Formulas

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Part I

Mathematics

Chapter 1

Vector Calculus

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \tag{1.1}$$

$$\vec{A} \cdot \vec{B} = ABcos\theta \tag{1.2}$$

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{1.3}$$

1.1 Triple Products

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \tag{1.4}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.5}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot C \tag{1.6}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{1.7}$$

1.2 Derivatives

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{1.8}$$

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \tag{1.9}$$

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{1.10}$$

$$\nabla \times \vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \tag{1.11}$$

$$\nabla'^2\left(\frac{1}{|\vec{r} - \vec{r'}|}\right) = -4\pi\delta(\vec{r} - \vec{r'}) \tag{1.12}$$

1.3 Product Rules

1.4 Cylindrical Coordinates

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{1.13}$$

1.5 Spherical Coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{1.14}$$

1.6 Green Functions

1.6.1 Green's Identities

Green's first identity:

$$\int_v \left(\phi\nabla'^2\psi + \nabla'\phi \cdot \nabla'\psi\right) d^3x' = \oint_s \phi\frac{\partial \psi}{\partial n} da' \tag{1.15}$$

Green's second identity or Green's Theorem:

$$\int_v \left(\phi\nabla'^2\psi - \psi\nabla'^2\phi\right) d^3x = \oint_s \left[\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right] da' \tag{1.16}$$

1.6.2 Green Functions

$$\nabla'^2 G(\vec{x}, \vec{x'}) = -4\pi\delta(\vec{x} - \vec{x'}) \tag{1.17}$$

$$G(\vec{x}, \vec{x'}) = \frac{1}{|\vec{x} - \vec{x'}|} + F(\vec{x}, \vec{x'}) \tag{1.18}$$

$$\nabla'^2 F(\vec{x}, \vec{x'}) = 0 \tag{1.19}$$

1.6.3 Boundary Conditions

Dirichlet:

$$\begin{gathered} \Phi(\vec{x}) \text{ specified on S} \\ G_D(\vec{x}, \vec{x}') = 0 \text{ for } \vec{x}' \text{ on S} \end{gathered} \tag{1.20}$$

Neumann:

$$\begin{gathered} \frac{\partial \Phi}{\partial n} \text{ specified on S} \\ \frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}') = -\frac{4\pi}{S} \text{ for } \vec{x}' \text{ on S} \end{gathered} \tag{1.21}$$

Chapter 2

Linear Algebra

2.1 Determinant Properties

Theorem 2.1.1. *The determinant of a matrix A is non-zero if and only if A is invertible*

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)} \tag{2.1}$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i \tag{2.2}$$

Determinant is product of eigenvalues

$$\det(AB) = \det(A)\det(B) \tag{2.3}$$

2.2 Orthogonal Matrices

Definition 2.2.1. An nxn real matrix A is orthogonal if the columns of A are orthonormal.

Theorem 2.2.1. *Every orthogonal matrix is invertible.*

Theorem 2.2.2. *The inverse of an orthogonal matrix orthogonal.*

$$A^T A = I \tag{2.4}$$

$$A^T = A^{-1} \tag{2.5}$$

$$\det(A) = \pm 1 \tag{2.6}$$

2.3 Transpose Properties

$$(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T \tag{2.7}$$

$$(\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T\boldsymbol{A}^T \tag{2.8}$$

$$\det\left(\boldsymbol{A}^T\right) = \det(\boldsymbol{A}) \tag{2.9}$$

2.4 Linear Vector Spaces

Definition 2.4.1. Linear Vector Space. A linear vector space V is a collection of objects $|1\rangle, |2\rangle, ..., |V\rangle, ... |W\rangle, ...,$ called vectors, for which there exists

- A definite rule for forming the vector sum, denoted $|V\rangle + |W\rangle$
- A definite rule for multiplication by scalars $a, b, ...,$ denoted $a|V\rangle$ with the following features:
    1. Closure under addition: $|V\rangle + |W\rangle \in V$
    2. Scalar multiplication is distributive in the vectors: $a(|V\rangle + |W\rangle) = a|V\rangle + a|W\rangle$
    3. Scalar multiplication is distributive in the scalars: $(a+b)|V\rangle = a|V\rangle + b|V\rangle$
    4. Scalar multiplication is associative $a(b|V\rangle) = ab|V\rangle$
    5. Addition is commutative: $|V\rangle + |W\rangle = |W\rangle + |V\rangle$
    6. Addition is associative: $|V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$
    7. There exists a null vector $|0\rangle$ obeying $|V\rangle + |0\rangle = |V\rangle$
    8. For every vector there exists an inverse under addition, $|-V\rangle$, such that $|V\rangle + |-V\rangle = |0\rangle$

Definition 2.4.2. Field. The numbers $a, b, ...$ are called the field over which the vector space is defined.

