The Clock Model and Its Relationship with the Allan and Related Variances

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[2] J. W. Chaffee, "Relating the Allan variance to the diffusion coefficients of a linear stochastic differential equation model for precision oscillators," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 34, pp. 655–658, Nov. 1987.

***Abstract*—The clock errors are modeled by stochastic differential equations (SDE) and the relationships between the diffusion coefficients used in SDE and the Allan variance, a typical tool used to estimate clock noise, are derived. This relationship is fundamental when a mathematical clock model is used, for example in Kalman filter, noise estimation, and clock prediction activities.**

I. Introduction

The clock errors can be suitably modeled by means of stochastic processes obeying to stochastic differential equations (SDE) [1]–[3], and this is of particular importance for the evaluation of the impact of clock noises on some complex systems, for the prediction and simulation of clock behavior, and for the insertion of clock data in some filtering technique (e.g., the Kalman filter). This is of particular interest for the satellite navigation systems, such as GPS [4] and the currently under project Galileo, for the evaluation and dimensioning of telecommunication networks, for sizing memory buffers, and for the metrological purpose of timekeeping and dissemination. In some cases it may be of interest to know the probability that a certain time after synchronization the clock error has exceeded a certain threshold of permissible error, and this quantity can be directly computed from a correct modeling of the clock errors by means of suitable SDE and stochastic processes [5]. The clock noises involved in the SDE are generally introduced through a covariance matrix that contains diffusion coefficients, which are, in mathematical language, the variances of the Wiener processes driving the fundamental noises [6], [7]. In the time and frequency community, conversely, the clock noises are in general characterized by means of variances evaluated on successive differences of frequency values. The most common definition of variance used in this context is the Allan variance [8]. But similar tools were proposed as the structure functions [9], the Hadamard variance [10] or the combined use of some of them [11], [12].

In order to use mathematical clock models, the knowledge of the relationships between such variances and the diffusion coefficients appearing in the covariance matrices is thus necessary. Some previous works faced this evaluation [2], [13] in particular contexts or by including some particular assumptions that simplify the calculations; in particular, in [2] a simplifying assumption was incorrectly taken leading to a false result that propagated in the literature. In this paper we present a general derivation of the relationships between the different measures of variance, which can be applied in a wide context. In our approach, we do not introduce any simplifying assumptions. The aim is to give a more general description of these relationships to allow mathematicians to follow the role of any variables in the clock models, and clock model users can extract the evaluations necessary to their application with the confidence of using a robust computation with general validity. In other works [14], [15] this same problem is addressed by another point of view evaluating the autocorrelation, variances, and cross correlation of the clock noises starting from the spectral analysis and using the appropriate transfer function that transforms a white noise in a colored noise. In that context also the flicker noise was considered. Because the flicker noise cannot be modeled by a finite order state model to be inserted in a SDE, we are not dealing with it in this work. Conversely, in [14], the flicker noise was treated by assuming its existence only on a certain frequency region and, therefore, an approximated evaluation was given.

In Section II we present a three state clock mathematical model described by means of a SDE that can be exactly solved. An iterative form for this SDE, of particular interest for the following calculation and for the applications, is also introduced. In Sections III and IV we present two alternative ways to relate the diffusion coefficients to the Allan or related variances, pointing out the more delicate points and some subcases that may be of interest to the users.

II. The Clock Model

We consider a three-state clock model as partly described and used in different cases [1]–[4], [13], [16]. It represents a generalization of the most common two-state model [3], [4], in which a random walk on the frequency drift, sometimes referred as random run [16], and a linear variation in time of the frequency drift are added. This additional noise and drift may be useful in modeling some types of clocks, but they represent a clock signal whose frequency value is not given by the integration of stationary increments and this will cause, as we will demonstrate, that in those cases the Allan variance will depend on the epoch t and not only on the observation time τ as usually assumed.

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The three-state clock model appears as:

$$\begin{cases} dX_1(t) = (X_2(t) + \mu_1)\,dt + \sigma_1 dW_1(t) \\ dX_2(t) = (X_3(t) + \mu_2)\,dt + \sigma_2 dW_2(t), \quad t \geq 0 \\ dX_3(t) = \mu_3 dt + \sigma_3 dW_3(t), \end{cases} \tag{1}$$

with initial conditions:

$$\begin{cases} X_1(0) = c_1 \\ X_2(0) = c_2 \\ X_3(0) = c_3 \end{cases},$$

where $\{W_i(t), t \geq 0\}$, $i = 1,2,3$ are three independent, one-dimensional standard Wiener processes (standard Brownian motion), each one defined as a Gaussian process with stationary independent increments such that $W(t) - W(s) \sim N(0, t-s)$ and $W(0) = 0$ [6]. This implies $W(t) \sim N(0,t)$. The expression $\sim N(0,t)$ indicates that the variable has a Gaussian distribution with zero mean and variance t; therefore, both the increments and the Wiener processes have Gaussian distribution as indicated. The Wiener process is often referred to as the integral of white noise $\xi(t)$ and is formally written as $dW(t) = \xi(t)dt$.

