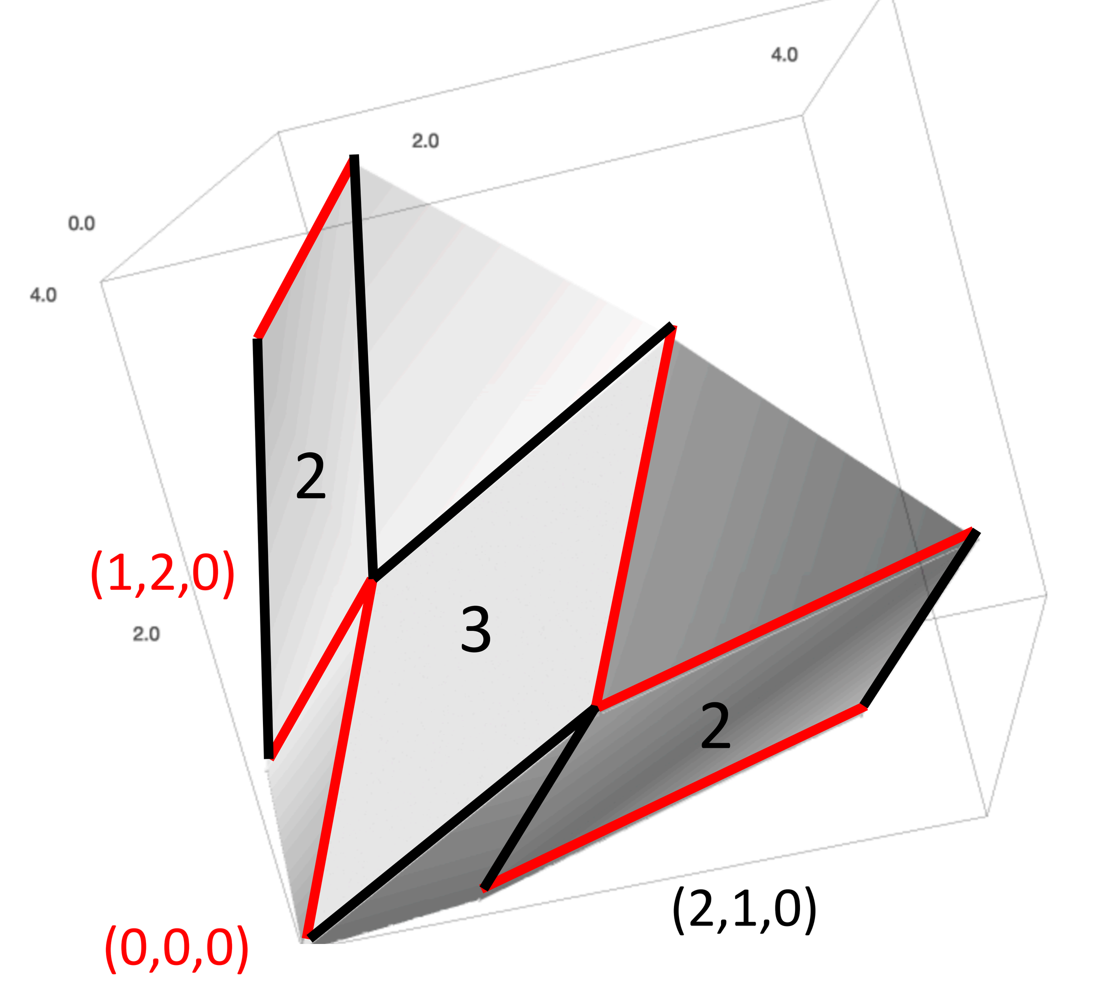
$$\text{System} = \begin{cases} x_1^2x_2 + 1 \\ x_1^2 + 1 \end{cases}$$