Theorem 2.4.1. *The above axioms of vector spaces imply:*

1. $|0\rangle$ *is unique*

2. $0\,|V\rangle = |0\rangle$

3. $|-V\rangle = -\,|V\rangle$

4. $|-V\rangle$ *is the unique additive inverse of* $|V\rangle$

2.4.1 Linear Independence

Definition 2.4.3. Linearly Independent. A set of vectors $\{|i\rangle\}$ is said to be linearly independent if $\sum_{i=1}^{n} a_i\,|i\rangle = |0\rangle$ only when all $a_i = 0$.

Definition 2.4.4. Dimension. A vector space has a dimension n if it can accommodate a maximum of n linearly independent vectors.

Theorem 2.4.2. *Any vector* $|V\rangle$ *in an* n*-dimensional space can be written as a linear combination of* n *linearly independent vectors* $|1\rangle \ldots |n\rangle$*.*

Theorem 2.4.3. *The vector expansion* $|V\rangle = \sum_{i}^{n} v_i\,|i\rangle$ *is unique.*

2.4.2 Inner Product Space

Definition 2.4.5. Inner Product Space. A vector space with an inner product is called an inner product space where the inner product has the following properties:

- $\langle V|W\rangle = \langle W|V\rangle^*$ (skew-symmetric)
- $\langle V|V\rangle \geq 0$ and $\langle V|V\rangle = 0$ iff $|V\rangle = |0\rangle$ (positive semidefiniteness)
- $\langle V|(a\,|W\rangle + b\,|z\rangle)\rangle = \langle V|aW + bZ\rangle = a\,\langle V|W\rangle + b\,\langle V|Z\rangle$ (linearity in ket)

2.4.3 Basis of a Vector Space

Definition 2.4.6. Basis. A set of n linearly independent vectors in an n-dimensional space is called a basis.

Definition 2.4.7. Components of a Vector. The coefficients of expansion v_i of a vector in terms of a linearly independent basis $\{|i\rangle\}$ are called the components of the vector in that basis.

$$\langle i|j\rangle = \delta_{ij} \text{ for an orthonormal basis} \tag{2.10}$$

$$\sum_{i=1}^{n} |i\rangle \langle i| = I \text{ for a complete basis} \tag{2.11}$$

2.5 Dual Spaces

2.5.1 Projection Operator

$|i\rangle \langle i|$ is the projection operator that projects onto the vector $|i\rangle$

$$|V\rangle = \sum_i |i\rangle \langle i|V\rangle \tag{2.12}$$

The adjoint of the vector $|V\rangle = \sum_i v_i |i\rangle$ is:

$$\langle V| = \sum_i \langle i| v_i^* \tag{2.13}$$

The adjoint of the vector $|V\rangle = \sum_i |i\rangle \langle i|V\rangle$ is:

$$\langle V| = \sum_i \langle V|i\rangle \langle i| \tag{2.14}$$

2.5.2 Schwarz and Triangle Inequality

$$|\langle V|W\rangle| \leq |V||W| \text{ Schwarz Inequality} \tag{2.15}$$

$$|V + W| \leq |V| + |W| \text{ Triangle Inequality} \tag{2.16}$$

2.6 Matrix Inverse

2.6.1 2x2 Matrix Inverse

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{2.17}$$

2.6.2 3x3 Matrix Inverse

$$A^{-1} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \begin{vmatrix} e & h \\ f & i \end{vmatrix} & \begin{vmatrix} g & d \\ i & f \end{vmatrix} & \begin{vmatrix} d & g \\ e & h \end{vmatrix} \\ \begin{vmatrix} h & b \\ i & c \end{vmatrix} & \begin{vmatrix} a & g \\ c & i \end{vmatrix} & \begin{vmatrix} g & a \\ h & b \end{vmatrix} \\ \begin{vmatrix} b & e \\ c & f \end{vmatrix} & \begin{vmatrix} d & a \\ f & c \end{vmatrix} & \begin{vmatrix} a & d \\ b & e \end{vmatrix} \end{pmatrix} \tag{2.18}$$