The variable X_1 represents the phase deviation, the second component X_2 is a part of the clock frequency deviation (i.e., what is generally called the random walk component), and X_3 represents part of the so-called frequency drift or aging. This last component may be given by a constant value or, as in this case, by a linear variation with time considered via μ_3 plus a random term given by $W_3(t)$. The white and random walk frequency noises, as named in time metrology, that give ~~false~~ rise to a Wiener and an integrated Wiener process on the clock phase, are represented by $W_1(t)$ and $W_2(t)$, respectively. The frequency deviation, often indicated by $y(t)$ in metrological literature [8], results to be $\dot{X}_1$ and not X_2, as erroneously referred to in some cases.

The constants σ_1, σ_1, and σ_3 represent the diffusion coefficients of the three noise components and give the intensity of each noise. The exact relationship between these diffusion coefficients and the Allan variance is the objective of our work.

The constants μ_1, μ_2, and μ_3 can be interpreted as drift terms for the three Wiener processes, but in the clock terminology they represent what is generally referred to as the deterministic phenomena driving the atomic clock errors. In particular, μ_1 is related to the constant initial frequency offset, often indicated by y_0, and μ_2 represents what is generally indicated by a and named frequency aging or drift. In more familiar metrological notations, we would have $c_2 + \mu_1 = y_0$ and $c_3 + \mu_2 = a$, and μ_3 is the linear coefficient of the time variation of the frequency drift a. We will return to the relationships between the mathematical and metrological notation later.

We can rewrite (1) in a matrix form:

$$d\mathbf{X}(t) = (\mathbf{F}\,\mathbf{X}(t) + \mathbf{M})dt + \mathbf{Q}\,d\mathbf{W}(t), \quad t \geq 0, \tag{2}$$

where $\mathbf{F}$ and $\mathbf{Q}$ are the following 3×3 matrices:

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \tag{3}$$

and $\mathbf{M}$, $d\mathbf{X}(t)$, $d\mathbf{W}(t)$ are:

$$\mathbf{M} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad d\mathbf{X}(t) = \begin{bmatrix} dX_1(t) \\ dX_2(t) \\ dX_3(t) \end{bmatrix}, \quad d\mathbf{W}(t) = \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix}. \tag{4}$$

Expression (2) is a strictly linear stochastic differential equation. Hence, it is possible to obtain the solution in a closed form ([6], [7]):

$$\mathbf{X}(t) = \mathbf{\Phi}_{t,0}\mathbf{X}(t_0) + \mathbf{B}_t\mathbf{M} + \mathbf{G}_t, \tag{5}$$

where the transition matrix $\mathbf{\Phi}_{t,0}$ and $\mathbf{B}_t$ are deterministic 3×3 matrices defined by:

$$\begin{aligned} \mathbf{\Phi}_{t,s} &= e^{\mathbf{F}(t-s)} = \mathbf{I} + \mathbf{F}(t-s) + \mathbf{F}^2\frac{(t-s)^2}{2} \\ &= \begin{bmatrix} 1 & t-s & \dfrac{(t-s)^2}{2} \\ 0 & 1 & t-s \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{6}$$

and

$$\mathbf{B}_t = \begin{bmatrix} t & \dfrac{t^2}{2} & \dfrac{t^3}{6} \\ 0 & t & \dfrac{t^2}{2} \\ 0 & 0 & t \end{bmatrix}, \tag{7}$$

where $\mathbf{I}$ is the identity matrix, the transition matrix $\mathbf{\Phi}_{t,0}$ satisfies the equation $\mathbf{\Phi}_{t,s} = \mathbf{\Phi}_{t-s,0}$, and the vector $\mathbf{G}_t$ is a 3×1 stochastic vector that represents the innovation of the stochastic process $\mathbf{X}(t)$:

$$\mathbf{G}_t = \begin{bmatrix} \sigma_1 W_1(t) + \sigma_2 \int_0^t (t-s)dW_2(s) + \sigma_3 \int_0^t \dfrac{(t-s)^2}{2} dW_3(s) \\ \sigma_2 W_2(t) + \sigma_3 \int_0^t (t-s)dW_3(s) \\ \sigma_3 W_3(t) \end{bmatrix} \sim N(\mathbf{0}, \mathbf{\Sigma}_t). \tag{8}$$