Vector: (0, 0, 1)

$$\text{System} = \begin{cases} x_1^2x_2 + 1 \\ x_1x_2^2 + 1 \end{cases}$$

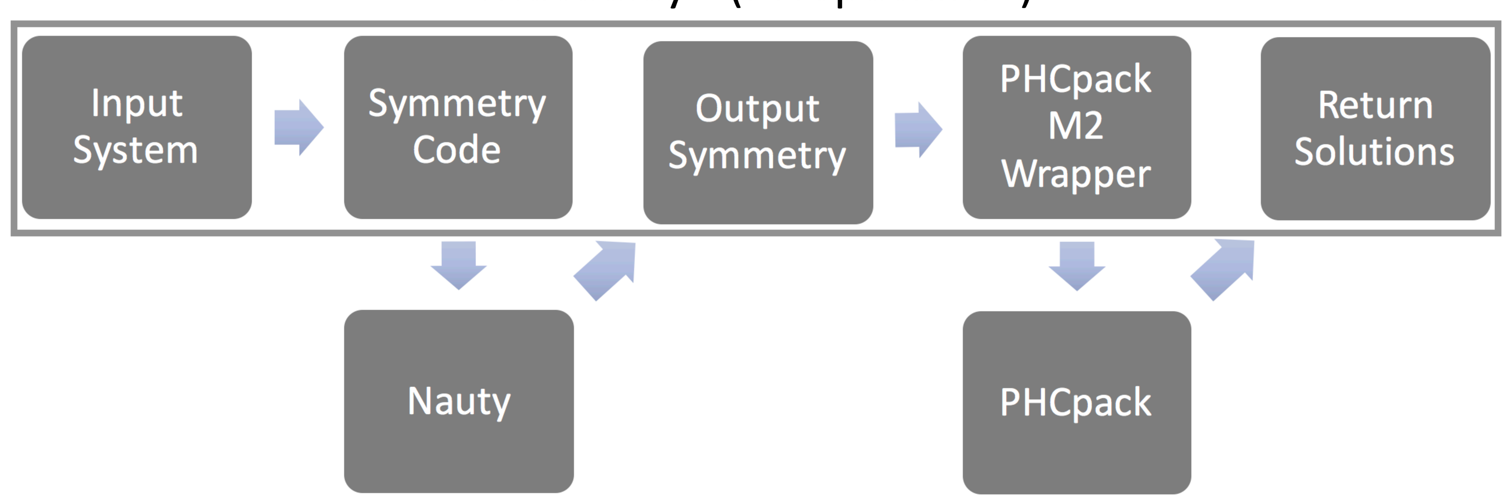
Vector: (2, -1, 2)

$$\text{System} = \begin{cases} x_2^2 + 1 \\ x_1x_2^2 + 1 \end{cases}$$



Implementation Overview

Macaulay2 (PHCpack.m2)



Experimental Results

The results were produced from Elementary Symmetric Polynomial family tests. An n=3 example can be seen below:

$$x_1 + x_2 + x_3 = 0$$
$$x_1x_2 + x_1x_3 + x_2x_3 = 0$$
$$x_1x_2x_3 - 1 = 0$$

n	Symmetric / Static Lifting
3	0.7623
4	0.6791
5	0.6627
6	0.8242

Tests were run on a Macbook Pro fitted with a 2.5GHz Intel Core i7 and 16gb of 2133 MHz RAM.

$$x_1 + x_2 + x_3 + x_4 + x_5$$
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1$$
$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2$$
$$x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3$$
$$x_1x_2x_3x_4x_5 - 1$$

The number of paths in the above n=5 cyclic system is reduced from 70 to 10 by the exploitation of symmetry.

References/Acknowledgements

This project was funded by the Illinois Institute of Technology College of Science Undergraduate Research Stipend.

1. B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Mathematics of Computation*, 64(212):1541-1555, 1995.
2. Grayson, D. R., Stillman, M. E., 2006. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.
3. McKay, B.D. and Piperno, A., *Practical Graph Isomorphism, II*, Journal of Symbolic Computation, 60 (2014), pp. 94-112
4. J. Verschelde and K. Gatermann. Symmetrical newton polytopes for solving sparse polynomial systems. Advances in Applied Mathematics, 16(1):95-127, 1995.