2.7 Operators

2.7.1 Commutators

$$[\Omega, \Lambda\theta] = \Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta \tag{2.19}$$

$$[\Lambda\Omega, \theta] = \Lambda[\Omega, \theta] + [\Lambda, \theta]\Omega \tag{2.20}$$

2.7.2 Inverses

The inverse of a product of operators is the product of the inverses in reverse:

$$(\Omega\Lambda)^{-1} = \Lambda^{-1}\Omega^{-1} \tag{2.21}$$

2.7.3 Matrix Elements of an Operator

$$\langle i|j'\rangle = \langle i|\,\Omega\,|j\rangle = \Omega_{ij} \tag{2.22}$$

The matrix element Ω_{ij} is the i^{th} component of the j^{th} basis vector after it has been transformed by Ω.

2.7.4 Active and Passive Transformations

Under a change of basis by the unitary operator U

$$|V\rangle \to U\,|V\rangle \;\text{ (active)} \tag{2.23}$$

$$\Omega \to U^{\dagger}\Omega U \;\text{ (passive)} \tag{2.24}$$

2.7.5 Trace of an Operator

$$Tr(A) = \sum_i A_{ij} \tag{2.25}$$

Properties: permutations are cyclic, the trace of an operator is independent of basis

$$Tr(AB) = Tr(BA) \tag{2.26}$$
$$Tr(ABC) = Tr(BCA) = Tr(CAB) \tag{2.27}$$
$$Tr(A) = Tr(U^\dagger AU) \tag{2.28}$$

2.7.6 Hermitian Operators

Definition 2.7.1. Hermitian Operator: An operator Ω is Hermitian if $\Omega^\dagger = \Omega$

Definition 2.7.2. Anti-Hermitian Operator: An operator Ω is anti-Hermitian if $\Omega^\dagger = -\Omega$

Hermitian operators are like real numbers, anti-Hermitian operators are like complex numbers. Any operator can be decomposed into Hermitian and anti-Hermitian parts:

$$\Omega = \frac{\Omega + \Omega^\dagger}{2} + \frac{\Omega - \Omega^\dagger}{2} \tag{2.29}$$

Theorem 2.7.1. *The eigenvalues of a Hermitian operator are real valued.*

Theorem 2.7.2. *To every Hermitian operator Ω, there exists (at least) a basis consisting of its orthonormal eigenvectors. The operator is diagonal in this eigenbasis and has its eigenvalues as its diagonal entries.*

Theorem 2.7.3. *The eigenvectors of a hermitian operator belonging to distinct eigenvalues are orthogonal.*

Theorem 2.7.4. *The eigen vectors of a hermitian operator span the space.*

2.7.7 Unitary Operators

Definition 2.7.3. Unitary Operator: An operator U is unitary if
$UU^\dagger = I$

Theorem 2.7.5. *The columns of a unitary matrix are orthonormal and the rows of a unitary matrix are orthonormal as well.*

Theorem 2.7.6. *The eigenvalues of a unitary operator are complex numbers of unit modulus.*

Theorem 2.7.7. *The eigenvectors of a unitary operator are mutually orthogonal.*

2.7.8 Diagonalization of Hermitian Matrices

If Ω is a Hermitian matrix, there exists a unitary matrix U (built out of the eigenvectors of Ω) such that

$$U^\dagger \Omega U \tag{2.30}$$

is diagonal.

Theorem 2.7.8. *if Ω and Λ are two commuting Hermitian operators, there exists (at least) a basis of common eigenvectors that diagonalizes them both. The common eigenbasis is unique if only one of Ω or Λ are degenerate in some subspace. If both Ω and Λ are degenerate in some subspace, the common eigenbasis is not unique.*

Theorem 2.7.9. *One can always find, for a finite n, a set of operators $\{\Omega, \Lambda, \Gamma, ...\}$ that commute with each other and define a unique eigenbasis that is shared by all operators in the set.*

Definition 2.7.4. Complete set of commuting operators: A set of commuting operators $\{\Omega, \Lambda, \Gamma, ...\}$ that share a unique eigenbasis.