Hence, $\mathbf{G}_t$ is distributed as a Gaussian random vector with zero mean and covariance matrix $\mathbf{\Sigma}_t$, depending on the final time t:

$$\begin{aligned} \mathbf{\Sigma}_t &= E[\mathbf{G}_t\mathbf{G}_t^T] \\ &= \begin{bmatrix} \sigma_1^2 t + \sigma_2^2\dfrac{t^3}{3} + \sigma_3^2\dfrac{t^5}{20} & \sigma_2^2\dfrac{t^2}{2} + \sigma_3^2\dfrac{t^4}{8} & \sigma_3^2\dfrac{t^3}{6} \\ \sigma_2^2\dfrac{t^2}{2} + \sigma_3^2\dfrac{t^4}{8} & \sigma_2^2 t + \sigma_3^2\dfrac{t^3}{3} & \sigma_3^2\dfrac{t^2}{2} \\ \sigma_3^2\dfrac{t^3}{6} & \sigma_3^2\dfrac{t^2}{2} & \sigma_3^2 t \end{bmatrix}. \end{aligned} \tag{9}$$

where $E[.]$ represents the expectation value and the superscript T denotes the transposed matrix. Note that the only stochastic part of (5) is the vector $\mathbf{G}_t$. Because the solution $\mathbf{X}(t)$ is a linear combination of Gaussian processes, it can be written as a Gaussian process itself with mean:

$$E[\mathbf{X}(t)] = \begin{bmatrix} c_1 + (c_2+\mu_1)\,t + (c_3+\mu_2)\dfrac{t^2}{2} + \mu_3\dfrac{t^3}{6} \\ c_2 + (c_3+\mu_2)\,t + \mu_3\dfrac{t^2}{2} \\ c_3 + \mu_3 t \end{bmatrix}, \tag{10}$$

and covariance matrix $\mathbf{\Gamma_x}(t) = \mathbf{\Sigma}_t$.

We note that the covariance matrix is not affected by the presence of the drifts μ_1, μ_2, and μ_3 connected with the three Wiener processes W_1, W_2, and W_3.

To simplify the notation without losing generality in the following we set $\mu_1 = \mu_2 = 0$. With this hypothesis the solution (5) expressed by means of its scalar components becomes:

$$\begin{cases} X_1(t) = c_1 + c_2\,t + c_3\dfrac{t^2}{2} + \mu_3\dfrac{t^3}{6} + \sigma_1 W_1(t) \\ \quad +\sigma_2 \int_0^t (t-s)dW_2(s) + \sigma_3 \int_0^t \dfrac{(t-s)^2}{2} dW_3(s) \\ X_2(t) = c_2 + c_3\,t + \mu_3\dfrac{t^2}{2} + \sigma_2 W_2(t) \\ \quad +\sigma_3 \int_0^t (t-s)dW_3(s) \\ X_3(t) = c_3 + \mu_3 t + \sigma_3 W_3(t) \end{cases}, \tag{11}$$

where the integrals in X_1 and X_2 can be rewritten, integrating by parts, in the following equivalent ways:

$$\begin{aligned} \int_0^t (t-s)dW(s) &= \int_0^t W(s)ds, \\ \int_0^t \frac{(t-s)^2}{2} dW(s) &= \int_0^t (t-s)W(s)ds \\ &= \int_0^t \int_0^s W(v)dvds. \end{aligned} \tag{12}$$

By using a notation more familiar in the metrological literature, we can express the solution (11) as:

$$\begin{cases} X_1(t) = x(0) + y(0)t + a\dfrac{t^2}{2} + \mu_3\dfrac{t^3}{6} + \sigma_1 W_1(t) \\ \qquad + \sigma_2 \displaystyle\int_0^t (t-s)dW_2(s) + \sigma_3 \int_0^t \frac{(t-s)^2}{2} dW_3(s) \\ X_2(t) = c_2 + a\,t + \mu_3\dfrac{t^2}{2} + \sigma_2 W_2(t) \\ \qquad + \sigma_3 \displaystyle\int_0^t (t-s)dW_3(s) \\ X_3(t) = c_3 + \mu_3 t + \sigma_3 W_3(t) \end{cases}, \tag{13}$$

where we can recognize the initial phase and frequency values $x(0)$ and $y(0)$, the frequency drift a, plus the additional frequency drift linear variation coefficient μ_3. The Wiener noises have different integration order accordingly to their effect on the frequency drift, frequency, or phase.

It is possible to write the solution (5) in an iterative form by a discretization of the time t. Let us consider a fixed interval $[0, T]$ and an equally spaced partition $0 \equiv t_0 < t_1 < \cdots < t_N \equiv T$ and let us denote with $\tau = t_{k+1} - t_k$, for each $k = 0, 1, \ldots N-1$, the resulting discretization step. We can express the solution at time t_{k+1} in terms of the solution at time t_k in matrix form:

$$\mathbf{X}(t_{k+1}) = \mathbf{\Phi}_{t_{k+1},t_k}\mathbf{X}(t_k) + \mathbf{B}_\tau \mathbf{M} + \mathbf{J}_k, \tag{14}$$

that in components becomes:

$$\begin{cases} X_1(t_{k+1}) = X_1(t_k) + X_2(t_k)\tau + X_3(t_k)\dfrac{\tau^2}{2} \\ \quad +\mu_3\frac{\tau^3}{6} + J_{k,1} \\ X_2(t_{k+1}) = X_2(t_k) + X_3(t_k)\tau + \mu_3\dfrac{\tau^2}{2} + J_{k,2} \\ X_3(t_{k+1}) = X_3(t_k) + \mu_3\tau + J_{k,3} \end{cases} \tag{15}$$

