- without constraint:

$$T = \frac{1}{2}\sum_i m_i \dot{q}_i^2 \implies T = \frac{1}{2}\dot{\vec{q}} \cdot M\dot{\vec{q}} \qquad M^i{}_j = \delta_{ij} m_j$$

$$m_i \ddot{q}_i = \underset{\text{real force}}{Q_i} \iff M\ddot{\vec{q}} = \vec{Q} \iff \ddot{\vec{q}} = \underset{M^{-1}}{W}\vec{Q}$$

constraint: $\vec{C}(\vec{q}) = \dot{\vec{C}}(\vec{q}) \overset{!}{=} \vec{0}$ (1)

find constraint force $\hat{Q}$ that guarantees $\ddot{\vec{C}} = 0$ when added to real force

$$\dot{\vec{a}}(\vec{b}) = \underbrace{\left(\frac{\partial a^i}{\partial b_j}\right)^i{}_j}_{=:J} \dot{\vec{b}}$$

$$\dot{\vec{C}}(\vec{q}) = J\dot{\vec{q}} \overset{!}{=} \vec{0}$$

$$\ddot{\vec{C}}(\vec{q}) = \dot{J}\dot{\vec{q}} + J\ddot{\vec{q}} = \dot{J}\dot{\vec{q}} + J\,\overbrace{W(\vec{Q} + \hat{Q})}^{\text{acceleration caused by real + constraint force}}$$

$$\dot{J}\vec{\dot{q}} := \left(\sum_{ijk} \frac{\partial^2 C^i}{\partial q_j \partial q_k} e^i \otimes e_j \otimes e_k\right)(\cdot, \vec{\dot{q}}, \vec{\dot{q}})$$

$$(2) \quad \ddot{\vec{C}}(\vec{q}) = \dot{J}\vec{\dot{q}} + J\,W(\vec{Q} + \hat{Q})$$

$$(3) \quad \dot{\vec{C}} = 0 \Rightarrow J\vec{\dot{q}} \overset{(1)}{=} 0$$

(4) virtual work

conditions

$$\dot{T} = \sum_i m_i \dot{q}_i \ddot{q}_i = (M\vec{\ddot{q}})\vec{\dot{q}} = MW(\vec{Q} + \hat{Q})\vec{\dot{q}}$$

$$= \vec{Q}\vec{\dot{q}} + \hat{Q}\vec{\dot{q}} \overset{(4)}{\Longrightarrow} \hat{Q}\vec{\dot{q}} = \vec{0}$$

$$\Rightarrow J\vec{\dot{q}} = 0 \text{ and } \hat{Q}\vec{\dot{q}} = 0$$

$$J^i_j = \frac{\partial C^i}{\partial q^j} \qquad (J^T)^i_j = \frac{\partial C^j}{\partial q^i} \Rightarrow \vec{J}^T_j = \nabla_{\vec{q}} C^j$$

$\Rightarrow$ the vectors $\vec{J}^T_j$ are normal vectors of the hypersurface $\vec{C} = \text{const.}$

$\Rightarrow$ any legal velocity must not have any component in direction of any $\vec{J}^T_j$

$\Rightarrow$ constraint forces prohibit development of components in $\vec{J}^T_j$ direction

$\Rightarrow$ (5) $\hat{Q} = J^T \vec{\lambda}$ ← Lagrange-Multiplier

- combat numerical drift with spring and dampening:

$$\ddot{\vec{C}} = -k_s \vec{C} - k_d \dot{\vec{C}}$$

$$\overset{(2)}{\Leftrightarrow} -k_s \vec{C} - k_d \dot{\vec{C}} = \dot{\vec{J}} \dot{\vec{q}} + J W (\vec{Q} + \hat{Q})$$

$$\Leftrightarrow \quad J W \hat{Q} = -\dot{\vec{J}} \dot{\vec{q}} - J W \vec{Q} - k_s \vec{C} - k_d \dot{\vec{C}}$$

$$\overset{(5)}{\Leftrightarrow} \boxed{J W J^T \vec{\lambda} = -\dot{\vec{J}} \dot{\vec{q}} - J W \vec{Q} - k_s \vec{C} - k_d \dot{\vec{C}}}$$

solve for $\vec{\lambda}$