2.8 Eigenvectors and Eigenvalues

Each operator has eigen vectors associated with it:

$$\Omega |V\rangle = \omega |V\rangle \tag{2.31}$$

$$(\Omega - \omega I) |V\rangle = |0\rangle \tag{2.32}$$

$$\sum_j (\Omega_{ij} - \omega \delta_{ij}) v_j = 0 \tag{2.33}$$

which gives the eigenvectors if the eigenvalue is known.
$(\Omega - \omega I)^{-1}$ does not exist so the eigenvalue problem cannot be directly solved:

$$|V\rangle = (\Omega - \omega I)^{-1} |0\rangle = |0\rangle \tag{2.34}$$

2.8.1 Characteristic Equation and Polynomial

Characteristic equation

$$\det(\Omega - \omega I) = 0 \tag{2.35}$$

gives the eigenvalues.
The characteristic equation is a polynomial equation containing the characteristic polynomial:

$$P^n(\omega) = \sum_{m=0}^{n} c_m \omega^m = 0 \tag{2.36}$$

Roots of the characteristic polynomial are the eigenvalues.

2.8.2 Theorems

Theorem 2.8.1. *The eigenvalues of an operator Ω are covariant, they have the same value in any basis.*

Theorem 2.8.2. *A Hermitian or Unitary operator in $V^n(C)$ has n eigenvalues.*

2.9 Functions of Operators

Definition 2.9.1. c-number (or classical number): Refers real or complex numbers which are commuting.

Definition 2.9.2. q-number(or quantum number): Refers to operators which in general do not commute.

If only one q-number is present in an equation, everything commutes and it can be treated as a c-number.
If Ω is an operator:

$$e^{\Omega} = \sum_{n=1}^{\infty} \frac{\Omega^n}{n!} \tag{2.37}$$

If Ω is Hermitian, then in its eigenbasis:

$$e^{\Omega} = \begin{bmatrix} \sum_{m=0}^{\infty} \frac{\omega_1^m}{m!} & & \\ & \ddots & \\ & & \sum_{m=0}^{\infty} \frac{\omega_n^m}{m!} \end{bmatrix} \tag{2.38}$$

If $\theta(\lambda)$ is the operator:

$$\theta(\lambda) = e^{\lambda\Omega} \tag{2.39}$$

$$\frac{d\theta(\lambda)}{d\lambda} = \Omega e^{\lambda\Omega} = \theta(\lambda)\Omega \tag{2.40}$$

$$\theta(\lambda) = cexp\left(\int_0^{\lambda} \Omega d\lambda'\right) = cexp(\Omega\lambda) \tag{2.41}$$

2.10 Infinite Dimensional Vector Spaces

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)dx \text{ inner product} \tag{2.42}$$

$$\langle x|x'\rangle = \delta(x-x') \text{ normalization condition of basis vectors} \tag{2.43}$$

2.11 Derivative Operator

$$D\left|f\right\rangle = \left|df/dx\right\rangle \tag{2.44}$$

$$\langle x| D |x'\rangle = D_{xx'} = \delta'(x-x') = \delta(x-x')\frac{d}{dx'} \tag{2.45}$$

The derivative operator is not hermitian. $K = -iD$ is Hermitian in the space of functions obeying:

$$-ig^*(x)f(x)\Big|_a^b = 0 \tag{2.46}$$

Chapter 3

Lie Algebra

$$X_i = -i \frac{\partial M(\vec{\theta})}{\partial \theta_i}\bigg|_{\vec{\theta}=0} \text{ (Generators)} \tag{3.1}$$

The generators contain all information about the group.

$$M^{gen}(\theta_1, \theta_2, ...) = e^{i\theta^i X_i} \tag{3.2}$$

The generators give the group elements

$$[X_i, X_j] = iC^k_{i,j} X_k \text{ (Lie Algebra)} \tag{3.3}$$

Must be true for M^{gen} to be a valid representation of the group
$C^k_{i,j}$ (structure constants) are independent of the group representation
This is for passive transformations, for active transformations $i \to -i$

Notes:

1. The representations of the generators and the group elements belong to different spaces and behave differently
2. The Lie Algebra is not a group

3.1 SU(2)

$$h = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \{a, b, c, d\} \in \mathbb{C} \tag{3.4}$$

8 parameters

$$h^\dagger h = I \implies h^\dagger = h^{-1} \tag{3.5}$$

Four equations so now 4 parameters

$$h = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \tag{3.6}$$

Proof: $h^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = h^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$