where $J_{k,1}$, $J_{k,2}$, and $J_{k,3}$ are the three components of the vector $\mathbf{J}_k$ (see (16) next page), where the variance-covariance matrix $\mathbf{Q}$ is:

$$\mathbf{Q} = \begin{bmatrix} \sigma_1^2\tau + \sigma_2^2\dfrac{\tau^3}{3} + \sigma_3^2\dfrac{\tau^5}{20} & \sigma_2^2\dfrac{\tau^2}{2} + \sigma_3^2\dfrac{\tau^4}{8} & \sigma_3^2\dfrac{\tau^3}{6} \\ \sigma_2^2\dfrac{\tau^2}{2} + \sigma_3^2\dfrac{\tau^4}{8} & \sigma_2^2\tau + \sigma_3^2\dfrac{\tau^3}{3} & \sigma_3^2\dfrac{\tau^2}{2} \\ \sigma_3^2\dfrac{\tau^3}{6} & \sigma_3^2\dfrac{\tau^2}{2} & \sigma_3^2\tau \end{bmatrix}. \tag{17}$$

We note that the vector $\mathbf{J}_k$, which is called innovation, is a tridimensional Gaussian random variable whose covariance matrix $\mathbf{Q}$ is independent of the instant t_k.

Iterative type solution results are useful for further processing of the data as the definition of a time scale, or the use of auto-regressive integrated moving average (ARIMA) techniques, Kalman filtering or other signal processing methods or the evaluation and prediction of the clock behavior. The iterative solution also is useful for simulative purposes. In particular, the solution (15) allows to simulate the solution process in an exact and manageable way.

The iterative form (15) also will be useful in successive sections, in which the Allan variance is considered.

III. The Clock Model and Allan Variance

In this section we determine the relation between the diffusion coefficients characterizing the process $\mathbf{X}(t)$ and the Allan variance. The Allan variance is defined as [8]:

$$\begin{aligned} \sigma_y^2(\tau) &= \frac{1}{2}E\left[\left(\overline{Y}_{k+1} - \overline{Y}_k\right)^2\right] \\ &= \frac{1}{2}\left(E\left[\overline{Y}_{k+1}^2\right] + E\left[\overline{Y}_k^2\right] - 2E\left[\overline{Y}_{k+1}\overline{Y}_k\right]\right), \end{aligned} \tag{18}$$

where

$$\overline{Y}_k = \frac{1}{\tau}\int_{t_k}^{t_{k+1}} y(t)dt, \tag{19}$$

here $y(t)$ represents the instantaneous value of the fractional frequency deviation that in our notation is given by $\dot{X}_1$. Actually, the usual definition of the Allan variance

$$\mathbf{J}_k = \begin{bmatrix} \Big[\sigma_1 \left(W_1\left(t_{k+1}\right) - W_1\left(t_k\right)\right) + \sigma_2 \int_{t_k}^{t_{k+1}} \left(W_2(s) - W_2\left(t_k\right)\right) ds \\ +\sigma_3 \int_{t_k}^{t_{k+1}} \left(t_{k+1} - s\right)\left(W_3(s) - W_3\left(t_k\right)\right) ds\Big] \\ \sigma_2 \left(W_2\left(t_{k+1}\right) - W_2\left(t_k\right)\right) + \sigma_3 \int_{t_k}^{t_{k+1}} \left(W_3(s) - W_3\left(t_k\right)\right) ds \\ \sigma_3 \left(W_3\left(t_{k+1}\right) - W_3\left(t_k\right)\right) \end{bmatrix} \sim N(\mathbf{0}, \mathbf{Q}) \tag{16}$$

in (18) contains the time average $< \cdots >$ instead of the expectation value $E[.]$ [8], because it is, in general, evaluated on signal $y(t)$ with stationary increments. Because this is not always the case (as in this three-state model), we use the definition with the expectation value E for a more general treatment.

It is worth mentioning that the relationship between the Allan variance and the diffusion coefficients can be derived using two different expressions of the Allan variance, containing the average frequency or the time offset, respectively. We present both methods: the first one, that follows immediately, is the most demanding in terms of calculation, but it allows one to trace step by step the integrated effect of the different noises, and it was the method followed in [2] from which we want to outline our controversy. The second approach, followed in [13], will be delineated in Section IV.

In our notation (19) is equivalent to:

$$\overline{Y}_k = \frac{1}{\tau} \int_{t_k}^{t_{k+1}} \dot{X}_1(t) dt, \tag{20}$$

where:

$$\dot{X}_1(t) = \frac{dX_1}{dt} = c_2 + c_3\, t + \mu_3 \frac{t^2}{2} + \sigma_1 \frac{dW_1(t)}{dt} + \sigma_2 W_2(t) + \sigma_3 \int_0^t (t-s) dW_3(s), \tag{21}$$

is the derivative of the phase component.

To have an evaluation of (18), we are interested in the following stochastic quantity:

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \dot{X}_1(t)dt &= \int_{t_k}^{t_{k+1}} \left(c_2 + c_3\, t + \mu_3 \frac{t^2}{2}\right) dt \\ &+ \sigma_1 \int_{t_k}^{t_{k+1}} dW_1(t) + \sigma_2 \int_{t_k}^{t_{k+1}} W_2(t)dt \\ &+ \sigma_3 \int_{t_k}^{t_{k+1}} \left(\int_0^t (t-s)dW_3(s)\right) dt \\ &= c_2\tau + c_3 \frac{t_{k+1}^2 - t_k^2}{2} + \mu_3 \frac{t_{k+1}^3 - t_k^3}{6} \\ &+ \sigma_1 \left(W_1\left(t_{k+1}\right) - W_1\left(t_k\right)\right) \\ &+ \sigma_2 \int_{t_k}^{t_{k+1}} W_2(t)dt \\ &+ \sigma_3 \int_{t_k}^{t_{k+1}} \left(\int_0^t (t-s)dW_3(s)\right) dt, \end{aligned} \tag{22}$$

remembering that $\tau = t_{k+1} - t_k$. Because (22) is a linear combination of Gaussian random variables, it is distributed as a Gaussian random variable with mean:

$$E\left(\int_{t_k}^{t_{k+1}} \dot{X}_1(t)dt\right) = c_2\tau + c_3 \frac{t_{k+1}^2 - t_k^2}{2} + \mu_3 \frac{t_{k+1}^3 - t_k^3}{6}. \tag{23}$$

The three noises W_1, W_2, and W_3 that affect the process are independent. Hence, the variance of the process can be obtained as the sum of the following variances:

$$\begin{aligned} Var\left(\sigma_1\left(W_1\left(t_{k+1}\right) - W_1\left(t_k\right)\right)\right) &= \sigma_1^2 Var\big(W_1\left(t_{k+1}\right) \\ &\quad - W_1\left(t_k\right)\big) \\ &= \sigma_1^2\tau, \end{aligned} \tag{24}$$

$$\begin{aligned} Var\left(\sigma_2 \int_{t_k}^{t_{k+1}} W_2(t)dt\right) &= \sigma_2^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} (s \wedge t) ds dt \\ &= \sigma_2^2 \left(\frac{\tau^3}{3} + t_k\tau^2\right), \end{aligned} \tag{25}$$

obtained remembering that:

$$E\left(W(t)W(s)\right) = s \wedge t \equiv \min(s, t), \tag{26}$$

and

$$\begin{aligned} &Var\left(\sigma_3 \int_{t_k}^{t_{k+1}} \left(\int_0^t (t-u)dW_3(u)\right) dt\right) \\ &= \sigma_3^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \left(\int_0^t \int_0^s (t-u)(s-v)\delta(u-v)dvdu\right) dsdt \\ &= \sigma_3^2 \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_0^{t\wedge s} (t-u)(s-u)dudsdt \\ &= \sigma_3^2 \left(\frac{\tau^5}{20} + \frac{\tau^4 t_k}{4} + \frac{\tau^3 t_k^2}{2} + \frac{\tau^2 t_k^3}{3}\right), \end{aligned} \tag{27}$$

since:

$$E[dW(u)dW(v)] = \delta(u-v)dudv, \tag{28}$$

where $\delta(.)$ is a Dirac delta function.

In this way we obtain:

$$\begin{aligned} Var\left[\int_{t_k}^{t_{k+1}} \dot{X}_1(t)dt\right] &= \sigma_1^2\tau + \sigma_2^2 \left(\frac{\tau^3}{3} + t_k\tau^2\right) \\ &+ \sigma_3^2 \left(\frac{\tau^5}{20} + \frac{\tau^4 t_k}{4} + \frac{\tau^3 t_k^2}{2} + \frac{\tau^2 t_k^3}{3}\right). \end{aligned} \tag{29}$$

From (29) we notice that the expression for the variance does not depend only on the integration interval τ, as it is dependent on the endpoints of the interval $[t_k, t_{k+1}]$. Then, we can conclude that the integrals of $\dot{X}_1\left(t_k\right)$ (i.e.,

$\overline{Y}_k$, for each t_k), form a sequence of neither independent nor identically distributed Gaussian random variables.

These particular features and their correct treatment without introducing simplifying assumptions allow the general evaluation of the relationship between Allan variance and diffusion coefficient that was not possible in [2], and this motivates our controversy and the demanding computation reported here.