(Four equations above are $d = a^*$, $c = -b^*$)

$$\det(h) = 1 \tag{3.7}$$

$$|a|^2 + |b|^2 = 1 \tag{3.8}$$

One constraint so now 3 parameters ($\boldsymbol{R^3}$)

Only 3 free parameters so SU(2) is a three dimensional group.

$$\boxed{\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}} \tag{3.9}$$

Parameterized by $\mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)$

3.1.1 Generators of SU(2)

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1,\ X_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2,\ X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \tag{3.10}$$

These are the Pauli Matrices

3.1.2 Lie Algebra of SU(2)

$$[X_i, X_y] = i\epsilon_{ijk} X_k \tag{3.11}$$

Same algebra as SO(3)

3.1.3 Topology

If $a = x + iy$, $b = z + ir$ then

$$\det(h) = x^2 + y^2 + z^2 + r^2 = 1 \tag{3.12}$$

and the topology is S^3, the unit 3-sphere in $\boldsymbol{R^4}$

3.1.4 Lie Algebra

$$[X_i, X_j] = i\epsilon_{ijk}X_k \tag{3.13}$$

3.1.5 Basis

Basis of SU(2) is given by

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.14}$$

3.2 SO(3)

$$h = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \tag{3.15}$$

9 parameters

$$\det(h) = 1 \tag{3.16}$$

3 parameters

$$h^T = h^{-1} \tag{3.17}$$

6 parameters

Representation $M(\theta_1, \theta_2, \theta_3) = M_1(\theta_1)M_2(\theta_2)M_3(\theta_3)$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_1) & -sin(\theta_1) \\ 0 & sin(\theta_1) & cos(\theta_1) \end{pmatrix} \; M_2 = \begin{pmatrix} cos(\theta_2) & 0 & -sin(\theta_2) \\ 0 & 1 & 0 \\ sin(\theta_2) & 0 & cos(\theta_2) \end{pmatrix} \; M_3 = \begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.18}$$

Generators:

$$X_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \; X_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; X_3 = -i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.19}$$

3.2.1 Lie Algebra

Structure constants C_{ij}^k are the Levi-Civita tensor:

$$[X_i, X_j] = i\epsilon^{\text{`}} ijkX_k \tag{3.20}$$

$$[X_1, X_2] = iX_3,\ [X_2, X_3] = iX_1,\ [X_3, X_1] = iX_2 \tag{3.21}$$

3.3 SO(1, 3)

$$h^\dagger gh = g \tag{3.22}$$

Generalized orthogonality condition, g is the metric

Chapter 4

Dirac Delta Function

The dirac delta function is even

$$\delta(x - x') = \delta(x' - x) \tag{4.1}$$

Derivative of the dirac delta function:

$$\delta'(x - x') = \frac{d}{dx}\delta(x - x') = -\frac{d}{dx'}\delta(x - x') \tag{4.2}$$

$$\int \delta'(x - x') f(x') dx' = \frac{d}{dx} f(x) \tag{4.3}$$

$\delta'(x - x')$ is an odd function

$$\frac{d^n \delta(x - x')}{dx^n} = \delta(x - x') \frac{d^n}{dx'^n} \tag{4.4}$$

Definition of the dirac delta function:

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x - x')} dk \tag{4.5}$$

Chapter 5

Fourier Analysis

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \tag{5.1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(k) dk \tag{5.2}$$

Part II

Physics

Chapter 6

Electrodynamics

6.1 Poisson Equation

$$\nabla^2 \Phi = -\rho/\epsilon_0 \tag{6.1}$$

Integral form ($R = |\vec{x} - \vec{x}'|$):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho(\vec{x}')}{R} d^3x' + \frac{1}{4\pi} \oint_s \left[\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] da' \tag{6.2}$$

6.1.1 Solutions to the Poisson Equation

Dirichlet boundary conditions:

$$\begin{aligned} &G_D(\vec{x}, \vec{x}') = 0 \text{ for } \vec{x}' \text{ on } S \\ &\Phi(\vec{x}) = \frac{1}{4\pi} \int_V \rho(\vec{x}') G_D(\vec{x}, \vec{x}') d^3x' - \frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G_D}{\partial n'} da' \end{aligned} \tag{6.3}$$