From (23) and (29) we get that the quantity:

$$\begin{aligned}\overline{Y}_k &= \frac{1}{\tau}\int_{t_k}^{t_{k+1}} \dot{X}_1(t)dt \\ &= c_2 + c_3\frac{t_{k+1}^2 - t_k^2}{2\tau} + \mu_3\frac{t_{k+1}^3 - t_k^3}{6\tau} \\ &\quad + \frac{\sigma_1}{\tau}\left(W_1\left(t_{k+1}\right) - W_1\left(t_k\right)\right) + \frac{\sigma_2}{\tau}\int_{t_k}^{t_{k+1}} W_2(t)dt \\ &\quad + \frac{\sigma_3}{\tau}\int_{t_k}^{t_{k+1}}\left(\int_0^t (t-s)dW_3(s)\right)dt,\end{aligned} \tag{30}$$

is Gaussian with mean and variance given by:

$$\begin{aligned}E\left(\overline{Y}_k\right) &= c_2 + c_3\frac{t_{k+1}^2 - t_k^2}{2\tau} + \mu_3\frac{t_{k+1}^3 - t_k^3}{6\tau}, \\ Var\left(\overline{Y}_k\right) &= \frac{\sigma_1^2}{\tau} + \sigma_2^2\left(\frac{\tau}{3} + t_k\right) \\ &\quad + \sigma_3^2\left(\frac{\tau^3}{20} + \frac{\tau^2 t_k}{4} + \frac{\tau\, t_k^2}{2} + \frac{t_k^3}{3}\right),\end{aligned} \tag{31}$$

respectively.

Remembering that $E\left(Y^2\right) = Var(Y) + (E(Y))^2$, by using (31), we obtain $E\left(\overline{Y}_k^2\right)$ and $E\left(\overline{Y}_{k+1}^2\right)$, to be used for the evaluation of (18).

Using (30) and remembering the independency of the three noises W_1, W_2, and W_3, we obtain:

$$\begin{aligned}E\left(\overline{Y}_k\overline{Y}_{k+1}\right) &= \left[c_2 + c_3\frac{t_{k+1}^2 - t_k^2}{2\tau} + \mu_3\frac{t_{k+1}^3 - t_k^3}{6\tau}\right] \\ &\quad\cdot\left[c_2 + c_3\frac{t_{k+2}^2 - t_{k+1}^2}{2\tau} + \mu_3\frac{t_{t+2}^3 - t_{k+1}^3}{6\tau}\right] \\ &\quad + E\left(\frac{\sigma_2^2}{\tau^2}\int_{t_k}^{t_{k+1}} W_2(t)dt\int_{t_k+1}^{t_{k+2}} W_2(s)ds\right) \\ &\quad + E\Bigg(\frac{\sigma_3^2}{\tau^2}\int_{t_k}^{t_{k+1}}\int_0^t (t-u)dW_3(u)dt \\ &\quad\cdot\int_{t_k+1}^{t_{k+2}}\int_0^s (s-v)dW_3(v)ds\Bigg),\end{aligned} \tag{32}$$

that gives:

$$\begin{aligned}E\left(\overline{Y}_k\overline{Y}_{k+1}\right) &= \left[c_2 + c_3\frac{t_{k+1}^2 - t_k^2}{2\tau} + \mu_3\frac{t_{k+1}^3 - t_k^3}{6\tau}\right] \\ &\quad\cdot\left[c_2 + c_3\frac{t_{k+2}^2 - t_{k+1}^2}{2\tau} + \mu_3\frac{t_{k+2}^3 - t_{k+1}^3}{6\tau}\right] \\ &\quad + \sigma_2^2\frac{\left(t_k + t_{k+1}\right)}{2} \\ &\quad + \sigma_3^2\left(\frac{5\tau^3}{24} + \frac{3\tau^2 t_k}{4} + \tau\, t_k^2 + \frac{t_k^3}{3}\right),\end{aligned} \tag{33}$$

obtained using relations (26) and (28).

If we compute the quantity:

$$\frac{1}{2}\left(E\left(\overline{Y}_{k+1}^2\right) + E\left(\overline{Y}_k^2\right) - 2E\left(\overline{Y}_{k+1}\overline{Y}_k\right)\right) \tag{34}$$

that appears in the Allan variance (18), we obtain a time dependent Allan variance, named $\sigma_y^2(t,\tau)$ given by:

$$\begin{aligned}\sigma_y^2\left(t_k,\tau\right) &= \frac{1}{2}\left(E\left(\overline{Y}_{k+1}^2\right) + E\left(\overline{Y}_k^2\right) - 2E\left(\overline{Y}_{k+1}\overline{Y}_k\right)\right) \\ &= \left(\frac{\sigma_1^2}{\tau} + \frac{\sigma_2^2\tau}{3} + \frac{\sigma_3^2\tau^3}{20}\right) + \sigma_3^2\left(\frac{\tau^3}{3} + \frac{\tau^2 t_k}{2}\right) \\ &\quad + \frac{\tau^2}{2}\left[c_3 + \mu_3\left(\tau + t_k\right)\right]^2,\end{aligned} \tag{35}$$

where the dependence on t_k clearly appears.

If $\mu_3 = \sigma_3 = 0$ (i.e., the frequency drift is a constant), we get the following usual expression for the Allan variance:

$$\sigma_y^2(\tau) = \frac{\sigma_1^2}{\tau} + \frac{\sigma_2^2\tau}{3} + \frac{\tau^2}{2}c_3^2, \tag{36}$$

that depends only on the time increment τ and is in agreement with [2]. The last term describes the known effect of the frequency drift a by recalling that the following relationship holds $a = c_3$.

However, if we add a third state to the clock model to account for a possible random walk on the frequency drift (random run) or a possible linear variation of it, we obtain that the Allan variance depends not only on τ but also on t_k. The presence of stochastic or deterministic variation on the frequency drift is not common on the experimentally observed clock data. Nevertheless, because a generalization to a third state seems in same cases interesting—as for example to allow the modelization of flicker noise by means of a linear combination of Markov processes [17] or for other purposes—here we point out that, in those cases, the involved noise and drift terms describe a clock signal with a frequency not obtained by the accumulation of stationary increments. Therefore, the Allan variance (acting on frequency increments) loses one of its interesting peculiarities of depending only on τ and not on t_k, and the relationship with the diffusion coefficients should be carefully examined.

Recently, the definition of a time-dependent Allan variance to be used on nonstationary signal has been proposed [18], and it is in agreement with the use we introduced in (35).

[18] L. Galleani and P. Tavella, "The characterization of clock behavior with the dynamic Allan variance," in *Proc. Joint Meeting Eur. Freq. Time Forum and IEEE Freq. Contr. Symp.*, May 2003, pp. 239–244.

IV. Another Approach

Another approach [13] is useful and simpler when one is interested in determining the relationship between the Allan variance and the diffusion coefficient.

Remembering that the average frequency deviation may be written in terms of phase deviation as:

$$\overline{Y}_k = \frac{1}{\tau}\int_{t_k}^{t_{k+1}} \dot{X}_1(t)dt = \frac{1}{\tau}\left(X_1\left(t_{k+1}\right) - X_1\left(t_k\right)\right), \tag{37}$$

we can write the Allan variance in the following form:

$$\begin{aligned}\sigma_y^2(\tau) = \frac{1}{2\tau^2}E\Big[\Big(&\left(X_1\left(t_{k+2}\right) - X_1\left(t_{k+1}\right)\right)\\ &- \left(X_1\left(t_{k+1}\right) - X_1\left(t_k\right)\right)\Big)^2\Big].\end{aligned} \tag{38}$$

Using the iterative solution (15) we get:

$$\begin{aligned}\Delta &= \left(X_1\left(t_{k+2}\right) - X_1\left(t_{k+1}\right)\right) - \left(X_1\left(t_{k+1}\right) - X_1\left(t_k\right)\right)\\ &= \tau^3\mu_3 + \tau^2 X_3\left(t_k\right) + J_{k+1,1}\\ &\quad + \left[-J_{k,1} + \tau J_{k,2} + \frac{\tau^2}{2}J_{k,3}\right],\end{aligned} \tag{39}$$

where $J_{k,1}$, $J_{k,2}$, and $J_{k,3}$ are the three components of the innovation vector $\mathbf{J}_k$ defined in Section II. The quantity Δ is a Gaussian random variable with mean:

$$E[\Delta] = \tau^2\left[c_3 + \mu_3\left(t_k + \tau\right)\right]. \tag{40}$$

and variance:

$$Var[\Delta] = 2\sigma_1^2\tau + 2\sigma_2^2\frac{\tau^3}{3} + \sigma_3^2\left(\frac{23}{30}\tau^5 + \tau^4 t_k\right), \tag{41}$$

which can be obtained by observing that the stochastic quantities involved in Δ are independent and with the following variances:

$$\begin{aligned}Var\left[X_3\left(t_k\right)\right] &= \sigma_3^2 t_k,\\ Var\left[J_{k+1,1}\right] &= \sigma_1^2\tau + \sigma_2^2\frac{\tau^3}{3} + \sigma_3^2\frac{\tau^5}{20},\\ Var\left[-J_{k,1} + \tau J_{k,2} + \frac{\tau^2}{2}J_{k,3}\right] &= \left[-1\ \tau\ \frac{\tau^2}{2}\right]\mathbf{Q}\begin{bmatrix}-1\\ \tau\\ \frac{\tau^2}{2}\end{bmatrix}\\ &= \sigma_1^2\tau + \sigma_2^2\frac{\tau^3}{3} + \sigma_3^2\frac{43}{60}\tau^5,\end{aligned} \tag{42}$$

where $\mathbf{Q}$ is the variance-covariance matrix of the vector $\mathbf{J}_k$.

We then get that the time-dependent Allan variance may be written as:

$$\begin{aligned}\sigma_y^2\left(t_k,\tau\right) &= \frac{1}{2\tau^2}E\left[\Delta^2\right] = \frac{1}{2\tau^2}\left(Var[\Delta] + E[\Delta]^2\right)\\ &= \left(\frac{\sigma_1^2}{\tau} + \frac{\sigma_2^2\tau}{3} + \frac{\sigma_3^2\tau^3}{20}\right) + \sigma_3^2\left(\frac{\tau^3}{3} + \frac{\tau^2 t_k}{2}\right)\\ &\quad + \frac{\tau^2}{2}\left[c_3 + \mu_3\left(\tau + t_k\right)\right]^2\end{aligned} \tag{43}$$

according with the previous result (35). We notice that this alternative approach is more direct and requires less computation.