Neumann boundary conditions:

$$\begin{aligned} &\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x}') = -\frac{4\pi}{S} \text{ for } \vec{x}' \text{ on } S \\ &\Phi(\vec{x}) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0} \int_V \rho(\vec{x}') G_N(\vec{x}, \vec{x}') d^3x' + \frac{1}{4\pi} \oint_S \frac{\partial \Phi}{\partial n'} G_N da' \end{aligned} \tag{6.4}$$

6.1.2 Potential Energy

$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3x \tag{6.5}$$

$$W = \frac{\epsilon_0}{2} \int \left| \vec{E} \right|^2 d^3x \tag{6.6}$$

Notes:

1. 6.6 includes self energies

6.2 Special Relativity

6.2.1 Constant Acceleration

τ is proper time, x is coordinate distance, t is coordinate time

$$x(\tau) = \frac{c^2}{a}\left(cosh\frac{a\tau}{c} - 1\right) \tag{6.7}$$

$$t(\tau) = \frac{c}{a}sinh\frac{a\tau}{c} \tag{6.8}$$

$$c = 3 \times 10^8 \text{ m/s} \tag{6.9}$$

$$\beta = \pi \times 10^7 \text{ s/yr} \tag{6.10}$$

$$x_{lyr} = \frac{x_m}{c\beta} \tag{6.11}$$

$$\tau_{yr} = \frac{\tau_{sec}}{\beta} \tag{6.12}$$

For 1g of acceleration, $a = g$

$$x_{lyr}(\tau) = \frac{c^2}{g\beta c}cosh\left(\frac{g\beta\tau_{yr}}{c} - 1\right) \tag{6.13}$$

$$g\beta \approx c \tag{6.14}$$

$$x_{lyr}(\tau) \approx cosh\left(\tau_{yr} - 1\right) \tag{6.15}$$

$$t_{yr}(\tau_{yr}) \approx sinh\left(\tau_{yr}\right) \tag{6.16}$$

Chapter 7

Quantum Mechanics

7.1 Theorems

Theorem 7.1.1. *There is no degeneracy in on-dimensional bound states.*

Theorem 7.1.2. *The eigenfunctions of a real valued Hamiltonian can always be chosen pure real in the coordinate basis.*

Theorem 7.1.3. *The Hermiticity of the Hamiltonian is preserved under a unitary change of basis, the reality of the Hamiltonian is not.*

7.2 Operators

	Position Space	Momentum Space
Position Operator	x	$i\hbar\frac{\partial}{\partial p}$
Momentum Operator	$-i\hbar\frac{\partial}{\partial x}$	p

Part III

Misc

7.3 Eigenvalues and Eigenvectors of Rotation Matrix

The rotation matrix

$$\Omega = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$$

has eigenvectors and eigenvalues

$$|\omega_1\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix},\ \omega_1 = e^{-i\theta} \text{ and } |\omega_2\rangle = \tfrac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix},\ \omega_2 = e^{i\theta}$$

A vector can be expressed as a linear combination of the eigenvectors

$$\begin{aligned} |\psi\rangle &= c_1 |\omega_1\rangle + c_2 |\omega_2\rangle \\ |\psi\rangle &= \langle\omega_1|\psi\rangle |\omega_1\rangle + \langle\omega_2|\psi\rangle |\omega_1\rangle \\ |\psi\rangle &= (|\omega_1\rangle \langle\omega_1| + |\omega_2\rangle \langle\omega_2|) |\psi\rangle \\ |\omega_1\rangle \langle\omega_1| + |\omega_2\rangle \langle\omega_2| &= I \end{aligned}$$

The image of $|\psi\rangle$ is

$$\begin{aligned} \Omega |\psi\rangle &= (\Omega |\omega_1\rangle \langle\omega_1| + \Omega |\omega_2\rangle \langle\omega_2|) |\psi\rangle \\ &= (\omega_1 |\omega_1\rangle \langle\omega_1| + \omega_2 |\omega_2\rangle \langle\omega_2|) |\psi\rangle \end{aligned}$$

So

$$\boxed{\Omega = \omega_1 |\omega_1\rangle \langle\omega_1| + \omega_2 |\omega_2\rangle \langle\omega_2|}$$

For the rotation matrix

$$\Omega = \frac{e^{-i\theta}}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} \begin{pmatrix} -i & 1 \end{pmatrix} + \frac{e^{i\theta}}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix} \begin{pmatrix} i & 1 \end{pmatrix}$$

$$\boxed{\Omega = e^{-i\theta} \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + e^{i\theta} \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}}$$