This method can be extended easily to other kinds of variances, for example, to the Hadamard variance that is defined as [10]:

$$\begin{aligned}{}_H\sigma_y^2(\tau) = \frac{1}{6\tau^2}E\Big[\Big(&\left(X_1\left(t_{k+3}\right) - X_1\left(t_{k+2}\right)\right)\\ - 2\left(X_1\left(t_{k+2}\right) - X_1\left(t_{k+1}\right)\right) &+ \left(X_1\left(t_{k+1}\right) - X_1\left(t_k\right)\right)\Big)^2\Big].\end{aligned} \tag{44}$$

Proceeding with a computation similar to the case of the Allan variance, we get:

$${}_H\sigma_y^2(\tau) = \frac{\sigma_1^2}{\tau} + \frac{\sigma_2^2\tau}{6} + \frac{11}{120}\sigma_3^2\tau^3 + \frac{\mu_3^2\tau^4}{6}, \tag{45}$$

that is in complete agreement with the result found in [13], but it is generalized to the cases of three noises and drifts μ_1, μ_2, and μ_3. We notice that Hadamard variance depends only on τ, even if we consider the complete three-state model. Actually, it is known that the Hadamard variance, using a higher order difference on the frequency values, is insensitive to the frequency drift given by μ_2.

The evaluation of the Hadamard variance could proceed with the longer approach presented in Section III leading to the same result (45).

The two presented methods also can be applied to other more general structure functions [9] or to a different kind of combined variances as in the multivariance analysis [11], [12].

V. Practical Application

In different contexts, a mathematical model of the clock errors is necessary. We quoted [1], [4], [13], [14], [16] as the most known applications, in which, above all, the use of a Kalman filter is addressed. But a clock model and its iterative solution (15) may be used also for simulation purposes, and for estimation and prediction purposes in more wide system such as, for example, the orbit and clock estimation algorithm of a navigation system [15], [19].

In all these cases, the clock noise parameters to be inserted in the covariance matrix $\mathbf{Q}$ (3) are necessary; and, in general, one knows the noise characterization in terms of Allan variance.

The first point to address is the identification of the dominant type of noise. This is possible by observing the slope of the Allan variance plot with respect to the observation interval τ [8]. The identification of one or more dominant noises allows the sizing and identification of the corresponding $\mathbf{Q}$ matrix and state model (1). Then the values of the noise parameters in $\mathbf{Q}$ are to be estimated. To make an example, let us suppose we have to deal only with white frequency noise related to the σ_1 parameter in the matrix $\mathbf{Q}$. From (36), without any other noises and frequency drift, we obtain that the relationship between the Allan variance and σ_1 is given by:

$$\sigma_y^2(\tau) = \frac{\sigma_1}{\tau}. \tag{46}$$

This is a simple situation. From the estimated Allan variance at a certain interval τ, we can easily infer σ_1 inverting (46).

Let us suppose we are dealing with a clock affected by two noises: white frequency and random walk frequency, without frequency drift (i.e., $c_3 = 0$) and all the other noise parameters are zero except σ_1 and σ_2. The relationship to be used from (36) becomes:

$$\sigma_y^2(\tau) = \frac{\sigma_1}{\tau} + \frac{\sigma_2^2 \tau}{3}. \tag{47}$$

The Allan variance results from the sum of the two noises. From Allan variance plot we can identify a region in which each noise is dominant; for example, in the case of a Cesium clock, the white frequency noise may be dominant for observation interval from 1 second to 1 day. Let us indicate with $\sigma_y^{2^{WF}}(\tau_1)$ the Allan deviation value mostly due to white frequency noise at the observation interval τ_1, (46) can be applied to estimate σ_1. Similarly, on another observation interval τ_2 for which the random walk frequency noise is dominant, the σ_2 parameter may be estimated by inverting the relationship:

$$\sigma_y^{2RWF}(\tau_2) = \frac{\sigma_2^2 \tau_2}{3}, \tag{48}$$

with analogous notation. In some other cases, some noise may not be clearly appearing, the Allan Variance estimates then can be used to infer the superior limit to the value σ_1 or σ_2. In the case of additional noise and drift components, the complete expression (35) should be used.

VI. Conclusions

By the explicit writing of the complete three-state clock model by means of stochastic differential equations, it was possible to infer the exact relationship between the Allan variance and the diffusion coefficients describing the white frequency, random walk frequency, and random walk frequency drift (random run) noises affecting a clock signal. The computation was carried out by also using deterministic drifts on the clock phase up to the third order. Two computation procedures are described that also can be used for evaluating similar relationships for other Allan-type variances of particular interest in different domains of time and frequency applications. The obtained relationships can be used in the different applications in which a clock model is necessary to exactly deal with the clock noises usually estimated in terms of Allan variances.

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